Cavity solitons and localized patterns in a finite-size optical cavity

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In appropriate ranges of parameters, laser-driven nonlinear optical cavities can support a wide variety of optical patterns, which could be used to carry information. The intensity peaks appearing in these patterns are called cavity solitons and are individually addressable. Using the Lugiato-Lefever equation to model a perfectly homogeneous cavity, we show that cavity solitons can only be located at discrete points and at a minimal distance from the edges. Other localized states which are attached to the edges are identified. By interpreting these patterns in an information coding frame, the information capacity of this dynamical system is evaluated. The results are explained analytically in terms of the tail characteristics of the cavity solitons. Finally, the influence of boundaries and of cavity imperfections on cavity solitons are compared.

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I. INTRODUCTION

Cavity solitons (CSs) are dissipative localized optical structures that develop in the transverse plane of nonlinear optical cavities. Most often, they result from a subcritical Turing bifurcation [1,2], which makes them similar to localized structures appearing in reaction-diffusion systems [3], electric discharge systems [4], magnetic fluids [5], solid and fluid mechanics [6,7], and other contexts. Many optical setups have demonstrated the existence of CSs experimentally, each time with refined precision [8–12], but from a practical point of view, one of the most interesting configurations is the semiconductor cavity. Indeed, thanks to the fast optical response of semiconductor materials, CSs can be envisaged as carriers of optical information for telecommunication purposes [13]. Another promising setting in this regard is the fiber cavity, where CSs are localized in time rather than space [14]. When CSs develop in the transverse plane of a cavity, there is the notion that they can be switched on and off at arbitrary places, and that their localized character makes them immune to boundary effects. Moreover, if it were not for unavoidable spatial imperfections which attract and trap them, they would stay forever where they were excited [11,13,15]. This would be true if the system were infinite.

Recent results suggest that this might not be the case anymore when the cavity size is finite, even if it is much larger than the CS width [16]. In fact, we show here that boundaries can still have a strong influence on CSs in that case. Mainly, they drive CSs to a set of discrete locations that differ from where they were generated, even in the absence of any imperfection. This sets a fundamental limit on the number of binary digits that could be encoded with CSs, which is the information capacity of the system.

As we will show, the physics behind this phenomenon involves two dynamical effects: the tail interaction between CSs and the pinning effect. The former effect was described in rather general terms for dissipative systems that admit localized structures and which are translation invariant [17,18]. The latter is associated to the existence of two spatial scales, which become infinitely separated in some limit. It was conjectured in [19] but was explained only later through phase space [20–22] and asymptotic studies [23,24]. The pinning effect is responsible for the locking of fronts between a homogeneous solution and a spatially periodic pattern over a finite range of parameters. This, in turn, is closely associated to the snaking bifurcation diagram that is typical of localized patterns and CSs which are currently actively studied in one dimension (1D) [25–27] and two dimensions (2D) [28,29].

This paper is organized as follows: The mathematical model is presented in Sec. II and studied numerically in Sec. III. In Sec. IV, we bring out some scaling laws associated to the location and the dynamics of CSs, which allows us to explain the numerical observations in the previous section. Since imperfections are always present in an optical cavity, we compare the pinning effects of background noise and boundaries in Sec. V. Finally, we conclude.

II. CAVITY MODEL

The model we study is the Lugiato-Lefever model [1] for the intracavity field $E$. In dimensionless form, it reads

$$\frac{\partial E}{\partial t} = E_I - (1 + i \theta - |E|^2)E + \frac{i}{2} \frac{\partial^2 E}{\partial x^2} - i E \frac{|E|^2}{\Gamma} ,$$

where the two control parameters are the injection-field amplitude $E_I$ and the cavity detuning $\theta$. The nonlinearity in this equation is of the Kerr type, whereby the light intensity slightly raises the index of refraction. Many variants of the Lugiato-Lefever model exist, each with a different nonlinear term. It should be pointed out that the semiconductor nonlinearity is slightly more complicated than in Eq. (1) [15]. However, it is well known that, qualitatively, the dynamics depends little on the details of the nonlinearity; see, for instance, [30]. What is necessary for the present discussion is to have a subcritical Turing instability. At this instability, branches of CS solutions bifurcate from the homogeneous solution. The homogeneous steady state, $E = E_s$, of (1) is implicitly given in terms of $I = |E_s|^2$ by

$$E_I = \sqrt{I} \sqrt{1 + (\theta - I)^2} , \quad E_s = \frac{E_I (I)}{1 + i (\theta - I)} .$$

and it is customary to replace the driving amplitude $E_I$ by the reference intracavity intensity $I$ as a control parameter. $I$ is the background intensity around which cavity solitons develop. At
the edges of the cavity, several choices of boundary conditions can be envisaged. One possibility is to assume that $E$ vanishes there. However, since $E = 0$ is not a solution of (1); this is likely to induce boundary layers in the intensity profile. Here, we intend to show that the influence of boundaries on the CS does not rely on the existence of such boundary layers. For this reason, we assume that $E$ is given by (2) at $x = \pm \Gamma/2$.

III. NUMERICAL STUDY

We study Eq. (1) numerically in domains with parameter values that are known to yield localized patterns if the cavity is infinitely broad [30]. As suggested by experiments [12], we choose a domain size $\Gamma$ close to 90 (i.e., large enough to contain ten CSs). To obtain a one-peak CS, we add a localized disturbance to the homogeneous solution (2) and integrate (1) in time until a stable steady state is reached. Figure 1(a) shows one such CS. It consists of a main peak and a set of attenuated maxima that make up an oscillatory tail. As the intensity oscillations become small, the tail assumes the simple form

$$|E|^2 - I \propto e^{\pm x/\mu} \cos(2\pi x/\lambda + \text{const.}).$$

(3)

The wavelength $\lambda$ and the attenuation length $\mu$ above are closely associated to the Turing bifurcation. Indeed, from the figure, we estimate that $\lambda \approx 8.99$ and $\mu \approx 4.92$. This is consistent with the linear stability analysis near the Turing bifurcation point $I = 1$, which predicts that $\lambda \sim 2\pi/\sqrt{\lambda - \theta}$ and $\mu \sim \sqrt{4 - 2\theta}/\sqrt{1 - \theta}$. In all the figures, we keep the same values of $I$ and $\theta$ so as to preserve the values of $\lambda$ and $\mu$. To locate the CS, we compute

$$\langle x \rangle = \frac{\int x|E(x)|^2 - I dx}{\int |E(x)|^2 - I dx}.$$

(4)

We start our investigation with one-peak CSs. To this end, we compute the temporal evolution of a large number of initial conditions where the steady state (2) is locally perturbed. More precisely, we take

$$E(x,0) = E_s \left[1 + \frac{1}{2} i \frac{2\pi x}{\lambda} + \phi_0 \right] \cosh\left(\frac{x - x_0}{\lambda}\right),$$

where $\phi_0$ and $x_0$ are free parameters. The complex factor in front of the perturbation is suggested by the linear stability analysis [31]. Such a localized perturbation rapidly leads to a one-peak CS. which then slowly drifts. At each time step, the motion is monitored by $\langle x \rangle$, defined above. As can be seen in Fig. 1(b), all trajectories are attracted to a finite set of points. The separation between those is very close to $\lambda/2$, which is also the width of their basin of attraction. In fact, a Newton-Raphson resolution of the time-independent Lugiato-Lefever equation reveals that stationary solutions exist at approximately every multiple of $\lambda/4$ from the outermost solution toward the center of the domain, but half of them are unstable. The temporal scales in Fig. 1(b) indicate that the dynamics is very slow. The time to reach a stable equilibrium increases exponentially with the distance to the boundary. Indeed, this motion is so slow that one could easily—and mistakenly—neglect it.

The interest of CSs as information-carrying objects is demonstrated in Fig. 2. The various stable patterns shown demonstrate that peaks can potentially be written at any of the stable positions of Fig. 1(b). One restriction, however, is that peaks should be an integer number of $\lambda$ apart. On the other hand, a given aggregate of CSs can be shifted as a whole by $\lambda/2$; compare Figs. 2(b) and 2(e).

Figure 1(b) also indicates that initial disturbances that are too close to the boundary do not develop into stable, one-peak CSs. Instead, satellite peaks grow on the exterior side, eventually leading to a localized pattern that fully overlaps with the boundary. These patterns typically consist of more than one peak; see Fig. 3. However, contrary to localized patterns made of CSs, the peaks inside such a structure cannot be switched.

FIG. 1. (a) One-peak CS computed for $I = 0.964$, $\theta = 1.5$, and $\Gamma = 88.86$. $\lambda$ and $\mu$ are the wavelength and attenuation length, respectively, of the oscillatory tail of the CS. (b) Trajectories of one-peak CS.

FIG. 2. Examples of multiple CS configurations. Same parameters as in Fig. 1.
on and off at will: only the one that is closest to the center can be added or removed. For instance, we were unable to obtain a solution as in Fig. 3(b) with the middle peak removed. Hence these states appear to be fundamentally different from the usual CSs and we call them “edges states.” While CSs can in good part be explained in dynamical-system terms when the domain $[-\Gamma/2, \Gamma/2]$ becomes the real line, edge states clearly cannot. A full dynamical description of “edge state” is outside the scope of this paper; here, we note that the diversity of configurations accessible with these states is much less than with CSs. Indeed, the three possibilities shown in Fig. 3 could be replaced by $2^3$ if the peaks were usual CSs. Note that, while the peaks inside an edge state cannot be understood as CSs, it is possible to switch on a CS some distance away from an edge state. Let us also remark that these edge states are similar to some recently computed convection cells of binary mixtures in closed containers [32].

### IV. ASYMPTOTIC ANALYSIS

The results just described concerning CSs are fully consistent with previous analytical results derived for the Swift-Hohenberg equation [16]. Although the Swift-Hohenberg model is of gradient form and much simpler than (1), it allows one to understand the main features of Fig. 1(b). We now borrow some scaling arguments from the theory in [16]. These are derived in the limit where $2\pi \mu/\lambda \gg 1$ (i.e., close to the Turing bifurcation), but appear to be qualitatively robust even when this ratio is not very large.

As we have seen, the tail of a CS has the form given by Eq. (3). Let us consider a CS at a distance $\ell$ from a boundary. At the boundary, its amplitude is $O(\exp(-\ell/\mu))$. However, there is another distance $\ell$ to cover before boundary effects are felt by the CS. Hence, the CS affects itself through the boundary with a strength that decays as $\exp(-2\ell/\mu)$. Figure 1(b) suggests that, for the CSs that are closest to the boundary, the necessity to adjust to the boundary conditions has a repelling effect.

Secondly, within the CS, the interaction between the spatial oscillations and their envelope gives rise to a pinning effect, whose strength scales as

$$(\mu/\lambda)^2 \exp(-\pi^2 \mu/\lambda) \cos(2\pi \ell/\lambda + \varphi), \quad (5)$$

where $\varphi$ is the phase of the oscillations [23,24]. In general, two values of $\varphi$ are possible, which differ by $\pi$. Comparing the two competing effects—boundary repulsion and pinning—the minimal distance $\ell$ for which an equilibrium is reached requires the two exponentials above to balance [i.e., that $\ell \approx \pi^2 \mu^2/(2\lambda)$]. Therefore, the CSs are constrained to reside in a subdomain of size $L_c$, given by

$$L_c = \Gamma - 2\ell \approx \Gamma - \pi^2 \mu^2 / \lambda. \quad (6)$$

Using the numerical values reported above, we obtain $L_c \approx 62$, in good agreement with Fig. 1(b), where half of the domain is shown. Moreover the oscillatory nature of the pinning force and the two possible values of $\varphi$ explain that successive values of $\ell$ differ by $\lambda/4$ (and therefore the CS positions, too). On the other hand, the same oscillations in the pinning force means that the gradient of this force changes sign from one site to the next. Therefore, every other stationary position is unstable and stable positions are separated by $\lambda/2$. The periodic dependence in (5) is due to the periodic oscillations contained in the CS and is believed to be quite general [22,33].

Finally, our qualitative theory also explains the time scales needed for the CS to settle at a stable position. Since the motion is driven by the interaction with the boundaries, the timescale should scale as $\exp(2\ell/\mu)$. This is in full agreement with Fig. 1(b). Indeed, comparing the first and sixth stable positions, starting from the top of the figure, they are separated by $5\lambda/2$ and the ratio between the time scales to reach equilibrium should therefore be $\exp(5\lambda/2\mu)$. With the numerical values found for $\lambda$ and $\mu$, this factor is close to $10^5$. This gives support to the validity of our analysis.

We now turn to the information capacity $C$ of the cavity associated with CSs. We define the information capacity $C$ as the largest number of bits that can be encoded in the system. It is the logarithm in base 2 of the number of configurations that we can make with CSs. As we have discussed, while a single CS has a choice between locations separated by $\lambda/2$, any additional CS must be an integer number of $\lambda$ away from it. In effect, therefore, the number of available locations for CSs is approximately

$$N = L_c/\lambda \approx \Gamma/\lambda - (\pi \mu/\lambda)^2. \quad (7)$$

This yields $2^N$ possible configurations, and since each configuration can be shifted by $\lambda/2$, depending on the location of the first CS, we have $\Omega_N = 2^{N+1}$ allowed configurations. The information capacity associated to CSs is, therefore, $C = N+1$. The contribution of edge states, on the other hand, is comparatively minor. Indeed, suppose we can insert before any word made of CSs an edge state of length $\lambda$. The resulting capacity is $C = N+1 + \log_2 n$. Hence, the resulting capacity is $C = N+1 + \log_2 n$.

### V. IMPERFECT CAVITY

So far, we have been considering a perfectly smooth cavity. In practice, unevenness of the cavity length effectively results in a space-dependent detuning. This can be modeled by substituting $\theta \to \theta + \eta g(x)$ in Eq. (1), where $\eta \ll 1$. The impact of $\eta g(x)$ on the solution can be assessed by linearizing (1) around the steady homogeneous solution and computing the response $E - E_r$ to its various Fourier components. The worst possible inhomogeneity is of the resonant form $\eta \cos(2\pi x/\lambda)$. One easily derives in this case that $E - E_r \propto$
in thermoconvection in [32], Similar “wall states” were also discussed in Taylor-Couette flows [35], although outside the pinning range. The strong difference between a large and an infinite domain was also highlighted in that context.

On the other hand, we found that localized states are attracted to a discrete set of locations, contrary to previous intuition. The associated motion can be very slow, compared with the switching time of a CS. However, if CSs are to be used as a data buffer, then they may live long enough to drift significantly away from where they were generated. Importantly, the influence of boundaries is long ranged and not limited to a few CS oscillations, as often believed.

All our numerical observations could be well accounted for by simple theoretical considerations which are built upon the simpler Swift-Hohenberg model [16]. It is sometimes questioned whether predictions based on so simple a model as the Swift-Hohenberg model are reliable. At issue is the variational character of that equation, which implies the existence of a Lyapunov functional. This property is not shared by many physical models that describe pattern formation and homoclinic snaking. The present study, being conducted on a notoriously nonvariational model, shows that the general picture drawn from the Swift-Hohenberg model does not essentially depend on the existence of a Lyapunov functional. As we indicated in the introduction, what matters is the proximity of a subcritical Turing bifurcation, also called Hamiltonian-Hopf or reversible 1:1 resonance in dynamical system theory [20]. The present theory only appeals to $\lambda$ and $\mu$, which are the oscillation period and attenuation length of the tail of the localized structures, respectively. As such, it is easily transposable to other contexts, such as thermoconvection [7,36,37] and Marangoni convection [38] and may be relevant to recent investigations of the onset of turbulence in Couette [39] and channel flows [40], where unstable localized patterns have been identified. Finally, our reasoning can in principle be applied to 2D settings since the tail oscillations [33] and the pinning effect [28,29] are both present in that case. Needless to say, an extension of the present study to a 2D domain would require a discussion of the shape of the boundary. If the boundary is rounded, we expect this to matter only little if the radius of curvature is large compared to the minimal CS-boundary distance. On the other hand, studying the 2D case can be numerically challenging, since the dynamics is exponentially slow. Hence, to identify stable positions by simple time-stepping requires one to integrate the system theory [33] a torsional model, showing that the general picture drawn from the Swift-Hohenberg model does not essentially depend on the existence of a Lyapunov functional.

VI. CONCLUSION

In conclusion, the finite character of the cavity in which CSs develop brings a series of important corrections to the infinite-domain picture, or to the domain with periodic boundary conditions. On the one hand, we identified edge states, which are localized patterns that overlap with the boundary. Such states were also reported in fluid mechanics and may be relevant to recent investigations of the onset of turbulence in Couette [39] and channel flows [40], where unstable localized patterns have been identified. Finally, our reasoning can in principle be applied to 2D settings since the tail oscillations [33] and the pinning effect [28,29] are both present in that case. Needless to say, an extension of the present study to a 2D domain would require a discussion of the shape of the boundary. If the boundary is rounded, we expect this to matter only little if the radius of curvature is large compared to the minimal CS-boundary distance. On the other hand, studying the 2D case can be numerically challenging, since the dynamics is exponentially slow. Hence, to identify stable positions by simple time-stepping requires one to integrate the model for a very long time.

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