Bidirectional ring laser: Stability analysis and time-dependent solutions

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We provide a comprehensive analysis of the standard model of the bidirectional ring laser in which only one mode can be supported in each direction and in the limit that the polarization can be eliminated adiabatically. The interaction between the two counterpropagating modes can be derived and it is most naturally viewed as a coupling between them via scattering from a spatial grating formed in the population inversion. If the grating is a sufficiently small modulation of the spatial average of the population inversion, it can be approximated by a sinusoidal function. A systematic derivation of the model with only a sinusoidal grating for the homogeneously broadened case is presented that reveals and corrects errors in several previously published analyses. The stability of the steady-state solution is analyzed. The bidirectional steady-state solution is unstable and the unidirectional steady-state solutions may be stable or unstable depending on the parameters. The well-established result of bistability between the two modes when the cavity is tuned to resonance is recovered, a result that persists, in part, even when the losses between the two modes are different. With detuning, the unidirectional steady-state solutions can be destabilized, creating regions in the parameter space where only time-dependent solutions are found. For parameters characteristic of solid-state or molecular gas lasers, the instability can occur for very small detunings and even very close to the lasing threshold. Asymptotic limits of the instability boundaries for these parameters are presented. Additional results are derived formally for inhomogeneous broadening with a Lorentzian line shape and for arbitrary inhomogeneous linewidths. Explicit analytic results are presented in the limits of very small and very large inhomogeneous linewidths compared with the homogeneous linewidth. Time-dependent solutions in the homogeneously broadened case for relaxation rates appropriate to CO_2 lasers show that there are broad regions in the parameter space of gain and detuning for which the behavior is dynamically chaotic in the form of nearly "square-wave" alternation between the counterpropagating modes of the laser with irregular switching times. In other regions of that parameter space we find more irregular pulsations and pulse alternations. Regions of periodic pulsations also have been found.

I. INTRODUCTION

Bidirectional operation of ring lasers has been studied since the design of ring lasers themselves[1] because the almost inescapable reversibility of optical paths permits such lasers to operate with counterpropagating fields. Such lasers can be reduced to simple traveling-wave devices only after insertion of a sufficiently nonreciprocal element in the cavity such as, for example, a Faraday rotator together with polarization-dependent losses. Bidirectional ring lasers are also of considerable technical interest because of their use as gyroscopes wherein the frequencies of the counterpropagating modes are split by any rotation of the plane of the laser about an axis normal to the plane. Thus these lasers have been well studied for many years and the bibliography can be found by consulting review articles.[2,3] The bidirectional solid-state laser has been a special interest of Khanin and coworkers who have published several papers and reviews on the subject.[4,5]

We have returned to this problem for several reasons. First, recent experiments on bidirectional laser action in CO_2 ring lasers[6–8] provide the first significant experimental evidence of the dynamical behavior of ring lasers which are appreciably detuned from resonance under conditions where the models can be simplified by adiabatic elimination of the polarization. Almost all previous experiments for lasers in which the polarization relaxation rate $\gamma_\perp$ greatly exceeded the cavity decay rate κ were for solid-state lasers of such large linewidth that appreciable detuning was never practical because the frequency spacing between adjacent cavity modes was much less than $\gamma_\perp$, leading to mode hopping instead of detuning. Second, the experimental results indicate that the CO_2 ring laser can easily show complex dynamical evolution of the type known as deterministic chaos, behavior that also merits study, particularly because the time scales of the real-time pulsations are amenable to careful measurement. Third, the experimental results indicate a variety of characteristic pulsation frequencies which seem to be of different orders of magnitude, a factor not previously found in theoretical studies. Finally, there is disagreement in the literature about the equations that should be used to model this system, a debate for which we feel we can now propose a resolution.

In Sec. II we provide some general historical background and a summary of previous work. In Sec. III we offer a careful derivation of the equations for the homogeneously broadened bidirectional ring laser in which we resolve previous disagreements. In Sec. IV we present the steady-state solutions of the model and a stability analysis of these steady states for arbitrary losses in the two directions and for arbitrary detuning. Numerical solutions showing the time-dependent behavior in the region where the steady states are unstable are presented in Sec. V while the effects of inhomogeneous broadening are considered in Sec. VI. We conclude with some observations on work that remains to be done.

II. HISTORICAL BACKGROUND

Among the earliest theoretical studies of the homogeneously broadened bidirectional ring laser are those of Lisitsyn and Troshin;[9] Moss, Killick, and de la Perelle;[10] Tang, Statz, De Mars, and Wilson;[11] Aronowitz;[1] Basov, Morozov, and Oraevskii;[12] Bespalov and Gaponov;[13] Livshitz and Tsikunov;[14] Ostrovskii;[15] Zhelnov, Kazantsev, and Smirnov;[16] Morozov;[17] Petrun'kin *et al.*;[18] and Zeiger and Fradkin.[19] Meanwhile, experiments were begun with solid-state lasers by Hercher, Young, and Smoyer;[20] Geusic, Marcos, and van Uiltert;[21] and Klimontovich, Landa, and Lariontsev.[22]

Inhomogeneously broadened bidirectional ring lasers were first modeled by Walsh and Kemeny;[23] Roess;[24] Heer;[25] Lamb;[26] Aronowitz;[1] Aronowitz and Collins;[27] Zaitsev;[28] Zeiger and Fradkin;[19] and Zhelnov, Kazantsev, and Smirnov.[29] Corresponding experiments were done by Heer;[30] Moss, Killick, and de la Perrelle;[10] Aronowitz;[1] Bagaev, Kuznetsov, Toritskii, and Troshin;[31] Aronowitz and Collins;[27] Belenov, Markin, Morozov, Oraevskii, and Lebedev;[32] Hutchings, Winocur, Durrett, Jacobs, and Zingery;[33] Zaitsev;[28] and Lisitsyn and Troshin.[9] With numerous studies in the intervening years, it is readily apparent that the problem we are addressing has more than 20 years of history and tradition.

The steady-state solutions of the homogeneously broadened laser model are relatively simple to envision physically. Besides the trivial solution in which both modes have zero intensity, there are two unidirectional solutions (which are equivalent to the usual single-mode laser solutions for one mode and zero intensity in the other mode) and a bidirectional solution. The stability of these solutions depends on the values of the parameters such as the gain, detuning, and relaxation rates of the variables and on other physical complications which might be added to the model such as backscattering from the mirrors forming the laser cavity, backscattering from an external mirror, modulation of a parameter, etc. Hence, consideration of the general problem of the bidirectional laser requires the selection of parameter values and physical complications, careful derivation of a model which is then simplified as much as can be justified physically, and then pursuit of stability analyses and numerical solutions.

Studies can generally be classified by the type of laser used in the experiments and by the appropriate model that corresponds to that type of laser. The important distinctions have to do with the relative magnitudes of the relaxation rates of the fields, the polarization in the medium, and the population inversion of the medium. Irrespective of the relaxation rates, an important feature of the bidirectional laser models is that the presence of counterpropagating fields creates a spatial modulation of the population inversion. This modulation creates an additional coupling between the counterpropagating fields in addition to the coupling that results from simple competition for the homogeneously distributed gain of the medium. The coupling of the two fields through interaction with a spatially modulated gain can be viewed as a kind of Bragg scattering of one field into the other, and the importance of this kind of coupling was noted very early[16,17,32,34] and reemphasized in later studies by Hambenne and Sargent[35,36] and Kühlke and coworkers.[37–39]

When the relaxation rates of the material variables are much larger than the relaxation rates for the fields, one can adiabatically eliminate the equations for the material variables leaving only coupled equations for the amplitudes or intensities of the two interacting modes. Such an approach is appropriate to dye lasers. Because of its simplicity it was pursued as a generic model in many early studies and further approximated in the limit of lasers operating near threshold.[1,19,40,41] More recently it has been used for comparisons with dye laser experiments by Mandel and co-workers,[42–50] by Kühlke and co-workers,[37–39,51] and others.[52] It can be readily shown[38,39,45] that in this limit the unidirectional steady states are stable when they exist (above their respective lasing thresholds) and that the bidirectional solution is always unstable. Models without the adiabatic elimination of the material variables, and without approximations which limit them to weak fields, have been analyzed in a few cases and the same stability characteristics of the steady-state solutions were found.[53,54] Irregular switching between the two unidirectional solutions for dye lasers has been observed experimentally, but it has been demonstrated that this effect is induced by stochastic noise in the system.[48–50] Dynamical switching in the operation of dye lasers (or in the corresponding models with the adiabatic elimination of all material variables) has been observed when it is induced by backscattering or reflection from an external mirror when there is a mismatch in phase between the unidirectional field and the reinjected field.[39,41,51,52,55]

For solid-state lasers [Nd:YAG (yttrium aluminum garnet) and ruby] and certain molecular gas lasers (N_2O and CO_2) the relaxation rate for the population inversion is much less than the relaxation rate for the fields, while the relaxation rate for the polarization is much higher than the relaxation rate for the fields.[56] For solid-state lasers the typical rates for the population inversion, field, and polarization are 10^3 to 10^5, 10^7, and 10^{12} s^{-1}, respectively, while for low-pressure CO_2 lasers typical rates are of order 10^3, 10^7, and 10^8 s^{-1}, respectively. For models used to describe these lasers, it is common to adiabatically eliminate only the polarization. The resulting model for the bidirectional ring laser has equations for the two field amplitudes and the spatially varying population inversion. For resonantly tuned lasers this model has been considered by Klochan, Kornienko, Kravtsov, Lariontsev, and Shelaev[57] and by Mandel and Abraham.[54] In the latter paper it was shown that there are seven eigenvalues for the stability of the unidirectional solutions, three governing the coupling of perturbations of the lasing mode and the medium and four governing the coupling of the perturbations of the extinguished mode and the medium. The lasing mode has the stability characteristics of a single-mode laser operating with the same relaxation rates. Perturbations of the steady-state solutions lead to weakly damped oscillations at a frequency which is of the order of the geometric mean of the field and population decay rates. Perturbations of the extinguished mode also lead to weakly damped oscillations but in this case the frequency of the oscillations differs from those of the lasing mode by a factor of the square root of 2, an effect noted earlier in a truncated model by Khanin and co-workers[4,5,58–60] and confirmed by experimental results.[4,5,22,58–60] Further work for more general parameter values in a model that maintains the full spatial structure of the population inversion has yet to be done.

Instead, a common approximation is made, in which one expands the population inversion in a spatial Fourier series with a fundamental wavelength equal to one-half of the wavelength of lasing fields. This series is then truncated after the first harmonic term, so that the population inversion is then modeled by amplitudes for the uniform component of the inversion and the first spatially varying term. This approximation has a long history and results in equations for seven real variables when the laser cavity is detuned from the resonance of the medium (the two field amplitudes are then complex, the uniform inversion amplitude is real, and the amplitude of the spatial grating becomes complex). Early versions of this model are reported by Basov, Morozov, and Oraevskii;[12] Bespalov and Gaponov;[13] Belenov;[61] Livshitz and Tsikunov;[14] Morozov;[17] Petrun'kin *et al.*;[62] Zeiger and Fradkin;[19] and Moss, Killick, and de la Perelle.[10] In resonance, as for the more general model and the dye laser model already discussed, the two steady states are stable, resulting in bistable operation and noise-induced switching. With detuning, the steady states can be destabilized leading to regular, irregular, or chaotic switching between directions of lasing with ringing at the relaxation oscillation frequencies.[4,17,19,57,63–69]

Though the seven-equation model is rather well established, changes in notation over the years and from one group to another make quick comparisons rather difficult or tedious. Because of our previous interest in this problem and previous contributions, we have noted that several authors of derivations have erred in applying the combined approximations of the adiabatic elimination of the polarization and the harmonic truncation of the spatial dependence of the population inversion.[6–8,64] Because of the familiarity of two of us (N.B.A. and G.L.L.) with the derivations in the earlier three of these papers, we can point out, as a caution, the procedure that led to the error in the equations. Specifically, the error resulted from expanding the full set of equations for the polarization, field, and inversion in harmonic series and keeping only the first harmonic term for each variable before proceeding with the adiabatic elimination of the polarization. The resulting equations differ from those obtained by fully adiabatically eliminating the polarization before doing the harmonic expansion and truncation for the population inversion variable. We have since determined that the correct equations can be obtained when the harmonic expansion and truncation is done before the adiabatic elimination only if the polarization is kept to its first two harmonic terms before the adiabatic elimination is carried out. Because of this error in the equations, the numerical results for the time-dependent solutions presented in the papers cited above are erroneous as well. However, we note that the solutions presented there are extremely similar to solutions generated for the same parameter values with the correct equations; so at least for these parameter values, the error in the equations did not introduce qualitatively spurious behavior.

Results of stability analyses of the seven-equation model have been reported by a number of authors (Refs. 4, 5, 12, 13, 15, 17, 54, 57, 59, 61, 64, and 68-75). However, many of these results were obtained for special cases or for special limits. In Sec. IV we will present the full stability analysis (including nonreciprocity in the losses for the two modes) together with asymptotic limits which reveal both the general structure of the stability boundaries and specific results which may be useful for comparisons with experimental or numerical results.

Experiments on solid-state lasers that are adequately described by the "seven-equation model" have been reported by Antisiferov, Krivoshchekov, Pivtsov, and Folin;[76] Clobes and Brienza;[77] Golyaev, Evtyukhov, Kaptsov, and Smyshlyaev;[64] Geusic, Marcos, and van Uiltert;[21] Hercher, Young, and Smoyer;[20] Khandokhin and Khanin;[59–60] Khanin;[4] Klimontovich, Landa, and Lariontsev;[22] Klochan, Kornienko, Kravtsov, Lariontsev, and Shelaev;[57] Klochan, Kornienko, Kravtsov, Lariontsev, and Shelaev;[78] Kornienko, Kravtsov, Naumkin, and Prokhorov;[79] Kornienko, Kravtsov, and Shelaev;[80] Krivashchekov and Stupak;[81] Mak and Ustyugov;[82] Marowsky and Kaufman;[83] Schroder, Stein, Frölich, Fugger, and Welling;[84] and Tang, Statz, DeMars, and Wilson.[11] Regular switching was noted in some cases[80,82] and irregular switching has also been seen[4,5,57,60,75,77,78] (in Refs. 4, 5, 77, and 78, the irregular pulsing was attributed to chaotic or autostochastic behavior). A common observation was that detuning of the laser caused a change in the pulsing frequency.[57,64,77,78,80,82] As noted earlier, the difference in the relaxation oscillation frequencies of the strong lasing mode and the nearly extinguished mode were visible in the power spectrum of the laser signal because both are induced by noise perturbation of the system and, through nonlinear coupling of the variables, both then become visible as modulations of the strong lasing mode.

The recently reported results from experiments on a bidirectional CO_2 laser[6–8] included bistable behavior when the laser was resonantly tuned, while with detuning there was evidence for both regular and chaotic square-wave-type switching, in-phase and out-of-phase giant pulsing (auto-Q switching) of the two modes, and giant pulses followed by relaxation oscillations, as different parameters were changed. In order to make comparisons with many of these results, we will use relaxation rates and parameter values appropriate to CO_2 in our numerical simulations.

Experimental results have also been reported for bidirectional ring lasers for which adiabatic elimination is not appropriate.[85–90] Among these experiments, time-dependent behavior has been observed by Abraham *et al.*[85] for He-Xe, an inhomogeneously broadened transition, and by Klische and Weiss[86] for far-infrared (FIR) lasers. In both cases regular and irregular (seemingly chaotic) pulsations were observed.

As a further note, we should comment on the controversy about the possibility that bidirectional ring lasers might have different losses in the two directions. In principle, it is easy to achieve nonreciprocal losses if the laser resonator contains a Faraday rotator and polarization selective losses. However, in the absence of a specific nonreciprocal element, it is still possible for such effects to occur. Discussions have been vigorous about nonreciprocal losses or nonreciprocal coupling between the waves.[22,76,91–99] We note that while the model approximates the system as two counterpropagating plane waves, in reality, the waves have approximately Gaussian transverse profiles which vary with passage through the medium or apertures and with reflection from spherically curved mirrors. The nonlinear self-focusing effects, together with the selective effects of apertures and curved mirrors that are not placed symmetrically in the cavity, are sufficient to cause some nonreciprocity in the losses for fields traveling in different directions.

With this as a possibility, we have expanded our studies to the nonreciprocal case and have found a more complex phase diagram. In the nonreciprocal case, one must draw separate instability boundaries for the two unidirectional solutions. Thus there are regions where only one of the two unidirectional solutions exists. There are also regions near the excitation threshold where only one unidirectional mode is stable, even when both solutions exist. However, aside from a few small regions of anomalous behavior, we find that the qualitative patterns of the instabilities in the parameter space and the phenomena of the time-dependent solutions are relatively insensitive to changing from the case of reciprocal losses to the case of a small degree of nonreciprocity.

III. GENERAL FORMULATION

We consider a ring cavity containing a gain medium which is modeled as a homogeneously broadened and inverted two-level system with the atomic transition frequency ω_A. The laser is assumed to oscillate at the single frequency ω_L with a slowly varying amplitude, while the nearest cavity resonance frequency is ω_C. As in Ref. 53, we assume that the longitudinal mode spacing is large compared with the gain bandwidth of the medium and we assume that the laser operates on a single mode.

The laser characteristics are obtained by solving the Maxwell-Bloch equations which are extended to incorporate the counterpropagating nature of the cavity field. Making the usual plane-wave, slowly-varying-amplitude, and uniform-field approximations and following the approach and notation of Mandel and Agarwal,[53] the semiclassical equations are

$$i(\partial_t+\kappa_j)\beta_j(t)-\omega_C\beta_j(t)=\frac{N}{L}\int_0^L dx\, g_j^*(x)\alpha(x,t)\;, \tag{3.1a}$$

$$i(\partial_t+\gamma_\perp)\alpha(x,t)-\omega_A\alpha(x,t)=-D(x,t)\sum_j g_j(x)\beta_j(t)\;, \tag{3.1b}$$

$$i(\partial_t+\gamma_\parallel)D(x,t) =i\sigma\gamma_\parallel+2\sum_j[g_j(x)\alpha^*(x,t)\beta_j(t)-\text{c.c.}]\;, \tag{3.1c}$$

where $\beta_j(t)$ is the slowly-varying-field envelope, $\alpha(x,t)$ is the induced polarization, N is the total number of atoms and L is the length of the medium, $D(x,t)=(N_+-N_-)/(N_++N_-)$, and $\sigma\gamma_\parallel$ is the pumping rate for D. The mode index $j=1,2$ signifies the forward and the backward wave, respectively. The quantities $N_+(x,t)$, $N_-(x,t)$, κ_1, κ_2, $\gamma_\parallel$, and $\gamma_\perp$ correspond to the number of atoms per unit volume in the upper and lower states, the decay rates of the electric field amplitudes for the two waves, and the longitudinal and transverse relaxation rates, respectively. The coupling constant is

$$g_j(x)=-i(2\pi\omega_A)^{1/2}(\mu/\hbar)\exp[(-1)^{3-j}ikx]\;, \tag{3.2}$$

where $kc=\omega_L$, μ is the dipole matrix element, and x is measured along the cavity perimeter. Equations (3.1) fully incorporate the effects of spatial hole burning arising from the superposition of the counterpropagating waves.

Using the following decomposition for fields and polarization:

$$\xi_j(t)\equiv e^{i\omega_C t}\beta_j(t)$$

and

$$P_j(x,t)\equiv g_j^*(x)\alpha(x,t)e^{i\omega_C t}\;,$$

we can rewrite Eqs. (3.1) as

$$i(\partial_t+\kappa_j)\xi_j(t)=\frac{N}{L}\int_0^L dx\, P_j(x,t)\;, \tag{3.3a}$$

$$i(\partial_t+\gamma_\perp)P_j(x,t)+(\omega_C-\omega_A)P_j(x,t) =-|g|^2D(x,t)(\xi_j+\xi_{3-j}e^{(-1)^j2ikx})\;, \tag{3.3b}$$

$$i(\partial_t+\gamma_\parallel)D(x,t)=i\sigma\gamma_\parallel+2\sum_j(P_j^*\xi_j-P_j\xi_j^*)\;. \tag{3.3c}$$

In dimensionless variables given by

$$E_j \equiv \left[\frac{\gamma_\parallel \gamma_\perp}{4|g|^2}\right]^{-1/2} \zeta_j ,$$

$$\mathcal{P}_j \equiv \sigma^{-1}\left[\frac{|g|^2\gamma_\parallel}{4\gamma_\perp}\right]^{-1/2} P_j ,$$

$$\mathcal{D} \equiv D\sigma^{-1} ,$$

and for a new scaled time, $\tau = \kappa_1 t$, we can rewrite Eqs. (3.3) as

$$i(\partial_\tau + K_j)E_j(\tau) = \frac{A}{L}\int_0^L dx\, \mathcal{P}_j(x,\tau) , \quad (3.4a)$$

$$i(\partial_\tau + d_\perp)\mathcal{P}_j(x,\tau) + \theta\mathcal{P}_j(x,\tau) = -d_\perp \mathcal{D}(x,\tau)(E_j + E_{3-j}e^{(-1)^j 2ikx}) , \quad (3.4b)$$

$$i(\partial_\tau + d_\parallel)\mathcal{D}(x,\tau) = id_\parallel + \frac{d_\parallel}{2}\sum_j(\mathcal{P}_j^* E_j - \mathcal{P}_j E_j^*) , \quad (3.4c)$$

where

$$K_j \equiv \frac{\kappa_j}{\kappa_1}, \quad A \equiv \frac{\sigma N|g|^2}{\gamma_\perp \kappa_1}, \quad d_\parallel \equiv \frac{\gamma_\parallel}{\kappa_1}, \quad d_\perp \equiv \frac{\gamma_\perp}{\kappa_1} ,$$

and

$$\theta \equiv \frac{\omega_C - \omega_A}{\kappa_1} .$$

As we will consider cases where $d_\perp \gg d_\parallel, K_j$, we have two ways in which to proceed to simplify the equations. Either we perform a harmonic expansion for the population inversion and the polarization, eliminate adiabatically the polarization, and truncate the infinite hierarchy of equations, or we can first eliminate the polarization, then perform a harmonic expansion for the population inversion, and then truncate the hierarchy of equations. These two methods give the same final result, but, as the second one is shorter, we will follow it. Setting $\partial_\tau \mathcal{P}_j = 0$ and solving Eq. (3.4b) for $\mathcal{P}_j$, we find

$$\mathcal{P}_j = \frac{\mathcal{D}(E_j + E_{3-j}e^{(-1)^j 2ikx})}{1+\Delta^2}(i - \Delta) , \quad (3.5)$$

where

$$\Delta = \frac{\theta}{d_\perp} = \frac{\omega_C - \omega_A}{\gamma_\perp} .$$

The equations for the field and the population inversion are given by

$$(\partial_\tau + K_j)E_j = \frac{\tilde{A}}{L}(1+i\Delta)E_j\int_0^L \mathcal{D}(x,\tau)dx + \frac{\tilde{A}}{L}(1+i\Delta)E_{3-j}\int_0^L \mathcal{D}(x,\tau)e^{(-1)^j 2ikx}dx , \quad (3.6a)$$

$$(\partial_\tau + d_\parallel)\mathcal{D}(x,\tau) = d_\parallel - \tilde{d}_\parallel \mathcal{D}(x,\tau)(E_1E_1^* + E_2E_2^*) - \tilde{d}_\parallel(E_1E_2^* e^{2ikx} + E_1^*E_2 e^{-2ikx}) \times \mathcal{D}(x,\tau) , \quad (3.6b)$$

where we have defined $\tilde{A} \equiv A/(1+\Delta^2)$ and $\tilde{d}_\parallel \equiv d_\parallel/(1+\Delta^2)$. To derive the harmonic expansion, we define

$$\mathfrak{D}(n,\tau) \equiv \frac{1}{L}\int_0^L \mathcal{D}(x,\tau)e^{i2nkx}dx , \quad (3.7)$$

where n is an integer. The field equations then become

$$(\partial_\tau + 1)E_1 = \tilde{A}E_1\mathfrak{D}(0,\tau)(1+i\Delta) + \tilde{A}E_2\mathfrak{D}^*(1,\tau)(1+i\Delta) , \quad (3.8a)$$

$$(\partial_\tau + K_2)E_2 = \tilde{A}E_2\mathfrak{D}(0,\tau)(1+i\Delta) + \tilde{A}E_1\mathfrak{D}(1,\tau)(1+i\Delta) , \quad (3.8b)$$

and for the first few components $D(n,\tau)$ of the inversion, we get

$$(\partial_\tau + d_\parallel)\mathfrak{D}(0,\tau) = d_\parallel - \tilde{d}_\parallel\mathfrak{D}(0,\tau)(E_1E_1^* + E_2E_2^*) - \tilde{d}_\parallel[E_1E_2^*\mathfrak{D}(1,\tau) + E_1^*E_2\mathfrak{D}^*(1,\tau)] , \quad (3.9a)$$

$$(\partial_\tau + d_\parallel)\mathfrak{D}(1,\tau) = -\tilde{d}_\parallel\mathfrak{D}(1,\tau)(E_1E_1^* + E_2E_2^*) - \tilde{d}_\parallel[E_1E_2^*\mathfrak{D}(2,\tau) + E_1^*E_2\mathfrak{D}(0,\tau)] , \quad (3.9b)$$

$$(\partial_\tau + d_\parallel)\mathfrak{D}(2,\tau) = -\tilde{d}_\parallel\mathfrak{D}(2,\tau)(E_1E_1^* + E_2E_2^*) - \tilde{d}_\parallel[E_1E_2^*\mathfrak{D}(3,\tau) + E_1^*E_2\mathfrak{D}(1,\tau)] . \quad (3.9c)$$

We introduce the approximation, $\mathfrak{D}(n,\tau) \simeq 0$ $(n \geq 2)$. This approximation automatically cuts the coupled hierarchy of the equations and truncates the equations to include only terms in $\mathfrak{D}(1,\tau)$ and $\mathfrak{D}(0,\tau)$. Examining the equations we see that the coupling to higher-order terms is via a factor $E_1E_2^*$. The truncation can be easily justified if $|E_1E_2^*| \ll 1$. However, while this may be a sufficient condition, it is not necessary condition. We examined some of our numerical solutions of the truncated hierarchy of equations when it was at least occasionally true during the time evolution that $|E_1E_2^*| \sim 1$ and found that $|D(1,\tau)| \ll D(0,\tau)$, suggesting that the harmonic series rapidly converges and that it can probably be truncated without significant impact.

The equations for the inversion are then given by

$$(\partial_\tau + d_\parallel)\mathfrak{D}(0,\tau) = d_\parallel - \tilde{d}_\parallel\mathfrak{D}(0,\tau)(E_1E_1^* + E_2E_2^*) - \tilde{d}_\parallel[E_1E_2^*\mathfrak{D}(1,\tau) + E_1^*E_2\mathfrak{D}^*(1,\tau)] , \quad (3.10a)$$

$$(\partial_\tau + d_\parallel)\mathfrak{D}(1,\tau) = -\tilde{d}_\parallel\mathfrak{D}(1,\tau)(E_1E_1^* + E_2E_2^*) - \tilde{d}_\parallel E_1^*E_2\mathfrak{D}(0,\tau) . \quad (3.10b)$$

The sets of Eqs. (3.8) and (3.10) are the final truncated equations used to study the stability of stationary states.

In summary, the equations which we shall study are

$$(\partial_\tau + 1)E_1 = (1+i\Delta)\tilde{A}(E_1\mathfrak{D}_0 + E_2\mathfrak{D}_1^*) , \quad (3.11a)$$

$$(\partial_\tau+K)E_2=(1+i\Delta)\tilde{A}(E_2\mathfrak{D}_0+E_1\mathfrak{D}_1)\ , \quad (3.11b)$$

$$(\partial_\tau+d_\parallel)\mathfrak{D}_0=d_\parallel-\tilde{d}_\parallel\mathfrak{D}_0(|E_1|^2+|E_2|^2)-\tilde{d}_\parallel(E_1E_2^*\mathfrak{D}_1+\text{c.c.})\ , \quad (3.11c)$$

$$(\partial_\tau+d_\parallel)\mathfrak{D}_1=-\tilde{d}_\parallel\mathfrak{D}_1(|E_1|^2+|E_2|^2)-\tilde{d}_\parallel E_1^*E_2\mathfrak{D}_0\ , \quad (3.11d)$$

with the simplified notation $K\equiv K_2$ and $\mathfrak{D}_n\equiv\mathfrak{D}(n,\tau)$.

IV. STEADY STATES AND STABILITY

The set of Eqs. (3.11) has four solutions which can be classified as follows.

(i) The trivial solution,

$$E_1=E_2=0\ . \quad (4.1)$$

(ii) The first unidirectional solution,

$$I_1\equiv|E_1|^2=A-(1+\Delta^2),\quad E_2=0\ , \quad (4.2)$$

which exists for

$$A\geq 1+\Delta^2\ .$$

(iii) The second unidirectional solution,

$$I_2\equiv|E_2|^2=\frac{A}{K}-(1+\Delta^2),\quad E_1=0\ , \quad (4.3)$$

which exists for $A\geq K(1+\Delta^2)$.

(iv) The bidirectional solution,

$$I_1=\frac{1}{2(K^2+1)}\{A-2(K^2+1)(1+\Delta^2)+[A^2+4(1+\Delta^2)^2(1+K^2)]^{1/2}\}\ , \quad (4.4)$$

$$I_2=KI_1+(K-1)(1+\Delta^2)\ .$$

We choose κ_2 to be the greater of the two decay rates, so that $K\geq 1$. Therefore we shall refer to (4.2) as the "strong-mode solution" and to (4.3) as the "weak-mode solution," because the solution (4.2) has a lower threshold value of A and a stronger output intensity, relative to solution (4.3), for a fixed value of A. When $K<1$, the unidirectional solution with $I_1\neq 0$ [Eq. (4.2)] becomes the weak-mode solution, whereas the solution with $I_2\neq 0$ [Eq. (4.3)] becomes the strong-mode solution. Our restriction to the case $K\geq 1$ is not a limitation because the stability properties for the strong-mode solution for $K>1$ are similar (in fact, identical, after a suitable rescaling of the variables) to the stability properties of the weak-mode solution for $K<1$.

A. Linear stability of the trivial solution

It is simple to verify that the trivial solution (4.1) is stable if and only if

$$\tilde{A}<1\ . \quad (4.5)$$

B. Linear stability of the strong-mode solution

To study the stability of the solution (4.2) we seek solutions of Eqs. (3.11) of the form

$$E_1=I_1^{1/2}e^{i\Delta\tau}+\epsilon\xi_1e^{i\Delta\tau}e^{\lambda\tau}+O(\epsilon^2)\ ,$$

$$E_2=\epsilon\xi_2e^{i\Delta\tau}e^{\lambda\tau}+O(\epsilon^2)\ ,$$

$$\mathfrak{D}_0=\tilde{A}^{-1}+\epsilon\eta_0e^{\lambda\tau}+O(\epsilon^2)\ ,$$

$$\mathfrak{D}_1=\epsilon\eta_1e^{\lambda\tau}+O(\epsilon^2)\ ,$$

where ξ_1, ξ_2, η_0, and η_1 are $O(1)$ quantities, and ϵ is the small parameter. To first order in ϵ this leads to a seventh-order characteristic equation for λ which factors into a cubic and quartic. The cubic governs the stability of the strong mode, I_1, coupled to $\mathfrak{D}_0$ and the quartic governs the stability of the mode which is suppressed, I_2, coupled to $\mathfrak{D}_1$.

The cubic is exactly the equation governing the stability of a single-mode laser with adiabatic elimination of the polarization. For the cubic there is a zero eigenvalue corresponding to the marginal stability of the phase of the oscillating mode and the other two eigenvalues are given by the quadratic equation,

$$\lambda^2+\lambda d_\parallel\tilde{A}+2d_\parallel(\tilde{A}-1)=0\ . \quad (4.6)$$

When $\tilde{A}>1$, the solutions of (4.6) always have negative real parts and thus there is no instability which originates by perturbations in the manifold of the variables coupled to the oscillating mode.

The quartic which governs the stability of the suppressed mode factors into two complex conjugate quadratic equations, one of which can be written as

$$\lambda^2+\lambda(d_\parallel\tilde{A}+K-1)+d_\parallel(K\tilde{A}-1)+i\Delta d_\parallel(\tilde{A}-1)=0\ . \quad (4.7)$$

When $\tilde{A}>1$, both of the solutions of Eq. (4.7) have negative real parts under the condition

$$\Delta^2<\frac{(\tilde{A}K-1)(d_\parallel\tilde{A}+K-1)^2}{d_\parallel(\tilde{A}-1)^2}\ . \quad (4.8)$$

When the equality holds, a Hopf bifurcation takes place with a characteristic frequency Ω (pure imaginary eigenvalue) given by

$$\Omega=[d_\parallel(\tilde{A}_cK-1)]^{1/2}\ . \quad (4.9)$$

The critical values of detuning and pump, Δ_c and $\tilde{A}_c$ (the values on the Hopf bifurcation boundary), can be derived from Eq. (4.8) and are given by

$$\Delta_c^2d_\parallel(\tilde{A}_c-1)^2=(K\tilde{A}_c-1)(d_\parallel\tilde{A}_c-1+K)^2\ . \quad (4.10)$$

A plot of $\tilde{A}_c$ versus Δ_c^2 is displayed in Fig. 1. It shows that a minimum amount of detuning is necessary to destabilize the strong mode of oscillation. When this minimum detuning $\Delta_{\min}^2$ is surpassed, there exists a finite domain of the pump parameter for which the strong mode is unstable. Note, however, that even for $\Delta^2>\Delta_{\min}^2$ the strong mode is stable near the lasing threshold and at sufficiently high values of the pump.

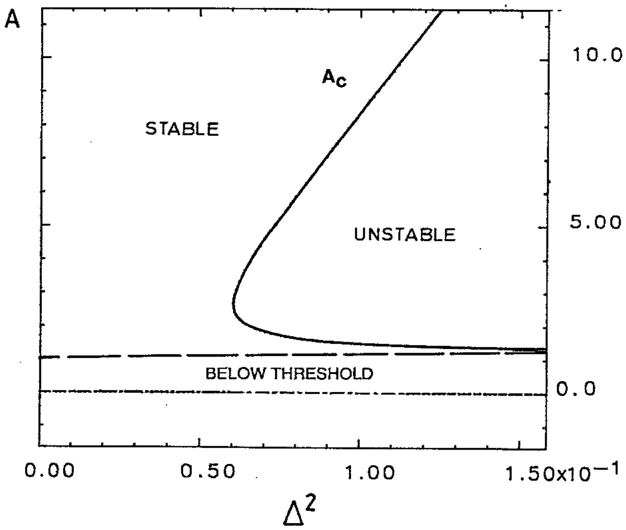


FIG. 1. Stability of the strong-mode steady-state solutions in the (A,Δ^2) parameter plane for $K=1.005$ and $d_\parallel=0.01$. The critical curve of the change of stability given by A_c denotes the locus in parameter space of the Hopf bifurcation.

C. Linear stability of the weak-mode solution

Following the procedure used in the analysis of the strong-mode solution, we find that the characteristic equation for the stability analysis of the weak-mode solution also factors into a cubic governing the oscillating mode (in this case the weak mode I_2) coupled to $\mathfrak{D}_0$, and two quadratic equations (one the complex conjugate of the other) governing the suppressed mode (in this case the strong mode I_1) coupled to $\mathfrak{D}_1$. As before, we obtain a quadratic equation for the stability of the oscillating mode I_2,

$$\lambda^2+\lambda d_\parallel \tilde{A}_2+2d_\parallel K(\tilde{A}_2-1)=0, \quad \tilde{A}_2=\frac{A}{K(1+\Delta^2)} , \tag{4.11}$$

and another quadratic (and its complex conjugate) which govern the stability of the suppressed mode I_1,

$$\lambda^2+\lambda(d_\parallel \tilde{A}_2+1-K)+d_\parallel(\tilde{A}_2-K)+iK\Delta d_\parallel(\tilde{A}_2-1)=0 . \tag{4.12}$$

When $\tilde{A}_2>1$ (a necessary condition for the existence of the weak-mode solution), Eq. (4.11) has solutions whose real parts are always negative so that they cannot induce instability. However, as for the strong-mode solution, the suppressed mode is not always stably suppressed. Details of the stability conditions resulting from Eq. (4.12) are discussed in the following sections.

D. Special cases—exact results

1. *Equal losses*

When $K=1$ there is no longer a distinction between the strong-mode solution and the weak-mode solution. The condition for a Hopf bifurcation, Eqs. (4.10), then becomes

$$\Delta_c^2(\tilde{A}_c-1)=d_\parallel \tilde{A}_c^2 , \tag{4.13a}$$

which has explicit solutions:

$$A_c(\pm)=\frac{1+\Delta^2}{2d_\parallel}[\Delta^2\pm(\Delta^4-4\Delta^2 d_\parallel)^{1/2}] . \tag{4.13b}$$

This leads to a stability boundary (common to both modes) which is similar to Fig. 1, with a minimum detuning $\Delta_{\min}^2=4d_\parallel$, corresponding to a pump parameter $A_c=2(1+4d_\parallel)$. Hence there is bistability (both directions of propagation can sustain a stable c.w. solution) when $1+\Delta^2<A<A_c(-)$ and $A>A_c(+)$ and no stable steady solution in either direction when $A_c(-)<A<A_c(+)$.

2. *No detuning*

In the absence of detuning $(\Delta=0)$ we can easily solve analytically for the stability conditions for both strong- and weak-mode solutions. Since we keep $K>1$, but otherwise arbitrary, it is easy to verify that the strong-mode solution is always stable. For the weak mode, the solution of Eq. (4.12) is

$$\lambda(\pm)=\tfrac{1}{2}\{K-1-d_\parallel A_2 \pm[(K-1-d_\parallel A_2)^2-4d_\parallel(A_2-K)]^{1/2}\} . \tag{4.14}$$

Stability requires $\mathrm{Re}[\lambda(\pm)]<0$, i.e., the two conditions

$$A>A_A\equiv K^2 , \tag{4.15a}$$

and

$$A>A_B\equiv K(K-1)/d_\parallel . \tag{4.15b}$$

There are two cases.

Case A: $K(1-d_\parallel)<1$ [or $d_\parallel>(K-1)/K$]; then only the first condition is relevant because the second is automatically satisfied.

Case B: $K(1-d_\parallel)>1$ [or $d_\parallel<(K-1)/K$]; then only the second condition is relevant because the first condition is automatically satisfied.

Thus we see that with enough pumping, the weak-mode solution may be stabilized and both unidirectional solutions are stable.

Hence in the case $\Delta=0$, Eqs. (4.15) represent the conditions for stable operation of the weak mode and the larger of the two values of A_A and A_B is the threshold for stable operation. This may be significantly higher than the condition $A>K$ (the "above threshold condition" for the existence of the weak-mode solution) which might have been expected to lead to stable steady-state operation from the stability condition of the trivial solution (4.5). Competition between the two modes destabilizes the weaker-mode solution unless there is sufficiently strong pumping.

3. *Weak-mode solution stability with detuning*

The weak-mode solution is stable if the following three conditions are simultaneously satisfied:

$$\text{(i)}\quad \tilde{A}_2 > 1\ , \tag{4.16a}$$

$$\text{(ii)}\quad \tilde{A}_2 > (K-1)/d_\parallel\ , \tag{4.16b}$$

$$\text{(iii)}\quad \Delta^2 < \frac{(\tilde{A}_2 - K)(\tilde{A}_2 d_\parallel - K + 1)^2}{d_\parallel K^2 (\tilde{A}_2 - 1)^2}\ . \tag{4.16c}$$

Equality in the expression for Δ^2 in Eq. (4.16c) signals a Hopf bifurcation with a frequency Ω given by

$$\Omega = [d_\parallel(\tilde{A}_{2c} - K)]^{1/2}\ ,$$

where the boundary for the Hopf bifurcation and $\tilde{A}_{2c}$ are defined from Eq. (4.16c) by the equation

$$\Delta_c^2 d_\parallel K^2 (\tilde{A}_{2c} - 1)^2 = (\tilde{A}_{2c} - K)(\tilde{A}_{2c} d_\parallel - K + 1)^2\ . \tag{4.17}$$

The conditions on A in Eqs. (4.16a) and (4.16b), which restrict the boundary given by Eq. (4.17) in some cases, indicate when the Hopf bifurcation corresponds to an overall change in the stability of the solutions. The boundary of the instability coincides with Eq. (4.10) when $K=1$ as it should since in that case the parameters for the two modes are the same. When $K>1$ the possibilities for Eq. (4.17) are somewhat richer than for Eq. (4.10). [It should be noted, however, that the bifurcation diagram which we obtain from Eq. (4.17) and $K>1$ is the same as the bifurcation diagram which would be obtained from Eq. (4.10) with $K<1$. However, the terminology "strong-mode solution" and "weak-mode solution" relies on $K>1$ and so we consider only this case without loss of generality.] Three domains appear in the $(d_\parallel, K)$ plane as shown in Fig. 2.

(i) In the domain labeled I there exists for all Δ, K, and $d_\parallel$ a critical pump parameter $\tilde{A}_{2H} > 1$ such that the weak-mode solution is stable for $\tilde{A}_2 > \tilde{A}_{2H}$ and unstable for $1 < \tilde{A}_2 < \tilde{A}_{2H}$. In Figs. 3(a) and 3(b) we show how the various parameters of the laser modify the position of the boundary of stability.

(ii) In the domain labeled III, for fixed Δ there may be either one or three boundaries of Hopf bifurcations, but only a single boundary of stability, because when there are three boundaries, the lower two Hopf bifurcations occur while the other eigenvalue has a positive real part. The solution is unstable at low pump parameters and eventually becomes stable as $\tilde{A}_2$ increases (Fig. 4).

(iii) In the domain labeled III′, for fixed Δ there is the possibility of one or three values for Hopf bifurcations, all of which represent changes of stability. This differs from domain III in that in this case the Hopf boundary

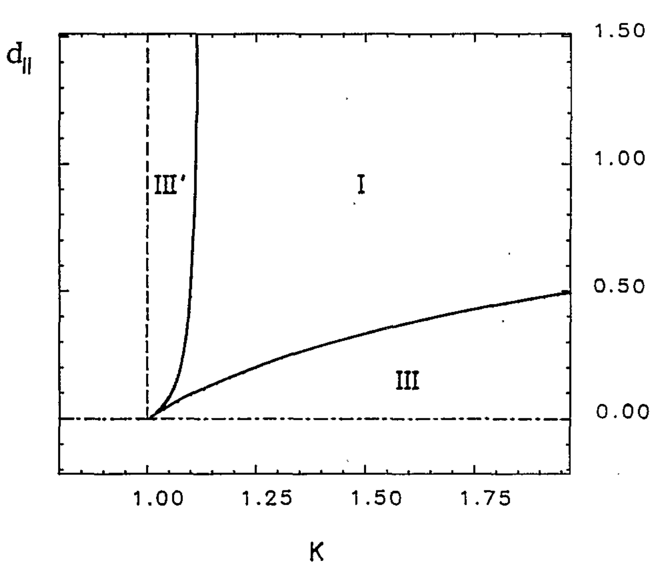


FIG. 2. Regions in the parameter plane of $(d_\parallel, K)$ for different types of stability diagrams and Hopf bifurcation boundaries for the weak-mode steady-state solution. $d_\parallel = \gamma_\parallel/\kappa_1$ and $K = \kappa_2/\kappa_1$ are both adimensional quantities.

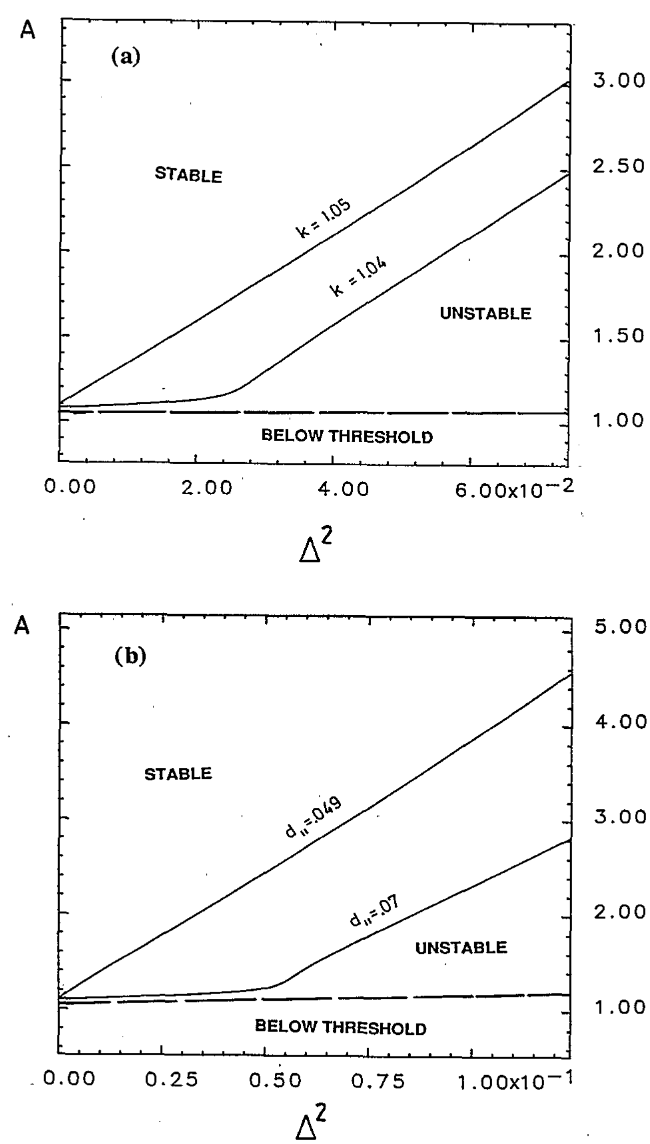


FIG. 3. Stability of the weak-mode steady-state solutions in the (A, Δ^2) plane under conditions appropriate to region I: (a) two values of K for $d_\parallel = 0.05$ and (b) two values of $d_\parallel$ for $K = 1.05$. The steady-state solutions are stable above the boundaries.

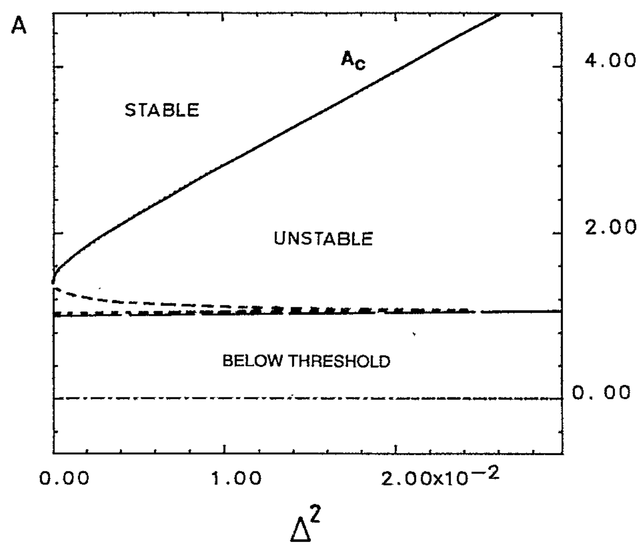


FIG. 4. Solid line and short-dashed lines mark the boundaries of Hopf bifurcations from the weak-mode solutions in the (A,Δ^2) plane for parameters appropriate to region III in Fig. 2: $K=1.014$ and $d_{\parallel}=0.01$. The large-dashed line slightly above $A=1.0$ marks the weak-mode threshold. The weak-mode steady-state solution is stable only above the solid-line boundary.

is always a stability boundary and in that, at resonance $(\Delta=0)$, there is only one Hopf bifurcation instead of the three found in region III (see Fig. 5).

When $(K-1)/K>d_{\parallel}$, the solution falls in domain III and the Hopf bifurcations boundary defined by Eq. (4.17) intersects the $\Delta=0$ axis at two points in the (A,Δ) plane: $A=K^2$ and $A=K(K-1)/d_{\parallel}$. The weak-mode solution is stable under condition (4.16c) only for A values above the second intersection. This corresponds to case B discussed under Sec. IV C 2.

When $(K-1)/K<d_{\parallel}$, the solution falls in domain I or domain III′ and the Hopf bifurcation boundary defined by Eq. (4.17) intersects the $\Delta=0$ axis at only one point in the (A,Δ) plane: $A=K^2$. The weak-mode solution is stable under condition (4.16c) for A values above this intersection. This corresponds to case A discussed in Sec. IV C 2.

E. Asymptotic results—nearly reciprocal losses and small $d_{\parallel}$

In this section we consider two limiting cases suggested by the experimental values of K and $d_{\parallel}$. When the losses are significantly nonreciprocal $(K\gg 1)$, the strong-mode solution is, of course, favored. The largest domain of both unidirectional solutions being stable occurs when K is very near unity. To explore this region we define a smallness parameter ϵ via

$$K=1+\epsilon,\quad 0<\epsilon\ll 1\ . \tag{4.18}$$

Two cases can be considered, depending on the values of $d_{\parallel}$. For low-pressure CO_2 lasers, $d_{\parallel}$ is very small and we shall therefore introduce the expansion,

$$d_{\parallel}=\epsilon d+O(\epsilon^2)\ . \tag{4.19}$$

In this case the strong-mode solution has its stability boundary, Eq. (4.10), which can be solved to give two

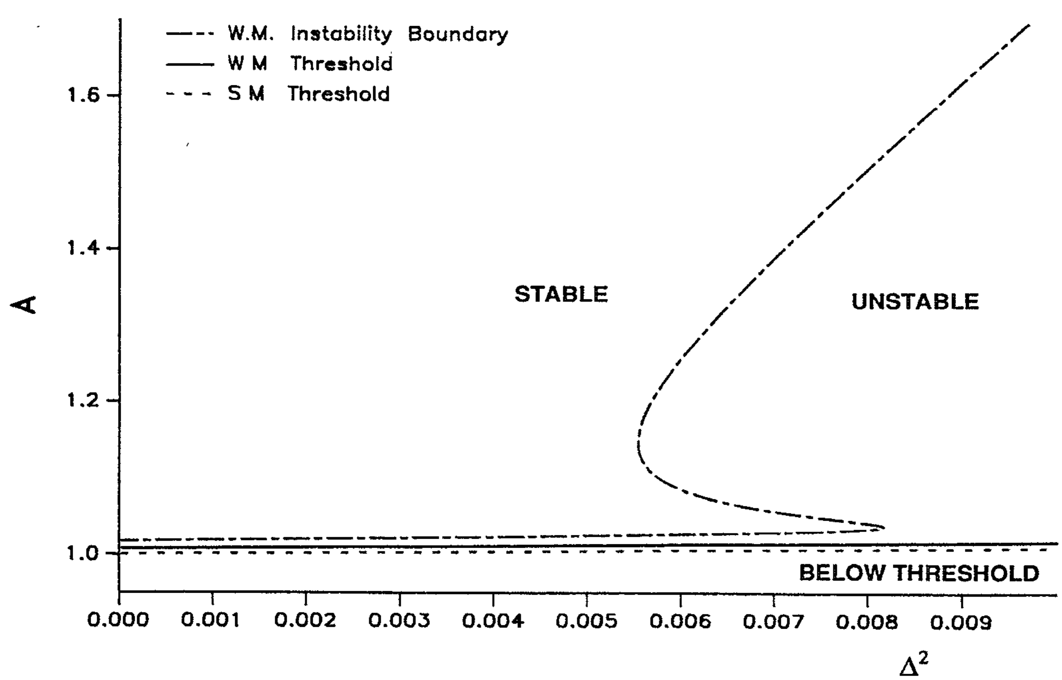


FIG. 5. Stability of the weak-mode solution in the (A,Δ^2) plane for parameters appropriate to region III′: $K=1.0085$ and $d_{\parallel}=0.01$. Also shown for reference are the thresholds for the strong-mode solution and the weak-mode solution.

physical roots

$$A_c(\pm)=\frac{1+\Delta^2}{2\epsilon d}\{\Delta^2-2\epsilon\pm\Delta[\Delta^2-4\epsilon(d+1)]\}+O(\epsilon) \tag{4.20}$$

and the unphysical root $\tilde{A}_c=1-\epsilon+O(\epsilon^2)$. Hence, instability requires that

$$\Delta^2>\Delta^2_{\min}=4\epsilon(d+1) \ . \tag{4.21}$$

Furthermore, when $\Delta^2=O(1)$, the two solutions (4.20) have the following behavior:

$$A_c(-)=(1+\Delta^2)\left\{1+\epsilon\frac{(1+d)^2}{2d\Delta^2}\left[1+\left[1+\frac{4\Delta^2 d}{(d+1)^2}\right]^{1/2}\right]+O(\epsilon^2)\right\} , \tag{4.22}$$

$$A_c(+)=\frac{\Delta^2(\Delta^2+1)}{\epsilon d}+O(1) , \tag{4.23}$$

which are the two branches of Fig. 1. The expression for $A_c(-)$ in Eq. (4.22), which is carried to higher order in ϵ, cannot be derived directly from Eq. (4.20). Requiring terms $O(\epsilon^2)$, it can be found using Eq. (4.10).

The study of the stability boundary of the weak mode is more complex, as can be expected from the consideration of the diagram displayed in Fig. 2. We first assume that (4.18) and (4.19) hold, which define the domain of variation for K and $d_\parallel$. When $\Delta^2=O(1)$, the weak-mode stability condition (4.17) has always at least one solution which is

$$A_{2c}(1)=\frac{\Delta^2(\Delta^2+1)}{\epsilon d}+O(1) , \tag{4.24}$$

which coincides with a stability boundary of the strong mode (4.23).

Let

$$d(\pm)=1+2\Delta^2\pm2\Delta(1+\Delta^2)^{1/2} \ . \tag{4.25}$$

When $d(-)<d<d(+)$, Eq. (4.24) is the only boundary. Otherwise, a pair of additional solutions of (4.17) becomes relevant and is given by

$$A_{2c}(\pm)=(1+\Delta^2)\left[1+\epsilon\left\{1+\frac{(d-1)^2}{2d\Delta^2}\left[1\pm\left[1-\frac{4\Delta^2 d}{(d-1)^2}\right]^{1/2}\right]\right\}+O(\epsilon^2)\right] . \tag{4.26}$$

The occurrence of three boundaries corresponds to domain III when $d<d(-)$ and to domain III′ when $d>d(+)$, whereas the domain where only $A_{2c}(1)$ is a physical solution corresponds to domain I of Fig. 2.

From the solution (4.24) we see that the series in powers of ϵ will no longer converge when Δ^2 is comparable to ϵ. We can analyze this case separately as follows. Let the detuning be defined by

$$\Delta^2=\epsilon\delta+O(\epsilon^2), \quad \delta>0 \ . \tag{4.27}$$

Then the stability boundary condition (4.17) always has at least the solution

$$A_{2c}(1)=1+\epsilon(2+\delta)+O(\epsilon^2) \ . \tag{4.28}$$

Furthermore, if the condition

$$\delta+4(1-d)>0 \tag{4.29}$$

is verified, then two other boundaries appear,

$$A_{2c}(\pm)=\frac{1}{2d}\{\delta+2\pm[(\delta+2)^2-4(1+\delta d)]^{1/2}\}+O(\epsilon) \ . \tag{4.30}$$

We note, however, that in the limit as $\Delta=0$, there is only the single boundary as discussed earlier after Eqs. (4.15). Whatever the order of magnitude of Δ^2 is, the distinction between domains III and III′ still rests on the inequalities (4.16) which imply, with the definitions (4.18) and (4.19), that when there are three critical values of the pump parameter, the domain III corresponds to $d<1$, whereas the domain III′ corresponds to $d>1$.

For lasers with sufficiently small nonreciprocity in the losses, we have $K=1+\epsilon$ and $d_\parallel$ is not necessarily small on the ϵ scale. Let us consider lasers for which

$$d_\parallel=O(\epsilon^0) \ . \tag{4.31}$$

Using this definition as well as (4.18) it is easy to verify that the strong mode is stable for pump parameters A outside the range (A_-,A_+) where

$$A_\pm=\frac{1+\Delta^2}{2d_\parallel}[\Delta^2\pm(\Delta^4-4d_\parallel\Delta^2)^{1/2}]+O(\epsilon) \ . \tag{4.32}$$

This domain of instability requires, again, a finite amount of detuning

$$\Delta^2>\Delta^2_{\min}, \quad \Delta^2_{\min}=4d_\parallel \ . \tag{4.33}$$

A similar analysis of the weak mode indicates that there are three boundaries:

$$A_{2c}(1)=(1+\Delta^2)(1+2\epsilon)+O(\epsilon^2) , \tag{4.34}$$

$$A_{2c}(\pm)=\frac{1+\Delta^2}{2d_\parallel}[\Delta^2\pm(\Delta^4-4d_\parallel\Delta^2)^{1/2}]+O(\epsilon) \ . \tag{4.35}$$

F. Summary

We see from this asymptotic analysis that the influence of the three parameters, Δ, $d_{\parallel}$, and K can be described as follows.

(i) In all cases the weak-mode stability domain is embedded in the strong-mode stability domain. Hence the weak-mode stability domain also represents a domain of bistability (both unidirectional modes are stable). These features are displayed in Figs. 6–8.

(ii) The bistable domain near threshold decreases when either Δ or K increases, whereas the bistable domain increases with increasing $d_{\parallel}$.

(iii) Though the existence of large amplitude, time-dependent solutions cannot be predicted by linear stability analyses, it is our experience with numerical simulations that pulsations are observed only when both steady-state solutions are unstable; hence, only in the region where the strong mode is unstable.

V. NUMERICAL SOLUTIONS OF TIME-DEPENDENT BEHAVIOR

The equations of motion for the homogeneously broadened ring laser model [Eqs. (3.11)] were integrated for parameters typical for the CO_2 laser used in earlier experiments.[6–8] We take $\gamma_{\parallel}=2.5\times10^3$, $\kappa_1\sim\kappa_2=1.4\times10^7$, and $K=1.0007$. As the values of κ_1 and κ_2 are very close, the thresholds and instability boundaries of the two unidirectional solutions are nearly the same. For precision, we show the instability boundary for the strong-mode solution (the more stable of the two modes), in the A,Δ^2 parameter plane in Fig. 9. In this plane we have labeled the points for which we have run extremely long time solutions to the differential equations. Integration was done using a midpoint rule with variable time step.[100] The different relaxation rates of the variables gave several very different time scales to the resulting solutions, and these required us to use an integration step size maximum of 100 normalized time units. Particularly for small detunings, the solutions frequently did not converge to asymptotic behavior until after 10^5 time units (equivalent to almost 0.01 sec of actual laser operation).

Selected examples of the time-dependent behavior are shown in Fig. 10 at various detuning values for $A=4.0$. We observe that for detunings just slightly larger than the lower critical value, $\Delta_c=0.034$, the laser shows a nearly square-wave switching type of behavior, alternating between unidirectional lasing action in one direction and unidirectional lasing action in the other direction.

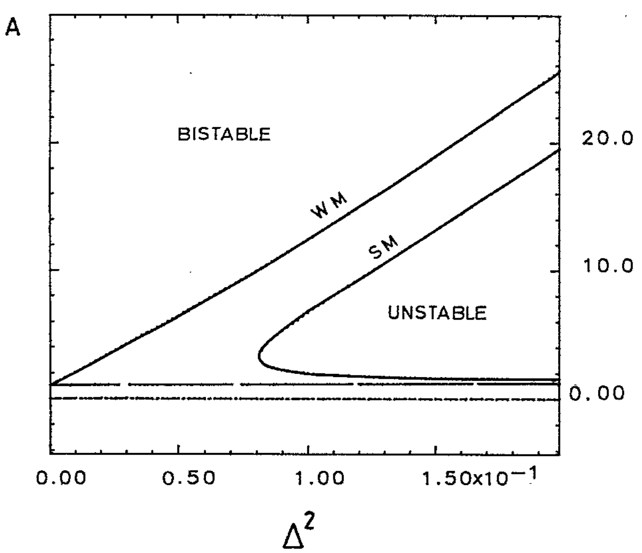


FIG. 6. Combined domains of stability of both the weak mode (WM) and strong mode (SM) for $K=1.01$ and $d_{\parallel}=0.01$. Long-dashed line indicates the strong-mode threshold and the weak-mode threshold (1% higher). The domain above the solid line boundary of the weak-mode boundary corresponds to a bistable domain.

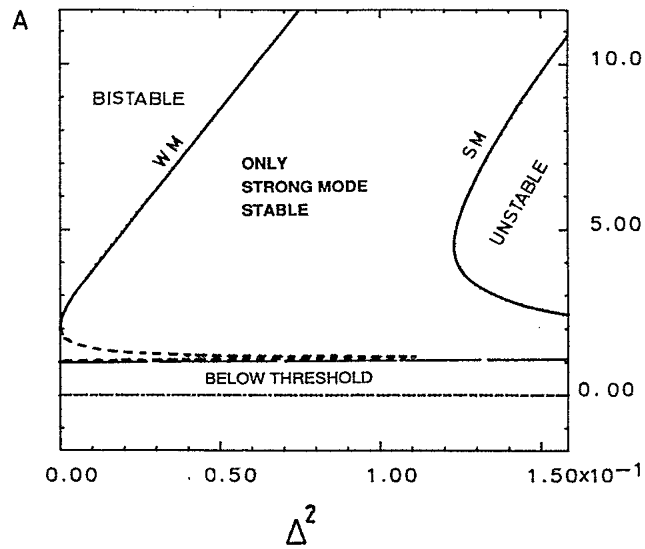


FIG. 7. Same as in Fig. 6 but for $K=1.02$ and $d_{\parallel}=0.01$.

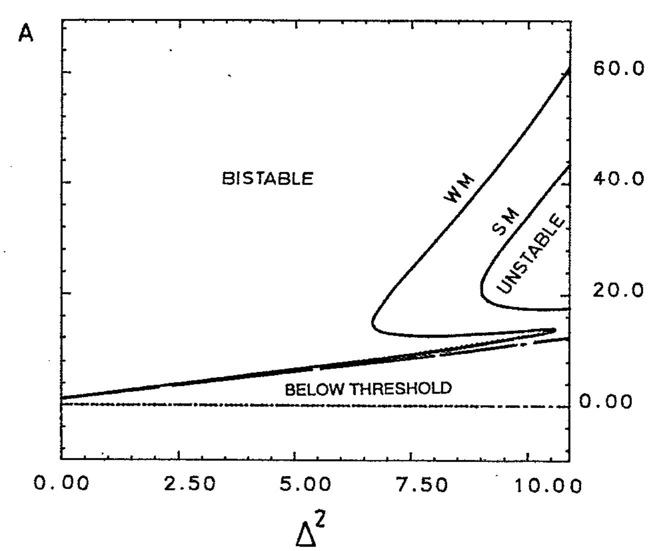


FIG. 8. Same as in Fig. 6 but for $K=1.05$ and $d_{\parallel}=2.0$.

The time between switches is as long as hundreds of thousands of roundtrip times for parameters near the instability boundary. In Fig. 11 we plot the average switching rate (average frequency of the square waves) as a function of Δ near Δ_c, showing the power-law dependence as the critical value is approached. This indicates that the time-dependent behavior shares the boundary of stability of the steady-state solutions found from the stability analysis. Hence, we do not expect to find coexistence of a stable steady state and a time-dependent state in this region of the phase space.

For small values of Δ which appear to give "clean" square-wave pulsations, we show expanded scale versions in Figs. 12(a) and 12(b) which indicate that the mode switching on and the mode switching off relax toward their respective values of the steady-state solution (on-mode intensity $I = A - 1 - \Delta^2$; off-mode intensity $I = 0$) by small amplitude, damped oscillations at frequencies which are approximately those given by the imaginary parts of the eigenvalues of the stability analysis. The

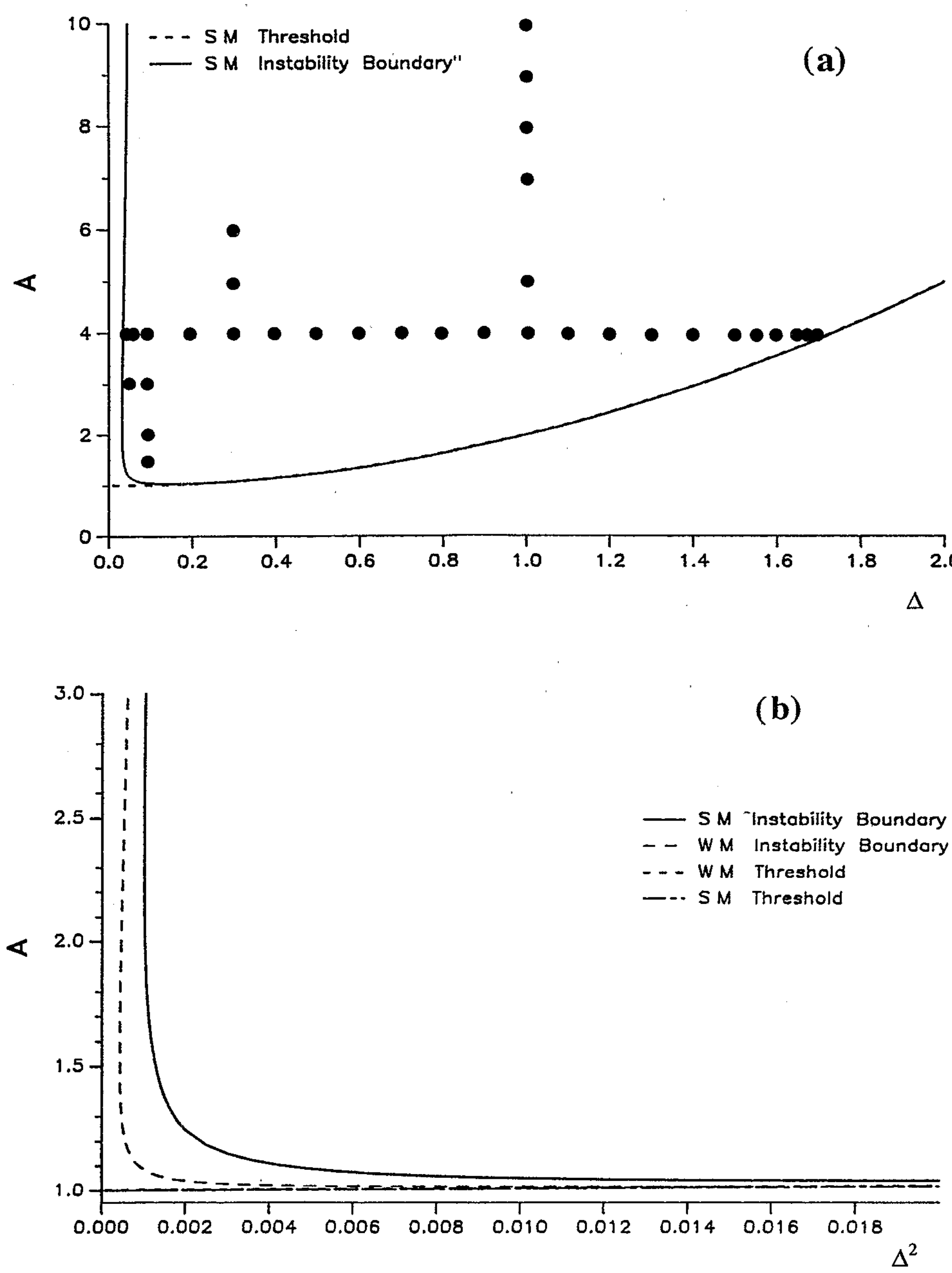


FIG. 9. (a) Stability boundary for the strong-mode solution for parameters appropriate to a CO_2 laser: $\kappa_1 = 1.4 \times 10^7$, $\kappa_2 = 1.4001 \times 10^7$, $\gamma_\parallel = 2.5 \times 10^3$, $K = 1.0007$. Dots indicate parameter values for which numerical integration studies were used to find time-dependent solutions. (b) Expanded version for small pump and detuning values showing strong- and weak-mode thresholds (nearly identical because K is near 1) and the instability boundaries for both modes.

two frequencies differ by almost exactly the square root of 2 as predicted for the difference in eigenfrequencies for the two factors in the stability analysis on resonance.[5,54] For the decaying mode, the mean intensity and the amplitude of the oscillations decrease until the mean intensity arrives sufficiently close to zero; then the mean intensity begins to grow while the amplitude of the oscillations continues to decrease. This indicates that

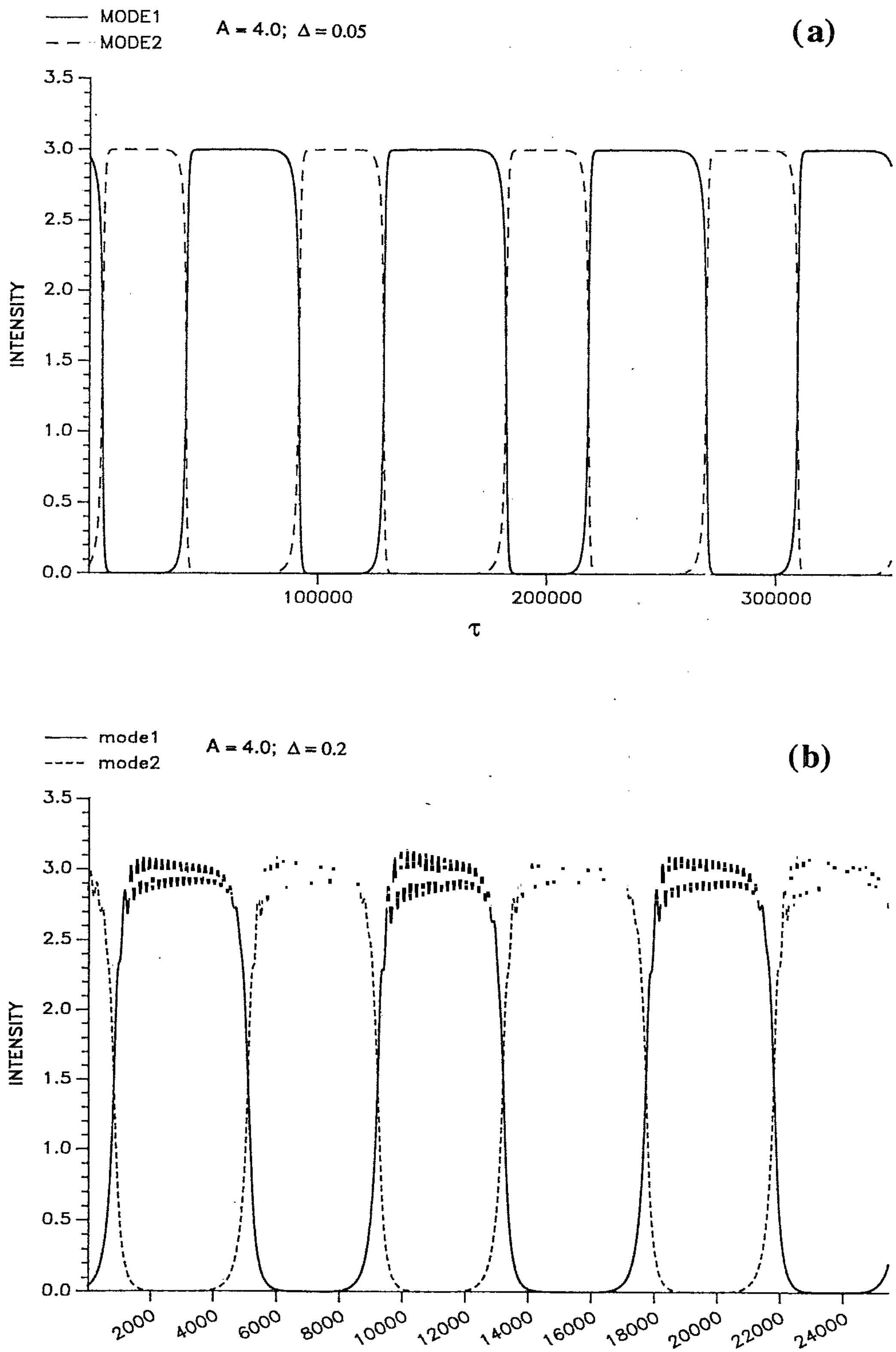


FIG. 10. Time dependence of I_1 and I_2 for $A=4.0$ at various detunings. $\tau=\kappa_1 t$ is the adimensional time variable.

the solution has discovered new directions in the phase space which are unstable. About where the mean intensity of the near-zero mode begins to increase, the corresponding modulation of the strong mode appears to suffer one or more abrupt jumps or phase shifts and the ensuing modulation becomes quasiperiodic, reflecting the simultaneous modulation by two frequencies differing by the square root of 2.

The behavior in the vicinity of the (now unstable) fixed points suggests that the stable and unstable manifolds of

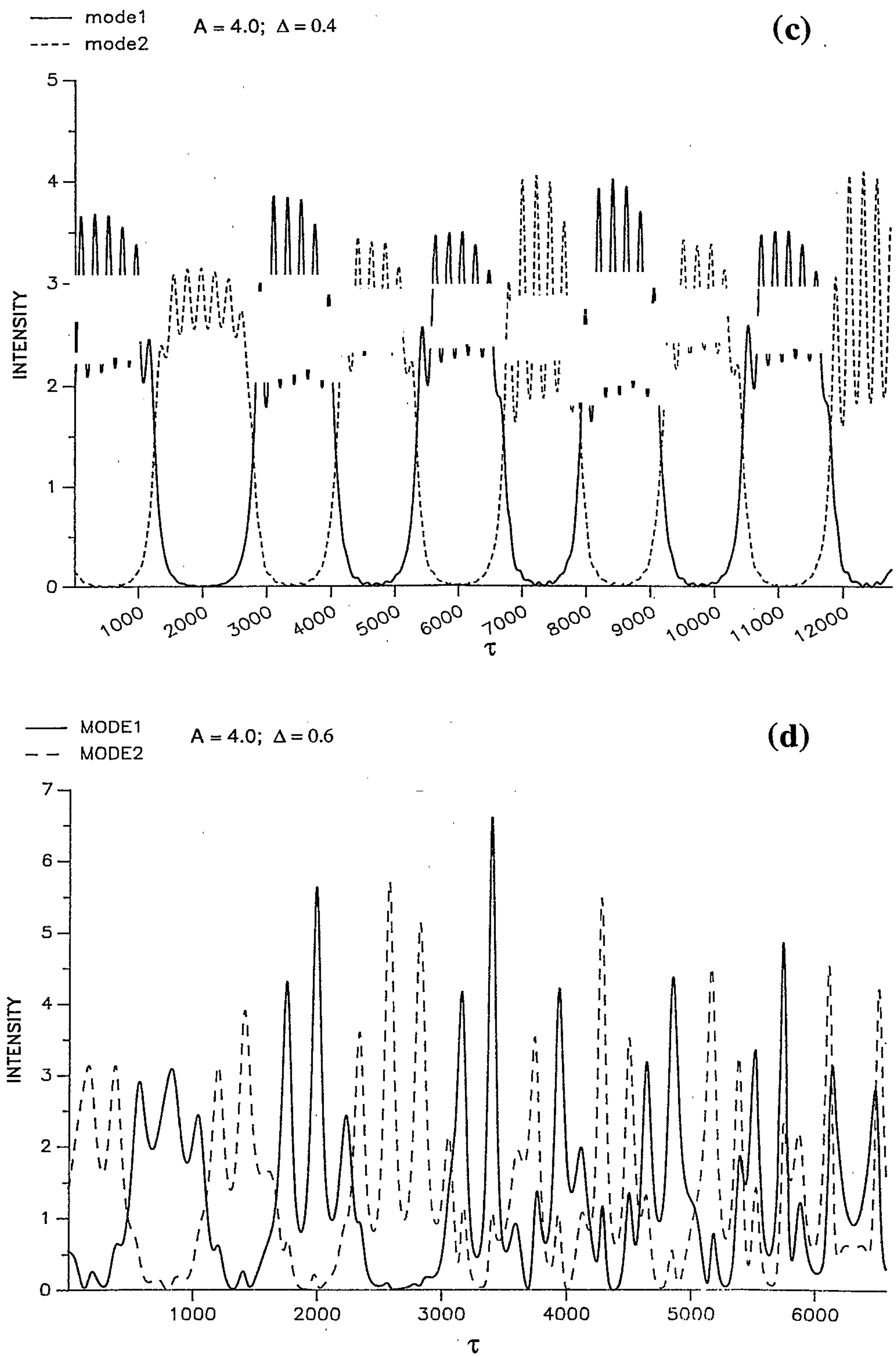


FIG. 10. (*Continued*).

the system are composed, respectively, of four stable dimensions (including attracting foci in two separate pairs of dimensions), and two unstable dimensions which lead to a spiraling away from the fixed point. This agrees with the picture generated by the linear stability analysis.

The square-wave switching never appears to be exactly periodic, rather there are residual fluctuations in the times between switches, although the fluctuations seem to be reduced as the detuning is reduced towards the instability threshold. This appears to be evidence of deterministic chaos and efforts are underway to measure and

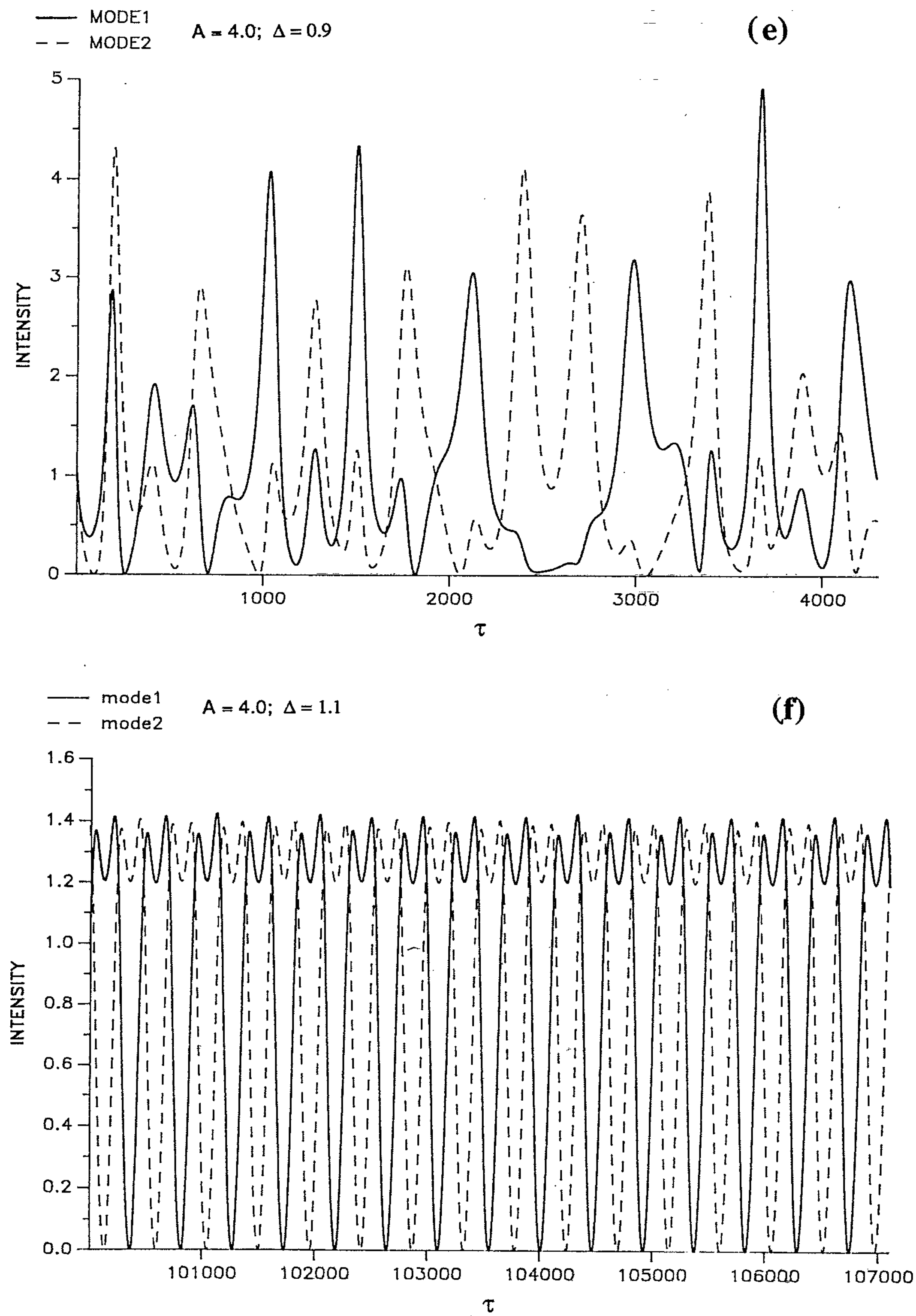
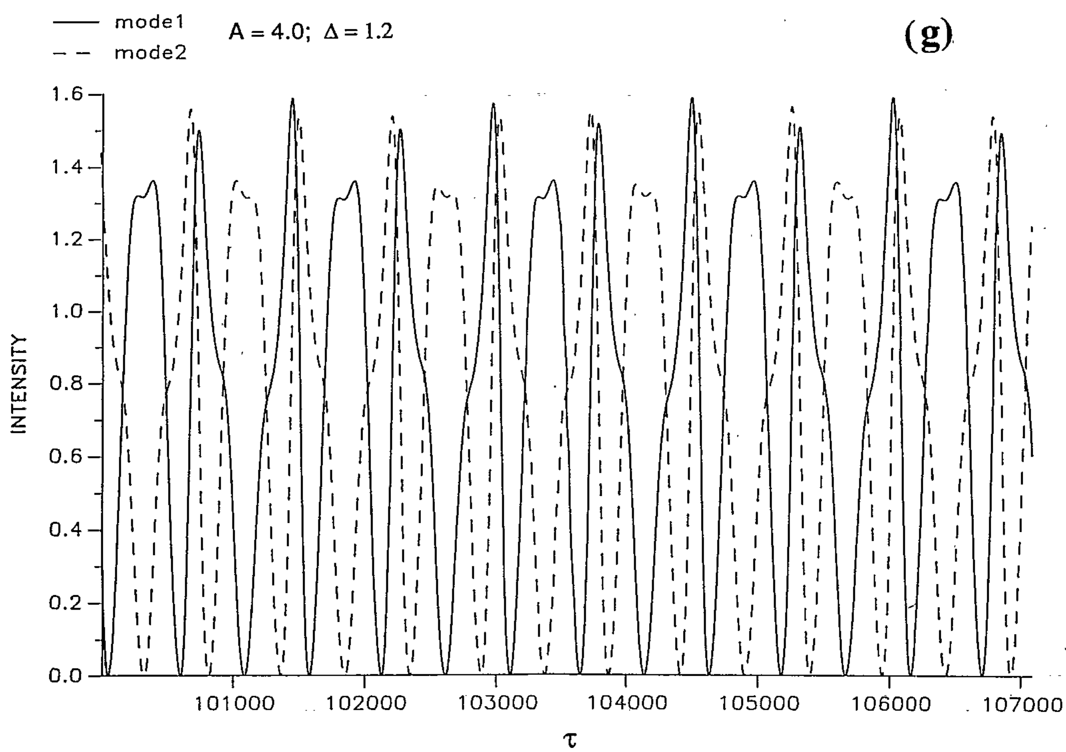


FIG. 10. (*Continued*).

characterize the nature of the fluctuations and to confirm that their origin lies, as suspected, in deterministic effects.

With increasing detuning, the depth of modulation of the mode which is on increases. For $\Delta \geq 0.6$ the square-wave mode of switching breaks down entirely in favor of a more irregular pulsation pattern. For $0.6 < \Delta < (A-1)^{1/2}$ the pulsations are generally irregular and chaotic; however, periodic oscillations have been observed in a small region near $\Delta \sim 1.0-1.2$.

We have also sampled the time-dependent solutions along a line in the parameter space given by fixed detun-



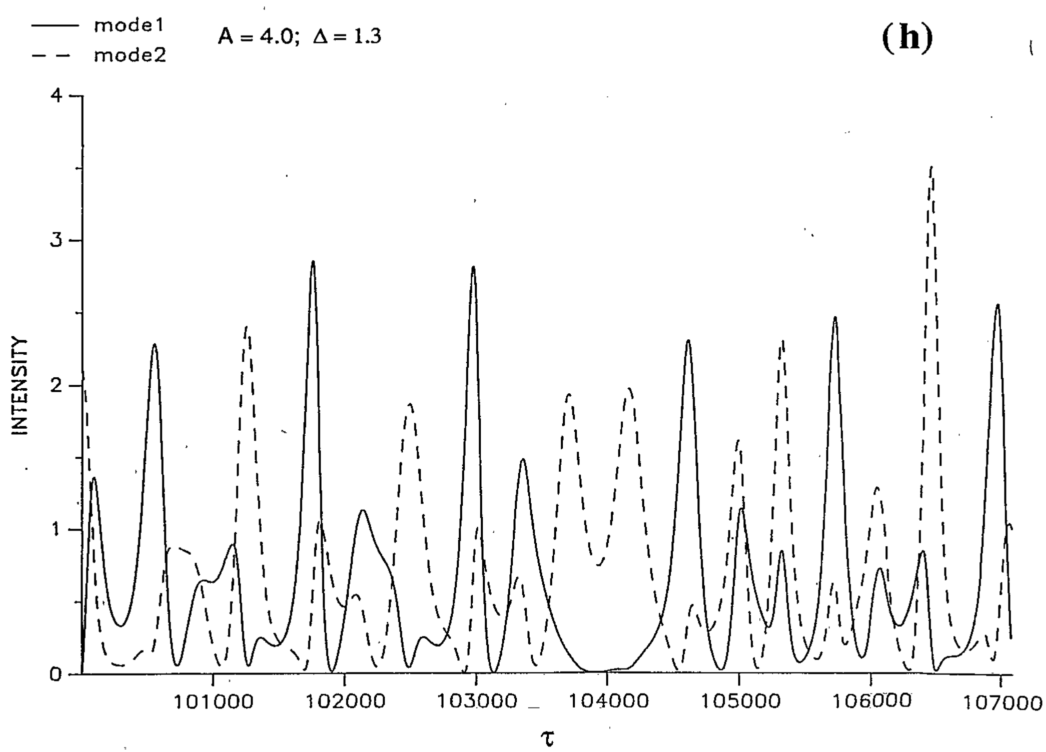


FIG. 10. (*Continued*).

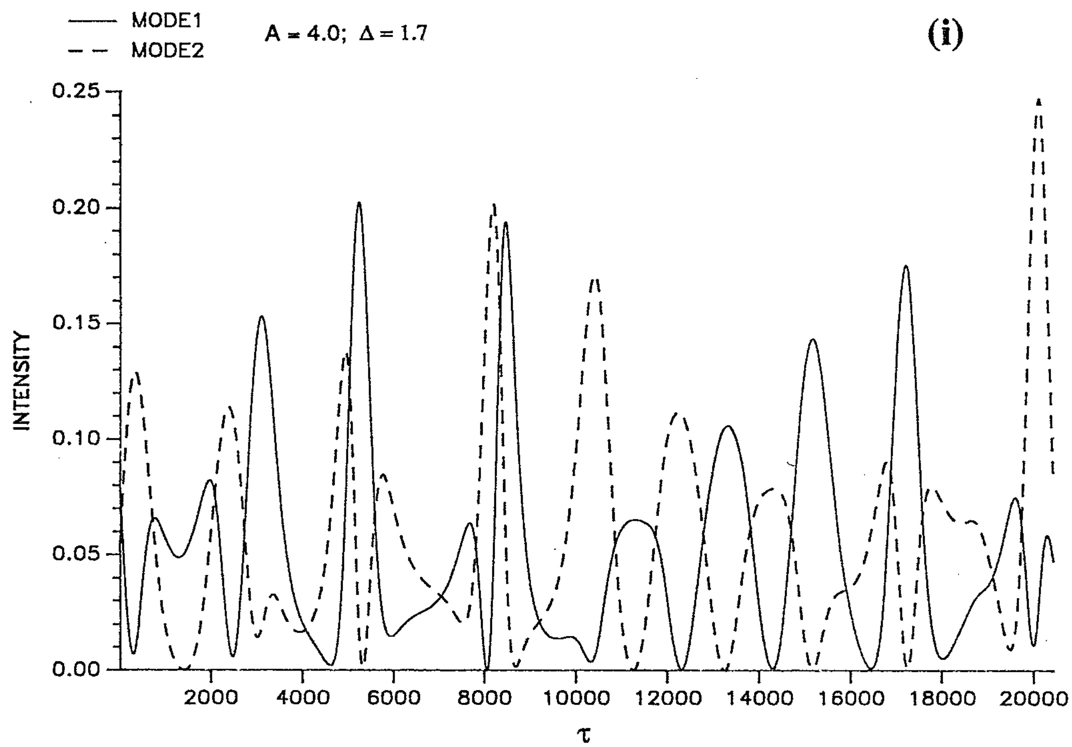


FIG. 10. (*Continued*).

ing and different values of the pump parameter. For $A=4.0$ and $\Delta=0.1$ the behavior is of the square-wave switching type. The switching rate decreases with decreasing pump values and there is again evidence (seen in Fig. 13) of a power-law scaling of the switching rate as the instability boundary is approached. This again indicates that this portion of the stability boundary in the (A,Δ^2) parameter space of the strong-mode steady-state solution is also the stability boundary of the time-dependent solutions.

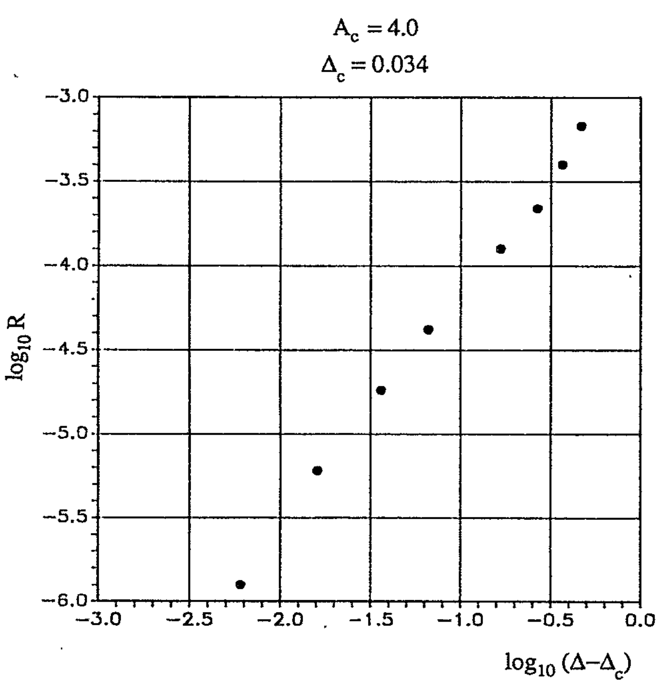


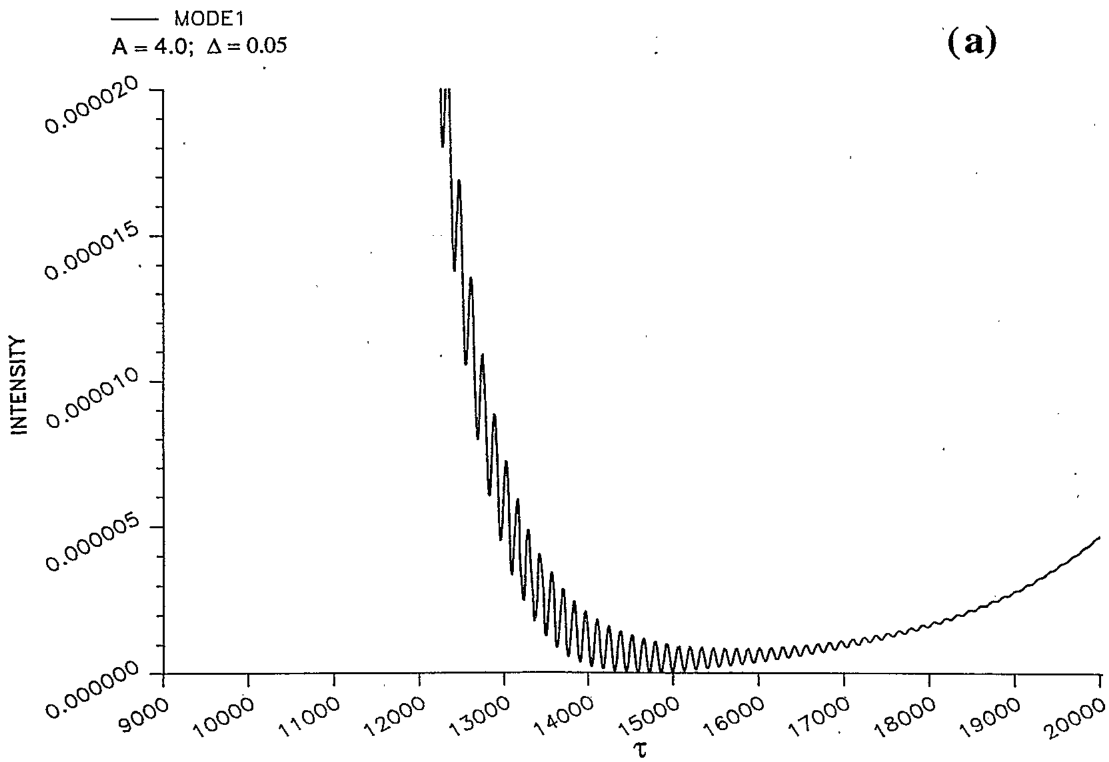
FIG. 11. Critical slowing down of square-wave switching is shown by a log-log plot of the inverse of the mean time between switches R vs the difference between the detuning and its critical value on the instability boundary. $A=4.0$.

For $\Delta=1.0$, as shown in Fig. 14, we see that for $7.0 < A < 10.0$, the irregular pulsing of lower pump values gives way to a form of regular, double-peaked pulses. For $A=7.0$, 8.0, and 9.0 it appears that the double-peaked pulses alternate regularly between two different pulse shapes with the differences between the two different shapes increasing with increasing pump. For even larger values of A, the periodic behavior is replaced again by chaotic pulsing.

VI. INFLUENCE OF INHOMOGENEOUS BROADENING

In this section we shall try to assess how inhomogeneous broadening affects the boundaries of the stability of the single-mode solutions derived in Sec. IV. We shall use a typical model[101] in which the relative weight of individual two-level atomic groups with their center frequencies ω is given by a Lorentzian distribution centered around a most probable frequency ω_A, with a characteristic width u measured in units of $\gamma_\perp$. In this case Eqs. (3.1) are replaced by

$$i(\partial_t+\kappa_j)\beta_j(t)-\omega_C\beta_j(t) = \frac{N}{L}\int_0^L dx \int_{-\infty}^{+\infty} d\omega\, g_j^*(x,\omega)h(\omega)\alpha(\omega,x,t) ,$$



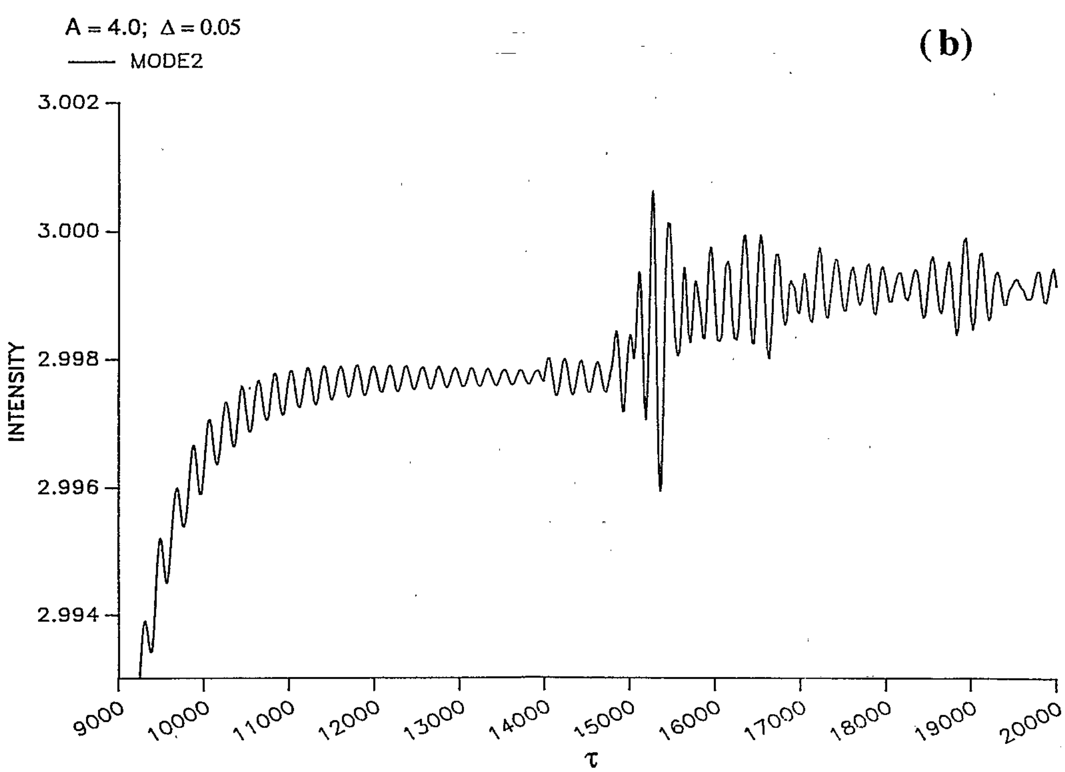


FIG. 12. Expanded scale plots of I_1 and I_2 as the time-dependent solutions pass near one of the steady-state solutions for $A=4.0$ and $\Delta=0.05$. Note that the ringing frequencies on approach toward the steady-state solution differ by a factor of about 1.4 and that as the solution leaves the vicinity of the steady-state solution a quasiperiodic modulation (a combination of both frequencies) appears on the mode near the nonzero steady-state value.

$$i(\partial_t+\gamma_\perp)\alpha(\omega,x,t)-\omega\alpha(\omega,x,t) = -D(\omega,x,t)\sum_j g_j(x,\omega)\beta_j(t) ,$$

$$i(\partial_t+\gamma_\parallel)D(\omega,x,t) = i\sigma\gamma_\parallel+2\sum_j [g_j(x,\omega)\alpha^*(\omega,x,t)\beta_j(t)-\text{c.c.}] ,$$

where

$$\gamma_\perp h(\omega)\equiv f(z)\equiv\frac{u}{\pi(z^2+u^2)}, \quad z\equiv\frac{\omega_A-\omega}{\gamma_\perp} .$$

Proceeding as in Sec. III for the adiabatic elimination of the polarization and the harmonic truncation we obtain the generalization of Eqs. (3.11):

$$(\partial_\tau+1)E_1=A\int_{-\infty}^{+\infty}dz\, f(z)\frac{1+i\delta}{1+\delta^2}(E_1\mathcal{D}_0+E_2\mathcal{D}_1^*) , \tag{6.1a}$$

$$(\partial_\tau+K)E_2=A\int_{-\infty}^{+\infty}dz\, f(z)\frac{1+i\delta}{1+\delta^2}(E_2\mathcal{D}_0+E_1\mathcal{D}_1) , \tag{6.1b}$$

$$(\partial_\tau+d_\parallel)\mathcal{D}_0=d_\parallel-\tilde{d}_\parallel(I_1+I_2)\mathcal{D}_0-\tilde{d}_\parallel(E_1E_2^*\mathcal{D}_1+\text{c.c.}) , \tag{6.1c}$$

$$(\partial_\tau+d_\parallel)\mathcal{D}_1=-\tilde{d}_\parallel(I_1+I_2)\mathcal{D}_1-\tilde{d}_\parallel E_1^*E_2\mathcal{D}_0 , \tag{6.1d}$$

with $\delta\equiv z+\Delta$, where $\Delta=(\omega_C-\omega_A)/\gamma_\perp$ as before, and where the material variables and δ are functions of z.

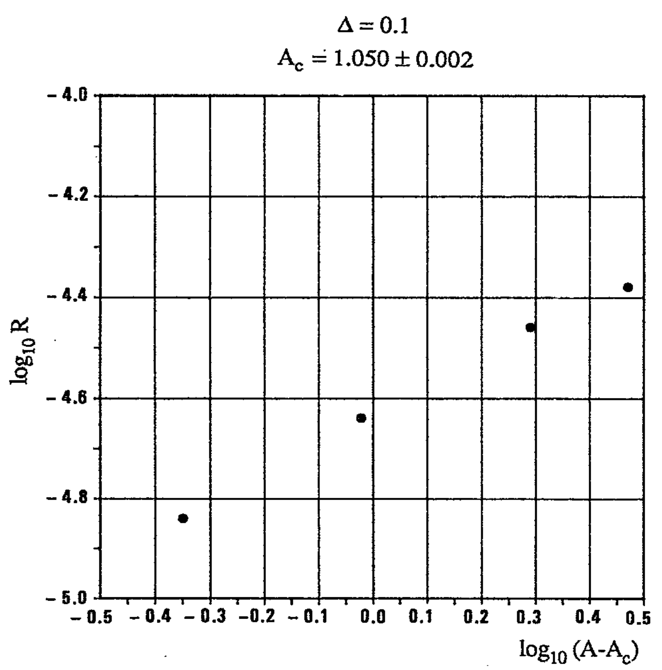


FIG. 13. Critical slowing down is also demonstrated in a log-log plot of the inverse of the mean time between switches vs the excess pump above the critical value on the stability boundary for a fixed value of the detuning equal to 0.1.

To find the steady-state solutions of Eqs. (6.1), we introduce the polar decomposition $E_j=\sqrt{I_j}\exp(i\theta_j)$ and define the steady-state solution(s) by the conditions,

$$\partial_\tau I_j=0,\quad \partial_\tau\mathcal{D}_j=0,\quad \partial_\tau\theta_j=\Lambda_j,\quad \partial_\tau\Lambda_j=0 ,$$

where Λ_j is the laser frequency referenced to ω_A in units of $\gamma_\perp$. As in the homogeneously broadened case, we again find four steady-state solutions.

(i) The trivial solution,

$$E_1=E_2=0 . \tag{6.2}$$

(ii) The strong-mode solution given by

$$1=\frac{A}{\sqrt{1+I_1}}\frac{u+\sqrt{1+I_1}}{\Delta^2+(u+\sqrt{1+I_1})^2} , \tag{6.3a}$$

$$\Lambda_1=A\frac{\Delta}{\Delta^2+(u+\sqrt{1+I_1})^2} , \tag{6.3b}$$

together with $I_2=0$. Note that I_1 will be positive only when A exceeds a threshold value $A_{1\text{thr}}$ given by

$$A_{1\text{thr}}=(u+1)\left[1+\left[\frac{\Delta}{1+u}\right]^2\right] . \tag{6.4a}$$

At threshold the frequency is

$$\Lambda_{1\text{thr}}=\frac{\Delta}{u+1} , \tag{6.4b}$$

which indicates the additional mode pulling of the laser frequency toward the atomic resonance as is usually associated with inhomogeneously broadened transitions.

(iii) The weak-mode solution given by

$$K=\frac{A}{\sqrt{1+I_2}}\frac{u+\sqrt{1+I_2}}{\Delta^2+(u+\sqrt{1+I_2})^2} , \tag{6.5a}$$

with the frequency given by

$$\Lambda_2=A\frac{\Delta}{\Delta^2+(u+\sqrt{1+I_2})^2} , \tag{6.5b}$$

and $I_1=0$. The intensity I_2 will be non-negative provided A exceeds a threshold $A_{2\text{thr}}$ given by

$$A_{2\text{thr}}=KA_{1\text{thr}} . \tag{6.6a}$$

At this threshold the laser frequency is given by

$$\Lambda_{2\text{thr}}=K\Lambda_{1\text{thr}} . \tag{6.6b}$$

(iv) The fourth steady-state solution is the bidirectional solution for which both I_1 and I_2 are nonzero. Although the integrals can be performed explicitly, the resulting coupled equations are too complex to be of practical use and therefore will not be written.

We now consider the stability questions. As expected, the trivial solution is easily shown to be stable provided $A < A_{1\text{thr}}$ when $K\geq 1$.

A. Linear stability of the strong-mode solution

A linear stability of the strong-mode steady solution is ruled by a pair of characteristic equations as in the homogeneously broadened case. The stability of the oscillating mode I_1 is determined by the equation,

$$\lambda^2-2d_\parallel=-2d_\parallel A\frac{u+\gamma}{\gamma[\Delta^2+(u+\gamma)^2]}\ , \tag{6.7}$$

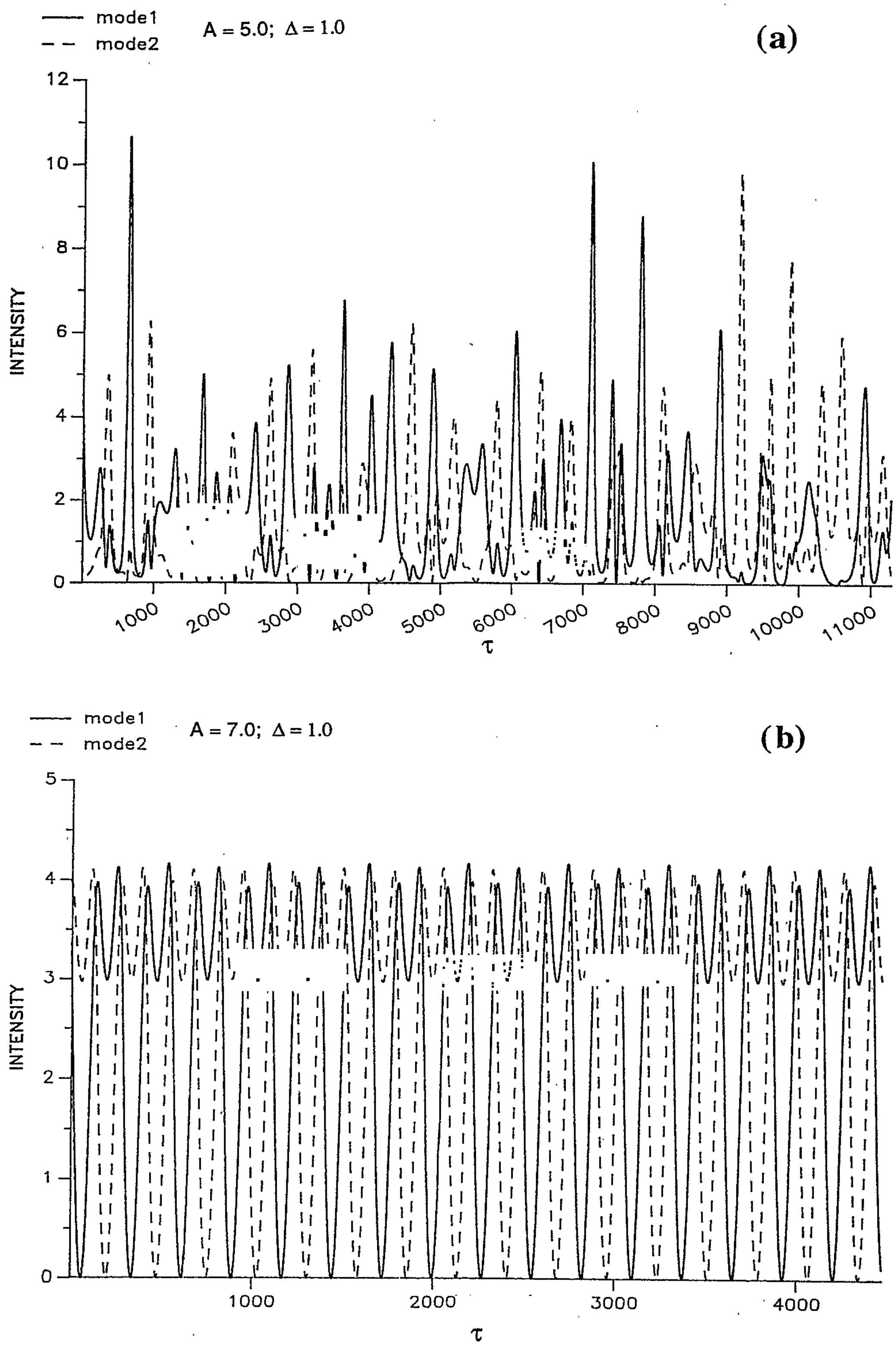


FIG. 14. Selected time-dependent solutions for the detuning fixed at 1.0 for various pump values showing various degrees of "period doubled" solutions in a window between chaotic behavior for higher and lower pump values.

whereas the stability of the suppressed mode is governed by the equation,

$$\lambda^2+\lambda(K-1)-d_{\parallel}(1+i\Lambda_1)=-\frac{d_{\parallel}A}{\gamma}\frac{u+\gamma(1+i\Delta)}{\Delta^2+(u+\gamma)^2} \quad (6.8)$$

and its complex conjugate. In both equations we have used the definitions,

$$\gamma\equiv\left[\frac{\lambda+d_{\parallel}J^2}{\lambda+d_{\parallel}}\right]^{1/2}, \quad J^2\equiv 1+I_1 \quad (6.9)$$

and the relations,

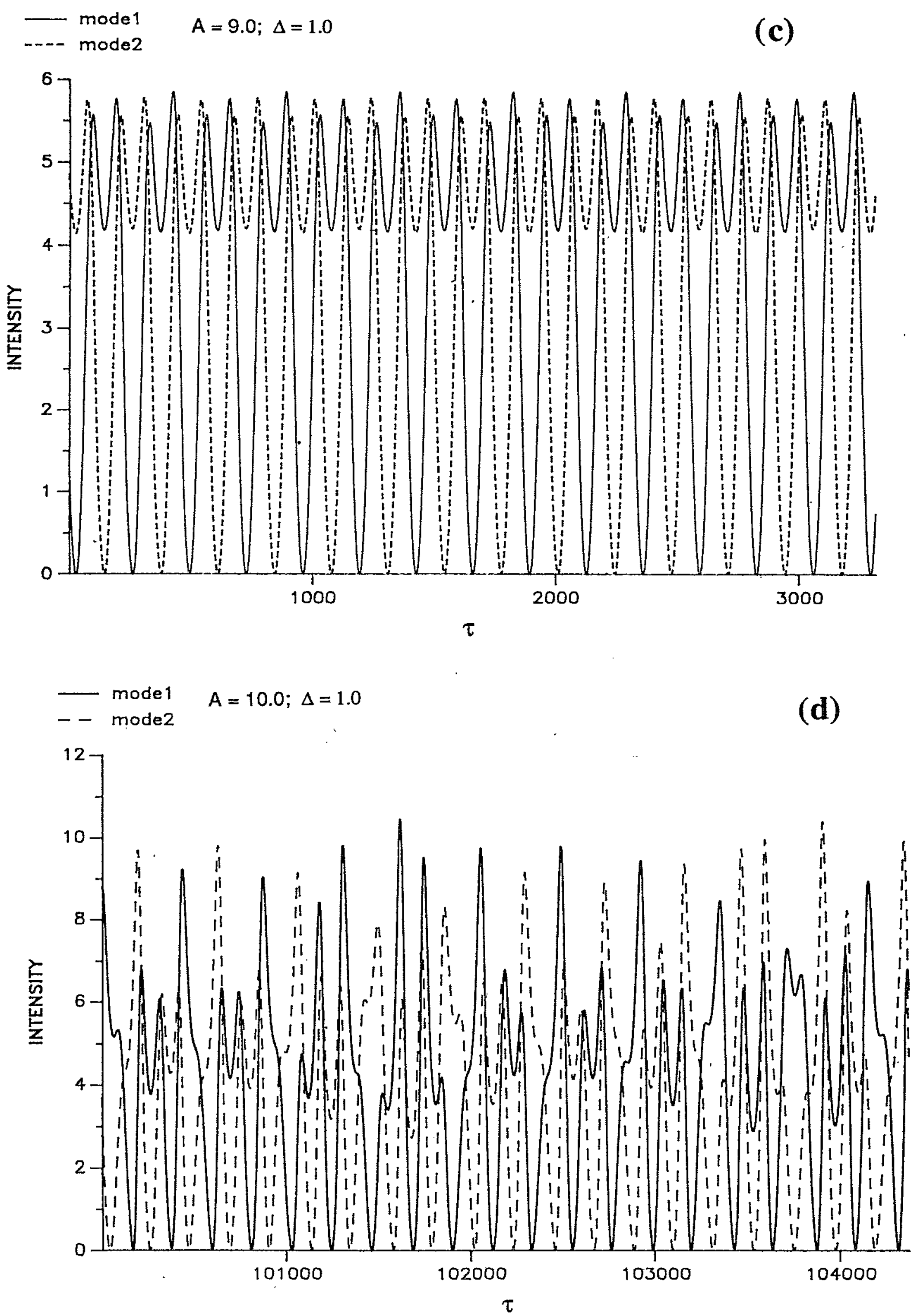


FIG. 14. (*Continued*).

$$A = J(J+u) + \Delta^2 \frac{J}{J+u} ,$$
$$\Lambda_1 = \Delta \frac{J}{J+u} , \tag{6.10}$$

which follow from the steady-state equations, Eq. (6.3). No general solution has been obtained from Eqs. (6.7) and (6.8). However, we analyze two limiting cases.

(i) We first consider the case

$$d_{\parallel} = \epsilon, \quad 0 < \epsilon \ll 1 . \tag{6.11}$$

When u, Δ, and J are $O(1)$ functions, we can show that the solution of the characteristic equation (6.7) is of the form

$$\lambda = \epsilon^{1/2}\lambda_0 + \epsilon\lambda_1 + O(\epsilon^{3/2}) , \tag{6.12}$$

where

$$\lambda_0 = \pm i \left[\frac{2(J-1)}{\Delta^2 + (u+1)^2} \right]^{1/2} \left[(u+1)(J+u+1) + u\frac{\Delta^2}{J+u} \right]^{1/2} , \tag{6.13}$$

$$\lambda_1 = \frac{-J(J+1)}{4} \frac{[\Delta^2 + (J+u)^2][u\Delta^2 + (u+1)^2(u+2)]}{[\Delta^2 + (u+1)^2][u\Delta^2 + (u+1)(J+u)(J+u+1)]} < 0 . \tag{6.14}$$

These results, which indicate that (6.7) leads to no instability in the limit (6.11), present some interesting features. First, we mention that the expression given in (6.14) for λ_1 holds only if $\lambda_0 \neq 0$; otherwise, λ_1 also vanishes at the lasing threshold $A = A_{1\text{thr}}$ corresponding to $J=1$. Second, we notice that the inhomogeneous broadening has not modified the fact that for $d_{\parallel} \ll 1$ the steady solution is only weakly stable; a perturbation from the steady state will be damped very slowly, the dominant relaxation time diverging like $1/\epsilon$. Furthermore, in the limit of large inhomogeneous broadening, we have

$$\lambda_0 = \pm i\sqrt{2(J-1)} + O(1/u) ,$$
$$\lambda_1 = \frac{-J(J-1)}{4} + O(1/u) , \tag{6.15}$$

which are independent of the detuning Δ. This turns out to be a general property, namely, that in the large-u limit the characteristic roots λ no longer depend on the detuning.

(ii) Another case which has been investigated is the domain defined by

$$u = 1/\epsilon, \quad 0 < \epsilon \ll 1 , \tag{6.16}$$

where $d_{\parallel}$, Δ, and J are $O(1)$ functions. In this domain the expansion of the characteristic root λ in a power series of ϵ is given by

$$\lambda = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2) \tag{6.17}$$

and the leading term satisfies the equation,

$$\lambda_0^2 - 2d_{\parallel} = -2d_{\parallel}J/\gamma , \tag{6.18}$$

with γ defined in (6.9). It can be shown that the roots of (6.18) always have a negative real part. For $d_{\parallel} < 2$ the two roots are complex conjugates whereas for $d_{\parallel} > 2$ they are real. Thus in this case also there is no source of instability.

We now consider the stability properties of the suppressed mode of the strong-mode solution, whose stability is ruled by the Eq. (6.8). Various domains of the parameter space have been analyzed.

(i) Firstly, we consider

$$d_{\parallel} = \epsilon; \quad K = 1 + a\epsilon + O(\epsilon^2), \quad \Delta^2 = 0 , \tag{6.19}$$

with the specifications $0 < \epsilon \ll 1$ and $a > 0$. In this domain the characteristic roots have the form

$$\lambda = \epsilon^{1/2}\lambda_0 + \epsilon\lambda_1 + O(\epsilon^2) , \tag{6.20}$$

where

$$\lambda_0 = \pm i \left[\frac{(J-1)(J+u+1)}{u+1} \right]^{1/2} ,$$

and

$$\lambda_1 = -1/2 \left[a + \frac{J(J+1)(J+u)(u+2)}{2(u+1)(J+u+1)} \right] , \tag{6.21}$$

which describe damped oscillations but no instabilities.

(ii) Secondly, we consider

$$d_{\parallel} = \epsilon, \quad K = 1 + a\epsilon + O(\epsilon^2), \quad \Delta = \epsilon^{1/2}p + O(\epsilon) . \tag{6.22}$$

Correspondingly, the characteristic equation has solutions of the form

$$\lambda(\pm) = \pm\epsilon^{1/2}\lambda_0 + \epsilon(\lambda_1 \pm \lambda_2) + O(\epsilon^2) , \tag{6.23}$$

where

$$\lambda_0 = i \left[\frac{(J-1)(J+u+1)}{u+1} \right]^{1/2} ,$$
$$\lambda_1 = -\frac{2a(u+1)(J+u+1) + J(J+u)(J+1)(u+2)}{4(J+u+1)(u+1)} ,$$
$$\lambda_2 = \frac{p}{2} \left[\frac{J-1}{(u+1)(J+u+1)} \right]^{1/2} \frac{J(J+2u+1)}{(J+u)(u+1)} . \tag{6.24}$$

In addition, $\lambda^*(\pm)$ are also characteristic roots of the complete stability problem. This time an instability can occur because $\lambda_1 \pm \lambda_2$ may vanish (for $p < 0$ and $p > 0$, respectively). The condition $\lambda_1 \pm \lambda_2 = 0$ is an implicit equation for A_c, the critical value of the pump which signals the onset of a Hopf bifurcation. The implicit equation has been solved numerically and Fig. 15 shows how the

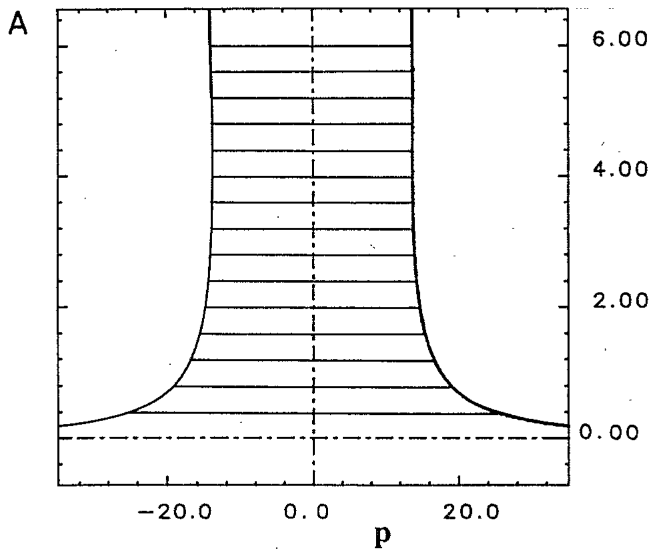


FIG. 15. Stability boundaries of the strong-mode solution vs cavity detuning for $a=1.0$ and $u=2.0$. The internal domain is the region of stability.

critical intensity depends on p. Furthermore, we find that the stability domain increases as either a or u is increased.

(iii) Thirdly, we consider

$$u=1/\epsilon, 0<\epsilon\ll 1,\quad d_{\parallel}=O(1),\quad K-1=a=O(1)\ . \tag{6.25}$$

The expansion for the characteristic root is

$$\lambda=\lambda_0+\epsilon\lambda_1+O(\epsilon^2)\ , \tag{6.26}$$

where λ_0 is the solution of

$$\lambda_0^2+a\lambda_0-d_{\parallel}=-d_{\parallel}J/\gamma\ . \tag{6.27}$$

For $a>0$ this equation leads to no instability point. This result is consistent with the property, already mentioned, that stability increases with increasing u and $d_{\parallel}$.

B. Linear stability of the weak-mode solution

The linear stability of the weak mode is ruled by a pair of characteristic equations. For the nonzero intensity I_2 we find

$$\lambda^2-2d_{\parallel}K=-2d_{\parallel}K\left[J(J+u)+\Delta^2\frac{J}{J+u}\right]\times\frac{u+\gamma}{\gamma[\Delta^2+(u+\gamma)^2]}\ , \tag{6.28}$$

whereas the stability of the suppressed mode I_1 is governed by

$$\lambda^2+(1-K)\lambda-d_{\parallel}K\left[1+i\Delta\frac{J}{J+u}\right]=-d_{\parallel}K\left[J(J+u)+\Delta^2\frac{J}{J+u}\right]\frac{u+\gamma(1+i\Delta)}{\gamma[\Delta^2+(u+\gamma)^2]} \tag{6.29}$$

and its complex conjugate. For these equations we use the definitions,

$$\gamma\equiv\left[\frac{\lambda+d_{\parallel}J^2}{\lambda+d_{\parallel}}\right]^{1/2},\quad J^2\equiv 1+I_2\ . \tag{6.30}$$

As in the preceding sections, no instability of the nonzero intensity has been found in the domains $d_{\parallel}=\epsilon$, $u=O(1)$, and $u=1/\epsilon, d_{\parallel}=O(1)$. By contrast, the zero intensity I_1 can become unstable. We again analyze three domains.

(i) The first domain is

$$d_{\parallel}=\epsilon\ll 1,\quad \Delta=0,\quad K=1+a\epsilon+O(\epsilon^2),\quad a>0\ . \tag{6.31}$$

The solutions of the characteristic equation (6.29) have the following structure:

$$\lambda=\pm\epsilon^{1/2}i\lambda_0+\epsilon\lambda_1+\cdots$$

and one can easily show that

$$\lambda_0=\left[\frac{(J-1)(J+u+1)}{u+1}\right]^{1/2},\qquad \lambda_1=\frac{1}{2}\left[a-\frac{J(J+u)(J+1)(u+2)}{2(u+1)(J+u+1)}\right]\ . \tag{6.32}$$

Quite clearly an instability is possible and occurs for a J_c such that $\lambda_1=0$. This leads to a cubic with only one real physical solution ($J_c>1$) for $u\geq O(1)$. The solution begins at $a=1$ and is monotonically increasing function of a. This means that to have an instability with $\Delta=0$ there must be some nonreciprocity of the losses ($K\geq 1+\epsilon$, where $d_{\parallel}=\epsilon$). A somewhat surprising property, which is related to the perfect tuning condition, is that the stability domain shrinks when u increases. This is because in perfect tuning ($\Delta=0$) the effect of increasing u is to decrease the gain at $\Delta=0$ since $f(z)$, the inhomogeneous line shape, is a normalized function.

(ii) In order to assess the influence of a small detuning we now consider the domain

$$d_{\parallel}\equiv\epsilon\ll 1,\quad \Delta=\epsilon^{1/2}p+O(\epsilon)\ ,\qquad K=1+a\epsilon+O(\epsilon^2),\quad a>0\ . \tag{6.33}$$

The roots of the characteristic equation (6.29) now become

$$\lambda=\pm i\epsilon^{1/2}\lambda_0+\epsilon(\lambda_1\pm\lambda_2)+O(\epsilon^{3/2})\ ,$$

where λ_0 and λ_1 are given by (6.32), while

$$\lambda_2=\frac{p}{2}\lambda_0\frac{J(J+2u+1)}{(J+u)(u+1)(J+u+1)}\ . \tag{6.34}$$

Again an instability will develop for $\lambda_1\pm\lambda_2=0$. Two cases have to be considered, depending on the value of $a>0$. When $0<a<1$, the stability boundaries are shown on Fig. 16, which also contains the strong-mode stability boundary. The weak mode is stable in the domain containing the $p=0$ line. For $a=1$ and $a>1$, Figs. 17 and 18, respectively, display the boundaries of stability for both the weak and strong modes. It is in-

teresting to note that for $a > 1$, the weak mode is stable only above the point where the two boundaries cross. As appears clearly on Figs. 16, 17, and 18, the stability domain of the weak mode is always embedded in the stability domain of the strong mode. Hence, the weak-mode stability automatically implies bistability between the two modes. At this point we can discuss the influence of the inhomogeneous width. For $u=0$ and when condition (6.33) [which corresponds to (4.18), (4.19), and (4.27)] is satisfied, there are three domains giving one (I) or possibly three (III and III′) Hopf bifurcations for fixed Δ^2 as A is varied:
domain III′,

$$K-1<O(d_{\parallel}) ,$$

domain I,

$$K-1=O(d_{\parallel}) ,$$

and domain III,

$$K-1>O(d_{\parallel}) .$$

These domains have been analyzed in Sec. IV and are displayed in Figs. 3–5. In all cases there is at least one bifurcation for $\Delta=0$. By contrast, when $u=O(1)$ we find that there will never be three Hopf bifurcations; rather there will be either one (I) or possibly two (II) under the following conditions:
domain II,

$$K-1\le d_{\parallel} ,$$

and domain I,

$$K-1>d_{\parallel} .$$

Domain II denotes a stability diagram qualitatively similar to that displayed in Fig. 1 but this time for the weak mode. This means that now there is no bifurcation for

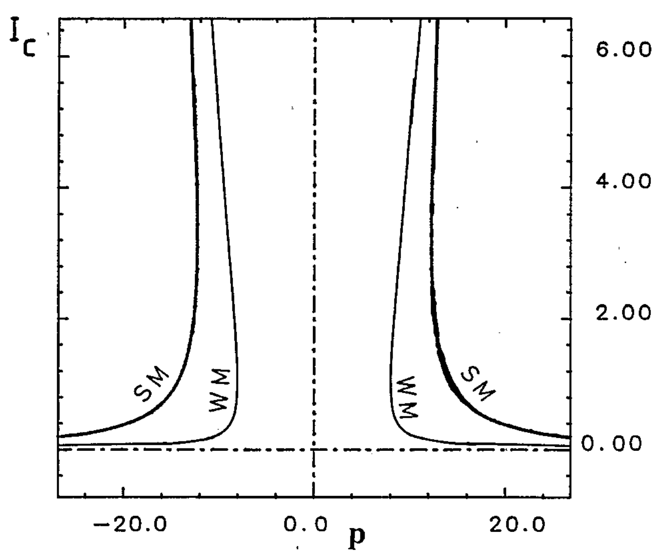


FIG. 16. Stability boundaries of the weak- and strong-mode solutions vs cavity detuning for $a=0.5$ and $u=2.0$. The solutions are stable in the domain containing $p=0.0$.

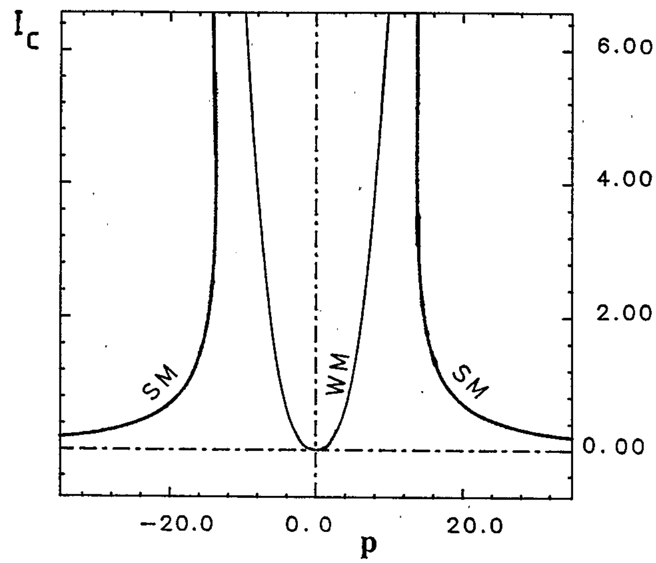


FIG. 17. Stability boundaries of the weak- and strong-mode solutions vs cavity detuning for $a=1.0$ and $u=2.0$. The solutions are stable in the domain containing $p=0.0$.

sufficiently small Δ^2 even though there was always such a bifurcation when $u=0$. Hence we see that an increase of u simplifies the bifurcation diagram by reducing the number of possible instabilities. This is related to the fact that for $u=0$ the intricate part of the diagrams occurs for small values of the pump parameter. As u increases, the values of A which lead to $J\ge 1$ (making the solution physical) also increase. For instance, the condition $J=1+\alpha\epsilon$ leads to $A=1+u+\Delta^2/(u+1)+O(\epsilon)$. Hence the intricate domain of multiple bifurcations still exists but becomes nonphysical because it corresponds to $J<1$. Likewise an increase in u leads to an increased domain of stability and therefore of bistability. This is in contrast to the $\Delta=0$ case and results because initially

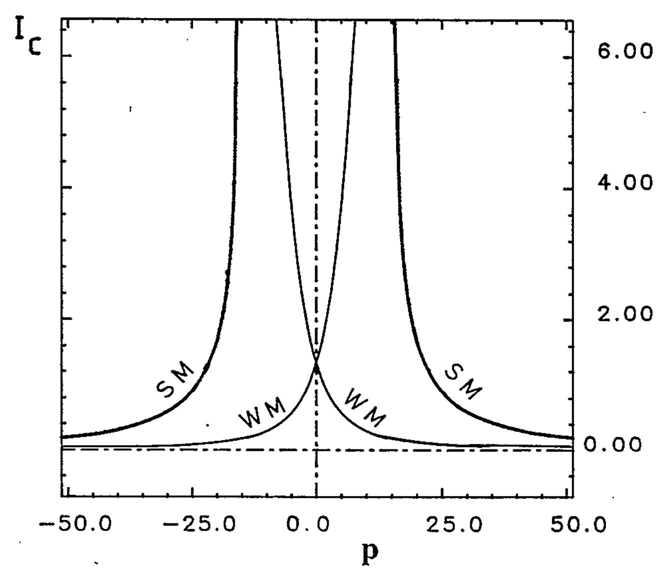


FIG. 18. Stability boundaries of the weak- and strong-mode solutions vs cavity detuning for $a=2.0$ and $u=2.0$. The solutions are stable in the domain containing $p=0.0$. For the weak-mode solution the domain of stability is the upper region.

the increase of u leads to an increase in the gain at intermediate detunings.

(iii) Finally, when

$$u=1/\epsilon \gg 1,\quad d_{\parallel}=O(1)\ ,\Delta \ \text{arbitrary}\ ,$$
$$K=1+a+O(\epsilon)\ , \tag{6.35}$$

the roots of the characteristic equation (6.29) assume the form

$$\lambda=\lambda_0+\epsilon\lambda_1+O(\epsilon^2)$$

and

$$\lambda_0^2-a\lambda_0-d_{\parallel}(a+1)=-d_{\parallel}(a+1)J\left[\frac{\lambda_0+d_{\parallel}}{\lambda_0+d_{\parallel}J^2}\right]^{1/2} .$$

The roots of this equation do not lead to an instability.

VII. DISCUSSION AND SUMMARY

Some words of caution about our assumptions and analysis and about the limitations of the model are in order. First we should note that Mandel and Agarwal[53] have shown that the bidirectional solution is unstable on resonance and we expect and have assumed that this holds out of resonance as well for two reasons: (1) It has been shown by Erneux[102] that stability or instability of steady-state solutions depend on the symmetry properties of the coefficients of the nonlinear terms in the equations, which are not changed by detuning or inhomogeneous broadening of the type considered here; and (2) the numerical solutions show no evidence of even transient affinity for the bidirectional solution.

We should also note that the harmonic truncation is not valid if a deep (and thereby nonsinusoidal) spatial grating is formed in the population inversion. This should happen if I_1 and I_2 are both strong [on the order of the saturation intensity $(I_1 I_2 \sim 1)$] for a sufficiently long time to saturate the inversion. For example, the harmonic truncation would not be valid for a cw bidirectional solution. A test of the magnitude of the coefficients of the first spatial harmonic in cases where pulsations give $I_1 I_2 > 1$ is shown in Fig. 19. We see that the amplitude of the grating never exceeds a few percent of the average inversion. Hence we are confident that for the parameter values explored here the approximations are good ones.

Based on our analysis we also recommend that special care be taken in truncating the equations to a single harmonic term in cases of counterpropagating waves. As errors have occurred in other derivations, we emphasize that accuracy (at least to first order) is guaranteed here because the polarization was fully eliminated before the harmonic expansion was undertaken. Expansions of both the polarization and population in truncated harmonic series require that, in lowest order, two harmonic terms be kept in the polarization if the correct behavior is to be preserved. This has been shown explicitly in the resonant case by comparison between the stability eigenvalues of the truncated models with the exact untruncated results of Mandel and Agarwal.[53] It is natural to assume that the result may be extended to the out-of-resonance case. Furthermore, the first two terms in the

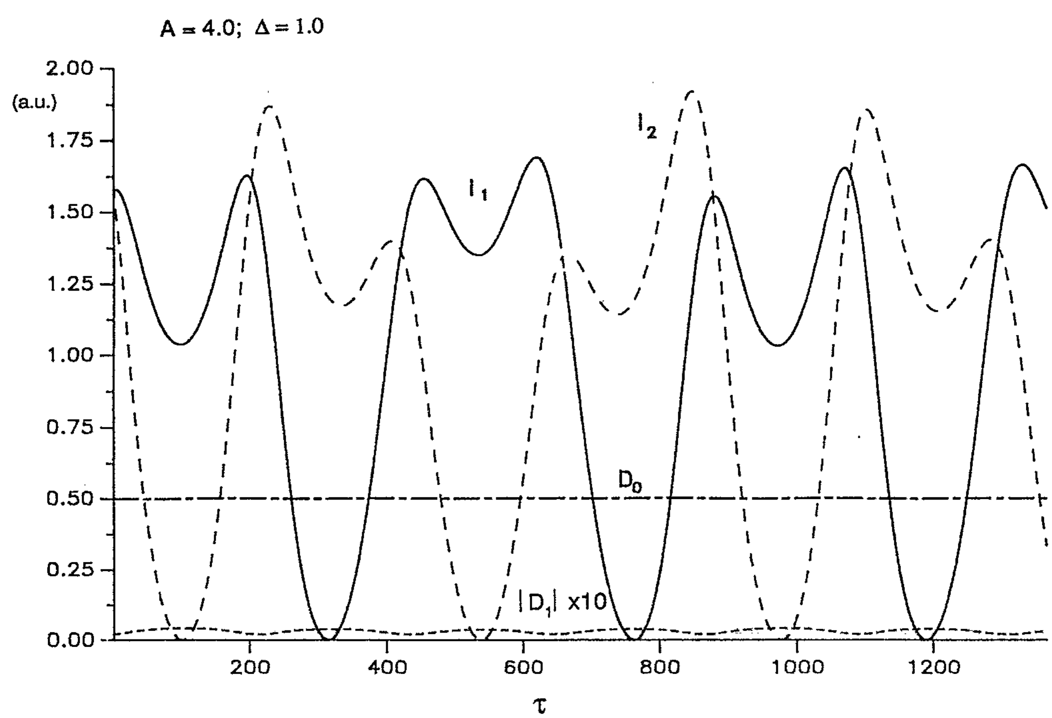


FIG. 19. Plot of I_1, I_2, $\mathfrak{D}_0$, and $\mathfrak{D}_1$ (enlarged by a factor of 10) for the time-dependent solution from the series taken at $A=4.0$ and $\Delta=1.0$ (for which other selected examples are given in Fig. 10) showing that the modulus of the grating never exceeds 1% of the average population inversion.

polarization are required if the harmonic truncation followed by adiabatic elimination of the polarization is to give the same results for the first two terms in the population inversion expansion that are obtained by first eliminating the polarization entirely and then truncating.

We should also stress that the instabilities reported here arise from Hopf bifurcations of the steady states under the conditions of sufficient detuning [Eqs. (4.33) or (4.29)]. It should be emphasized that some stability results obtained for $\Delta=0$ are destroyed as soon as $\Delta=O(\epsilon)$. This means that the condition $\Delta=0$ can lead to singular results which will be difficult to check experimentally. The stability of the seven-dimensional steady state is determined by two factors of the secular equation which are related, respectively, to the stability of the operating mode of the field (coupled to the average population inversion—three equations) and the stability of the suppressed mode which has zero intensity (coupled to fluctuations in the zero-amplitude population grating—four equations, which include as a parameter the intensity of the operating mode). The strong mode has the stability of the usual single-mode laser in the rate equation limit; that is to say, it is always stable, though with relaxation oscillations for suitable parameters. This means that infinitesimal perturbations of the field of the strong mode or of the average population inversion will not induce instabilities. The destabilization occurs due to a suitable perturbation of the field of the suppressed mode and the grating of the population inversion in the presence of a sufficient detuning. As the full seven variables are coupled by nonlinear terms, the growth of the grating and the weak-mode field ultimately destabilize the strong mode as well.

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