TECHNICAL NOTE
MULTICOMPONENT SYSTEMS WITH TIME-DEPENDENT TRANSITION RATES. A NOTE ON THE REPRESENTATION AND NUMBER OF SYSTEM STATES

P. E. LABEAU
Service de Métrologie Nucléaire, Université Libre de Bruxelles, Avenue F. D. Roosevelt, 50, 1050 Bruxelles, Belgium
(Received 23 August 1997)

Abstract—Many applications in reliability and risk assessment ask for a time-dependent analysis. In most cases, the Markovian treatment, based on the assumption of constant transition rates between states, is used. However, a realistic modeling should take into account the time at which a component entered its current state, because transition rates actually depend on the time elapsed in this state. When multicomponent systems are considered, the chronological order of the last transition times of the components affects the calculation of the system reliability characteristics. This fact entails a multiplication of the number of system states, compared with the classical Markovian analysis. This work presents a systematic counting of all possible system configurations. © 1998 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

The application of the Markovian theory to reliability is a well-known issue (Pagès and Gondran, 1980). Based on the assumption that the future evolution of a system only depends on its present situation, and not on its past history, it allows to write a simple balance equation for the probability for the system to be in one of its states. The Markovian hypothesis boils down to taking constant transition rates, and thus an exponential distribution for the time the system will leave its current state.

Nonetheless, when e.g. aging or wearout effects are accounted for, the time at which a state was entered cannot be considered as irrelevant anymore. The transition rate $\gamma_i$ out of a state $i$ must then be written as $\gamma_i(t - t')$ or even $\gamma_i(t, t')$, where $t'$ is the time at which state $i$ was entered. This time dependency in the hazard functions cannot be introduced as such in the Markovian equations, since these were based on the assumption that $t'$ had no importance. A general framework for systems with time-dependent transition rates was recently
propounded (Dubi and Gurvitz, 1995). Based on the concept of event density, it gives proper allowance to the entry of a system in a state rather than its being in a state. General integral equations for the event densities are then deduced from logical considerations, and reliability characteristics can be expressed as functionals of the event densities.

As long as a single time is sufficient to identify the entry of a system into a given state, only one integral equation is possible for each state, and the application of this transport-like theory keeps simple. In the case of multicomponent systems however, the time at which each component entered its respective state could have to be taken into account. The chronological order of these times of entry influences the future evolution of the system, this fact being equivalent to an increase in the number of system states. This situation is further explicited in the next section, which also reminds the representation of states of a multicomponent installation introduced in Dubi and Gurvitz (1995). We then present a systematic way of calculating the ratio between the number of states of a system with time-dependent transition rates and the same quantity for a Markovian system. The value of this ratio is obtained for some simple numerical examples, before some concluding remarks are given.

2. REPRESENTATION OF THE STATES OF A MULTICOMPONENT SYSTEM

Let us consider, like in Dubi and Gurvitz (1995), a system with two components, each of them presenting two states denoted 0 and 1. Let \( t^k, k = 1, 2 \), be the time at which component \( k \) entered its current state, and assume \( t^1 < t^2 \). If we are interested in the transitions out of the current state of the system at time \( t > t^2 \), then it is obvious that component 1 remained in the same state during the time interval \([t^1, t^2]\). Therefore, the pdf \( f_s(t) \) of the time at which the system leaves its current state is conditional to this fact. To illustrate this situation, let us consider the transition \((0, 0) \rightarrow (0, 1)\). Let \( R_k(t - t') \) the state reliability of component \( k \) in state \( j \), i.e. the probability that it will remain in state \( j \) up to time \( t \) given that it entered this state at time \( t' \), and \( \lambda_j(t - t') \) the corresponding hazard function. Then, for this transition, the pdf is

\[
f_s((0, 0) \rightarrow (0, 1); t^1, t^2; t) = \begin{cases} 
R_0^1(t - t^1) \cdot R_0^2(t - t^2) \lambda_0^2(t - t^2) & \text{if } t^1 < t^2 < t \\
R_0^1(t - t^1) \cdot \frac{R_0^2(t - t^2)}{R_0^2(t^1 - t^2)} \lambda_0^2(t - t^2) & \text{if } t^2 < t^1 < t.
\end{cases}
\]

From this expression, we understand that the future evolution of the system depends on the order in which the last transitions of all components took place. It thereby means that the usual representation \((i, j)\) of a system state is incomplete. In one way or another, we must be able to introduce in the definition of the state one of these three circumstances: \( t^1 < t^2, t^1 > t^2 \) and \( t^1 = t^2 \). The following representation is propounded in Dubi and Gurvitz (1995):

- \((i | j)\) denotes that component 1 entered state \( i \) before component 2 entered state \( j \);
- \((j | i)\) indicates that the opposite order of transitions took place;
- \((i | j)\) means that both components entered their respective state at the same time.
In the general case of a \( n \)-component system, a matrix-type description is necessary: each row corresponds to a component, while the column denotes its position in the ordering of the last transitions. If component \( i \) presents \( k_i \) states, then the total number of system states in the Markovian theory is:

\[
\alpha_M = \sum_{i=1}^{n} k_i.
\]

(2)

When time-dependent transition rates are taken into account, this number has to be multiplied by a factor \( f \) showing in how many ways successions of transitions can lead to a state \((i_1...i_n)\). The purpose of the next section is to estimate this factor.

3. COUNTING THE NUMBER OF SYSTEM STATES

Depending on assumptions of independence of components, or on the repair or maintenance policy, situations where several components enter their respective state at the same time could be either forbidden or compulsory. This would lead to a reduction of the total number of states. But we are interested in the most general case, for which any number of components could have had their last transition simultaneously. We then define a group of components as the subset of those components which entered their respective state at the same time. The number \( l \) of such groups in the system can take all integer values between 1 and \( n \). The size of group \( i \) is denoted by \( n_i \), and the groups are ranked in such a way that:

\[
n \geq n_1 \geq n_2 \geq ... \geq n_l \geq 1.
\]

(3)

In order to estimate \( f \), we must determine: — the number of chronological orderings of the transition times characteristic of the groups; — the number of ways of distributing \( n \) components among \( l \) groups of given size \( n_i \), \( i = 1...l \); — the different values of \( n_i \), summing up to \( n \), for a given number of groups, and sum these results on all possible values of \( l \).

The first step in these calculations is very simple: as we have \( l \) different transition times, there exist \( l! \) different ways to order them. The number of ways of dividing the \( n \) components in \( l \) groups is given by the number of combinations of \( n_i \), \( i = 1...l \), in \( n \), which equals \( n!/\prod_{i=1}^{l} n_i! \). But this result is conditional to the fact that all values of the \( n_i \)'s are different. Indeed, if several groups present the same size, the permutations between these groups must not be considered here, since they were already accounted for in determining the number of orderings of the different transition times. The number of combinations of the components in groups must then be divided by a factor \( D \) corresponding to the number of situations counted twice (or more). This quantity can be calculated in the following way. Let \( g(n_1...n_l)(\leq l) \) be the number of different values among the \( n_i \)’s, and \( d_j \) the number of groups having the size number \( j \), \( j = 1...g(n_1...n_l) \). Then, we simply have:

\[
D = \prod_{j=1}^{g(n_1...n_l)} d_j!
\]

(4)
Another simple way to obtain \( D \) is built on the ranking of the groups by non-increasing size. Let \( e(1) = 1 \). For \( i = 2 \ldots l \):

\[
\text{if } n_i = n_{i-1} \text{ then } e(i) = e(i - 1) + 1 \\
\text{else } e(i) = 1.
\]  

(5)

We then check that:

\[
D = \prod_{i=1}^{l} e(i).
\]  

(6)

Up to now, we have assumed a given number \( l \) of groups and a given set \( (n_1 \ldots n_l) \) of their sizes. The preceding results must be summed on all values of \( l \), and on all values of the \( n_i \)'s satisfying the following conditions:

\[
\begin{align*}
\sum_{i=1}^{l} n_i &= n \\
n &\geq n_1 \geq n_2 \geq \ldots \geq n_l \geq 1.
\end{align*}
\]  

(7)

Let \( M(a/b) \) be the smallest integer greater than or equal to \( a/b \). Consider first the admissible values of \( n_1 \). As group number 1 is the largest one, \( n_1 \) is at least equal to \( M(n/l) \). On the other hand, since each group must at least contain one component, \( n_1 \) may not exceed \( n - (l-1) \). Therefore, we have:

\[
M(n/l) \leq n_1 \leq n - (l - 1).
\]  

(8)

Once \( n_1 \) is fixed, the restrictions on the values of \( n_2 \) are the following:

- \( n_2 \) is the largest group size among the \((l-1)\) groups including the \((n-n_1)\) remaining components;
- there must be at least one component in the last \((l-2)\) groups;
- \( n_2 \) may not exceed \( n_1 \).

The interval of \( n_2 \) is then the following:

\[
M\left(\frac{n-n_1}{l-1}\right) \leq n_2 \leq \min(n_1, n - n_1 - (l - 2)).
\]  

(9)

In the same way, we obviously have:

\[
M\left(\frac{n - \sum_{j=1}^{i-1} n_j}{l - (i - 1)}\right) \leq n_i \leq \min(n_{i-1}, n - \sum_{j=1}^{i-1} n_j - (l - i)), i = 2 \ldots l - 1
\]  

(10)

and

\[
n_i = n - \sum_{i=1}^{l-1} n_i.
\]  

(11)
We are now able to write the final expression of factor $f$:

$$f = \sum_{l=1}^{n} \frac{n!}{n_1! \cdots n_l!} \sum_{n_1=M(n/l)}^{\min(n_1,n-n_l-(l-2))} \cdots \sum_{n_{l-1}=M\left(\frac{n-n_l-(l-2)}{l-1}\right)}^{\min(n_{l-2},n_l-1)} \frac{n!}{\prod_{i=1}^{l} e(i)n_i!}$$  \hspace{1cm} (12)

We check for instance that a two-component system corresponds to a value of $f = 3$ ($t^1 < t^2$, $t^1 > t^2$, $t^1 = t^2$). Four components give $f = 75$, distributed as follows:

- all components entered their current state simultaneously (1 possibility);
- two groups of two components can be obtained in 6 different manners;
- three of the components had their last transition at the same time (8 combinations);
- the components are divided in three groups: $n_1 = 2$, $n_2 = n_3 = 1$ (36 possibilities);
- all components have independent transitions (4! combinations).

## 4. CONCLUSIONS

When time-dependent transition rates are considered, the time at which the current state of the system was entered is of prime importance in its future evolution. If a $n$-component system is analysed, the definition of system states will depend on at most $n$ times of entry. Compared with the ideal Markovian case, this situation entails a multiplication by a factor $f$ of the number of system configurations.

The value of $f$ was estimated in a general way for a $n$-component system. Simple numerical examples showed that the expression of $f$ given in Dubi and Gurvitz (1995) was underestimated. Though in actuality the value of $f$ can be limited by assumptions on the components independence or by the choice of a repair policy, this counting reinforces the conclusion that reliability calculations of multicomponent systems should make use of numerical techniques unaffected by the number of states. Monte Carlo simulation should then be preferred to deterministic methods.

*Acknowledgement*—The author would like to acknowledge the support of a ULB grant.

## REFERENCES