Optimal replacement policy for a series system with obsolescence

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SUMMARY

Most maintenance policies assume that failed or used components are replaced with identical units. Actually, such a hypothesis neglects the possible obsolescence of the components. When a new, more reliable and less consuming technology becomes available, a decision has to be made as for the replacement strategy to be used: old-type components can all be immediately replaced, or new-type units can be introduced progressively, each time a corrective action is undertaken. Partly corrective, partly preventive policies can also be envisioned.

This work tackles this issue in the case of a series system made of \(n\) identical and independent components with a constant failure rate. It provides, under given modelling assumptions, the fully analytical expression of the mean total cost induced by each possible strategy, as well as the optimal replacement policy, as a function of the problem parameters. This is performed by accounting for different costs for preventive or corrective replacements, with some economical dependence between replacements, different energy consumption rates for old-type and new-type components as well as a discount rate.

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1. INTRODUCTION

Many papers devoted to the optimization of preventive or corrective maintenance policies assume that failed or used components are always replaced by identical items. Actually, most components are subject to technological obsolescence: new components can appear on the market with the same (or even higher) capabilities but smaller failure rates and/or lower energy consumption. Managers then face an important question: how to optimally schedule the replacement of old-type units by new-type ones? Is it worth preventively replacing still working, old-type components by new-type ones?

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This question was considered in Reference [1] in the case of one single component subject to ageing, which can be either periodically maintained or replaced by a technologically more advanced unit.

A different perspective of this issue was given in Reference [2], where the possible occurrence of a technological change of an equipment is accounted for in the definition of an optimal overhaul-replacement policy, with inspections taking place at given discrete time periods. Various economical aspects were treated, such as the overhauling cost, function of the equipment age, the revenue of the corresponding process, tax effects, inflation, etc.

Another approach [3] consists in comparing replacement and utilization costs, the latter increasing with the component age.

Obsolescence is often considered in spare parts inventories [4–6], for rarely needed parts, which are likely to become obsolescent before being used. Models like those cited in reference usually consider elements such as a demand rate of the spare components, ordering time and delivery delays and the random lifetime of the current technology before the occurrence of challengers, but they do not require a precise definition of the characteristics of the new generation of components.

The present work investigates the issue of obsolescence and replacement in the case of a system of components, once given challenger units have become available. It focuses on the particular case of a series system made of \( n \) identical and independent components with constant failure rates, which can be instantaneously replaced by new-type ones. The latter are assumed (without any loss of generality) to become available at time 0. No preventive replacement is performed at the beginning; a new-type component is introduced in the system only to replace an old-type unit that has failed. This goes on until \( K – 1 \) corrective actions of this kind have been performed \((1 \leq K \leq n)\). When the \( K \)th old-type unit fails, all the remaining old-type units are also preventively replaced with new-type ones, thereby leading to the simultaneous replacement of \( n – K + 1 \) old-type components. This strategy will be referred to as strategy \( K \). Note that strategy \( n \) corresponds to a purely corrective approach, since it simply consists in replacing failed units without any preventive action. As for strategy 1, all units are changed as soon as one fails. Besides, we also consider the so-called ‘strategy 0’, where all the old-type units are replaced with new-type ones as soon as they appear on the market, i.e. at time 0. The object of our study is to look for the optimal strategy \( K_{\text{opt}} \), which minimizes a given cost function, which is defined later on, with respect to a certain mission time, say \( t \).

Such a problem was actually propounded and studied in Reference [7]. This work takes into account different costs for preventive and corrective maintenance, as well as a discount rate. Yet these developments rely on some assumptions and approximations, which are not always acceptable. Firstly, the calculation of the cost function is made under a large mission time assumption. In other words, it is assumed that a sufficient time interval has elapsed and that all replacements to be done in strategy \( K \) have already taken place before the end of the mission. This prevents the manager from getting a genuine estimation of the cost induced by a given strategy on short and medium mission times, especially for large values of \( K \). Secondly, no economical dependence in the replacement costs is accounted for, i.e. the cost of performing \( m \) simultaneous replacements is taken proportional to \( m \). Yet one of the main approximations of this reference is the following: all along the calculations, the random sojourn time of the system in a state is substituted by its mean, instead of using its full distribution. As for the best strategy, the optimal \( K \) is only numerically computed.
In this paper, the assumption on the mission time is relaxed, and the actual distributions of the sojourn times are used through all calculations. Besides, the optimal value of $K$ is analytically provided, depending on the mission time and on the problem parameters. As for the modellization, an economical dependence in case of simultaneous replacements has been introduced. Moreover, an additional improvement in the technology is accounted for here: it is expressed by a constant rate different for each technology. This additional feature does not bring many more mathematical difficulties but it might be useful to model different improvements in the technology. For instance, one might think of a higher production rate or of a lower energy consumption rate for the new-type units. This last interpretation has been chosen here.

This paper is organized as follows. Section 2 presents the modelling assumptions of the problem and the notations used. The expression of the mean total cost entailed by the application of each possible strategy is given in Section 3. The next section deals with the determination of the optimal replacement policy, and is followed in Section 5 by a numerical case illustrating the results. In Section 6 some concluding remarks and perspectives are given. Finally, some technical results which are used in Sections 3 and 4 are given in Appendix A.

2. ASSUMPTIONS—NOTATIONS

Let $\lambda_1$ and $\lambda_2$ be the failure rates of the old- and new-type components, respectively. We recall that both failure rates are assumed to be constant, i.e. ageing is not taken into account for both kinds of units.

For $0 \leq K \leq n$, let $\mathbb{E}(C_K([0, t]))$ be the mean cost of exploitation of the system on the interval $[0, t]$ when strategy $K$ is used. As already noted, a discount rate, say $i_r$, is taken into account and all the costs are thus calculated at time 0.

As for the mean replacement cost, denoted by $\mathbb{E}(CR_K([0, t]))$, we assume that each solicitation of the repair team entails a fixed cost, say $r$. Besides, each corrective and preventive replacement involves a supplementary cost, say $c_f$ and $c_p$, respectively, to be added to $r$. Discounting the costs at time 0, we obtain:

- cost of replacement of a single component in case of failure at time $u$ such that $0 \leq u \leq t$: $(r + c_f)(1 + i_r)^{-u}$,
- cost of simultaneous preventive replacements of $n$ components at time 0: $r + nc_p$,
- cost of simultaneous preventive replacements of $j$ components and corrective replacement of one component at time $u$ such that $0 \leq u \leq t$: $(r + j c_p + c_f)(1 + i_r)^{-u}$.

As for the mean energy consumption cost, denoted by $\mathbb{E}(CE_K([0, t]))$, we assume a constant consumption rate on each interval $[u, u + du]$, to be discounted at time 0. Besides, a new-type unit is assumed to have a smaller consumption rate than an old-type one. We obtain:

- energy consumption of $j$ new-type units on $[u, u + du]: j \eta (1 + i_r)^{-u} du$. This leads to the following cost on $[t_1, t_2]$ (included in $[0, t]$):

\[
\int_{t_1}^{t_2} \eta j (1 + i_r)^{-u} du = \frac{\eta j}{\ln(1 + i_r)} (1 + i_r)^{-t_2} - (1 + i_r)^{-t_1}
\]
• Energy consumption of \( j \) old-type units on \([t_1, t_2]\) (included in \([0, t]\)):

\[
\int_{t_1}^{t_2} (\eta + v)(1 + i_r)^{-w} \, dt = j \eta + v \frac{\ln(1 + i_r)}{(1 + i_r)^{-t_1} - (1 + i_r)^{-t_2)}
\]

with \( \eta, v \geq 0 \).

In our notations, we have \( \mathbb{E}(C_K([0, t])) = \mathbb{E}(CR_K([0, t])) + \mathbb{E}(CE_K([0, t])) \).

When strategy \( n \) is used (i.e. no preventive replacement is undertaken), let \( 0 < T_1 < T_2 < \cdots < T_n \) be the failure times of the old-type units. The random variable \( T_1 \) is exponentially distributed with parameter \( n \lambda_1 \), \( T_2 - T_1 \) is exponentially distributed with parameter \( (n - 1) \lambda_1 \), \( T_{K+1} - T_K \) is exponentially distributed with parameter \( (n - K) \lambda_1 \), etc., all these time intervals being independent of each other.

When strategy \( K \) is used (with \( K > 1 \)), only failed old-type units are replaced at times \( T_1, T_2, \ldots, T_{K-1} \), whereas new-type components are substituted into all the \( n - K + 1 \) remaining old-type units at time \( T_K \).

We shall use the following notations:

\[
a = \frac{r + c_f}{c_p}
\]

\[
h = \left( \frac{r + c_f}{c_p} \right) \left( 1 - \frac{\lambda_2}{\lambda_1} \right) - 1 = a \left( 1 - \frac{\lambda_2}{\lambda_1} \right) - 1
\]

\[
b = hc_p + \frac{v}{\lambda_1} = \frac{1}{\lambda_1} [(r + c_f)(\lambda_1 - \lambda_2) - \lambda_1 c_p + v]
\]

\[
a = \frac{n \lambda_1 + \ln(1 + i_r)}{\lambda_1}
\]

Besides, symbols \( \Gamma, B \) and \( I_x \) will stand for the gamma, beta and incomplete beta functions, respectively. Finally, we shall use the function \( R \) defined as

\[
R(t, v, w) = \lambda_1 \int_0^t e^{-\lambda_1 u}(1 - e^{-\lambda_1 w})^v \, du = B(v, w)(1 - I_{e^{-\lambda_1 w}}(v, w))
\]

for all \( t \geq 0, v, w > 0 \).

3. COST FUNCTION

We now have to compute the mean cost induced by the application of strategy \( K \) on \([0, t]\). If it can quite easily be expressed in the case \( K = 0 \), it is far from trivial in the general case. Actually, the analysis of the problem is made easier by accounting for the following remark: the mixture of both types of components in the system is the same before time \( T_K \) and after time \( T_{K+1} \), when either strategy \( K \) or \( K + 1 \) is used. The difference of (mean) costs associated with these two strategies can then be restricted to their difference on \([T_K, T_{K+1}] \cap [0, t] \), which highly simplifies the calculation. This leads us to calculate first the cost induced by strategy \( 1 \), then calculate the difference of costs between strategies \( K \) and \( K + 1 \) and finally derive the cost in the general case. Note that we also have to treat separately the cost associated with strategy 0 because it is not a
special case of the general strategy. The results are summarized in Theorem 1 just below, then the different proofs are presented.

3.1. Mean total costs

Theorem 1

\[ \mathbb{E}(C_0([0,t])) = nC_p + r + n((r + c_f)\lambda_2 + \eta) \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)} \]  

(5)

For 1 ≤ K ≤ n:

\[ \mathbb{E}(C_K([0,t])) = n((r + c_f)\lambda_2 + \eta) \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)} + c_pKC_n^K(n - K)R(t, nz - K + 1, K) \]

\[ + \sum_{j=1}^{K} (c_p + b)C_n^jR(t, nz - j + 1, j) \]  

(6)

As a special case:

\[ \mathbb{E}(C_n([0,t])) = n((r + c_f)\lambda_2 + \eta) \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)} + n(c_p + b) \frac{1 - e^{-(n(z-1)+1)i_r t}}{n(z-1) + 1} \]  

(7)

3.2. Strategy 0

When strategy 0 is used, the cost on [0, t] is due to:

- the energy consumption of the n new-type units on [0, t], which is equal to

\[ \mathbb{E}(CE_0([0,t])) = n\eta \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)} \]

- the preventive replacement of n components at time 0, which is equal to \( nc_p + r \),

- the corrective replacements of the new-type units (among n) which fail on [0, t].

Let \( U_1, U_2, \ldots, U_j, \ldots \) be exponentially distributed random variables, all of parameter \( n\lambda_2 \), independent of each other. Symbols \( U_1, U_2, \ldots, U_j, \ldots \) represent the time intervals between successive failures on [0, t]. We obtain

\[ \mathbb{E}(CR_0([0,t])) = nc_p + r + (r + c_f)E \left( \sum_{j=1}^{\infty} \left( \left( 1 + i_r \right)^{-U_j} \prod_{i=1}^{j} \left( 1 + i_r \right)^{-(U_{i+1} + \cdots + U_j)} \right) \right) \]

\[ = nc_p + r + (r + c_f)E \left( \sum_{j=1}^{\infty} \left( 1 + i_r \right)^{-(U_{j+1} + \cdots + U_j)} \right) \]

(8)

after reduction. The first result of Lemma A.4 (see Appendix A) then provides us with the first result of Theorem 1.
3.3. Strategy 1

We now calculate the cost on \([0, t]\) induced by strategy 1.

Lemma 2

\[
E(C_1([0, t])) = (nc_p + b) \frac{1 - e^{-n\lambda_1 t}}{\alpha} + n(r + c_f)\hat{\lambda}_2 + \eta \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)}
\]

Proof

The energy consumption costs induced by strategy 1 on \([0, t]\) are:

- the energy consumption on \([0, \inf(t, T_1)]\) of the \(n\) old-type components,
- the energy consumption on \([T_1, t]\) of the \(n\) new-type components if \(t \geq T_1\).

We obtain

\[
E(CE_1([0, t])) = \mathbb{E}\left(n \frac{\eta + v}{\ln(1 + i_r)} (1 - (1 + i_r)^{-\inf(t, T_1)})\right) + \mathbb{E}\left(\frac{nn\eta}{\ln(1 + i_r)} ((1 + i_r)^{-T_1} - (1 + i_r)^{-t})1_{t \geq T_1}\right)
\]

Splitting the first term here above in cases \(T_1 > t\) and \(T_1 \leq t\), and using Lemma A.3 (see Appendix A), we obtain

\[
E(CE_1([0, t])) = \frac{n}{\ln(1 + i_r)} \left[\eta + v - \frac{v}{\ln(1 + i_r)} (1 + i_r)^{-T_1} - \mathbb{P}(T_1 > t) - v\mathbb{E}((1 + i_r)^{-T_1})1_{t \geq T_1}\right]
\]

after reduction.

Let us now look at the replacement costs. If \(t < T_1\), no replacement is performed on \([0, t]\). If \(t \geq T_1\), the replacement costs are due to:

- the corrective replacement and the \(n - 1\) preventive replacements at time \(T_1\),
- the corrective replacements on \([T_1, t]\), because of failures among the \(n\) new-type components, which has to be computed.

Let \(U_1, U_2, \ldots U_j, \ldots\) be exponentially distributed random variables, all of parameter \(n\hat{\lambda}_2\), independent of each other. Using Lemma A.3 and taking \(\mu = n\hat{\lambda}_2\) in Lemma A.4
(see Appendix A), we obtain

\[
\mathbb{E}(CR_1([0, t])) = (r + c_f + (n - 1)c_p)\mathbb{E}((1 + i_r)^{-T_1} \mathbf{1}_{(T_1 < t)})
\]

\[
+ (r + c_f)\mathbb{E}\left( \sum_{j=1}^{+\infty} \left( (1 + i_r)^{-j(T_1 + U_1 + \cdots + U_j)} \mathbf{1}_{(T_1 + U_1 + \cdots + U_j < t)} \right) \right)
\]

\[
= (r + c_f + (n - 1)c_p)\mathbb{E}((1 + i_r)^{-T_1} \mathbf{1}_{(T_1 < t)})
\]

\[
+ (r + c_f)\mathbb{E}\left( \sum_{j=1}^{+\infty} (1 + i_r)^{-j(T_1 + U_1 + \cdots + U_j)} \mathbf{1}_{(T_1 + U_1 + \cdots + U_j < t)} \right)
\]

\[
= (r + c_f + (n - 1)c_p) \left( 1 - \frac{e^{-\lambda_2 t}}{\lambda_2 \alpha} + (r + c_f) n \lambda_2 \frac{1 - (1 + i_r)^{-t} \left( \ln(1 + i_r) - 1 - \frac{e^{-\lambda_2 t}}{\lambda_2} \right)}{\ln(1 + i_r)} \right)
\]

\[
= c_p(n + h) \frac{1 - e^{-\lambda_2 t}}{\lambda_2 \alpha} + (r + c_f) n \lambda_2 \frac{1 - (1 + i_r)^{-t} \left( \ln(1 + i_r) - 1 - \frac{e^{-\lambda_2 t}}{\lambda_2} \right)}{\ln(1 + i_r)}
\]

after reduction, using the definition of \( h \), see (2). This gives the result, using parameter \( b \) defined in (3). \( \square \)

3.4. Difference of costs between strategies \( K \) and \( K + 1 \)

We first calculate the difference of energy consumption costs between strategies \( K \) and \( K + 1 \) on \([0, t]\).

Lemma 3

For \( 1 \leq K \leq n - 1 \):

\[
\mathbb{E}(CE_{K+1}([0, t])) - CE_K([0, t])) = (n - K) \frac{v}{\lambda_1} C_n^K R(t, n \alpha - K, K + 1)
\]

Proof

As we already noticed in Section 1, the mixture of both types of components in the system is the same before time \( T_K \) and after time \( T_{K+1} \) for both strategies \( K \) and \( K + 1 \). This implies that the difference of costs on \([0, t]\) between these two strategies is null when \( t < T_K \). We then restrict the study to the case \( t \geq T_K \) and calculate the difference of costs on \([T_K, \inf(t, T_{K+1})]\).

When \( t \geq T_K \), there are \( n \) new-type components in the system when strategy \( K \) is applied, whereas there are \( K \) new-type and \( n - K \) old-type ones for strategy \( K + 1 \) on \([T_K, \inf(t, T_{K+1})]\). Consequently, the corresponding difference of energy consumption costs is due to the difference in energy consumption on \([T_K, \inf(t, T_{K+1})]\) of \( n - K \) components which are old-type.
in strategy $K+1$, and new-type in strategy $K$. We obtain
\[
\mathbb{E}(CE_{K+1}([0, t]) - CE_K([0, t])) = \frac{(n - K)v}{\ln(1 + \sigma_r)} \mathbb{E}(1_{1[\inf(T_K - (1 + \sigma_r))]}(1 + \sigma_r)^{-T_K} - (1 + \sigma_r)^{-\inf(T_{K+1})})
\]
\[
= \frac{(n - K)v}{\ln(1 + \sigma_r)} \left[ \mathbb{E}(1_{1[T_K \leq t]}(1 + \sigma_r)^{-T_K} - (1 + \sigma_r)^{-\inf(T_{K+1})}) \right]
\]
\[
= \frac{(n - K)v}{\ln(1 + \sigma_r)} \left[ E_C^K R(t, n\sigma - K + 1, K) - (1 + \sigma_r)^{-t} E_C^n(1 - e^{-\lambda t})^K e^{-\lambda t(n - K)} \right]
\]
\[
= \frac{(n - K)v}{\ln(1 + \sigma_r)} \left[ (n\sigma - K)R(t, n\sigma - K, K + 1) + (1 - e^{-\lambda t})^K e^{-(n\sigma - K)\lambda t} \right]
\]
because of (A2) and Lemma A.3.

Using (A3) and noting that $(n - K)C_n^K = (K + 1)C_n^{K+1}$, we now have
\[
\mathbb{E}(CE_{K+1}([0, t]) - CE_K([0, t]))
\]
\[
= \frac{(n - K)v}{\ln(1 + \sigma_r)} C_n^K \left[ -(1 + \sigma_r)^{-t}(1 - e^{-\lambda t})^K e^{-\lambda t(n - K)} - (n - K)R(t, n\sigma - K, K + 1) \right]
\]
which gives the result, after reduction. \hfill \Box

Let us now treat the difference of replacement costs between strategies $K$ and $K + 1$.

**Lemma 4**

For $1 \leq K \leq n - 1$:
\[
\mathbb{E}(CR_{K+1}([0, t])) - \mathbb{E}(CR_K([0, t]))
\]
\[
= c_p(n - K)C_n^K \left[ -(1 - e^{-\lambda t})^K e^{-(n\sigma - K)\lambda t} + (h - n(\sigma - 1))R(t, n\sigma - K, K + 1) \right]
\]
\[
= c_p(n - K)C_n^K \left[ (n - K + h)R(t, n\sigma - K, K + 1) - KR(t, n\sigma - K, K + 1, K) \right]
\]
(both expressions are used below).

**Proof**

Just as for the energy consumption cost, we can restrict the study to the case $t \geq T_K$ and calculate the difference of costs on $[T_K, \inf(t, T_{K+1})]$, which is due to:

- the $n - K$ preventive replacements at time $T_K$ in case of strategy $K$ (to be added to the corrective replacement, which is common to both strategies),
- the corrective replacements on $[T_K, \inf(t, T_{K+1})]$, because of failures among the $n$ (respectively, $K$) new-type components for strategy $K$ (respectively, $K + 1$),
- the corrective replacement and the $n - K - 1$ preventive replacements at time $T_{K+1}$ when $T_{K+1} \leq t$ in case of strategy $K + 1$.  

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Let $U_1, U_2, \ldots, U_j, \ldots$ be exponentially distributed random variables, all of parameter $n\lambda_2$, independent of each other and independent of $T_K$. We define a second set of such random variables, $U'_1, U'_2, \ldots, U'_j, \ldots$, with parameter $K\lambda_2$ and independent of $T_K$ and $T_{K+1}$.

We obtain

$$\mathbb{E}(CR_{K+1}([0, t]) - CR_K([0, t]))$$

$$= -(n - K)c_p\mathbb{E}((1 + i_r)^{-T_K}1_{\{T_k \leq t\}})$$

$$- ac_p\mathbb{E}\left(\sum_{j=1}^{+\infty} \left(\left(1 + i_r\right)^{-(T_k + U_1)} + \left(1 + i_r\right)^{-(T_k + U_1 + U_2)} + \cdots + \left(1 + i_r\right)^{-(T_k + U_1 + \cdots + U_j)}\right)\right)$$

$$+ ac_p\mathbb{E}\left(\sum_{j=1}^{+\infty} \left(\left(1 + i_r\right)^{-(T_k + U'_1)} + \left(1 + i_r\right)^{-(T_k + U'_1 + U'_2)} + \cdots + \left(1 + i_r\right)^{-(T_k + U'_1 + \cdots + U'_j)}\right)\right)$$

$$+ c_p(n - K - 1 + a)\mathbb{E}(1_{\{T_{K+1} \leq t\}}(1 + i_r)^{-T_{K+1}})$$

with $ac_p = r + cf$, see (1). This expression may be reduced similar to what has been done for strategies 0 or 1. Using Lemmas A.3, A.4 and $(n - K)C_n = (K + 1)C_{n+1}$, the result follows after some reduction, however.

3.5. Mean total cost for strategy $K$

We now derive the cost in the general case. The results are displayed in Theorem 1 at the beginning of this section. They can be derived with straightforward calculations from Lemmas 2–4 and from (9), starting from

$$\mathbb{E}(C_K([0, t])) = \mathbb{E}(C_1([0, t])) + \sum_{j=1}^{K-1} \mathbb{E}(CE_{j+1}([0, t]) - CE_j([0, t]))$$

for all $1 \leq K \leq n$.

When $K = n$, expression (6) can be compacted to (7) thanks to (A4) (see Appendix A).

4. OPTIMAL STRATEGY

This section provides the optimal value of $K$, say $K_{opt}$, as a function of the mission time and of the problem parameters. These results are summarized in Theorem 5, just below. The proof has been split into different parts. Firstly, we compare strategies 0 and 1, which allows us to limit the study to the case $n \geq 2$ later on. Secondly, we compare strategies 1, 2, \ldots, $n$ and show that the best one is always either strategy 1 or $n$, which are then compared. Finally, we compare strategies 0 and $n$ before concluding.

4.1. Result

Theorem 5

For $n = 1$:

1. If $b \leq 2r + (r - 1)c_p$ (or $v + (r + cf)(\lambda_1 - \lambda_2) \leq (r + c_p)(\lambda_1 + \ln(1 + i_r))$), then $K_{opt} = 1$
2. If \( b > ax + (a - 1)c_p \): let \( t_1 = 1/\lambda_1 \ln((c_p + b)(b - (a - 1)c_p - ax)) \). Then
\[ K_{opt} = 1 \text{ for } t \leq t_1 \]
\[ K_{opt} = 0 \text{ for } t > t_1 \]

For \( n \geq 2 \):

1. If \( b \leq c_p n(a - 1) \) (or \( v + (r + c)(\lambda_1 - \lambda_2) < c_p (\lambda_1 + \ln(1 + i_r)) \)), then \( K_{opt} = n \).
2. If \( b > c_p n(a - 1) \): let \( t_0 \) be the single \( t > 0 \) such that
\[ n(c_p + b) \frac{1 - e^{-(n(a - 1) + 1)\lambda_1 t}}{n(a - 1) + 1} - (nc_p + b) \frac{1 - e^{-nx\lambda_1 t}}{x} = 0 \]

(a) If \( b \leq ax + n(a - 1)c_p \) (or \( vn + (n + r + c)(\lambda_1 - \lambda_2) < (n + c_p)(n\lambda_1 + \ln(1 + i_r)) \)), then:
\[ K_{opt} = n \text{ for } t \leq t_0 \]
\[ K_{opt} = 1 \text{ for } t > t_0 \]
(b) If \( b > ax + n(a - 1)c_p \): let
\[ t_1 = \frac{1}{\lambda_1 n x} \ln \left( \frac{nc_p + b}{b - n(a - 1)c_p - ax} \right) \]
and \( t_2 = \frac{1}{(n(a - 1) + 1)\lambda_1} \ln \left( \frac{n(c_p + b)}{nb - r - (nc_p + r)n(a - 1)} \right) \)

(i) If \( t_2 < t_0 \) (or \( t_1 < t_0 \)), then
\[ K_{opt} = n \text{ for } t \leq t_2 \]
\[ K_{opt} = 0 \text{ for } t_2 < t \]
(ii) If \( t_2 \geq t_0 \) (or \( t_1 \geq t_0 \)), then
\[ K_{opt} = n \text{ for } t \leq t_0 \]
\[ K_{opt} = 1 \text{ for } t_0 < t \leq t_1 \]
\[ K_{opt} = 0 \text{ for } t > t_1 \]

Remark 6
For \( n \geq 2 \), it is quite noticeable that strategies 2, 3, ..., \( n - 1 \) are never optimal. Namely, the best is always to choose among the three following strategies: either preventively replace all components as soon as the new-type ones appear on the market (strategy 0), wait for a first failure and then preventively replace all the remaining components at that time (strategy 1) or perform no preventive replacement at all (strategy \( n \)). Also, as intuitively expected, we can note that, when the mission time is short, strategy \( n \) is always optimal, and preventive replacements would increase the cost. On the contrary, under some conditions depending on the problem parameters (see Theorem 5 for details), when the mission time is large, it is better to preventively replace all old-type units very quickly, either at time 0 or at the first failure.

4.2. Comparison between strategies 0 and 1

Lemma 7
If \( b \leq ax + n(a - 1)c_p : E(C_1([0, t])) \leq E(C_0([0, t])) \) for all \( t \geq 0 \).
If \( b > x r + n(x - 1)c_p \):

\[
\mathbb{E}(C_1([0, t])) \leq \mathbb{E}(C_0([0, t])) \iff t \leq t_1 = \frac{1}{\lambda_1 n x} \ln \left( \frac{nc_p + b}{b - n(x - 1)c_p - x} \right)
\]

**Proof**

The result follows from the study of the variations of \( h_1(t) \) with

\[
h_1(t) = \mathbb{E}(C_1([0, t])) - \mathbb{E}(C_0([0, t])) = (nc_p + b) \frac{1 - e^{-n\lambda_1 t}}{x} - nc_p - r
\]

(see (5) and Lemma 2).

We now assume \( n \geq 2 \) up to the end of this section.

4.3. Comparison between strategies 1, 2, \ldots, \( n \)

**Lemma 8**

The best strategies among 1, 2, \ldots, \( n \) is always 1 or \( n \), namely,

\[
\min(\mathbb{E}(C_n([0, t])), \mathbb{E}(C_1([0, t]))) \leq \mathbb{E}(C_K([0, t]))
\]

for all \( 1 \leq K \leq n \).

The proof of this result is a little more technical and has been postponed to Appendix A.

We now have to compare strategies 1 and \( n \).

**Lemma 9**

- If \( b \leq c_p n(x - 1) \), then \( \mathbb{E}(C_n([0, t])) \leq \mathbb{E}(C_1([0, t])) \) for all \( t \).
- If \( b > c_p n(x - 1) \), there is a single \( t_0 \geq 1/(n - 1)\lambda_1 \) \( \ln (nc_p + b)/(c_p + b) \) such that

\[
nc_p + b \frac{1 - e^{-(n(x-1)+1)\lambda_1 t_0}}{n(x - 1) + 1} - (nc_p + b) \frac{1 - e^{-n\lambda_1 t_0}}{x} = 0
\]

and \( \mathbb{E}(C_n([0, t])) \leq \mathbb{E}(C_1([0, t])) \iff t \leq t_0 \).

**Proof**

The first case is already included in the proof of the previous lemma. The second case easily follows from the study of the variations of function \( g(t) = \mathbb{E}(C_n([0, t]) - C_1([0, t])) \) with

\[
g(t) = nc_p + b \frac{1 - e^{-(n(x-1)+1)\lambda_1 t_0}}{n(x - 1) + 1} - (nc_p + b) \frac{1 - e^{-n\lambda_1 t}}{x}
\]

4.4. Comparison between strategies 0 and \( n \)

**Lemma 10**

If \( nb \leq r + (nc_p + r)n(x - 1) \), then \( \mathbb{E}(C_n([0, t])) \leq \mathbb{E}(C_0([0, t])) \)

\[
nb > r + (nc_p + r)n(x - 1):
\]

\[
\mathbb{E}(C_n([0, t])) \leq \mathbb{E}(C_0([0, t])) \iff t \leq t_2 = 1/[\lambda_1 + \ln(1 + ir)] \ln nc_p + b)/[nb - r - (nc_p + r)n(x - 1)].
\]
Proof

The result follows from the study of the variations of function $h_2$ with

$$h_2(t) = \mathbb{E}(C_n([0, t])) - \mathbb{E}(C_0([0, t]))$$

$$= \frac{\lambda}{\lambda + \ln(1 + i_p)} \left[ nb - r - (nc_p + r)n(\alpha - 1) - n(c_p + b)e^{-(\lambda + \ln(1 + i)))t} \right]$$

(see (5) and (7)).

4.5. Discussion

Let us summarize the different results.

If $b \leq c_p n(\alpha - 1)$, then $nb \leq r + (nc_p + r)n(\alpha - 1)$ and we can derive from Lemmas 9 and 10 that $K_{\text{opt}} = n$.

We now assume $b > c_p n(\alpha - 1)$.

From Lemma 9, we know that: $K_{\text{opt}} = 0$ or $n$ if $t \leq t_0$ and $K_{\text{opt}} = 0$ or $1$ if $t > t_0$.

If $b \leq x r + n(\alpha - 1)c_p$;

We derive from Lemma 7 that $K_{\text{opt}} = 1$ for $t \geq t_0$. Besides, if $nb \leq r + (nc_p + r)n(\alpha - 1)$, Lemma 10 then implies $K_{\text{opt}} = n$ for $t \leq t_0$. If $nb > r + (nc_p + r)n(\alpha - 1)$, the conclusion is the same.

Indeed, according to Lemma 7, we know that $h_1(t) \leq 0$, for all $t$. As a particular case, $h_1(t_0) = h_2(t_0) = 0 = h_2(t_2)$ (due to $g(t_0) = h_2(t_0) - h_2(t_2)$). As $h_2$ is non-decreasing on $\mathbb{R}^+$, this implies $t_0 \leq t_2$. We now derive from Lemma 10 that $K_{\text{opt}} = n$ for $t \leq t_0$.

We now assume $b > x r + n(\alpha - 1)c_p$. We then have $nb > r + (nc_p + r)n(\alpha - 1)$ too. Besides, $t_1$ and $t_2$ are always on the same side of $t_0$. Indeed, as $g(t_0) = 0 = h_2(t_0) - h_1(t_0)$, we know that $h_1(t_0)$ and $h_2(t_0)$ have the same sign. Moreover, functions $h_1$ and $h_2$ are non-decreasing on $\mathbb{R}^+$ and are zero at $t_1$ and $t_2$, respectively. Instants $t_1$ and $t_2$ are thus on the same side of $t_0$. As a consequence:

- If $t_2 < t_0$ (or $t_1 < t_0$), Lemma 10 implies that $K_{\text{opt}} = n$ for $t \leq t_2$ and $K_{\text{opt}} = 0$ for $t_2 < t \leq t_0$.

  Lemma 7 implies $K_{\text{opt}} = 0$ for $t > t_0$.

- If $t_2 > t_0$ (or $t_1 > t_0$), Lemma 10 implies that $K_{\text{opt}} = n$ for $t \leq t_0$. Lemma 7 implies that $K_{\text{opt}} = 0$ for $t > t_1$ and $K_{\text{opt}} = 1$ for $t_0 < t \leq t_1$.

All these results have been summed up in Theorem 5 at the beginning of this section.

5. A NUMERICAL CASE

All the cases of Theorem 5 have been numerically tested with Matlab. Here we present an example illustrating one of these cases. We recall that the incomplete beta function, the gamma function and its logarithm (gammaln) are implemented in Matlab. Expressions with products or ratios of gamma functions have been computed with gammaln instead of gamma to prevent overflow (using gamma(x) = exp(gammaln(x))).

We take

$$c_f = 0.05; c_p = 0.0001; r = 0.012; \lambda_1 = 0.0015; \lambda_2 = 0.0011$$

$$n = 100; i = 0.025; \eta = 0.000001; \nu = 0.000005$$
We derive

\[
b - c_p n(x - 1) \approx 0.0181 > 0 \\
b - ar - n(x - 1)c_p \approx 0.0041 > 0 \\
t_0 \approx 6.0534 < t_1 \approx 11.2853 \\
t_0 \approx 6.0534 < t_2 \approx 8.2041
\]

This situation corresponds to the case 2(b)(ii) of Theorem 5.

The time evolution of the mean cost associated with each strategy is presented in Figures 1 and 2, with two different time windows.

We can check in Figures 1 and 2 that for \( t < t_0 \approx 6.0534 \), \( K_{\text{opt}} = 100 \), for \( t_0 \approx 6.0534 < t < t_1 \approx 11.2853 \), \( K_{\text{opt}} = 1 \), and for \( t \geq t_1 \approx 11.2853 \), \( K_{\text{opt}} = 0 \), as forecast by Theorem 5.
6. CONCLUSIONS AND PERSPECTIVES

This paper has provided the analytical treatment of a simple case of an important industrial issue, i.e. the optimization of the replacement strategy of components subject to technological obsolescence. The so-called strategy $K$ was defined as the one-by-one introduction of the challenger components in the system on a corrective basis, up to the $K$th failure of an old-type component, and as the preventive replacement of the remaining $n - K$ old-type units at the time of this $K$th failure. An additional strategy 0, corresponding to the preventive replacement of all units as soon as the challengers become available, was also considered. Assuming both types of units had constant failure rates, the exact expression of the mean cost entailed by each possible strategy was obtained, while accounting for a discount rate and for different consumption rates for both kinds of components. The determination of the optimal replacement policy was also performed analytically: it was shown that, no matter what value the problem parameters take, there are only three possible optimal strategies, i.e. 0, 1 and $n$. Observing that hybrid corrective–preventive policies are never optimal for $K \geq 2$ was not an intuitive result of the study.

Analytical expressions could be obtained in this work assuming a.o. a series system, no ageing of both old- and new-type units, no common cause failures, instantaneous replacements and a full compatibility of the new components. Obviously, the modelling which has been adopted is nothing but a first step in the treatment of this kind of problems. The special case dealt with in this work could be generalized in many ways.

For instance, series systems are usually made of different components, and this leads to many more possible system evolutions and possible replacement strategies. It should also be noted that the assumption of a series system namely implies that:

- the replacement of a failed component must be carried out immediately, in order to ensure the shortest interruption in the service provided by the system;
- the cost associated with the failure of a component is always the same, since it corresponds to the unavailability of the whole system;
- the competition between the replacement of a failed component and the failure of another unit is not critical to the system working, since each component failure entails that of the whole system; therefore, given a sufficiently large team to perform the replacements, relaxing the assumption of instantaneous replacements does not add much to the modelling in this case.

Dealing with a more complicated system structure modifies these observations: components are not always critical to the system working; replacements could sometimes be postponed if exploiting the system in a degraded mode is accepted, and preventive replacements can be limited to the subsystem which the failed component belongs to. Replacement delays will also have much more importance in this case, since additional failures are likely to affect the system behaviour on these time intervals.

Although other types of structures are likely to be at least partly analytically treated (such as $k$-out-of-$n$ systems with passive or active redundancy e.g.), limitations will soon appear as the system becomes more complex. Yet the allowance to be given to the possible ageing of the components is likely to be a more important ‘threat’ on the analytical approach.
An implicit hypothesis which has been used so far is that of compatibility of the new components with the system they are included in. This assumption also deserves some comments. Even if new-type components should display higher performances, their inclusion in the system could weaken it, at least at the beginning. Indeed, a new, unfamiliar installation procedure could delay the restarting of the system after a replacement, or unexpected adaptations of connections would have to be undertaken. The likelihood of such side effects is expected to decrease significantly with time and experience, as vendors provide additional information on their products and the technical staff gets used to the new technology. Yet, as far as the authors know, such aspects are barely accounted for in the literature on maintenance and replacement models. Developments at this level are thus required.

Up to now, no constraint on the time period on which the new technology has to be fully installed has been introduced. This approach does not seem appropriate in many application domains, where successive generations of components follow each other quickly. Be it the random arrival time of a new generation of components, or a legal delay for the full implementation of all the new-type units, a maximum time window to switch to the new technology should be taken into account.

Future work should therefore give allowance to these more realistic situations. This will soon imply to give up the analytical work and to resort to a tool like Monte-Carlo simulation. Yet analytical solutions such as those obtained for the particular case treated in this paper will turn out to be useful in validating the simulation approach.

APPENDIX A

Here we give some technical results, useful for the computation of the mean cost $C_K([0,t])$ (see Section 3).

Lemma A.1

$T_K$ admits $f_K$ and $F_K$ for probability density and cumulative distribution functions, respectively, with

$$f_K(t) = \lambda_1 K C_n^K e^{-(n-K+1)\lambda_1 t}(1 - e^{-\lambda_1 t})^{K-1}$$ (A1)

for all $1 \leq K \leq n$ and

$$F_{K+1}(t) = F_K(t) - C_n^K (1 - e^{-\lambda_1 t})^K e^{-\lambda_1 (n-K)}$$ (A2)

for all $1 \leq K \leq n - 1$.

Proof

The first result is easy to prove recursively using $T_K = (T_K - T_{K-1}) + \cdots + (T_2 - T_1) + T_1$ and the fact that $T_{i+1} - T_i$ follows an exponential distribution with rate $(n-i)\lambda_1$, for all $0 \leq i \leq n - 1$ (with $T_0 = 0$). Noting that $(n-K)C_n^K = (K+1)C_n^{K+1}$, an integration by parts on $F_{K+1}(t) = \int_0^t \lambda_1 (K+1)C_n^{K+1} e^{-(n-K)\lambda_1 u}(1 - e^{-\lambda_1 u})^{K} du$ easily gives the second result. \qed
Lemma A.2
For all \( t \geq 0, v > 0 \) and \( w > 1 \):

\[
vR(t, v, w) = (w - 1)R(t, v + 1, w - 1) - (1 - e^{-\lambda t})^{w-1}e^{-\lambda t}
\]  
(A3)

\[
\sum_{j=1}^{n} C_{n-j}^j R(t, n\lambda - j + 1, j) = \frac{1 - e^{-(n\lambda + 1)\lambda t}}{n(\lambda - 1) + 1}
\]  
(A4)

Proof
Those equalities are direct consequences of the definition of \( R \) and of the properties of the beta and incomplete beta functions.

Lemma A.3

\[
\mathbb{E}(I_{T_k \leq t}(1 + i_r)^{-T_k}) = KC_n^k R(t, n\lambda - K + 1, K)
\]

where \( C_n^k \) is the binomial coefficient.

As a special case:

\[
\mathbb{E}(I_{T_k \leq t}(1 + i_r)^{-T_k}) = \frac{1 - e^{-n\lambda\lambda t}}{\lambda}
\]

Proof
Due to the density function of \( T_k \) (see Lemma A.1), this result is a direct consequence of the definition of \( R \).

Lemma A.4

Let \( U_1, U_2, \ldots, U_j, \ldots \) be exponentially distributed random variables of parameter \( \mu \) for all \( U_j \), all independent of each other and independent of \( T_k \). Then,

\[
\mathbb{E} \left( \sum_{j=1}^{+\infty} I_{U_1 + \cdots + U_j \leq t} (1 + i_r)^{U_1 + \cdots + U_j} \right) = \mu \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)}
\]  
(A5)

\[
\mathbb{E} \left( \sum_{j=1}^{+\infty} I_{T_k + U_1 + \cdots + U_j \leq t} (1 + i_r)^{(T_k + U_1 + \cdots + U_j)} \right) = \mu \frac{1 - (1 + i_r)^{-t}}{\ln(1 + i_r)} - \frac{1 - e^{-\lambda t\lambda n t}}{n\lambda n \lambda t}
\]  
(A6)

\[
\mathbb{E} \left( \sum_{j=1}^{+\infty} I_{T_k + U_1 + \cdots + U_j \leq \inf(T_k, t)} (1 + i_r)^{(T_k + U_1 + \cdots + U_j)} \right) = \frac{\mu}{\lambda} C_n^k R(t, n\lambda - K, K + 1)
\]  
(A7)

Proof
As \( U_1 + \cdots + U_j \) has an Erlang distribution \( \Gamma(j, \mu) \), the deduction of (A5) is straightforward.
As for (A6), we can write
\[
\mathbb{E}\left(\sum_{j=1}^{+\infty} 1_{\{T_{j+1} = \ldots = T_j < t\}} (1 + \lambda t)^{-(T_{j+1} + \ldots + T_j)}\right) = n\lambda_1 \int_0^t (1 + \lambda t)^{-u} \mathbb{E}\left(\sum_{j=1}^{+\infty} 1_{\{U_1 + \ldots + U_j < t-u\}} (1 + \lambda (t-u))^{-(U_1 + \ldots + U_j)}\right) e^{-n\lambda_1 u} \, du
\]
because \(T_1\) is exponentially distributed with parameter \(n\lambda_1\) and independent of all \(U_j\)'s. Using (A5) and the definition of \(\alpha\) (see (4)), Equation (A6) follows from straightforward calculations.

As for the last equation, we can write
\[
\mathbb{E}\left(\sum_{j=1}^{+\infty} 1_{\{T_j + U_{j+1} + \ldots + U_j < \inf(t,T_{j+1})\}} (1 + \lambda t)^{-(T_j + U_{j+1} + \ldots + U_j)}\right) = \mathbb{E}\left((1 + \lambda t)^{-T_k} \sum_{j=1}^{+\infty} 1_{\{U_1 + \ldots + U_j < t-T_k\}} (1 + \lambda t)^{-(U_1 + \ldots + U_j)} 1_{\{t < (T_{j+1}-T_k) + T_k\}}\right)
\]
\[+ \mathbb{E}\left((1 + \lambda t)^{-T_k} \sum_{j=1}^{+\infty} 1_{\{U_1 + \ldots + U_j \leq T_{j+1} - T_k\}} (1 + \lambda t)^{-(U_1 + \ldots + U_j)} 1_{\{t \geq (T_{j+1}-T_k) + T_k\}}\right)\]

Since \(T_k\) and \(T_{k+1} - T_k\) are independent and using (A5), we obtain
\[
\mathbb{E}\left(\sum_{j=1}^{+\infty} 1_{\{T_j + U_{j+1} + \ldots + U_j < \inf(t,T_{j+1})\}} (1 + \lambda t)^{-(T_j + U_{j+1} + \ldots + U_j)}\right) = \mathbb{E}\left((1 + \lambda t)^{-T_k} \frac{1 - (1 + \lambda t)^{-(t-T_k)}}{\ln(1 + \lambda t)} 1_{\{T_k \leq t < (T_{k+1} - T_k) + T_k\}}\right)
\]
\[+ \mathbb{E}\left((1 + \lambda t)^{-T_k} \frac{1 - (1 + \lambda t)^{-(T_{k+1} - T_k)}}{\ln(1 + \lambda t)} 1_{\{t \geq (T_{k+1}-T_k) + T_k\}}\right)
\]
\[= \frac{\mu}{\ln(1 + \lambda t)} \mathbb{E}\left(1_{\{T_k \leq t\}} - (1 + \lambda t)^{-T_k} 1_{\{T_k \leq t \leq T_{k+1}\}} - (1 + \lambda t)^{-T_{k+1}} 1_{\{t \geq T_{k+1}\}}\right)\]
\[= \frac{\mu}{\ln(1 + \lambda t)} \left(KC_n^K R(t, nx - K + 1, K) - C_n^K (1 - e^{-\lambda_1 t})^K e^{-\lambda_1 (nx - K)}
\right.\]
\[\left. - (K + 1)C_n^{K+1} R(t, nx - K, K + 1)\right)\]
thanks to Lemmas A.1 and A.3, after reduction.
We finally derive (A7) using \((n-K)C_n^K = (K+1)C_n^{K+1}\) and (A3). \(\square\)
Proof of Lemma 8

For \(1 \leq K \leq n - 1\), let

\[
g_K(t) = \frac{1}{(n-K)C_n^K} [E(C_{K+1}([0,t])) - E(C_K([0,t]))]
\]

\[= R(t, n\alpha - K, K + 1) [b - c_p n(\alpha - 1)] - c_p [1 - e^{-t_1 K} e^{-(n\alpha-K)t_1}]
\]

(A8)
after reduction (see Lemma 3 and (8)).

We already see that, if \(b \leq c_p n(\alpha - 1)\), then \(g_K(t) \leq 0\), i.e. \(E(C_{K+1}([0,t])) \leq E(C_K([0,t]))\) for all \(t > 0, 1 \leq K \leq n - 1\) and the best strategy among 1, 2, \ldots, \(n\) is \(n\).

Now, let us assume \(b > c_p n(\alpha - 1)\) (which implies \(b \geq 0\)). It is easy to check that:

\[
g_K(t) = \lambda_1 e^{-(n\alpha-K)t_1} [1 - e^{-t_1 K} e^{-(n\alpha-K)t_1}]
\]

Let \(t_K' = (1/\lambda_1) \ln((nc_p + b)/(b + (n-K)c_p))\). There is a single \(t_K' \geq t_K'\) such that \(g_K(t_K') = 0\) and such that \(E(C_{K+1}([0,t])) \leq E(C_K([0,t])) \Rightarrow t \leq t_K'\).

Let us prove that \(t_K''\) is non-decreasing.

We know that \(t_{K+1}' \geq t_K' \geq t_K^n\) and \(t_K'' \geq t_K''\). As \(g_K\) is non-decreasing on \([t_K', +\infty[\), it is enough to prove \(g_K(t_{K+1}') \leq g_K(t_{K+1}''\) to obtain \(t_K'' \leq t_{K+1}'\).

Due to (A8) and (A3), we have

\[
g_{K+1}(t) = \frac{1}{n\alpha - K - 1} \left[ (K+1)(b - c_p n(\alpha - 1)) R(t, n\alpha - K, K + 1) 
\right. 
- (b + c_p (n - K - 1))(1 - e^{-t_1 K} e^{-(n\alpha-K)t_1})
\]

after reduction.

Using \(g_{K+1}(t_{K+1}''\) = 0, we may now substitute \((K+1)(b - c_p n(\alpha - 1)) R(t_{K+1}'', n\alpha - K, K + 1)\) with \((b + c_p (n - K - 1))(1 - e^{-t_1 K} e^{-(n\alpha-K)t_1})\) in (A8). Hence, we obtain

\[
g_{K}(t_{K+1}'') = \frac{1}{K+1} [b + c_p (n - K - 1)] - (b + c_p n) e^{-t_1 K''} (1 - e^{-t_1 t_{K+1}'\}) e^{-(n\alpha-K)t_1} t_{K+1}'\)
\]

and \(t_{K+1}'' \geq t_K'',\) for all \(1 \leq K \leq n - 1\).

It is now easy to conclude:

- If \(t \geq t_{n-1}'\), then \(t \geq t_K'\) for all \(1 \leq K \leq n - 1\). This implies

\[E(C_n([0,t])) \geq E(C_{n-1}([0,t])) \geq \cdots \geq E(C_2([0,t])) \geq E(C_1([0,t]))\]

- If \(t \leq t_K'\), then \(t \leq t_K''\) for all \(1 \leq K \leq n - 1\). Therefore,

\[E(C_n([0,t])) \leq E(C_{n-1}([0,t])) \leq \cdots \leq E(C_2([0,t])) \leq E(C_1([0,t]))\]

- If \(t_1' \leq \cdots < t_{K+1}'' \leq t < t_{K+1}' < \cdots < t_{n-1}'\), then

\[E(C_{K+1}([0,t])) \geq E(C_K([0,t])) \geq \cdots \geq E(C_2([0,t])) \geq E(C_1([0,t]))\]

and

\[E(C_n([0,t])) \geq E(C_{n-1}([0,t])) \geq \cdots \geq E(C_{K+2}([0,t])) \geq E(C_{K+1}([0,t]))\]

In all cases, the minimal cost is always reached for strategy 1 or \(n\).
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