Steady-State Finite Element Analysis of a Salient-Pole Synchronous Machine in the Frequency Domain

J. Gyselinck, L. Vandevelde, J. Melkebeek, and W. Legros

Abstract—In this paper the authors study the steady-state finite element simulation of synchronous machines. A recently proposed harmonic balance method is briefly described and applied to a salient-pole synchronous machine at no-load. The frequency domain approach is shown to produce results that agree well with those obtained with plain time stepping, however with a reduction of the computation time. Particular attention is paid to the damping effect of the eddy currents in the solid poles.

Index Terms—Finite element methods, harmonic analysis, synchronous machines, eddy currents.

I. INTRODUCTION

The steady-state finite element (FE) simulation of rotating electrical machines is mostly carried out in the time domain (time stepping), in spite of the possibly very long transient to first step through. The frequency domain approach, using the Harmonic Balance Finite Element (HBFE) method [1], citelu is not widely adopted as it requires the cumbersome assembly and resolution of a single but very large system of nonlinear algebraic equations. Furthermore, rotation cannot be easily accounted for. Besides the particular case of slip transformation modelling of induction machines [3], few frequency domain simulation techniques for rotating machines, in which saturation and slot harmonics are rigorously taken into account, have been reported in literature [4].

In [5] the authors have presented a novel HBFE method for 2D models of rotating electrical machines, in which rotation is allowed for by means of a moving band. In the following section, the method is outlined considering the 2D eddy current field problem in the cross-section of the rotating machine. In the third section, its application to a synchronous machine at no-load is studied in detail.

II. HARMONIC BALANCE WITH ROTATION

A. 2D eddy current problem

We consider a classical 2D eddy current problem in a domain \( \Omega \) in the \( xy \)-plane [6]. In the subdomain \( \Omega_s \), the current density \( j = j(x, y, t) \) (along the \( z \)-axis) is a given function of time \( t \). In the complementary conducting domain \( \Omega_c \), eddy currents may be induced. The magnetic field strength \( B \) and the magnetic induction \( \mathbf{B} \), the \( z \)-component of which vanishes, are to be calculated. The magnetic constitutive law is given. For isotropic materials, it reads \( B = \nu(b) \mathbf{B} \), where the scalar relativity \( \nu \) is a function of the magnitude of the \( b \)-vector. In the conducting domain \( \Omega_c \), the electrical field \( \mathbf{E} \) and the induced current density \( j_z \), both directed along the \( z \)-axis, are linked by Ohm’s law \( j_z = \sigma \mathbf{E} \) where \( \sigma \) is the electrical conductivity. Initial and boundary conditions are supplied as well.

The problem is commonly formulated in terms of the magnetic vector potential, which can be chosen along the \( z \)-axis: \( \mathbf{A} = a(x, y, t) \mathbf{1}_z \). This way, the induction \( \mathbf{B} = \text{curl} \mathbf{A} \) in \( \Omega \) and the electrical field \( \mathbf{E} = -\partial \mathbf{A} / \partial t \) in \( \Omega_c \) automatically satisfy the magnetic Gauss law \( \text{div} \mathbf{B} = 0 \) and Faraday’s law \( \text{curl} \mathbf{E} = -\partial \mathbf{B} / \partial t \).

The numerical solution of the eddy current problem requires a space and a time discretisation.

B. 2D FE model with fixed geometry

Let us first consider the case where the geometry and the space discretisation do not vary in time.

B.1 Space and time discretisation

A FE discretisation of \( \Omega \) in e.g. first order triangular elements leads to a piecewise linear approximation of the potential:

\[
\alpha(x, y) = \sum_{n=1}^{\#n} \alpha_n(t) \alpha_n(x, y),
\]

where \( \alpha_n(x, y) \) is the basis function associated with the \( n \)-th node in the FE mesh, and \( \#n \) is the total number of nodes.

Adopting the Galerkin approach, Ampère’s law \( \text{curl} \mathbf{A} = j \) is weakly imposed by weighing it with the \( \#n \) basis functions \( \alpha_n \). This yields a system of \( \#n \) first order differential equations:

\[
S A + T \frac{dA}{dt} = J(t),
\]

where \( A(t) \) is a column matrix containing the \( \#n \) nodal values of the potential. The square and sparse stiffness matrix \( S \) and conductivity matrix \( T \) depend on the relativity \( \nu \) and the conductivity \( \sigma \) respectively. The right hand side column matrix \( J(t) \) depends on the given current density.

B.2 Harmonic time discretisation

Let us now consider a time periodic problem. The current excitation \( J(t) \) varies periodically in time, with fundamental frequency \( f \) and period \( T = 1/f \). The periodic time variation of \( A(t) \) is approximated by a truncated Fourier series. The corresponding time basis functions \( H(t) \) are \( \sqrt{2} \cos(2\pi mf t) \) and

The research was carried out in the frame of the Inter-University Attraction Poles for fundamental research funded by the Belgian federal government.

J. Gyselinck and W. Legros are with the Institut Montefiore, Department of Electrical Engineering, University of Liège, Sart Tilman B28, B-4000 Liège (e-mail: johan.gyselinck@ulg.ac.be).

L. Vandevelde and J. Melkebeek are with the Electrical Energy Laboratory (EELAB), Ghent University, Belgium. The former is Postdoctoral Fellow with the Fund for Scientific Research—Flanders (Belgium) (FWO-Vlaanderen).
\(-\sqrt{2} \sin(2\pi mf t)\) for each nonzero frequency \(mf\), and a constant function 1 for the dc component.

It should be noticed that the harmonic basis functions are orthonormal:

\[
\frac{1}{T} \int_{0}^{T} H_k(t) H_l(t) \, dt = \delta_{k,l}.
\]

The harmonic time discretisation of \(A(t)\) can thus be written as

\[
A(t) = \sum_{k=1}^{\#h} A^{(t)} H_k(t),
\]

where \(\#h\) is the number of harmonic basis functions \(H_k(t)\), i.e. twice the number of nonzero frequencies considered, plus 1 if the dc term is taken into account.

This results in a total of \#n\#h degrees of freedom for the time and the space discretisation of \(a(x, y, t)\). An equal number of algebraic equations can be obtained by applying the Galerkin approach to the system of \#n\#h differential equations (2), i.e. weighing it with the \#h basis functions \(H_k(t)\) in the fundamental period \([0, T]\):

\[
\frac{1}{T} \int_{0}^{T} H_k(t) \left( S A + T \frac{dA}{dt} - J(t) \right) \, dt = 0.
\]

The resulting system of \#n\#h algebraic equations can be written as

\[
(S_H + X_H) A_H = J_H
\]

with

\[
A_H = \begin{bmatrix} A^{(1)} \\ \vdots \\ A^{(#h)} \end{bmatrix} \quad \text{and} \quad J_H = \begin{bmatrix} J^{(1)} \\ \vdots \\ J^{(#h)} \end{bmatrix}.
\]

\(J_H\) contains the harmonic components of the current excitation:

\[
J_H^{(t)} = \frac{1}{T} \int_{0}^{T} J(t) H_i(t) \, dt.
\]

The matrices \(S_H\) and \(X_H\) etc. can be partitioned into blocks \(S_H^{(k,l)}\) and \(X_H^{(k,l)}\), where \((k,l)\) refers to a pair of harmonic functions, i.e. \(H_k(t)H_l(t)\). From the orthonormality of the basis functions, it follows that, in the linear case, the matrix \(S_H\) has a diagonal block structure:

\[
S_H^{(k,l)} = \frac{1}{T} \int_{0}^{T} S H_k(t)H_l(t) \, dt = \delta_{k,l} S.
\]

Most blocks of \(T_H\) are zero as well. Only the off-diagonal blocks that correspond to the cosine and the sine basis function of the same frequency \(mf\) are nonzero:

\[
X_H^{(k,l)} = -\frac{1}{T} \int_{0}^{T} T \frac{dH_k}{dt} H_l(t) \, dt = \pm 2\pi mf T,
\]

where \(mf\) is the nonzero frequency of \(H_k(t)\) and \(H_l(t)\).

For linear systems, there is no coupling between different frequencies. Due to the eddy currents, the two harmonic components of each nonzero frequency are coupled.

Magnetic saturation causes all harmonics considered in the nonlinear region to be coupled. It can be taken into account as proposed in [7]. The system of nonlinear algebraic equations is straightforwardly solved by means of the Newton-Raphson (NR) method. Hereto, for each NR iteration and for each element situated in a nonlinear region of the FE domain, the differential reluctivity tensor \(\frac{\partial h}{\partial B}\) multiplied by each pair of harmonic basis functions needs to be integrated over \([0, T]\). These integrals are evaluated numerically.

C. 2D FE model with moving band

C.1 Space discretisation

For modelling rotating electrical machines, the FE domain \(\Omega\) is commonly split up into three complementary subdomains: a “stator” \(\Omega_s\), a “rotor” \(\Omega_r\) and the so-called moving band \(\Omega_{mb}\) (see Fig. 1).

\[
S = S^s + S^{mb} + S^r,
\]

\[
T = T^s + T^r.
\]
Considering the number of the nodes (11), the stiffness matrices can be further partitioned as follows:

\[
S^s = \begin{bmatrix}
S^s_{oo} & S^s_{so} & 0 & 0 \\
S^s_{so} & S^s_{oo} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
S^r = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & S^r_{ir} & S^r_{rr} & 0 \\
0 & S^r_{i_r} & S^r_{rr} & 0
\end{bmatrix}
\]

(14)

and

\[
S^{sm}(\theta) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & S^s_{or}(\theta) & S^s_{or}(\theta) & 0 \\
0 & S^s_{or}(\theta) & S^s_{or}(\theta) & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

(15)

When disregarding material nonlinearities (i.e., considering a constant reluctivity \(\nu\) and constant conductivity \(\sigma\)), the matrices produced by the stator and the rotor are time-invariant. Only the contribution of the moving band depends on the rotor position \(\theta(t)\).

C.2 Harmonic time discretisation

In a time periodic problem, the current excitation \(J(t)\) and the rotor position \(\theta(t)\) (modulo \(2\pi\)) vary periodically in time. The sets of frequencies that occur in the stator and the rotor frame are different. (This is shown hereafter for synchronous machines.)

Considering thus \(#h_s\) harmonic functions for the \(#n_s + #n_o\) nodal values in the stator and on its boundary, and \(#h_r\) harmonic functions for the \(#n_i + #n_s\) nodal values in the rotor and on its boundary, a total of \((#n_s + #n_o)\#h_s + (#n_i + #n_r)\#h_r\) degrees of freedom is obtained. An equal number of algebraic equations is obtained by weighing the differential equations with the \(#h_s + #h_r\) harmonic functions. This leads to integrals of the different stiffness matrix blocks as in (9). Only for those blocks that are \(\theta\)-independent, the orthonormality of the harmonic basis functions can be exploited. For the blocks of \(S^{sm}\) produced by the moving band, we obtain

\[
S^{mb}(k,l)_{oo} = \frac{1}{T} \int_0^T S^{mb}_{oo}(\theta(t)) H_{sk}(t) H_{sl}(t) \, dt,
\]

(16)

\[
S^{mb}(k,l)_{oi} = \frac{1}{T} \int_0^T S^{mb}_{oi}(\theta(t)) H_{sk}(t) H_{rl}(t) \, dt,
\]

(17)

\[
S^{mb}(k,l)_{io} = \frac{1}{T} \int_0^T S^{mb}_{io}(\theta(t)) H_{rk}(t) H_{sl}(t) \, dt,
\]

(18)

\[
S^{mb}(k,l)_{ii} = \frac{1}{T} \int_0^T S^{mb}_{ii}(\theta(t)) H_{rk}(t) H_{rl}(t) \, dt,
\]

(19)

where \(H_{sk}(t)\) and \(H_{rl}(t)\) are the \(k\)-th and the \(l\)-th harmonic basis function defined in the stator and the rotor respectively.

In practice, the time integrals (16–19) are evaluated numerically, considering a number of discrete time instants \(t_i\) in \([0, T]\). For each rotor position \(\theta(t_i)\), the static stiffness matrix of the moving band is calculated, and the ensuing contribution to the global HB system matrix is effected, considering all the relevant pairs of harmonic basis functions.

III. APPLICATION TO THE NOLOAD OPERATION OF SYNCHRONOUS MACHINES

As an application example of HBFE modelling, we consider the no-load operation of a salient-pole synchronous machine. First the spectrum and the spatial distribution of the magnetic fields in the stator and the rotor are derived by means of the rotating field theory.

A. Spectrum and spatial distribution of the magnetic fields in stator and rotor

We consider a salient-pole synchronous machine having \(N_p\) pole pairs and \(N_s\) stator slots. \(N_s\) is a multiple of \(N_p\). The rotor is running at synchronous speed \(\Omega = \frac{2\pi f}{N_p}\).

The magnetic quantities, such as the induction and the reluctivity, can be written as a sum of travelling waves. Here, we consider a circle of radius \(r\) in the cross-section of the machine. The angular positions on this circle in the stator and the rotor frame, denoted \(\theta_s\) and \(\theta_r\) respectively, are related as follows:

\[
\theta_r = \theta_s - \theta_s - \frac{2\pi f}{N_p} t.
\]

(20)

Using the complex notation, the radial (or tangential) component of the induction can be written as:

\[
b(r, \theta_s, t) = \sum_k \Re \{b_k(r) \exp[j(2\pi \lambda_s k f t - \kappa_s \theta_s)]\},
\]

(21)

\[
b(r, \theta_r, t) = \sum_k \Re \{b_k(r) \exp[j(2\pi \lambda_k r f t - \kappa_k \theta_r)]\},
\]

(22)

where the spatial harmonic orders \(\lambda_{s,k}\) and \(\lambda_{r,k}\) are the same in the stator and the rotor reference frame, while the respective time harmonic orders \(\kappa_{s,k}\) and \(\kappa_{r,k}\) are different. From (20), it follows that the latter are related by

\[
\lambda_{r,k} = \lambda_{s,k} - \frac{\kappa_s}{N_p}.
\]

(23)

The further analysis is restricted to the no-load operation of the synchronous machine (zero stator currents). In the rotor reference frame, the harmonic orders of the induction are given by

\[
\lambda_{r,k} = -m N_s / N_p,
\]

(24)

\[
\kappa_{k} = (1 + 2n) N_p + m N_s,
\]

(25)

where \(m\) and \(n\) are integers (0, \pm 1, \ldots). The fundamental orders correspond to \(m = n = 0\). The terms comprising \(m N_s\) are due to stator slotting.

Note that the orders (24)-(25) include saturation as well. This can be seen as follows. The reluctivity \(\nu\) is an even function of the induction. Consequently, the harmonic orders of the former are given by

\[
\lambda_{r,k} = -2m' N_s / N_p,
\]

(26)

\[
\kappa_{k} = 2(1 + 2m') N_p + 2m' N_s.
\]

(27)

By combining the orders of the induction (24)-(25) and those of the reluctivity (26)-(27), we obtain the general expression of the orders of the magnetic field strength:

\[
\lambda_{r,k} = -(m + 2m') N_s / N_p,
\]

(28)

\[
\kappa_{k} = (1 + 2(1 + m + 2m')) N_p + (m + 2m') N_s.
\]

(29)

These orders indeed comply with the expressions (24)-(25).
From (24)-(25), it follows that the time and spatial harmonic orders of the induction in the stator reference frame are given by

\[ \lambda_{s,k} = 1 + 2n, \]  
\[ \kappa_k = (1 + 2n) N_p + mN_s. \] (30)  

At no load, the time harmonics in the stator reference frame are given by odd numbers (30) and in the rotor reference frame by multiples of \( N_s / N_p \) (24). The voltage induced in the stator windings thus only contains odd harmonics.

The induction waves for which \( n \) in (30)-(31) is a multiple of \( N_s / 2N_p \) are so-called stator slot harmonics. Their orders are given by

\[ \lambda_{s,k} = 1 + m' N_s N_p, \] \[ \kappa_k = N_p + m' N_s + mN_s. \] (32)  

As they may have (relatively) high frequencies and as their winding factor is the same as that of the fundamental wave (\( \kappa = N_p \)), the stator slot harmonics may be clearly present in the stator no load voltage.

**B. Application to a particular machine**

**B.1 Description of the machine and the 2D FE model**

The no load operation of a 4-pole 50 Hz 220 V/380 V 35 kVA synchronous machine is considered. The geometry of the cross-section and the two-layer winding scheme of the stator phases are shown in Fig. 2. Each phase winding comprises 80 turns in series.

The stator has \( N_s = 60 \) rectangular slots (of width 6.7 mm). The mean airgap radius equals 129 mm, while the airgap width varies between 2.1 mm and 2.5 mm. The axial length of the machine is 180 mm.

By considering the anti-periodicity of the magnetic vector potential, only one pole of the cross-section has to be discretised. The FE mesh used for all TD and HB calculations is shown in Fig. 3.

![Fig. 2. One pole of the synchronous machine (with indication of the negative and/or negative phase belts of phases A, B and C, and with location of point 1 in a stator tooth and points 2 and 3 in the rotor pole)](image)

The second difficulty concerns the large time constant related
to the conductivity of the rotor pole. If the initial conditions are too far from the steady-state solution, in which the stator slot harmonics are considerably attenuated, the transient phenomenon decays very slowly. Therefore, a HB solution is used as initial condition for the TDc calculation. Even then it takes several periods for the induction in the interior of the rotor pole to reach steady-state.

Fig. 5 shows an flux pattern obtained with TDn. (The one obtained with TDc is quasi identical.)

Fig. 6 shows the measured and the calculated waveform of the no-load voltage. The effect of the induced currents (TDc vs. TDn) is clearly visible. The spectrum of the three waveforms is depicted in Fig. 7.

Besides the fundamental 50 Hz component, the 5th harmonic is accurately predicted by the two time-stepping simulations. The very small 3rd harmonic is present in both the measured and calculated waveforms. The 11th harmonic is accurately calculated, while the 7th harmonic is considerably overestimated. The main difference between the measured and calculated spectrum is obviously the stator slot harmonics of order 30±1. By taking into account the induced currents in the solid rotor poles, these harmonics are considerably reduced, but still greater than the measured ones. A further reduction can be obtained with a finer rotor pole mesh, obviously at the expense of a much higher computational cost.

B.3 Harmonic balance simulations

Three different HB calculations, denoted HB1, HB2 and HB3, are carried out. The sets of considered frequencies are

- HB1 : 0 and 1
- HB2 : 0 and 1, 3, 5, 7
- HB3 : 0 and 1, 3, 5, 7, 9, 11, 13

In the rotor, only the dc component is taken into account, while in the stator, the fundamental 50 Hz component is considered, as well as a number of odd harmonics, up to the 13th. Eddy currents in the rotor poles are thus ignored, but so are the stator slot harmonics. (The HB3 solution has served as initial condition for the TDc calculation.)

Some harmonic components (0, 1, 3, 5, 7) of the flux pattern are shown in Fig. 8.

The no-load voltage waveforms obtained with the two time-stepping simulations and the three HB calculations are depicted in Fig. 9. Apart from the fact that the HB waveforms do not contain the stator slot harmonics, the former converge well to the time-stepping waveforms.

The time-stepping and HB waveforms of the radial component of induction in the points 1, 2 and 3 (indicated in Fig. 2) are depicted in Figs. 10 to 12. Notice again the good convergence of the HB results to those obtained with time stepping. The Figs. 11 and 12 also evidence that the eddy currents prevent the higher harmonics from penetrating the rotor: in the interior
of the pole (point 3), they are very small.

Fig. 9. No-load voltage waveform (1/2 of period) obtained with time stepping (TDn and TDc) and harmonic balance (HB1, HB2 and HB3)

B.4 Calculation time

All time-stepping and HB calculations have been carried out on a Pentium III 750 MHz. Both the time-stepping and the HB systems of algebraic equations have been solved by means of GMRES with ILU preconditioning, after renumbering with the reverse Cuthill McKee algorithm [8]. As the fill-in (average number of nonzero entries per row) increases with the number of considered frequencies, it is important for the GMRES convergence and the computational cost to set the fill-in and dropping parameters of the preconditioning to a appropriate value.

Fig. 10. Waveform (1/2 of period) of radial induction in point 1

Fig. 11. Waveform of radial induction in point 2

The TDn computation takes about 1500 s (360 steps, 3 to 4 NR iterations per step). The transient TDc computation is more expensive, especially as several periods, say 4, have be simulated to reach quasi-steady-state (3000 s per period, 360 time steps per period, 6 NR iterations per time step).

The HB1, HB2 and HB3 calculations take 50 s, 280 s and 650 s respectively (with 8, 5 and 4 NR iterations). The number of NR iterations decreases because the HB1 and the HB2 solution serve as initial condition for HB2 and HB3 respectively.

IV. CONCLUSIONS

In this paper the authors have studied the steady-state finite element simulation of synchronous machines. A recently proposed harmonic balance method has been outlined and applied to a salient-pole synchronous machine at no-load. The frequency domain results converge well to those obtained with plain time stepping. The harmonic balance approach requires less computation time and, unlike the time domain approach, it takes the strong damping effect of the eddy currents in the solid rotor poles implicitly into account.

In further research, the load operation will be simulated using both time stepping and the proposed HB method.

REFERENCES