ON LINEAR MODELS WITH RATIONAL EXPECTATIONS WHICH ADMIT A UNIQUE SOLUTION

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Based on a generalization of Doob's theorem, the method used in this paper is applied to derive the unique reduced form of a general linear model containing rational expectations of the current endogenous variables made in several previous periods. In this procedure there is no need for assumptions on the structure of the policy instruments.

1. Introduction

The derivation of reduced forms of rational expectations models is much discussed in the economic literature. Many authors [e.g. Blanchard (1979), Gourieroux et al. (1982), McCallum (1981), Shiller (1978) and Taylor (1977)] have been confronted with the presence of an infinity of solutions when models contain expectations of future endogenous variables. In a recent paper Broze et al. (1982) develop a method based on a generalization of Doob's theorem which permits to derive the reduced form for a large class of models of this kind.

Models without expectations of future variables are also of interest. In this case it is established that, except in some degenerate cases, the reduced form is unique. Various solution techniques have been suggested [Aoki and Canzoneri (1979) and Visco (1981)]. Visco (1981) solves a general linear model containing expectations of the present endogenous variables made in several previous periods, but he supposes from the start that the policy instrument vector x has a known feedback structure.

In this paper we also consider a general model but, using the mathematical background developed in Broze et al. (1982), we need no particular assumption on the structure of the stochastic process x . However, unlike to what happens when models contain expectations of future variables, the martingales we introduce at the beginning of the procedure disappear when the final unique reduced form is established. Moreover, we show that the solution obtained by Visco is a particular case of ours in which the 'feedback assumption' is made.

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The solution technique applied in this paper allows to introduce the assumption on the structure of the policy instrument at the very end of the procedure, i.e., after the reduced form of the model has been established. Therefore it becomes possible to simulate the effects of various policies (simple or multiple feedback with different parameters, exogenous policies, etc.). In section 2, we give the expression of the reduced form written as a sum of the 'perfect foresight' solution of the model and a 'stochastic' term. This expression is valid for any locally integrable vector x .

2. The model

We consider the following general model:¹

$$y_t = \sum_{k=1}^p B_k y_{t-k} + \sum_{i=1}^q A_i y_{t-i} + Cx_t + u_t, \quad (2.1)$$

where:

(1) The information available at time t is given by the set of the present and past values of the vector processes y and x ,

$$I_t = (y_t, y_{t-1}, \dots, x_t, x_{t-1}, \dots).$$

We will call $\mathcal{J}$ the increasing sequence of σ -algebras (I_t) .

(2) The expectations are 'rational', i.e.,

$$y_{t|t-k} = E[y_t | I_{t-k}].$$

(3) The processes $y=(y_t)$ and $x=(x_t)$ are integrable and the perturbation process $u=(u_t)$ satisfies the usual assumptions,

$$E[u_t] = 0, \quad \forall t \quad \text{and} \quad E[u_t | I_{t-k}] = 0, \quad \forall t, \forall k > 0.$$

y is the vector of endogenous variables (noted z by Visco) and x can denote a vector of either policy instruments or exogenous variables; the only requirement on x is the integrability, i.e., the existence of a finite $E[x_t]$ for all t . Note that the weaker assumption of local integrability, i.e., the existence of a finite $E[x_t | I_{t-1}]$ for all t , would be sufficient for further developments.

A generalized Doob's theorem [see e.g. Broze et al. (1982)] shows that,

¹This model is the same as Visco's (1981, p. 356) general model but the lagged values of $y=(y_t)$ appear explicitly. Moreover, when $B_2 = \dots = B_p = 0$, it can be solved easily by a projection technique [see Chow (1980, p. 48)].

for any integrable and adapted process $g=(g_t)_{t \in \mathbb{N}}$, there exist a martingale $M=(M_t)_{t \in \mathbb{N}}$ and an integrable predetermined process $P=(P_t)_{t \in \mathbb{N}}$ such that, for any $t \in \mathbb{N}$, $g_t = M_t + P_t$. We apply this theorem to the $(p+1)$ vector processes defined by (y_t) , $(E[y_{t+1} | I_t])$, $\dots$, $(E[y_{t+p} | I_t])$, which are obviously integrable and adapted to $\mathcal{J}$.

$$\begin{aligned} y_t &= M_t^0 + P_t^0, \\ E[y_{t+1} | I_t] &= M_t^1 + P_t^1, \\ &\vdots \\ E[y_{t+p} | I_t] &= M_t^p + P_t^p, \end{aligned} \quad (2.2)$$

where $M^0, M^1, \dots, M^p$ are martingales and $P^0, P^1, \dots, P^p$ are integrable predetermined processes.

Proposition. If $M^0, M^1, \dots, M^p$ and $P^0, P^1, \dots, P^p$ are the processes defined by (2.2), then for any $k \in \{1, \dots, p\}$

$$M_t^k + P_t^k = y_{t+k} + \sum_{j=0}^{k-1} (M_{t+k-1-j}^j - M_{t+k-j}^j) \text{ w.p.1.}^2 \quad (2.3)$$

This proposition will be established in the appendix by proving two preliminary lemmas.

Using (2.2) and (2.3), (2.1) becomes

$$\begin{aligned} V_p y_t &= \sum_{k=1}^p B_k \sum_{j=0}^{k-1} (M_{t-j-1}^j - M_{t-j}^j) + \sum_{i=1}^q A_i y_{t-i} \\ &\quad + Cx_t + u_t \\ &= \sum_{j=0}^{p-1} W_{j+1} (M_{t-j-1}^j - M_{t-j}^j) + \sum_{i=1}^q A_i y_{t-i} + Cx_t + u_t, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} V_0 &= I, \\ V_j &= I - \sum_{k=1}^j B_k \quad \text{for } j \in \{1, \dots, p\}, \\ W_j &= \sum_{k=j}^p B_k. \end{aligned} \quad (2.5)$$

²Most of the equalities which appear in this paper will denote equalities with probability one.

Degenerate cases appear when at least one V_j is singular. In these cases, when some necessary conditions on the structure of x are fulfilled, there exists an infinite set of reduced forms for (2.1). Otherwise (2.1) has no solution.

Let us consider the non-degenerate case. To eliminate the martingales in (2.4) we project (2.1) successively on $I_{t-1}, I_{t-2}, \dots, I_{t-p}$ and, by (2.2), obtain for $j \in \{1, \dots, p\}$ ³

$$V_j(M_{t-j}^j + P_{t-j}^j) = \sum_{k=j+1}^p B_k(M_{t-k}^k + P_{t-k}^k) + \sum_{i=j}^q A_i y_{t-i} + \sum_{i=1}^{j-1} A_i(M_{t-i}^{j-i} + P_{t-i}^{j-i}) + CE[x_t | I_{t-j}]. \quad (2.6)$$

We subtract each of these projections from the preceding one and use Lemma 1 (see the appendix) and (2.2),

$$V_j(M_{t-j}^j - M_{t-j-1}^j) = \sum_{i=1}^j A_i(M_{t-i}^{j-i} - M_{t-i-1}^{j-i}) + C(E[x_t | I_{t-j}] - E[x_t | I_{t-j-1}]), \quad (2.7)$$

or

$$V_j(M_{t-j}^j - M_{t-j-1}^j) = K_j(M_{t-j}^0 - M_{t-j-1}^0) + \sum_{i=1}^j L_{i,j} C w_{t-j+1,i}, \quad (2.8)$$

where the matrices K_j and $L_{i,j}$ are determined recursively by

$$K_0 = I, \quad K_j = A_j + \sum_{i=1}^{j-1} A_j V_{j-i}^{-1} K_i, \quad (2.9)$$

$$L_{j,j} = I, \quad L_{i,j} = \sum_{k=1}^{j-i} A_k V_{j-k}^{-1} L_{i,j-k} \quad \text{for } i < j, \quad (2.10)$$

and

$$w_{i,s} = E[x_t | I_{t-s}] - E[x_t | I_{t-s-1}]. \quad (2.11)$$

³We adopt the convention $\sum_{i=a}^b u_i = 0$ when $a > b$.

Using (2.1) and (2.2) we write

$$M_{t-j}^0 - M_{t-j-1}^0 = u_{t-j} + C w_{t-j,0}, \quad (2.12)$$

and substituting first (2.12) in (2.8) and then (2.8) in (2.4) we obtain the unique reduced form of (2.1),⁴

$$y_t = \sum_{i=1}^q V_p^{-1} A_i y_{t-i} + V_p^{-1} C x_t + V_p^{-1} u_t - \sum_{j=0}^{p-1} V_p^{-1} W_{j+1} V_j^{-1} \left(K_j C w_{t-j,0} + K_j u_{t-j} + \sum_{i=1}^j L_{i,j} C w_{t-j+1,i} \right). \quad (2.13)$$

It is then possible to separate (2.13) into the reduced form of the 'perfect foresight version' of the model and the so-called 'stochastic part' Φ_t^p ,

$$y_t = \sum_{i=1}^q V_p^{-1} A_i y_{t-i} + V_p^{-1} C x_t + \Phi_t^p. \quad (2.14)$$

Φ_t^p describes a serially correlated error structure.⁵ Moreover, the influence of the policy instruments x in the 'stochastic part' of the reduced form of (2.1) is completely included in the bivariate stochastic process w which denotes differences between successive expectations of the same value of the policy vector.

3. Visco's model

The model considered by Visco (1981) is

$$y_t = A_1 y_{t-1} + \sum_{i=1}^p B_i y_{t-i} + C x_t + u_t, \quad (3.1)$$

⁴It is clearly possible to eliminate the lagged values of y in (2.13) and express y_t (for $t \geq q$) given q initial conditions $y_0, \dots, y_{q-1}$.

$$y_t = V_p^{-1} \sum_{i=1}^q F_{q-i} y_{q-i} - \sum_{k=0}^{i-q} \sum_{j=0}^{p-1} V_p^{-1} W_{j+1} V_j^{-1} K_j C_k (C w_{t-j-k,0} + u_{t-j-k}) - \sum_{k=0}^{i-q} \sum_{j=0}^{p-1} \sum_{i=0}^j V_p^{-1} W_{j+1} V_j^{-1} L_{i,j} C_k C w_{t-j+1-k,i} + \sum_{k=0}^{i-q} V_p^{-1} C_k (C x_{t-k} + u_{t-k}).$$

The coefficients C_k and F_{q-i} are appropriately determined in terms of the coefficients of the model (i.e., B_k, A_i, C).

⁵If $q < p$ we suppose that A_i is zero for $i > q$. This is of course a non-restrictive assumption.

with the 'feedback assumption' on x_t ,

$$x_t = G_t y_{t-1} + v_t, \quad (3.2)$$

where (G_t) is a deterministic sequence included in the information and $v = (v_t)$ is a perturbation.

We solve model (3.1) by the previous technique, without any assumption on x . We have

$$K_1 = A_1, \quad (3.3)$$

$$K_j = A_1 V_{j-1}^{-1} K_{j-1} \quad \text{for } j > 1,$$

$$L_{j,j} = I, \quad (3.4)$$

$$L_{i,j} = A_1 V_{j-1}^{-1} L_{i,j-1} \quad \text{for } i < j,$$

$$w_{t-j,0} = v_{t-j}, \quad (3.5)$$

$$w_{t-j+i,i} = G_{t-j+i} (M_{t-j}^{j-1} - M_{t-j-1}^{j-1}). \quad (3.6)$$

Introducing (3.2) in the reduced form, we obtain (after a lot of algebra) Visco's result, i.e.,

$$y_t = V_p^{-1} A_1 y_{t-1} + V_p^{-1} C x_t + \Phi_t^p, \quad (3.7)$$

where

$$\begin{aligned} \Phi_t^p &= \sum_{j=0}^{p-1} V_p^{-1} W_{j+1} V_j^{-1} K_j (C v_{t-j} + u_{t-j}) \\ &\quad + \sum_{j=1}^{p-1} \sum_{i=1}^j V_p^{-1} W_{j+1} V_j^{-1} L_{i,j} C W_{t-j+i,i} + V_p^{-1} u_t \end{aligned} \quad (3.8)$$

$$= u_t - V_p^{-1} W_1 C v_t + \sum_{i=1}^{p-1} R_{t,i} (u_{t-i} + C v_{t-i}). \quad (3.9)$$

The values of the matrices $R_{t,i}$ are the same as Visco's R_t^i (p. 359); they depend on the coefficients of the model (3.1) and on the deterministic part of the policy G .

4. Concluding remarks

The method used in this paper allows to express the reduced form of

models with expectations of the current endogenous variables made in several previous periods. In Broze et al. (1982) models containing expectations of future endogenous variables are considered. More generally, the procedure using a generalization of Doob's theorem is applicable to any linear model containing rational expectations of the form $y_{t+s|t-i}$ ($s \geq 0, i > 0$). The number of arbitrary martingales which will appear in the reduced form will be the highest value of s .

This paper shows that Visco's results on model (3.1) and (3.2) can be generalized to model (2.1). The conclusion is then that for non-degenerate linear model with rational expectations the reduced form appears to be the sum of the solution of the 'perfect foresight version' of the model and a serially correlated error term in which the influence of the policy instruments can be described by perturbations due to differences between successive expectations of these instruments.

Appendix: Proof of proposition (2.4)

Two lemmas on the processes introduced in (2.2) are first established.

Lemma 1. For any $k \in \{1, \dots, p\}$,

$$M_t^k + P_t^k = M_{t+1}^{k-1} + P_{t+1}^{k-1}. \quad (A.1)$$

Proof. Using (2.2) and a property of conditional expectations, we obtain, successively,

$$\begin{aligned} M_t^k + P_t^k &= E[y_{t+k} | I_t] \\ &= E[E[y_{t+k} | I_{t+1}] | I_t] \\ &= E[M_{t+1}^{k-1} + P_{t+1}^{k-1} | I_t]. \end{aligned} \quad (A.2)$$

Since $M^{k-1} = (M_t^{k-1})$ is a martingale and $P^{k-1} = (P_t^{k-1})$ is a predetermined process, it follows that

$$E[M_{t+1}^{k-1} | I_t] = M_t^{k-1}, \quad (A.3)$$

$$E[P_{t+1}^{k-1} | I_t] = P_{t+1}^{k-1}. \quad (A.4)$$

Introducing (A.3) and (A.4) in (A.2) we establish (A.1). $\square$

Lemma 2. For any $k \in \{1, \dots, p\}$,

$$P_t^k = P_{t+k}^0 + \sum_{j=1}^k (M_{t+k-j}^{j-1} - M_{t+k-j}^j). \quad (\text{A.5})$$

Proof. (A.5) will be proved by induction. For $k=1$ we have by (A.1)

$$P_t^1 = P_{t+1}^0 + (M_t^0 - M_t^1). \quad (\text{A.6})$$

Let us now suppose that

$$P_t^{k-1} = P_{t+k-1}^0 + \sum_{j=1}^{k-1} (M_{t+k-1-j}^{j-1} - M_{t+k-1-j}^j). \quad (\text{A.7})$$

(A.1) implies that

$$P_t^k = M_t^{k-1} - M_t^k + P_{t+1}^{k-1}. \quad (\text{A.8})$$

Then from (A.7) and (A.8),

$$\begin{aligned} P_t^k &= M_t^{k-1} - M_t^k + P_{t+1}^0 + \sum_{j=1}^{k-1} (M_{t+k-1-j}^{j-1} - M_{t+k-1-j}^j) \\ &= P_{t+k}^0 + \sum_{j=1}^k (M_{t+k-j}^{j-1} - M_{t+k-j}^j). \quad \square \end{aligned}$$

Proposition. For any $k \in \{1, \dots, p\}$,

$$M_t^k + P_t^k = y_{t+k} + \sum_{j=0}^{k-1} (M_{t+k-1-j}^j - M_{t+k-j}^j). \quad (\text{2.4})$$

Proof. (A.5) implies that

$$M_t^k + P_t^k = M_t^k + P_{t+k}^0 + \sum_{j=1}^k (M_{t+k-j}^{j-1} - M_{t+k-j}^j). \quad (\text{A.9})$$

By adding and subtracting M_{t+k}^0 in the right-hand side of (A.9),

$$M_t^k + P_t^k = P_{t+k}^0 + M_{t+k}^0 + \sum_{j=0}^{k-1} (M_{t+k-1-j}^j - M_{t+k-j}^j).$$

Since

$$P_{t+k}^0 + M_{t+k}^0 = y_{t+k},$$

the result is proved. $\square$

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