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The Banzhaf Index in Complete and Incomplete Shareholding Structures: A New Algorithm

Marc Levy

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Abstract:

In this global world many firms present a complex shareholding structure with indirect participation, such that it may become difficult to assess a firm's controllers. Furthermore, if there are numerous dominant shareholders, the control can be shared between them. Determining who has the most influence often is a difficult task. To measure this influence, game theory allows modeling voting game and computing the Banzhaf index. This paper first offers a new algorithm to compute this index in all structures and suggests some modelisations of the floating shareholder. Then, our model is applied to a real case study: The French Group Lafarge. This exemplary case demonstrates how the float's structure and hidden coalition can impact the power relationship between dominant shareholders.

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1. Introduction:

The direct owners of a firm take decisions by voting. Basically, the shareholders' voting power derives from their ownership share. Complete corporate structures are present in many countries around the world (Aytac, 2007; Chapelle, 2001; Cheung *et al.*, 2005; Du and Day, 2005; La porta *et al.*, 1999; Zattoni, 1999). However, in those structures indirect ownership gives rise to control tunnelling (Brioschi, 1989; Bebchuck *et al.*, 2000; Biebuyck *et al.,* 2005). Extending previous work and introducing Monte Carlo simulations in ownership consolidation, this paper offers a new algorithm to evaluate voting powers at stake in any complex structure.

An ownership structure is represented by a weighted graph in which the firms and the shareholders are the vertexes. The sources of the graph are the ultimate shareholders. We seek to assess the control power exerted on selected "target firms". The remaining firms of the graph act as intermediaries. In today's global economy, most firms are likely interconnected to some extent. In other words, the corporate world could be represented in a single huge graph. However, in practice, interest is focused on the control of specific target firms and the graph is limited to significant shareholders. Nonetheless, due to the size of the actual corporate groups and pyramids, the graphs under consideration are often very large. Therefore, adequate computation techniques are needed to efficiently address the control issue.

To compute control power in ownership graphs, the literature proposes three methodological approaches. First, the weakest link model, suggested by Claessens *et al*. (2000) and used by Faccio and Lang (2002), allocates to any corporate chain (i.e., a chain going from one ultimate shareholder to a target firm) the voting rights associated to the smallest weight encountered along this chain. The total voting share of the ultimate shareholder in the target firm is then the sum of the control shares associated with all relevant chains. Second, the matrix consolidation model, developed by Chapelle and Szafarz (2005), is based on a control threshold (typically 50%). Full control is allocated to any shareholder who reaches the voting right threshold. After this first step, the method proceeds by consolidating indirect shareholdings thanks to the algorithm proposed by[1] Chapelle and Szafarz (2007). This top-down method is also designed to identify all firms controlled by a given shareholder. It is therefore particularly relevant when the graph includes one dominant shareholder. The third method is based on the Banzhaf (1965) index that measures the probability for a shareholder to make a decisive vote in the General Assembly (see Owen, 1995; Crama *et al.*, 2003). In line with Mann and Shapley (1960) who apply the Banzhaf index to political

[1] See also Flath (1992) and Ellerman (1991) for similar techniques.

elections, Crama and Leruth (2007) offer a Monte Carlo algorithm adapted to corporate ownership and control. This bottom-up approach identifies the ultimate shareholders that control a given target firm. The use of Monte-Carlo simulations is justified by the large number of sources in the graph.

Drawing on the last two methods, this article offers a new algorithm to compute consolidated shares of control in a corporate group. It combines the top-down and bottom-up approaches in order to gain efficiency. More precisely, it first uses the matrix consolidation method (with a 50% threshold) in order to concentrate on the relevant shareholders and thus reduce the number of sources to be considered. Then, it derives the coincidence matrix and computes the Banzhaf index. While the algorithm proposed by Crama and Leruth (2007) requires a full-scale simulation for each pair made of an ultimate shareholder and a target firm, our algorithm computes the Banzhaf index with a single simulation, which simultaneously applies to all ultimate shareholders and all target firms in the graph. Moreover, we provide a generalization of the algorithm to target firms with unknown shareholders (incomplete structure) constituting its float. We introduce several float specifications and compare them with real-life examples.

Section 2 defines the voting game associated with a corporate structure. Section 3 describes the new algorithm for ownership structures with known shareholders only (complete structures). Section 4 extends the algorithm to incomplete structures. Section 5 applies the generalized algorithm to the French corporate group Lafarge. Finally, Section 6 concludes this paper.

2. Ownership Structure and Control Consolidation with the Banzhaf Index

This section introduces the graph representation of an ownership structure and the associated voting game (Levy, 2009). The following notions are defined: direct ownership, integrated ownership, direct and indirect weighted majority voting game, and the Banzhaf index.

2.1 Graph of the Ownership Structure:

An ownership structure is described through its graph of direct shareholdings $G = (V, S, A)$, where V is the vertex set representing the different firms and shareholders, $S \subseteq V \times V$ is the arc set, and $A = \left(a_{ij}\right)$ is the matrix of direct ownership, with a_{ij} a real number between 0 and 1 giving the fraction of shares that firm i owns in j. We denote by n the number of vertexes in the graph. Vertexes $1, 2, ..., s$ are assumed to be the sources, or ultimate shareholders, the remaining vertexes $s+1, s+2, ..., n,$ being the other firms under consideration. The vertex *i* is a *predecessor* of *j* or *j* is a successor of *i* if $a_{ij} \neq 0$.

From matrix *A*, Baldone *et al.* (1998) compute the integrated matrix, $Y = (y_{ij})$, where y_{ij} is the sum of all participations (direct and indirect) of i in j:

$$Y = \sum_{i=1}^{\infty} A^i = (I - A)^{-1} A \tag{1}$$

Most publicly traded companies have a large number of unknown shareholders that constitute the float. The float of any non-source firm j is given by:

$$\tilde{a}_j = 1 - \sum_{i=1}^{n} a_{ij} \tag{2}$$

An ownership structure is said *complete* if there is no float which means that:

$$\forall j \in \{s+1, s+2, ..., n\}: \tilde{a}_j = 0$$

2.2 The Banzhaf Index

Game theory offers two indexes for measuring control power: The Shapley and Shubik (1954) index of a player is the fraction of the possible voting sequences in which the player casts the deciding vote, i.e. the vote that first guarantees passage or failure. The Banzhaf (1965) index is the probability for a player to have a decisive vote[2].

The Shapley index used by Leech (2002), and Bilbao *et al.* (2002), depends on the number of ordering in which a player is pivotal (for a given order of the players we suppose all players placed before him vote 1 and after him vote 0). The use of this index needs to compute all the possible orders of the players and so cannot easily be adapted to ownership structure with float.

The Banzhaf index was initially developed by political scientists as a measure of the influence of the different states in the American Electoral College. Indeed, this index evaluates the probability for a player to have a decisive vote considering that each player randomly votes “0” or “1”.

[2] Mathematical properties of the Banzhaf index can be found in Owen 1978 and 1995), Dubey and Shapley (1979), Gambarelli and Owen (1994), Dubey *et al.* (2005)

Consider the binary voting game where a set N composed of n players are asked to vote: "0" or "1". If $S \subseteq N$, let $v(S) \in \{0,1\}$ be the result of the vote in which all players of S vote 1 and the other players vote 0. A subset $S \subseteq N$ is a winning coalition if $v(S) = 1$.

The Banzhaf index for player i is given by (see Banzhaf, 1965):

$$B(i) = \frac{1}{2^{n-1}} \sum_{S \subset N \setminus \{i\}} \left(v(S \cup \{i\}) - v(S) \right) \tag{3}$$

Election results obviously depend on the enforced decision rules. In ownership structures, decision rules may be complex as they are influenced by the links existing among the agents and firms composing the associated graph. To make those rules explicit, we define direct and indirect voting games.

2.3 The Crama and Leruth (2007) Approach in Complete Ownership Structure

In their approach, Crama and Leruth (2007) make a distinction between direct voting games which concern a firm and its direct shareholders on the one hand and indirect voting games which take into account all firms of the graph.

In the direct voting game, including one firm and its direct shareholders, the decision rule corresponds to the weighted majority voting. Each player has a weight proportional to his number of shares in the company at stake. So, in a complete ownership structure, player i has a weight $w_i \geq 0$, such that $\sum_{i=1}^{n} w_i = 1$ and his vote can be either 0 or 1. The result x_i of the vote is given by the function $g : \{0,1\}^n \to \{0,1\}$ that gives to each player $X = (x_1, x_2, x_3, \ldots, x_n) \in \{0,1\}^n$ the result:

$$g(X) = \begin{cases} 1 & \text{si} \quad \sum_{i=1}^{n} w_i x_i \geq 0.5 \\ 0 & \text{si} \quad \sum_{i=1}^{n} w_i x_i < 0.5 \end{cases}. \tag{4}$$

In indirect voting games as defined by Crama and Leruth (2007), the vote of a firm depends on its predecessors. In a complete ownership structure, all predecessors of any firm are known. Let $X = (x_1, x_2, ..., x_n) \in \{0,1\}^n$ be the votes of the n firms in the graph and $\{i_1, i_2, ..., i_l\}$ be the set of the l predecessor of j. The indirect voting game is then defined as:

$$v_j(X) = g_j\left(v_{i_1}(X), v_{i_2}(X), ..., v_{i_l}(X)\right)$$
$$= \begin{cases} 1 & \text{if} \quad \sum_{m=1}^{l} a_{i_m j} v_{i_m}(X) \geq 0.5 \\ 0 & \text{if} \quad \sum_{m=1}^{l} a_{i_m j} v_{i_m}(X) < 0.5 \end{cases} \tag{5}$$

Under this formulation, if j is a source, its vote x_j is either 0 or 1. If j is not a source, its vote x_j is defined by its predecessor's votes which means that $x_j = v_j(X)$.

This last equation defines the decision rule in the indirect voting game. Only the source of the graph can choose their own vote. From this decision rule, if j is a source and t the target firm, the definition of the Banzhaf index of j in t can be extended as:

$$Z_G(j,t) = \frac{1}{2^{n-1}}\left(\sum_{X\in\{0,1\}^n : x_j=1} v_t(X) - \sum_{X\in\{0,1\}^n : x_j=0} v_t(X)\right) \tag{6}$$

Computing the exact value of (6) can turn out to be technically difficult since it requires an algorithm of complexity $O(2^n)$. For this reason Crama and Leruth (2007) suggest using a Monte-Carlo algorithm based on the following approximation:

$$Z_G(j,t) \approx \frac{1}{2^{|S|}}\left(\sum_{X\in S : x_j=1} v_t(X) - \sum_{X\in S : x_j=0} v_t(X)\right) \tag{7}$$

Where S is a random sub-set of $\{0,1\}^n$ with cardinal $|S|$.

Formula (7) is to be applied sequentially to each pair composed of a source and a target firm, while our new algorithm simultaneously computes the Banzhaf index for all sources and targets of the graph.

2.4 The Coincidence Matrix Approach:

In order to compute in one simulation all the Banzhaf indexes for each pair of source-target firm (j,t), we propose a new concept of the coincidence matrix. This matrix stocks the frequency of identical votes for any two firms in the graph. It thus gives the probability of two firms to vote in the same way. From this matrix, all the Banzhaf indexes of all ultimate shareholders in all target firms can easily be deduced. As this matrix can be computed with one simulation, all the Banzhaf indexes in the graph can be computed in one simulation.

If the number of ultimate shareholders is relatively small (less than 20), the exact value of the Banzhaf index between any source and any target firm can be computed exactly. The algorithm computes the coincidence matrix C, in which c_{ij} is the probability that $v_i(X) = v_j(X)$.

The coincidence matrix C of a complete pyramidal ownership structure is defined as $$c_{ij} = \frac{1}{2^s} \sum_{X_s \in \{0,1\}^s} \left(v_i(X).v_j(X) + (1 - v_i(X).(1 - v_j(X)) \right) \qquad (8)$$ *where* $X_s \in \{0,1\}^s$ *represents the votes of the sources in the graph.*

If i and j are independent then $c_{ij} = 0.5$ and $0.5 < c_{ij} \leq 1$ else. From this definition, the symmetry of the coincidence matrix can easily be proved.

The link between the Banzhaf index and the coincidence matrix can be deduced from the properties of probability[3]. Let j be a player and t a target firm. Then,

$$\begin{aligned} c_{jt} &= P\left(v_j(X) = v_t(X)\right) \\ &= P\left(v_j(X) = v_t(X) \middle| j \text{ is not decisif in } t\right).P\left(j \text{ is not decisif in } t\right) \\ &\quad + P\left(v_j(X) = v_t(X) \middle| j \text{ is decisif in } t\right).P\left(j \text{ is decisif in } t\right) \\ &= \frac{1}{2}.(1 - z_{jt}) + 1.z_{jt} \end{aligned}$$

Where z_{jt} represents the Banzhaf index of j in t. We can infer that:

$$z_{jt} = 2.c_{jt} - 1 \qquad (9)$$

The Banzhaf index matrix Z can now be computed with $$Z = (2C - L).D \qquad (10)$$ *Where* C *is the coincidence matrix,* L *is a* $n \times n$ *matrix of 1,* D *is such that* $d_{ij} = 1$ *if i is*

[3]Equation (9) can also be deduced directly from definition (6) and (8).

a predecessor of j (here we consider that i is not a predecessor of itself) and 0 else, and the product is a Hadamard product[4].

The multiplication of $(2C-L)$ by D ensures that $z_{ij}=0$ if i is not a predecessor of j. This must be done because C is symmetric and does not take into account the orientation of the arcs in the graph.

2.5 Extension to Incomplete Ownership Structure

In practice, it is often very hard to obtain all the data on all firms present in an ownership structure, especially if they are quoted on the stock market. As the float can impact the control structure, it needs to be modelled.

However, implicitly, equation (5) excludes the presence of float and cross-ownership. In order to extend the definition of indirect voting games to situations with a float, we denote by f_j the fraction of the float of j that votes "1". Consequently, f_j is a random variable taking values between 0 and 1. This allows us to define a new game as:

$$v_j(X)=g_j\left(v_{i_1}(X),v_{i_2}(X),\ldots,v_{i_l}(X)\right)$$
$$=\begin{cases}1 & \text{if} \quad \sum_{m=1}^{l} a_{i_m j} v_{i_m}(X)+\tilde{a}_j f_j \geq 0.5 \\ 0 & \text{if} \quad \sum_{m=1}^{l} a_{i_m j} v_{i_m}(X)+\tilde{a}_j f_j < 0.5\end{cases} \tag{11}$$

This new formulation of voting game can now be used in formula (6), (7) and (8), to compute the new Banzhaf indexes. As proposed by Crama and Leruth (2007), the algorithm will consider f_j as a random variable.

3 Algorithm in Complete Ownership Structure

This section suggests an algorithm that is an extension of Crama and Leruth's algorithm (2007). We propose here two improvements. The first one is the use of Chapelle and Szafarz's algorithm (2005), the matrix consolidation method, which allows us to simplify the graph by suppressing shareholders with no power on the firm. The second improvement is

[4] The Hadamard product for two matrix of same dimensions is defined as $(A.B)_{ij}=A_{ij}.B_{ij}$

the introduction of the coincidence matrix which enables us to compute the Banzhaf index of all ultimate shareholders in all target firms in the graph with only one simulation.

So, the proposed algorithm is composed of two parts. The first part simplifies the graph using the matrix consolidation method, with a threshold of 50%, while the second part computes the Banzhaf index for the remaining firms thanks to a coincidence matrix.

3.1 First Part: Matrix Consolidation

In order to simplify the graph to shareholders that may have an impact in the general assembly of the firm, we first applied the Chapelle and Szafarz's algorithm (2005) to the ownership structure with a threshold of 50%. This algorithm gives full control to a shareholder who reachs more than 50% of voting rights and, in presence of a controlling shareholder, it removes the other shareholders' voting rights. This will decrease the number of shareholders in the graph and thus simplify the simulation.

As a starting point, we consider the matrix of direct ownership $A=\left(a_{ij}\right)$ of direct ownership, with a_{ij} corresponding to the fraction of shares that firm i owns in j. The algorithm contains three steps:

Step 1

We define matrix D^0 of direct control by:

$$d_{ij}^{0}=\begin{cases}1 & \text{if } i \text{ control directy } j \text{ or } a_{ij}>0.5\\ 0 & \text{if } \exists k\neq i: k \text{ control directly } j\\ a_{ij} & \text{otherwise}\end{cases}$$

This first step will give the control to direct shareholders that hold more than 50% of voting rights and cancel the voting right of shareholder with less than 50% of voting rights in presence of one controlling shareholder.

Step 2

Starting from D^0, we build matrix $B^{(1)}$ defined by:

$$b_{ij}^{(1)}=\begin{cases}1 & \text{if } \exists \text{ a chain of 1 in } D^0 \text{ from } i \text{ to } j\\ 0 & \text{if } \exists k\neq i: \exists \text{ a chain of 1 in } D^0 \text{ from } k \text{ to } j\\ d_{ij}^{0} & \text{otherwise}\end{cases}$$

This second step identifies chains of full direct control. It gives control of the firm at the end of this chain to the firm at the beginning of the chain. This step also discards the intermediary firm in the chain.

Step 3

Step 3 is iterative. From matrix $B^{(h)}$, it builds first matrix $D^{(h)}$, then matrix $B^{(h+1)}$ in the following way:

$$d_{ij}^{(h)} = \begin{cases} 1 & \text{if } b_{ij}^{(h)} + \sum_{l:b_{il}^{(h)}=1} b_{lj}^{(h)} > 0.5 \\ 0 & \text{if } \exists k \neq j : \text{if } b_{ik}^{(h)} + \sum_{l:b_{il}^{(h)}=1} b_{lk}^{(h)} > 0.5 \\ b_{ij}^{(h)} & \text{otherwise} \end{cases}$$

and

$$b_{ij}^{(h+1)} = \begin{cases} 1 & \text{if } \exists \text{ a chain of 1 in } D^{(h)} \text{ from } i \text{ to } j \\ 0 & \text{if } \exists k \neq i : \exists \text{ a chain of 1 in } D^{(h)} \text{ from } k \text{ to } j \\ d_{ij}^{(h)} & \text{otherwise} \end{cases}$$

This iterative step will successively give the control to a firm that reaches more than 50% of integrated voting rights, and then give the control to the firm at the beginning of a chain of control. The iterations stop when $B^{(h+1)} = B^{(h)}$. The last matrix *B* contains the number one if and only if the Banzhaf index of *i* in *j* is equal to one.

3.2 Second Part: Coincidence and Banzhaf Index Matrix Computation in Complete Graph

The previous definitions allow building an exact algorithm to compute the Banzhaf index matrix of a complete ownership structure. This section simulates all possible votes of sources $X_s^0 = (x_1^0, x_2^0, ..., x_s^0) \in \{0,1\}^s$ and deduces the Banzhaf indexes.

Step 1:

The algorithm generates all $X_s^0 = (x_1^0, x_2^0, ..., x_s^0) \in \{0,1\}^s$ and completes them with 0 to establish the vector $X^0 = (x_1^0, x_2^0, ..., x_n^0) = (x_1^0, x_2^0, ..., x_s^0, 0, 0, ..., 0) \in \{0,1\}^n$.

Step 2:

From each initial value of $X^0 = (x_1^0, x_2^0, ..., x_n^0)$, we compute the final result $\left(v_1(X), v_2(X), ..., v_n(X)\right)$. The iterative calculation defines $X^1 = (x_1^1, x_2^1, ..., x_n^1)$ with $x_i^1 = g_i(X^0)$ and X^{k+1} with $x_i^{k+1} = g_i(X^k)$ (see equation (5)). The iterations stop when $X^{k+1} = X^k$.

Step 3

For each vector $X_s^0 \in \{0,1\}^s$, we associate to the result $\left(x_1, x_2, ..., x_n\right)$ a matrix S such that $s_{ij} = \begin{cases} 1 & \text{if } x_i = x_j \\ 0 & \text{if } x_i \neq x_j \end{cases}$.

Step 4

The coincidence matrix C, defined by (6), is obtained by taking the average value of S. From this last matrix the Banzhaf index matrix is computed with:

$$z_{ij} = \begin{cases} 2.c_{ij} - 1 & \text{if } i \text{ is a predecessor of } j \\ 0 & \text{if } i \text{ is not a predecessor of } j \end{cases}$$

If i is a source and j is another firm, the typical element of matrix Z, z_{ij}, represents the Banzhaf index of i in j.

Remark:

By taking all the $X_s^0 = (x_1^0, x_2^0, ..., x_s^0) \in \{0,1\}^s$, the algorithm computes the exact value of the Banzhaf index. However, as the complexity of the algorithm is $O(2^n)$, if the number of sources is great, the algorithm can be adapted to a Monte-Carlo algorithm (Crama and Leruth, 2007) by taking a random sample of X_s^0.

3.3 An Example

The figure 1 shows the example of Crama and Leruth (2007, 883), which contains seven sources (numbered from 1 to 7) and three other firms (from 8 to 10).

The coincidence matrix C, and the Banzhaf index matrix Z corresponding to this structure are:

$$C = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.5 & 0.625 \\ 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.5 & 0.625 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.5 & 0.625 \\ 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.625 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.75 & 0.625 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.75 & 0.625 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.75 \\ 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.75 \\ 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 1 & 0.75 \\ 0.625 & 0.625 & 0.625 & 0.625 & 0.625 & 0.625 & 0.75 & 0.75 & 0.75 & 1 \end{pmatrix}$$

and

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The last matrix gives a Banzhaf index of 0.5 for shareholder 1,2 and 3 in firm 8 (and for shareholder 4, 5 and 6 in firm 9), and a Banzhaf index of those 6 first shareholders of 0.25 in 10. Shareholder 7 has a Banzhaf index of 0.5 in 10. This matrix also gives the Banzhaf index of 0.5 for intermediate firms 8 and 9 in 10.

If we consider that shareholder 3 and 4 merge in firm 3' as shown in figure 2, then the new matrixes are:

$$C = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.5 & 0.625 \\ 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.5 & 0.625 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 \\ 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.75 & 0.625 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.75 & 0.625 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.688 \\ 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 1 & 0.625 & 0.844 \\ 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.5 & 0.625 & 1 & 0.844 \\ 0.625 & 0.625 & 0.75 & 0.625 & 0.625 & 0.688 & 0.844 & 0.844 & 1 \end{pmatrix}$$

and

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.375 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.688 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.688 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Banzhaf index of shareholder 7 has decreased from 0.5 to 0.375 because 8 and 9, having a common shareholder, have a greater probability to vote the same way. Furthermore, the computed index given by the matrix of 8 and 9 does not correspond to a Banzhaf index. Indeed, the computed index is based on the number of coincidence between 8 (or 9) and 10, but since 8 and 9 have a common shareholder, they have a higher probability to vote the same way and the number of coincidence between 8 and 10 is higher than if 8 and 9 were independent. Thus, the computed index is higher than the Banzhaf index of this intermediate firm. As all sources are independent, their computed indexes in all firms of the graph correspond to their Banzhaf index.

To summarize, the Banzhaf index matrix provides for all sources of the structure their Banzhaf indexes in all firms of the graph. In a pyramidal tree structure (for each pair of vertices (i,j) there is at most one path that goes from i to j), it also calculates the Banzhaf index of intermediate firms.

To avoid having indexes that are not Banzhaf indexes, we will restrict the matrix to a $s \times (n-s)$ matrix, where the lines represent the sources and the columns represent the other firms of the graph. Consequently, we ignore the indexes of intermediate firms.

4 Extension to Incomplete Graphs

As some firms of the graph are quoted on the stock market, a fraction of the shareholders are unknown. The float denoted by $\tilde{a}_i$ is the fraction of unknown shareholders. Most models neglect this float, but in some cases it should not be neglected. Indeed, some important shareholders can hide participations within the float, or small shareholders can coalise with the help of a lobbying agency or an external shareholder can take significant participations within the float. All those possible phenomena can happen for firms having an important float.

This section proposes to extend the model to the case of many unknown shareholders. As suggested by Crama and Leruth (2007), the idea is to model the float by a random variable representing the fraction of the float voting 1.

4.1 Stochastic Float:

In this subsection, we propose different mathematical modeling of the float and we try to identify to which behavior of the float it corresponds.

The model considers the part of this float that votes 1, f_i, as a random variable with value between 0 and 1. Because the Banzhaf index is based on a probability of ½ for each player to vote 0 or 1, the f_i must have an average value of ½ and be symmetric with respect to ½ (this means that $P(f_i = x) = P(f_i = 1-x)$). The choice of those random variables influences the value of the computed Banzhaf index and models a structure or behaviour of the float. Below, we suggest some potential float structures.

A first possible choice is to imagine that the entire float votes the same way. So $P(f_i = 1) = P(f_i = 0) = 1/2$ and 0 for all other value of f_i. This supposition gives a low value of the Banzhaf index for the other shareholders and is not very realistic since most of the float does not take part in the vote, and since different parts of the float have different interests. Moreover, if the float is bigger than 50%, then the float has full power and the other shareholders lose their power.

A second possible case in point is to consider that the float does not participate in the vote. This is similar to supposing that half of the float votes 1 and the other half votes 0 (so the votes of the float do not matter). This gives $P\left(f_i = 1/2\right) = 1$ and $P\left(f_i \neq 1/2\right) = 0$. The Banzhaf indexes of important firms will in most cases take high values. This model ignores the float which is not realistic if some shareholders among the float take part in the general assembly. Furthermore, in the case where the vote among important shareholders is close to the equilibrium, this model can give biased results. For example, if a firm only has two known shareholders with 25% and 24% of voting right respectively, then the measure of the Banzhaf index allocates all the power to the owner with 25% of the shares. Nonetheless, if some unknown shareholders vote in the same way as other medium-sized shareholders, then the vote of the biggest shareholder could not be followed.

A third option is to contemplate the float as the sum of m small shareholders with same participation. In this case, we obtain a binomial distribution, $f_i \sim \frac{B(m,1/2)}{m}$ and have a variance of $\frac{1}{4m}$. A high value of m tends to correlate with the second case while a value of m=1 corresponds to the first case. The choice of m may be crucial to model the reality. A high value of m will give high power to the known shareholders while a low value of n will decrease their power. Furthermore, the value of m could depend on the size of the float. Such a choice allows modelling the float as entities with a given value of voting right for each. For instance, if the float consists in shareholders with each 1% of voting right, then $m = round(100 * \tilde{a}_i)$.

A fourth possibility is to envisage that the percentage of the float that votes 1 is a uniform variable. This entails $P\left(x < f_i < x + \Delta x\right) = \Delta x$. This scenario will give the float a relatively high probability that most of the float votes in the same way. This supposition is well-adapted if most of the float takes part in the vote, and has similar interests. As demonstrated in the appendix, this modelling can be visualized as a float with an infinite number of shareholders with one that owns 1/2 of the float, a second ¼, a third 1/8, etc.

Alternative choices of float distribution certainly exist. The float can be modelled with a Gaussian distribution $f_i \sim N(1/2, \sigma)$, but the tail of the distribution must be treated carefully (because $0 \leq f_i \leq 1$). Another possibility is to separate the float as the sum of one part (β) that does not take part in the vote and another part ($1-\beta$) that does. In this case $f_i \sim \frac{\beta}{2} + (1-\beta)\tilde{f}_i$ where $\tilde{f}_i$ can be any distribution described above.

To illustrate the float's impact, figure 3 shows the value of the Banzhaf index in the case where one main shareholder owns a fraction of the share x and the float owns $1-x$. One first observes that with a binomial variable the Banzhaf index is a step function. Secondly, a high value of m implies a high value of the Banzhaf index for the main shareholder. Finally, the uniform distribution entails lower Banzhaf index values. This is due to the relatively high probability that most floating shareholders vote in the same way.

4.2 The Algorithm:

As the float is a random variable that can take many values (even an infinite number of values if it is a continuous variable), the algorithm must take random samples of those variables, and so become a Monte-Carlo algorithm. Therefore, in the previous algorithm, step 1 will change, and will take for each $X_s^0 = (x_1^0, x_2^0, ..., x_s^0) \in \{0,1\}^s$ M random samples of $F = (f_{s+1}, f_{s+2}, ..., f_n) \in [0,1]^{n-s}$ representing the fractions of the float voting 1. The step 2 of the algorithm will still converge as we can consider an incomplete graph with a float as a complete graph where each float is considered as two sources of the graph, one voting 0 and the other voting 1.

4.3 Algorithm Convergence:

If we consider the Monte-Carlo algorithm, in which we take N random samples of $X_s^0 \in \{0,1\}^s$ and of $F = (f_{s+1}, f_{s+2}, ..., f_n) \in [0,1]^{n-s}$, the value of the computed c_{ij} is the random variable $\frac{B(N, c_{ij})}{N}$. The variance of this variable is $\frac{N.c_{ij}.(1-c_{ij})}{N?} \leq \frac{1}{4N}$. So the variance of the computed $z_{ij} = 2.c_{ij} - 1$ is smaller than $\frac{1}{n}$ (and equal to $\frac{1}{n}$ only if $z_{ij} = 0$). The error can thus be approximated to $\frac{1}{\sqrt{n}}$. If we take all the $X_s^0 \in \{0,1\}^s$, and random samples of $F = (f_{s+1}, f_{s+2}, ..., f_n) \in [0,1]^{n-s}$, the error becomes smaller and depends on the sizes of the floats.

5 Application to the Lafarge Group

To illustrate the model and show how the behaviour and organization of the float can impact the controlling structure of a firm, we consider the French group Lafarge. This group is a CAC40 quoted French firm active in building material. Its main products are cement, aggregate, concrete and gypsum. The sales of the group was 15.9 Billions€ in 2009 (and 19 Billions€ in 2008), and a net income of 736 million€ (against 1.6 Billion€ in 2008). At the end of 2009, the capitalizing of the group was 16.5 B€. The control of this quoted group is particularly interesting to analyze because we have direct shareholders with significant voting rights such as GBL with 19.5, or NNS holding with 12.5%. and an important float of 50.8

The ownership structure is represented in figure 4. The data were collected from the software Amadeus (dating from May 2008) of Bureau Van Dijck and from the websites of the firms (updated in May 2010). The analysis simulates two scenarios with five different float representations (making a total of ten simulations). In the first scenario, we assume that all French public institutions act independently, while in the second scenario, we suppose that they all vote in the same way. As far as the float is concerned, the first model excludes it from the voting. The second model considers that the float vote is a random variable with uniform distribution. The third model splits the float into 100 small shareholders, each with 1% of the remaining float. The fourth float is composed of shareholders, each with 1% of the total number voting rights while the last model supposes that the float is made of 10 identical shareholders.

Lafarge's largest direct shareholder is the Belgian group GBL. This holding is jointly controlled by Albert Frere and the Desmarais family. Albert Frere is a Belgian businessman born in 1924 who started his fortune with the commerce of steel. Paul Desmarais is a Canadian businessman born in 1927. In 1981, Frere and Desmarais built together the Swiss financial holding Pargesa that acquired the GBL group. The pyramidal structure of their holding is mainly based on six industrial firms: Imerys, Total, Lafarge, Pernod Ricard, GDF-Suez and Suez Environment. The second largest shareholder of Lafarge is NNS Holding mainly active in the business of construction industry, and owned in full by Nassef Sawiris, an Egyptian businessman.

Figure 3 offers a full picture of the companies involved in Lafarge ownership. It includes 40 firms and shareholders, among which 22 are sources. Some shareholders (e.g. small shareholder of Credit Agricole) are omitted because they do not really participate in the voting game since they have shares that are fully controlled by another shareholder. The graph presents a small cycle between Total and the CNP, and a longer cycle of 7 firms between Total-CNP, Agesca,…,GBL. In principle, the algorithm should not work, but in this case, as CNP is fully controlled by another shareholder the algorithm will provide the Banzhaf indexes.

In the first scenario (French Institutions are independent), the first part of the algorithm produces the reduced graph, displayed in figure 5, with 17 vertexes. The 13 sources include, e.g. Albert Frere, Power Corporation du Canada, Nassef Sawiris, and French public institutions (Caisse des Dépots et Consignations (CDC), Banques Populaires Caisse d'Epargne (BPCE), SAS Rue la Boétie[5]). As Parjointco controls GBL, it benefits from all its voting rights.

In the second scenario, CDC, BPCE and SAS Rue de la Boétie are controlled by the French state. The first part of the algorithm then leads to the graph in figure 6 including 13 vertexes and 10 sources.

Whatever the scenario, Albert Frere shares the control of GBL with the Desmarais family thanks to a pyramidal structure. He fully controls CNP (more than 50% voting rights) leaving no control at all to Total and its shareholders.

The second part of the algorithm computes the Banzhaf indexes. Five float representations are considered for each company with incomplete structure. 1) The share of float voting "1" is equal to ½. 2) The float votes are randomly drawn from a uniform probability distribution. 3) The float is split into 100 identical shareholders whose votes are randomly drawn from a binomial variable. 4) A binomial variable from which we consider a float composed of small shareholders with 1% of the total numbers of voting right. 5) The float is split into 10 shareholders with the same voting rights. In all cases, floats from different companies vote independently from each other.

The vote simulations for two shareholders with 50% voting rights each randomly picks the winner (probability ½ for each one). Because the votes of the float are established through a Monte-Carlo algorithm taking 10000 random values, the resulting error is below 0.1%. We omit to consider the float as a single shareholder. Otherwise, as a largely held company, Lafargue would be controlled by its float, which is unrealistic.

The simulation results are displayed in table 1. The first column gives integrated ownership. The next five double columns correspond to the five float representations. Each column is divided into two sub-columns representing, respectively, scenario 1 (French institutions acting independently), and scenario 2 (French institutions controlled by the French State). The last line of the table shows the control power of GBL. Although GBL is not a source, the algorithm provides its Banzhaf index.

The integrated ownership or cash flow rights given in the first column takes into account direct and indirect participations. This integrated ownership takes into account the

[5] which is actually controlled by Caisse Communale

cross-ownership. Normally our algorithm does not work in presence of cross-ownership, but in the present case, as CNP is fully controlled by Albert Frere, Total loses all its voting right in CNP after the first simplification.

The second column shows the Banzhaf index if the float does not take part in the vote. In the first sub-column, the French institutions are independent while in the second they are all controlled by the French State. A surprising result is that the Banzhaf index of Nassef Sawiris increases with the apparition of the French state as a big shareholder. This is due to the fact that with independent French institutions, Nassef Sawiris needed to find a lot of partners to form a coalition that would counterbalance GBL. However, with coalised French institutions (together those institution have 5.98% of voting rights), if Nassef Sawiris cooperates with the French State, he can more easily beat a coalition with GBL. In this last case, to find partners among small shareholders becomes crucial. With a non-influent float, we observe a transfer of 8.8% in the Banzhaf index from GBL to Nassef Sawiris as the French State controls its institutions. Nonetheless, in both scenarios, Albert Frere and Demarais Family stay the most influent shareholders.

With a uniform float, the Banzhaf index of GBL strongly decreases while the Banzhaf index of Nassef Sawiris stays to a relatively high value. Consequently, Nassef has a higher Banzhaf index than Albert Frere and Power Corporation du Canada. Furthermore, if the French institutions act as a coalition, the Banzhaf index stays more or less unchanged. In fact, this float assumes the existence of one hidden shareholder with 25% of voting right (half of the float), one with 12.5% of voting right (1/4 of the float), one with 6.25%, etc. With such float it is much easier to find partners to form a winning coalition beating the biggest shareholder. Because big shareholders exist in such a float, the apparition of a coalised French State does not so much impact the Banzhaf index of other shareholders. Therefore, with a uniform float, the results are less impacted by hidden coalition, unknown data or change in the ownership structure (for example if one shareholder increases his participation or an external investor enters in the game). Furthermore, for all direct shareholders, the computed Banzhaf index with this float is close to twice its participation (figure 3 shows that with a unique shareholder a uniform float gives a smaller Banzhaf index than twice its participation).

In the last columns, we consider the float as composed of small independent shareholders with same participation. In the fourth column we assume that each float is composed of hundred of shareholders (thus for Lafarge they all have 0.5% of voting rights), in the fifth column the floats are composed of shareholders with about 1% of voting rights, while the last column considers floats made of 10 shareholders (with 5% of voting rights for Lafarge). We observe that with an increasing size of shareholders in the float, the Banzhaf index of GBL decreases, while the Banzhaf index of Nassef Sawiris increases. This is due to the fact that it becomes easier for Nassef Sawiris to find partners to form a winning coalition. With a float composed of 10 shareholders, we observe that the Banzhaf index of Albert Frere, Power Corporation du Canada, and Nassef Sawiris are very close. Furthermore, in this last case, if the French institutions are independent, then Albert Frere and Power Corporation du Canada have a slightly better index than the Egyptian Businessman, while with controlled French institutions Nassef Sawiris has a slightly better index.

A last general observation is that the more the float is dispersed (or does not take part in the voting game), the greater the difference between the indexes with independent and coalised French institutions.

These analyses show that the float can strongly impact the control relation between shareholders. To reach in a winning coalition, Albert Frere needs to cooperate with the Demarais Family. Furthermore, if it possible for Nassef Sawiris to form a winning coalition with big partners, he can obtain a higher Banzhaf index than the Belgian businessman. The answer to the question “who is the most influent ultimate shareholder of Lafarge” clearly is the following: “It depends on the float structure and existing coalition in it.”

An important remark can be made about the float assumptions. Indeed, some floats are considered to have big shareholders. This hypothesis is probably wrong, but we can identify three phenomena that can appear among the float and justify it. 1) The first is that small shareholders can unite with the help of a small-shareholders’ lobbying agency. This union has the same behaviour than one big shareholder. 2) The second phenomenon corresponds to external investors taking important participations in the target firm. 3) The last concerns existing shareholders who can increase their participations in the target firm. This would equate a coalition with an important shareholder in the float.

6 Conclusions:

The first part of this paper proposes a new algorithm to measure the control of firms in any ownership structure. The algorithm computes the Banzhaf index in two steps: the first step simplifies the structure by using the matrix consolidation method and the second deducts the Banzhaf index from the coincidence matrix. The Banzhaf index presents the following advantages: 1) it allows one to compare the extent of control exerted between different shareholders when this control is shared. 2) The result can be interpreted as the probability for a shareholder to have a decisive vote. 3) The index can be used to mathematically prove properties of control in ownership structures. The example demonstrates the interest of this index.

The algorithm allows to compute the Banzhaf index of all ultimate shareholders with high accuracy in all the firms of the graph with only one Monte-Carlo simulation while the previous computation required one simulation for each pair of ultimate shareholder and target firm. The main advantage of the new algorithm is to be applicable to large ownership structures, and to assess for all ultimate shareholders their extent of control in all target firms of the group. Nevertheless, the results regarding intermediate firms must be interpreted with care as their computed index does not represent their real influence.

The second part of this article models the float and shows by means of an example how the float's behaviour can impact the control relation between shareholders. In our empirical illustration, the float is first modelled as non-intervening in the game, then, as a uniform variable, and lastly as a binomial random variable. The float behaviour changes the probability for different important shareholders to find winning coalitions. Not only can the value of the shareholders' indexes change, but their order of importance as well.

The main limitation of the algorithm comes from the exclusion of cross-ownership in most cases. In the example of the Lafarge Group, cross-ownership was not a problem since the group Total loses all his voting right in CNP as Albert Frere fully controls CNP. Nonetheless, if this were not the case, the algorithm would cycle to infinite in some cases (see Crama and Leruth 2007). The answer to the control question in cross-ownership structures still remains a challenge that demands further research.

The innovative algorithm proposed in this paper constitutes a significant step for further research in ownership structure analyses. Indeed, it offers a good tool to evaluate control, especially when control is shared. In addition, this model can easily be applied to large-scale data.

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Annexe:

This appendix shows that a uniform float can be seen as a float with an infinite number of shareholders with one that owns 1/2 of the float, a second ¼, a third 1/8, etc. The following theorem proves it:

> *Theorem:*
>
> *If* $\forall i \in \mathbb{N}, X_i \sim B(1/2)$ *Bernoulli distribution with probability ½, then* $Z = \sum_{i=1}^{\infty} \frac{1}{2^i} X_i \sim U(0,1)$ *a uniform distribution on (0,1).*

This theorem is a consequence of the following lemma:

> *Lemma:*
>
> *If the random variable* X *has Bernoulli distribution with a probability of ½ and* Y *a uniform distribution on* $(0,1)$ *then the random variable* $Z = \frac{X+Y}{2}$ *has a uniform distribution on* $(0,1)$. *Mathematically, if* $X \sim B(1/2)$, *and if* $Y \sim U(0,1)$, *then* $Z = \frac{X+Y}{2} \sim U(0,1)$.

Proof of the lemma:

If $z < Z < z + \Delta z$ with $z < 0.5$, then it means that $X = 0$ and $z < \frac{Y}{2} < z + \Delta z$. So if $0 < z < 0.5$, then

$$P(z < Z < z + \Delta z) = P(X = 0 \text{ and } z < \frac{Y}{2} < z + \Delta z) = P(X = 0).P(2z < Y < 2z + 2\Delta z)$$

$$= \frac{1}{2}.2\Delta z = \Delta z$$

If $z < Z < z + \Delta z$ with $1 > z \geq 0.5$, then it means that $X = 1$ and $z < \frac{1+Y}{2} < z + \Delta z$ or $2.z - 1 < Y < 2.z - 1 + 2.\Delta z$. So if $z \geq 0.5$, then

$$P(z < Z < z + \Delta z) = P(X = 0 \text{ and } 2.z - 1 < Y < 2.z - 1 + 2.\Delta z)$$

$$= P(X = 1).P(2z - 1 < Y < 2z - 1 + 2\Delta z) = \frac{1}{2}.2\Delta z = \Delta z$$

So in any case, if $0 < z < 1$, $P(z < Z < z + \Delta z) = \Delta z$ and 0 else, which means that $Z \sim U(0,1)$.

Proof of the theorem:

From this lemma, we can deduce that if $Z \sim U(0,1)$ and all random variable $X_1, X_2, X_3, \ldots$ have independent Bernoulli distributions $B(1/2)$, and all $Y_1, Y_2, Y_3, \ldots$ have independent uniform distributions $U(0,1)$, then

$$Z = \frac{1}{2}.X_1 + \frac{1}{2}.Y_1 = \frac{1}{2}.X_1 + \frac{1}{4}.X_2 + \frac{1}{4}.Y_2 = \ldots = \sum_{i=1}^{n} \frac{1}{2^i} X_i + \frac{1}{2^n} Y_n \quad \forall n \in \mathbb{N}$$

If $n \to \infty$ the equality becomes $Z = \sum_{i=1}^{\infty} \frac{1}{2^i} X_i$.

Figure 1: Theoretical Example

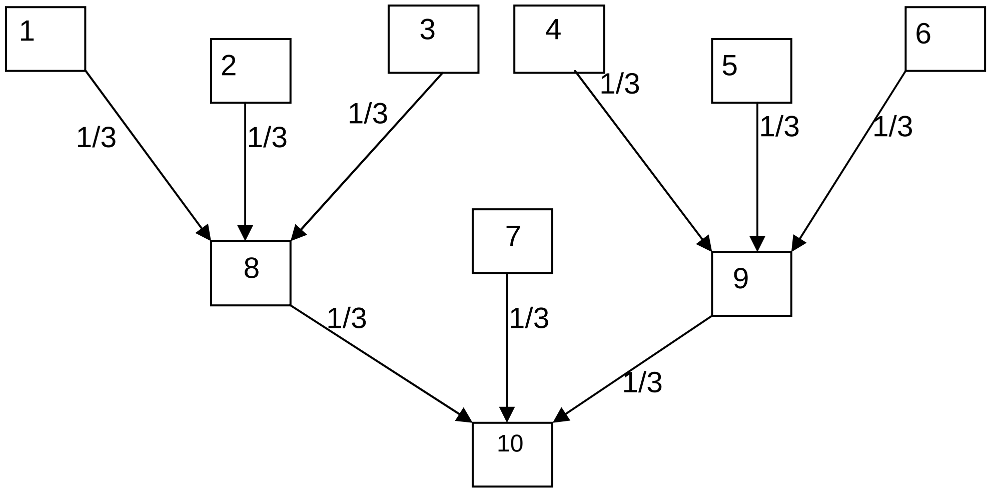

Figure 2: Example 1 after the fusion 3 and 4

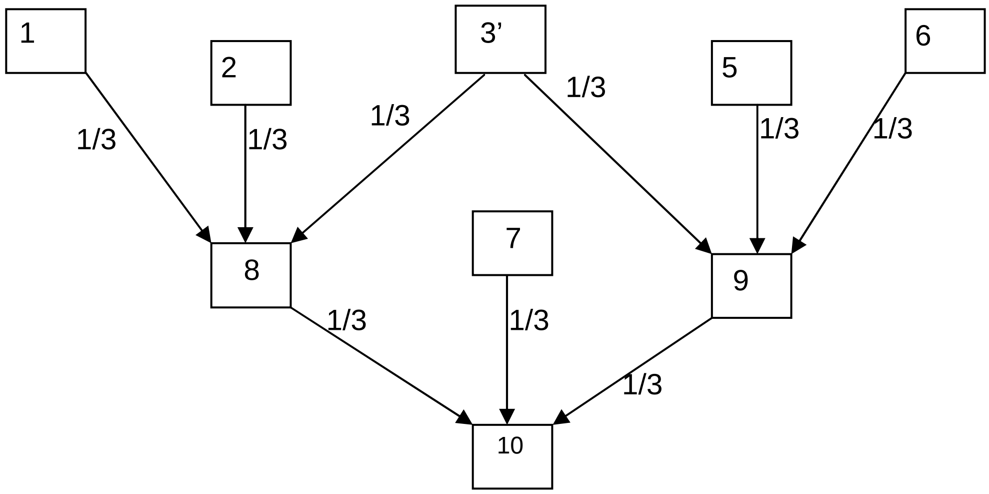

Figure 3: Banzhaf index of a shareholder with a participation x for different float distributions

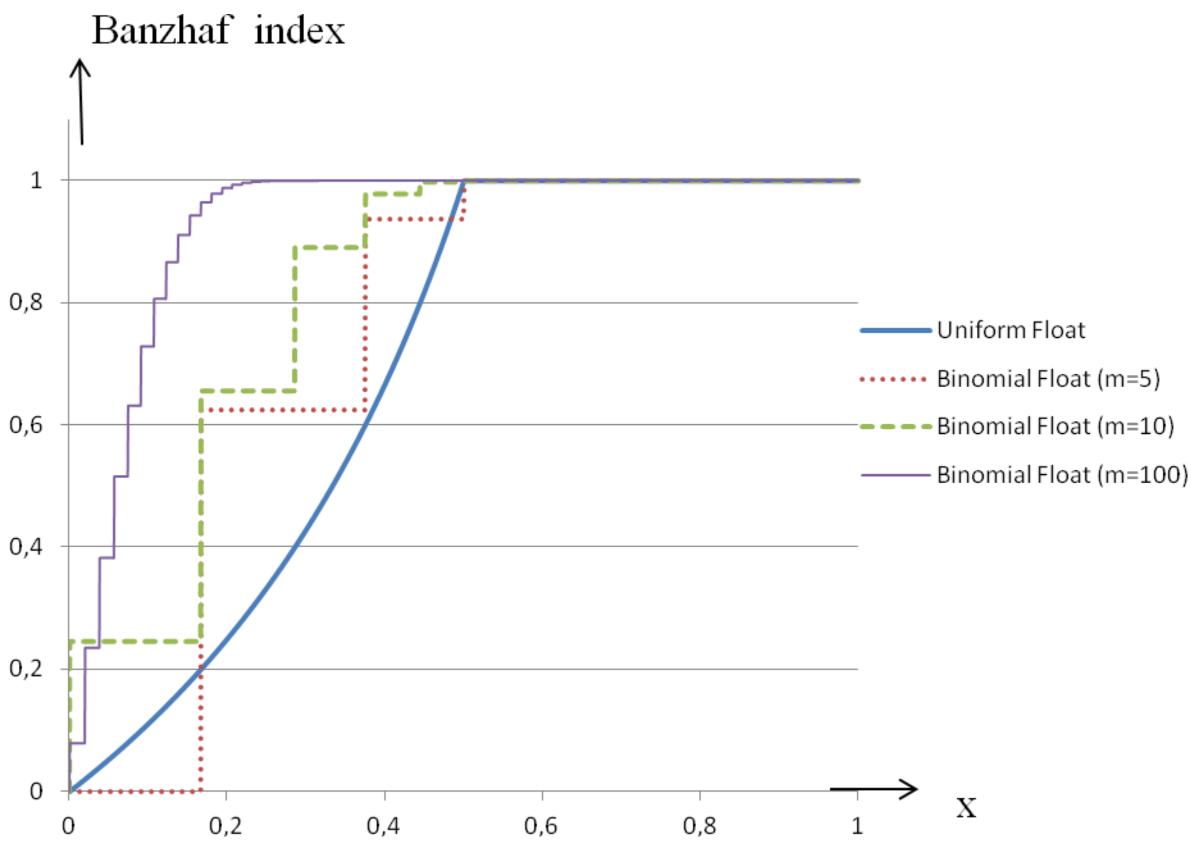

Figure 4: Direct ownership structure of Lafarge

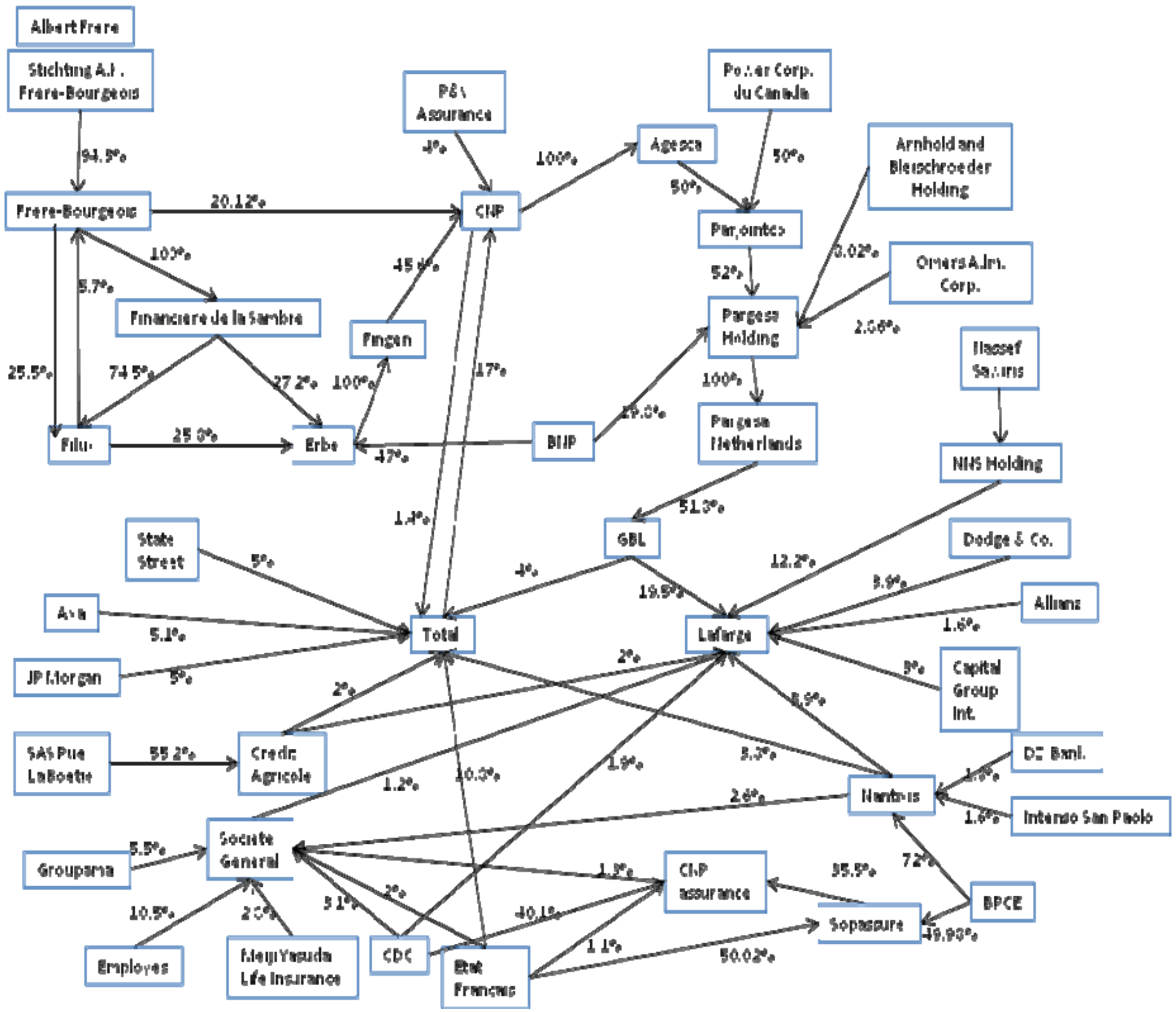

Figure 5: Controlling ownership structure with independent French institutions

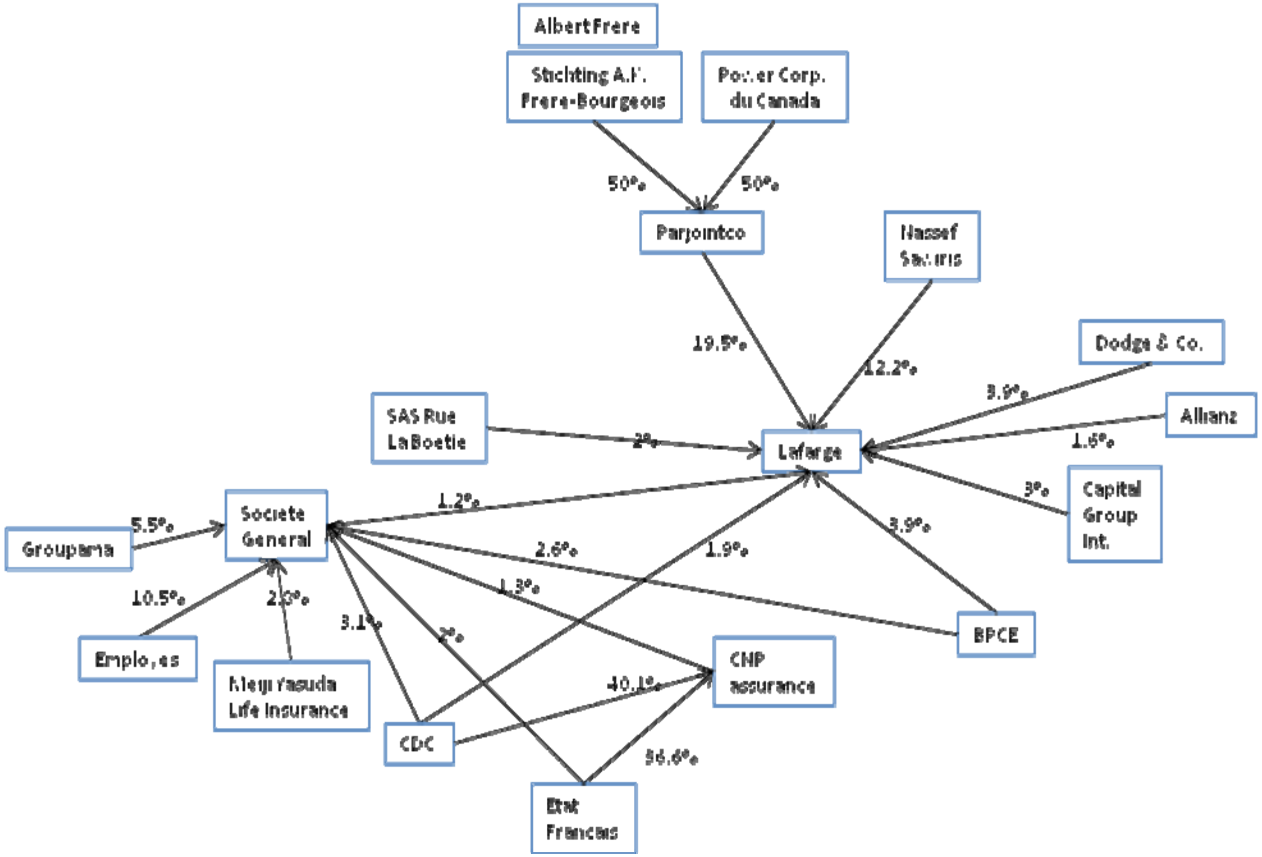

Figure 6: Controlling ownership structure with controlled French institutions (by the French State)

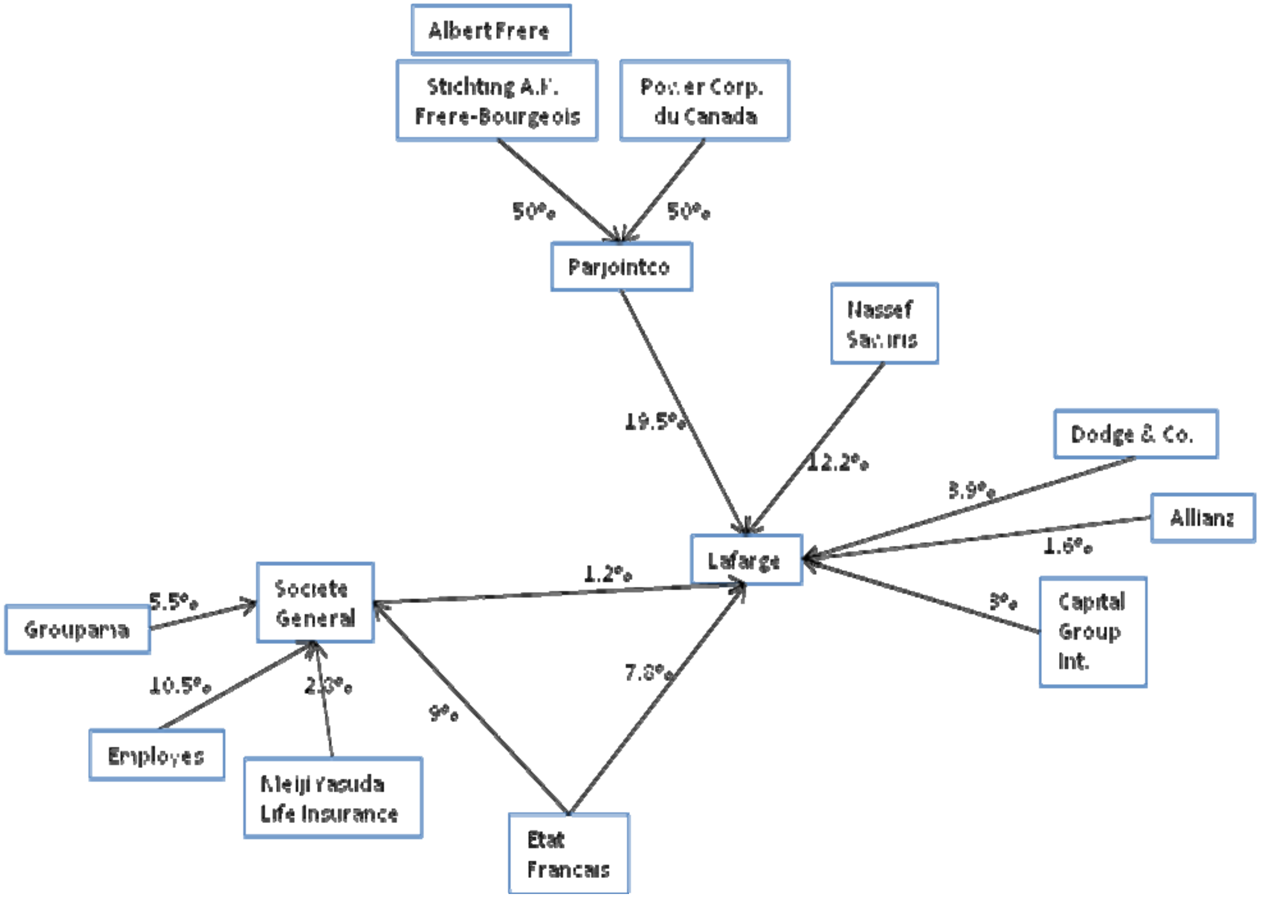

Table 1: Results of the Banzhaf index in the different scenario

		Non influent float		Uniform float		Float bin(100)		Float (1%)		Float (10)	
	Int. Own.	no-coal	Coal	no-coal	Coal	no-coal	Coal	no-coal	Coal	no-coal	Coal
Stichting A.K. Frere-Bourgeois	1,17%	41,2%	35,5%	19,7%	19,2%	39,0%	35,8%	37,9%	35,2%	31,3%	29,8%
Power Corp. du Canada	2,63%	41,2%	35,5%	19,7%	19,2%	39,1%	35,8%	38,0%	35,3%	31,3%	29,8%
Nassef Sawiris	12,20%	17,7%	28,5%	24,6%	24,0%	21,9%	28,3%	24,0%	29,2%	30,5%	30,7%
Dodge & Co.	3,90%	12,6%	18,4%	7,9%	7,7%	12,8%	11,9%	11,9%	11,0%	9,8%	10,7%
Capital Group Int.	3,00%	10,6%	6,6%	6,1%	5,9%	9,7%	9,0%	9,2%	8,2%	7,5%	6,9%
Allianz	1,60%	4,4%	6,6%	3,2%	3,1%	5,1%	4,9%	4,2%	3,6%	3,9%	4,5%
Groupama	0,07%	0,9%	2,7%	0,2%	0,2%	0,9%	1,0%	0,9%	1,0%	0,5%	0,7%
Employés (SG)	0,13%	3,8%	2,7%	0,4%	0,4%	2,5%	2,0%	2,1%	2,1%	1,0%	1,1%
Meiji Yasuda Life Insurance	0,03%	0,9%	0,0%	0,1%	0,1%	0,5%	0,5%	0,5%	0,5%	0,3%	0,2%
SAS Rue La Boétie	1,11%	8,1%		4,0%		6,4%		6,1%		5,1%	
CDC (Caisse des Dépots et Consignations)	1,94%	11,8%		5,4%		10,8%		9,3%		7,2%	
French State	0,08%/5.98%	0,3%	28,5%	1,2%	15,6%	0,6%	26,8%	1,4%	25,9%	1,4%	21,1%
BPCE	2,85%	12,8%		8,0%		13,2%		12,3%		9,9%	
GBL	19,50%	82,3%	71,5%	39,4%	38,4%	78,1%	71,5%	75,9%	70,4%	62,7%	59,5%