Capillary waves in the subcritical nonlinear Schrödinger equation

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We expand recent results on the nonlinear Schrödinger equation with cubic-quintic nonlinearity to show that some solutions are described by the Bernoulli equation in the presence of surface tension. As a consequence, capillary waves are predicted and found numerically at the interface between regions of large and low amplitude.

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In a recent article, Novoa, Michinel, and Tomasini derived a Young-Laplace equation to describe a certain class of stationary solutions of the nonlinear Schrödinger equation (NLS) with cubic and quintic nonlinearities [1]. Light beams, therefore, can sometimes take on the attribute of a liquid when propagating in nonlinear media. This occasional liquidlike nature of light was noted by several authors [2,3]. The abovementioned solutions of the NLS, on the other hand, are also propagating in nonlinear media. This occasional liquidlike phase is given by

\[ \phi(x,y,z) = e^{i\xi_0 z} + \Psi_1(x,y) + \Psi_2(x,y) + \cdots, \]

where \( \xi_0 \) is a time variable for quantum condensates.

As explained in Ref. [1] and equivalently in Ref. [4], a "liquid" phase exists for solutions of the form \( \Psi = |A|e^{-i \mu z} \) when the wave number \( \mu \) approaches a critical value \( \mu_{\infty} = -3\sqrt{2}/16a \). Inside this liquid phase, \( |A| \approx |A_{\infty}| = 3\sqrt{2}/4a \), while outside, in the "gaseous" phase, \( |A| \approx 0 \). To explore the limit \( \mu \rightarrow \mu_{\infty} \) further, we first rescale the space variables as \( x' = x/\ell, y' = y/\ell, z' = z/\ell^2 \), with \( \ell = (2\sqrt{2}/3a)^{1/2} \) and write \( \Psi = |A_{\infty}|a(x')\exp[-i\mu_{\infty}z - i\phi(x')] \). We thus obtain

\[ \frac{\partial \phi}{\partial z'} = -\frac{a - 3\ell^2 a_0}{4} + \nabla^2 a - a|\nabla \phi|^2, \quad \frac{\partial a}{\partial z'} - \nabla a \cdot \nabla \phi - a\nabla^2 \phi, \]

where \( \nabla = (\partial/\partial x', \partial/\partial y') \). After this rescaling, the reference solution marking the transition between a liquid and a gaseous phase is given by

\[ a_0 = (1 + e^{i\phi})^{-1/2}, \quad \phi = 0. \]

Let us perturb this solution by locating the boundary between the two phases at \( x' = \eta, \) where \( \eta \) can depend slowly on \( y' \) and \( z' \). To this end, we introduce a small parameter \( \epsilon \) and seek a solution in the form \( a = a_0(x') + \epsilon \phi_1(X,Y,Z) + \cdots, \phi = \epsilon \phi_1(X,Y,Z) + \cdots \), where \( X = x' - \eta(Y,Z) \) and \( (Y,Z) = (e^{i\phi}, e^{i\phi'}) \). The first nontrivial equation arises at \( O(\epsilon) \) in Eq. (3) and yields, simply,

\[ \frac{\partial \phi_1}{\partial X} = -\frac{\partial \eta}{\partial Z} \quad \text{or} \quad \frac{\partial \phi}{\partial x'} = -\frac{\partial \eta}{\partial z'}. \]

In the hydrodynamical interpretation of \( \phi \) as a velocity potential, the above expression can be viewed as a kinematic boundary condition. On the other hand, at \( O(\epsilon^2) \), we find

\[ \frac{\partial^2 a_1}{\partial X^2} - \frac{1 - 12a_0^2 + 15a_0^4}{4} a_2 = -2a_0 \left[ \frac{\partial \phi_1}{\partial Z} + \frac{1}{2} \left( \frac{\partial \eta}{\partial Z} \right)^2 \right] + a_0 \frac{\partial \phi_1}{\partial X} \frac{\partial \eta}{\partial Y} - \frac{\partial^2 a_0}{\partial X^2} \left( \frac{\partial \eta}{\partial Y} \right)^2. \]

Denoting the right-hand side above by \( G \), the solvability condition is \( \int_{\infty}^{\infty} a_0 \delta X = 0 \), which gives

\[ \frac{\partial \phi_1}{\partial Z} + \frac{1}{2} \left( \frac{\partial \eta}{\partial Z} \right)^2 + \frac{1}{8} \frac{\partial^2 a_0}{\partial X^2} \frac{\partial \eta}{\partial Y} = 0, \]

or

\[ \frac{\partial \phi}{\partial z'} + \frac{1}{2} \left( \frac{\partial \eta}{\partial z'} \right)^2 + \frac{1}{8} \frac{\partial^2 a_0}{\partial X^2} \frac{\partial \eta}{\partial Y} = 0. \]

This is a Bernoulli equation, in which the second term has the meaning of a kinetic energy. The Young-Laplace equation can be deduced from it with \( a_2 = 0 \) and \( \phi = p' z' \), where \( p' \) is a rescaled pressure.1 Note that the term \( \nabla^2 a \) is equivalent to \( \frac{\partial \phi}{\partial z'} \frac{\partial \phi}{\partial z'} \) in Ref. [1] has not been neglected in deriving Eq. (6).

Surface tension is associated to capillary waves in hydrodynamics [6]. Let us investigate this possibility in the present situation. Inside the "liquid" phase, \( \phi \approx 1 \) and \( \nabla^2 \phi \approx 0 \). Hence, small-amplitude waves are approximately given by \( \phi = Fe^{ikr} \sin(k(y' - c z')) \). On the other hand, from the kinematic condition (5) at the interface, \( \eta = -(F/c) \cos(k(y' - c z')) \). Finally, neglecting quadratic terms in \( F \), Eq. (6) yields the dispersion relation \( c = \sqrt{k/\beta} \), which is typical of capillary waves. To check this prediction, Eqs. (2) and (3) were integrated in \( z' \) with a locally curved interface separating the "liquid" and "gaseous" phases as initial condition. The result, shown in Fig. 1, confirms that short wavelengths travel faster than longer ones. Equally, the capillary oscillations of large liquid drop states observed numerically in Ref. [5] should be obtainable by combining \( \nabla^2 \phi \approx 0 \) with (5) and (6) at the boundary along the lines in Ref. [6] (p. 245).

1To compare with the pressure \( p \) defined in Ref. [1], \( p = (9\eta^2/8\delta^2)p' \).
To conclude this Brief Report, we discuss the conditions where such a phenomenon can be observed. From the point of view of model (1), all that this requires is taking an initial condition that is close enough (after rescaling) to Eq. (4), i.e., \( |\Psi|^2 \approx 3y^2/4\delta \) in the center and a sharp transition to \( \Psi = 0 \) at the boundary of the condensed phase. In the context of Bose-Einstein condensate a frequently quoted candidate is the \(^7\)Li condensate, for which two-body phase interactions are attractive and three-body interaction are repulsive [7–9]. The phase transition is described in detail in Refs. [7,8], in the presence of a trapping potential. We assume that the presence of such a trapping potential does not affect the validity of the theoretical argument present here, at least in its broad lines. Hence the present discussion should be applicable where [7,8] are valid. On the optical side, a “condensed” phase may mean a rather large beam power density. Such a high intensity (in W/cm\(^2\)) can be attained by compressing in time a light pulse carrying a given energy. This is described by an equation of the type

\[
\frac{\partial \psi}{\partial z} + \frac{\alpha}{2} \sqrt{\psi} + k n_2 |\psi|^2 \psi - k n_4 |\psi|^4 \psi - \frac{\beta}{2} \frac{\partial^2 \psi}{\partial T^2} = 0,
\]

(7)

where \( k \) is the optical wave number, \( \alpha = 1/n_0 k \), \( n_0 \) is the refraction index, \( n_2 \) and \( n_4 \) are the nonlinear refraction indices, \( \beta \) is the dispersion coefficient, and \( T \) is time in moving with the pulse. The required power density is thus \( 3n_2/4n_4 \). Given a pulse duration \( \Delta T \) to achieve this power density, chromatic dispersion effects become significant over a distance \( L_\beta = \sqrt{\beta \Delta T} \). On the other hand, the nonlinear effects described here require a propagation length \( L_{NL} \approx 40\ell^2 = 40 \times (2n_4/3kn_2^2) \), where the factor 40 comes from consideration of Fig. 1. In order for the chromatic dispersion to be negligible, one must ensure that \( L_\beta \gg L_{NL} \). In addition, \( 3n_2/4n_4 \) must be low enough for other physical processes, such as ionization, not to occur; besides, this intensity must be less than the damage threshold of the optical components of the experimental setup. All these conditions seem to be achievable in the experiment described in Ref. [10], which studies filamentation in CS\(_2\). For the data in that article, \( 3n_2/4n_4 \approx 11 \times 10^{11} \text{W/cm}^2 \) and \( L_{NL} \approx 0.8 \text{mm} \). Other optical media are also possible, particularly colloid quantum dot nanocrystals, which have been characterized in Refs. [11,12]. In that case the nonlinearity is rather of the form

\[
\frac{n_2 |\psi|^2}{1 + |\psi|^2/I_s} \approx n_2 |\psi|^2/(1 - |\psi|^2/I_s),
\]

where \( I_s \) is a saturation intensity and the condensed phase corresponds to \( |\psi|^2 \approx 3I_s/4 \). For quantum dot PbSe, \( I_s = 46 \text{MW/cm}^2 \). Finally, the possibility of liquidlike dynamics of light beam propagating in a four-level medium was discussed in Ref. [13].

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