Which Optimal Design For Lottery Linked Deposit Accounts?

M. Pfiffelmann

Lottery-linked deposit accounts (LLDAs) are financial assets that provide an interest rate determined by a lottery. These accounts that combine savings and lottery have become very popular in recent years and in a number of countries (Guillen and Tschoegel). However, their existence cannot be explained in the framework of the expected utility model. Their popularity can only be understood in light of behavioral finance studies, especially if individual preferences are described by Kahneman and Tversky's cumulative prospect theory (1992). Actually, this theory provides a good explanation for the emergence of these deposit accounts by integrating simultaneously risk-averse and risk-seeking behaviors. In this paper, we propose a behavioral analysis of these financial assets by assuming that investors' individual preferences obey cumulative prospect theory. We study how the structure of prizes of the LLDAs should be framed to appeal to and attract many investors. Our aim is thus to determine the optimal design of these financial assets.

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WHICH OPTIMAL DESIGN FOR LOTTERY LINKED DEPOSIT ACCOUNTS?

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Résumé:

Les comptes d’épargne associés à des loteries sont des actifs financiers dont les intérêts sont déterminés par une loterie. Ces dernières années, ces actifs qui combinent ainsi épargne et loterie sont devenus très populaires dans de nombreux pays (Guillen et Tschoegel). Leur existence ne peut cependant pas être expliquée par la théorie traditionnelle de l’utilité espérée. Par contre, leur popularité peut être comprise grâce aux apports de la finance comportementale et spécialement dans le cas où la théorie des perspectives cumulative de Tversky et Kahneman (1992) décrit les préférences individuelles des agents. En effet, cette théorie procure une bonne explication à l’émergence de ces instruments financiers en intégrant simultanément des comportements de recherche de risque et de riscophobie. Dans cet article, nous proposons une analyse comportementale de ce type d’actifs financiers en supposant que les préférences des investisseurs sont représentées par la théorie de perspectives. Nous étudions comment structurer les prix de la loterie de manière à attirer de nombreux investisseurs. Notre objectif est de déterminer par ce biais le design optimal de ces comptes d’épargne.

Abstract:

Lottery-linked deposit accounts (LLDAs) are financial assets that provide an interest rate determined by a lottery. These accounts that combine savings and lottery have become very popular in recent years and in a number of countries (Guillen and Tschoegel). However, their existence cannot be explained in the framework of the expected utility model. Their popularity can only be understood in light of behavioral finance studies, especially if individual preferences are described by Kahneman and Tversky’s cumulative prospect theory (1992). Actually, this theory provides a good explanation for the emergence of these deposit accounts by integrating simultaneously risk-averse and risk-seeking behaviors. In this paper, we propose a behavioral analysis of these financial assets by assuming that investors’ individuals preferences obey cumulative prospect theory. We study how the structure of prizes of the LLDAs should be framed to appeal to and attract many investors. Our aim is thus to determine the optimal design of these financial assets.

Jel Classification: D81, G11
1 Introduction

Expected Utility Theory has been considered for several decades to be a benchmark for describing decision making under risk. According to this normative model of rational choices, attitude towards risk is entirely characterized by the shape of the utility function. In economics and finance, it is generally assumed that risk-averse behavior is modeled by a concave utility function. However, this assumption of strict risk-aversion has recently been seriously questioned. On the one hand, individuals’ preferences for insurance lead to a risk-averse behavior. On the other hand, acceptance of gambling indicates risk-seeking behavior. Two conflicting behavioral choices are therefore observed. Such behavioral choices are also observed in the financial field. More precisely, lottery linked deposit accounts (LLDA) have recently become very popular in many countries (Guillen and Tschoegl 2002). These deposit accounts are financial assets that provide an interest rate determined by a lottery. Their existence cannot be explained in the framework of expected utility models since a risk-averse investor would accordingly always prefer to get the expected value of a lottery rather than participate in the gamble. Gambling is therefore inconsistent with expected utility theory. However, the popularity of these financial assets can be better understood by behavioral finance studies. Indeed, Rank Dependent Expected Utility (Quiggin 1982) and Cumulative Prospect Theory (Tversky, Kahneman 1992) provide a good explanation for the emergence of these deposit accounts by integrating simultaneous risk-averse and risk-seeking behaviors.

Pfiffelmann and Roger (2005) have shown how investors with individual preferences described by Kahneman and Tversky’s prospect theory can be attracted by such financial assets. They compared two LLDDAs: the Premium Savings Bonds issued by the British Treasury and the Savings Account MMmax launched by the French insurance company “les Mutuelles du Mans”. The strong asymmetry of the Premium Bonds makes this financial asset more attractive than the Savings Account MMmax. In fact, the Premium Bonds propose a very substantial potential gain associated with an infinitesimal winning probability. Investors are attracted to this financial asset because they dream of earning a large amount of money. This desire drives them to overweight this low probability of winning. Furthermore, Pfiffelmann and Roger pointed out that a modification of the payment structure ("ceteris paribus") for the lottery of the MMmax account in the sense of an increase in the positive asymmetry could improve the appeal of the account.
The purpose of the present paper is to determine the optimal design of an LLDA given that all investors have identical individual preferences as per Kahneman and Tversky. We are investigating on the "best" structure of payments of these type of financial assets. First, we analyze the case of the two particular financial assets described previously (the MMmax account and the Premium Saving Bonds). In order to optimize the lottery design, we first minimize the anticipated cost of the issuer under the investor’s participation constraint. An other way to formulate this problem is to maximize the satisfaction of investors given an issuer’s cost equal to the risk-free interest rate. In both cases, the optimization program leads to an optimal and very relevant structure of payment. In fact, the results show that the lottery should be strongly asymmetrical. The explanation lies in the tendency individuals have to overweight the extremely low probability of the desired outcome. In the next step, we generalize the results and analyze the problem of the optimal design of LLDA. Our aim is now to determine the optimal payment structure and the associated optimal probabilities. Our results allow us to discuss the shape of the Kahneman and Tversky’s weighting function. Links with the behavioral portfolio theory (Shefrin and Statman 2000) can be established.

The paper is structured as follow: In section II, we review the main characteristics of the Cumulative Prospect Theory developed by Kahneman and Tversky. Section III presents the MMmax account and Premium Savings Bonds and relates previous results. In section IV, we establish the optimal design of these LLDAs. In section V, we discuss the optimal design of LLDAs in a general framework and the shape of the weighting function. The paper concludes in section VI with a summary of our findings.

2 Review of Prospect Theory

In this section, we introduce the Cumulative Prospect Theory by first reviewing the main observations established by Tversky and Kahneman.

2.1 Main observations

- Utility is defined over gains and losses rather than over final asset position. So risky prospects are evaluated relatively to a reference point. This reference point corresponds to the asset position one expects to reach. This point often corresponds
to status quo.

- Individuals do not use objective probabilities when evaluating risky prospects. They transform objective probabilities via a weighting function. They overweight the probabilities of extreme outcomes (events at the upper tail of the distribution). Conversely, they underweight outcomes with average probabilities.

- "The difference in value between a gain of 100 and a gain of 200 appears to be greater than the difference between a gain of 1100 and a gain of 1200." (Kahneman and Tversky 1979). The marginal utility of the value of the gains is decreasing. The same observation is relevant for losses.

- Experiments have identified a gain-loss asymmetry and have indicated a greater sensitivity to losses than to gains. In fact, the loss of $100 creates a distress greater than the satisfaction generated by the gain of the same amount of money.

- Experimental evidence has established a “fourfold pattern of risk attitudes”: risk aversion for most gains and low probability losses, and risk seeking for most losses and low probability gains.

2.2 Modelisation

In this section we briefly present the Cumulative Prospect Theory formalized by Tversky and Kahneman in 1992. In a first version (1979) Kahneman and Tversky supposed that people transform individual probabilities directly via a weighting function. This assumption leads to preferences that violate the first order stochastic dominance criteria. In the cumulative version, they took into account Quiggin and Yaari’s work and applied the probability weighting function to the cumulative probability distribution. Therefore, while attitude towards risk is fully characterized by the value function under Expected Utility Theory, under Cumulative Prospect Theory, attitude towards risk is determined simultaneously by the value function and the cumulative weighing function.

Consider a prospect $X$ defined by:

$$X = ((x_i, p_i) | i = -m, ..., n)$$

with $x_{-m} < x_{-m+1} < ... < x_0 = 0 < x_1 < x_2 < ... < x_n$.

We have mentioned previously that gains and losses are evaluated differently by individuals. In order to take into account this assumption, the evaluation function $V$ of a prospect $X$ is defined by:
\[ V(X) = V(X^+) + V(X^-) \]

where \( X^+ = \max(X; 0) \) et \( X^- = \min(X; 0) \).

We set:

\[ V(X^+) = \sum_{i=0}^{n} \pi_i^+ v(x_i) \]
\[ V(X^-) = \sum_{i=-m}^{0} \pi_i^- v(x_i) \]

where \( v \) is a strictly increasing value function defined with respect to a reference point satisfying \( v(x_0) = v(0) = 0 \).

\( \pi^+ = (\pi_0^+, \ldots, \pi_n^+) \) and \( \pi^- = (\pi_{-m}^-, \ldots, \pi_0^-) \) are the weighting functions for gains and losses respectively defined by:

\[ \pi_i^+ = w^+(p_i) \]
\[ \pi_{-m}^- = w^-(p_{-m}) \]
\[ \pi_i^+ = w^+(p_i + \ldots + p_n) - w^+(p_{i+1} + \ldots + p_n) \text{ with } 0 \leq i \leq n - 1 \]
\[ \pi_i^- = w^-(p_{-m} + \ldots + p_i) - w^-(p_{-m} + \ldots + p_{i-1}) \text{ with } -m \leq i \leq 0 \]

with \( w^+(0) = 0 = w^-(0) \) and \( w^+(1) = 1 = w^-(1) \)

Consider \( F \) the cumulative distribution function of \( X \). We notice that \( w^+(p_i) \) is applied to the decumulative distribution function of \( X \) (i.e.)

\[ w^+(\sum_{j=1}^{n} p_j) = w^+(1 - F(x_{i-1})) \]

whereas \( w^-(p_i) \) is applied to the cumulative distribution function of \( X \)

\[ w^-(\sum_{j=-m}^{i} p_j) = w^-(F(x_i)) \]

Tversky and Kahneman (1992) proposed the following functional form for the value function:

\[ v(x) = \begin{cases} 
  x^\alpha & \text{if } x > 0 \\
  -\lambda(-x)^{-\beta} & \text{if } x < 0 
\end{cases} \]

For \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \) the value function \( v \) is concave over gains and convex over losses (observation 3). It is kinked at the origin and steeper for losses than for gains (observation 4). The parameter \( \lambda \) describes the degree of loss aversion (Kobberling and Wakker 2005). Based on experimental evidence Tversky
and Kahneman estimated the values of the parameters $\alpha$, $\beta$, and $\lambda$. $\alpha = \beta = 0.88$ and $\lambda = 2.25$.

They proposed the following functional form for the weighting function:

$$w^+(p) = \frac{p^{\gamma^+}}{[p^{\gamma^+} + (1 - p^{\gamma^+})]^{1/\gamma^+}} \quad w^-(p) = \frac{p^{\gamma^-}}{[p^{\gamma^-} + (1 - p^{\gamma^-})]^{1/\gamma^-}}$$

For $\gamma < 1$, this functional form integrates the overweighting of low probabilities and the greater sensitivity for changes in probabilities for extremely low and extremely high probabilities. The weighting function is concave near 0 and convex near 1.

Tversky and Kahneman estimated the parameters $\gamma^+$ and $\gamma^-$ as 0.61 and 0.69.

3 Description of LLDAs

LLDA are financial assets that combine savings with a lottery. The lottery only affects the coupon provided by the LLDAs’ issuer. The interest rate of this type of assets can thus range from zero to a very high value. However, the investor’s principal is not affected by the lottery and is then totally secured. In this section we describe two LLDAs: The MMmax account and the Premium Saving Bonds.

3.1 MMmax

MMmax is a savings account launched by the French insurance company "les Mutuelles du Mans" in November 2003. This account provides a fixed interest rate of 2.5% plus the chance to win a price in a quarterly lottery. This lottery makes it possible to win additional interest of 5, 10 or 20%. The interest rate of the MMmax account, thus, ranges from 2.5% to 22.5% per year. The expected return (the expected cost for the issuer) of the account is random. There are four draws per year, but investors can only participate in two of them. However, even if investors do participate in two draws per year they cannot win more than one prize a year. So, if an investor wins a prize in the first draw, he will not be granted permission to participate in a second draw. The minimum investment authorized is 150 euros per contract while the maximum is 1500 per contract.

Table 1 displays the interest rates offered by the insurance company and the associated probabilities.

The expected value of the return is 3.564% per year.
Table 1: interest rates and corresponding probabilities

<table>
<thead>
<tr>
<th>return</th>
<th>2.5%</th>
<th>7.5%</th>
<th>12.5%</th>
<th>22.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>0.81</td>
<td>0.171</td>
<td>0.0171</td>
<td>0.0019</td>
</tr>
</tbody>
</table>

Pfiffelmann and Roger applied prospect theory to the MMmax account. They calculate the utility and the ponderation for each possible gain and loss. Table 2 shows the results of the evaluation of the deposit.\(^1\)

Table 2: Results of the evaluation

<table>
<thead>
<tr>
<th>probability</th>
<th>return</th>
<th>final asset</th>
<th>(v(x))</th>
<th>(w(p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.81</td>
<td>2.5%</td>
<td>1025</td>
<td>-18.02</td>
<td>67.81%</td>
</tr>
<tr>
<td>0.171</td>
<td>7.5%</td>
<td>1075</td>
<td>25.33</td>
<td>17.55%</td>
</tr>
<tr>
<td>0.0171</td>
<td>12.5%</td>
<td>1125</td>
<td>52.12</td>
<td>5.77%</td>
</tr>
<tr>
<td>0.0019</td>
<td>22.5%</td>
<td>1225</td>
<td>100.92</td>
<td>2.12%</td>
</tr>
</tbody>
</table>

They established that investors would prefer a regular savings account providing a fixed interest rate of 3.564% to the MMmax account \((V(X) < 0)\). An explanation of these results lies on the fact that the skewness of the lottery is not strong enough. In fact, its positive asymmetry is not sufficiently important to offset loss aversion. In particular, the amount of the jackpot is too low to involve a risk-seeking behavior. In light of these facts, it can be established that an increase in the asymmetry of the account would increase its appeal. Actually, a modification of the structure of payments of the MMmax account could improve its attractiveness. In order to increase the asymmetry we can reduce the fixed interest rate to 2.4% and increase the amount of the jackpot to 75% \(^2\). The application of the prospect theory to this modified MMmax account leads to a positive evaluation \((V(X) > 0)\). In this setting case investors would prefer the modified MMmax account to a regular savings account which provides a fixed interest rate of 3.564%. The analysis given here suggests that a modification of the structure of payments can improve the attractiveness of the account.

\(^1\)The reference point corresponds to the initial wealth capitalized at the expected value of the account. The initial wealth considered is 1000 euros.

\(^2\)In order to keep the comparability of the results, an expected cost of 3.564% is kept.
3.2 Premium Saving Bonds

Premium Bonds are investments in which instead of getting an interest rate, investors have the chance to win tax-free prizes. For each pound invested, investors receive one bond. Each bond automatically enrolls the holder in a monthly lottery. The prize fund for each draw is shared between three prize bands (higher value, medium value and lower value) and prizes range from £50 to £1 million. These bonds are backed by the Treasury. The minimum purchase is £100 and the maximum is £30000.

Each month’s prize fund is calculated relatively to one month’s interest determined by the Treasury. For example, in February, 2006 the interest rate determined by the Treasury was 0.25% (3% pa). So the prizes were allocated in such a way that the expected value of the lottery was 0.25%.

Table 3 shows the percentage share of the fund allocated to each prize band, together with the number of prizes, their value and the corresponding probabilities of winning for February 2006.

<table>
<thead>
<tr>
<th>Prize band</th>
<th>Prize value</th>
<th>Number of prizes</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higher value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7% of prize fund</td>
<td>£1 million</td>
<td>2</td>
<td>6.7693×10⁻¹¹</td>
</tr>
<tr>
<td></td>
<td>£100 000</td>
<td>6</td>
<td>2.0308×10⁻¹⁰</td>
</tr>
<tr>
<td></td>
<td>£50 000</td>
<td>13</td>
<td>4.4001×10⁻¹⁰</td>
</tr>
<tr>
<td></td>
<td>£25 000</td>
<td>26</td>
<td>8.8002×10⁻¹⁰</td>
</tr>
<tr>
<td></td>
<td>£10 000</td>
<td>64</td>
<td>2.1662×10⁻⁹</td>
</tr>
<tr>
<td></td>
<td>£5 000</td>
<td>126</td>
<td>4.2642×10⁻⁹</td>
</tr>
<tr>
<td>Medium value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6% of prize fund</td>
<td>£1 000</td>
<td>1 772</td>
<td>5.9976×10⁻⁸</td>
</tr>
<tr>
<td></td>
<td>£500</td>
<td>5 316</td>
<td>1.7903×10⁻⁷</td>
</tr>
<tr>
<td>Lower value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>87% of prize fund</td>
<td>£100</td>
<td>61 530</td>
<td>2.0826×10⁻⁶</td>
</tr>
<tr>
<td></td>
<td>£50</td>
<td>1 162 176</td>
<td>3.9336×10⁻⁵</td>
</tr>
<tr>
<td>Total value</td>
<td>£73.9 million</td>
<td>1 231 031</td>
<td></td>
</tr>
</tbody>
</table>

An investor with individual preferences as described by Kahneman and Tversky would prefer holding Premium Saving Bonds rather than regular bonds that provide a fixed 3% annual interest rate (V(X) = 0.27 > 0). Unlike the MMmax account the asymmetry of the lottery is strong enough to attract investors. In fact, the amount of the jackpot is very large (£1 million). Investors with individual
preferences obeying prospect theory overweight this event because they dream of growing rich in a sizeable way. More precisely, the weighting of the event "gain of £1 million" is $6.2608 \times 10^{-7}$ whereas the probability is $6.7693 \times 10^{-11}$ (approximately 10 000 times more).

4 Optimal design for MMmax and Premium Bonds.

In this paper, we study framing in the design of financial assets. We analyze how the structure of payments of LLDAs should be framed to be optimal. In this section, we start by examining two particular cases of the MMmax account and the Premium Saving Bonds.

4.1 MMmax account

Our aim is to determine the optimal design of the MMmax account. In this section we will keep the draw’s modalities used by the "Mutuelles du Mans" since November 2003. We consider a fixed explicit interest rate noted $y_1$ (associated to a probability of 0.81) and three additional interest rates determined by a quarterly lottery ($z_2$, $z_3$ and $z_4$). The probabilities associated to these additional interest rates are respectively 0.171, 0.0171 and 0.0019. Table 4 summarizes these different elements.

Table 4: interest rates and corresponding probabilities

<table>
<thead>
<tr>
<th>return</th>
<th>$y_1$</th>
<th>$y_1 + z_2$</th>
<th>$y_1 + z_3$</th>
<th>$y_1 + z_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>0.81</td>
<td>0.171</td>
<td>0.0171</td>
<td>0.0019</td>
</tr>
</tbody>
</table>

4.1.1 Minimization problem

In order to optimize the lottery design, we minimize the anticipated costs of the issuer under the investor’s participation constraint. In fact, we want to find the payment structure which provides the smallest cost for the insurance company. At the same time, this account should be attractive for investors. Since utility is defined by gains and losses, a risky prospect is evaluated relatively to a reference point. We denote $r$ as the reference point. So a savings account is attractive if investors would prefer this account rather than one which pays an annual interest rate of $r \%$. A payoff less than the initial wealth capitalized at a rate of $r$ is considered as a loss.
and a payoff greater than the initial wealth capitalized at a rate of $r$ is considered as a gain. In our problem we consider that $y_1$ is inferior to $r$ and that $y_1 + z_2$, $y_1 + z_3$ and $y_1 + z_4$ are superior to $r$. Actually, investors get $y_1$ when they loose the lottery and $y_1 + z_2$, $y_1 + z_3$ or $y_1 + z_4$ when they win. The evaluation function of investors can be written as:

$$V(X) = \pi_1 [-2.25(r - y_1)^\alpha] + \pi_2 (y_1 + z_2 - r)^\alpha + \pi_3 (y_1 + z_3 - r)^\alpha + \pi_4 (y_1 + z_4 - r)^\alpha$$

where $\pi_1 = w^-(p_1) = w^- (0.81) = 0.6781$

$\pi_2 = w^+(p_2 + p_3 + p_4) - w^+(p_3 + p_4) = w^+(0.19) - w^+(0.019) = 0.1755$

$\pi_3 = w^+(0.019) - w^+(0.0019) = 0.0577$

$\pi_4 = w^+(0.0019) = 0.0211$

$\alpha = 0.88$

This function represents the investor’s participation constraint. In fact, $V(X) > 0$ means that investors would choose the savings account and then accept the contract proposed by the issuer.

Thus the constrained minimization problem is:

$$\text{Min}_{y_1, z_2, z_3, z_4} \quad 0.81y_1 + 0.171(y_1 + z_2) + 0.0171(y_1 + z_3) + 0.0019(y_1 + z_4)$$

subject to

$$\pi_1 [-2.25(r - y_1)^\alpha] + \pi_2 (y_1 + z_2 - r)^\alpha + \pi_3 (y_1 + z_3 - r)^\alpha + \pi_4 (y_1 + z_4 - r)^\alpha \geq 0$$

$$r - y_1 > 0$$

$$y_1 + z_2 - r \geq 0$$

$$y_1 + z_3 - r \geq 0$$

$$y_1 + z_4 - r \geq 0$$

$$y_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0$$

The solution of the system is:

$$y_1 = 0$$

$$z_2 = r[1 + \left(\frac{0.171}{\alpha \times \pi_2}\right)^{1/\alpha} \times \left(\frac{2.25 \times \pi_1}{G}\right)^{1/\alpha}]$$

$$z_3 = r[1 + \left(\frac{0.0171}{\alpha \times \pi_3}\right)^{1/\alpha} \times \left(\frac{2.25 \times \pi_1}{G}\right)^{1/\alpha}]$$

$$z_4 = r[1 + \left(\frac{0.0019}{\alpha \times \pi_4}\right)^{1/\alpha} \times \left(\frac{2.25 \times \pi_1}{G}\right)^{1/\alpha}]$$

with $G = \pi_2^{1/\alpha} \times \left(\frac{\alpha}{0.171}\right)^{1/\alpha} + \pi_3^{1/\alpha} \times \left(\frac{\alpha}{0.0171}\right)^{1/\alpha} + \pi_4^{1/\alpha} \times \left(\frac{\alpha}{0.0019}\right)^{1/\alpha}$
In 2005, long term interest rates were on average around 3.5% and short term interest rates were around 2.2%. In light of these facts, we have considered that the return investors expect (the reference point \( r \)) is 3%.

The optimal structure of payments is represented in table 5.

Table 5: optimal structure of payments of MMmax account

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( z_2 )</th>
<th>( z_3 )</th>
<th>( z_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>3.000001%</td>
<td>3.018984%</td>
<td>391.708923%</td>
</tr>
</tbody>
</table>

With this structure of payments, the expected cost of the issuer is 1.31%.

When investors make decisions according to cumulative prospect theory, they would accept a 1.31% interest rate only if the lottery is strongly asymmetrical. In fact, the optimization program leads to an optimal structure of payments highly skewed: no more fixed interest rate, but in compensation an amount of jackpot (\( z_4 \)) that is very high.

In order to optimize the lottery design, we have minimized the anticipated costs of the issuer under the investor’s participation constraint. We have obtained an expected cost of 1.31% (which represents also the expected return of the investment). But even if investors are willing to accept a lower expected return than they would receive on an regular savings account, can we imagine a financial institution that provides an expected return of 1.31%? Actually, most investors attracted by these financial assets are small individual investors. In Europe, this type of investor is well protected by consumer protection organizations. And it is doubtful that these organizations would allow banks to provide a financial asset that pays less than the risk-free interest rate.

4.1.2 Maximization problem

The problem of the optimal design of LLDAs can be formulated differently. An alternative is to find the structure of payments that maximizes the satisfaction of investors (represented by \( V(X) \)) given an expected return (cost) equal to the risk-free interest rate (\( r_f \)). In this framework, the savings account will be optimal for investors; and thus very attractive to them. And as many investors will subscribe to the contract proposed by the bank, the administrative and management costs will be spread over more clients. This will inevitably lead to economies of scale.
Consequently, "les Mutuelles du Mans" will have two advantages. On one hand, the issuer’s expected cost will be the risk-free interest rate (so a very low). On the other hand, he will benefit from the economies of scale.

In our problem, we still consider $r$ as a reference point. Thus a payoff inferior to $r$ is considered as a loss and a payoff superior to $r$ is considered as a gain. As previously noted, we consider that $y_1$ is inferior to $r$ and that $y_1 + z_2$, $y_1 + z_3$ and $y_1 + z_4$ are superior to $r$.

The constrained maximization program is:

$$
\max_{y_1, z_2, z_3, z_4} \pi_1 \left[-2.25(r - y_1)^a\right] + \pi_2(y_1 + z_2 - r)^a + \pi_3(y_1 + z_3 - r)^a + \pi_4(y_1 + z_4 - r)^a
$$

subject to

$$
0.81y_1 + 0.171(y_1 + z_2) + 0.0171(y_1 + z_3) + 0.0019(y_1 + z_4) = r_f
$$

$$
r - y_1 > 0
$$

$$
y_1 + z_2 - r \geq 0
$$

$$
y_1 + z_3 - r \geq 0
$$

$$
y_1 + z_4 - r \geq 0
$$

$$
y_1 \geq 0, z_2 \geq 0, z_3 \geq 0, z_4 \geq 0
$$

The solution of the system is:

$$
y_1 = 0
$$

$$
z_2 = r + \left[\frac{0.171}{\alpha \times \pi_2} \frac{1}{\alpha - 1} \times \left(\frac{r_f - 0.19r}{G}\right)\right]
$$

$$
z_3 = r + \left[\frac{0.0171}{\alpha \times \pi_3} \frac{1}{\alpha - 1} \times \left(\frac{r_f - 0.19r}{G}\right)\right]
$$

$$
z_4 = r + \left[\frac{0.0019}{\alpha \times \pi_4} \frac{1}{\alpha - 1} \times \left(\frac{r_f - 0.19r}{G}\right)\right]
$$

with

$$
G = \left(\frac{0.171}{\alpha \times \pi_2}\right) \frac{1}{\alpha - 1} \times 0.171 + \left(\frac{0.0171}{\alpha \times \pi_3}\right) \frac{1}{\alpha - 1} \times 0.0171 + \left(\frac{0.0019}{\alpha \times \pi_4}\right) \frac{1}{\alpha - 1} \times 0.0019
$$

As we did previously, we consider a 3% reference point ($r$). The risk-free interest rate is equal to 2.2% (euribor 3 months).

The optimal structure of payments is represented in table 6. We obtain $V(X) = 0.0701$.

The results are similar to those obtained previously: the design of the MMmax account is optimal when the structure of payments is strongly asymmetrical. That
means that investors exhibit a preference for extremely positively skewed financial accounts. Actually, under cumulative prospect theory, investors overweight the tails of a probability distribution. The solution derived here is consistent with the results of an experiment realized by Shapira and Venezia. The aim of the study was to determine the tradeoffs between size and frequency of lottery prizes. Subjects in the experiment were willing to accept a decrease in the values of small and medium prizes in order to increase the amount of the jackpot as much as possible.

The structure of payments derived here is quite similar to the structure of the Premium Saving Bonds. Actually, these bonds are highly skewed: they do not provide a fixed interest rate but offer the chance to win £1 million.

### 4.2 Premium Bonds

The Premium Saving Bonds are very popular financial assets in the United Kingdom. In a previous section, we explained why these bonds are attractive. Our aim in this section is to increase this attractiveness. We point out that the expected return of these bonds are determined by the Treasury. For example, in February 2006, the Treasury determined a 0.25% interest rate. Therefore the cost born by the issuer is deterministic and fixed. However, the National Savings and Investments (the issuer) can increase its benefits by improving the design of the bonds. Actually, an improvement of the structure of payments would attract more investors. Thus, the issuer could benefit from economies of scale. In this study we have kept the amounts of prizes offered by the National and Savings (the prizes still range from £50 to £1 million). Our goal is to determine the number of winning prizes of the lottery which maximizes the satisfaction of investors given an expected return of 3%. Table 7 presents the results of the maximization program.

We still consider a reference point of 3% (0.25% per month).

The third column displays the number of prizes offered by the National Savings and Investments in February 2006. The fourth column represents the number of prizes the lottery should offer each month to be optimal.

---

Table 6: optimal structure of payments of MMmax account

<table>
<thead>
<tr>
<th>$y_1$</th>
<th>$z_2$</th>
<th>$z_3$</th>
<th>$z_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>3.000002%</td>
<td>3.041979%</td>
<td>860.5176%</td>
</tr>
</tbody>
</table>

---

3The optimization program has been solved by excel solver.
Table 7: Optimal design of Premium Saving Bonds

<table>
<thead>
<tr>
<th>Prize band</th>
<th>Prize value</th>
<th>Number of prizes</th>
<th>Optimal number of prizes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higher value</td>
<td>£1 million</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>£100 000</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>£50 000</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>£25 000</td>
<td>26</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>£10 000</td>
<td>64</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>£5 000</td>
<td>126</td>
<td>5</td>
</tr>
<tr>
<td>Medium value</td>
<td>£1 000</td>
<td>1 772</td>
<td>796</td>
</tr>
<tr>
<td></td>
<td>£500</td>
<td>5 316</td>
<td>5625</td>
</tr>
<tr>
<td>Lower value</td>
<td>£100</td>
<td>61 530</td>
<td>61 556</td>
</tr>
<tr>
<td></td>
<td>£50</td>
<td>1 162 176</td>
<td>1 162 181</td>
</tr>
<tr>
<td>Total value</td>
<td>£73.9 million</td>
<td>1 231 031</td>
<td>1 230 188</td>
</tr>
</tbody>
</table>

A modification of the distribution of the lottery prizes affects the attractiveness of the bonds. The results derived here suggest that investors are willing to accept a decrease in the probability of winning medium prizes in order to increase the number of higher prizes. An explanation of this result lies on the claim that investors are motivated by hope: people dream of growing rich in a sizeable way. In order to have a chance to access at higher social standing, they are willing to take risk and give up to security. This explains why the National Savings and Investments should modify the structure of payments of the Premium Bonds by reducing the number of medium prizes in order to increase the number of high prizes. This modification would improve the attractiveness of the bonds (the function of evaluation $V(X)$ goes from 0.27 to 0.317).

The results derived here are consistent with "Behavioral Portfolio Theory" developed by Shefrin and Statman (2000). According to Lopes, three factors have to be taken into consideration in investors’ problem of choice: security, potential and aspiration. Aspiration relates to a goal, a target value to reach. Security and potential relate to the principal emotions that operate on investors. On the one hand, investors are driven by fear and wish security for their wealth (they want to avoid poverty). But on the other hand, they are willing to take risk in order to grow rich (concept of potential). Shefrin and Statman have developed a portfolio theory by using Lopes’ theory of choice under uncertainty. They also integrate the mental
accounting structure from Kahneman and Tversky prospect theory. Actually most
investors combine low aspiration and high aspiration levels. Thus, they act as if they
create mental accounts and apply specific decision rules to each account, segregating
their portfolio into distinct mental accounts. The optimal portfolio in Behavioral
Portfolio Theory is different from the mean variance optimal portfolio. According
to Shefrin and Statman, each investor determines a probability of acceptable ruin
and a target value that he wants to reach. Then he secures his wealth to this target
value in almost every state. The number of states secured is determined according
to the acceptable probability of ruin each investor has previously established. The
remaining wealth is invested in one state. Investors wish to grow rich and bet on
one particular state. If this state occurs, the portfolio holder will receive much
more money than the usual coupon he would have received in the other states. The
payoff of this optimal portfolio can be viewed as a combination of bonds (risky or
not) and a lottery ticket. The optimal structure of payments we have obtained fits
perfectly into this result. Let’s consider an investor with an aspiration level equal
to the face value of the Premium Bonds and with a probability of ruin equal to
zero. In this case, the optimal structure of payments of these bonds is quite similar
to the optimal portfolio which would be obtained in Behavioral Portfolio Theory
framework. Actually, a portfolio composed by Premium Bonds is safe. In every
states, bondholders will always get back the face value of the bonds. Investor’s
principal is secured. Their wish of security is therefore satisfied. Then there are
few states where they can win a prize offered by the lottery. Moreover in one of
these states bondholders can win a jackpot. This chance to win a large amount
of money meets everyone’s desire to grow rich in a sizeable way. Our results are
thus consistent with the conclusions of the behavioral portfolio theory developed by
Shefrin and Statman.

5 Optimal design of LLDAs

We have determined the optimal design of two particular LLDAs: the MMmax
account and the Premium Saving Bonds. Our aim now is to discuss the optimal
design of any LLDA.
5.1 Optimization problem

In this study, we consider a one dollar face value bond. Its coupons are fully determined by a lottery. In fact, each bond automatically enrolls the investor in a lottery which provides four prizes \((y_1, y_2, y_3, y_4)\). We assume that the first prize \((y_1)\) is less than the gain the investor would have obtained if he had invested the face value of the bond (here 1$) at a rate of \(r \%\) (the reference point) (so \(y_1 < r\)). We also assume that the three other prizes are greater than this point. The probabilities associated to the payoffs are respectively \((p_1, p_2, p_3, p_4)\). The cost of the issuer is deterministic.

We now extend the previous analysis and examine the optimal payment structure and the associated optimal probabilities.

\[
\begin{align*}
&\text{Max}_{y_1, y_2, y_3, y_4, p_1, p_2, p_3, p_4} \\
&-2.25 \times w^-(p_1) \times (r - y_1)^\alpha + [w^+\left(\sum_{j=2}^{4} p_j\right) - w^+\left(\sum_{j=3}^{4} p_j\right)] \times (y_2 - r)^\alpha + [w^+\left(\sum_{j=3}^{4} p_j\right) - w^+(p_4)] \times (y_3 - r)^\alpha + w^+(p_4) \times (y_4 - r)^\alpha
\end{align*}
\]

subject to

\[
\begin{align*}
p_1 \times y_1 + p_2 \times y_2 + p_3 \times y_3 + p_4 \times y_4 &= r_f \\
p_1 + p_2 + p_3 + p_4 &= 1 \\
r - y_1 &> 0 \\
y_2 - r &\geq 0 \\
y_3 - r &\geq 0 \\
y_4 - r &\geq 0 \\
y_1 \leq y_2 \leq y_3 \leq y_4 \\
y_1 \geq 0, \ y_2 \geq 0, \ y_3 \geq 0, \ y_4 \geq 0 \\
p_1 \geq 0, \ p_2 \geq 0, \ p_3 \geq 0, \ p_4 \geq 0
\end{align*}
\]

where \(w^+(p) = \frac{p^{0.61}}{[p^{0.61} + (1 - p^{0.61})]^{1/0.61}}\) and \(w^-(p) = \frac{p^{0.69}}{[p^{0.69} + (1 - p^{0.69})]^{1/0.69}}\).

The maximization problem is now more complex: both probabilities and payments must be determined. In order to discuss the optimal design of LLDAs we have realized some simulations. Table 8 displays the results we have obtained with a reference point of 0.03.

The evaluation seems to increase infinitely when the jackpot \((y_4)\) tends towards infinity with an infinitesimal chance of winning \((p_4\) tends toward zero).
Table 8: Optimal Design of LLDAs

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0</td>
<td>0.03025</td>
<td>1</td>
<td>19</td>
<td>0.9207</td>
<td>0.0759</td>
<td>0.0019</td>
<td>0.0013</td>
</tr>
<tr>
<td>0.3232</td>
<td>0</td>
<td>0.03012</td>
<td>1</td>
<td>120</td>
<td>0.9231</td>
<td>0.0758</td>
<td>0.00086</td>
<td>0.00022</td>
</tr>
<tr>
<td>0.41</td>
<td>0</td>
<td>0.03012</td>
<td>1</td>
<td>250</td>
<td>0.9232</td>
<td>0.0758</td>
<td>0.00086</td>
<td>0.000107</td>
</tr>
<tr>
<td>1.25</td>
<td>0</td>
<td>0.03012</td>
<td>1</td>
<td>10000</td>
<td>0.923</td>
<td>0.0758</td>
<td>0.000864</td>
<td>2.68 $\times 10^{-6}$</td>
</tr>
</tbody>
</table>

The solution derived here suggests that an infinite jackpot associated to an extremely small probability should be sufficient to attract lots of investors. Thus, the amount of the jackpot would be the principal determinant of risk-seeking behavior, and the role of the second prizes would be minor.

But why have we obtained such a result? The explanation lies on the shape of the weighting function.

5.2 Weighting function

Number of experiments have underlined the tendency of individuals to overweight small probabilities. This overweighting of small probabilities induces risk seeking behavior in the domain of gains. The probability weighting function permits probabilities to be weighted nonlinearly. Lots of empirical studies have been done to determine the shape of the weighting function (Wu and Gonzalez 1996, Prelec 1998, Klika and Weber 2001). In our study we have used the weighting function formed by Kahneman and Tversky. This function is inverse S-shaped (concave in the range $(0, p^*)$ and convex in the range $(p^*, 1)$).

![Figure 1: Kahneman and Tversky’s weighting function](image)

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Its slope tends towards infinity when the probabilities are extremely small (near 0). This explains why we have found a infinity jackpot with an infinitesimal chance of winning. The optimal solution we have obtained is a corner solution.

We find the same type of results with the weighting function proposed by Prelec. This function is characterized by:

\[ w^+(p) = \exp[-\beta^+(-\ln(p))\alpha] \quad \text{and} \quad w^-(p) = \exp[-\beta^-(-\ln(p))\alpha] \]

The evaluation seems to increase infinitely when the jackpot \( y_4 \) tends towards infinity with an infinitesimal chance of winning \( p_4 \) tends toward zero.

This result is quite unrealistic. Undoubtedly investors are willing to take risks in order to enrich themselves considerably. However, an infinite jackpot associated to a more or less null probability is an extreme result; consequently, we should question the shape of the weighting function.

6 Discussion

In this study, we have investigated the optimal design of lottery-linked-deposit-accounts given that all investors have identical Kahneman and Tversky’s individual preferences. Actually, the popularity of these financial assets cannot be understood in light of traditional models of rational choices such as Expected Utility Theory. In fact, it has been observed that investors transform objective probabilities via a weighting function, which overweights the tails of a probability distribution. This observation cannot be integrated in the framework of Expected Utility Theory. Under Cumulative Prospect Theory, people express risk seeking behavior for low probability gains. We have showed that such a behavior can explain the development of these financial accounts. Using first two particular LLDAs (MMmax account and Premium Saving Bonds), we have determined the optimal structure of payment of these financial assets. Then our aim was to determine the optimal design of any LLDA. The analysis presented here suggests that the lottery should be strongly asymmetrical. In order to attract a maximum number of investors, banks should provide financial assets that are very positively skewed. The optimal structure of payments derived in this study are then consistent with Behavioral Portfolio Theory developed by Shefrin and Statman (2000). On the one hand, LLDAs are "riskless": the lottery does not affect the principal but only the interest rate provided by the issuer. Thus, the security desire of investors (driven by fear) is fulfilled. On the
other hand, investors can win a large amount of money. The possibility of winning a large jackpot makes investors (driven this time by potential) dream of and meets their expectations of wealth.

We have also pointed out that the satisfaction of investors seems to increase infinitely when the jackpot tends towards infinity with an infinitesimally chance of winning. An infinite jackpot associated to an extremely small probability should be sufficient to attract a lot of investors. We have suggested that an explanation of the results could lie on the form of the weighting function. In our study, we have used an inverse S-shape weighting function (the one formed by Kahneman and Tversky) because this function provides a good explanation of risk taking behavior. However, the slope of this function tends towards infinity when the probability of winning is extremely small. Thus, concluding such results is not amazing. But is it realistic to imagine a financial asset that provides only an infinite gain with an infinitesimally chance of winning?

According to this results, we plan to extend our study in several ways. First, we have considered in our study a financial asset that provides four payments. This decision was arbitrary. Why not five, six or three payments? A possible extension to our work is to determine the number of prizes which would be optimal for investors.

The problem of the corner solution raises a number of questions for further investigations. First, we will use another weighting function in order to determine the optimal design of LLDA. Our aim is to form a new weighting function whose slope does not tend towards infinity near zero. Then, we plan to realize new experiments on the form of the weighting function and test the effect of jackpot in a context of savings. Actually, most of experiments on the shape of the weighting functions were realized in a context of games. Nevertheless, people’s behavior are not identical in a context of games and in a context of savings. We can imagine that in a context of game earning a fixed 1% interest rate does not represent much for people. Actually, the principal point is to not loose money. But, in a context of savings a fixed 1% interest rate can strengthen the investor’s expectations of security.

References


Appendix

\[ \begin{align*}
\text{Min} & \quad 0.81 y_1 + 0.171(y_1 + z_2) + 0.0171(y_1 + z_3) + 0.0019(y_1 + z_4) \\
\text{subject to} & \\
\pi_1 [-2.25(r - y_1)^a] + \pi_2(y_1 + z_2 - r)^a + \pi_3(y_1 + z_3 - r)^a + \pi_4(y_1 + z_4 - r)^a & \geq 0 \\
\quad r - y_1 & > 0 \\
\quad y_1 + z_2 - r & \geq 0 \\
\quad y_1 + z_3 - r & \geq 0 \\
\quad y_1 + z_4 - r & \geq 0 \\
\quad y_1 & \geq 0, \quad z_2 \geq 0, \quad z_3 \geq 0, \quad z_4 \geq 0
\end{align*} \]

Form the Lagrangian

\[ L(y_1, z_2, z_3, z_4, \lambda_1, \lambda_2, \ldots, \lambda_9) = 0.81 y_1 + 0.171(y_1 + z_2) + 0.0171(y_1 + z_3) + 0.0019(y_1 + z_4) \]

\[ - \lambda_1 [-2.25(r - y_1)^a \pi_1 + \pi_2(y_1 + z_2 - r)^a + \pi_3(y_1 + z_3 - r)^a + \pi_4(y_1 + z_4 - r)^a] \]

\[ + \pi_4(y_1 + z_4 - r)^a] - \lambda_2(r - y_1) - \lambda_3(y_1 + z_2 - r) - \lambda_4(y_1 + z_3 - r) \]

\[ - \lambda_5(y_1 + z_4 - r) - \lambda_6 y_1 - \lambda_7 z_2 - \lambda_8 z_3 - \lambda_9 z_4 \]

Then, there exist multipliers \( \lambda_1^*, \lambda_2^*, \ldots, \lambda_9^* \) such that:

(1) \( \frac{\partial L}{\partial y_1} = 0 \)

(2) \( \frac{\partial L}{\partial z_2} = 0 \)

(3) \( \frac{\partial L}{\partial z_3} = 0 \)

(4) \( \frac{\partial L}{\partial z_4} = 0 \)

(5) \( \lambda_1 [-2.25(r - y_1)^a \pi_1 + \pi_2(y_1 + z_2 - r)^a + \pi_3(y_1 + z_3 - r)^a + \pi_4(y_1 + z_4 - r)^a] = 0 \)

(6) \( \lambda_2(r - y_1) = 0 \)

(7) \( \lambda_3(y_1 + z_2 - r) = 0 \)

(8) \( \lambda_4(y_1 + z_3 - r) = 0 \)

(9) \( \lambda_5(y_1 + z_4 - r) = 0 \)

(10) \( \lambda_6 y_1 = 0 \)

(11) \( \lambda_7 z_2 = 0 \)

(12) \( \lambda_8 z_3 = 0 \)

(13) \( \lambda_9 z_4 = 0 \)

(14) \( \lambda_1, \lambda_2, \ldots, \lambda_9 \geq 0 \)

(15) \( -2.25(r - y_1)^a \pi_1 + \pi_2(y_1 + z_2 - r)^a + \pi_3(y_1 + z_3 - r)^a + \pi_4(y_1 + z_4 - r)^a \geq 0 \)

(16) \( r - y_1 \geq 0 \)

(17) \( y_1 + z_2 - r \geq 0 \)

(18) \( y_1 + z_3 - r \geq 0 \)

(19) \( y_1 + z_4 - r \geq 0 \)

(20) \( y_1 \geq 0 \)

(21) \( z_2 \geq 0 \)

(22) \( z_3 \geq 0 \)

(23) \( z_4 \geq 0 \)
\[
(1) \frac{\partial L}{\partial y_1} = 0.81 + 0.171 + 0.0171 + 0.0019 - \lambda_1[-2.25 \times \alpha (r - y_1)^{\alpha-1} + \lambda_2 - \lambda_3 - \lambda_4 \\
- \lambda_5 - \lambda_6 + \pi_1 + \pi_2 \times \alpha(y_1 + z_2 - r)^{\alpha-1} + \pi_3 \times \alpha(y_1 + z_3 - r)^{\alpha-1} \\
+ \pi_4 \times \alpha(y_1 + z_4 - r)^{\alpha-1}] = 0
\]
\[
(2) \frac{\partial L}{\partial z_2} = 0.171 - \lambda_1[\pi_2 \times \alpha(y_1 + z_2 - r)^{\alpha-1}] - \lambda_3 - \lambda_7 = 0
\]
\[
(3) \frac{\partial L}{\partial z_3} = 0.0171 - \lambda_1[\pi_3 \times \alpha(y_1 + z_3 - r)^{\alpha-1}] - \lambda_4 - \lambda_8 = 0
\]
\[
(4) \frac{\partial L}{\partial z_4} = 0.0019 - \lambda_1[\pi_4 \times \alpha(y_1 + z_4 - r)^{\alpha-1}] - \lambda_5 - \lambda_9 = 0
\]

Equation (1) is not defined for \( r - y_1 = 0, \ y_1 + z_2 - r = 0, \ y_1 + z_3 - r = 0 \) and \( y_1 + z_3 - r = 0 \). That case will be treated separately. For the moment we consider \( r - y_1 > 0, \ y_1 + z_2 - r > 0, \ y_1 + z_3 - r > 0, \ y_1 + z_3 - r > 0 \).

Look at the case \( \lambda_7 > 0 \). Condition 11 implies \( z_2 = 0 \). By condition 17, \( y_1 + 0 - r > 0 \) that involves \( y_1 > r \) - a contradiction to equation (16). Since the assumption \( \lambda_7 > 0 \) leads to a contradiction, we conclude that \( \lambda_7 = 0 \). Similar arguments show that \( \lambda_8 \) and \( \lambda_9 \) are null too.

As we are not interested by the cases \( r - y_1 = 0, \ y_1 + z_2 - r = 0, \ y_1 + z_3 - r = 0, \ y_1 + z_3 - r = 0 \) conditions 6, 7, 8 and 9 lead to \( \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0. \nSo we have : \( \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_7 = \lambda_8 = \lambda_9 = 0 \)

- Look at the case \( \lambda_6 > 0 \).

Then using equation (10), we see that \( y_1 = 0 \).

We can rewrite the system:
\[
(1) 1 - \lambda_6 - \lambda_1[-2.25 \times \alpha \times r^{\alpha-1} \pi_1 + \pi_2 \times \alpha (z_2 - r)^{\alpha-1} + \pi_3 \times \alpha (z_3 - r)^{\alpha-1} + \pi_4 \times \alpha (z_4 - r)^{\alpha-1}] = 0
\]
\[
(2) 0.171 - \lambda_1[\pi_2 \times \alpha (z_2 - r)^{\alpha-1}] = 0
\]
\[
(3) 0.0171 - \lambda_1[\pi_3 \times \alpha (z_3 - r)^{\alpha-1}] = 0
\]
\[
(4) 0.0019 - \lambda_1[\pi_4 \times \alpha (z_4 - r)^{\alpha-1}] = 0
\]
\[
(5) \lambda_1[-2.25 \times r^{\alpha} \times \pi_1 + \pi_2 (z_2 - r)^{\alpha} + \pi_3 (z_3 - r)^{\alpha} + \pi_4 (z_4 - r)^{\alpha}] = 0
\]
(6) \( \lambda_2 = 0 \)  
(7) \( \lambda_3 = 0 \)  
(8) \( \lambda_4 = 0 \)  
(9) \( \lambda_5 = 0 \)  
(10) \( \lambda_6 y_1 = 0 \)  
(11) \( \lambda_7 = 0 \)  
(12) \( \lambda_8 = 0 \)  
(13) \( \lambda_9 = 0 \)  
(14) \( \lambda_1, \lambda_6 \geq 0 \)  
(15) \[-2.25 \times r^\alpha \times \pi_1 + \pi_2(z_2 - r)^\alpha + \pi_3(z_3 - r)^\alpha + \pi_4(z_4 - r)^\alpha \geq 0 \]  
(16) \( r - y_1 > 0 \)  
(17) \( z_2 - r > 0 \)  
(18) \( z_3 - r > 0 \)  
(19) \( z_4 - r > 0 \)  
(20) \( y_1 = 0 \)  
(21) \( z_2 > 0 \)  
(22) \( z_3 > 0 \)  
(23) \( z_4 > 0 \)  

We know that \( \pi_2, \pi_3, \pi_4 > 0 \). We also know that \( \alpha > 0 \) and that \( z_2 > r, z_3 > r \) and \( z_4 > r \). So by conditions 2,3 and 4 \( \lambda_1 > 0 \). Then condition 5 implies that:

\[-2.25 \times r^\alpha \times \pi_1 + \pi_2(z_2 - r)^\alpha + \pi_3(z_3 - r)^\alpha + \pi_4(z_4 - r)^\alpha = 0.\]

It follows now that:

\[(a) \frac{1 - \lambda_6}{\lambda_1} = 2.25 \times \alpha \times r^{\alpha - 1} \pi_1 + \pi_2 \times \alpha (z_2 - r)^{\alpha - 1} + \pi_3 \times \alpha (z_3 - r)^{\alpha - 1}
+ \pi_4 \times \alpha (z_4 - r)^{\alpha - 1} \]

\[(b) -2.25 \times r^\alpha \times \pi_1 + \pi_2(z_2 - r)^\alpha + \pi_3(z_3 - r)^\alpha + \pi_4(z_4 - r)^\alpha = 0 \]

\[(c) \pi_2 \times \alpha (z_2 - r)^{\alpha - 1} = \frac{0.171}{\lambda_1} \]

\[(d) \pi_3 \times \alpha (z_3 - r)^{\alpha - 1} = \frac{0.0171}{\lambda_1} \]

\[(e) \pi_4 \times \alpha (z_4 - r)^{\alpha - 1} = \frac{0.0019}{\lambda_1} \]

Plugging equations (c), (d), (e) into equation (a) we obtain:

\[2.25 \times \alpha \times r^{\alpha - 1} \pi_1 + \frac{0.171}{\lambda_1} + \frac{0.0171}{\lambda_1} + \frac{0.0019}{\lambda_1} = \frac{1 - \lambda_6}{\lambda_1} \]

\[\iff \lambda_1 = 0.81 - A \times \lambda_1 \]

with \( A = 2.25 \times \alpha \times r^{\alpha - 1} \pi_1 \)

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Plugging equations (c), (d), (e) into equation (b) we obtain:

\[2.25 \times r^\alpha \times \pi_1 = \pi_2 \left( \frac{0.171}{\alpha \times \lambda_1 \times \pi_2} \right)^{\alpha/(\alpha-1)} + \pi_3 \left( \frac{0.0171}{\alpha \times \lambda_1 \times \pi_3} \right)^{\alpha/(\alpha-1)} + \pi_4 \left( \frac{0.0019}{\alpha \times \lambda_1 \times \pi_4} \right)^{\alpha/(\alpha-1)}\]

so we have: \[\lambda_1 = \left( \frac{2.25 \times r^\alpha \times \pi_1}{G} \right)^{(1-\alpha)/\alpha}\]

with \[G = \pi_2^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.171} \right)^{\alpha/(1-\alpha)} + \pi_3^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.0171} \right)^{\alpha/(1-\alpha)} + \pi_4^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.0019} \right)^{\alpha/(1-\alpha)}\]

Plugging \(\lambda_1\) into equations (c), (d) and (e), we obtain:

\[z_2 = r \left[ 1 + \left( \frac{0.171}{\alpha \times \pi_2} \right)^{\frac{1}{1-\alpha}} \times \left( \frac{2.25 \times \pi_1}{G} \right)^{1/\alpha} \right]\]

\[z_3 = r \left[ 1 + \left( \frac{0.0171}{\alpha \times \pi_3} \right)^{\frac{1}{1-\alpha}} \times \left( \frac{2.25 \times \pi_1}{G} \right)^{1/\alpha} \right]\]

\[z_4 = r \left[ 1 + \left( \frac{0.0019}{\alpha \times \pi_4} \right)^{\frac{1}{1-\alpha}} \times \left( \frac{2.25 \times \pi_1}{G} \right)^{1/\alpha} \right]\]

with \[G = \pi_2^{\frac{1}{1-\alpha}} \times \left( \frac{\alpha}{0.171} \right)^{\frac{\alpha}{1-\alpha}} + \pi_3^{\frac{1}{1-\alpha}} \times \left( \frac{\alpha}{0.0171} \right)^{\frac{\alpha}{1-\alpha}} + \pi_4^{\frac{1}{1-\alpha}} \times \left( \frac{\alpha}{0.0019} \right)^{\frac{\alpha}{1-\alpha}}\]

- Look at the case \(\lambda_6 = 0\).

The system is:

1. \[1 = \lambda_1 \left[ -2.25 \times \alpha \times \left( r - y_1 \right)^{\alpha - 1} \pi_1 + \pi_2 \times \alpha \left( y_1 + z_2 - r \right)^{\alpha - 1} + \pi_3 \times \alpha \left( y_1 + z_3 - r \right)^{\alpha - 1} + \pi_4 \times \alpha \left( y_1 + z_4 - r \right)^{\alpha - 1} \right]\]

2. \[0.171 - \lambda_1 \left[ \pi_2 \times \alpha \left( z_2 - r \right)^{\alpha - 1} \right] = 0\]

3. \[0.0171 - \lambda_1 \left[ \pi_3 \times \alpha \left( z_3 - r \right)^{\alpha - 1} \right] = 0\]

4. \[0.0019 - \lambda_1 \left[ \pi_4 \times \alpha \left( z_4 - r \right)^{\alpha - 1} \right] = 0\]

5. \[\lambda_1 \left[ -2.25 \times \left( r - y_1 \right)^{\alpha} \times \pi_1 + \pi_2 \left( y_1 + z_2 - r \right)^{\alpha} + \pi_3 \left( y_1 + z_3 - r \right)^{\alpha} + \pi_4 \left( y_1 + z_4 - r \right)^{\alpha} \right] = 0\]

6. \(\lambda_2 = 0\)

7. \(\lambda_3 = 0\)

8. \(\lambda_4 = 0\)

9. \(\lambda_5 = 0\)
(10) \( \lambda_6 = 0 \)  \hspace{1em} (11) \( \lambda_7 = 0 \)  \hspace{1em} (12) \( \lambda_8 = 0 \)  \hspace{1em} (13) \( \lambda_9 = 0 \)  \hspace{1em} (14) \( \lambda_1 \geq 0 \)

(15) \(-2.25 \times (r - y_1)^{\alpha} \times \pi_1 + \pi_2 (y_1 + z_2 - r)^{\alpha} + \pi_3 (y_1 + z_3 - r)^{\alpha} + \pi_4 (y_1 + z_4 - r)^{\alpha} \geq 0\)

(16) \( r - y_1 > 0 \)  \hspace{1em} (17) \( y_1 + z_2 - r > 0 \)

(18) \( y_1 + z_3 - r > 0 \)  \hspace{1em} (19) \( y_1 + z_4 - r > 0 \)

(20) \( y_1 \geq 0 \)  \hspace{1em} (21) \( z_2 \geq 0 \)  \hspace{1em} (22) \( z_3 \geq 0 \)  \hspace{1em} (23) \( z_4 \geq 0 \)

We know that \( \pi_2, \pi_3, \pi_4 > 0 \). We also know that \( \alpha > 0 \) and that \( y_1 + z_2 > r, y_1 + z_3 > r \) and \( y_1 + z_4 > r \). So by conditions 2,3 and 4 \( \lambda_1 > 0 \). Then condition 5 implies that:

\[-2.25 \times (r - y_1)^{\alpha} \times \pi_1 + \pi_2 (y_1 + z_2 - r)^{\alpha} + \pi_3 (y_1 + z_3 - r)^{\alpha} + \pi_4 (y_1 + z_4 - r)^{\alpha} = 0\]

It follows now that:

(a) \( \frac{1}{\lambda_1} = 2.25 \times \alpha \times (r - y_1)^{\alpha - 1} \times \pi_1 + \pi_2 \times \alpha (y_1 + z_2 - r)^{\alpha - 1} + \pi_3 \times \alpha (y_1 + z_3 - r)^{\alpha - 1} + \pi_4 \times \alpha (y_1 + z_4 - r)^{\alpha - 1} \)

(b) \(-2.25 (r - y_1)^{\alpha} \times \pi_1 + \pi_2 (y_1 + z_2 - r)^{\alpha} + \pi_3 (y_1 + z_3 - r)^{\alpha} + \pi_4 (y_1 + z_4 - r)^{\alpha} = 0\)

(c) \( \pi_2 \times \alpha (y_1 + z_2 - r)^{\alpha - 1} = \frac{0.171}{\lambda_1} \)

(d) \( \pi_3 \times \alpha (y_1 + z_3 - r)^{\alpha - 1} = \frac{0.0171}{\lambda_1} \)

(e) \( \pi_4 \times \alpha (y_1 + z_4 - r)^{\alpha - 1} = \frac{0.0019}{\lambda_1} \)

Plugging equations (c), (d), (e) into equation (a) we obtain:

\[2.25 \times \alpha \times (r - y_1)^{\alpha - 1} \times \pi_1 + \frac{0.171}{\lambda_1} + \frac{0.0171}{\lambda_1} + \frac{0.0019}{\lambda_1} = \frac{1}{\lambda_1}\]

\( \iff (r - y_1)^{\alpha - 1} = \left( \frac{0.81}{\lambda_1 \times \pi_1 \times \alpha \times 2.25} \right)^{1/(\alpha - 1)} \) (f)
Plugging equations (c), (d), (e) and (f) into equation (b) we obtain:

\[ 2.25 \times \pi_1 \times \left( \frac{0.81}{\lambda_1 \times \pi_1 \times \alpha 	imes 2.25} \right)^{\alpha/(\alpha-1)} = \pi_2 \times \left( \frac{0.171}{\alpha \times \lambda_1 \times \pi_2} \right)^{\alpha/(\alpha-1)} + \pi_3 \times \left( \frac{0.0171}{\alpha \times \lambda_1 \times \pi_3} \right)^{\alpha/(\alpha-1)} + \pi_4 \times \left( \frac{0.0019}{\alpha \times \lambda_1 \times \pi_4} \right)^{\alpha/(\alpha-1)} \]

\[ 0 = \lambda_1^{\alpha/(1-\alpha)} \times \left[ -2.25 \times \pi_1^{1/(1-\alpha)} \times \left( \frac{2.25 \times \alpha}{0.81} \right)^{\alpha/(\alpha-1)} + \pi_2^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.171} \right)^{\alpha/(\alpha-1)} + \pi_3^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.0171} \right)^{\alpha/(\alpha-1)} \right] \times \left( \frac{\alpha}{0.0171} \right)^{\alpha/(\alpha-1)} + \pi_4^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.0019} \right)^{\alpha/(\alpha-1)} \]

So \( \lambda_1^{\alpha/(1-\alpha)} \times H = 0 \)

where \( H = -2.25 \times \pi_1^{1/(1-\alpha)} \times \left( \frac{2.25 \times \alpha}{0.81} \right)^{\alpha/(\alpha-1)} + \pi_2^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.171} \right)^{\alpha/(\alpha-1)} + \pi_3^{1/(1-\alpha)} \times \left( \frac{\alpha}{0.0171} \right)^{\alpha/(\alpha-1)} \times \left( \frac{\alpha}{0.0019} \right)^{\alpha/(\alpha-1)} \]

In our study \( \pi_1 = 0.6781, \pi_2 = 0.1755, \pi_3 = 0.0577, \pi_4 = 0.0211, \alpha = 0.88 \).

So \( H \sim 384200 \neq 0 \).

As \( \lambda_1^{\alpha/(1-\alpha)} \times H = 0 \), \( \lambda_1 \) should be equal to zero. Or we previously found that \( \lambda_1 > 0 \).

Since the assumption that \( \lambda_6 = 0 \) leads to a contradiction, we conclude that \( \lambda_6 > 0 \).

The solution of the constrained minimization problem is:

\[
\begin{align*}
\mathbf{y}_1 &= 0 \\
\mathbf{z}_2 &= r[1 + \left( \frac{0.171}{\alpha \times \pi_2} \right)^{1/(1-\alpha)} \times \left( \frac{2.25 \times \pi_1}{\alpha} \right)^{1/(1-\alpha)}] \\
\mathbf{z}_3 &= r[1 + \left( \frac{0.0171}{\alpha \times \pi_3} \right)^{1/(1-\alpha)} \times \left( \frac{2.25 \times \pi_1}{\alpha} \right)^{1/(1-\alpha)}] \\
\mathbf{z}_4 &= r[1 + \left( \frac{0.0019}{\alpha \times \pi_4} \right)^{1/(1-\alpha)} \times \left( \frac{2.25 \times \pi_1}{\alpha} \right)^{1/(1-\alpha)}] \\
\text{with } G &= \pi_2^{\frac{1}{\alpha}} \times \left( \frac{\alpha}{0.171} \right)^{\frac{1}{\alpha}} + \pi_3^{\frac{1}{\alpha}} \times \left( \frac{\alpha}{0.0171} \right)^{\frac{1}{\alpha}} + \pi_4^{\frac{1}{\alpha}} \times \left( \frac{\alpha}{0.0019} \right)^{\frac{1}{\alpha}} \]
\end{align*}
\]

For the maximization problem resolution details are available from the author.