Pricing and Hedging Asian Basket Spread Options

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Abstract

In this paper we consider the problem of pricing a general Asian basket spread option. We develop approximations formulae based on comonotonicity theory and moment matching methods. We compare their relative performances and explain how to choose the best approximation technique as a function of the Asian basket spread characteristics. We also give the Greeks for our proposed methods. In the last section we extend our results to options denominated in foreign currency.

1 Introduction

We consider a security market consisting of \( m \) risky assets and a risk-less asset with rate of return \( r \). We assume that under the risk-neutral measure the price process dynamics are given by

\[
dS_{jt} = rS_{jt}dt + \sigma_j S_{jt}dB_{jt},
\]

where \( \{B_{jt} : t \geq 0\} \) is a standard Brownian motion associated with asset \( j \). Further we assume that the asset prices are correlated according to

\[
\text{corr}(B_{jt_v},B_{st_s}) = \rho_{ji}\min(t_v,t_s).
\]

Given the above dynamics, the price of the \( j^{th} \) asset at time \( t_i \) equals

\[
S_{jt_i} = S_{j}(0)e^{(r - \frac{\sigma_j^2}{2})t_i + \sigma_j B_{jt_i}},
\]

With this in hand we can define an Asian basket spread as

\[
S = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S_{jt_i},
\]

where \( a_j \) is the weight given to asset \( j \) and \( \varepsilon_j \) its sign in the spread. We assume that \( \varepsilon_j = 1 \) for \( j = 1, \ldots, p \), \( \varepsilon_j = -1 \) for \( j = p + 1, \ldots, m \), where \( p \) is an integer such that \( 1 \leq p \leq m - 1 \) and \( t_0 < t_1 < t_2 < \cdots < t_n = T \). The price of an Asian basket spread with exercise price \( K \) at \( t_0 = 0 \) can be defined as

\[
e^{-rT}E_Q(S - K)_+,
\]

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where $E_Q$ represents the expectation taken with respect to the risk-neutral measure $Q$. In what follows we will simply write $E$ for the expectation under the risk-neutral measure.

Examples of such contracts can be found in the energy markets. The basket spread part may for example be used to cover refinement margin (crack spread) or the cost of converting fuel into energy (spark spread). While the Asian part (the temporal average) avoids the problem common to the European options, namely that speculators can increase gain from the option by manipulating the price of the assets near maturity.

Since the density function of a sum of non-independent log-normal random variables has no closed-form representation, there is no closed-form solution for the price of a security whenever $m > 1$ or $n > 1$ within the Black and Scholes framework. Therefore one has to use an approximation method when valuating such a security. It is always possible to use Monte Carlo techniques to get an approximation of the price. However such techniques are rather time-consuming. Furthermore financial institutions also need approximations of the hedge parameters in order to control the risk, which further increases the computation time. This explains why the research for a closed-form approximation has become an active area.

Some special cases of the above formula have been extensively studied. For example if we set $m = 1$ and $n > 1$ we have an Asian option. Approximation formulae for this kind of derivatives can be found in Kaas et al. (2000), Thompson (2002), Nielsen and Sandmann (2003), Vanmaele et al. (2006), Vyncke et al. (2004), Lord (2006) and Zhou and Wang (2007). If $m > 1$ and $n = 1$ we have a basket option. See Deelstra et al. (2004), Vanmaele et al. (2004) and Zhou and Wang (2007) for basket options where all the assets have a positive weight. And Borovkova et al. (2007), Castellacci and Siclari (2003) for the case of basket spread options. Finally setting $m = 2$, $n = 1$ and $p = 1$ we end up with a spread option. Pretty accurate approximation formulae for spread options can be found in the paper of Alexander and Scourse (2004), Bjerksund and Stensland (2006), Carmona and Durrleman (2003a, 2003b), and Li et al. (2006). However few papers develop methods that can be used in the case of an Asian basket spread. Castellacci and Siclari (2003), Borovkova et al. (2007) and Carmona and Durrleman (2005) are the only we are aware of.

In this paper we start by deriving approximation formulae for expression (5) using comonotonic bounds. We derive four different approximations: the upper, the improved upper, the lower and the intermediary bound. We also try to approximate the security price with the help of moment matching techniques. We improve the hybrid moment matching method of Castellacci and Siclari (2003) and propose an extension of the method developed by Borovkova et al. (2007). We explain which method should be used depending on the basket characteristics. We also provide closed-form formulae for the Greeks of our selected approximation techniques. We explain how our results can be adapted in order to deal with options written in foreign currency (compo and quanto options).

The paper is composed as follows. In section 2, we construct a price approximation using comonotonic sums. In section 3, we develop some moment matching methods. Section 4 studies the relative performance of the methods we developed. In section 5 we derive the Greeks for our best performing approximation. Section 6 deals with options in foreign currency. Finally section 7 concludes.

2 Comonotonic Approximations

We start this section by recalling some results on comonotonicity from Dhaene et al. (2002a, 2002b). The reader is referred to those papers for a full treatment of this theory. As in Dhaene et al. (2002a) we will start by defining comonotonicity on a set of $\mathbb{R}^n$.

**Definition 1.** $A \subseteq \mathbb{R}^n$ is said to be a comonotonic set if for any $x$ and $y$ in $A$, either $x \leq y$ or $y \leq x$ holds.$^1$

$^1$The notation $\leq$ is used for componentwise order, i.e. $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$. 


With this in hand we can define a comonotonic random vector as follows:

**Definition 2.** A random vector \( X = (X_1, \ldots, X_n) \) is said to be comonotonic if it has a comonotonic support.

In actuarial science it is common to encounter sums of the form \( S = \sum_{i=1}^{n} X_i \) where the marginal distribution of each \( X_i \) is known but the dependency structure between the \( X_i \)'s is unknown or too difficult to work with. In such a case comonotonicity theory allows us to find the joint distribution of the \( X_i \)'s, given their marginal one, with the smaller (larger) sum in the convex order sense (we note this order as \( \leq_{cx} \)). Put it differently, we could replace the original random vector by its comonotonic counterparts \( S^\ell \) and \( S^c \) which are such that

\[
E[g(S^\ell)] \leq E[g(S)] \leq E[g(S^c)],
\]

for any convex function \( g(\cdot) \). From this it follows that

\[
E(S^\ell - K)_+ \leq E(S - K)_+ \leq E(S^c - K)_+,
\]

for all \( K \in \mathbb{R} \). Thus we see that comonotonicity allows us to find bounds for expressions like (5). Below we will see that comonotonic sums give us closed-form formulae for such approximations. Traditionally comonotonicity is used in the pricing of Asian or Asian basket options (see Vanmaele et al. (2006) and Deelstra et al. (2004)). To our knowledge, this is the first time that this approach has been used in order to approximate basket spread or Asian basket spread options.

### 2.1 Comonotonic Upper Bound

It can be shown that the convex largest sum of the components of a random vector \( X \) is given by the following comonotonic sum (see Dhaene et al. 2002b):

\[
S^c = \sum_{i=1}^{n} F^{-1}_{X_i}(U),
\]

where the distribution function of each \( X_i \) is non-decreasing and left-continuous, and \( F^{-1}_{X_i}(p) \) is defined as

\[
F^{-1}_{X_i}(p) = \inf \{ x \in \mathbb{R} \mid F_{X_i}(x) \geq p \}, \quad p \in (0,1).
\]

Kaas et al. (2000) showed that the inverse distribution function of a sum of comonotonic random variables is equal to the sum of the marginal inverse distribution functions. Assuming that the marginal distributions are strictly increasing we can recover the cumulative distribution function (cdf) of the comonotonic sum using:

\[
x = F^{-1}_{S^c}(F_{S^c}(x)) = \sum_{i=1}^{n} F^{-1}_{X_i}(F_{S^c}(x)), \quad F^{-1}_{S^c}(0) < x < F^{-1}_{S^c}(1).
\]

The next theorem, of which the proof can be found in Dhaene et al. (2002b) and Kaas et al. (2000), will be useful in what follows.

**Theorem 1.** The stop-loss premiums of the comonotonic sum \( S^c \) of the random vector \( X \) are given by

\[
E[(S^c - K)_+] = \sum_{i=1}^{n} E \left[ \left( X_i - F^{-1}_{X_i}(F_{S^c}(K)) \right)_+ \right],
\]

for \( F^{-1}_{S^c}(0) < K < F^{-1}_{S^c}(1) \).

Below we will also use the following proposition
**Proposition 1.** If $Y$ is log-normal $(\mu, \sigma^2)$, then for any $K > 0$ we have,

$$E(Y - K)_+ = e^{\mu + \frac{\sigma^2}{2}} \Phi(K_1) - K \Phi(K_2),$$
$$E(Y - K)_- = e^{\mu + \frac{\sigma^2}{2}} \Phi(-K_1) - K \Phi(-K_2),$$

where $K_1$ and $K_2$ are given by

$$K_1 = \frac{\mu + \sigma^2 - \ln(K)}{\sigma}, \quad K_2 = K_1 - \sigma,$$

where $\Phi$ is the cdf of the standard normal distribution.

In what follows we will denote the inverse function of a standard normal cdf by $\Phi^{-1}$. With Theorem 1 and Proposition 1 at hand, it becomes easy to derive an upper bound to expression (5).

**Proposition 2.** A comonotonic upper bound to the price of a derivative of the type (5) when the underlying dynamics are described by (3) is given by

$$e^{-rt} E(S^n - K)_+ = e^{-rt} \left[ \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \varepsilon_j a_j S_j(0) e^{r \tau_i} \Phi(Y_{ji} - \Phi^{-1}(F_{S^n}(K))) - K(1 - F_{S^n}(K)) \right],$$

where $F_{S^n}(K)$ can be found by solving the following equation:

$$K = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \varepsilon_j a_j S_j(0) e^{(r - \frac{\sigma^2}{2}) \tau_i + Y_{ji} \Phi^{-1}(F_{S^n}(K))},$$

and where $Y_{ji} = \varepsilon_j \sigma_j \sqrt{\tau_i}$.

**Proof**

The underlying is of the type

$$S = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \varepsilon_j a_j S_j(0) e^{(r - \frac{\sigma^2}{2}) \tau_i + \sigma_j B_{ji}}.$$

Thus using Proposition 1 from Kaas et al. (2000), its comonotonic counterpart is given by

$$F_{S^n}^{-1}(U) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \varepsilon_j a_j S_j(0) e^{(r - \frac{\sigma^2}{2}) \tau_i + \varepsilon_j \sigma_j \sqrt{\tau_i} \Phi^{-1}(U)},$$

where $U$ is uniformly distributed over $(0,1)$. Then by means of Theorem 1 we can rewrite the upper bound as

$$e^{-rt} \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \varepsilon_j E(a_j S_j(0) e^{(r - \frac{\sigma^2}{2}) \tau_i + \varepsilon_j \sigma_j \sqrt{\tau_i} \Phi^{-1}(U)} - K_{ij} \varepsilon_j),$$

where

$$K_{ij} = a_j S_j(0) e^{(r - \frac{\sigma^2}{2}) \tau_i + \varepsilon_j \sigma_j \sqrt{\tau_i} \Phi^{-1}(F_{S^n}(K))},$$

(6)

where $(X)_{\varepsilon_j}$ is equal to the function $\max(X,0)$ if $\varepsilon_j = 1$ and $\min(X,0)$ if $\varepsilon_j = -1$. The result then follows from Proposition 1.

**Remark** Formulae (6)-(7) provide a natural interpretation to the comonotonic upper bound. Indeed these formulae show that we could write the upper bound as a linear combination of call and put options on the initial underlying with different maturities and strike prices given by $K_{ij}$. This result can be linked to the literature on static hedging, see for example Simon et al. (2000), Dhaene et al. (2002), Hobson et al. (2005) and Chen et al. (2007).
2.2 Improved Comonotonic Upper Bound

It is possible to sharpen the above upper bound by conditioning the distribution of the vector $X$ on some random variable $\Lambda$. Assume that $\Lambda$ is a random variable whose distribution is known, and such that the distribution of the $X_i$ conditionally on $\Lambda$ is known. If we further assume that the cumulated density function is continuous and strictly increasing, then we have the following theorem from Dhaene et al. (2002):

**Theorem 2.** Let $U$ be uniform $(0,1)$ distributed random variable independent of $\Lambda$. Then we have

$$S = \sum_{i=1}^{n} X_i \leq cx \quad S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U) \leq cx \quad S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U).$$

In what follows we will consider the following conditioning variable:

$$\Lambda = \sum_{j=1}^{m} \sigma_j a_j S_j(0) B_j T. \tag{8}$$

Numerical simulations showed that in the case of positively correlated assets this conditioning variable produces the sharpest bounds. Using all this, we can derive the following proposition:

**Proposition 3.** The improved comonotonic upper bound (ICUB) of the price of a derivative of the type (5) when the underlying dynamics are given by (3) is

$$e^{-rT} E[(\mathbb{S}^c - K)_{+} | \Lambda]$$

$$= e^{-rT} \int_{0}^{1} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j \sigma_j S_j(0) \epsilon e^{(r-\sigma^2/2)t_i + A_{ji}(u) + Y_{ji} \Phi^{-1}(F_{\mathbb{S}^c|U=u}(K))} du - K (1 - F_{\mathbb{S}^c}(K)),$$

where we can recover $F_{\mathbb{S}^c|U=u}(K)$ by solving

$$K = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j \sigma_j \sqrt{T_i} \Phi^{-1}(u),$$

with

$$F_{\mathbb{S}^c}(K) = \int_{0}^{1} F_{\mathbb{S}^c|U=u}(K) du,$$

and where for $\gamma_{ji}$ being the correlation between $B_{ji}$ and the conditioning variable $\Lambda$,

$$A_{ji}(u) = \gamma_{ji} \sigma_j \sqrt{T_i} \Phi^{-1}(u), \quad Y_{ji} = \varepsilon_j \sigma_j \sqrt{1 - \gamma_{ji}^2} \sqrt{T_i}.$$

**Proof**

First notice that the conditional distribution of the underlying is

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j \sigma_j S_j(0) \epsilon e^{(r-\sigma^2/2)t_i + \gamma_{ji} \sigma_j \sqrt{T_i} \Phi^{-1}(u) + \epsilon_j \sigma_j \sqrt{1 - \gamma_{ji}^2} \sqrt{T_i} \Phi^{-1}(V)},$$

where $U = (\Lambda - E(\Lambda))/\sigma_\Lambda$ and $V$ are independent and are uniformly distributed on $(0,1)$. Then, using the tower property:

$$E[(\mathbb{S}^c - K)_{+}] = E[E[(\mathbb{S}^c - K)_{+} | \Lambda]], \tag{9}$$

the rest follows by applying Theorem 1 and Proposition 1. \qed
2.3 Comonotonic Lower Bound

Assume that there exists a conditioning variable $\Lambda$ such that the distribution of $X_i$ conditionally on $\Lambda$ is known for each $i$. Then from Kaas et al. (2000) we know that the following random variable provides a lower bound in the convex order sense:

$$S^\ell = E[S|\Lambda].$$

Furthermore assume the conditioning variable $\Lambda$ is such that for all $i$ $E[X_i|\Lambda]$ is a non-decreasing (or non-increasing, which can be dealt with in a similar way) and a continuous function of $\Lambda$ for each $i$. And if we further assume that the cdf of $E[X_i|\Lambda]$ is continuous and strictly increasing, then we can recover the distribution function of $S^\ell$ from:

$$\sum_{i=1}^n E[X_i|\Lambda = F^{-1}_\Lambda(x)] = x, \quad x \in (F^{-1}_{S^\ell}(0), F^{-1}_{S^\ell}(1)). \quad (10)$$

In such a case we can also use Theorem 1 and write the stop-loss premium as

$$E(S^\ell - K)_+ = \sum_{i=1}^n E[(E[X_i|\Lambda] - E[X_i|\Lambda = F^{-1}_\Lambda(K))]_+]$$

for all $K \in (F^{-1}_{S^\ell}(0), F^{-1}_{S^\ell}(1))$.

The most difficult part consists in finding a conditioning variable $\Lambda$ such that all the conditional expectations are non-decreasing (or non-increasing). In the case of a general Asian basket spread with positively correlated Brownian motion there is no obvious choice. Indeed finding such a conditioning variable would require a numerical optimization procedure. We would need to find the conditioning variable $\Lambda$ which is a linear combination of the Brownian motion such that it maximizes the lower bound under a comonotonicity constraint. Considering the high dimensionality of the problem this would quickly become impossible. This is why we choose another approach. Instead of taking the correlation between the Brownian motion as given and start looking for a conditioning variable such that $S^\ell$ is comonotone, we take a specific conditioning variable and then determine which set of correlation coefficients satisfy in order to have comonotonicity.

In what follows we will take the following conditioning variable

$$\Lambda = \sum_{j=1}^m \varepsilon_j B_j T, \quad (11)$$

which produces comonotonic conditional expectation vectors as long as the correlation coefficients (2) satisfy

$$\text{sign}(\sum_{j=1}^m \varepsilon_j \rho_{ji}) = \varepsilon_i. \quad (12)$$

One should note that such a condition always hold for simple spreads. The above discussion leads to the following proposition:

**Proposition 4.** Under the assumption that the correlation coefficients satisfy (12), a comonotonic lower bound to the price of a derivative of type (5) when the underlying dynamics are described by (3) is given by:

$$e^{-r T} E(S^\ell - K)_+ = e^{-r T} \left[ \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \varepsilon_j a_j S_0 e^{r T_i} \Phi(\sigma_j \sqrt{T_i} \gamma_{ji} - \Phi^{-1}(F_{S^\ell}(K))) - K(1 - F_{S^\ell}(K)) \right]$$

$^2$Note that a non-decreasing conditional expectation vector is equivalent to the requirement $\varepsilon_j = \text{sign}(\gamma_{ji})$, where $\gamma_{ji}$ is the correlation between the $j^{th}$ Brownian motions at time $i$ and the conditioning variable.
where $\gamma_{ji}$ is the correlation between $B_{jt_i}$ and the conditioning variable given in (11), and $F_S^\ell(K)$ can be found by solving (10).

**Proof**

Simply start by noting that

\[
S^t = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_j a_j S_0 e^{\left(r - \frac{(\gamma_{ji} \sigma_j)^2}{2}\right) t_i + \gamma_{ji} \sigma_j \sqrt{t_i} \Phi^{-1}(U)}.
\]

The proof then follows from the above discussion and an application of Proposition 1.

For the choice in (11) we arbitrary chose to use $B_{jT}$. But one could also choose any other time $t_k$ in $[0, T]$ and then maximize the lower bound over all those $t_k$.

2.4 Comonotonic Intermediary Bound

As explained above it is indeed fairly difficult to find a conditioning variable that produces a comonotone conditional expectation vector. In such a case one can always build an approximation using the following procedure. Start by choosing a first conditioning variable, $\Lambda_1$. Then construct the conditional expectation vector $E[X | \Lambda_1]$.

Once this is done, we choose a second conditioning variable $\Lambda_2$ and construct an ICUB of this conditional expectation vector. The advantage of this procedure is that we surely end up with a comonotone vector. The drawback is that we do not know whether our approximation is an upper or a lower bound. In our computations we choose $\Lambda_1$ and $\Lambda_2$ to be

\[
\Lambda_1 = \sum_{j=1}^{m} \sigma_j a_j S_0 B_{jT} \quad \text{and} \quad \Lambda_2 = \sum_{j=1}^{m} B_{jT}.
\]

This choice of $\Lambda_1$ is justified by the fact that this conditioning variable seemed to be a good first order approximation of $\text{Var}(S|\Lambda)$. And this choice of $\Lambda_2$ was yielding one of the best ICUB. Since it can be shown that (see Vanmaele et al. (2004))

\[
S^t \leq_S S^{\text{int}} \leq_S S^{\text{ic}},
\]

where $S^t$ is the (non-comonotonic) lower bound based on conditioning variable $\Lambda_1$ and $S^{\text{ic}}$ is the improved comonotonic upper bound based on $\Lambda_2$, we are reducing the possible range of fluctuation of our approximation.

**Proposition 5.** The intermediary bound to the price of a derivative of the type (5) when the underlying dynamics are described by (3), is given by:

\[
e^{-rT} \left[E(S^{\text{int}} - K)_+\right] = e^{-rT} \int_0^1 \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_j a_j S_j(0) e^{\left(r - \frac{(\gamma_{ji} \sigma_j)^2}{2}\right) t_i + A_{ji}(u) \Phi(Y_{ji} - \Phi^{-1}(F_S^{\text{int}|U=u}(K)))} du \]

\[
- e^{-rT} K (1 - F_S^{\text{int}}(K)),
\]

where $\gamma_{\Lambda_1,\Lambda_2}$ is the correlation between the conditioning variables, and $\gamma_{ji}$ is the correlation between the first conditioning variable $\Lambda_1$ and $B_{jt_i}$. $F_S^{\text{int}|U=u}(K)$ can be recovered by solving

\[
K = \frac{1}{n} \sum_{j=1}^{m} \sum_{i=1}^{n} \epsilon_j a_j S_j(0) e^{\left(r - \frac{(\gamma_{ji} \sigma_j)^2}{2}\right) t_i + A_{ji}(u) + Y_{ji} \Phi^{-1}(F_S^{\text{int}|U=u}(K))},
\]

\footnote{The only restriction set on this variable is that it is normally distributed and that $(\Lambda_1, \Lambda_2)$ is bivariate normally distributed.}
where
\[ F_{\text{diff}}(K) = \int_0^1 F_{\text{diff}|U=u}(K) du, \]
and
\[ A_{ji}(u) = \gamma_1 \Lambda_2 \gamma_{ji} \sigma_j \sqrt{u} \Phi^{-1}(u), \quad Y_{ji} = \varepsilon_j \sqrt{1 - \gamma_1^2 \Lambda_2 |\gamma_{ji}| \sigma_j \sqrt{u}}. \]

Proof
The proof is completely analogous to the one of Theorem 7 in Vanmaele et al. (2004).

\[ \square \]

3 Moment Matching Methods

Another approximation technique is the so-called moment matching method. The idea is to replace the original distribution of the underlying by a law with the same first moments as the original law and whose stop-loss premium has a closed-form expression. In this section we describe three different moment matching techniques that can be used to price Asian basket spread options.

3.1 Hybrid Moment Matching Method

An obvious way of attacking the problem is to use the hybrid moment matching method, see for example Castellacci and Siclari (2003). The idea is to reduce the Asian basket spread option pricing problem to a spread option pricing problem. To do so, we start by splitting the underlying \( S \) in two parts (one containing all the assets with a positive sign, denoted as \( S_1 \), another containing those with a negative sign, denoted as \( S_2 \)) and moment match each term, separately, with a log-normal random variable. Once this is done we are left with the problem of approximating a spread option. This is a well studied problem for which pretty accurate approximations are available. In this section we improve the classical hybrid moment matching method by using new spread approximation techniques. We will see later that one of these approximations turns out to be extremely useful when we need to recover the Greeks of such an Asian basket option.

More formally, hybrid moment matching allows us to rewrite (5) as:
\[ E(\tilde{S}_1 - \tilde{S}_2 - K)_+, \]
where \( \tilde{S}_j \) is a log-normal random variable with mean \( \mu_j \) and variance \( \sigma_j^2 \) given by:
\[ \mu_j = 2 \ln(m_{1j}) - \frac{1}{2} \ln(m_{2j}), \quad \sigma_j^2 = \ln(m_{2j}) - 2 \ln(m_{1j}), \quad j = 1, 2. \]

Here \( m_{1j} \) and \( m_{2j} \) are the first and second moments of the sum \( S_j \) described above. We will also need the correlation coefficient \( \rho \) between \( \ln(\tilde{S}_1) \) and \( \ln(\tilde{S}_2) \) to compute the spread approximation. We use the following equation to recover \( \rho \) from equality of the crossmoments (crm):
\[ E[S_1 S_2] = E[\tilde{S}_1 \tilde{S}_2], \]
namely:
\[ \text{crm} := \sum_{j=p+1}^m \sum_{l=1}^p \sum_{i,s=1}^n \frac{1}{n^2} a_j a_l E[S_{lt} S_{ls}] = e^{\mu_1 + \mu_2 + \frac{1}{2} \sigma_1^2 + \frac{1}{2} \sigma_2^2 + \sigma_1 \sigma_2 \rho}. \]

Finally we have to approximate the resulting spread. We will use two different approximations, first we use the method proposed by Li et al. (2006) whose formula is given in the next proposition. We choose this method since it does not require any optimization and performs remarkably well compared to other techniques like the ones of Bjerksund and Stensland (2006) or Carmona and Durrleman (2003a).
Proposition 6. Let $K \geq 0$ and $|\rho| < 1$. Let $y_0$ be any real close to 0. The spread options price $\Pi$ under the general jointly-normal returns setup is given by

$$\Pi = e^{\frac{1}{2} \sigma_1^2 + \mu_1 - rT} I_1 - e^{\frac{1}{2} \sigma_2^2 + \mu_2 - rT} I_2 - Ke^{-rT} I_3.$$ 

The integrals $I_i$’s are approximated to second order in $\varepsilon$ as

$$I_i \approx J_0(C^i, D^i) + J_1(C^i, D^i) \varepsilon + \frac{J_2(C^i, D^i)}{2} \varepsilon^2,$$

where the function $J_i$’s are defined as

$$J_0(u, v) = \Phi \left( \frac{u}{\sqrt{1 + v^2}} \right),$$

$$J_1(u, v) = 1 + \frac{u^2 + 2v^2 - 1}{(1 + v^2)^{5/2}} \phi \left( \frac{u}{\sqrt{1 + v^2}} \right),$$

$$J_2(u, v) = \frac{(6 - 6u^2)v^2 + (21 - 2u^2 - u^4)v^4 + 4(3 + u^2)v^6 - 3}{(1 + v^2)^{11/2}} u \phi \left( \frac{u}{\sqrt{1 + v^2}} \right),$$

where $\phi$ is the probability density function of a standard normal and the arguments $C^i$, $D^i$, and $\varepsilon$ are given by

$$C^1 = C^3 + D^3 \rho \sigma_1 + 2\sigma_1^2 \varepsilon^2 + \sqrt{1 - \rho^2} \sigma_1,$$

$$D^1 = D^3 + 2\sigma_1 \rho \varepsilon,$$

$$C^2 = C^3 + D^3 \sigma_2 + 2\sigma_2 \varepsilon^2,$$

$$D^2 = D^3 + 2\sigma_2 \varepsilon,$$

$$C^3 = \frac{1}{\sigma_1 \sqrt{1 - \rho^2}} \left( \mu_1 - \ln(R + K) + \frac{\sigma_2 R}{R + K} y_0 - \frac{\sigma_2^2 RK}{2(R + K)^2} y_0 \right),$$

$$D^3 = \frac{1}{\sigma_1 \sqrt{1 - \rho^2}} \left( \rho \sigma_1 - \frac{\sigma_2 R}{R + K} + \frac{\sigma_2^2 RK}{(R + K)^2} y_0 \right),$$

$$\varepsilon = -\frac{1}{2\sigma_1 \sqrt{1 - \rho^2}} \frac{\sigma_2^2 RK}{(R + K)^2}.$$

with $R = e^{\sigma_2 y_0 + \mu_2}$.

As a second approximation of the spread we choose the ICUB of expression (13). The reason for this choice will become clear in the next section where we will see that the improved comonotonic upper bound outperforms the approximation of Li et al. (2006). We derive the ICUB in the next proposition.

Proposition 7. Consider the following stop-loss premium

$$E(e^{\mu_1 + \sigma_1 X_1} - e^{\mu_2 + \sigma_2 X_2} - K)_+,$$

with $X_1$, $X_2$ two correlated standard normal random variables. The comonotonic improved upper bound of this spread is given by

$$\sum_{i=1}^2 \varepsilon_i \int_0^1 e^{\mu_i + A_i(u) + \frac{1}{2} Y_i^2} \Phi(Y_i - \Phi^{-1}(F_{G_i|U=U}(K))) du - K(1 - F_{G_i}(K)),$$

(16)

\[^{4}\text{When computing the approximation we followed Li et al. (2006) and set it equal to zero.}\]
where \( \varepsilon_1 = 1 \) and \( \varepsilon_2 = -1 \). And where \( F_{S_i|U=u} \) can be found by solving

\[
\sum_{i=1}^{2} \varepsilon_i e^{\mu_i + A_i(u) + Y_i \Phi^{-1}(F_{S_i|U=u}(K))} = K,
\]

with

\[
A_i(u) = \gamma_i \sigma_i \Phi^{-1}(u), \quad Y_i = \varepsilon_i \sqrt{1 - \gamma_i^2}, \quad F_{S_i}(K) = \int_0^1 F_{S_i|U=u}(K) \, du,
\]

where \( \gamma_i \) is the correlation between \( X_i \) and the conditioning variable.

**Proof**

See Proposition 3. \( \square \)

When performing the simulations we used the following conditioning variable

\[
\Lambda = e^{\mu_1 \sigma_1 X_1 + e^{\mu_2 \sigma_2 X_2}}, \quad (17)
\]

where \( X_1, X_2 \) are \( N(0,1) \) with correlation coefficient \( \rho \) determined through (15) and \( \mu_i \) and \( \sigma_i \) are given by (14).

We introduce the following decomposition in order to interpret the approximation error:

\[
E(S - K)_+ = \tilde{\Pi} + \Delta_1 + \Delta_2
\]

where \( \tilde{\Pi} \) is the chosen spread approximation. Thus \( \Delta_1 \) represents the error made by replacing our original basket with moment matched log-normal random variables, while \( \Delta_2 \) is the error originating from the analytical approximation of the resulting spread. Later we will see that when approximating the price of an Asian basket spread, the first part has the highest contribution to the error.

### 3.2 Shifted Log-Normal Moment Matching

Here we describe the methodology proposed by Borovkova et al. (2007) to price basket spread options. We start by introducing shifted log-normal random variables before discussing the methodology itself.

#### 3.2.1 Shifted Log-Normal Law

We say that the random variable \( Z \) follows a shifted log-normal law with positive skewness if it is of the form

\[
Z = e^{\mu + \sigma X} + \eta,
\]

where \( X \) is a standard normal random variable. A random variable \( Z \) follows a shifted log-normal law with negative skewness if it is of the type

\[
Z = -e^{\mu + \sigma X} - \eta,
\]

where \( X \) is again a standard normal random variable. After some computations we can derive the following probability density function for those variables

\[
f(z) = \frac{1}{\sqrt{2\pi}\sigma(z-\eta)} e^{-\frac{(\ln(z-\eta)-\mu)^2}{2\sigma^2}}, \quad z > \eta,
\]
for the positive skewness case, and

\[ f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln(-z-\eta) - \mu)^2}{2\sigma^2}}, \quad z < -\eta, \]

for the case with negative skewness.

The first three moments of a log-normal random variable with positive skewness are given by

\begin{align*}
M_1 &= \eta + e^{\mu + \frac{1}{2}\sigma^2} \\
M_2 &= \eta^2 + 2\eta e^{\mu + \frac{1}{2}\sigma^2} + e^{2\mu + 2\sigma^2} \\
M_3 &= \eta^3 + 3\eta^2 e^{\mu + \frac{1}{2}\sigma^2} + 3\eta e^{2\mu + 2\sigma^2} + e^{3\mu + \frac{3}{2}\sigma^2}.
\end{align*}

(18)

The first three moments of a negatively skewed random variable can be found from (18) by replacing \( M_1 \) and \( M_3 \) by \(-M_1\) and \(-M_3\).

### 3.2.2 Shifted Log-Normal Approximation

In their paper Borovkova et al. (2007) noticed that the shape of a basket spread option still resembles to a positively (negatively) skewed log-normal random variable\(^5\). The only noticeable difference comes from the fact that the basket options can take negative (positive) values. In order to accommodate for this fact they introduced a shift parameter. Their methodology goes as follows:

1. Start by computing the first three moments of your basket.
2. Compute the basket skewness index which is defined as

\[ \frac{E[(S - E[S])^3]}{(E[S]^2 - (E[S])^2)^{3/2}}. \]

3. If the skewness index is negative, moment match the basket spread with a negatively skew log-normal distribution. If the skewness is positive, use a positively skewed log-normal random variable.
4. Adjust the shift parameter of the matching distribution if needed. Indeed using numerical simulations Borovkova et al. (2007) noticed that when the moment matched distribution is such that \( \eta > 0 \) then one can improve the quality of the approximation by setting \( \eta = 0 \).
5. Compute the stop-loss premium of the matched random variable.

The stop-loss premium can be easily computed for shifted log-normal random variables. Some computations show that in the positively skewed case the stop-loss premium for \( K - \eta > 0 \) is

\[ e^{-rT} E(e^{\mu + \sigma X + \eta - K})_+ = e^{-rT} \left( e^{\mu + \frac{1}{2}\sigma^2} \Phi \left( \frac{\mu + \sigma^2 - \ln(K-\eta)}{\sigma} \right) - (K - \eta) \Phi \left( \frac{\mu - \ln(K - \eta)}{\sigma} \right) \right), \]

and in the negatively skewed case for \( -K - \eta > 0 \)

\[ e^{-rT} E(-e^{\mu + \sigma X - \eta - K})_+ = e^{-rT} \left( -e^{\mu + \frac{1}{2}\sigma^2} \Phi \left( -\frac{\mu - \sigma^2 + \ln(-K - \eta)}{\sigma} \right) - (K + \eta) \Phi \left( -\frac{\mu + \ln(-K - \eta)}{\sigma} \right) \right). \]

In their original paper Borovkova et al. (2007) also derived approximations for the Greeks. Since these formulae were cumbersome we choose not to report them here.

---

\(^5\)In their paper they only deal with basket spread options however their procedure can be directly implemented to Asian basket spread options at no cost.
3.3 Shifted Log-Extended Skew Normal Moment Matching

In this section, we develop an extension of the methodology introduced by Borovkova et al. (2007) and Zhou and Wang (2007). We consider the possibility of using a shifted log-extended skew normal random variable instead of a shifted log-normal random variable in order to perform moment matching. In doing so, we are combining both the approach of Borovkova et al. and of Zhou and Wang. Compared to the traditional shifted log-normal moment matching, this new method introduces two additional parameters giving us more moments to match. Compared to the Zhou and Wang approach, this new method allows us to consider baskets in which some assets have a negative weight. We start this section by giving a general overview about extended skew normal random variables and introducing a new version of it before explaining its implementation in option pricing problems.

3.3.1 Shifted Log-Extended Skew Normal Law

We say that $X$ is extended skew normally distributed with skewness parameters $\alpha$ and $\tau$ if it has the following probability density function:

$$
\psi(x, \alpha, \tau) = \phi(x) \frac{\Phi(\tau \sqrt{1 + \alpha^2 + \alpha z})}{\Phi(\tau)}, \quad \alpha, \tau \in \mathbb{R}.
$$

When the parameters $\alpha$ and $\tau$ are both equal to zero, we recover the standard normal distribution. Whenever one of them is different from zero, we have an extended skew normal distribution. This distribution family was first introduced by Azzalini (1985) and studied in details by Arnold et al. (1993). The presence of these two additional parameters allows us to model the asymmetry in the distribution.

A random variable $Z$ is said to be shifted log-extended skew normal with location parameter $\mu$, scale parameter $\sigma$, shift parameter $\eta$ and skewness parameters $\alpha$ and $\tau$ if it is of the type

$$
Z = e^{\mu + \sigma X + \eta},
$$

where $X$ is an extended skew normal random variable with skewness parameters $\alpha$ and $\tau$. In order to allow for negative values, we thus modified the specification of Zhou and Wang (2007) by adding the shift parameter $\eta$. We denote a shifted log-extended skew normal random variable by $\text{SLESN}(\mu, \sigma, \alpha, \tau, \eta)$. After some straightforward computation one can write the density of $Z$ as:

$$
\psi(z, \mu, \sigma, \alpha, \tau, \eta) = \frac{1}{(z - \eta)\sigma} \phi\left( \frac{\ln(z - \eta) - \mu}{\sigma} \right) \frac{\Phi(\tau \sqrt{1 + \alpha^2 + \alpha ((\ln(z - \eta) - \mu)/\sigma)})}{\Phi(\tau)}, \quad z > \eta.
$$

We will need the first five moments ($\hat{m}_1, \ldots, \hat{m}_5$) of $\text{SLESN}(\mu, \sigma, \alpha, \tau, \eta)$. After some straightforward computations we can compute and rewrite the moments as

$$
\begin{align*}
\hat{m}_1 &= M_1 + \eta \\
\hat{m}_2 &= M_2 + 2\eta\hat{m}_1 - \eta^2 \\
\hat{m}_3 &= M_3 + 3\eta\hat{m}_2 - 3\eta^2\hat{m}_1 + \eta^3 \\
\hat{m}_4 &= M_4 + 4\eta\hat{m}_3 - 6\eta^2\hat{m}_2 + 4\eta^3\hat{m}_1 - \eta^4 \\
\hat{m}_5 &= M_5 + 5\eta\hat{m}_4 - 10\eta^2\hat{m}_3 + 10\eta^3\hat{m}_2 - 5\eta^4\hat{m}_1 + \eta^5,
\end{align*}
$$

where $M_j$ is the $j^{th}$ moment of the corresponding log-extended skew normal given by:

$$
M_j = e^{j\mu + \frac{1}{2}j(\sigma^2)/2} \frac{\Phi(\tau + j\delta\sigma)}{\Phi(\tau)}, \quad \delta = \frac{\alpha}{\sqrt{1 + \alpha^2}}.
$$
Below we will need the negative SLESN(\(\mu, \sigma, \alpha, \tau, \eta\)) which is defined as

\[ Z = -e^{\mu + \sigma X} - \eta. \]

Its density can be derived in exactly the same way as for the SLESN, and its moments can be found from those of the SLESN by replacing \(M_1, M_3\) and \(M_5\) by \(-M_1 - M_3\) and \(-M_5\).

### 3.3.2 Pricing Asian Basket Spread Options

We followed the methodology introduced by Borovkova et al. (2007) but instead of using a shifted log-normal law variable we used a shifted log-extended skew normal law as our matching distribution. Thus we proceed as follows:

1. Start by computing the first five moments of the Asian basket.
2. Compute the Asian basket skewness index which is defined as

\[
E[(S - E[S])^3]\frac{1}{(E[S^2] - (E[S])^2)^{3/2}}.
\]

3. If the skewness index is negative, moment match the Asian basket spread with a negative shifted log-extended skew normal. If the skewness is positive use a positive shifted log-extended skew normal law.
4. Adjust the shift parameter of the matching distribution if needed.
5. Compute the stop-loss premium of the matched random variable.

The following theorem gives the stop-loss premium for a SLESN random variable.

**Proposition 8.** Let \(X\) be a SLESN(\(\mu, \sigma, \alpha, \tau, \eta\)) then

\[
E(X - K)_+ = e^{\mu + \frac{1}{2}\sigma^2} \Phi(\tau + \delta \sigma) \Psi(I_1; -\alpha; \tau + \delta \sigma) - (K - \eta)\Psi(I_2; -\alpha; \tau),
\]

where for \(K - \eta > 0\)

\[
I_1 = \frac{\mu + \sigma^2 - \ln(K - \eta)}{\sigma}, \quad I_2 = I_1 - \sigma,
\]

and \(\Psi\) is the cdf of an extended skew normal law.

**Proof**

Let, for \(K - \eta > 0\), \(A = \left\{ e^{\mu + \frac{1}{2}\sigma^2} \geq K - \eta \right\}\), then

\[
E(X - K)_+ = \int_A \left( e^{\mu + \sigma x} + \eta - K \right) \psi(x, \alpha, \tau) dx
\]

\[
= e^{\mu + \frac{1}{2}\sigma^2} \frac{\Phi(\tau + \delta \sigma)}{\Phi(\tau)} \int_{A - \sigma} \psi(x, \alpha, \tau + \delta \sigma) dx - (K - \eta) \int_A \psi(x, \alpha, \tau) dx
\]

\[
= e^{\mu + \frac{1}{2}\sigma^2} \frac{\Phi(\tau + \delta \sigma)}{\Phi(\tau)} \Psi(-I_1; -\alpha; \tau + \delta \sigma) - (K - \eta)\Psi(-I_2; -\alpha; \tau).
\]

where in the last step we used the property that \(1 - \Psi(x, \alpha, \tau) = \Psi(-x, -\alpha, \tau). \)

We will also need the following proposition.
Proposition 9. Let $X$ be a negative SLESN$(\mu, \sigma, \alpha, \tau, \eta)$ random variable then,

$$E(X - K)_+ = -e^{\mu + \frac{1}{2} \sigma^2} \frac{\Phi(\tau + \delta \sigma)}{\Phi(\tau)} \Psi(I_1; \alpha; \tau + \delta \sigma) - (K + \eta)\Psi(I_2; \alpha; \tau),$$

where for $-K - \eta > 0$

$$I_1 = \frac{\ln(-K - \eta) - \mu - \sigma^2}{\sigma} \quad I_2 = I_1 + \sigma,$$

and $\Psi$ is the cdf of a extended skew normal law.

Proof

Immediate from the proof of Proposition 8. \qed

Notice that if we set $\tau = 0$ and $\alpha = 0$ we recover the formula from Borovkova et al. (2007). If we set $\eta = 0$ then we recover the formula from Zhou and Wang (2007).

Remark 1 It may seem strange to look at the skewness of the Asian basket when performing moment matching. After all, we introduced the SLESN in order to model this skewness. Thus normally all the skewness from the original distribution should be embedded in the matched parameters $\alpha$ and $\tau$. Indeed the reason is more computational than theoretical. If we try to match a positively SLESN to a Asian basket whose skewness is negative our algorithm fails. The failure is due to the shift parameter who is converging to $-\infty$ in order to associate a positive density to the negative (positive) value of the axis. Taking a negative SLESN law as matching distribution resolves the problem in this case.

Remark 2 Solving the system in (19) poses some serious problems since it contains five non-linear equations. Fortunately some simple manipulations in the spirit of those introduced in Zhou and Wang (2007) allow us to avoid this problem. We proceed as follows: start by taking the logarithm of the first equation in (20) for $j = 1$ and $j = 2$, which leads to

$$\mu + \frac{1}{2} \sigma^2 = -\ln \frac{\Phi(\tau + \sigma \delta)}{M_1} + \ln \Phi(\tau),$$
$$\mu + \sigma^2 = -\ln \frac{\Phi(\tau + 2 \sigma \delta)}{M_2} + \frac{1}{2} \ln \Phi(\tau).$$

We can rewrite those two equations as

$$\mu = -2 \ln \frac{\Phi(\tau + \sigma \delta)}{M_1} + \frac{1}{2} \ln \frac{\Phi(\tau + 2 \sigma \delta)}{M_2} + \frac{3}{2} \ln \Phi(\tau),$$
$$\sigma^2 = 2 \ln \frac{\Phi(\tau + \sigma \delta)}{M_1} - \ln \frac{\Phi(\tau + 2 \sigma \delta)}{M_2} - \ln \Phi(\tau).$$

These equations will provide us the parameters $\mu$ and $\sigma^2$. Note that the right-hand side of those equations depends on the parameters $\eta$, $\tau$ and the product $\sigma \delta$. Let us denote $\theta = \sigma \delta$. Combining the remaining equations in (20) for $j = 3, 4, 5$ and using the above expression for $\mu$ and $\sigma^2$ yields.

$$\ln \frac{\Phi(\tau + 3 \theta)}{M_3} - 3 \ln \frac{\Phi(\tau + 2 \theta)}{M_2} + 3 \ln \frac{\Phi(\tau + \theta)}{M_1} = \ln \Phi(\tau),$$
$$\ln \frac{\Phi(\tau + 4 \theta)}{M_4} - 6 \ln \frac{\Phi(\tau + 2 \theta)}{M_2} + 8 \ln \frac{\Phi(\tau + \theta)}{M_1} = 3 \ln \Phi(\tau),$$
$$\ln \frac{\Phi(\tau + 5 \theta)}{M_5} + 15 \ln \frac{\Phi(\tau + \theta)}{M_1} = 10 \ln \frac{\Phi(\tau + 2 \theta)}{M_2} + 6 \ln \Phi(\tau).$$

(21)
When computing the matching distribution, we start by replacing the \( \hat{m}_j \) by the corresponding moments of \( S \). We solve the last three, non-linear, equations simultaneously in order to find \( \theta \), \( \eta \) and \( \tau \). Then we insert those parameters in the expression we found for \( \mu \) and \( \sigma^2 \). Finally, we recover \( \alpha \) using
\[
\alpha = \frac{\theta}{\sqrt{\sigma^2 - \theta^2}}.
\]

**Remark 3** If we set \( \tau = 0 \) then we are left with a skew normal distribution. This distribution will depend on only 4 parameters instead of 5. This means that when matching our Asian basket spread we only need to consider a system of 4 equations. These 4 equations can be resolved by solving simultaneously the first two equations in (22) in \( \theta \) and \( \eta \) and then using the relations (21) in order to recover \( \mu \) and \( \sigma \). The advantage of setting \( \tau = 0 \) is that we need to solve only 2 non-linear equations when determining the skewness and the shift. This reduces the complexity and can have an impact on the accuracy of our numerical procedure. The drawback is that we lose a parameter, meaning that we should lose precision when performing an approximation.

### 4 Numerical Simulations

In this section we give a number of numerical examples of spread, basket spread and Asian basket spread options. We study the relative performances of the approximation formulae developed above and give a general procedure for approximating spread options. When performing simulations we set the risk-less rate of return to 5 percent. The asset volatilities where chosen randomly between \([0.1,0.6]\) and their correlations between \([0.1,0.9]\). The option’s maturity is set equal to one year. Finally, when working with Asian options we took the average over the last 30 days, which is a common practice in energy markets. The tables containing simulations output can be found in the appendix.

#### 4.1 Spread Options

We start by considering spread options. We can draw the following conclusions:

- The comonotone upper bound offers a poor approximation to spread option prices. This failure is due to the combination of sign switching and a positive correlation between the assets which is generating a non-comonotonic spread.

- The lower bound yields poor approximations. Furthermore its accuracy is strongly influenced by the correlation between the assets. This failure should not come as a surprise since, as our conditioning variable choice was constrained by the comonotonicity requirement, we could not choose a conditioning variable maximizing \( \text{Var}(S|\Lambda) \).

- The intermediary bound has some stability problems, its performance being competitive only for some parametrization of the spread.

- The ICUB performs remarkably well. Its performance is comparable with that of the Li et al. approximation. In addition, we know that the obtained approximation represents an upper bound. Thus, our preference goes to the former method when pricing spread options. Note that this justifies the introduction of ICUB in hybrid moment matching (see section 3.1).

- The shifted log-normal moment matching performs poorly compared to the ICUB and Li et al. approximation.

---

\(^6\) We computed Li et al. approximation and the ICUB for several spread options. We compared their mean squared error and mean absolute deviation. Simulations showed that the ICUB slightly outperforms Li et al. approximation according to both criteria.
The performances of SLESN moment matching varies considerably. This is due to the complexity of the system we need to solve when matching the parameters. We encountered several problems with the optimization algorithm and in some cases the optimization outcome was strongly influenced by the initial value. Fixing one of the parameters equal to zero (see remark 3) did not solve the problem posed by optimization.

Because the problems we have pointed out concerning the comonotonic upper, lower, intermediary bounds and SLESN moment matching only worsen when working with more general Asian basket spread, we will not mention these approximations in what follows. From now on we will only focus on four approximation techniques: the ICUB, hybrid moment matching with Li et al.’s and ICUB approximation and shifted log-normal moment matching.

4.2 Basket Spread Options

A quick look at the simulation output allows us to discard two approximation techniques. First we see that the ICUB performance is clearly declining as the number of assets increases, and since this problem only worsens when dealing with Asian options we will no longer discuss this approximation. Second, we see that the hybrid moment matching associated with ICUB slightly outperforms hybrid moment matching associated with Li et al.’s approximation. Thus from now on when discussing, we will only consider the first hybrid moment matching method.

We are left with two candidates: the shifted log-normal and hybrid moment matching associated with the ICUB. It is indeed impossible to completely exclude one of these approximation techniques. The choice of the method will depend on the underlying characteristics. We can distinguish four possible cases:

1. The underlying has a positive skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) > 0$. Then the shifted log-normal approximation tends to outperform hybrid moment matching when the correlation between the assets in the positive (negative) part is low (by low we mean below 0.8) and the volatilities are low (see table 4).

2. The underlying has a positive skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) < 0$. In this case, numerical simulations tend to favor hybrid moment matching since its results clearly dominate the shifted log-normal approximation (see table 5).

3. The underlying has a negative skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) < 0$. Then the shifted log-normal approximation tend to outperform hybrid moment matching when the correlation between the assets in the positive (negative) part is low (by low we mean below 0.8) and the volatilities are low (see table 6).

4. The underlying has a negative skewness and $\sum_{j=1}^{m} a_j \varepsilon_j S_j(0) > 0$. In this case, numerical simulations tend to favor hybrid moment matching, the results of which clearly dominate the shifted log-normal approximation (see table 7).

As we see, the choice of the approximation will depend on the skewness, correlation, and initial values of the underlying. Simulation results for each of the basket configuration can be found in the appendix.

4.3 Asian Basket Spread Options

Finally we consider Asian basket spread options. First of all, remark that in the case of Asian spread options hybrid moment matching associated to the ICUB works remarkably well. This may be explained by the strong correlation between the components of the positive (negative) part of such options (due to the temporal dependency). Second, when dealing with more general Asian basket spread, hybrid moment matching seems to be slightly more efficient than shifted log-normal moment matching especially when the
The Greeks of the ICUB (denoted by Proposition 10. Li et al.’s approximation. For simple spreads we have the following proposition.

The accuracy of the ICUB based on this conditioning variable seems to be the same as the one obtained with (23), are given by

\[ \frac{F}{2} \]

computations we choose to replace this conditioning variable by \( \gamma \). Since the conditioning variable (8) is a function of \( S \) we choose. Computing the Greeks of the ICUB is a heavy task. The difficulties originate from the conditioning variable 5.1 Simple Spread Approximation

be computed as in Borovkova et al. (2007).

5 Option Greeks

volatilities are low. Our preferences clearly go to hybrid moment matching when dealing with Asian basket spread.

5 Option Greeks

In the previous section we saw that depending on the underlying characteristics it may be preferable to use hybrid moment matching or shift log-normal moment matching. Below we provide the Greeks for the hybrid moment matching approximation. In case of shifted log-normal moment matching, the Greeks can be computed as in Borovkova et al. (2007).

5.1 Simple Spread Approximation

Computing the Greeks of the ICUB is a heavy task. The difficulties originate from the conditioning variable we choose. Since the conditioning variable (8) is a function of \( S \) and \( \sigma \) we need to differentiate all the covariance and variance present in the approximation when computing the Greeks. In order to simplify the computations we choose to replace this conditioning variable by

\[ \Lambda = \sum_{j=1}^{2} B_{jT}. \] (23)

The accuracy of the ICUB based on this conditioning variable seems to be the same as the one obtained with Li et al.’s approximation. For simple spreads we have the following proposition.

**Proposition 10.** The Greeks of the ICUB (denoted by \( \Gamma \)) of a simple spread when the conditioning variable is (23), are given by

\[ \frac{\partial \Gamma}{\partial S_j(0)} = \int_{0}^{1} \varepsilon_j a_j e^{-\frac{\Lambda^2}{2} + A_i \Phi^{-1}(u)} \Phi(Y_j - F(u, K)) du \]

\[ \frac{\partial^2 \Gamma}{\partial S_j(0) \partial \sigma_p} = \int_{0}^{1} \varepsilon_j \varepsilon_p a_j a_p e^{(r-\frac{\sigma^2}{2})T + A_i \Phi^{-1}(u) + Y_j F(u, K)} \Phi(Y_j - F(u, K))^2 du \]

\[ \frac{\partial \Gamma}{\partial \sigma_j} = \sum_{i=1}^{2} \varepsilon_i a_i S_i(0) (-\gamma_i^2 \sigma_i T + \gamma_i \sqrt{T} \Phi^{-1}(u)) \frac{\partial \sigma_i}{\partial \sigma_j} \times \]

\[ \int_{0}^{1} e^{(r-\frac{\sigma^2}{2})T + A_i \Phi^{-1}(u) + Y_j F(u, K)} du \]

\[ + \sum_{i=1}^{2} a_i S_i(0) \frac{1 - \gamma_i^2}{2} \frac{\partial \sigma_i}{\partial \sigma_j} \sqrt{T} \int_{0}^{1} e^{A_i \Phi^{-1}(u) - \frac{\sigma^2}{2} T - \frac{F(u, K)^2}{2} + Y_j F(u, K)} du \]

\[ \frac{\partial \Gamma}{\partial \rho} = \sum_{i=1}^{2} \varepsilon_i a_i S_i(0) \int_{0}^{1} \frac{\sigma_i}{2} T \left( -\frac{\sigma_i}{\sigma_A} + \frac{\Phi^{-1}(u)}{\sigma} \right) e^{(r-\frac{\sigma^2}{2})T + A_i \Phi^{-1}(u) + Y_j F(u, K)} du \]

\[ - \sum_{i=1}^{2} \frac{\sigma_i \sqrt{T}}{4 \sqrt{2 \pi (1 - \gamma_i^2)}} a_i S_i(0) \int_{0}^{1} e^{(r-\frac{\sigma^2}{2})T + A_i \Phi^{-1}(u) - \frac{F(u, K)^2}{2} + Y_j F(u, K)} du \]

where \( F(u, K) = \Phi^{-1}(F_{S_i|U=u}(K)), A_i = \gamma_i \sigma_i \sqrt{T}, Y_i = \varepsilon_i \sigma_i \sqrt{T} \sqrt{1 - \gamma_i^2} \) with \( \gamma_i^2 = \frac{1 + \rho}{2} \).
5.2 Asian Basket Spread Approximation

In this case a simple trick allows us to compute the Greeks easily. Start by replacing the conditioning variable (17) by

\[ \Lambda = X_1 + X_2. \]

The parameters \( \sigma, \mu \) and \( \rho \), are determined according to

\[
\begin{align*}
\mu_j &= 2 \ln(m_{1j}) - \frac{1}{2} \ln(m_{2j}) \\
\sigma_j^2 &= \ln(m_{2j}) - 2 \ln(m_{1j}) \\
\rho &= \frac{\ln(crm) - \mu_1 - \mu_2}{\sigma_1 \sigma_2} - \frac{1}{2} \left( \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} \right),
\end{align*}
\]

Assume we want to recover the sensitivity of the price with respect to a parameter (denote this parameter by \( G \)). First of all, remark that

\[
\begin{align*}
\frac{\partial \mu_j}{\partial G} &= 2 \frac{\partial m_{1j}}{m_{1j}} - \frac{1}{2} \frac{\partial m_{2j}}{m_{2j}}, \quad \frac{\partial \sigma_j^2}{\partial G} = \frac{\partial m_{2j}}{m_{2j}} - 2 \frac{\partial m_{1j}}{m_{1j}}, \quad \frac{\partial \rho}{\partial G} = \frac{1}{\sigma_1 \sigma_2} \left( \frac{(1 - \gamma_2^2) \frac{\partial \sigma_j^2}{\partial G} - \sigma_j^2 \frac{\partial \rho}{\partial G}}{2} \right) - \frac{1}{2} \left( \frac{\partial \sigma_j^2}{\partial G} \sigma_2 - \sigma_j \frac{\partial \sigma_1}{\partial G} + \frac{\partial \sigma_1}{\partial G} \sigma_1 - \frac{\sigma_2 \partial \sigma_1}{\partial G} \right) \frac{\partial \sigma_1}{\partial G}.
\end{align*}
\]

From these computations one can easily write \( \partial \rho / \partial G \) as

\[
\frac{\partial \rho}{\partial G} = \frac{1}{\sigma_1 \sigma_2} \left( \frac{1}{\text{crm}} \frac{\partial \text{crm}}{\partial G} - \frac{\partial \mu_1}{\partial G} - \frac{\partial \mu_2}{\partial G} \right) - \frac{\ln(crm) - \mu_1 - \mu_2}{(\sigma_1 \sigma_2)^2} \left( \sigma_2 \frac{\partial \sigma_1}{\partial G} + \sigma_1 \frac{\partial \sigma_2}{\partial G} \right) - \frac{1}{2} \left( \frac{\partial \sigma_1}{\partial G} \sigma_2 - \sigma_j \frac{\partial \sigma_1}{\partial G} + \frac{\partial \sigma_1}{\partial G} \sigma_1 - \frac{\sigma_2 \partial \sigma_1}{\partial G} \right) \frac{\partial \sigma_1}{\partial G}.
\]

With these elements in hand we can now recover an arbitrary Greek by differentiating (16) with respect to \( G \) and using the relations in (24) and (25):

\[
\begin{align*}
\frac{\partial \Gamma}{\partial G} &= e^{-rT} \sum_{i=1}^{2} \int_0^1 h_i e^{\mu_i + A_i \Phi^{-1}(u) - \frac{F(u,K)^2}{2}} + Y_i F(u,K) du \\
&\quad + e^{-rT} \sum_{i=1}^{2} \varepsilon_i \int_0^1 f_i(u) e^{\mu_i + A_i \Phi^{-1}(u) + \frac{\gamma_i^2}{2} \Phi(Y_i - F(u,K))} du
\end{align*}
\]

with

\[
\begin{align*}
h_i &= \frac{1}{\sigma_i \sqrt{8\pi (1 - \gamma_i^2)}} \left( (1 - \gamma_i^2) \frac{\partial \sigma_j^2}{\partial G} - \sigma_i^2 \frac{\partial \rho}{\partial G} \right) \\
f_i(u) &= \frac{\partial \mu_i}{\partial G} + \left( \frac{\sigma_i}{2 \sqrt{2(1 + \rho^2)}} \Phi^{-1}(u) - \frac{\sigma_i^2}{4} \right) \frac{\partial \rho}{\partial G} + \frac{1}{2} \left( 1 + \gamma_i \left( \frac{\Phi^{-1}(u)}{\sigma_i} - \gamma_i \right) \right) \frac{\partial \sigma_i^2}{\partial G}
\end{align*}
\]

and \( F(u, K) = \Phi^{-1}(F_{\mathbb{S} \in U = u}(K)), A_i = \gamma_i \sigma_i, Y_i = \varepsilon_i \sigma_i \sqrt{1 - \gamma_i^2}, \gamma_i^2 = \frac{1 + \rho}{2}. \)

Taking \( G \) to be \( \sigma, \rho \) or \( S_j(0) \) yields the Greeks.
6 Quanto and Compo Options

When the underlying is not denominated in the domestic currency, the option contains an additional risk, the exchange rate risk. In order to cover this risk, the buyer/seller may choose to modify the contract in order to include the exchange rate dynamics in it. Below we consider two ways of studying these quanto and compo options. We describe how previous results can be adapted in order to price quanto and compo Asian basket spread options. To our knowledge, this is the first time that such an extension is considered.

We assume that the exchange rate of the dollar (i.e. number of dollar for one euro) is given by

\[ dI_t^e/I_t^e = (r_e - r_s)dt + \sigma_e dB_t^e. \]

The options price dynamics, in dollar, are given by

\[ dS_t^j/S_t^j = r_j dt + \sigma_j dB_{jt}. \]

We will denote the correlation between \( B_{ti}^e \) and \( B_{jt}^j \) by \( \hat{\rho}_j \min(t_i, t_s) \).

6.1 Quanto Options

The option price is given by

\[ e^{-r_e T} E \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S_{jt_i}^e I_{ti}^e - K \right) . \]

This case is easy to deal with. Some straightforward computations allows us to rewrite the underlying as

\[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S(0)^e I_{ti}^e e^{(r_e - \frac{\sigma_j^2}{2})t_i + \sigma_j B_{jt_i}^j} , \]

where

\[ \hat{B}_{jt_i} = \frac{\sigma_j B_{jt_i}^j + \sigma_e B_{t_i}^e}{\hat{\sigma}_j} \]

\[ \hat{\sigma}_j = \sqrt{\sigma_j^2 + \sigma_e^2 + 2\sigma_j \sigma_e \hat{\rho}_j} \]

\[ \text{corr}(\hat{B}_{jt_i}, \hat{B}_{ts}) = \frac{\sigma_j \sigma_l \hat{\rho}_{jl} + \sigma_j \sigma_e \hat{\rho}_{je} + \sigma_l \sigma_e \hat{\rho}_{le} + \sigma_e^2_{\text{corr}} \min(t_i, t_s)}{\hat{\sigma}_j \hat{\sigma}_l} \sqrt{t_i t_s} \]

We can see that in this case the underlying is still log-normally distributed. Thus we can still apply hybrid moment matching methods but with the basket given in (26).

6.2 Compo Options

The option price is given by

\[ e^{-r_e T} E \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S_{jt_i}^j I_{ti}^j - K \right) . \]

This expression can be rewritten as

\[ e^{-r_s T} E(0) \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S(0)^j e^{(r_s - \frac{\sigma_j^2}{2})t_i + \sigma_j B_{jt_i}^j} - K \right) + e^{\sigma_e B_{t_i}^e - \frac{\sigma_e^2 T}{2}} . \]
Note that in the above expression there are $mn + 1$ normal random variables. As long as the correlation matrix of the $mn + 1$ random variables is not singular, we can compute a Cholesky decomposition as follows

$$
B_T^c = W_1 \\
B_{1t_1} = a_{111}W_1 + a_{112}W_2 \\
B_{1t_2} = a_{121}W_1 + a_{122}W_2 + a_{123}W_3 \\
B_{1t_3} = a_{131}W_1 + a_{132}W_2 + a_{133}W_3 + a_{134}W_4 \\
\cdots
$$

(29)

where $W_i$ are i.i.d. $N(0, 1)$ for $i = 2, 3, \ldots, mn + 1$. Using this we can write

$$
e^{-r_sT} I^c(0) \int \cdots \int \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S^g(0) e^{(r_s - \sigma_i^2)T_i + \sigma_j \sum_{h=1}^{mn+1} a_{jih} W_h - K} \right) \times
$$

(30)

$$
\times \frac{1}{(2\pi)^{mn+1/2} T^{1/2}} e^{-\frac{1}{2} \left( \frac{W_1 - \sigma \varepsilon T}{T} \right)^2 - \frac{W_2^2}{2} - \frac{W_{mn+1}^2}{2}} dW_1 dW_2 \ldots dW_{mn+1}.
$$

Setting $W_1 - \sigma \varepsilon T = \hat{B}_T^c$, we can rewrite the above expression as

$$
e^{-r_sT} I^c(0) E_V \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_j a_j S^g(0) e^{(r_s - \sigma_i^2)T_i + a_{j1i} \sigma_j \sigma \varepsilon T + T_j \hat{B}_{jti} - K} \right) \right],
$$

(31)

where $V$ is a new measure under which $\hat{B}_{jti}$ is a Brownian motion. We manage to rewrite the compo option as a sum of log-normal random variables. Thus once more we can use our previous methodology in order to price this option. Note that from the Cholesky decomposition the coefficient $a_{j1i}$ is equal to $\hat{\rho}_j \min(T, t_i)/T$.

7 Conclusion

In this paper, we found that the ICUB offers a good approximation of the price of spread options. We tried several approximation methods for Asian basket spread options and found that a combination of hybrid moment matching and shifted log-normal moment matching seems to work best. We developed formulae for the Greeks for the hybrid moment matching method with an ICUB approximation. We also showed how our methodology can easily be applied to the case of options in foreign currency.

References


Appendix

CUB = comonotonic upper bound  ICUB = improved comonotonic upper bound
CLB = comonotonic lower bound  CIntB = comonotonic intermediary bound
SLN = shifted log-normal approximation  SLESN = shifted log extended skew normal approximation
Li et al. = Li et al. spread approximation
MC = Monte Carlo price  S.E. = standard error
HybMMICUB = Hybrid moment matching with improved comonotonic upper bound
HybMMLi = Hybrid moment matching with Li et al. spread approximation

Spread Options

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Table 1: \( a = [1, -1], S(0) = [100, 200], \sigma = [0.6, 0.6], \rho_{12} = 0.28, \) 100 millions of paths (negative skewness).

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Table 2: \( a = [1, -1], S(0) = [100, 40], \sigma = [0.4, 0.17], \rho_{12} = 0.12, \) 60 millions of paths (positive skewness).

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Table 3: \( a = [1, -1], S(0) = [100, 50], \sigma = [0.6, 0.2], \rho_{12} = 0.89, \) 60 millions of paths (positive skewness).
Basket Spread Options

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Table 4: $a = [-1, -1, -1], S(0) = [100, 24; 46], \sigma = [0.4, 0.22, 0.3], \rho_{12} = 0.17, \rho_{13} = 0.91, \rho_{23} = 0.41, 300$ millions of paths (positive skewness).

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Table 5: $a = [-1, -1, -1, -1], S(0) = [100, 100, 50, 70], \sigma = [0.5, 0.15, 0.2, 0.17], \rho_{ij} = 0.9$ for all $i$ and $j$, 300 millions of paths (positive skewness).

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Table 6: $a = [-1, -1, -1, -1], S(0) = [100, 60, 40, 30], \sigma = [0.16, 0.23, 0.32, 0.43], \rho_{12} = 0.42, \rho_{13} = 0.5, \rho_{14} = 0.3, \rho_{23} = 0.24, \rho_{24} = 0.42, \rho_{34} = 0.35, 500$ millions of paths (negative skewness).

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</tr>
<tr>
<td>47.5</td>
<td>1.0323</td>
<td>0.7369</td>
<td>0.4861</td>
<td>0.4913</td>
<td>0.4026</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

Table 7: $a = [-1, -1, -1, -1], S(0) = [100, 63, 12], \sigma = [0.21, 0.34, 0.63], \rho_{12} = 0.87, \rho_{13} = 0.3, \rho_{23} = 0.43, 15$ millions of paths (negative skewness).
Asian Basket Spread Options

<table>
<thead>
<tr>
<th>Strike</th>
<th>SLN</th>
<th>HybMMICUB</th>
<th>MC</th>
<th>S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>20.8073</td>
<td>20.7637</td>
<td>20.7645</td>
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<tr>
<td>30</td>
<td>17.7711</td>
<td>17.6921</td>
<td>17.6931</td>
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</tr>
<tr>
<td>35</td>
<td>15.0630</td>
<td>14.9580</td>
<td>14.9591</td>
<td>0.00008</td>
</tr>
<tr>
<td>40</td>
<td>12.6769</td>
<td>12.5566</td>
<td>12.5579</td>
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<td>10.4747</td>
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<td>50</td>
<td>8.8065</td>
<td>8.6859</td>
<td>8.6873</td>
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<tr>
<td>55</td>
<td>7.2767</td>
<td>7.1671</td>
<td>7.1685</td>
<td>0.00006</td>
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</tbody>
</table>

Table 8: \[ a = [1, -1], S(0) = [100, 60], \sigma = [0.33, 0.25], \rho_{12} = 0.4, 400 \text{ millions of paths (positive skewness)}. \]

<table>
<thead>
<tr>
<th>Strike</th>
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<th>MC</th>
<th>S.E.</th>
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<tr>
<td>-200</td>
<td>61.7315</td>
<td>61.7593</td>
<td>61.7623</td>
<td>0.0002</td>
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<tr>
<td>-180</td>
<td>47.0642</td>
<td>47.0946</td>
<td>47.0975</td>
<td>0.0002</td>
</tr>
<tr>
<td>-160</td>
<td>33.9175</td>
<td>33.9449</td>
<td>33.9473</td>
<td>0.0001</td>
</tr>
<tr>
<td>-140</td>
<td>22.6666</td>
<td>22.6835</td>
<td>22.6853</td>
<td>0.0001</td>
</tr>
<tr>
<td>-120</td>
<td>13.6581</td>
<td>13.6572</td>
<td>13.6585</td>
<td>0.00008</td>
</tr>
<tr>
<td>-100</td>
<td>7.1111</td>
<td>7.0901</td>
<td>7.0908</td>
<td>0.00005</td>
</tr>
<tr>
<td>-80</td>
<td>2.9895</td>
<td>2.9558</td>
<td>2.9561</td>
<td>0.00003</td>
</tr>
</tbody>
</table>

Table 9: \[ a = [1, -1], S(0) = [100, 240], \sigma = [0.18, 0.35], \rho_{12} = 0.9, 400 \text{ millions of paths (negative skewness)}. \]

<table>
<thead>
<tr>
<th>Strike</th>
<th>SLN</th>
<th>HybMMICUB</th>
<th>MC</th>
<th>S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20.7157</td>
<td>20.5521</td>
<td>20.6014</td>
<td>0.00008</td>
</tr>
<tr>
<td>15</td>
<td>17.7346</td>
<td>17.5209</td>
<td>17.5549</td>
<td>0.00008</td>
</tr>
<tr>
<td>20</td>
<td>15.0677</td>
<td>14.8236</td>
<td>14.8411</td>
<td>0.00007</td>
</tr>
<tr>
<td>25</td>
<td>12.7097</td>
<td>12.4559</td>
<td>12.4577</td>
<td>0.00007</td>
</tr>
<tr>
<td>30</td>
<td>10.6478</td>
<td>10.4030</td>
<td>10.3907</td>
<td>0.00007</td>
</tr>
<tr>
<td>35</td>
<td>8.8635</td>
<td>8.6425</td>
<td>8.6186</td>
<td>0.00006</td>
</tr>
<tr>
<td>40</td>
<td>7.3342</td>
<td>7.1472</td>
<td>7.1146</td>
<td>0.00006</td>
</tr>
</tbody>
</table>

Table 10: \[ a = [1, -1, -1], S(0) = [100, 50, 25], \sigma = [0.35, 0.3, 0.25], \rho_{12} = 0.3, \rho_{13} = 0.8, \rho_{23} = 0.7, 300 \text{ millions of paths (positive skewness)}. \]

<table>
<thead>
<tr>
<th>Strike</th>
<th>SLN</th>
<th>HybMMICUB</th>
<th>MC</th>
<th>S.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>-40</td>
<td>3.6613</td>
<td>3.6643</td>
<td>3.6659</td>
<td>0.00004</td>
</tr>
<tr>
<td>-50</td>
<td>6.2375</td>
<td>6.2174</td>
<td>6.2191</td>
<td>0.00005</td>
</tr>
<tr>
<td>-60</td>
<td>9.7479</td>
<td>9.7098</td>
<td>9.7103</td>
<td>0.00007</td>
</tr>
<tr>
<td>-70</td>
<td>14.2134</td>
<td>14.1659</td>
<td>14.1661</td>
<td>0.00008</td>
</tr>
<tr>
<td>-80</td>
<td>19.5929</td>
<td>19.5450</td>
<td>19.5432</td>
<td>0.00008</td>
</tr>
<tr>
<td>-90</td>
<td>25.8016</td>
<td>25.7600</td>
<td>25.7580</td>
<td>0.00009</td>
</tr>
<tr>
<td>-100</td>
<td>32.7291</td>
<td>32.6977</td>
<td>32.6946</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

Table 11: \[ a = [1, 1, -1, -1], S(0) = [100, 50, 200, 20], \sigma = [0.18, 0.2, 0.25, 0.3], \rho_{12} = 0.6, \rho_{13} = 0.7, \rho_{14} = 0.8, \rho_{23} = 0.5, \rho_{24} = 0.75, \rho_{34} = 0.9, 300 \text{ millions of paths (negative skewness)}. \]