Asymptotic approximation of the Hitting-time and evaluation of a risky bond

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Abstract

In this paper we give an approximation for the density of the first –passage time through a boundary defined by smooth function \( S(t) \). The density is a solution of some Voltera integral and admits an expansion of the type Neumann series, the error of the approximation is précised and converges rapidly to zero. We will examine the case of a non homogeneous-time Brownian diffusion witch is related to the evaluation of many claims on financial asset. As application we give an approximation of the value of risky bond and option on the asset of levered firm.

1. Introduction

Many questions in finance are related to the crossing boundary problem. For instance the value of option on levered firm can’t be determined without information about the distribution of it’s default time. In general we can’t give an explicit solution of this distribution, unless when the kernel of it’s integral representation vanishes identically. Thus we must give an approximation solution with a best error. This error naturally depends on the kernel of the integral representation. The integral representation that we consider here is a kind of a classical Voltera integral:

\[
g(t) = g_0(t) + \int_0^t K(t, \tau)g(\tau)d\tau
\]

Under some suitable conditions on the kernel \( K \) and \( g_0 \) the existence and uniqueness of the solution \( g \) are guaranteed by the fixed point theorem.

One of the interesting application is to price the debt of a firm subject to default or similar instruments. One of the first models for pricing risky debt is developed by Merton (1974).And Since several models are proposed to valuing corporate debt. See for instance Geske (1977), Ingresoll (1977) , Leland (1994a, 1994b), Longstaff and Schwartz (1995), Leland and Toft(1996) and many others.

To simplify we consider a firm to have with a single liability with promised payoff \( K \) In the Merton’s model the bankruptcy cannot occur before maturity of the debt. Thus according to Black and Scholes (1973) , the corporate debt ca be viewed as a default-free bond minus a put option on the firm’s assets with strike price \( K \). Since the Merton’s model is simple and calculations are more flexible, many applications of Merton’s result are developed but financial distress is accompanied by costs and can lead to bankruptcy before maturity.
Thus pricing instruments using Merton’s model cannot be fire. See Franks and Torous (1994) for a study the effect of bankruptcy costs on asset prices.

The first passage time models introduced by Black and Cox (1976) and developed by many others (Leland, Longstaff…) assume that bankruptcy occurs when the total value of the firm crosses a specified level. The purpose of this work is to extend those frameworks.

We start from the model of Leland and Toft (1996). In their approach, interest rates are assumed to be constant. This is clearly unrealistic, since interest rates; in general depend on time and even stochastic. In the first section we relax this assumption by allow all parameters (interest rate, volatility...) to depend on time but deterministic and we give a closed-form to the risky debt witch generalize the result of Leland and Toft (1996). In the second section as Longstaff and Schwartz (1995), we allow the interest rates to be stochastic but we allow all parameters of both process of interest rates and firm’s value to depend on time.

1. Pricing Risky Zero Coupon risky bond

In this section and we suppose that the market is complete witch implies the existence of a risk-neutral probability measure Q that prices all assets in this market. We denote by B a risky zero coupon bond with face value F maturing at T, issued by a private firm whose total value \( V(t) \) of it’s asset follows, under Q, the geometric Brownian process

\[
    dV(t) = \left( r(t) - \lambda(t) \right) V(t) dt + \sigma(t) V(t) dW(t), \quad t \in [0, T], V(0) = V > 0
\]

Where \( r(t) \) design the instantaneous spot free rate witch may be stochastic, \( \lambda(t) \) is the payout rate to the investors .The volatility \( \sigma(t) \) is assumed to be a bounded positive continuous function.

1.1 Leland’s approach (Generalization).

Suppose that the firm is exposed to the default and must pays at the default time a fraction \( (1 - w)V_B \), \( 0 \leq w \leq 1 \) of the trigger level \( V_B \) and let \( \tau \) to be the first time that \( V(t) \) hits the level \( V_B \). The value of B at maturity T is then

\[
    B(T) = F, \quad \text{if} \quad \tau > T
\]

\[
    B(T) = (1 - \omega)V_B \exp \left( \int_\tau^{T} r(\theta)d\theta \right), \quad \text{if} \quad \tau \leq T
\]

Or equivalently

\[
    B(T) = F - \left( F - (1 - \omega)V_B \exp \left( \int_\tau^{T} r(\theta)d\theta \right) \right) \chi_{\tau \leq T} \quad (1.2)
\]

Where \( \chi_{\tau \leq T} \) is the characteristic function of the set \( \{\tau \leq T\} \).
Theorem 1.1. Suppose that the free interest rate is uncorrelated with the risk default then the arbitrage price of B at time $t \leq T$ is given by

$$B(t) = F(1 - G(t))\exp\left(-\int_t^r r(\theta)d\theta\right) + (1 - w)V_{\theta} \int_0^r g(s)\exp\left(-\int_t^s r(\theta)d\theta\right)ds$$

(1.3)

Where $G(t)$ is the distribution of the default time and $g(t)$ is the corresponding density given by

$$g(t) = g_0(t) + \int_0^t K(t, s)g(s)ds$$

with

$$g_0(t) = \frac{1}{\sqrt{2\pi} \int_0^t \sigma^2(\theta)d\theta} \exp\left(-\frac{1}{2\int_0^t \sigma^2(\theta)d\theta} \left[\ln\frac{V_{\theta}}{V} - \int_0^t r(\theta) - \frac{1}{2} \sigma^2(\theta) d\theta - \ln \left(r(t) - \frac{1}{2} \sigma^2(\theta)\right)\right]^2\right)$$

$$\times \left[-r(t) + \lambda(t) - \frac{\sigma^2(t)}{\int_0^t \sigma^2(\theta)d\theta} \ln \frac{V_{\theta}}{V} - \int_0^t \left(r(\theta) - \frac{1}{2} \sigma^2(\theta)\right)d\theta\right]$$

$$K(t, s) = \frac{1}{\sqrt{2\pi} \int_s^t \sigma^2(\theta)d\theta} \exp\left(-\frac{1}{2\int_s^t \sigma^2(\theta)d\theta} \left[\int_s^t \left(r(\theta) - \frac{1}{2} \sigma^2(\theta)\right)d\theta\right]^2\right)$$

$$\times \left[r(t) - \lambda(t) - \frac{\sigma^2(t)}{\int_s^t \sigma^2(\theta)d\theta} \int_s^t \left(r(\theta) - \frac{1}{2} \sigma^2(\theta)\right)d\theta\right]$$

Proof

In fact since the market is complete, the arbitrage price of B at time $t \leq T$ is given by

$$B(t) = E_Q\left(B(T)\exp\left(-\int_t^T r(\theta)d\theta\right)\mathcal{F}_{t+}\right)$$

Where $\mathcal{F}_{t+}$ is the natural filtration of the Brownian motion driven by $Q$.

Since the free interest rate is assumed to be uncorrelated with the risk default we obtain from (1.2)

$$B(t) = E_Q\left(\exp\left(-\int_t^T r(\theta)d\theta\right)\left[F - \left(F - (1 - \omega)V_{\theta} \exp\left(-\int_t^T r(\theta)d\theta\right)\right)\chi_{T<T}\right]\mathcal{F}_{t+}\right)$$

$$= F(1 - G(t))\exp\left(-\int_t^r r(\theta)d\theta\right) + (1 - w)V_{\theta} \int_0^r g(s)\exp\left(-\int_t^s r(\theta)d\theta\right)ds$$

(1.3)
The expression of $g_0(t)$ and $K(t,s)$ are obtained from (3.12) by taking $S(t) = V_B$, $x_0$ and 
$\mu(t) = r(t) - \lambda(t)$.

We assume now that $r(t)$, $\lambda(t)$ are continuous functions and $\sigma(t)$ is continuous satisfying
$\sigma_1 \leq \sigma(t) \leq \sigma_2$ for some constants $\sigma_1$, $\sigma_2 \in \mathbb{R}^+$. Under those conditions one can show easily
that the kernel $K(t,s)$ satisfies the condition
$|K(t,s)| \leq \frac{C}{\sqrt{t-s}}$

Now we can use the theorem 4.1 of appendix to give analytic approximation to the bond value by using an appropriate program.

**Remark 1.1.** Note that we can relax the continuous condition by replacing it by $L^2$-condition on $\sigma$ and $L^1$-condition on the other parameters. Thus discontinuous phenomena can be assumed.
It is also interesting to note that the kernel smooth on the diagonal, it vanishes identically. In fact using the Lebesgue point theorem to get

$$
\lim_{s \to r} \left( r(t) - \lambda(t) - \frac{\sigma^2(t)}{\int_s^t \sigma^2(\theta) d\theta} \int_s^t (r(\theta) - \lambda(\theta)) d\theta \right)
\lim_{s \to r} \left( r(t) - \lambda(t) - \frac{\sigma^2(t)}{\int_s^t \sigma^2(\theta) d\theta} \frac{1}{\int_s^t \sigma^2(\theta) d\theta} \int_s^t (r(\theta) - \lambda(\theta)) d\theta \right)
= \left( r(t) - \lambda(t) - \frac{\sigma^2(t)}{\sigma^2(t)} (r(t) - \lambda(t)) \right) = 0
$$
When all parameters are constant we obtain the following particular closed result due to Leland and al. (1996).

**Corollary 1.1 (Leland’s 1996).** Suppose that $r, \sigma$ and $\lambda$ are constant then

$$B(0) = F\left(1 - G(T)\right) \exp(-rT) + (1 - w)V_b H(T)$$

With

$$G(T) = N\left(\frac{-b-aT}{\sigma\sqrt{T}}\right) + \left(\frac{V}{V_b}\right)^{2a} N\left(\frac{-b+aT}{\sigma\sqrt{T}}\right)$$

$$H(T) = \int_0^r g(s) \exp(-rs) ds = \left(\frac{V}{V_b}\right)^{2a} N\left(\frac{-b-aT}{\sigma\sqrt{T}}\right) + \left(\frac{V}{V_b}\right)^{2a} N\left(\frac{-b+azT}{\sigma\sqrt{T}}\right)$$

$$b = \ln\left(\frac{V}{V_b}\right), a = r - \lambda - \frac{1}{2} \sigma^2 \text{ and } z = \sigma \sqrt{a^2 + 2r\sigma^2}.$$ 

**Proof**

Since $r, \sigma$ and $\lambda$ are constant and the boundary $S(t) = V_b$ is also constant, the kernel of the integral equation (3.11) appendix vanishes identically. Hence the density function $g(t)$ is given by

$$g(t) = \ln\left(\frac{V}{V_b}\right) \frac{1}{\sigma \sqrt{2\pi t}} \exp\left(-\frac{\ln\left(\frac{V}{V_b}\right) + \left(r - \lambda - \frac{1}{2} \sigma^2\right)t}{\sigma \sqrt{t}}\right)^2$$

Now using the simple identity that we can get from direct calculation

$$\frac{b}{\sigma \sqrt{2\pi t}} \exp\left(-\frac{1}{\sqrt{2\pi t}} \left[\frac{b + at}{\sigma \sqrt{t}}\right]^2\right) = N\left(\frac{-b-aT}{\sigma\sqrt{T}}\right) + \exp\left(-\frac{2ba}{\sigma^3}\right) N\left(\frac{-b+at}{\sigma\sqrt{T}}\right)$$

where $a, b$ are real constant and $N$ is the cumulative standard normal distribution, to conclude

$$\int_0^r \frac{b}{\sigma \sqrt{2\pi s^3}} \exp\left(-\frac{1}{2} \left[\frac{b + as}{\sigma \sqrt{s}}\right]^2\right) ds = N\left(\frac{-b-aT}{\sigma\sqrt{T}}\right) + \exp\left(-\frac{2ba}{\sigma^3}\right) N\left(\frac{-b+aT}{\sigma\sqrt{T}}\right) \quad (1.4)$$
this leads
\[
G(T) = N\left(\frac{-b-aT}{\sigma\sqrt{T}}\right) + \left(\frac{V}{V_b}\right)^{2a/\sigma} N\left(\frac{-b+aT}{\sigma\sqrt{T}}\right)
\]
with \( b = \ln\left(\frac{V}{V_b}\right) \) and \( a = r - \frac{1}{2} \sigma^2 \).

Using again the identity (1.4) after a suitable change of variable to obtain
\[
\int_0^T g(s) \exp(-rs) ds = \int_0^T \frac{b \exp(-rs) \exp\left(-\frac{1}{2}\left[\frac{b+\theta s}{\sigma\sqrt{s}}\right]^2\right)}{\sigma\sqrt{2\pi s^3}} ds
\]
\[
= \left(\frac{V}{V_b}\right)^{-a\sigma^2} N\left(\frac{-b-a\sigma^2}{\sigma T}\right) + \left(\frac{V}{V_b}\right)^{-a\sigma^2} N\left(\frac{-b+a\sigma^2}{\sigma T}\right)
\]
with \( z = \sigma\sqrt{a^2 + 2r\sigma^2} \).

Using now (1.3) to conclude.

\[\square\]

### 1.2 Longstaff-Schwartz approach (Generalization)

We assume here, without loss of generality, the face value of the risky zero coupon \( F = 1 \) and the payout rate \( \lambda(t) = 0 \).

In their model Longstaff and Schwartz (1995) suppose that the short risk interest follows a Vasicek process (Vasicek (1977)). We shall here to extend this result when the short interest rate follows the more general Hull-White Process, J.Hull and A.White(1990).

\[
\begin{align*}
\frac{dr(t)}{r(t)} = (\alpha(t) - \beta(t)r(t))dt + \eta(t)dZ_t \\
(H.W) \quad \alpha(t), \beta(t) and \eta(t) are integrable functions \quad (1.5)
\end{align*}
\]

where \( Z_t \) is a standard Brownian motion under Q correlated with standard Brownian motion \( W_t : dZ_t, dW_t = \rho dt, \rho \in \mathbb{R} \).

We adopt here Longstaff-Schwartz approach. We assume that B is a risky zero coupon bond with face value 1 that pays at maturity \( 1 - \omega \), \( 0 \leq w \leq 1 \), when the bankruptcy occurs. That is
\[
B(T) = (1-\omega) \chi_{\tau>T}
\]

Let \( P(t,T) \) to be the value at time \( t \) of a riskless bond maturing at \( T \), then as well known \( P(t,T) \) is given by
\[
P(t,T) = \exp\left(-r(t)C(t,T) - A(t,T)\right)
\]
where
\[
C(t, T) = A(t, T) = e^{b(t)} \int_0^t e^{-b(s)} ds = e^{b(t)} \gamma(t)
\]

\[
A(t, T) = \int_0^t \left( e^{b(s)} \alpha(s) \gamma'(s) - \frac{1}{2} e^{2b(s)} \sigma^2(s) \gamma^2(s) ds \right)
\]

\[
b(t) = \int_0^t \beta(s) ds
\]

(1.8)


We obtain from (1.7)

\[
dP(t, T) = r(t)P(t, T)dt - \sigma_p(t, T)P(t, T)dZ(t)
\]

(1.9)

where the volatility of the zero coupon bond \( P \) is \( \sigma_p(t, T) = \eta(t)C(t, T) \).

Now define new measure probability \( Q_T \), called forward probability measure introduced by Jamashidian, F (1989) by

\[
\frac{dQ_T}{dQ} = \frac{\int_0^\tau \exp(-r(s)) ds}{P(0, T)}
\]

and let

\[
dZ^Q_T = dZ_T + \sigma_p(t, T)dt
\]

(1.10)

Then by Cameron-Martin-Girsanov theorem \( Z^Q_T \) is a \( Q_T \)-brownian motion (see for instance Baxter,F., and A.Rennie (1996)) and the price \( v(t) \) at time \( t \) of any claim with a payoff \( X_T \) at date \( T \) is given by

\[
v(t) = P(t, T)E_{Q_T}(X_T \mid F_t)
\]

(1.11)

this gives us the price of the discounted risky bond

\[
B(t, T) = P(t, T) - P(t, T)Q_T(\{ \tau < T \})
\]

(1.12)

Under the forward probability measure we have

\[
\begin{align*}
\left\{ dr(t) = (\alpha(t) - \eta(t))\sigma_p(t, T) - \beta(t) r(t) \right\} dt + \eta(t) dZ^Q_t \\
\left\{ dV(t) = (r(t) - \rho \sigma_V(t) \sigma_p(t, T)) dt + \sigma_V(t) dB_t \right\}
\end{align*}
\]

(1.13)

where

\[
dB_t = \rho dZ^Q_t + \sqrt{1 - \rho^2} dZ_t
\]

define a \( Q_T \)-brownian motion and \( \sigma_V(t) \) design the volatility of the return of the total asset \( V \).

Integration of the first equation of (1.13) leads

\[
r(t) = e^{-\beta(t)} \left( r_0 + \int_0^t e^{\beta(s)} (\alpha(s) - \eta(s)\sigma_p(s, T)) ds + \int_0^t e^{\beta(s)} \eta(s) dZ^Q_s \right)
\]

From the first equation we obtain by Using Itô’s lemma

\[
d \ln V(t) = \left( r(t) - \rho \sigma_V(t)\sigma_p(t, T) - \frac{1}{2} \sigma_V^2(t) \right) dt + \sigma_V(t) dB_t
\]

(1.14)
with
\[ M(t,T) = e^{-b(t)} \left( n_0 + \int_0^t e^{b(s)} \left( \alpha(s) - \eta(s) \sigma_p(s,T) \right) ds - \rho \sigma_p(t) \sigma_p(t,T) \right) \]

Integrating equation (1.14) and using Fubini’s Theorem to get
\[
\ln V(t) = \ln V + \int_0^t \left( M(s,T) - \frac{1}{2} \sigma^2_v(s) \right) ds + \int_0^t e^{b(s)} b(t,s) \eta(s) dZ_s^Q + \int_0^t \sigma_v(s) dB_s
\]
with
\[ b(t,s) = \int_s^t e^{-b(\theta)} d\theta \]
Thus, \( \ln V(t) \) is normally distributed with mean
\[ \mu(t) = \ln V + \int_0^t \left( M(s,T) - \frac{1}{2} \sigma^2_v(s) \right) ds \]
and variance
\[ \sigma^2(t) = \int_0^t \left( e^{2b(s)} \eta^2(s) b^2(t,s) + \sigma^2_v(s) + 2 \rho \sigma_p(s) \eta(s) b(t,s) e^{b(s)} \right) ds \]
Now we can use the identities (3.14) and (3.15) of appendix to expressing \( Q^T \left\{ \tau < T \right\} \) and the theorem (4.1) to give an approximation of \( B(t,T) \).
Thus we have generalized the result Longstaff & Schwartz. Our method can be adapted to any process \( r(t) \). The analytic approximation of solutions is obtained with a controlled error. The numerical result can be obtained using an appropriate software program.
Appendix

1. Introduction

The integral representation that we consider here is a kind of a classical Voltera integral:

\[ g(t) = g_0(t) + \int_0^t K(t, \tau)g(\tau)d\tau \]

Under some suitable conditions on the kernel \( K \) and \( g_0 \) the existence and uniqueness of the solution \( g \) are guaranteed by the fixed point theorem.

2. Preliminaries

Next we recall the following classical theorem.

**Theorem 1.** Let \( X \) be a Banach space and let \( g_0 \) be a given element of \( X \). If \( A : X \rightarrow X \) is a bounded linear operator satisfying

\[ 1 + \sum_{n=1}^{\infty} \|A^n\| < \infty \]  

(2.1)

then the operator defined by

\[ A_{g_0}(x) = g_0 + A(x) \]  

(2.2)

has an unique fixed point given by

\[ x_0 = g_0 + \sum_{n=1}^{\infty} A^n(g_0). \]  

(2.3)

Consider now the Voltera integral equation

\[ g(t) = g_0(t) + \int_0^t K(t, \tau)g(\tau)d\tau, \quad t \in [0, T] \]  

(2.4)

we assume that the kernel satisfies the following conditions

\[ \begin{align*}
1) & \quad K(t, \tau) \text{ is a measurable function on } [0, T], \text{ for each } \tau \in [0, T] \\
2) & \quad K(t) = \text{ess sup}|K(t, \tau)|, \quad 0 \leq \tau \leq t \text{ is bounded on } [0, T]
\end{align*} \]
While \( g_0 \in X \) is fixed function with \( X \) is either \( C^0[0,T] \) or \( L^p[0,T], 1 \leq p \leq +\infty \).

We also assume when \( X = C^0[0,T] \) the following continuity condition

3) The map \( t \to K(t,.) \) is continuous from \([0,T] \to L^p[0,T] \).

An important class of kernels satisfying the above conditions are of the form

\[
K(t,\tau) = \frac{1}{\sqrt{2\pi(t-\tau)}} \left[ S'(t) - \frac{S(t)-S(\tau)}{t-\tau} \right] \exp \left( -\frac{(S(t)-S(\tau))^2}{2(t-\tau)} \right)
\]

where \( S(t) \in C^2[0,T] \). This class plays an important role to study the behaviour of the first passage time of the Brownian motion throw the absorbing boundary \( S(t) \). See for instance Durbin (1985).

Other questions related the kernel of the same type are also considered by D. O’regan (1996).

**Lemma 2.1**

*The operator \( A g(t) = \int_0^t K(t,\tau)g(\tau)d\tau \) is continuous on \( X \).*

**Proof**

Let \( g \in C^0[0,T] \), then

\[
|Ag(t) - Ag(s)| = \left| \int_0^t K(t,\tau)g(\tau)d\tau - \int_0^s K(s,\tau)g(\tau)d\tau \right|
\]

\[
= \left| \int_0^s \left[ K(t,\tau) - K(s,\tau) \right]g(\tau)d\tau + \int_s^t K(s,\tau)g(\tau)d\tau \right|
\]

\[
\leq \|K(t,\cdot) - K(s,\cdot)\|_{L^p}[0,T] \int_0^s |g(\tau)|d\tau + \sup_{\{0,T\}} \int_s^t |g(\tau)|d\tau
\]

This proves the continuity of \( A \) on \( X \) when \( X = C^0[0,T] \), \( X = L^p[0,T] \) or \( X = L^p[0,T] \).

On the other hand if we let \( g \in L^p, 1 < p < \infty \) and we design by \( p' \) the exponent conjugate of \( p \) then using Hölder inequality we get

\[
|Ag(t)| = \left| \int_0^t K(t,\tau)g(\tau)d\tau \right| \leq \left( \int_0^t |K(t,\tau)|^{p'}d\tau \right)^{1/p'} \left( \int_0^t |g(\tau)|^pd\tau \right)^{1/p}
\]

\[
\leq T^{1/p'} \sup_{0 \leq \tau \leq T} |K(t,\tau)| \|g\|_{L^p}
\]

which leads
\[ \|Ag\|_{L^p} \leq T \sup_{[0,T]} K(t) \|g\|_{L^p} \]

This assures the boundedness of \( A \).

\section*{Proposition 2.1}

The solution of Volterra equation is unique and it is given by the Neumann series
\[ g(t) = g_0(t) + \int_0^t R(t,s)g_0(s)ds \]
where the resolvent kernel is given by
\[ R(t,s) = \sum_{n=1}^{\infty} K_n(t,s) \]
with
\[ K_n(t,s) = \int_s^t K_{n-1}(t,\tau)K(\tau,s)d\tau, \quad n = 2,3,... \]
and
\[ K_1(t,s) = K(t,s) \]

\section*{Proof}
We have by Fubini theorem
\[ A^2g(t) = \int_0^t K(t,\tau)Ag(\tau)d\tau = \int_0^t K(t,\tau) \left( \int_0^\tau K(\tau,s)g(s)ds \right) d\tau \]
\[ = \int_0^t g(s) \left( \int_s^\tau K(t,\tau)K(\tau,s)d\tau \right) ds \]
hence the kernel of \( A^2 \) is given by
\[ K_2(t,s) = \int_s^t K(t,\tau)K(\tau,s)d\tau. \]
Iterating we get
\[ A^n g(t) = \int_s^t K_n(t,\tau)g(\tau)d\tau \]
with
\[ K_n(t,s) = \int_s^t K_{n-1}(t,\tau)K(\tau,s)d\tau, \quad n = 2,3,... \]
and
\[ K_1(t,s) = K(t,s) \]
From the second assumption on the kernel we obtain
\[ K_2(t,s) = \int_s^t K(t,\tau)K(\tau,s)d\tau \leq K(t) \int_s^t K(\tau)d\tau \]
\[ K_3(t,s) = \int_s^t K_2(t,\tau)K(\tau,s)d\tau \leq K(t) \int_s^t K(\tau)\left( \int_s^\tau K(v)dv \right) d\tau \]

Taking into account that

\[ K(\tau) = \frac{\partial}{\partial \tau} \int_s^\tau K(v)dv \]

and integrating by parts to obtain

\[ \int_s^\tau K(\tau)\left( \int_s^\tau K(v)dv \right) d\tau = \int_s^\tau K(\tau)\left( \int_s^\tau K(v)dv \right) d\tau 
= \int_s^\tau K(\tau)\left( \int_s^\tau K(v)dv - \int_s^\tau K(v)dv \right) d\tau \]

that is

\[ \int_s^\tau K(\tau)\left( \int_s^\tau K(v)dv \right) d\tau = \frac{1}{2} \left( \int_s^\tau K(\tau)d\tau \right)^2 \]

This leads

\[ |K_3(t,s)| \leq \frac{1}{2} K(t) \left( \int_s^\tau K(\tau)d\tau \right)^2 \]

Iterating to obtain

\[ |K_n(t,s)| \leq \frac{1}{(n-1)!} K(t) \left( \int_s^\tau K(\tau)d\tau \right)^{n-1} \]  \hspace{1cm} (2.5)

Now summing over \( n \) to conclude that

\[ \sum_{n=1}^\infty |K_n(t,s)| \leq \sum_{n=1}^\infty \frac{1}{(n-1)!} K(t) \left( \int_s^\tau K(\tau)d\tau \right)^{n-1} \leq K(t) \exp \left( \int_s^\tau K(\tau)d\tau \right) \]

This prove that

\[ 1 + \sum_{n=1}^\infty \| A^n \|_X < \infty \]

The fixed point theorem can be applied to achieve the proof.

\[ \blacksquare \]

**Remark.**

The same result hold if we replace the condition 2) by the following condition:

2') \( K(\tau) = \text{ess sup}_{|K(t,\tau)|}, \tau \leq t \leq T \) is bounded on \([0,T]\).

In this case, using a similar argument as before, we obtain
\[ \left| \sum_{n=1}^{\infty} K_n(t, s) \right| \leq \sum_{n=1}^{\infty} \frac{1}{(n-1)!} K(s) \left( \int_{s}^{t} K(\tau) d\tau \right)^{n-1} \]
\[ \leq K(s) \exp \left( \int_{s}^{t} K(\tau) d\tau \right) \]

Now consider an other important class of kernel satisfying

i) \( K(t, \tau) \) is a measurable function on \([0, T]\), for each \( \tau \in [0, T] \)

ii) \( \exists C > 0 \text{ s.t } |K(t, \tau)| \leq C|t-s|^{\alpha-1} , \ 0 < \alpha < 1. \)

iii) The map \( t \rightarrow K(t, \tau) \) is continuous from \([0, T] \rightarrow L^\alpha[0, T] \).

We restrict ourselves to the space \( X = C^0[0, T] \), the similar result holds for the \( L^\alpha \) space.

As before let \( K_n(t, s) \) to be the kernel defined inductively by:

\[ K_n(t, s) = \int_{s}^{t} K_{n-1}(t, \tau) K(\tau, s) d\tau, \ \ n = 2, 3, \ldots \]

and

\[ K_1(t, s) = K(t, s) \]

then using the well known identity :

\[ \int_{s}^{t} (t-\tau)^{\alpha-1}(\tau-s)^{\beta-1} d\tau = (t-s)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \]

where \( \Gamma(\alpha) = \int_{0}^{\infty} t^{\alpha-1}e^{-t} dt \) stands for the Gamma function, to get

\[ |K_2(t, s)| = \int_{s}^{t} |K_1(t, \tau) K(\tau, s)| d\tau \leq K^2 \int_{s}^{t} (t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1} d\tau \]
\[ \leq C^2(t-s)^{2\alpha-1} \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} \]

Iterating to obtain

\[ |K_n(t, s)| = \int_{s}^{t} |K_{n-1}(t, \tau) K(\tau, s)| d\tau \leq C^2 \int_{s}^{t} (t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1} d\tau \]
\[ \leq C^n(t-s)^{n\alpha-1} \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} \]  \hspace{1cm} (2.6)

thus
Now observe that for \( n > \frac{1}{\alpha} \), and \( C < 1 \)

\[
\Gamma(n\alpha) = \int_0^\infty t^{n\alpha-1} e^{-t} dt \geq \int_{CT(\alpha)}^\infty t^{n\alpha-1} e^{-t} dt \geq \Gamma^{n\alpha-1}(\alpha)(CT(\alpha))^{n\alpha-1} e^{-CT}
\]

and then

\[
C^n \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} T^{n(n-\alpha)} \leq \Gamma(\alpha) C^n(1-\alpha) e^{-CT}
\]

on the other hand for \( n > \frac{1}{\alpha} \), and \( C > 1 \)

\[
\Gamma(n\alpha) = \int_0^\infty t^{n\alpha-1} e^{-t} dt \geq \int_{\frac{T}{C^{\alpha}(\alpha)}}^\infty t^{n\alpha-1} e^{-t} dt \geq \left(C^{-1}T\Gamma(n\alpha)\right)^{n\alpha-1} e^{-CT}
\]

which implies

\[
C^n \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} T^{n(n-\alpha)} \leq \Gamma(\alpha) C^{n\alpha-1-n} e^{-CT}
\]

we conclude from this observation that

\[
\left| \sum_{n=1}^\infty K_n(t,s) \right| \leq C(t-s)^{\alpha-1}
\]

We have then proved the following result.

**Proposition 2.2**

*Under the condition i), ii) and iii) the Volterra equation admits an unique solution given by it’s Neumann series expansion and the corresponding resolvent \( R(s,t) \) satisfies*

\[
R(s,t) \leq C'(s-t)^{\alpha-1}
\]

3. The first passage time for the log-normal distribution.
Consider the process which follows under some probability measure $\mathbb{P}$

$$dX(t) = \mu(t)X(t)dt + \sigma(t)X(t)dB(t), \quad t \in [\tau, T], \tau \in \mathbb{R}$$

(3.1)

with values on $\mathbb{R}^+$, where $B(t)$ is a $\mathbb{P}$-Brownian motion.

We will assume for simplicity that the drift $\mu(t)$ and the volatility $\sigma(t)$ are continuous functions.

Denote by

$$F(x, t; y, \tau) = \mathbb{P}\{X(t) \leq x \mid X(\tau) = y\}$$

(3.2)

the transition distribution of $X(t)$. Then as well known $F$ must satisfies the Kolmogorov equation

$$\frac{\partial F}{\partial \tau} + \mu(\tau)y \frac{\partial F}{\partial y} + \frac{1}{2} \sigma^2(t)y^2 \frac{\partial^2 F}{\partial y^2} = 0$$

(3.3)

with the condition

$$F(x, t; y, t) = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x \geq y \end{cases}$$

(3.4)


**Theorem 3.1**

The solution of (3.3) and (3.4) is given by

$$F(x, t; y, \tau) = \Phi\left( \frac{1}{\delta(t, \tau)} \left[ \ln \frac{x}{y} + \int_{\tau}^{t} \left( \mu(\theta) - \frac{1}{2} \sigma^2(\theta) \right) d\theta \right] \right)$$

and the density function $f(x, t; y, \tau) = \frac{\partial}{\partial x} F(x, t; y, \tau)$ is given by

$$f(x, t; y, \tau) = \frac{1}{x \delta(t, \tau) \sqrt{2\pi}} \exp \left( -\frac{1}{2 \delta^2(t, \tau)} \left[ \ln \frac{x}{y} + \int_{\tau}^{t} \left( \mu(\theta) - \frac{1}{2} \sigma^2(\theta) \right) d\theta \right]^2 \right)$$

(3.5)

where

$$\delta^2(t, \tau) = \int_{\tau}^{t} \sigma^2(s)ds$$

and $\Phi$ is the cumulative normal distribution $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$.

**Proof**

Put for a fixed $x$ and $t$ $F(x, t; y, \tau) = u(z, s)$ with

$$z = \ln \frac{x}{y} + \int_{\tau}^{t} \left( \mu(\theta) - \frac{1}{2} \sigma^2(\theta) \right) d\theta$$

and $s = t - \tau$.

Then a straightforward calculation shows that $u$ satisfies
\[-\frac{\partial u}{\partial s} + \frac{1}{2} \sigma^2 (t-s) \frac{\partial^2 u}{\partial z^2} = 0 \quad (3.6)\]

with the condition

\[u \left( \ln \frac{x}{y}, 0 \right) = \begin{cases} 
0 & \text{if } x < y \\
1 & \text{if } x \geq y
\end{cases}\]

or equivalently

\[u(z,0) = \begin{cases} 
0 & \text{if } 0 < z \\
1 & \text{if } 0 \geq z
\end{cases} = H(-z) = \tilde{H}(z) \quad (3.7)\]

where H is Heaveside function.

Taking the Fourier transform of both sides of (3.6) we get

\[\frac{\partial \hat{u}}{\partial s}(\xi, s) + 2\pi^2 |\xi|^2 \sigma^2 (t-s) \hat{u}(\xi, s) = 0 \quad (3.8)\]

Integrating (3.8) with respect s and taking into account the condition (3.7) to obtain

\[\hat{u}(\xi, s) = (\tilde{H}^\prime)(\xi) \exp \left( -2\pi \sigma^2 (s-t) |\xi|^2 \right) \quad (3.9)\]

(We note that $(\tilde{H}^\prime)(\xi)$ is a tempered distribution as the Fourier transform of the tempered distribution $(\tilde{H})(\xi)$ and is given by $(\tilde{H})(\xi) = \frac{1}{2} \delta_z - \frac{1}{2\pi i} \text{vp} \frac{1}{\xi}$ where $\delta_z$ design the Dirac measure and $\text{vp} \frac{1}{\xi}$ is the principal distribution of $\frac{1}{\xi}$, the identity (3.9) makes then sense).

Now we can write

\[u(z, s) = (\tilde{H} * K)(z) \quad (3.10)\]

with

\[K(z) = \frac{1}{\delta \sqrt{2\pi}} \exp \left( -\frac{|z|^2}{2\delta^2} \right) \] is the inverse Fourier transform of $\exp \left( -2\pi \delta^2 |z|^2 \right)$. The convolution identity (3.10) gives

\[u(z, s) = \int_{\mathbb{R}} \tilde{H}(z - \xi) \exp \left( -\frac{|\xi|^2}{2\delta^2} \right) d\xi = \int_{\mathbb{R}} \exp \left( -\frac{|\xi|^2}{2\delta^2} \right) d\xi = \Phi(-\frac{z}{\delta})\]

this leads the first result of the theorem since

\[F(x,t;y,\tau) = u(\ln \frac{x}{y} + \int_{\tau}^{t} (\mu(\theta) - \frac{1}{2} \sigma^2(\theta)) d\theta, s)\]
Differentiating $F$ with respect $x$ to obtain $f$.

Now according to Buonocore and al. (1987) the density $g$ of the first-passage of the process $X(t)$ through a $C^1$-boundary $S(t)$ conditional upon $X(t_0) = x_0$ is a solution of the Volterra integral equation:

$$g(S(t), t; x_0, t_0) = 2\Psi(S(t), t; x_0, t_0) - 2\int_{t_0}^{t} \Psi(S(t), t; S(\tau), \tau) g(S(\tau), \tau; x_0, t_0) d\tau, \quad S(t_0) > x_0$$

where

$$2\Psi(S(t), t; y, \tau) = \frac{1}{2} f(S(t), t; y, \tau) \left[ S'(t) - \mu(t) x + \frac{3}{4} \frac{\partial}{\partial x} \left( x^2 \sigma^2(t) \right) \right]_{x=S(t)}$$

$$+ \left[ \frac{1}{2} \chi \sigma(t) \frac{\partial f}{\partial x} (x, t; y, \tau) \right]_{x=S(t)}$$

Using the expression of the density function given by theorem 3.1 the equation (3.12) becomes

$$2\Psi(S(t), t; y, \tau) = \frac{1}{\delta(t, \tau) \sqrt{2\pi}} \exp \left( -\frac{1}{2 \delta^2(t, \tau)} \left[ \ln \frac{S(t)}{y} - \int_{\tau}^{t} \left( \mu(\theta) - \frac{1}{2} \sigma^2(\theta) \right) d\theta \right]^2 \right)$$

$$\times \left[ \frac{S'(t)}{S(t)} - \mu(t) - \frac{\sigma^2(t)}{\delta^2(t, \tau)} \left( \ln \frac{S(t)}{y} - \int_{\tau}^{t} \mu(\theta) d\theta \right) \right]$$

We obtain finally

$$g(t) = g_0(t) + \int_{0}^{t} K(t, s) g(s) ds$$

with
Consider now the process
\[ dY(t) = \mu(t)dt + \sigma(t)dB(t), \quad t \in [\tau, T], \tau \in \mathbb{R} \]
and put \( X(t) = e^{Y(t)} \), then using Itô’s lemma to get
\[ dX(t) = \left( \mu(t) + \frac{1}{2}\sigma^2 \right) X(t)dt + \sigma(t)X(t)dB(t) \]
On the other hand we have
\[
\begin{align*}
Y(t) &= S(t) \leftrightarrow X(t) = e^{S(t)} \\
Y(t_0) &= y_0 \leftrightarrow X(t_0) = e^{y_0} = x_0
\end{align*}
\]
Hence the density \( g \) of the first passage time of \( Y(t) \) through \( S(t) \) conditional upon \( Y(t_0) = y_0 \) is given by
\[ g(t) = 2\Psi(e^{S(t)}, t; e^{y_0}, t_0) - 2 \int_{t_0}^{t} \Psi(e^{S(t)}, t; e^{y_0}, \tau) g(\tau)d\tau \] (3.14)
A straightforward calculation gives
\[
2\Psi(e^{S(t)}, t; e^y, \tau) = \frac{1}{\delta(t, \tau)\sqrt{2\pi}} \exp \left\{ -\frac{1}{2\delta^2(t, \tau)} \left[ S(t) - y - \int_{\tau}^{t} \left( \mu(\theta) + \frac{1}{2}\sigma^2(\theta) \right)d\theta \right]^2 \right\} \times \left[ S'(t) - \mu(t) - \frac{1}{2}\sigma^2(t) - \frac{\sigma^2(t)}{\delta^2(t, \tau)} \left( S(t) - y - \int_{\tau}^{t} \left( \mu(\theta) + \frac{1}{2}\sigma^2(\theta) \right) \right) \right] (3.15)
\]
If \( \mu(t) = 0 \) and \( \sigma(t) = 1 \) we obtain the well known formula (See for instance Durbin (1981), Ferebee (1982) and for other expression C.Jennen (1985)):
\[ g(t) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{S(t)^2}{2t} \right) \int_0^t \exp \left( -\frac{g(s)}{\sqrt{2\pi(t-s)}} \right) \left( S'(t) - \frac{S(t) - S(s)}{t-s} \right) ds \]

4. Approximated solutions and error bounds.

**Theorem 4.1**. Let \( K(t,s) \) to be the first the class satisfying the condition (I). and put

\[ g_m(t) = g_0(t) + \sum_{n=1}^{\infty} \int_0^t K_n(t,s)g_0(s)ds \]

then we have the following approximation

\[ |g_m(t) - g_0(t)| \leq \frac{t^{m+1}(t)}{m!} \int_0^t \left| g_0(s) \right| e^{K(t)(t-s)} ds \]

**Proof**

We have from the inequality (2.5)

\[ |g_m(t) - g_0(t)| \leq \sum_{n=1}^{\infty} \int_0^t \left| K_n(t,s) \right| \left| g_0(s) \right| ds \]

\[ \leq \sum_{n=1}^{\infty} \int_0^t \left| K(t) \right| \left| g_0(s) \right| ds \]

\[ \leq \sum_{j=0}^{\infty} \frac{t^{m+j}}{m!} \int_0^t \left| K(t) \right| (t-s)^j \left| g_0(s) \right| ds \]

\[ \leq \frac{t^{m+1}(t)}{m!} \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^t \left| g_0(s) \right| ds \]

\[ \leq \frac{t^{m+1}(t)}{m!} \int_0^t \left| g_0(s) \right| e^{K(t)(t-s)} ds \]

\[ \blacksquare \]

**Theorem 4.2**. Let \( K(t,s) \) to be the second class satisfying the condition (I) with \( 0 < \alpha < 1 \) and put

\[ g_s(t) = g_0(t) + \sum_{j=1}^{\infty} \int_0^t K_j(t,s)g_0(s)ds \]

then we have the following approximation
\[ |g_n(t) - g_0(t)| \leq \left\| g_0(s) \right\|_x \left( 2\pi \right)^{-\frac{1}{2}} t^\alpha C^n e^{-\alpha t^\frac{1}{2}} C^n e^{-\alpha t e^{-\frac{n}{2}} \mu^\alpha (\alpha+1)} \frac{(n-2)!\alpha}{((n-2)!\alpha)^{\alpha}} \]

for \( n \geq \max \left( 1 + \frac{1}{\alpha}, \frac{\log(2\pi Ct)}{\alpha} \right) \)

To compute the error bound for the second kernel satisfying we need some classical properties of the gamma function.

1. Binet (1839). There is a unique error function \( 0 < \mu(x) < \frac{1}{12x} \) such that
   \[ \Gamma(x) = \sqrt{2\pi} x^{\frac{1}{2} - x} e^{-x} e^{\mu(x)} \quad \text{for all} \quad x > 0 \]

2. Gauss (1812). \( \prod_{j=0}^{n-1} \left( x + \frac{j}{n} \right) = (2\pi)^{n/2} n^{1/2-n} \Gamma(nx), \quad n = 2, 3, \ldots \)

3. Mellin (1889). \( \Gamma(x+1) = x\Gamma(x) \).

From the Binet Formula we get
\[
\frac{\Gamma(x+1)}{\Gamma(x+s)} = \frac{(x+1)^{x+1/2} e^{-x+x} e^{\mu(x+1)}}{(x+s)^{x+s-1/2} e^{-x+s} e^{\mu(x+s)}} = \frac{(x+1)^{x+1/2} e^{-1-x+x} e^{\mu(x+1)-\mu(x+s)}}{(x+s)^{x+s-1/2} e^{-1-x+s} e^{\mu(x+s)}}
\]

As well known the Binet function \( \mu(x) \) is an analytic, Taylor expansion of order 1 leads then
\[ \mu(x+1) - \mu(x+\varepsilon) = (1-\varepsilon)\mu'(\theta) \]
for some \( \varepsilon + x < \theta < x + 1 \)

On the other hand
\( \mu'(\theta) \) is given by the well known Poisson Formula
\[
\mu'(\theta) = 2\int_0^1 \frac{tdt}{(t^2+1)(1-e^{-2\varepsilon})}
\]

See for instance R. Campbell (1966).

Since \( s + x < \theta < x + 1 \) we get then from (4.1)
\[ \mu'(\varepsilon + x) \leq \mu'(\theta) \leq \mu'(1 + x) \]

We have then prove the following result

**Proposition 4.1.** For every \( x > 0 \) and \( 0 < s < 1 \) we have
\[
\frac{(x+1)^{x+1/2} e^{-x-x} e^{\mu(x+1)}}{(x+s)^{x+s-1/2} e^{-x-s} e^{\mu(x+s)}} \leq \frac{\Gamma(x+1)}{\Gamma(x+\varepsilon)} \leq \frac{(x+1)^{x+1/2} e^{-1-x-x} e^{\mu(x+1)}}{(x+s)^{x+s-1/2} e^{-1-x-s} e^{\mu(x+s)}}
\]

**Proof of the Theorem**
From (2.6) we have
\[ |g(t) - g_n(t)| \leq \sum_{j=1}^{\infty} C_j \frac{\Gamma(\alpha(j))}{\Gamma(j \alpha)} \int_0^t (t-s)^{\alpha-1} |g_0(s)| \, ds \]

\[ \leq \sum_{m=0}^{\infty} C^{m+n} \frac{\Gamma(\alpha)^{n+m}}{\Gamma((m+n)\alpha)} \int_0^t (t-s)^{(m+n)\alpha-1} |g_0(s)| \, ds \]

on the other hand using the Gauss and Mellin formula to obtain

\[ \frac{\Gamma^{n+m}(\alpha)}{\Gamma((m+n)\alpha)} = (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \prod_{j=0}^{n+m-1} \Gamma\left(\alpha + \frac{j}{n+m}\right) \]

\[ = (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \prod_{j=0}^{n+m-1} \alpha^{-\frac{1}{2}} \Gamma(\alpha+1) \frac{\prod_{j=0}^{n+m-1} \alpha + \frac{j}{n+m} \Gamma(\alpha+1)}{\Gamma\left(\alpha + \frac{j}{n+m}\right)} \]

the proposition 4.1 gives then

\[ \frac{\Gamma^{n+m}(\alpha)}{\Gamma((m+n)\alpha)} \leq (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \prod_{j=0}^{n+m-1} \alpha^{-\frac{1}{2}} \left(\frac{\alpha + \frac{j}{n+m}}{\alpha + \frac{j}{n+m}}\right)^{1/2} \]

\[ \leq (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \prod_{j=0}^{n+m-1} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \]

\[ \leq (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \prod_{j=0}^{n+m-1} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \]

\[ \leq (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \prod_{j=0}^{n+m-1} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \]

\[ \leq (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \prod_{j=0}^{n+m-1} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \]

\[ \leq (2\pi)^{(n+m-1/2)} (n+m)^{(1/2-(n+m))\alpha} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \prod_{j=0}^{n+m-1} \left(\alpha + \frac{j}{n+m}\right)^{1/2} \]
\[
\leq (2\pi)^{(n+m-1)/2} (n+m)^{1/2} \alpha^{-((m+n)/(a+1))} \alpha^{(\alpha+1)^{n+m-1/2}} e^{\frac{n+m}{2}} e^{-\frac{(n+m-1)}{2}} \mu(\alpha+1)
\]

\[
\times \prod_{j=1}^{n+m-1} \left(1 + \frac{j}{n+m}\right)^{1/2} \left(1 + \frac{1}{n+m}\right)^{\alpha \mu(\alpha+1)}
\]

Using the usual inequalities
\[
e^{y+1/2} \leq 1 + \frac{1}{y} \quad \text{for} \quad y > 1 \quad \text{and} \quad 1 + y \leq e^y \quad \text{for} \quad y > 0
\]

to get

\[
\leq \prod_{j=1}^{n+m-1} \left(1 + \frac{j}{n+m}\right)^{1/2} \left(1 + \frac{1}{n+m}\right)^{\alpha \mu(\alpha+1)} \leq \prod_{j=1}^{n+m-1} \frac{e^{2(n+m)}}{e^{2(n+m+1/2)}} \leq \frac{e^{n+m-1}}{e^{2(n+m+1/2)}} = e^{n+m-1} \frac{(\alpha+1)(n+m-1)}{2(n+m+1/2)}
\]

hence

\[
\frac{\Gamma^{n+m}(\alpha)}{\Gamma((m+n)\alpha)} \leq \left(2\pi\right)^{(n+m-1)/2} (n+m)^{1/2} \alpha^{-((m+n)/(a+1))} \alpha^{(\alpha+1)^{n+m-1/2}} e^{\frac{n+m}{2}} e^{-\frac{(n+m-1)}{2}} \mu(\alpha+1)
\]

Now from the inequality
\[
(k-1)! \sqrt{ke^{-k}} < k^k \quad \forall \in \mathbb{R}_0
\]

we obtain

\[
\leq \left(2\pi\right)^{(n+m-1)/2} \alpha^{-((m+n)/(a+1))} \alpha^{(\alpha+1)^{n+m-1/2}} e^{\frac{n+m}{2}} e^{-\frac{(n+m-1)}{2}} \mu(\alpha+1)
\]

\[
\times e^{\frac{(a+1)(n+m-1)}{2(n+m+1/2)}} \sqrt{n+m} \leq \left(2\pi\right)^{(n+m-1)/2} \alpha^{-((m+n)/(a+1))} \alpha^{(\alpha+1)^{n+m-1/2}} e^{\frac{n+m}{2}} e^{-\frac{(n+m-1)}{2}} \mu(\alpha+1)
\]

\[
\times e^{\frac{(a+1)(n+m-1)}{2(n+m+1/2)}} \sqrt{n+m} \leq \left(2\pi\right)^{(n+m-1)/2} \alpha^{-((m+n)/(a+1))} \alpha^{(\alpha+1)^{n+m-1/2}} e^{\frac{n+m}{2}} e^{-\frac{(n+m-1)}{2}} \mu(\alpha+1)
\]

we get finally
\[ |g(t) - g_n(t)| \leq \sum_{m=1}^{\infty} C^{m+n} \frac{\Gamma(\alpha)^{m+n}}{\Gamma((m+n)\alpha)} \int_0^t (t-s)^{(m+n)\alpha-1} |g_0(s)| ds \leq \]

\[ \leq \|g_0(s)| \|_X \left( \frac{2\pi^{\frac{1}{2}}}{(n-2)!^{\alpha\alpha}} \right)^{\frac{1}{4}} e^{\frac{-\alpha(n-2)\alpha - 3}{2} e^{2} e^{-\frac{\alpha(n-2)(\alpha+1)}{2}}} \]

\[ \sum_{m=1}^{\infty} \frac{(2\pi)^{m} C^{m} \Gamma^m e^{-m(n+m)\alpha\alpha - \frac{m+n}{4} \alpha\alpha} e^{-\frac{\alpha(n+m-1)}{2}}}{(m!)^{(n+m)\alpha}} \]

The result follows by choosing \( n \geq \max \left( 1 + \frac{1}{\alpha^2}, \frac{\log \left( \frac{2\pi\alpha\alpha}{C^m} \right)}{\alpha} \right) \)

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