Approximating equity volatility

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The volatility estimation is a crucial problem for pricing derivative instrument. The traditional implied volatility approach induces the undesired smile\(^1\) effect is therefore inconsistent with the market reality. A second approach, more realistic, is due to Bensoussan, Crouhy and Galai (1995). They derive an extension of the Black-Scholes model where the stochastic nature of equity volatility \(\sigma\) is endogenous and arises from the impact of a change in the value of the firm’s asset on financial leverage. They give an analytic approximation for \(\sigma\) when the firm is financed by external funds such as debts. They supposed that the risk-free rate and the volatility of the return on the firm’s asset are constant. In this work, we will generalize this result by varying those parameters.

Keywords: Black-Scholes model; pricing derivative; volatility.

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Abstract
The volatility estimation is a crucial problem for pricing derivative instrument. The traditional implied volatility approach induces the undesired smile\textsuperscript{1} effect is therefore inconsistent with the market reality. A second approach, more realistic, is due to Bensoussan, Crouhy and Galai (1995). They derive an extension of the Black-Scholes model where the stochastic nature of equity volatility $\sigma$ is endogenous and arises from the impact of a change in the value of the firm's asset on financial leverage. They give an analytic approximation for $\sigma$ when the firm is financed by external funds such as debts. They supposed that the risk-free rate and the volatility of the return on the firm's asset are constant. In this work, we will generalize this result by varying those parameters.

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1 Introduction
In the Black-Scholes model the value of European option $f$ depends on the current time $t$, the value of underlying asset $S(t)$, the strike price $K$, the maturity $T$ and the volatility $\sigma$ of $S(t)$. The most important parameter is the volatility $\sigma$ since is not well known. Thus estimating volatility problem is crucial for pricing the option or any claim on the asset $S(t)$.
Since the price of the option is monotonic function of $\sigma$ one can derive numerically the value of this last, from the quoted prices, by inverting the Black-Scholes formula. This value is called the implied volatility and is noted $\sigma_{\text{impl}}$. But this

is not consistent with the market reality. The implied volatility does not correspond to the historical volatility. Actually the value of \( \sigma_{\text{impl}} \)

1. changes with the maturity \( T \) for fixed exercise price \( K \)

2. changes with the exercise price \( K \) for fixed maturity \( T \) (smile effect)

To take into account the first point R. Merton (1973) proposed to regard both the return on the asset \( \mu = \mu(t) \) and the volatility \( \sigma = \sigma(t) \) as functions of time. The smile effect appears more delicate, various models are proposed to extend the Black-Sholes model: diffusion with jump (see for instance R. Avesani and al. (1997)), diffusion in witch \( \sigma = \sigma(\bar{\sigma}, t, S(t)) \) where \( \bar{\sigma} \) is a parameter that we must estimate from observed data. According to Fourné and al (1997), whenever \( \sigma \) fluctuates between several close values, the observer sees a constant average volatility. But this constant character disappears on days when the price on the underlying asset makes large deviation. This suggests naturally to give a reasonable estimation of \( \sigma \) for large deviation of the underlying asset.

We will adopt here the Benssoussan and al (1995) approach in which they derive an extension of the Black-Scholes model where the volatility arises from the impact of a change in the value of the firm’s asset on financial leverage. In their model \( \sigma = \sigma(\bar{\sigma}, t, S(t)) \) and the parameter \( \bar{\sigma} \) is assumed to be constant. To make this more clear we take as point of reference the Black-Scholes model where total asset value of a firm, \( V(t) \) is the solution, in the risk-neutral probability, of the stochastic differential equation

\[
dV(t) = rV(t)dt + \sigma(t)\bar{\sigma}dW(t),
\]

where \( W(t) \) is a standard Brownian motion in \( \mathbb{R} \). The constants or functions \( r \) and \( \bar{\sigma} \) are known as being respectively the short term interest rate and the so-called volatility.

Let us suppose that the value of the firm’s equity \( S(V, t) \) is an European call option on the underlying asset \( V \). Then, as it is well known (we refer to Black-Scholes (1973), Black-Cox (1976) and Cox-Rubinstein (1985)), in order to avoid arbitrage opportunity, \( S \) must satisfy the Black-Scholes equation

\[
\frac{\partial S}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 S}{\partial V^2} + rV \frac{\partial S}{\partial V} - rS = 0 \quad \text{in} \quad \mathbb{R}^+ \times (0, T).
\]

with boundary conditions determined by the terminal data

\[
S(V, T) = (V - K)^+ \quad \text{in} \quad \mathbb{R}^+
\]

The value of this call is given by the Black-Scholes formula which is monotonic in \( V \), we can then express \( V \) as a function of \( S \), \( V = V(S, t) \). According to Itô’s lemma and the Black-Scholes equation, we have

\[
dS(t) = S(t)rdt + S(t)\sigma(S, t)dW(t)
\]
Where the instantaneous volatility $\sigma(S,t)$ of the return on claim $\sigma$ is given by

$$\sigma(S,t) = \tilde{\sigma} \frac{\partial S}{\partial V}(V(S,t),t) \frac{V(S,t)}{S}$$

Now the crucial question is how to estimate the volatility $\sigma$.

First observe that for large value of $V$

$$\frac{\partial S}{\partial V}(V(S,t)) = d_1(V(S,t)) \approx 1$$

the first temptation is to approach $\sigma(S,t)$ by the constant value $\tilde{\sigma}$. The error term is then

$$\sigma(S,T) - \tilde{\sigma} = \frac{\tilde{\sigma}}{S} (V \frac{\partial S}{\partial V} - S)$$

which gives the following error estimation for large value of $V$

$$|\sigma(S,t) - \sigma^*(S,t)| \leq \tilde{\sigma} K \exp(-r(T-t))$$

the second natural question is about the size of the approximation, that means

can we give an approximation with a smaller error?

Benoussan and al (1995) show that we can approximate $\sigma(S,t)$ with an error smaller than any polynomial function of $\frac{1}{S}$.
They also show, for a general claim, that under suitable conditions, the volatility can be approximated with error of order $o(\frac{1}{S})$.

In this work, we will extend this result whenever the volatility of $V$ and the rate of the interest depend on time. We show that the magnitude of the error depends actually on the terminal condition. We also give in the last section a simple proof of the Benoussan and al result which is more intuitive.

Note that in order to study the behavior of the volatility $\sigma$, assuming $\tilde{\sigma}$ constant, it is natural to consider the function

$$\psi(S,t) = S(\sigma(S,t) - \tilde{\sigma})$$

and when $\tilde{\sigma}$ depends on the time is more appropriate to consider the function

$$\psi(S,t) = S \frac{\sigma(S,t) - \tilde{\sigma}(t)}{\tilde{\sigma}(t)}$$

The expression of $\psi(S,t)$ viewed as function of $V$ is

$$\tilde{\psi}(V,t) = S \frac{\tilde{\sigma}(V,t) - \tilde{\sigma}(t)}{\tilde{\sigma}(t)}$$
It is now natural if we hope an approximation at least error of order $o\left(\frac{1}{S}\right)$ to take
\(\tilde{\psi}(S,t)\) as approximation of \(\psi(S,t)\) provided that \(|\tilde{\psi}(S,t) - \psi(S,t)|\) still small.
The approximation error in this case is
\[
\sigma^\ast(S,t) = \bar{\sigma}(t) + \bar{\sigma}(t)\frac{\tilde{\psi}(S,t)}{S}
\]
To finish this introduction we note that, as is mentioned by Fournée and al (1997) that the practitioner of the financial markets continue to use the Black-Scholes model with some modification. For instance the delta hedging \(\Delta(\sigma)\) is
used not according the theory, but replacing \(\sigma\) by \(\sigma_{impl}\). We can then think they can replace \(\sigma\) by any more adapted estimating volatility.

2 Model of security values

Let \(W\) be a Brownian motion with natural filtration \(\mathcal{F}\), defined on some probability space \((\Omega, \mathcal{F}, P)\) and consider a firm whose total asset value follows a diffusion stochastic equation
\[
dV(t) = V(t)\mu(t)dt + V(t)\sigma(t)dW(t)
\]
Where the drift \(\mu(t)\) is assumed to be a continuous function on \([0,T]\) and the volatility \(\sigma(t)\) will be \(C^1\)- function satisfying \(\sigma_1 < \sigma(t) < \sigma_2\), for all \(t \in [0, T]\) for some constants \(\sigma_1, \sigma_2 \in \mathbb{R}_o\).
We also assume that there is a riskless asset \(B(t)\) paying an interest rate \(r(t)\) s.t
\[
 dB(t) = r(t)B(t)dt, \quad B(0) = 1
\]
The equity of the firm will be denoted by \(S(t)\). Since the firm’s assets can also be financed by additional sources of funds as debt, the values \(S(t)\)and \(V(t)\) do not coincide in general. The fundamental hypothesis here is that there is a deterministic functional relation between \(S(t)\) and \(V(t)\):
\[
 S(t) = S(V(t), t)
\]
Under suitable condition we can show that there is martingale measure \(Q \sim P\) s.t
\[
 dV(t) = V(t)r(t)dt + V(t)\sigma(t)d\tilde{W}(t)
\]
the market \((B(t), V(t))\)is complete and the actualized value of a portfolio replicating the European claim \(S\) is a \(Q\)-martingale. where \(\tilde{W}(t)\) is a \(Q\)-Brownian motion (See for instance B.OKsendal (1998) and Musiela.M & M.Rutkowski (1999)). That is
\[
 S(V, t) = E_Q(\exp(-\int_t^T r(s)ds)S(V(T), T) | \mathcal{F}_t)
\]
By the Feynman-Kac formula, the process $S(t)$ satisfies then the following partial differential equation (PDE)

$$\frac{\partial S}{\partial t} + \frac{1}{2} \bar{\sigma}^2 V^2 \frac{\partial^2 S}{\partial V^2} + rV \frac{\partial S}{\partial V} - rS = 0 \quad (5)$$

With the terminal condition

$$S(V,T) = S_T(V)$$

Thus by the unique continuation theorem, (see appendix) $\frac{\partial S}{\partial V}(V,t) = 0$ can not occur on a open set, unless when $S$ vanishes identically.

We may then assume, by the implicit function theorem, that

$$\left\{ \begin{array}{l}
S = S(V(S,t),t) \quad \forall S \in \text{domain of } V(S,t) \\
V = V(S(V,t),t) \quad \forall S \in \text{range of } V(S,t)
\end{array} \right. \quad (6)$$

Using Itô’s calculus and the stochastic equation (2), we get

$$dS(t) = \left( \frac{\partial S}{\partial t} + \frac{1}{2} \bar{\sigma}^2 V^2 \frac{\partial^2 S}{\partial V^2} + rV \frac{\partial S}{\partial V} \right) dt + \bar{\sigma} \frac{\partial S}{\partial V}(V,t) d\tilde{W}(t)$$

hence

$$dS(t) = S(t)r(t)dt + \sigma(t)d\tilde{W}(t) \quad (7)$$

Here the instantaneous volatility of the return on equity $\sigma$ is given by

$$\sigma = \sigma(s,t) = \bar{\sigma} \frac{\partial S}{\partial V}(V(S,t),t) \frac{V(s,t)}{S} = \bar{\sigma} \frac{V(S,t)}{S} \frac{\partial S}{\partial S}(S,t) \quad (8)$$

From (6) we get

$$\left\{ \begin{array}{l}
\frac{\partial S}{\partial V} \frac{\partial V}{\partial S} = 1 \\
\frac{\partial V}{\partial S} \frac{\partial S}{\partial V} + \left( \frac{\partial V}{\partial S} \right)^2 \frac{\partial^2 S}{\partial V^2} = 0
\end{array} \right.$$

Using this last identity, the expression of $\sigma$ and equation (5) we conclude that $V$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2} \bar{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (9)$$

### 3 Approximating volatility

Let us denote by $\bar{\sigma}$ the volatility of return on equity viewed as a function of $V$, that is

$$\bar{\sigma} = \bar{\sigma}(V,t) = \bar{\sigma} \frac{\partial S}{\partial V} \frac{V}{S(V,t)} \quad (10)$$

We have then the correspondence

$$\left\{ \begin{array}{l}
\sigma(S,t) = \bar{\sigma}(V(S,t),t) \\
\bar{\sigma}(V,t) = \sigma(S(V,t),t)
\end{array} \right. \quad (11)$$
Proposition 1 Define

\[ \psi(S,t) = S \frac{\sigma(S,t) - \bar{\sigma}(t)}{\bar{\sigma}(t)} \]  

(12)

\[ \alpha(V,t) = \frac{\sigma(S,t)}{\bar{\sigma}(t)} \]  

(13)

then \( \psi \) and \( \alpha \) satisfy

\[ \frac{\partial \psi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \psi}{\partial S^2} + rS \frac{\partial \psi}{\partial S} - r\psi = 0 \]  

(14)

\[ \frac{\partial \alpha}{\partial t} + \frac{1}{2} \sigma^2 S^2 \alpha^2 \frac{\partial^2 \alpha}{\partial S^2} + S(\sigma^2 \alpha^2 + r) \frac{\partial \alpha}{\partial S} = 0 \]  

(15)

Proof

Put \( \varphi(S,t) = \ln V(S,t) \) then

\[ \frac{\partial^2 \varphi}{\partial S^2} = \frac{\partial}{\partial S} \left( \frac{\partial V}{\partial S} \frac{1}{V} \right) = \frac{\partial^2 V}{\partial S^2} \frac{1}{V^2} = \frac{\partial^2 V}{\partial S^2} V - \left( \frac{\partial \varphi}{\partial S} \right)^2 \]

hence

\[ \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 \varphi}{\partial S^2} + \left( \frac{\partial \varphi}{\partial S} \right)^2 \right) = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 V}{\partial S^2} \frac{1}{V} \]

\[ = -\frac{1}{V} \left( \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} - rV \right) = -\frac{\partial \varphi}{\partial t} - rS \frac{\partial \varphi}{\partial S} + r \]

or equivalently

\[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} S^2 \sigma^2 \left( \frac{\partial^2 \varphi}{\partial S^2} + \left( \frac{\partial \varphi}{\partial S} \right)^2 \right) + rS \frac{\partial \varphi}{\partial S} - r = 0 \]  

(16)

On the other hand, we have, since \( \frac{\partial \varphi}{\partial S} = \frac{1}{S} \frac{\sigma}{\bar{\sigma}} \)

\[ \frac{\partial^2 \varphi}{\partial S^2} = \frac{\partial}{\partial S} \left( \frac{1}{S} \frac{\sigma}{\bar{\sigma}} \right) = -\frac{\sigma}{S^2 \sigma^2} (\sigma + S \frac{\partial \sigma}{\partial S}) \]  

(17)

From (16) and (17) we get

\[ \frac{\partial \varphi}{\partial t} = \frac{1}{2} \sigma (\sigma - \bar{\sigma} + S \frac{\partial \sigma}{\partial S}) + r - r \frac{\bar{\sigma}}{\sigma} \]

Differentiating this last expression with respect \( t \) yields

\[ \frac{\sigma^2 \varphi}{\partial t} = \frac{1}{2} \sigma (2 \frac{\partial \sigma}{\partial S} - \bar{\sigma} + S \frac{\partial^2 \sigma}{\partial S^2}) + r \bar{\sigma} \frac{\partial \sigma}{\partial S} \]

But \( \frac{\partial \varphi}{\partial S} = \frac{1}{S} \frac{\sigma}{\bar{\sigma}} \), then differentiating again with respect \( t \) to obtain

\[ \frac{\sigma^2 \varphi}{\partial t \partial S} = \frac{1}{S} \frac{\partial \sigma}{\partial t} = \frac{1}{\sigma^2 S} \left( \frac{\partial \sigma}{\partial t} - \bar{\sigma} \frac{\partial \sigma}{\partial t} \right) \]
Equating to conclude
\[
\frac{1}{\sigma^2 S} \left( \frac{\partial \sigma}{\partial t} - \frac{\sigma}{\sigma} \frac{\partial \sigma}{\partial t} \right) = \frac{1}{2} \sigma (\sigma^2 \frac{\partial^2 \sigma}{\partial \sigma^2} + \sigma + S \frac{\partial \sigma}{\partial S} + \sigma \frac{\partial \sigma}{\partial S})
\]
thus
\[
\frac{\partial \psi}{\partial t} = \frac{1}{2} S^3 \sigma^2 \frac{\partial^2 \sigma}{\partial \sigma^2} + \frac{S^2}{\sigma} \sigma^2 \frac{\partial \sigma}{\partial S} (\sigma^2 + r) \frac{\partial \sigma}{\partial S} \quad (18)
\]
From the definition of \( \psi \) we get
\[
\frac{\partial \psi}{\partial S} = \frac{\sigma - \bar{\sigma}}{\sigma} + \frac{S \partial \sigma}{\partial S} \quad (19)
\]
and
\[
\frac{\partial^2 \psi}{\partial S^2} = \frac{2 \partial \sigma}{\partial S} + \frac{S \partial^2 \sigma}{\partial S^2} \quad (20)
\]
From (18), (19) and (20) we derive (14).

To prove (15) we note that \( \psi = \alpha S - S \), compute \( \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial S} \) and \( \frac{\partial^2 \psi}{\partial S^2} \) in terms of \( \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial S} \) and \( \frac{\partial^2 \alpha}{\partial S^2} \), replacing in equation (14) to obtain (15).

**Proposition 2** Define
\[
\tilde{\psi}(V,t) = S \frac{\tilde{\sigma}(V,t) - \bar{\sigma}(t)}{\bar{\sigma}(t)} \quad (21)
\]
\[
\tilde{\alpha}(V,t) = S \frac{\tilde{\sigma}(V,t)}{\bar{\sigma}(t)} \quad (22)
\]
then
\[
\frac{\partial \tilde{\psi}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \tilde{\psi}}{\partial V^2} + r V \frac{\partial \tilde{\psi}}{\partial V} - r \tilde{\psi} = 0 \quad (23)
\]
\[
\frac{\partial \tilde{\alpha}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \tilde{\alpha}}{\partial V^2} + V \left( \sigma^2 \frac{\partial \tilde{\alpha}}{\partial V} + r \right) \frac{\partial \tilde{\alpha}}{\partial V} = 0 \quad (24)
\]
**Proof**
Let \( \tilde{\varphi} = \ln S(V,t) \) so that
\[
\tilde{\sigma}(V,t) = \sigma V \frac{\partial \tilde{\varphi}}{\partial V}
\]
As in the proof of proposition 1 we can verify
\[
\frac{\partial \tilde{\varphi}}{\partial t} = -\frac{1}{2} \sigma V \frac{\partial \tilde{\varphi}}{\partial V} + \frac{1}{2} \sigma^2 \tilde{\varphi} - \frac{1}{2} \tilde{\varphi} \frac{\partial^2 \tilde{\varphi}}{\partial V^2} + r \tilde{\varphi}
\]
\[
\frac{\partial^2 \tilde{\varphi}}{\partial t \partial V} = -\frac{1}{2} \sigma V \frac{\partial^2 \tilde{\varphi}}{\partial V^2} - \frac{1}{2} \tilde{\varphi} \frac{\partial \tilde{\varphi}}{\partial V}
\]

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And since \( \frac{\partial \tilde{\phi}}{\partial V} = \frac{1}{2} \right \rightharpoonup t \), we obtain by differentiating this last expression with respect \( t \) and equating to deduce

\[
\frac{1}{\sigma^2 V} \left( \frac{\partial \tilde{\phi}}{\partial t} \tilde{\sigma} - \tilde{\sigma} \frac{\partial \tilde{\phi}}{\partial t} \right) = - \frac{1}{2} \sigma V \frac{\partial^2 \tilde{\phi}}{\partial V^2} \frac{r + \tilde{\sigma} \frac{\partial \tilde{\phi}}{\partial \tilde{V}}}{\sigma} \quad (25)
\]

Differentiating \( \tilde{\psi} \) with respect \( t \) and \( V \) yields

\[
\frac{\partial \tilde{\psi}}{\partial t} = \frac{\tilde{\sigma} - \bar{\sigma}}{\sigma} \frac{\partial \tilde{S}}{\partial t} + \frac{S}{\sigma} \left( \frac{\partial \tilde{\sigma}}{\partial \sigma} - \tilde{\sigma} \frac{\partial \tilde{\phi}}{\partial t} \right) \quad (26)
\]

\[
\frac{\partial \tilde{\psi}}{\partial V} = \frac{\tilde{\sigma} - \bar{\sigma}}{\sigma} \frac{\partial \tilde{S}}{\partial V} + \frac{S}{\sigma} \frac{\partial \tilde{\sigma}}{\partial V} \quad (27)
\]

\[
\frac{\partial^2 \tilde{\psi}}{\partial V^2} = \frac{\tilde{\sigma} - \bar{\sigma}}{\sigma} \frac{\partial^2 \tilde{S}}{\partial V^2} + \frac{2}{\sigma} \frac{\partial S}{\partial V} \frac{\partial \tilde{\sigma}}{\partial V} + \frac{S}{\sigma} \frac{\partial^2 \tilde{\phi}}{\partial V^2} \quad (28)
\]

From PDE (5) and (25) to (28) we obtain

\[
\frac{\partial \tilde{\psi}}{\partial t} = \frac{\tilde{\sigma} - \bar{\sigma}}{\sigma} \left( - \frac{1}{2} \sigma^2 V \frac{\partial^2 \tilde{S}}{\partial V^2} - rV \frac{\partial \tilde{S}}{\partial V} + rS \right)
+ S \left( - \frac{1}{2} \sigma^2 V \frac{\partial^2 \tilde{\phi}}{\partial V^2} - rV \frac{\partial \tilde{\sigma}}{\partial V} \frac{\partial \tilde{\phi}}{\partial \tilde{V}} \right)
= -rV \left( \frac{\tilde{\sigma} - \bar{\sigma}}{\sigma} \frac{\partial \tilde{S}}{\partial V} + \frac{S}{\sigma} \frac{\partial \tilde{\sigma}}{\partial V} \right)
- \frac{1}{2} \sigma^2 V^2 \left( \frac{\tilde{\sigma} - \bar{\sigma}}{\sigma} \frac{\partial^2 \tilde{S}}{\partial V^2} + \frac{S}{\sigma} \frac{\partial^2 \tilde{\phi}}{\partial V^2} \right)
+ \frac{S}{\sigma} \frac{\partial^2 \tilde{\phi}}{\partial V^2} \frac{\partial \tilde{\sigma}}{\partial V}
\]

which yields (23) since \( \tilde{\sigma} S = \sigma V \frac{\partial S}{\partial V} \)

The proof of (24) is straightforward.

**Remark 3.1** In order to solve the equations in propositions 1 and 2, the boundary conditions need to be specified. They can be derived from the given definitions when \( V(S,T) \) and \( S(V,T) \) are made explicit.

**Proposition 3**

\[
\tilde{\psi}(V,t) = \frac{\beta}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \left( \frac{x}{\delta^2} - 1 \right) \exp(-\frac{x^2}{2\delta^2}) S(V \exp(x + R + \frac{1}{2} \delta^2),T)dx \quad (29)
\]

\[
\frac{\partial \tilde{\psi}}{\partial V}(V,t) = \frac{1}{\delta^3 \sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\frac{x^2}{2\delta^2}) \frac{\partial S}{\partial V} (V \exp(x + R + \frac{1}{2} \delta^2),T)dx \quad (30)
\]

where

\[
R = R(t,T) = \int_0^T r(s)ds
\]
\[ \beta = \beta(t, T) = \exp(-R(t, T)) \]

\[ \delta^2 = \delta^2(t, T) = \int_t^T \sigma^2(s) ds \]

For the proof we need the following lemma

**Lemma 3.1** The solution of the PDF

\[ \frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 g}{\partial V^2} + rV \frac{\partial g}{\partial V} - rg = 0 \tag{31} \]

with a smooth boundary condition \[ g(V, T) \] is given by

\[ g(V, t) = \frac{\beta}{\delta \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\delta^2}\right) g(V \exp\left(x + R - \frac{1}{2} \delta^2\right), T) dx \tag{32} \]

**Proof of the lemma**

Put

\[ g(V, t) = u(x(V, t), \tau) \exp(-\int_t^T r(s) ds), \tag{33} \]

where

\[ \begin{cases} x(V, t) = \ln V + \int_t^T (r(s) - \frac{1}{2} \sigma^2(s)) ds \\ \tau(t) = T - t \end{cases} \tag{34} \]

Straightforward computation yields

\[ \frac{\partial g}{\partial t} = \frac{\partial u}{\partial x} (-r + \frac{1}{2} \sigma^2) - \frac{\partial u}{\partial \tau} + r S \]

\[ \frac{\partial g}{\partial V} = \frac{1}{V} \frac{\partial u}{\partial x} \]

\[ \frac{\partial^2 g}{\partial V^2} = -\frac{1}{V^2} \frac{\partial u}{\partial x} + \frac{1}{V^2} \frac{\partial^2 u}{\partial x^2} \]

Replacing in equation (5)

\[ \frac{\partial u}{\partial \tau} - \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{35} \]

We argue now as G.B.Folland (1976) by taking the Fourier transform of both side of (35) with respect to the x variable assuming that \( S(u(x, 0) = g(x p(x), T) = f(x) \) is sufficiently smooth, we obtain

\[ \begin{cases} \frac{\partial \hat{u}}{\partial \tau} + 2\pi^2 \sigma^2 \hat{u}(\xi, \tau) = 0 \\ \hat{u}(\xi, 0) = \hat{f}(\xi), \quad (\tau > 0) \end{cases} \tag{36} \]

The solution of (36) is

\[ \hat{u}(\xi, \tau) = \hat{f}(\xi) \exp(-2\pi^2 \delta^2 |\xi|^2); \quad (\tau > 0) \]
Thus \( u(x, \tau) = f \ast K_{\delta}(x) \), with \( K_{\delta}(x) = \frac{1}{\delta \sqrt{2\pi}} \exp(-\frac{|\xi|^2}{2\delta^2}) \). That means

\[
u(x, \tau) = \frac{1}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} f(x - \xi) \exp(-\frac{|\xi|^2}{2\delta^2}) d\xi
\]

With

\[
\delta^2 = \delta^2(t, T) = \int_{t}^{T} \bar{\sigma}^2(s) ds
\]

Since

\[
\begin{aligned}
u(x, 0) &= f(x) = g(\exp(x), T) \\
\tilde{g}(V, t) &= u(\ln(\exp(R - \frac{1}{2} \delta^2), \tau) \exp(-R)
\end{aligned}
\]

we get finally

\[
\tilde{g}(V, t) = \frac{\beta}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\frac{x^2}{2\delta^2}) \tilde{g}(V \exp(x + R - \frac{1}{2} \delta^2), T) dx
\]

**Proof of the Proposition 3**

The function \( \tilde{\psi} \) satisfies the PDE (31) with the boundary condition

\[
\tilde{\psi}(V, T) = V \frac{\partial S}{\partial V}(V, T) - S(V, T)
\]

we have then by lemma 3.1

\[
\tilde{\psi}(V, t) = \frac{\beta}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\frac{x^2}{2\delta^2}) \tilde{\psi}(V \exp(x + R - \frac{1}{2} \delta^2), T) dx
\]

Integrating by parts the first term of (39) and rearranging we get (29). To prove (30) we differentiate the both sides of (29) with respect to \( V \), to obtain

\[
\frac{\partial \tilde{\psi}}{\partial V}(V, t) = \frac{\beta}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\frac{x^2}{2\delta^2}) \exp(x + R - \frac{1}{2} \delta^2) \frac{\partial \tilde{\psi}}{\partial V}(V \exp(x + R - \frac{1}{2} \delta^2), T) dx
\]
Thus
\[ V \frac{\partial S}{\partial V}(V, t) = \exp(-R) \left( \frac{1}{\delta \sqrt{2\pi}} \right) \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\delta^2}\right) \frac{\partial S}{\partial x}(V\exp(x + R - \frac{1}{2}\delta^2), T) dx \]
\[ = \frac{\exp(-R)}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \frac{x}{\delta^2} \exp\left(-\frac{x}{2\delta^2}\right) S(V\exp(x + R - \frac{1}{2}\delta^2), T) dx \]
The proposition follows from this last expression, the explicit formula of $S(V, t)$ and from the fact that $\tilde{\psi} = V \frac{\partial S}{\partial V} - S$ \hfill \blacksquare

**Corollary 3.1** With the boundary condition $S(V, T) = (V - K)^+$, the functions $S$ and $\tilde{\psi}$ are given by

\[ S(V, t) = VN(d_1(V)) - K\beta N(d_2(V)) \tag{40} \]

With
\[ d_1(V) = \frac{\ln V}{K} + \frac{(R + \frac{1}{2}\delta^2)}{\delta}, \]
\[ d_2(V) = d_1(V) - \delta \]
\[ \tilde{\psi}(V, t) = K\beta N(d_2(V)) \tag{41} \]

**Proof**

Note that $\tilde{\psi}(V, T) = K\chi_{V > K}$, where $\chi_{V > K}$ is the characteristic function of the set $\{V > K\}$, we obtain then
\[ \tilde{\psi}(V, t) = \frac{\beta}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\delta^2}\right) \tilde{\psi}(V\exp(x + R - \frac{1}{2}\delta^2), T) dx \]
\[ = \frac{\beta}{\delta \sqrt{2\pi}} \int_{\exp(-x + R - \frac{1}{2}\delta^2) > \frac{V}{K}} \exp\left(-\frac{x^2}{2\delta^2}\right) dx \]
\[ = K\beta N(d_2(V)) \]
and
\[ V \frac{\partial S}{\partial V}(V, t) = V \frac{\beta}{\delta \sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\delta^2} + R - \frac{1}{2}\delta^2\right) \frac{\partial S}{\partial V}(V\exp(-x + R - \frac{1}{2}\delta^2), T) dx \]
\[ = V \frac{1}{\delta \sqrt{2\pi}} \int_{\exp(-x + R - \frac{1}{2}\delta^2) > \frac{V}{K}} \exp\left(-\frac{x^2 - \delta^2}{2\delta^2}\right) = N(d_1) \]
which gives
\[ S = V \frac{\partial S}{\partial V} - \tilde{\psi} = VN(d_1(V)) - K\beta N(d_2(V)) \]
\hfill \blacksquare
Theorem 3.1 Suppose there is a constant \( a \in \mathbb{R} \) such that for some \( n \in \mathbb{N} \), \( V^n(\frac{\partial}{\partial V}(V, T) - a) \) is bounded and vanishes at infinity then \( V^n(\frac{\partial}{\partial V}(V, t) - a) \) also vanishes at infinity \( \forall 0 < t < T \)

Proof
Put \( F(V, t) = S(V, t) - aV \), a straightforward calculation shows that \( F \) satisfies also the Black-Scholes equation

\[
\begin{aligned}
\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + rV \frac{\partial F}{\partial V} - rF &= 0 \\
F(V, t) &= S(V, t) - aV
\end{aligned}
\]

we have by lemma 4.1

\[
F(V, t) = \frac{\beta}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\delta^2}\right) F(V \exp(R - \frac{1}{2}\delta^2), T) \, dx
\]

From the assumptions of the theorem, it follows

\[
|V^n \frac{\partial F}{\partial V}(V, t)| \leq \frac{\beta}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\delta^2} - nx - nR + n\frac{1}{2}\delta^2\right)\frac{\partial F}{\partial V}(V \exp(R - \frac{1}{2}\delta^2)) \, dx < \infty
\]

and for any fixed \( x \) the integrand goes to zero as \( V \) goes to \( \infty \). The Lebesgue Dominated convergence Theorem implies the result

Corollary 3.2 Under the same condition in the last theorem and if in addition there is a constant \( C > 0 \) such that for large \( S \)

\[
\frac{\max(V(S, t), S)}{\min(V(S, t), S)} \leq C
\]

then there is a constant \( C' > 0 \) such that for large \( S \)

\[
|\tilde{\psi}(V(S, t), t) - \tilde{\psi}(S, t)| \leq C'S^{-n+1}
\]

Particularly

\[
\sigma^*(S, t) - \sigma(S, t) = o(S^{-n}) \text{ as } S \to +\infty
\]

where

\[
\sigma^*(S, t) = \bar{\sigma}(t) + \bar{\sigma}(t) \frac{\tilde{\psi}(S, t)}{S}
\]
proof
Fix a $\epsilon > 0$, then by assumption there is $S_1 > 0$ such that

$$S \geq S_1 \Rightarrow -\epsilon < V^n \left( \frac{\partial S}{\partial V}(V, t) - a \right) < \epsilon$$  \hspace{1cm} (43)

Dividing by $V^n$ and integrating between $\min(V(S, t), S)$ and $\max(V(S, t), S)$ to obtain

$$-\epsilon \int_{\min(V(S, t), S)}^{\max(V(S, t), S)} \frac{1}{x^n} dx \leq \int_{\min(V(S, t), S)}^{\max(V(S, t), S)} \left( \frac{\partial S}{\partial x}(x, t) - a \right) dx \leq \epsilon \int_{\min(V(S, t), S)}^{\max(V(S, t), S)} \frac{1}{x^n} dx$$

Or equivalently

$$| \int_S^{V(S, t)} (\frac{\partial S}{\partial x}(x, t) - a) dx | \leq | \epsilon \int_S^{V(S, t)} \frac{1}{x^n} dx |$$

It follows

$$| S(V(S, t), t) - aV(S, t) - S(S, t) + aS(S, t) | \leq \epsilon | \min(S, V(S, t))^{n-1} | V(S, t) - S | \leq \epsilon C | \min(S, V(S, t))^{n+1} | \leq \epsilon C \left( \frac{\max(V(S, t), S)}{\min(V(S, t), S)} \right)^{n-1} \frac{1}{\max(V(S, t), S)^{n-1}} \leq \epsilon C S^{n+1}$$

On the other hand we have

$$| \tilde{\psi}(V(S, t), t) - \tilde{\psi}(S, t) | = | V(S, t) \frac{\partial \tilde{\psi}}{\partial V}(V(S, t), t) - S - S \frac{\partial \tilde{\psi}}{\partial V}(S, t) + S(S, t) | = | V(S, t) \frac{\partial \tilde{\psi}}{\partial V}(V(S, t), t) - aV(S, t) - S \frac{\partial \tilde{\psi}}{\partial V}(S, t) + aS(S, t) - S + aV(S, t) + S(S, t) - aS(S, t) | \leq \epsilon | V(S, t)^{n+1} + S^{n+1} | + \epsilon C^{n+1} S^{n+1} | \leq \epsilon | \min(V(S, t), S)^{n+1} + S^{n+1} + C^{n+1} S^{n+1} | \leq \epsilon \left( \frac{\max(V(S, t), S)}{\min(V(S, t))} \right)^{n-1} \frac{1}{\max(V(S, t), S)^{n-1}} + S^{n+1} + C^{n+1} S^{n+1} \leq \epsilon C S^{n+1}$$
**Remark 3.2** When the claim $S$ is a European call, then it satisfies the assumptions in the theorem and the corollary, hence

$$\sigma^*(S, t) - \sigma(S, t) = o(S^{-n}) \text{ as } S \to +\infty$$

This a result of Bensoussan and al.

**Remark 3.3** We can obtain this last result by a simple argument. Suppose for simplicity that $r$ and $\sigma$ are constant and take $S + K\exp(-r(T - t))$ as approximation of $V$ which gives a candidate approximating

$$\sigma^*(S, t) = \frac{S + K\exp(-r(T - t))}{S} = \bar{\sigma} + \frac{K\exp(-r(T - t))}{S} \bar{\sigma}$$

The approximation error, in this case, is given by

$$\sigma^*(S, t) - \sigma(S, t) = \frac{1}{S} K\exp(-r(T - t))(N(d_2(V) - 1)$$

since

$$1 - N(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \exp\left(-\frac{z^2}{2}\right) dz$$

is essentially $\frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})$ near infinity, we obtain finally

$$| \sigma^*(S, t) - \sigma(S, t) | \leq \frac{1}{S} K\exp(-r(T - t))\exp(-bS^2)$$

for large $S$. Where $b$ is a nonnegative constant. This result still valid even if $r$ and $\bar{\sigma}$ are not constants.

**Remark 3.4** We note that $\frac{K\exp(-r(T - t))}{S} \bar{\sigma}$ is a particular solution of the nonlinear equation (24), when $r$ and $\bar{\sigma}$ are constant. This confirm that $\sigma^* = \bar{\sigma} + \frac{K\exp(-r(T - t))}{S} \bar{\sigma}$ is a good candidate. The same remark still valid when $r$ and $\bar{\sigma}$ depend on time.

**Conclusion**

In this study an analytic approximation of the stochastic volatility of a firm’s equity is obtained with a best error assuming the volatility of the total asset of the firm and the free risk rate depending on time. The same result is obtained for a general claim on an underlying asset with time-dependant volatility.
Appendix

Unique continuation result

Without loss of generality we restrict ourself to the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \exp(-\frac{|x-\xi|^2}{2})f(\xi)d\xi$$

of the heat equation with initial data $f(x)$. Now suppose $t > 0$ such that $\frac{\partial u}{\partial x} = 0$ on some open $A$ set of $\mathbb{R}_x$ then on $A$ we have for every integer $k$

$$\frac{\partial^k u}{\partial x^k} = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \frac{\partial^k}{\partial(x-\xi)^k} (\exp(-\frac{|x-\xi|^2}{2}))f(\xi)d\xi$$

$$= (-1)^k \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} H_k(x-\xi) \exp(-\frac{|x-\xi|^2}{2})f(\xi)d\xi = 0$$

where $H_k$ design the Hermite polynomial of order $k$. Multiplying the last integral by $\frac{x^k}{k!}(-1)^k$ with $x \in \mathbb{C}$ and summing over $k$ we get

$$\sum_{k=0}^{\infty} \int_\mathbb{R} H_k(x-\xi) \exp(-\frac{|x-\xi|^2}{2})f(\xi)d\xi = \int_\mathbb{R} \exp(-\frac{|x-\xi|^2}{2} + 2(x-\xi)\alpha - \alpha^2)f(\xi)d\xi$$

where the convergence takes place in $L^2((1 + \xi^2)^{-s/2}, d\xi)$ for some $s > 0$. See for instance M.Taylor. Taking $\alpha = iy$ with $y \in \mathbb{R}$, to obtain

$$\int_\mathbb{R} \exp(-\frac{|x-\xi|^2}{2} - i(y-\xi))f(\xi)d\xi = 0$$

hence

$$\mathcal{F}(f \exp(-\frac{(x-.)^2}{2}))(\xi) = 0$$

with $\mathcal{F}$ design the Fourier operator this is not possible unless if $f = 0$ and the $u$ must vanishes on $A$. We use now the unique continuation principle, see L.Hormander, to conclude that $u$ must vanishes identically on $\mathbb{R}_x$.

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References


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