Inverse Auctions: Injecting Unique Minima Into Random Sets

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We consider auctions in which the winning bid is the smallest bid that is unique. Only the upper-price limit is given. Neither the number of participants nor the distribution of the offers are known, so that the problem of placing a bid to win with maximum probability looks, a priori, ill-posed. We will see, however, that no more than two external (and almost compelling) arguments make the problem meaningful. By appropriately modelling the relationship between the number of participants and the distribution of the bids, we can then maximize our chances of winning the auction and propose a computable algorithm for placing our bid.

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1. INTRODUCTION AND MOTIVATION

We consider auctions in which only the upper-price limit V is known. We face a two-sided problem. Neither the number N of participants nor the distribution G of the offers is known. The auction winner is the one who places the smallest bid that is unique.

Viewed as an urn problem, there are N balls distributed among V urns; each ball is distributed according to the distribution G. With the knowledge of V but neither N nor G, we are able to place one additional ball. The placement should maximize the probability our ball resides in the leftmost (i.e., smallest-numbered) urn containing a single ball.

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On one hand, we want to model the auction in a convincing way, in terms of the expected behaviour of participants. On the other hand, we want to solve an optimization problem, i.e., our model should be tractable and should allow for asymptotic expansions, leading to a computable algorithm for placing our bid.

Our attack is based on arguing that G should be essentially geometric and that some information on the expected value $\mathbb{E}(N)$ of N and the variance $\mathbb{V}(N)$ of N can be obtained in practice. Under certain conditions concerning the relationship between G and N, we can compute the optimal placement of our bid in order to win the auction.

Poissonization (namely, changing the number N of balls from a fixed quantity into a random quantity with Poisson distribution and mean N) and dePoissonization (i.e. reconciling the Poissonized model with the original model, in which N is fixed) both play an important role in making our answers explicit.

We omit some of the calculations here, but we maintain a longer version of this report on our webpages for the interested reader (see [Bruss et al.]).

We are not aware of any closely related problems in the literature. The so-called "unique maximum problem for i.i.d. random variables" (see [Bruss and O'Cinneide 1990]), which has attracted interest, may sound somewhat similar. But this is a very different problem. (See also [Bruss and Grübel 2003] and [Kirschenhofer and Prodinger 1996] for more details and further references.) The unique maximum problem has no strategic component at all whereas in the problem we consider here, this is a major component. Note also that the minimum unique bid feature distinguishes our problem essentially from so-called "reverse auctions".

The auctions, in which the smallest unique bid is the winner, can be found in a variety of real-world auctions. The emerging importance of such auctions is chronicled on Wikipedia (see `http://en.wikipedia.org/wiki/Mobile_reverse_auction` and `http://en.wikipedia.org/wiki/Unique_bid_auction`) and in two working papers [Östling et al. ; Rapoport et al.]. According to the wikipedia terminology, our problem would be best described as a *sealed unique minimum bid auction with a random number of participants*. One successful website with such auctions is `http://www.limbo.com/unique`. We also mention the German websites `http://www.rtl.de/ratgeber/haus_900693.php` and `http://www.rtl.de/ratgeber/haus_900581.php`. We outline two other concrete examples.

Such auctions have been offered by a German real-estate company that conducts internet traumhaus (i.e., dreamhouse) auctions, which have attracted a great deal of interest. A house is put up for auction on the internet, with photos and an upper price estimation V. Offers can be made in the form of any nonnegative Euro/cents amount that does not exceed V. For instance, €0 or €1.47 are admissible offers; neither €1.471 nor $V + 1$ is valid. In one such auction on the internet, V was €350,000.00, and only the winning bidder had to pay. After a bidding period of several months, the bidder with the *smallest unique offer* wins the auction. The description of the rules is somewhat less transparent if all offers are bid at least twice. For this reason, we concentrate on the true objective, namely, to maximize the probability that our bid is the smallest unique bid.

Similar auctions or lotteries are now advertized in many places and seem to attract a lot of attention. At the Karlsberg brewery, with each purchase of a case

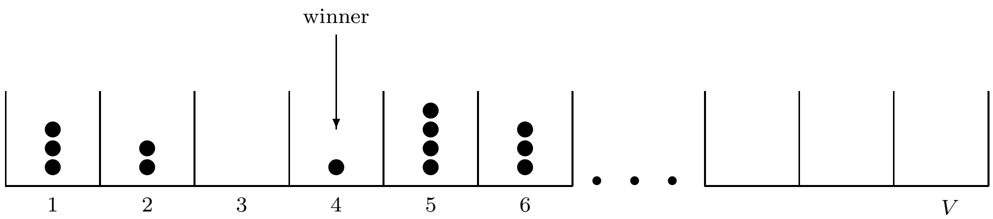


Fig. 1. The number of balls in urn K denotes the number of submitted offers of K cents. V is the upper offer limit (in cents). The offer of 4 cents is here the winner because it is the smallest *single* offer. Urns close to V are naturally very likely to be empty.

of the new "Urmild" beer, the buyer has an opportunity to make an offer for a sports car. In this case, the smallest single offer (if any) wins the car. Bids must be placed in Euro integer values.

The fact that such auctions appear in a variety of settings with the same basic principles is an indication that the advertizers have adequate intuition about the correct relationship between the upper bid limit and the number of bids placed, i.e., between the numbers of "urns and balls".

The Karlsberg brewery's sports car auction is likely to be an excellent advertizing campaign; moreover, it may directly increase sales of the "Urmild" beer, since each purchase of beer gives the purchaser the ability to make a new offer on the sports car. Hence the motivation for the Karlsberg brewery is very clear. In the traumhaus (dreamhouse) auction example, the value of the prize is much larger, so one might initially question whether such an auction can possibly be rewarding for the company who offers it. The company in this case is a real estate company, and advertizing aspects may again play a central role. We also note that bids can be placed by pay-phone, so several bids can be placed by the same person. Each bidder is allowed to make more than one offer overall, but at most one offer per day.

Additional rules concerning such auctions add further interest to the setup. For instance, in the above formulation of the traumhaus (dreamhouse) auction, only the auction winner has to pay the money that he or she bid. In another variation, each bid placed must actually be paid immediately, as well as some entrance fee; in this case, unsuccessful bids are returned to the bidder, but the entrance fee is not returned. This is reminiscent of buying a ticket in a lottery; the offers can be expected to be smaller than in the original version (i.e., without an entrance fee or deposit). Unlike a lottery, however, the ticket prices vary, and the buyer has full control of what ticket he chooses! We see a good chance of such auctions being very profitable for the company that offers them. It may be only a matter of time until lowest unique bid auctions are found even more prevalently on the internet. For this reason, we introduce an early analysis of such auctions.

2. DEFINITIONS AND OVERVIEW

Let N (corresponding to the number of bids placed in the auction) be a nonnegative integer-valued random variable with distribution function F. Let $X_1, X_2, \ldots$ (corresponding to the bids placed in the auction) be a sequence of discrete-valued, independent, identically distributed, nonnegative random variables. Given $N = n$, then the X_i's each have distribution function G_n. Our objective is to fix an additional value $X = X_{n+1}$ (corresponding to our own preferred bid in the auction) in such a way that $X = X_{n+1}$ has maximum probability of being the smallest among those values $X_1, X_2, \ldots, X_{n+1}$ that are unique. In other words, if

$$A_n = \left\{ X_k : \sum_{j=1}^{n} [\![X_j = X_k]\!] = 1 \right\}$$

denotes the set of the first n bids that are unique, then our goal is to find a value X in the support of G_n such that

$$X = \arg \max_{\{X \in \mathrm{supp} G_n\}} \mathbb{P}\left(\left\{ X < \min A_n \right\} \cap \left\{ \sum_{j=1}^{n} [\![X_j = X]\!] = 0 \right\} \right).$$

In other words, we want X to be distinct from all of the other bids (i.e., $\sum_{j=1}^{n} [\![X_j = X]\!] = 0$), and we want X to be smaller than all of the other unique bids (i.e., $X < \min A_n$). Moreover, we want to find an algorithm to efficiently compute an asymptotic value of X.

At the outset, we did not know the most appropriate distribution of the bids, so we used a variety of settings in our analysis, including some of the variations discussed above. For each model and each possible bid K, we calculate the probability of winning the auction when placing bid K. Section 3 proposes an urn model, where each urn corresponds to one offer. We are led to our first model, in which the offers are placed independently, each with geometric distribution (with fixed parameter p). For each possible bid K, we calculate two probabilities: (1.) the participants want to have a certain probability $P_1(K) = 1 - \eta$ of placing a bid that is unique, and (2.) the participant wants to maximize the probability $P_2(K)$ of winning by placing bid K, i.e., the probability that bid K is unique, and no lower bid is unique. Section 4 proves the unicity of the maximum. We investigated a selection of improved models; in Section 5 we present the best improvement that we found, in which the bids are again placed independently, each with geometric distribution $p = 1 - e^{-1/m^{\kappa}}$, for fixed $\kappa < 1$, where m denotes the average number of bids placed. This model inherently allows bidders to take advantage of the number of bids m that they believe will participate in the auction, based on past experience. Section 6 concludes the paper.

3. PROPOSING A MODEL

The essential difficulty of the problem is the fact that only the upper price limit V is known. In order to optimize the placement of our bid, we need to understand the total number of bids placed and the distribution of the bids; otherwise, the optimization problem is not well defined. Therefore, we first discuss aspects of the underlying stochastic model.

3.1 Probabilistic Model

We first note that, in order to make the real-world problem meaningful, the number of offers N and the cumulative distribution function G of the value of each bid cannot be chosen independently. The support of G (i.e., the discrete set of allowable bids) should have no gaps at all because, if G had a gap, then everyone would be interested in bidding at the smallest gap, which would solve the unicity problem already. Therefore, G should have support of the form $[1, V]$, (almost). So, $G(x)$ (the probability that a bid does not exceed x) is strictly increasing to 1 for $1 \leq x \leq V$.

By observing previous auctions, it should be possible to estimate $\mathbb{E}(N)$, the average number of bids placed. We model N as a binomial random variable with parameters n and $p = \mathbb{E}(N)/n$, where n denotes the approximate number of people who can read the auction on the internet. Let $N = I_1 + I_2 + \cdots + I_n$ where $I_k = 1$ if the kth reader will participate, and $I_k = 0$ otherwise. Then, by the Central Limit Theorem for sums of independent identically distributed random variables, we may suppose that N is approximately normal for large n. We want to be somewhat less restrictive, however, so we actually only require certain moment conditions on N (discussed in the analysis below).

Another crucial argument is that a memoryless property should hold. Indeed, keeping the large V in mind, someone who bids €100.17 would be just as willing to increase (upon advice) the bid to €100.89, as compared to somebody else who would increase a bid from €1.17 to €1.89. Exceptions to the memoryless property would be expected for large offers approaching the price limit V but this will be, as we shall see, of virtually no importance for our computations.

It is well-known that the geometric distribution is the only discrete memoryless distribution. The three conditions therefore strongly support the assumption that G should be modeled as a (truncated) geometric distribution. Here again we should point out that, if we want to maintain the memoryless property for the random variable describing the height of a bid, there is simply no alternative to the geometric distribution.

3.2 Bids Placed According To A Fixed Geometric Distribution

We first consider the case where the success parameter is fixed in the geometric distribution for G.

THEOREM 3.1. *Consider a unique-lowest-bid auction in which the bids are placed independently according to a geometric distribution with fixed probabilities of success and failure p and q, respectively. Let m and σ^2 denote the mean and variance of the number N of bids placed, not including our own bid. Use $U := (N - m)(p/q)$ to normalize the number of bids placed, and assume the kth moments of U satisfy $\mu_k = o(m^{k-2})\sigma^2$. Also, for simplicity, suppose U is integer-valued, with distribution $\rho(u)$. Our own bid is K. Let $\pi = pq^{K-1}$ denote the probability that a given bidder also bids K. The probability $P_1(K)$ that* no offer *coincides with K is*

$$P_1(K) \sim e^{-m\pi} + e^{-m\pi} \frac{(m\pi)^2}{2} \left[\frac{\sigma^2}{m^2} - \frac{1}{m} \right] + \dots . \tag{1}$$

The probability $P_2(K)$ of winning *when placing bid K, i.e., the probability that bid K is* unique *and there is* no lower unique bid *is*

$$P_2(K) \sim e^{-m\pi} \prod_{i=1}^{\infty} (1 - e^{-m\pi_i} m\pi_i), \tag{2}$$

where $\pi_i := pq^{K-1-i}$.

Before the proof of Theorem 1 we give three short examples to explain the assumptions. Throughout the proofs and examples below, we use $v = K - \log \frac{mp}{q}$ as a normalized representation of K. For ease of notation, sometimes we view P_1 and P_2 as functions of v instead of K; this is natural, since v and K are in a one-to-one correspondence.

EXAMPLE 3.1. *If* $p = q = 1/2$ *and* N *is Gaussian with mean and variance each equal to 1000, then a participant who wants to attain probability .8 of having a unique bid should bid* 12.

EXAMPLE 3.2. *More generally, consider the case where a participant wants to attain a certain probability* $P_1(K) = 1 - \eta$ *of having a unique bid. We define* $\beta := \frac{\sigma^2 p}{mq}$. *Set*

$$v = \tilde{v} + \frac{\gamma_0 q}{mp} + \dots,$$

with $\tilde{v} = \frac{\ln[-\ln(1-\eta)]}{\ln q}$ *and* $\gamma_0 = -\frac{\ln(1-\eta)}{2\ln q}\left[\beta - \frac{p}{q}\right]$.

Now we must find a particular bid v_n *such that* $P_1(v_n) = 1 - \eta$.

Example 3.1 was calculated using $\beta = 1$ *(hence* $\gamma_0 = 0$*),* $\eta = 0.2$*, which gives, from (1), a numerical value* $v_n = 2.18\dots$*, and* $\tilde{v} = 2.16\dots$*,* $\tilde{K} = \lfloor \log \frac{mp}{q} + \tilde{v} \rfloor = 12$.

EXAMPLE 3.3. *As in the proof of Theorem 3.1 below, we use* $u = (n - m)(p/q)$ *as a normalized version of the number of bids placed. We observe that*

$$P_2(K) \sim \sum_u \rho(u)(1-\pi)^n \left\{ \prod_{i=1}^{\infty} \left[1 - n\pi_i(1-\pi_i)^{n-1}\right] \right\}, \tag{3}$$

The range for i *should be* $1 \le i < K$*, but as proved in [Louchard et al. 2005], the error incurred by extending the range to* $1 \le i < \infty$ *is exponentially negligible.*

The numerical optimal value of (3), using the same conditions as Example 3.1 (i.e., $q = 1/2$ *and* $m = 1000$*) is given by* $v_n = 0.55\dots$*, with* $P_2(v_n) = 0.263\dots$*. A plot of* $P_2(v)$*, using (6), is given in Figure 2.*

To obtain $\max_v P_2(v)$, we first compute $\tilde{v}$ that yields a maximum in $F_0(v) := e^{-m\pi} \prod_{i=1}^{\infty}(1 - e^{-m\pi_i} m\pi_i)$ (notice that $F_0(v)$ is the first-order approximation of $P_2(v)$; this is discussed in the proof of Theorem 3.1 below). With the same choice of parameters from Example 3.1, a plot of $F_0(v)$ is given in Figure 3. This leads to $\tilde{v} = 0.5613032851\dots$, $F_0(\tilde{v}) = 0.2642452648\dots$, $\tilde{K} = \lfloor \log \frac{mp}{q} + \tilde{v} \rfloor = 10$.

A comparison between $P_2(v)$ and $F_0(v)$ is shown in Figure 4, where it seems that $F_0(v)$ dominates $P_2(v)$ on $[0, 2]$.

Then we set $\bar{v} = \tilde{v} + \frac{\gamma_1 q}{mp} + \dots$ and $P_2'(\bar{v}) = 0$. This gives to a value $\bar{v}$ that maximizes the expression for $P_2(v)$ given in (8), which we emphasize is second-order accurate. In other words, we write the expression for $P_2(v)$ from (8) as $P_2(v) = F_0(v) + \frac{F_1(v)q}{mp} + \cdots$, and we solve $F_0'(\bar{v}) + \frac{F_1'(\bar{v})}{m^*} = 0$, or $\gamma_1 = -\frac{F_1'(\tilde{v})}{F_0''(\tilde{v})}$, so that

$$P_2(\bar{v}) \sim F_0(\bar{v}) + \frac{F_1(\bar{v})q}{mp} \sim F_0(\tilde{v}) + \frac{F_1(\tilde{v})q}{mp}.$$

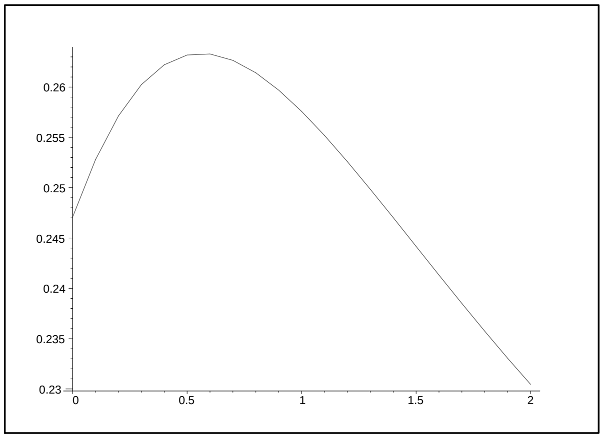


Fig. 2. $P_2(v)$. This graph presents the probability that the Kth urn is empty and all non-empty urns with number smaller than K contain more than one ball. Here K is scaled to $K = \log \frac{mp}{q} + v$.

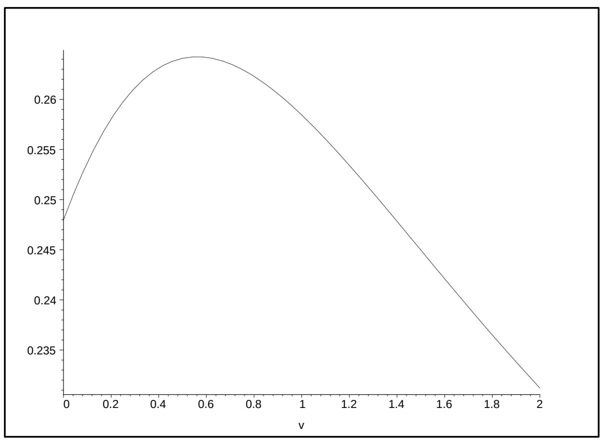
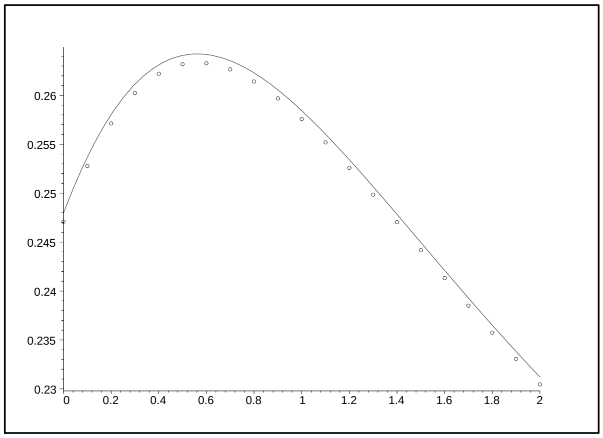


Fig. 3. $F_0(v)$

To summarize, the algorithm works as follows.

Algorithm 1

Input: p, m, β
Output: second order optimal value for $K : \bar{K}$
Solve $F_0'(\tilde{v}) = 0$;
Compute $\gamma_1 = -\frac{F_1'(\tilde{v})}{F_0''(\tilde{v})}$;
Compute $\bar{K} = \lfloor \log \frac{mp}{q} + \tilde{v} + \frac{\gamma_1 q}{mp} \rfloor$;
End



circle : $P_2(v)$
line : $F_0(v)$

Fig. 4. A comparison between $P_2(v)$ and $F_0(v)$

With our choice of parameters, we compute $\gamma_1 = 1.07903\ldots$. As K must be an integer, we see that the correction in our example is practically negligible.

Proof (of Theorem 3.1) First we prove (1). We use Poisson approximation to estimate $P_1(K)$. (See, for instance, Barbour et al. [Barbour et al. 1992].) Given there are exactly $N = n$ other offers, the condition probability that *no offer* coincides with our bid K is

$$\begin{aligned} P_1(K|N=n) &= (1-\pi)^n \\ &\sim \exp\left[-m\pi - m\pi\varepsilon - \frac{(m\pi)^2}{2m} + \mathcal{O}\left(\frac{1}{m}\right) + \mathcal{O}\left(\frac{\varepsilon}{m}\right)\right] \\ &\sim e^{-m\pi}\left[1 - m\pi\varepsilon - \frac{(m\pi)^2}{2m} + \frac{\varepsilon^2(m\pi)^2}{2} + \mathcal{O}\left(\frac{1}{m}\right) + \mathcal{O}\left(\frac{\varepsilon}{m}\right)\right], \quad (4) \end{aligned}$$

where the $\mathcal{O}$'s are functions of $m\pi$, and where ε is defined so that $n = m(1+\varepsilon)$. We obtain of course the Gumbel distribution as the dominant term.

Now we define $u = (n-m)(p/q)$. Summing (4) over all possible values of n (or equivalently, over all u) yields the unconditional probability that *no offer* coincides with K,

$$\begin{aligned} P_1(K) &\sim \sum_u \rho(u) e^{-m\pi}\left[1 - m\pi\frac{u}{mp/q} + \frac{(m\pi)^2}{2}\left[\left(\frac{u}{mp/q}\right)^2 - \frac{1}{m}\right] + \ldots\right] \\ &= e^{-m\pi} + 0 + e^{-m\pi}\frac{(m\pi)^2}{2}\left[\frac{\sigma^2}{m^2} - \frac{1}{m}\right] + \ldots, \quad (5) \end{aligned}$$

as desired.

Now we turn our attention to the proof of (2). As above, we use $u = (n-m)(p/q)$ as a normalized version of the number of bids placed. We observe that

$$P_2(K) \sim \sum_u \rho(u)(1-\pi)^n \left\{ \prod_{i=1}^{\infty} \left[1 - n\pi_i(1-\pi_i)^{n-1}\right] \right\}, \tag{6}$$

As noted above in Example 3.3, the range for i should be $1 \le i < K$, but as proved in [Louchard et al. 2005], the error incurred by extending the range to $1 \le i < \infty$ is exponentially negligible.

We must carefully study the effect of the dispersion of U around its mean 0. Recall $n = m(1+\varepsilon)$. We compute

$$\begin{aligned}
(1-\pi_i)^{n-1} &\sim \exp\left[-m\pi_i - m\pi_i\varepsilon - \frac{(m\pi_i)^2}{2m} + \frac{m\pi_i}{m} + \mathcal{O}\left(\frac{1}{m^2}\right) + \mathcal{O}\left(\frac{\varepsilon}{m}\right)\right] \\
&\sim e^{-m\pi_i}\Big[1 - m\pi_i\varepsilon - \frac{m\pi_i(-2+m\pi_i)}{2m} + \frac{\varepsilon^2(m\pi_i)^2}{2} \\
&\qquad + \mathcal{O}\left(\frac{1}{m^2}\right) + \mathcal{O}\left(\frac{\varepsilon}{m}\right)\Big].
\end{aligned}$$

Note that the $1/m$ term is different from the one in (4). This leads to

$$\begin{aligned}
P_2(K) &\sim e^{-m\pi} \sum_u \rho(u)\left[1 - m\pi\varepsilon + \frac{(m\pi)^2}{2}\varepsilon^2 - \frac{(m\pi)^2}{2m} + \dots\right] \\
&\quad \times \Big\{ \prod_{i=1}^{\infty} \Big(1 - e^{-m\pi_i}\Big[1 - m\pi_i\varepsilon + \frac{(m\pi_i)^2}{2}\varepsilon^2 \\
&\quad - \frac{m\pi_i}{2m}\left[-2 + m\pi_i\right] + \dots\Big] m\pi_i[1+\varepsilon]\Big)\Big\}.
\end{aligned} \tag{7}$$

This shows that the dispersion of K around $\log m$ is $\mathcal{O}(1)$. All these sums are easily shown to converge. So we get finally

$$\begin{aligned}
P_2(K) &\sim e^{-m\pi} \prod_{i=1}^{\infty}(1 - e^{-m\pi_i} m\pi_i) \\
&\quad + \frac{1}{m^*} e^{-m\pi} \prod_{i=1}^{\infty}(1 - e^{-m\pi_i} m\pi_i) \Big\{ \beta \Big[\sum_1^{\infty} \frac{-e^{-m\pi_i} m\pi_i(-m\pi_i + \frac{(m\pi_i)^2}{2})}{1 - e^{-m\pi_i} m\pi_i} \\
&\quad + \frac{1}{2}\left(\sum_1^{\infty} \frac{-e^{-m\pi_i} m\pi_i(-m\pi_i + 1)}{1 - e^{-m\pi_i} m\pi_i}\right)^2 - \frac{1}{2}\sum_1^{\infty}\left(\frac{-e^{-m\pi_i} m\pi_i(-m\pi_i + 1)}{1 - e^{-m\pi_i} m\pi_i}\right)^2 \\
&\quad - m\pi \sum_1^{\infty} \frac{-e^{-m\pi_i} m\pi_i(-m\pi_i + 1)}{1 - e^{-m\pi_i} m\pi_i} + \frac{(m\pi)^2}{2}\Big] \qquad (8) \\
&\quad - \frac{p(m\pi)^2}{2q} + \frac{p}{2q} \sum_1^{\infty} \frac{e^{-3m\pi_i} m\pi_i[-2 + m\pi_i]}{1 - e^{-m\pi_i} m\pi_i} \Big\}, \qquad (9)
\end{aligned}$$

where we recall $\beta := \frac{\sigma^2 p}{mq}$. ■

4. UNIQUENESS OF THE MAXIMUM

Please note that we cannot assure so far, that the candidate for this maximum is unique. Let us work with the simple case $p = 1 - \frac{1}{e}$ and $q = \frac{1}{e}$. Then

$$F_0(v) = \exp[-m\pi] \prod_{i=1}^{\infty} \left[1 - \exp\left[-m\pi_i\right] m\pi_i\right].$$

We will deal with the logarithm,

$$\ln F_0(v) = -m\pi + \sum_{i=1}^{\infty} \ln\left[1 - \exp\left[-m\pi_i\right] m\pi_i\right], \tag{10}$$

and observe the following:

i) Numerically we see that $\ln F_0(v)$ possesses a maximum at $\tilde{v} = 0.7983134948\ldots$ with $\ln F_0(\tilde{v}) = -1.024695735\ldots$. A plot of $\ln F_0(v)$ for $-0.2 \le v \le 4$ is given in Figure 5. We replaced the range of i in (10) with $1 \le i \le 30$.

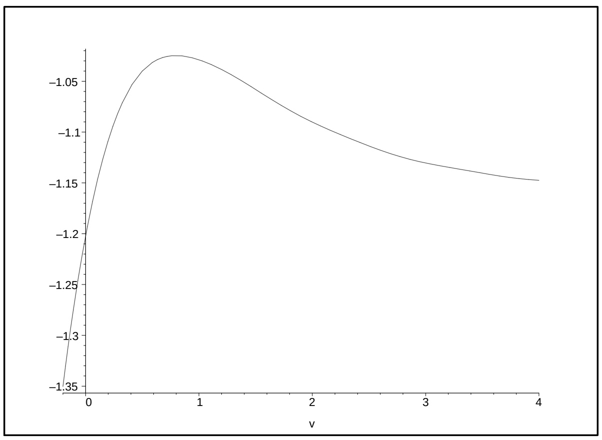


Fig. 5. $\ln F_0(v)$ for $-0.2 \le v \le 4$

ii) In fact, we obtain an excellent approximation of $\ln F_0$ for $0 \le v \le 4$ by using only $1 \le i \le 5$. The error incurred by ignoring the terms for $i \ge 6$ is less than $0.0045767767\ldots$. This justifies the use of $1 \le i \le 30$ in (10) for numerical computations.

iii) For $v < 0$, we have

$$\ln F_0(v) \le \ln F_0(0) = -1.202264688\ldots$$

iv) For $v \ge 4$, we can practically limit the sum in (10) to $\max\{\lfloor v\rfloor - 13,\ 1\} \le i \le \lfloor v\rfloor + 2$, which entails an asymptotic periodicity. The error incurred by ignoring the other values of i is less than $1.31\ldots 10^{-4}$. A plot of $4 \le \ln F_0(v) \le 12$ is given in Figure 6, showing the asymptotic periodicity for large v. We could analyze the periodicity in detail with Mellin transforms (see, for instance Flajolet et al. [Flajolet et al. 1995], or Szpankowski [Szpankowski 2001]) but we will not pursue this matter further here.

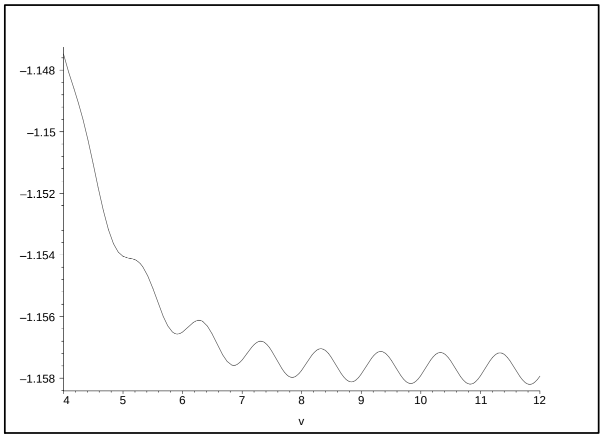


Fig. 6. $\ln F_0(v)$ for $4 \leq v \leq 12$

v) We conclude that the maximum (unique or not) occurs for some v with $0 \leq v \leq 1$.

vi) A similar analysis shows that $\ln F_0(v)$ is concave down for $0 \leq v \leq 1$, proving the unicity. A plot of $\partial_v^2 \ln F_0(v)$ is given in Figure 7.

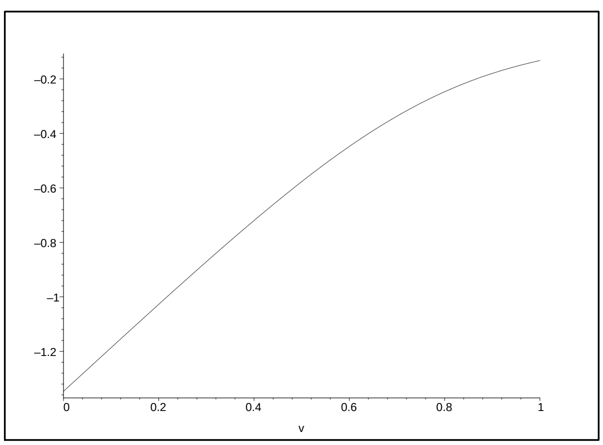


Fig. 7. $\partial_v^2 \ln F_0(v)$ for $0 \leq v \leq 1$

vii) Finally, the effect of $\frac{F_1(v)q}{pm}$, for large m, does not destroy the maximum unicity.

5. IMPROVED PROBABILISTIC MODEL

We now let p and q depend on n, where our problem stays as before. We did explore the possibility $q(n) = e^{-C/n}$ where C is constant, but this ultimately led to an absurdity, so we do not discuss the case $q(n) = e^{-C/n}$ here. Instead, we note that the absurdity can be removed by using $q(n) = e^{-C\delta(n)}$, by carefully choosing $\delta(n)$. For simplicity, let $C = 1$. We change the scale into

$$K = \varphi_0(m) + \varphi_1(m)v,$$

and we want $m\pi = mp(m)q(m)^{K-1} = e^{-v}$, where m again denotes the average of the number of bids N. Restricting attention first to the dominant term yields

$$m\pi \sim m\delta(m)e^{-\delta(m)[\varphi_0(m)+\varphi_1(m)v]}. \tag{11}$$

So we choose $\varphi_1(m) = \frac{1}{\delta(m)}$ and $m\delta(m)e^{-\delta(m)\varphi_0(m)} = 1$. For instance, if $\delta(m) = 1/m^\kappa$ for $\kappa < 1$, then we have $m^{1-\kappa}e^{-\varphi_0(m)/m^\kappa} = 1$, or equivalently $\varphi_0(m) = (1-\kappa)m^\kappa \ln(m)$; also $\varphi_1(m) = m^\kappa \ll \varphi_0(m)$ and $m^\kappa \ll m$. Now (11) leads to e^{-v}, as expected.

Therefore, throughout this section, we consider the model in which the bids are placed independently, each with GEOM($p(n)$) distribution, with $q(n) = e^{-1/n^\kappa}$ and $p(n) = 1 - q(n)$, for fixed $\kappa < 1$. So the probability that a given participant bids i is $\pi_i := p(n)q(n)^{i-1}$. If a total of k bids are placed, each with GEOM($p(n)$) distribution, then we define $X_i(k, n) = 1$ if bid i appears exactly once among the k bids (i.e., bid i is unique), and $X_i(k, n) = 0$ otherwise. Let $X(k, n) = \sum_{i=1}^{\infty} X_i(k, n)$. So $X(k, n)$ denotes the number of unique bids.

Theorem 5.1. *Consider an auction in which m and σ^2 are the mean and the variance of the number of bids placed on average. If n independent bids are placed, each with GEOM($p(n)$) distribution, where $q(n) = e^{-1/n^\kappa}$ and $p(n) = 1-q(n)$. The average number of unique bids is $\mathbb{E}(X(n, n)) = n^\kappa + O(n^{-1})$. The second moment is $\mathbb{E}(X^2(n, n)) = n^{2\kappa} + \frac{3}{4}n^\kappa + O(n^{-1})$.*

In the proof of Theorem 5.1, we will utilize Jacquet and Szpankowski's the Diagonal dePoissonization Theorem from [Jacquet and Szpankowski 1998]:

Diagonal DePoissonization Lemma. (Jacquet and Szpankowski, [Jacquet and Szpankowski 1998])

Let $\widetilde{F}_n(\tau)$ be a sequence of Poisson transforms of $f_{k,n}$ which is assumed to be a sequence of entire functions of τ. Consider a polynomial cone $\mathcal{C} := \{\tau = x + iy : |y| \le x\}$. Let the following two conditions hold for some $A > 0$, B, and $\alpha > 0$, β, and γ:

(I) For $\tau \in \mathcal{C}$ and $|\tau| \le 2n$,

$$|\widetilde{F}_n(\tau)| \le Bn^\beta\,,$$

(O) For $\tau \notin \mathcal{C}$ and $|\tau| = n$,

$$|\widetilde{F}_n(\tau)e^\tau| \le n^\gamma \exp(n - An^\alpha)\,.$$

Then, for large n,

$$f_{n,n} = \widetilde{F}_n(n) + O(n^{\beta-1})$$

and more generally, for every nonnegative integer m,

$$f_{n,n} = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \widetilde{F}_n^{\langle j \rangle}(n) + O(n^{\beta-m-1})$$

where $\widetilde{F}_n^{\langle j \rangle}(n)$ denotes the jth derivative of $\widetilde{F}_n(\tau)$ at $\tau = n$, and where the b_{ij} are defined by $B_j(x) = \sum_i b_{ij} x^i$ and $B_j(x) = [y^j](e^{-xy}(1+y)^x)$. (The relation of the coefficients b_{ij} to Poisson-Charlier polynomials and the Laguerre polynomials is also described briefly in [Jacquet and Szpankowski 1998].)

Proof (of Theorem 5.1) To obtain the asymptotics of $\mathbb{E}(X(n,n))$, we consider a model with a Poisson number N_τ of bids, where N_τ has mean τ. We write $\widetilde{X}_i(\tau,n) = 1$ if bid i appears exactly once among the N_τ bids (i.e., bid i is unique), and $\widetilde{X}_i(\tau,n) = 0$ otherwise. Let $\widetilde{X}(\tau,n) = \sum_{i=1}^{\infty} \widetilde{X}_i(\tau,n)$. So $\widetilde{X}(k,n)$ denotes the number of unique bids, when a total of N_τ bids are placed.

We let $\widetilde{M}_n(\tau) := \mathbb{E}(\widetilde{X}(\tau,n))$ and $\widetilde{U}_n(\tau) := \mathbb{E}(\widetilde{X}^2(\tau,n))$ denote the first and second moments of $\widetilde{X}(\tau,n)$, so

$$\widetilde{M}_n(\tau) = \sum_{k=0}^{\infty} \frac{e^{-\tau}\tau^k}{k!} \mathbb{E}(X(k,n)) = \sum_{i=1}^{\infty} \tau\pi_i e^{-\tau\pi_i},$$

$$\widetilde{U}_n(\tau) = \sum_{k=0}^{\infty} \frac{e^{-\tau}\tau^k}{k!} \mathbb{E}(X^2(k,n)) = \sum_{i=1}^{\infty} \tau\pi_i e^{-\tau\pi_i} \Big(1 + \sum_{j\neq i} \tau\pi(j) e^{-\tau\pi(j)}\Big). \quad (12)$$

Since $\mathbb{E}(X(n,n))$ satisfies conditions (I) and (O) the Diagonal dePoissonization Lemma (with $\beta = 1$) for $\widetilde{F}_n(\tau) = \widetilde{M}_n(\tau)$, then, for every nonnegative integer m,

$$\mathbb{E}(X(n,n)) = \sum_{i=0}^{m} \sum_{j=0}^{i+m} b_{ij} n^i \, \partial_\tau^j \mathbb{E}(\widetilde{X}(\tau,n)) \Big|_{\tau=n} + O(n^{-m}).$$

In particular, when $m = 1$,

$$\mathbb{E}(X(n,n)) = \mathbb{E}(\widetilde{X}(n,n)) - \frac{1}{2} n \widetilde{M}_n''(n) + \frac{1}{3} n \widetilde{M}_n'''(n) + O(n^{-1}). \quad (13)$$

We note that $\widetilde{M}_n^{\langle j \rangle}(n) = \sum_{i=1}^{\infty} (-\pi_i)^j e^{-n\pi_i}(n\pi_i - j)$. Each of the correction terms in (13) decay exponentially in terms of n. In particular, the $\frac{1}{3} n \widetilde{M}_n'''(n)$ term above is not only $O(n^{-1})$ but in fact is exponentially small in terms of n. Thus (13) simplifies to

$$\mathbb{E}(X(n,n)) = \mathbb{E}(\widetilde{X}(n,n)) - \frac{1}{2} n \widetilde{M}_n''(n) + \cdots . \quad (14)$$

We use Euler-Maclaurin summation to compute

$$\mathbb{E}(\widetilde{X}(n,n)) = \sum_{i=1}^{\infty} n\pi_i e^{-n\pi_i} = \int_1^{\infty} n\pi_i e^{-n\pi_i} \, di - \frac{1}{2} n\pi_i e^{-n\pi_i} \Big|_{i=1}^{\infty} + \cdots . \quad (15)$$

The integral evaluates to

$$n^\kappa (1 - \exp(n(e^{-1/n^\kappa} - 1))) = n^\kappa - n^\kappa \exp(-n^{1-\kappa}) + \cdots ;$$

also, the lower-order terms, and all correction terms, decay exponentially in terms of n.

Thus, the expectation in the Poisson model is

$$\mathbb{E}(\widetilde{X}(n,n)) \sim n^{\kappa}. \tag{16}$$

Next, we use Euler-Maclaurin summation to compute the correction between $\mathbb{E}(X(n,n))$ and $\mathbb{E}(\widetilde{X}(n,n))$, namely

$$-\frac{1}{2}n\widetilde{M}_n''(n) \sim -\frac{1}{2}n\int_1^{\infty}(-\pi_i)^2 e^{-n\pi_i}(n\pi_i - 2)\,di = \Theta(n^{1-\kappa}\exp(-n^{1-\kappa})),$$

and thus decays exponentially in terms of n. Thus, the expectation in the dependent model is

$$\mathbb{E}(X(n,n)) \sim n^{\kappa}.$$

In summary, $\mathbb{E}(X(n,n))$ and $\mathbb{E}(\widetilde{X}(n,n))$ are each asymptotically n^{κ}, and the difference between $\mathbb{E}(X(n,n))$ and $\mathbb{E}(\widetilde{X}(n,n))$ is at most $O(n^{-1})$.

Checking conditions (I) and (O) of the Diagonal dePoissonization Lemma for $\widetilde{F}_n(\tau) = \widetilde{U}_n(\tau)$ and $f_{k,n} = \mathbb{E}(X^2(k,n))$, we see that Since (I) and (O) are both satisfied (with $\beta = 2$). Thus, for every nonnegative integer m,

$$\mathbb{E}(X^2(n,n)) = \sum_{i=0}^{m}\sum_{j=0}^{i+m} b_{ij}n^i\, \partial_\tau^j \mathbb{E}(\widetilde{X}^2(\tau,n))\Big|_{\tau=n} + O(n^{1-m}).$$

In particular, when $m = 2$,

$$\mathbb{E}(X^2(n,n)) = \mathbb{E}(\widetilde{X}^2(n,n)) - \frac{1}{2}n\widetilde{U}_n''(n) + \frac{1}{3}n\widetilde{U}_n'''(n) + \frac{1}{8}n^2\widetilde{U}_n''''(n) + O(n^{-1}). \tag{17}$$

By similar reasoning to the average case analysis, the second moment in the Poisson model is

$$\mathbb{E}(\widetilde{X}^2(n,n)) \sim n^{2\kappa} + \frac{3}{4}n^{\kappa}, \tag{18}$$

and the error terms have exponential decay in terms of n. Also, similar to the average case, we use Euler-Maclaurin summation obtain the second moment in the dependent model:

$$\mathbb{E}(X^2(n,n)) = n^{2\kappa} + \frac{3}{4}n^{\kappa} + O(n^{-1}).$$

In particular, $\mathbb{E}(X^2(n,n))$ and $\mathbb{E}(\widetilde{X}^2(n,n))$ each are asymptotically $n^{2\kappa} + \frac{3}{4}n^{\kappa}$. The difference between $\mathbb{E}(X^2(n,n))$ and $\mathbb{E}(\widetilde{X}^2(n,n))$ is at most $O(n^{-1})$. ■

5.1 Location of the optimum bid

Now we consider the probability $P_2(K)$ of winning when placing bid K, or equivalently, the probability that bid K is unique and that there is no lower unique bid. As in Theorem 5.1, m and σ^2 are the mean and the variance of the number of placed bids . A total of n bids (not including our own bid) are independently placed, each with GEOM($p(n)$) distribution, with $q(n) = e^{-1/n^\kappa}$ and $p(n) = 1 - q(n)$, for fixed $\kappa < 1$. So the probability that a given participant bids i is $\pi_i := p(n)q(n)^{i-1}$. As above, we recall that v is a rescaling of K, by using $K = \varphi_0(m) + \varphi_1(m)v$, where $\varphi_0(m) = (1-\kappa)m^\kappa \ln(m)$ and $\varphi_1(m) = m^\kappa \ll \varphi_0(m)$.

EXAMPLE 5.1. *To illustrate the location of the optimum bid, consider the plot of $P_2(v)$ using (6), with $\kappa = 1/2$ and $m = 1000$, is given in Figure 8. This leads to $v_n = -1.24\ldots$ and $P_2(v_n) = 0.0117\ldots$. In our derivations below, we also numerically obtain $\tilde{v} = -1.24\ldots$, $\tilde{P}_2 \sim 0.01164\ldots$. This corresponds to the bid of $\tilde{K} = 70$.*

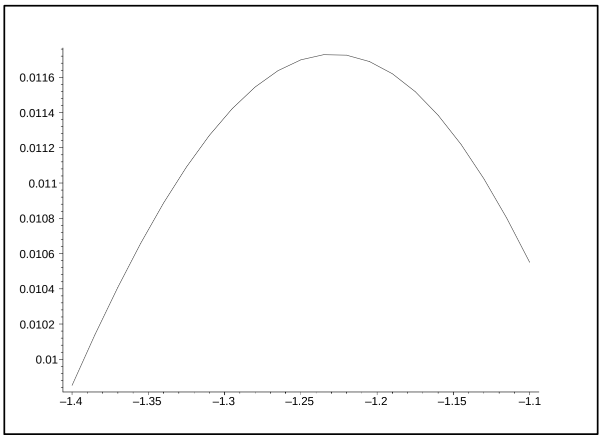


Fig. 8. $P_2(v)$ for the case $q(n) = e^{-1/n^\kappa}$. Here the maximum is taken for $\tilde{K}$ around 70.

THEOREM 5.2. *In general, consider the probability $P_2(K)$ of winning when placing bid K, or equivalently, the probability that bid K is unique and that there is no lower unique bid. As in Theorem 5.1, m bids are placed on average. A total of n bids (not including our own bid) are independently placed, each with GEOM($p(n)$) distribution, with $q(n) = e^{-1/n^\kappa}$ and $p(n) = 1-q(n)$, for fixed $\kappa < 1$. So the probability that a given participant bids i is $\pi_i := p(n)q(n)^{i-1}$. As above, we recall that v is a rescaling of K, by using $K = \varphi_0(m)+\varphi_1(m)v$, where $\varphi_0(m) = (1-\kappa)m^\kappa \ln(m)$ and $\varphi_1(m) = m^\kappa \ll \varphi_0(m)$.*

Then we obtain

$$\tilde{P}_2 \sim \frac{1}{m^\kappa} e^{-1}. \tag{19}$$

Proof Limiting ourselves to the dominant term, we have

$$P_2(v) \sim \exp\left[-e^{-v}\right] \prod_{i=1}^{\infty} \left\{1 - \exp[-e^{-(v-i/m^\kappa)}]e^{-(v-i/m^\kappa)}\right\}. \tag{20}$$

Numerical experiments show that we must take $v \ll 0$. This entails $e^{-(v-i/m^\kappa)} \gg 1$ and hence $\exp\left[-e^{-(v-i/m^\kappa)}\right] \ll 1$; also

$$\ln\left\{1 - \exp[-e^{-(v-i/m^\kappa)}]e^{-(v-i/m^\kappa)}\right\} \sim -\exp\left[-e^{-(v-i/m^\kappa)}\right] e^{-(v-i/m^\kappa)}.$$

Using Euler-Maclaurin and setting $(v - i/m^\kappa) = \eta$, yields

$$P_2(v) \sim \exp\left[-e^{-v}\right] \exp\left[-m^\kappa \int_{-\infty}^{v} \exp\left[-e^{-\eta}\right] e^{-\eta} d\eta\right]. \tag{21}$$

By taking logarithms, maximizing $P_2(v)$ leads to

$$e^{-\tilde{v}} - m^\kappa \exp[-e^{-\tilde{v}}] e^{-\tilde{v}} = 0,$$

or $\tilde{v} = -\ln[\ln(m^\kappa)] \ll 0$, as expected. Now we want $\tilde{P}_2 = P_2(\tilde{v})$. Set $\eta = \tilde{v} - \tau$. From (21) we obtain

$$-m^\kappa \int_{-\infty}^{\tilde{v}} \exp\left[-e^{-\eta}\right] e^{-\eta} d\eta = -m^\kappa \int_{\tau=0}^{\infty} \exp[-e^{-\tilde{v}+\tau}] e^{-\tilde{v}+\tau} d\tau.$$

Using $e^\tau = u$ yields

$$-m^\kappa \int_0^\infty \exp\left[-\ln(m^\kappa) e^\tau\right] \ln(m^\kappa) e^\tau d\tau = -m^\kappa \int_{u=1}^{\infty} \ln(m^\kappa) \exp(-\ln(m^\kappa) u) du,$$

and hence, after a straightforward computation $-m^\kappa \exp(-\ln(m^\kappa)) = -1$. So, finally

$$\tilde{P}_2 \sim \frac{1}{m^\kappa} e^{-1}. \tag{22}$$

Remark: We have tried to understand whether the factor $1/e$ has a simple explanation, but we are not able to find an analogy of the problem to a problem of sequential selection. We also note $|\tilde{v}| \ll \varphi_1(m)$ and $\tilde{K} = \varphi_0(m) + \varphi_1(m)\tilde{v} = m^\kappa[(1-\kappa)\ln(m) - \ln(\ln(m)) - \ln(\kappa)] \gg 1$. ■

A correction to the dominant term can be computed. We derive, with $\varepsilon = u/m$,

$$\begin{aligned} P_0(v) \sim \exp[-e^{-v}] \Bigg[& 1 - \frac{e^{-v}}{2m^\kappa} - e^{-v}\varepsilon[1 - \kappa + \kappa v] \\ & + e^{-v}\frac{\varepsilon^2}{2}\big[3\kappa^2 v - \kappa v - \kappa^2 - \kappa^2 v^2 \\ & + \kappa + e^{-v} + 2e^{-v}\kappa v - 2e^{-v}\kappa + e^{-v}\kappa^2 v^2 - 2e^{-v}\kappa^2 v + e^{-v}\kappa^2\big] \\ & + \mathcal{O}\left(\frac{1}{m^{2\kappa}}\right) + \mathcal{O}\left(\frac{1}{m}\right) + \mathcal{O}\left(\frac{\varepsilon}{m}\right)\Bigg]. \end{aligned}$$

We see here that *the dominant term is related to the* $\frac{1}{m^\kappa}$ *term.* This is *independent from* $\beta = \frac{\sigma^2 p}{mq}$. Similarly, we obtain (up to the required precision) $(1-\pi)^{n-1} \sim (1-\pi)^n$, and

$$n\pi \sim e^{-v}\left[1 + \frac{1}{m^{2\kappa}} + \varepsilon[1 - \kappa + \kappa v] - \kappa\frac{\varepsilon^2}{2}[-1 + \kappa v^2 + v - 3\kappa v + \kappa]\right].$$

So, finally, the dominant term bracketed in $P_2(v)$ (see (20)) becomes

$$\left\{\prod_{i=1}^{\infty}\left(1 - \exp\left[-e^{-(v-i/m^\kappa)}\right]\left[1 - \frac{e^{-(v-i/m^\kappa)}}{2m^\kappa}\right] e^{-(v-i/m^\kappa)}\left[1 + \frac{1}{2m^\kappa}\right]\right)\right\}. \tag{23}$$

and

$$P_2(v) \sim \exp\left[-e^{-v}\right]\left[1-\frac{e^{-v}}{2m^\kappa}\right] \times \left\{\prod_{i=1}^{\infty}\left(1-\exp\left[-e^{-(v-i/m^\kappa)}\right]\left[1-\frac{e^{-(v-i/m^\kappa)}}{2m^\kappa}\right]e^{-(v-i/m^\kappa)}\left[1+\frac{1}{2m^\kappa}\right]\right)\right\} \tag{24}$$

We must now compute the solution $\bar{v}$ of $\partial_v \ln(P_2(v)) = 0$. Proceeding as above, this leads to

$$1+\frac{1}{2m^\kappa} - m^\kappa \exp\left[-e^{-\bar{v}}\right]\left[1-\frac{e^{-\bar{v}}}{2m^\kappa}\right]\left[1+\frac{1}{2m^\kappa}\right] = 0. \tag{25}$$

Taking logarithms in (23) only induces an extra term of order $\exp[-2e^{-\tilde{v}}]e^{-2\tilde{v}} = \mathcal{O}\left(\frac{\ln(m^\kappa)^2}{m^{2\kappa}}\right)$ which does not affect (25). Set therefore $e^{-\bar{v}} = \ln(m^\kappa) + \eta$. This gives the dominant equation $1+\frac{1}{2m^\kappa} - e^{-\eta}\left[1-\frac{\ln(m^\kappa)+\eta}{2m^\kappa}\right]\left[1+\frac{1}{2m^\kappa}\right] = 0$, which is asymptotically equivalent to

$$1+\frac{1}{2m^\kappa} - (1-\eta)\left[1-\frac{\ln(m^\kappa)}{2m^\kappa}\right]\left[1+\frac{1}{2m^\kappa}\right] = 0.$$

Therefore $\eta \sim -\frac{\ln(m^\kappa)}{2m^\kappa}$, i.e. $e^{-\bar{v}} \sim \ln(m^\kappa)\left[1-\frac{1}{2m^\kappa}\right]$. Equivalently,

$$\bar{v} \sim -\ln[\ln(m^\kappa)] + \frac{1}{2m^\kappa} = \tilde{v} + \frac{1}{2m^\kappa}.$$

So we conclude that

$$\bar{K} \sim \tilde{K} + 1/2.$$

As K must be an integer, we see that the correction is practically negligible with our choice of parameters. To summarize, the algorithm works as follows.

Algorithm 2

Input: C, κ, m
Output: second order optimal value for $K : \bar{K}$
Compute $\tilde{v} = -\ln[\ln(m^\kappa)]$;
Compute $\tilde{K} = \varphi_0(m) + \varphi_1(m)\tilde{v}$;
Compute $\bar{K} = \lfloor \tilde{K} + 1/2 \rfloor$;
End

6. CONCLUSION

The problem studied in this paper may be seen as a game where an unknown number of players participate. Each player can choose infinitely many actions, in the sense that he/she can choose a distribution in an (uncountable) set of distributions according to which to place his/her offer. The only constraint is that the support of the chosen distribution must be on the set of integers between 1 and V. Hence, the only approach to such a problem is to assume that individual strategic behaviour (to maximize the probability of placing the minimum single offer) leads to a common distribution for all players. We gave several good reasons why a (truncated) geometric distribution should model the situation more suitably than other choices, although clearly our arguments depend more on exclusion of unreasonable distributions than on actual preferences. Our next step was to assume some knowledge about $\mathbb{E}(N)$ and the variance of N, because, as we argued, with no information on N whatsoever, we still would have an ill-posed problem.

With these assumptions, the problem is sufficiently well defined to allow the search for an optimum. In practice, finding the optimum is seemingly only possible via asymptotic expansions and algorithms for which we can give, as we have seen, explicit answers. We conclude that there is no good rule of thumb for the optimal choice, i.e., that is there is no easy answer without calculation. On the other hand, in many cases, as exemplified in Section 6.1, the effort is clearly rewarding. The optimum is frequently distinguishably better than a random choice in regions we may think of as being reasonable.

The algorithms we have given in this paper are very simple, because our whole setup is so far confined to deal with the problem in which we obtain no further information during the auction. This is a *non-sequential* setup. As soon as we have *some* sequential information about bids and numbers of bidders—and this is already the case in some versions on the internet—we can update the information and thus are confronted with a sequential problem. But the structure is then the same. Our algorithms indicate the place where the new information is needed. Most sequential problems are generally very hard, and we see no promising access to such problems other than by algorithms.

A final word. The general difficulty of the problem has one game-specific advantage. Since these results would probably be perceived as "too mathematical" by the large majority of participants, we do not expect any serious danger of distortion by publishing them.

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