A Multi-Curve HJM Factor model for pricing and risk management

Tobias Bienek†, Griselda Deelstra‡, Andreas Lichtenstern† and Rudi Zagst†

†Department of Mathematics, Technical University of Munich, Munich, Germany
‡Department of Mathematics, Université libre de Bruxelles, Brussels, Belgium

Abstract

In this paper, we introduce a multi-curve model under the historical probability based upon multiplicative relative spreads, inspired by the HJM and affine factor approaches, which implies positive and ordered spreads. In particular, we focus upon δ_i-XIBOR relative (instantaneous) forward rates and appropriate XIBOR HJM drift constraints, and we describe the dynamics of the different forward rates and spreads under different measure changes (including forward measures). We introduce an explicit model satisfying both the XIBOR HJM drift constraints as well as the property of positive and ordered spreads. We demonstrate the flexibility of this model for derivative pricing by focusing upon the price of a caplet and of options with a payoff based upon XIBOR forward prices with different tenors. We perform on one hand a calibration of the model based upon cap prices. On the other hand, we do an estimation of a spread curve in our proposed model under the historical probability by using a Kalman filter approach. Numerical results are included, and they confirm that the model performs very well.

1 Introduction

The crisis that hit the financial markets in 2008 had an enormeous influence upon interest-rate modelling. Since August 2007, spreads between interbank offered rates, like e.g. LIBOR and EURIBOR rates, and overnight-indexed swap (OIS) rates have been significant, see e.g. Figure 1 which presents historical spreads between EURIBOR forward rates and EONIA OIS rates for different tenors from Jan. 2006 to Feb. 2011. In particular, the assumption of a single interest-rate curve that could be used both for discounting and for generating future cash flows does no longer hold and leads to arbitrage opportunities. This led immediately after the financial crisis to the introduction of the class of multiple-curve interest-rate models, taking into account the fact that interbank "long-term" rates (typically one, three or six months) are riskier than short-term rates (typically one day). In the following, the interbank rates will be refered to as XIBOR rates (see e.g. [Bianchetti, 2010], [Gallitschke et al., 2017] and [Cuchiero et al., 2019]) and in Section 2, we will introduce the related notations. In the literature, there exist different modeling approaches of both the rates and the spreads. The spread can be defined as an additive spread such as in e.g. [Mercurio, 2010] or as a multiplicative spread such as first proposed by [Henrard, 2007]. When one first models the OIS and XIBOR rates simultaneously, one easily obtains tractable pricing formulas, but it is more difficult to guarantee the positivity of the spread.

Figure 1 illustrates that historical spreads are (mostly) positive, are correlated and that they are ordered in function of the tenors. In this paper, we will present a multiple-curve model starting from a Heath-Jarrow-Morton (HJM) factor model, based on multiplicative relative forward spreads and affine processes, which satisfies these constraints, and which can be used both for risk management and for pricing, since we will model it under both the real-world measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$.
The recent literature on multi-curve models has grown very rapidly since the financial crisis. Different modeling approaches can be distinguished in multi-curve modeling, such as short-term interest-rate models, Heath-Jarrow-Morton (HJM) models, LIBOR market models (LMM) and pricing kernel models. We refer to [Bianchetti and Morini, 2013], [Grbac and Runggaldier, 2015], [Henrard, 2014],

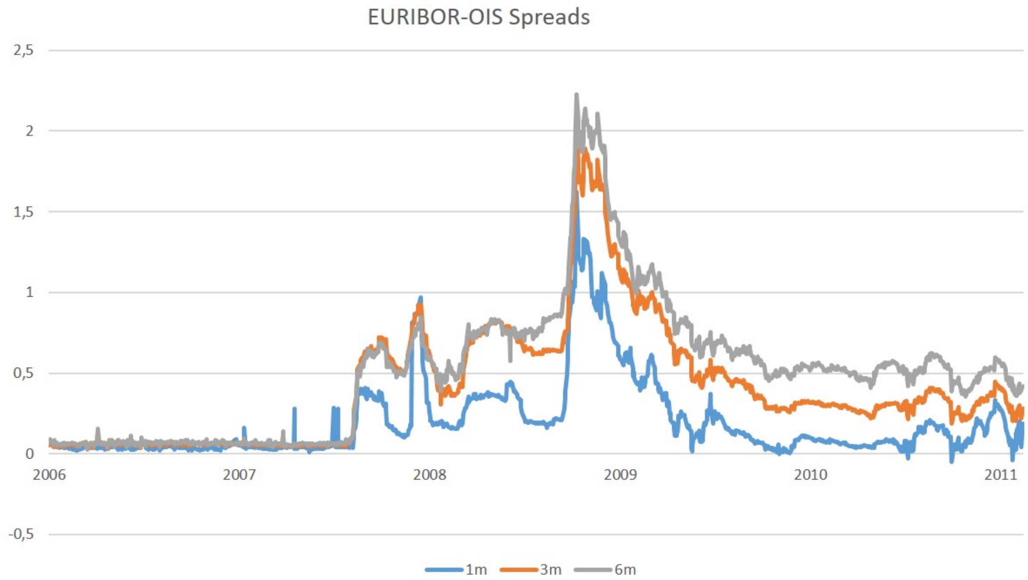

Figure 1: Historical EURIBOR-OIS spreads in % for different tenors from Jan. 2006 to Feb. 2011. The EURIBOR forward rates are the 1m-, 3m-, and 6m-EURIBOR (Bloomberg ticker EUR001M INDEX, EUR003M INDEX, EUR006M INDEX) and the OIS rates are the 1m-, 3m-, and 6m-EONIA OIS rates (Bloomberg ticker EUSWEA, EUSWEC, EUSWEF). The plotted spreads are the differences between the corresponding xm-EURIBOR and the xm-EONIA OIS rate for x = 1m, 3m and 6m.

[Cuchiero et al., 2016], [Macrina and Mahomed, 2018] and [Alfeus et al., 2020] for an overview of the literature on multi-curve models. We mention here only an incomplete list of papers closely related to our study. As already mentioned, multiplicative spreads for modeling multiple curves have been first considered in [Henrard, 2007]. In [Grbac et al., 2015] the dynamics of OIS and LIBOR rates are specified according to the methodology of affine LIBOR models and imply positive spreads. This non-negativity of the spreads is automatically ensured by using the framework of the affine LIBOR models proposed by [Keller-Ressel et al., 2013]. [Cuchiero et al., 2016] provide an HJM approach in a general semimartingale setting to model the term structure of multiplicative spreads between FRA rates and simply compounded OIS risk-free forward rates and they show how to construct models such that multiplicative spreads are greater than one and ordered with respect to the tenor's length. [Zhong, 2018] incorporates the LMM with a multiplicative basis which can be considered as a continuously compounding version of the excess return of the forward LIBOR rate over the OIS forward rate for the reset period as modeled in [Henrard, 2010]. [Cuchiero et al., 2019] models a general numéraire process and multiplicative spreads between XIBOR rates and simply-compounded OIS rates as functions of an underlying affine process. This model ensures ordered positive spreads and an exact fit to the initially observed term structures. Their general framework leads to tractable valuation formulas for caplets and swaptions and embeds all existing multi-curve affine models. Following [Grbac and Runggaldier, 2015] and [Miglietta, 2015], [Konikov and McClelland, 2020] focus upon fictitious processes playing the role of the spread between the instantaneous forward δ_i-XIBOR curve over the δ_{i-1}-XIBOR curve or between the δ_1-XIBOR curve and the OIS curve, which we call δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t, T)$ in this paper (see Definition 4). Whereas [Grbac and Runggaldier, 2015] and [Miglietta, 2015] used level-independent volatility functions for (Gaussian) spread processes, [Konikov and McClelland, 2020] propose level-dependent volatility functions as a mean for imposing lower bounds. They state no-arbitrage drift restrictions for the spread curve processes in their settings and are able to solve these drift restrictions in an explicit way. They further specify a multi-curve model in the spirit of [Cheyette, 1996] and derive a Markov representation of the relative forward spreads with two state variables which can both be expressed as (stochastic) integrals of the relative short spreads. Among other results, they derive swaption prices and present a practical calibration strategy of multi-curve models based upon a mix of historical and implied data. The main difference between our paper and [Konikov and McClelland, 2020] lies in the fact that we specify the δ_i-XIBOR relative (instantaneous) forward rates in our explicit model as affine functions of independent Cox-Ingersoll-Ross (CIR) factor processes. The factor structure allows us to significantly simplify the conditions to ensure positive and ordered relative spreads as well as the pricing and fitting to market data.

In this paper, we consider a filtered probability space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_t,\ \mathbb{P})$ where $\mathbb{P}$ is the historical probability, also called the physical or real-world measure. Notice that we start describing the multi-curve interest-rate model under the historical probability $\mathbb{P}$, whereas most papers start studying the model under the risk-neutral probability $\mathbb{Q}$. We concentrate upon a model with multiplicative relative spreads, inspired by the HJM and affine factor approaches, which allows for positive and ordered spreads, as well as tractable valuation formulas for some derivatives with optionality features. We will introduce the notion of *δ_i-positive multi-curve models* which summarizes the properties of positive spreads and ordered relative spreads. We start by formulating an HJM model under $\mathbb{P}$ and $\mathbb{Q}$ for both the OIS and XIBOR rates. For the latter, we will focus upon δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ as in [Konikov and McClelland, 2020]. We derive appropriate XIBOR HJM drift constraints under the measure $\mathbb{P}$ in the presence of correlations and describe the dynamics of the different forward rates and (relative) spreads under different measure changes (including forward measures). In particular, we introduce the probability measure under which the relative spreads are martingales. We further propose equivalent properties to check the δ_i-positivity, and as a result we explain why level-independent volatilities cannot lead to ordered spreads. Armed with these insights, we specify an explicit δ_i-positive model (under $\mathbb{P}$ and $\mathbb{Q}$) based upon affine spread factor processes, which satisfies the XIBOR HJM drift constraint as well as the δ_i-positivity constraints. Therefore, we use a Hull-White model for the OIS short rates since the OIS rate can turn negative, and we assume a constant market risk parameter for the OIS rates for defining $\mathbb{Q}$. On the other hand, we base the δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ as affine functions upon independent Cox-Ingersoll-Ross (CIR) factor processes, which are assumed to be independent from the OIS rates as well. Since solutions to CIR processes remain positive and $f_{i/i-1}(t,T)$ can be interpreted as related to spreads, this choice facilitates to satisfy the δ_i-positivity. Moreover, the δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ have a time-dependent mean-reversion and level-dependent volatilities. Surprisingly, the data allowed us to assume the same dynamics for the δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ under $\mathbb{P}$ and $\mathbb{Q}$, an observation that merits to be studied in more detail in future research. This feature simplifies the XIBOR HJM drift constraints in a very practical way. Besides, by the assumed independence of the OIS rates and the XIBOR relative forward rates, the dynamics of the δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ remain unchanged under the forward measures or the annuity measure (to price swaptions). In this explicit model, it is easy to price linear financial products as well as interest-rate derivatives with optionality features. As an example, we derive the price of a caplet and of options with a payoff based upon XIBOR forward prices with different tenors[1].

We note that our model can be recovered as a special case of the all-encompassing paper [Cuchiero et al., 2019], in particular it is an affine short rate multi-curve model. The δ_i-positivity of our model follows by construction and it fulfills their general Proposition 3.7. Our modeling startpoint of relative spreads and δ_i-XIBOR relative (instantaneous) forward rates in a multiplicative multi-curve framework leads, however, in our particular settings to very intuitive proofs, semi-explicit pricing formulae, easy implementation, and excellent estimation results. Since one of the main contributions of this paper is to focus upon a joint modeling under $\mathbb{P}$ and $\mathbb{Q}$, we detail estimation methods under both measures. We focus on one hand upon the calibration of a cap, and on the other hand, we do an estimation of a spread curve in our proposed

[1] Following the lines of [Konikov and McClelland, 2020] and the related SSRN-id6073 version, approximate quasi-explicit expressions for swaption prices can also be derived using some usual "freezing" methods and Fourier inversion techniques.

model under the historical probability by using a Kalman filter approach, see e.g. [Chatterjee, 2005] and [Filipovic and Trolle, 2013].

Multi-curve research papers usually focus on the risk-neutral $\mathbb{Q}$-measure because of its role in pricing. However from the point of risk management, which has gained a lot in importance since the financial crisis, a joint $\mathbb{P}$ and $\mathbb{Q}$ modeling is important. We refer for a review of research concerned with joint $\mathbb{P}$ and $\mathbb{Q}$ measures from a finance-oriented point of view to the papers of e.g. [Hull et al., 2014], [Steinrücke et al., 2015] or [Stein, 2016], and from an insurance-oriented point of view to the papers of e.g. [Van Dijk et al., 2018], [Diez and Korn, 2020] and [Berninger and Pfeiffer, 2021]. We summarize here only some of the motivations.

As an example in credit risk, it is well-known that the credit valuation adjustment (CVA) should be calculated under the $\mathbb{Q}$-measure, because it is concerned with valuation, whereas exposure at default (EAD) and potential future exposure (PFE) are concerned with scenario analysis and should be calculated under the $\mathbb{P}$-measure. As another example, we refer to the derivation of the solvency capital by using an internal model in the framework of Solvency II, where real-world simulations over a one-year horizon are performed, which are then for valuation purposes typically combined with risk-neutral pricing (by using $\mathbb{Q}$) given the real-world realizations under $\mathbb{P}$. Other examples of a joint $\mathbb{P}$ and $\mathbb{Q}$ modeling include models for inflation-linked bonds and an integrated market approach of stock and bond markets (see e.g. [Zagst et al., 2007]), defaultable bond pricing (see e.g. [Antes et al., 2008]), mortgage-backed securities (see e.g. [Kolbe and Zagst, 2007]), and credit derivative pricing with stochastic recovery (see e.g. [Höcht and Zagst, 2010]).

Finally, we want to underline that an approach for modelling multi-curve yield spreads, both under $\mathbb{P}$ and $\mathbb{Q}$, can be useful in different settings where spreads appear in a natural way. Examples include amongst others models in default risk, foreign-exchange risk, liquidity risk and inflation risk.

This paper is organized as follows. In Section 2, we start by defining the different rates that will be used to model a multiple yield-curve model, in particular the multiplicative XIBOR spreads and multiplicative relative XIBOR spreads, and the notion of δ_i-positivity. Next, a multi-curve HJM framework is introduced based upon δ_i-XIBOR relative forward curves $f_{i/i-1}(t,u)$, the appropriate HJM drift constraints are derived and the dynamics of the different forward prices and spreads are studied under different measure changes, including the forward-neutral probability. Finally, some equivalent properties to check the δ_i-positivity are obtained in this HJM framework. Section 3 is devoted to the detailed description of the dynamics of the multi-curve HJM factor model which satisfies the δ_i-positivity. Section 4 provides pricing formulae of caplets and options on XIBOR forward prices. Section 5 contains a calibration of the model under $\mathbb{Q}$. In Section 6, a Kalman filter approach is adapted for the estimation of the spread parameters under $\mathbb{P}$ and for filtering out the spread factor process. The last section concludes the paper. Appendices contain the proofs of the main results.

2 The Multi-Curve XIBOR Model

2.1 Definitions

We first introduce some notational conventions used throughout the paper. As mentioned above, we consider a filtered probability space $(\Omega,\, \mathcal{F},\, (\mathcal{F}_t)_t,\, \mathbb{P})$ where $\mathbb{P}$ is the historical probability, also called the physical or real-world measure. We fix $T > 0$ and some maturities $T_i = T + \delta_i$ $(i = 1, \ldots, N)$ with discrete tenors $0 < \delta_1 < \ldots < \delta_N$, where δ_i is typically $1, 3$ or 6 months. We assume that we are given (risk-free) discount bonds with maturities $\tau \in [0, T_N]$ and prices $P_0(t,\tau)$ at time $t \in [0,T]$, and a cash account with value $P_0(t)$.

The risk-neutral measure $\mathbb{Q}$ follows as usual from the martingale characterization of discounted bond prices $\{\frac{P_0(t,\tau)}{P_0(t)}\}_{t\in[0,\tau]}$. For some maturity $\tau \in [0, T_N]$, we further define the τ-forward measure $\mathbb{Q}^\tau$ by its (conditional) density with respect to the risk-neutral measure $\mathbb{Q}$:

$$\frac{d\mathbb{Q}^\tau}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} := \frac{P_0(t,\tau)}{P_0(t)}\frac{P_0(0)}{P_0(0,\tau)}. \tag{1}$$

We recall that δ_i***-OIS (simple) forward rates*** are defined for $t \in [0,T]$ as

$$L_i^D(t,T) := L^D(t,T,T_i) := \frac{1}{\delta_i}\left(\frac{P_0(t,T)}{P_0(t,T_i)} - 1\right), \tag{2}$$

and the δ_i***-OIS forward prices*** are determined by

$$P_i^D(t,T) := 1 + \delta_i \cdot L_i^D(t,T) = \frac{P_0(t,T)}{P_0(t,T_i)}. \tag{3}$$

It is well-known that $L_i^D(t,T)$ and $P_i^D(t,T)$ are $\mathbb{Q}^{T_i}$-martingales.

The δ_i***-XIBOR (simple) forward rates*** follow from the no-arbitrage condition of FRA contracts (see e.g. [Mercurio, 2010]) and can be determined as follows for $t \in [0,T]$:

$$L_i(t,T) := L(t,T,T_i) := \mathbb{E}_{\mathbb{Q}^{T_i}}\left[L(T,T_i) \,\Big|\, \mathcal{F}_t\right], \tag{4}$$

where $L(T,T_i)$ is the spot XIBOR rate with tenor δ_i at time T.
Furthermore, we focus upon the δ_i***-XIBOR forward prices***

$$P_i^L(t,T) := 1 + \delta_i \cdot L_i(t,T), \tag{5}$$

and we notice that also $L_i(t,T)$ and $P_i^L(t,T)$ are by definition $\mathbb{Q}^{T_i}$-martingales.
We further recall the definition of multiplicative ***XIBOR spreads*** $P_i^S(t,T)$ for $t \in [0,T]$, namely

$$P_i^L(t,T) := P_i^D(t,T) \cdot P_i^S(t,T), \tag{6}$$

and the fact that the spreads $P_i^S(t,T)$ are $\mathbb{Q}^T$-martingales (see e.g. [Grbac and Runggaldier, 2015] p. 133). We now introduce the definition of relative XIBOR spreads.

Definition 1 (Relative XIBOR spreads) *For time $t \in [0,T]$, we define multiplicative* ***relative XIBOR spreads*** *$P_{i/i-1}^S(t,T)$ through the 'telescope' relationship*

$$P_i^S(t,T) =: \prod_{j=1}^{i} P_{j/j-1}^S(t,T), \tag{7}$$

for $i \geq 1$.

Remark 1 *For $i = 2, \ldots, N$, the relative XIBOR spreads can be expressed in the following ways:*

$$P_{i/i-1}^S(t,T) = \frac{P_i^S(t,T)}{P_{i-1}^S(t,T)} = \frac{P_{i-1}^D(t,T)}{P_{i-1}^L(t,T)}\frac{P_i^L(t,T)}{P_i^D(t,T)} = \frac{P_{i/i-1}^L(t,T)}{P_{i/i-1}^D(t,T)} = \frac{P_i^L(t,T)}{P_{i-1}^L(t,T)}\frac{P_0(t,T_i)}{P_0(t,T_{i-1})} \tag{8}$$

where

$$P_{i/i-1}^L(t,T) = \frac{P_i^L(t,T)}{P_{i-1}^L(t,T)} \quad \textit{and} \quad P_{i/i-1}^D(t,T) = \frac{P_i^D(t,T)}{P_{i-1}^D(t,T)} \tag{9}$$

denote respectively the relative XIBOR forward prices and the relative OIS forward prices.

As mentioned in the introduction, empirical data shows that the additive spread between XIBOR rates and OIS rates is positive and increasing in tenor, which is equivalent with the facts that both the spreads $P_i^S(t,T)$ as well as the relative spreads $P_{i/i-1}^S(t,T)$ should be greater or equal than 1, which we will call δ_i-positivity.

Definition 2 (δ_i-positivity) *A multi-curve interest-rate model satisfies the property of δ_i-positivity if for all tenors $\delta_i = T_i - T$*

$$
\begin{aligned}
P_i^S(t,T) &\geq 1 \qquad (10)\\
P_{i/i-1}^S(t,T) &\geq 1 \qquad (11)
\end{aligned}
$$

for all $t \leq T$.

Remark 2 *It is easy to check that condition* (11) *implies the condition of* (10) *if $P_1^S(t,T) \geq 1$. We however prefer to state both conditions within this definition since we check the first condition* (10) *as a first step, and since in literature, different models only satisfy this first condition.*

2.2 The Multi-Curve HJM Framework

In the following, we develop an HJM-type multi-curve model and derive the relevant measure changes and price dynamics. In order to focus upon the main results, the proofs and some intermediate results can be found in Appendix A.

The model under $\mathbb{P}$

We introduce the OIS forward rates $f_0(t,T)$ with dynamics

$$df_0(t,T) = \mu_0(t,T)\,dt + \sigma_0(t,T)\,dW^{\mathbb{P}}(t), \qquad (12)$$

where $W^{\mathbb{P}}$ is an $(N+1)$-dimensional standard Brownian motion under $\mathbb{P}$, such that the (risk-free) OIS-bond prices follow from the OIS forward rates

$$P_0(t,T) = \exp\left(-\int_t^T f_0(t,u)\,du\right). \qquad (13)$$

The *(OIS) short-rate* $r(t)$ is obtained by $r(t) = f_0(t,t)$.
We further introduce XIBOR-bonds $P_i(t,T)$ (*pseudo discount bonds*) as in e.g. Section 3.2.2.1 of [Grbac and Runggaldie
by assuming that these fictitious objects are not traded assets and that

$$P_i^L(t,T) = \frac{P_i(t,T)}{P_i(t,T_i)}, \qquad (14)$$

in order to establish the natural relationship between bonds and (simple) forward rates as in (3). We also introduce the concept of relative XIBOR-bonds $P_{i/i-1}(t,T)$ defined by

$$P_{i/i-1}(t,T) = \frac{P_i(t,T)}{P_{i-1}(t,T)}. \qquad (15)$$

In the following definition we introduce δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ in order to capture more easily empirical observations with respect to different tenors.

Definition 3 (δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$) *We define δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$ such that the set of relative XIBOR-bonds $P_{i/i-1}(t,T)$ introduced in (15) equals*

$$P_{i/i-1}(t,T) \; = \; \exp\left(-\int_t^T f_{i/i-1}(t,u)\,du\right), \tag{16}$$

with $f_{0,-1} := f_0$. We assume that these δ_i-XIBOR relative (instantaneous) forward rates are determined by the following dynamics

$$df_{i/i-1}(t,T) \; = \; \mu_i(t,T)\,dt + \sigma_i(t,T)\,dW^{\mathbb{P}}(t), \tag{17}$$

for $i = 1,\dots,N$, where the volatility $\sigma_i(t,T) = \big(\sigma_{i,0}(t,T),\dots,\sigma_{i,N}(t,T)\big)$ is an $(N+1)$-dimensional row vector of progressively measurable processes and the drift $\mu_i(t,T)$ is itself a progressively measurable process.

Note that [Konikov and McClelland, 2020] propose a very similar framework but since they focus rather immediately upon a model which turns out to be very practical for pricing and calibration, they start under a risk-neutral measure $\mathbb{Q}$ and assume independence between the $f_{i/i-1}(t,T)$. In our (independent) work, we start under the historical measure and we allow for a general dependent framework for studying different measure changes, related HJM-drift conditions and conditions for δ_i-positivity. In the next section where we will present our practical δ_i-positivity model, however, we will focus upon more restrictive settings.

The XIBOR-bonds $P_i(t,T)$ introduced in (14) equal

$$P_i(t,T) \; = \; \exp\left(-\sum_{j=0}^{i}\int_t^T f_{j/j-1}(t,u)\,du\right) = \exp\left(-\int_t^T f_i(t,u)\,du\right), \tag{18}$$

with $f_i(t,u) := \sum_{j=0}^{i} f_{j/j-1}(t,u)$. We further denote for the bond volatilities

$$v_i(t,T) \; = \; \sum_{j=0}^{i}\int_t^T \sigma_j(t,u)\,du \tag{19}$$

and the drifts

$$m_i(t,T) \; = \; \sum_{j=0}^{i}\int_t^T \mu_j(t,u)\,du \tag{20}$$

for $i = 0,\dots,N$. The integral should be understood componentwise, such that v_i is still an $(N+1)$-dimensional row vector. The $\mathbb{P}$-dynamics of the OIS- and XIBOR-bonds are then easily determined.

Changing from $\mathbb{P}$ to $\mathbb{Q}$ and $\mathbb{Q}^T$

Let us first define the standard $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ by using Girsanov's theorem as

$$W^{\mathbb{Q}}(t) \; = \; \int_0^t \gamma(s)\,ds + W^{\mathbb{P}}(t), \tag{21}$$

where $\gamma(t) = \big(\gamma_0(t),\dots,\gamma_N(t)\big)^\top$ is a progressively measurable square-integrable process taking values in $\mathbb{R}^{(N+1)}$. Then, the $\mathbb{Q}$-dynamics of the OIS-bonds and the HJM drift condition

$$m_0(t,T) - \frac{1}{2}\big\|v_0(t,T)\big\|^2 \; = \; v_0(t,T)\,\gamma(t) \tag{22}$$

are well-known.
The change from the risk-neutral measure $\mathbb{Q}$ to the forward measure $\mathbb{Q}^T$ is also well-known (see e.g. [Zagst, 2002] p. 119), namely under $\mathbb{Q}^T$

$$W^{\mathbb{Q}^T}(t) \;=\; \int_0^t v_0(s,T)^\top \, ds \;+\; W^{\mathbb{Q}}(t), \tag{23}$$

is a standard Brownian motion. Note that this measure change is defined purely in terms of the OIS-rates $f_0(t,T)$.

Under the OIS drift condition (22), the dynamics of the OIS forward prices $P_i^D(t,T)$ are easily determined and confirmed to be $\mathbb{Q}^{T_i}$-martingales. Because the XIBOR forward prices $P_i^L(t,T)$ must also be a $\mathbb{Q}^{T_i}$-martingale, we need to impose a *XIBOR drift condition.* We summarize these important results in the following theorem.

Theorem 2.1 ($\mathbb{Q}^{T_i}$-martingales)

1. *The OIS forward prices $P_i^D(t,T)$ are $\mathbb{Q}^{T_i}$-martingales (for $i = 1,\dots,N$):*

$$dP_i^D(t,T) \;= P_i^D(t,T)\,\Big(v_0(t,T_i) - v_0(t,T)\Big)\, dW^{\mathbb{Q}^{T_i}}(t). \tag{24}$$

2. *The XIBOR forward prices $P_i^L(t,T)$ are $\mathbb{Q}^{T_i}$-martingales under the following XIBOR drift condition (for $i = 1,\dots,N$):*

$$m_i(t,T_i) - m_i(t,T) \;=\; -\frac{1}{2}\big\|v_i(t,T_i) - v_i(t,T)\big\|^2 \;+\; \big(v_i(t,T_i) - v_i(t,T)\big)\big(\gamma(t) + v_0(t,T_i)^\top\big). \tag{25}$$

Indeed, then

$$dP_i^L(t,T) \;=\; P_i^L(t,T)\,\Big(v_i(t,T_i) - v_i(t,T)\Big)\, dW^{\mathbb{Q}^{T_i}}(t). \tag{26}$$

Note that this XIBOR drift condition (25), which we assume to hold in the following, does not determine the coefficients $\mu_i(t,T)$ in a unique manner and this in contrast to the classical HJM condition, see also e.g. [Grbac and Runggaldier, 2015] Section 3.2.2.1 (for $i = 1$ and for the model determined under $\mathbb{Q}$). We underline that the XIBOR drift condition (25) is derived under the historical measure $\mathbb{P}$ and in a setting of correlated δ_i-XIBOR relative (instantaneous) forward rates $f_{i/i-1}(t,T)$, $i = 0,\dots,N$, and therefore includes analogous XIBOR drift conditions as in e.g. [Konikov and McClelland, 2020].

By setting

$$\bar{m}_i(t,T) \;=\; \sum_{j=1}^{i} \int_t^T \mu_j(t,u)\, du \tag{27}$$

and

$$\bar{v}_i(t,T) \;=\; \sum_{j=1}^{i} \int_t^T \sigma_j(t,u)\, du, \tag{28}$$

the XIBOR drift condition can further be rewritten as in the following proposition, which will be useful below.

Proposition 2.2 (Adjusted XIBOR drift condition)
The XIBOR drift condition (25) can be rewritten for $i = 1,\dots,N$ as

$$\begin{aligned}&\bar{m}_i(t,T_i) - \bar{m}_i(t,T)\\ &\quad= -\frac{1}{2}\big\|\bar{v}_i(t,T_i) - \bar{v}_i(t,T)\big\|^2 + \big(\bar{v}_i(t,T_i) - \bar{v}_i(t,T)\big)\big(\gamma(t) + v_0(t,T_i)^\top\big).\end{aligned} \tag{29}$$

The XIBOR drift condition is also useful for deriving the dynamics of the XIBOR spreads $P_i^S(t,T)$ under $\mathbb{Q}^T$:

Theorem 2.3 **($\mathbb{Q}^T$-martingale)**
Under the XIBOR drift condition in (25)*, the XIBOR spreads $P_i^S(t,T)$ in (6) are $\mathbb{Q}^T$-martingales and their $\mathbb{Q}$-dynamics are given by*

$$dP_i^S(t,T) \;= P_i^S(t,T)\;(\bar{v}_i(t,T_i) - \bar{v}_i(t,T))\;dW^{\mathbb{Q}^T}(t) \tag{30}$$

for $i = 1,\ldots,N$.

We now define a new probability $\mathbb{Q}^T_{i/i-1}$ via the Radon-Nikodym derivative $L^i(t,T)$ with respect to the forward measure $\mathbb{Q}^T$:

$$L^i(t,T) \;:=\; \frac{d\mathbb{Q}^T_{i/i-1}}{d\mathbb{Q}^T}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t \tilde{\gamma}_S^{i-1}(s,T)dW^{\mathbb{Q}^T}(s) - \frac{1}{2}\int_0^t \left\|\tilde{\gamma}_S^{i-1}(s,T)\right\|^2 ds\right) \;=\; \frac{P^S_{i-1}(t,T)}{P^S_{i-1}(0,T)} \tag{31}$$

where $\tilde{\gamma}_S^i(t,T) := (\bar{v}_i(t,T_i) - \bar{v}_i(t,T)))^\top$. Under this probability measure, the relative XIBOR spreads $P^S_{i/i-1}(t,T)$ are martingales.

Theorem 2.4 **($\mathbb{Q}^T_{i/i-1}$-martingale)**
Assuming the XIBOR drift condition in (25)*, the relative XIBOR spreads $P^S_{i/i-1}(t,T)$ are martingales under the probability measure $\mathbb{Q}^T_{i/i-1}$ defined by* (31) *and their dynamics under $\mathbb{Q}^T_{i/i-1}$ are given by*

$$\begin{aligned} dP^S_{i/i-1}(t,T) \;&= P^S_{i/i-1}(t,T)\left[\,\Big(\bar{v}_i(t,T_i) - \bar{v}_i(t,T)\Big) \;-\; \Big(\bar{v}_{i-1}(t,T_{i-1}) - \bar{v}_{i-1}(t,T)\Big)\,\right]dW^{\mathbb{Q}^T_{i/i-1}}(t), \\ &= P^S_{i/i-1}(t,T)\left[\,\tilde{\gamma}_S^i(t,T) \;-\; \tilde{\gamma}_S^{i-1}(t,T)\,\right]dW^{\mathbb{Q}^T_{i/i-1}}(t), \end{aligned} \tag{32}$$

for all $t \leq T$, where $\tilde{\gamma}_S^i(t,T) := (\bar{v}_i(t,T_i) - \bar{v}_i(t,T)))^\top$ and where

$$dW^{\mathbb{Q}^T_{i/i-1}}(t) := dW^{\mathbb{Q}^T}(t) - \tilde{\gamma}_S^{i-1}(t,T)dt. \tag{33}$$

2.3 Properties of δ_i-XIBOR relative forward rates

From the definition of OIS- and XIBOR-bonds in (13) and (18), it immediately follows that

$$\begin{aligned} P_i^S(t,T) \;&=\; \frac{P_i^L(t,T)}{P_i^D(t,T)} \;=\; \frac{\exp\left(\sum_{j=0}^i \int_T^{T_i} f_{j/j-1}(t,u)\,du\right)}{\exp\left(\int_T^{T_i} f_0(t,u)\,du\right)} \\ &=\; \exp\left(\sum_{j=1}^i \int_T^{T_i} f_{j/j-1}(t,u)\,du\right) \;=\; \exp\left(\int_T^{T_i} \bar{f}_i(t,u)\,du\right) \end{aligned}$$

with $\bar{f}_i(t,u) := \sum_{j=1}^i f_{j/j-1}(t,u)$, which shows that the multiplicative spreads are clearly correlated. This result also indicates that the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,u)$ can be related to relative forward spreads.

Since the goal is to describe a model under $\mathbb{P}$ satisfying the δ_i-positivity constraints (10)-(11), we need to formulate constraints upon the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,u)$.

Proposition 2.5 (δ_i-positivity constraints) *The δ_i-positivity constraints are equivalent to the following conditions in function of the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,u)$:*

$$1)\ \ P_i^S(t,T)\ \geq\ 1 \quad \Leftrightarrow \quad \sum_{j=1}^{i} \int_T^{T_i} f_{j/j-1}(t,u)\, du\ \geq\ 0 \tag{34}$$

$$2)\ \ P_{i/i-1}^S(t,T)\ \geq\ 1 \quad \Leftrightarrow \quad P_i^S(t,T) \text{ and therefore } \sum_{j=1}^{i} \int_T^{T_i} f_{j/j-1}(t,u)\, du \text{ are non-decreasing in } i, \tag{35}$$

for $i = 1, \ldots, N$.

Proof These statements follow immediately from (34) and (8). ■

Using Fubini's theorem and the XIBOR drift condition in (29), the constraint (34) can be reformulated. Indeed,

$$\begin{aligned}
&\sum_{j=1}^{i} \int_T^{T_i} f_{j/j-1}(t,u)\, du \\
&\quad = \sum_{j=1}^{i} \int_T^{T_i} f_{j/j-1}(0,u)\, du \ +\ \sum_{j=1}^{i} \int_T^{T_i} \int_0^t \mu_j(s,u)\, ds\, du \ +\ \sum_{j=1}^{i} \int_T^{T_i} \int_0^t \sigma_j(s,u)\, dW^{\mathbb{P}}(s)\, du \\
&\quad = \sum_{j=1}^{i} \int_T^{T_i} f_{j/j-1}(0,u)\, du \ +\ \int_0^t \left(\bar{m}_i(s,T_i) - \bar{m}_i(s,T)\right) ds \ +\ \int_0^t \left(\bar{v}_i(s,T_i) - \bar{v}_i(s,T)\right) dW^{\mathbb{P}}(s) \\
&\quad = \sum_{j=1}^{i} \int_T^{T_i} f_{j/j-1}(0,u)\, du \ -\ \frac{1}{2} \int_0^t \left\|\bar{v}_i(s,T_i) - \bar{v}_i(s,T)\right\|^2 ds \ +\ \int_0^t \left(\bar{v}_i(s,T_i) - \bar{v}_i(s,T)\right) dW^{\mathbb{Q}^T}(s),
\end{aligned} \tag{36}$$

which must be non-negative for all $t \in [0,T]$.

It is clear that this condition is not satisfied for $\bar{v}_i(s,T)$, $v_0(s,T)$ and $\gamma(s)$ being deterministic. This can also be observed from (30) and the fact that the dynamics of the XIBOR spreads $P_i^S(t,T)$ under different probablities are given by

$$\begin{aligned}
dP_i^S(t,T) &= P_i^S(t,T)\ \left(\bar{v}_i(t,T_i) - \bar{v}_i(t,T)\right)\, dW^{\mathbb{Q}^T}(t) \\
&= P_i^S(t,T)\ \left(\bar{v}_i(t,T_i) - \bar{v}_i(t,T)\right)\, \left(v_0(t,T)^\top\, dt \ +\ dW^{\mathbb{Q}}(t)\right) \\
&= P_i^S(t,T)\ \left(\bar{v}_i(t,T_i) - \bar{v}_i(t,T)\right)\, \left(v_0(t,T)^\top\, dt \ +\ \gamma(t)^\top\, dt \ +\ dW^{\mathbb{P}}(t)\right)
\end{aligned} \tag{37}$$

In the case of deterministic $\bar{v}_i(t,T)$, $v_0(t,T)$ and $\gamma(t)$, the XIBOR spreads $P_i^S(t,T)$ have a lognormal distribution under $\mathbb{P}$ (as well as under $\mathbb{Q}$ and $\mathbb{Q}^T$) and therefore the constraint (34) does not hold in these settings.

3 The δ_i-positive Multi-Curve HJM factor model

In this section, we will present a particular case of the model satisfying explicitly the constraints (34)-(35), as well as the XIBOR drift condition in (29).

We assume that the (OIS) short-rate r is given by the classical Hull-White model

$$dr(t)\ =\ \left(\theta_0(t) + \eta_0\,\lambda - \kappa_0\, r(t)\right) dt \ +\ \eta_0\, dW_0^{\mathbb{P}}(t), \tag{38}$$

with $W_0^{\mathbb{P}}$ the first component of the $(N+1)$-dimensional standard $\mathbb{P}$-Brownian motion introduced in the previous section; with deterministic function $\theta_0(t)$, and constants κ_0, η_0 and λ. The function $\theta_0(t)$ is chosen so as to exactly fit the initial term structure of (OIS) interest rates (see e.g. [Zagst, 2002] p. 135). It is well-known that in this way, (OIS) short-rates are allowed to turn negative as is observed in practice.

The OIS forward rate then has dynamics

$$df_0(t,T) \;=\; \mu_0(t,T)\,dt \;+\; \sigma_0(t,T)\,dW^{\mathbb{P}}(t) \tag{39}$$

with

$$\begin{aligned}\sigma_0(t,T) \;&=\; \Big(\eta_0\, e^{-\kappa_0(T-t)},\; 0,\; \dots,\; 0\Big) \quad \text{and}\\ \mu_0(t,T) \;&=\; \frac{\eta_0^2}{\kappa_0}\,(1-e^{-\kappa_0(T-t)})\,e^{-\kappa_0(T-t)} + \eta_0\, e^{-\kappa_0(T-t)}\lambda.\end{aligned} \tag{40}$$

We further assume the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,T)$ to be of the following affine form

$$f_{i/i-1}(t,T) \;=\; \alpha_i(t,T) \;+\; \beta_i(t,T)\,s_i(t) \tag{41}$$

with deterministic functions $\alpha_i(t,T)$ and $\beta_i(t,T)$ and with the (independent) factor processes $s_i(t)$ determined for all $i=1,\dots,N$ by the following CIR-processes

$$ds_i(t) \;=\; (\theta_i - \kappa_i\, s_i(t))\,dt \;+\; \eta_i\sqrt{s_i(t)}\,dW_i^{\mathbb{P}}(t), \quad s_i(0) \;=\; s_i^0 \tag{42}$$

where $s_i^0, \kappa_i, \eta_i, \theta_i \in (0,\infty)$. We recall that the solutions to CIR processes remain positive, which will facilitate to choose the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,T)$ to satisfy (34). In the following, we will refer to the factor process defined in (42) as the δ_i-relative spread factor process and its parameters as the spread factor parameters. We have chosen the relative spreads to be independent for technical reasons, and in particular since the vector of processes $(s_1,\dots,s_N)$ is then an afffine process, which are known to be tractable.

The dynamics of the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,T)$ are then given by

$$\begin{aligned}df_{i/i-1}(t,T) \;&=\Big(\,\partial_t\alpha_i(t,T) + \theta_i\;\beta_i(t,T) + (\partial_t\beta_i(t,T) - \kappa_i\;\beta_i(t,T))\;s_i(t)\,\Big)\;dt\\ &\qquad +\;\eta_i\;\beta_i(t,T)\;\sqrt{s_i(t)}\;dW_i^{\mathbb{P}}(t)\\ &= \mu_i(t,T)\;dt \;+\; \sigma_i(t,T)\;dW^{\mathbb{P}}(t)\end{aligned} \tag{43}$$

where

$$\begin{aligned}\sigma_i(t,T) \;&=\; \Big(0,\;\dots,\;0,\;\eta_i\;\beta_i(t,T)\;\sqrt{s_i(t)}\;,\;0,\;\dots,\;0\Big),\\ \mu_i(t,T) \;&=\; \Big(\;\partial_t\alpha_i(t,T) + \theta_i\;\beta_i(t,T) + (\partial_t\beta_i(t,T) - \kappa_i\;\beta_i(t,T))\;s_i(t)\;\Big).\end{aligned} \tag{44}$$

In the following, we define the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ by

$$\gamma(t) \;=\; \Big(\lambda,\; 0,\; \dots,\; 0\Big)^{\top}. \tag{45}$$

Our numerical results convinced us that this choice seems appropriate for our database. Further research should confirm whether this choice is always compatible with different data sets, but in any case, this choice and our framework is very convenient since they have several important implications. Indeed, in this modeling setup where we assumed above that also the OIS rates and all the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,T)$ are independent from each other, only the OIS rates will be influenced by the changes of measures from $\mathbb{P}$ to $\mathbb{Q}$ and $\mathbb{Q}^T$. The dynamics of the δ_i-XIBOR relative forward rates $f_{i/i-1}(t,T)$ remain the same under all measures. Furthermore, the XIBOR drift condition (29) simplifies to a more tractable equality.

By inserting the coefficients above into the XIBOR drift condition (29), we obtain

$$\begin{aligned} &\sum_{j=1}^{i} \int_T^{T+\delta_i} \mu_j(t,u)\, du \\ &\quad = -\frac{1}{2} \sum_{j=1}^{i} \left\| \int_T^{T+\delta_i} \sigma_j(t,u)\, du \right\|^2 + \left(\sum_{j=1}^{i} \int_T^{T+\delta_i} \sigma_j(t,u)\, du \right) \left(\gamma(t) + \int_t^{T+\delta_i} \sigma_0(t,u)^\top\, du \right) \\ \Leftrightarrow\ &\sum_{j=1}^{i} \left(\int_T^{T+\delta_i} \partial_t \alpha_j(t,u)\, du \,+\, \theta_j \int_T^{T+\delta_i} \beta_j(t,u)\, du \,+\, s_j(t) \int_T^{T+\delta_i} \left(\partial_t \beta_j(t,u) - \kappa_j\, \beta_j(t,u) \right) du \right) \\ &\quad = -\frac{1}{2} \sum_{j=1}^{i} \left(\eta_j \sqrt{s_j(t)} \int_T^{T+\delta_i} \beta_j(t,u)\, du \right)^2 \end{aligned} \tag{46}$$

where we used the fact that the vectors $\gamma(t)$ and $\sigma_0(t,T)$ are zero in the components $i = 1, \ldots, N$.

By comparing the coefficients of $(1,\, s_1(t), \ldots, s_N(t))$ in (46), we obtain the following system of equations

$$0 \;=\; \sum_{j=1}^{i} \left(\int_T^{T+\delta_i} \partial_t \alpha_j(t,u)\, du \,+\, \theta_j \int_T^{T+\delta_i} \beta_j(t,u)\, du \right) \tag{47}$$

for $i = 1, \ldots, N$, and

$$0 \;=\; \partial_t \left(\int_T^{T+\delta_i} \beta_j(t,u)\, du \right) \,-\, \kappa_j \left(\int_T^{T+\delta_i} \beta_j(t,u)\, du \right) \,+\, \frac{1}{2}\, \eta_j^2 \left(\int_T^{T+\delta_i} \beta_j(t,u)\, du \right)^2 \tag{48}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, i$.

Introducing the notation

$$B_{ij}(t,T) \;:=\; \int_T^{T+\delta_i} \beta_j(t,u)\, du, \tag{49}$$

one easily notices that equation (48) is a Riccati ODE with solution

$$B_{ij}(t,T) \;=\; \frac{2\,\kappa_j}{\eta_j^2} \left(\frac{1}{1 + \kappa_j\, C_{ij}(T)\, e^{-\kappa_j t}} \right) \;\geq\; 0 \tag{50}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, i$, where the positive function $C_{ij}(T)$ can be chosen arbitrarily, but such that for fixed j it is non-increasing in i (and $B_{ij}(t,T)$ thereby non-decreasing in i), e.g.

$$C_{ij}(T) \;=\; \frac{1}{e^{-\kappa_j T} - e^{-\kappa_j (T+\delta_i)}}. \tag{51}$$

Making the assumption that all terms in (47) are null, we further obtain the equation system

$$0 \;=\; \partial_t \left(\int_T^{T+\delta_i} \alpha_j(t,u)\, du \right) \,+\, \theta_j \left(\int_T^{T+\delta_i} \beta_j(t,u)\, du \right) \tag{52}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, i$.
We notice that this equation system will lead to just one possible solution and that there might be other solutions of (47), but we will show in the numerical sections that the solution proposed below leads to

practical estimations and results.

Indeed, a solution of (52) is given by

$$\alpha_j(t,u) \;=\; \theta_j \int_t^T \beta_j(s,u)\,ds \;+\; \phi_j(u), \tag{53}$$

such that

$$\begin{aligned} A_{ij}(t,T) \;&:=\; \int_T^{T+\delta_i} \alpha_j(t,u)\,du \\ &=\; \theta_j \int_t^T B_{ij}(s,T)\,ds \;+\; \int_T^{T+\delta_i} \phi_j(u)\,du \\ &=:\; \underbrace{\frac{2\,\kappa_j\,\theta_j}{\eta_j^2}\left[(T-t)+\frac{1}{\kappa_j}\ln\left(\frac{1+\kappa_j\,C_{ij}(T)\,e^{-\kappa_j T}}{1+\kappa_j\,C_{ij}(T)\,e^{-\kappa_j t}}\right)\right]}_{=:\,F_{ij}(t,T)} \;+\; \underbrace{\int_T^{T+\delta_i} \phi_j(u)\,du}_{=:\,\Phi_{ij}(T)} \end{aligned} \tag{54}$$

for $i = 1,\ldots,N$ and $j = 1,\ldots,i$. Note that the function $F_{ij}(t,T)$ is non-negative and non-decreasing in i, if the function $C_{ij}(T)$ is chosen as discussed above.
The deterministic functions $\Phi_{ij}(T)$ can be used to fit the initial term structure of spreads. From (34) (41), (49) and (54), we conclude that

$$P_i^S(0,T) \;=\; \exp\left(\sum_{j=1}^{i}\Big(A_{ij}(0,T) \;+\; B_{ij}(0,T)\,s_j^0\Big)\right), \tag{55}$$

and in particular

$$\ln P_i^S(0,T) \;-\; \sum_{j=1}^{i} B_{ij}(0,T)\,s_j^0 \;-\; \sum_{j=1}^{i} F_{ij}(0,T) \;=\; \sum_{j=1}^{i}\Phi_{ij}(T) \tag{56}$$

for $i = 1,\ldots,N$. Note that $\Phi_{ij}(T)$ needs to be non-negative and non-decreasing in i (for a given j and T), such that the spreads $P_i^S(t,T)$ are above 1 and non-decreasing in i as well in order to satisfy the δ_i-positivity constraints as stated in (34)-(35). This places constraints on the parameters s_i^0, κ_i, η_i and θ_i.

We notice that we are not interested in finding an explicit expression of $\beta_j(t,u)$ satisfying (49) nor in an explicit expression of $\alpha_j(t,u)$ in (54) since for derivative pricing on XIBOR and/or XIBOR spreads, the knowledge of $A_{ij}(t,T)$ and $B_{ij}(t,T)$ is enough in this framework.

We conclude our main results about the XIBOR spreads $P_i^S(t,T)$ in the following theorem.

Theorem 3.1 (δ_i-positive affine factor model)
In this model, the XIBOR spreads $P_i^S(t,T)$ have the affine form

$$P_i^S(t,T) \;=\; \exp\left(\sum_{j=1}^{i}\int_T^{T+\delta_i} f_{j/j-1}(t,u)\,du\right) \;=\; \exp\left(\sum_{j=1}^{i}\left(A_{ij}(t,T)+B_{ij}(t,T)\,s_j(t)\right)\right) \tag{57}$$

for $i = 1,\ldots,N$, with $A_{ij}(t,T)$ and $B_{ij}(t,T)$ given in (50), (51) and (54). For appropriate choices of $\Phi_{ij}(T)$ such that $\Phi_{ij}(T)$ is non-negative and non-decreasing in i for all $j = 1,\ldots,i$, the δ_i-positivity constraints (10)-(11) are satisfied, namely

$$P_i^S(t,T) \;\geq\; 1 \text{ and } P_{i/i-1}^S(t,T) \;\geq\; 1. \tag{58}$$

Remark 3 *We note that this model is an affine short rate multi-curve model as defined in [Cuchiero et al., 2019], see their Proposition 13.16(ii) and equation (57) above. We constructed the δ_i-positivity of our model by choosing $A_{ij}(t,T)$ and $B_{ij}(t,T)$ in (50), (51) and (54) to be non-negative and non-decreasing in i for all $j = 1,\ldots,i$. Moreover, since $P_i^S(t,T)$ in (57) depends only on the spread factors $s_j(t)$ for $j = 1,\ldots,i$, it is easy to see that the conditions of the general Proposition 3.7 in [Cuchiero et al., 2019] are fulfilled.*

4 Derivatives with optionality features

The pricing formulae of linear products such as swaps and basis swaps, and the related rates such as swap rates and basis swap rates, follow from the results in e.g. [Cuchiero et al., 2019]. Indeed, these derivations are based upon the facts that $P_i^L(t,T)$ are $\mathbb{Q}^{T_i}$-martingales as well as that by the definition

$$P_i^L(t,T) \;=\; P_i^D(t,T)\cdot P_i^S(t,T). \tag{59}$$

In this section, we therefore concentrate on financial instruments with optionality features such as caps and some options on spreads. In particular, as an example, we first focus upon semi-explicit pricing formulae for caplets in the explicit δ_i-positive affine factor model of Section 3. Section 5 will demonstrate the applicability of the analytical formulae by carrying out a calibration to market data.
In Section 4.2, we will concentrate upon the pricing of an option with payoff

$$\Big(P_0(U,T_i)P_i^L(U,T) - KP_0(U,T_{i-1})P_{i-1}^L(U,T)\Big)_+$$

to be payed out at U with $U \le T$, which gives protection against increasing relative XIBOR spreads, as will be explained below.

4.1 Caplet pricing

Let us recall that we fixed $T > 0$ and maturities $T_l = T + \delta_l$ $(l = 1,\dots,N)$ with discrete tenors $0 < \delta_1 < \ldots < \delta_N$.
The caplet price with strike $K > \frac{-1}{\delta_l}$, underlying $L_l(t,T) = L(t,T,T_l)$ and payoff $(L_l(T,T)-K)_+$ to be payed out at T_l in this setup reads as

$$\mathrm{Cpl}_l(0) = \delta_l\, P_0(0,T_l)\,\mathbb{E}_{\mathbb{Q}^{T_l}}\Big[(L_l(T,T)-K)_+\Big], \tag{60}$$

which equals by using (5) and (6)

$$\mathrm{Cpl}_l(0) = P_0(0,T_l)\,\mathbb{E}_{\mathbb{Q}^{T_l}}\Big[\Big(P_l^D(T,T)P_l^S(T,T) - K_l\Big)_+\Big], \tag{61}$$

with strike $K_l = 1 + \delta_l K > 0$. The caplet price can then be obtained by a Fourier inverse method, see e.g. [Carr and Madan, 1999].

Theorem 4.1 (Caplet)
The caplet price with strike $K > \frac{-1}{\delta_l}$, underlying $L_l(t,T) = L(t,T,T_l)$ and payoff $(L_l(T,T)-K)_+$ to be payed out at T_l equals

$$Cpl_l(0) = \frac{\delta_l\, P_0(0,T_l)\, e^{-\epsilon k_l}}{\pi}\int_0^\infty e^{-i\omega k_l}\,\frac{\Upsilon_{0,T,l}\,(i\omega+\epsilon+1)}{(i\omega+\epsilon)^2+(i\omega+\epsilon)}d\omega, \tag{62}$$

with $k_l = \ln(1+\delta_l K)$, $i = \sqrt{-1}$, $\epsilon > 1$ a dampening factor, and

$$\begin{aligned}\Upsilon_{0,T,l}\,(v) = g_l(v,T)\,\exp\left(-vB_0(T,T_l)\left(r_0e^{-\kappa_0 T} + \; e^{-\kappa_0 T}\int_0^T \tilde\theta_0(s)e^{\kappa_0 s}ds\right)\right)\\ \times\exp\left(v^2B_0(T,T_l)^2\eta_0^2\frac{1-e^{-2\kappa_0 T}}{2\kappa_0}\right)\prod_{j=1}^{l} e^{-\bar A_{lj}(0,T)-\bar B_{lj}(0,T)\;s_j(0)},\end{aligned} \tag{63}$$

with $g_l(v,T) = \exp\left(v(A_0(T,T) - A_0(T,T_l) + \sum_{j=1}^l A_{lj}(T,T))\right)$,

$$\bar A_{lj}(t,T) \;=\; \frac{2\theta_j}{\eta_j^2}\ln\left(\frac{1-\bar C_{lj}e^{\kappa_j t}}{1-\bar C_{lj}e^{\kappa_j T}}\right), \tag{64}$$

and

$$\bar{B}_{lj}(t,T) = \frac{2\kappa_j}{\eta_j^2}\left(\frac{1}{1-\bar{C}_{lj}e^{\kappa_j t}} - 1\right)$$

where

$$\bar{C}_{lj} = \frac{e^{-\kappa_j T} v B_{lj}(T,T)\frac{\eta_j^2}{2}}{v B_{lj}(T,T)\frac{\eta_j^2}{2} - \kappa_j}.$$

Proof See Appendix B. ■

4.2 Option pricing on XIBOR forward prices

In this subsection we focus upon an option with payoff

$$\Big(P_0(U,T_i)P_i^L(U,T) - KP_0(U,T_{i-1})P_{i-1}^L(U,T)\Big)_+ \tag{65}$$

to be payed out at U with $U \leq T$. Note that the first term equals the numerator of the relative XIBOR spreads in (8) whereas the second term equals K times its denumerator. This option can therefore be useful if one fears an increase in the relative XIBOR spreads (by e.g. an increase of the numerator in comparison with the denominator). Another economical interpretation of the usefulness of this payoff (65) can be given by noticing that both the first term and the corresponding factor in the second term are XIBOR forward prices $P_i^L(U,T)$ and $P_{i-1}^L(U,T)$ multiplied by resp. a zero-coupon with maturity date T_i and T_{i-1}. Indeed, the XIBOR forward prices $P_i^L(U,T)$ and $P_{i-1}^L(U,T)$ are determined at time U and represent payments to be done at resp. T_i and T_{i-1} (for receiving resp. $L_i(T,T)$ and $L_{i-1}(T,T)$ at resp. T_i and T_{i-1}), and the multiplication with resp. $P_0(U,T_i)$ and $P_0(U,T_{i-1})$ discounts these payments to the maturity time of the option, namely U. This option gives protection in the situation where discounted XIBOR forward rates of tenor δ_i, namely $P_0(U,T_i)P_i^L(U,T)$, are larger than $KP_0(U,T_{i-1})P_{i-1}^L(U,T)$ corresponding with tenor δ_{i-1}, and therefore in the situation that the relative XIBOR spreads are increasing. Theorem 4.2 shows that the price of this option boils down to determining an option on the relative XIBOR spreads by using the probability measure $\mathbb{Q}^T_{i/i-1}$ defined in (31), see Eq. (66).

Theorem 4.2 (Option pricing on XIBOR forward prices)
The price $C(t,U,T,i)$ at time t of an option with payoff

$$\Big(P_0(U,T_i)P_i^L(U,T) - KP_0(U,T_{i-1})P_{i-1}^L(U,T)\Big)_+$$

to be payed out at U with $t \leq U \leq T$ equals

$$C(t,U,T,i) = P_0(t,T)P_{i-1}^S(t,T)\,\mathbb{E}_{\mathbb{Q}^T_{i/i-1}}\left[\Big(P_{i/i-1}^S(U,T) - K\Big)_+ \,\Big|\, \mathcal{F}_t\right]. \tag{66}$$

Proof See Appendix C. ■

Using the fact that the relative XIBOR spreads $P_{i/i-1}^S(t,T)$ are martingales under the probability measure $\mathbb{Q}^T_{i/i-1}$ and in particular, by exploiting their dynamics under $\mathbb{Q}^T_{i/i-1}$ given in (32), $C(t,U,T,i)$ can be easily derived using Monte-Carlo simulations.
We notice that if $\tilde{\gamma}_S^i(t,T)$ can be assumed to be deterministic, then $P_{i/i-1}^S(t,T)$ clearly has a lognormal law and explicit option pricing formulae can be derived, but then the δ_i-positivity constraint (34) is not satisfied (when $v_0(t,T)$ and $\gamma(t)$ are deterministic, see e.g. (36)).

5 Calibration to EURIBOR caplets

In the following, we consider a single spread curve ($N = 1$) with tenor $\delta_1 = 6$ months, which is the most commonly used tenor for derivative products in the EURIBOR market. To estimate model parameters under the risk-neutral measure $\mathbb{Q}$, we obtained historical values for the necessary discount curves. The data contains EONIA OIS and EURIBOR discount factors for fixed discrete maturities $\{\tau_m\}_{m=1}^M$, where $M = 15$ and $\tau_m = m$ years for $m = 1, \ldots, M$. The historical XIBOR spreads P_1^S can be stripped from the OIS discount curve (P_1^D) and EURIBOR discount curve (P_1^L) using the relationship

$$P_1^S(t_k, t_k + \tau_m) \;=\; \frac{P_1^L(t_k, t_k + \tau_m)}{P_1^D(t_k, t_k + \tau_m)} \tag{67}$$

for $k = 1, \ldots, K$ and $m = 1, \ldots, M$. Discount factors with intermediate maturities $\tau_m + \delta_1$ are obtained from the coarser spaced historical data using the Nelson-Siegel interpolation method. From the historical spreads, the (δ_1-relative) spread factor process s_1 is not directly observable, but implicitly depends on the choice of parameters κ_1, θ_1 and η_1 through the relationship

$$P_1^S(t, T) \;=\; \exp\left(A_{11}(t, T) + B_{11}(t, T)\, s_1(t)\right) \tag{68}$$

with $A_{11}(t, T)$ and $B_{11}(t, T)$ in resp. (54) and (50).

We calibrate the model of Section 3 to a 10 year ATM cap on the 6m EURIBOR, where the single caplets are priced according to the caplet formulas derived in Section 4.1. Due to missing derivative prices for the EONIA OIS rates and no free of charge access to market cap prices, we calibrate the model to the following data and parameters. First, caps are quoted in terms of their implied flat volatility by the market, given the respective configuration setup (tenor, expiration, underlying, strike, etc.). Due to the restricted availability we exemplarily use an implied volatility of 24.5% (Black lognormal volatility) reported by [Moreni and Pallavicini, 2014] (Figure 6, p. 209) as of February 15, 2011, for a 10 year ATM cap on the 6m EURIBOR with nineteen caplets. Similar market quotes for the implied volatility can also be found in [Hull and White, 2015], [Hull, 2015] or [Crispoldi et al., 2015]. Figure 2 illustrates the corresponding OIS and EURIBOR spot rate curves.

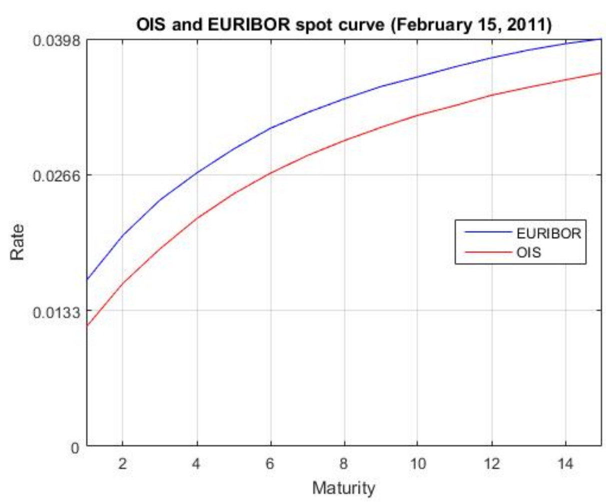

Figure 2: EONIA OIS and EURIBOR spot rate curves for different maturities as of February 15, 2011. Source: Bloomberg.

Furthermore, the OIS parameters κ_0 and η_0 are set to $\kappa_0 = 0.10$ and $\eta_0 = 0.01$. This selection almost coincides with the reported numbers in [Zagst, 2002] or [Russo and Torri, 2019]. Moreover, we impose the following constraints on the spread parameters in (42):

$$10^{-10} \leq \kappa_1, \theta_1, \eta_1, s_1^0 \leq 10^0. \tag{69}$$

In particular, all parameters are constrained to be positive. The calibration routine minimizes the mean squared error (MSE) of the observed market price to the model price.

Following Section 3, we further impose the additional constraint that the function Φ_{11} must remain non-negative, i.e.

$$\ln P_1^S(0,T) - B_{11}(0,T)\, s_j^0 - F_{11}(0,T) = \Phi_{11}(T) \geq 0. \tag{70}$$

The caplet pricing formula implied by the model is implemented via formula (62), and therefore by implementing the inverse Fourier transform of the characteristic function.

The output from the constrained calibration, given the fixed OIS parameters ($\kappa_0 = 0.10$, $\eta_0 = 0.01$) which are not part of the calibration, is

$$\kappa_1 = 0.2115,\ \theta_1 = 0.0004,\ \eta_1 = 0.9938,\ s_1^0 = 0.0131. \tag{71}$$

The cap market price to fitted model price ratio is furthermore given by 99.9999990%. This shows that market prices are thereby adequately reproduced by the model. Moreover, for every caplet we receive a value for $\Phi_{11}(T)$ that are all positive and do not exceed a value of 0.0009, hence are pretty small and close to zero. Finally, Table 1 summarizes all relevant calibration input parameters and output results.

OIS:	$\kappa_0 = 0.1000$	$\eta_0 = 0.0100$		
Cap configuration:	tenor: 6 months	maturity: 10 years	strike: ATM	implied vol: 24.5%
Calibration output:	$\kappa_1 = 0.2115$	$\theta_1 = 0.0004$	$\eta_1 = 0.9938$	$s_1^0 = 0.0131$
Cap prices:	Market price: 7.9410%	Model price: 7.9410%		

Table 1: Caplet calibration input parameters and output results.

6 Empirical estimation under $\mathbb{P}$

In the following, we present an empirical estimation approach based on the Kalman filter for a single spread curve ($N = 1$) with tenor $\delta_1 = 6$ months. The generalization to the case $N > 1$ is straightforward. We refer the reader interested in the use of a Kalman filter approach for the parameter estimation under $\mathbb{P}$ of the OIS Vasicek model to e.g. [Schmid, 2004] (pages 278ff).

6.1 Data

To estimate model parameters under the physical measure $\mathbb{P}$, we obtained historical time series of discount curves from Bloomberg. The data spans the time frame from Jan. 2006 through February 15, 2011 ($K = 1336$ daily time steps $\{t_k\}_{k=1}^K$ with $t_k - t_{k-1} = \Delta t = \frac{1}{250}$ years for $k = 2, \ldots, K$), which was the date used for the cap calibration in Section 5, and contains EONIA OIS and EURIBOR discount factors for fixed discrete maturities $\{\tau_m\}_{m=1}^M$, where $M = 30$ and $\tau_m = m$ years for $m = 1, \ldots, M$.

Figure 3 presents the log-spreads $\ln P_1^S(t_k, t_k+1), \ln P_1^S(t_k, t_k+3)$ and $\ln P_1^S(t_k, t_k+5)$ as implied by the market data[2]. In the following subsections we will explain how a Kalman filter approach can be applied to filter out the (δ_1-relative) spread factor process $s_1(t)$, which is also called filtered spread, and how optimal parameters can be estimated. The resulting XIBOR log-spread time series implied by the Kalman filter approach are also shown in Fig. 3 for maturities $m = 1$, 3 and 5, which clearly shows that the obtained results are very consistent with the market data.

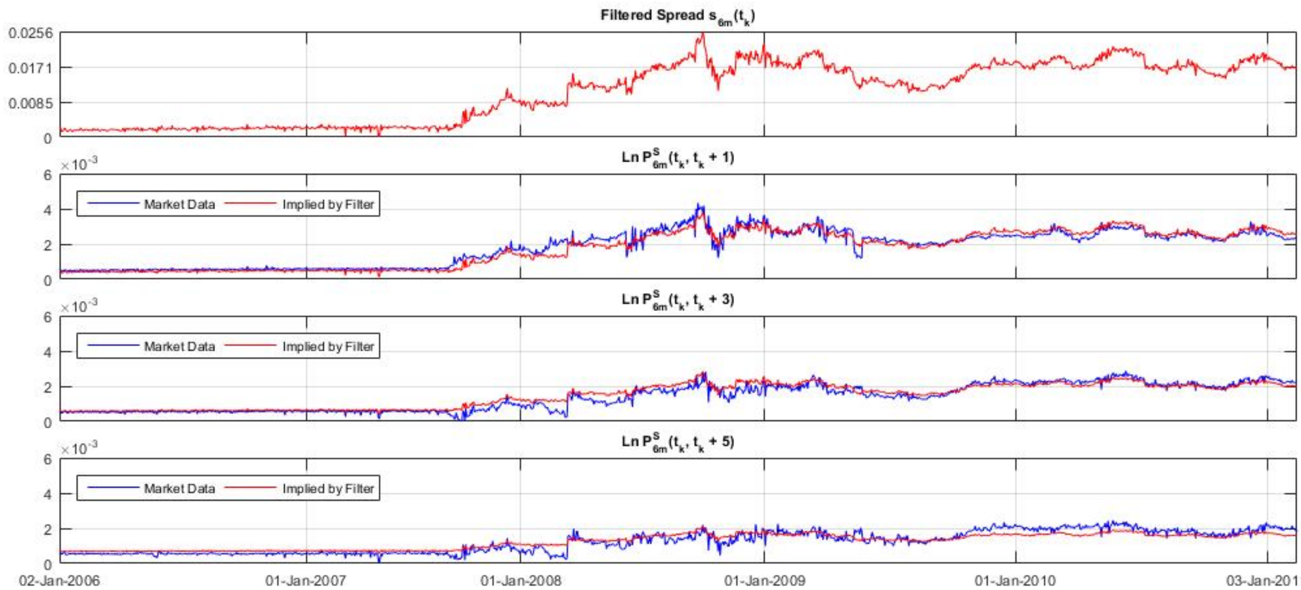

Figure 3: Results obtained from the Kalman filter estimation. The top plot shows the filtered spread factor time series $\hat{s}$ for the optimal parameter combination $(\kappa_1, \theta_1, \eta_1) = (0.2665, 0.0011, 0.9922)$. The bottom three plots show the log-spreads $\ln P_1^S(t_k, t_k+1)$, $\ln P_1^S(t_k, t_k+3)$ and $\ln P_1^S(t_k, t_k+5)$ as implied by the market data, the optimal parameter combination and the filtered spread factor time series $\hat{s}$.

6.2 Empirical Estimation using the Kalman Filter

From the historical XIBOR spread time series, the (δ_1-relative) spread factor process $s_1(t)$ is not directly observable, but implicitly depends on the choice of parameters κ_1, θ_1 and η_1 through the relationship

$$P_1^S(t,T) \;=\; \exp\left(A_{11}(t,T) + B_{11}(t,T)\, s_1(t)\right). \tag{72}$$

Using a Kalman filter approach for CIR processes, it is possible to obtain combined empirical estimates of both the parameters and the spread factor process time series.

For that matter, we choose

$$C_{11}(T) \;=\; \frac{1}{e^{-\kappa_1 T} - e^{-\kappa_1 (T+\delta_1)}},$$

such that,

$$\begin{aligned} B_{11}(t,T) \;&\equiv\; B_{11}(T-t) \;=\; \frac{2\,\kappa_1}{\eta_1^2}\left(\frac{1}{1+\kappa_1\left(e^{-\kappa_1(T-t)} - e^{-\kappa_1(T+\delta_1-t)}\right)^{-1}}\right) \quad \text{and} \\ A_{11}(t,T) \;&\equiv\; A_{11}(T-t) \;=\; \frac{2\,\kappa_1\,\theta_1}{\eta_1^2}\left[(T-t) + \frac{1}{\kappa_1}\ln\left(\frac{1+\kappa_1\left(1-e^{-\kappa_1\delta_1}\right)^{-1}}{1+\kappa_1\left(e^{-\kappa_1(T-t)} - e^{-\kappa_1(T+\delta_1-t)}\right)^{-1}}\right)\right] \end{aligned} \tag{73}$$

[2] In Figure 3, the notation $\ln P^S_{6\mathrm{m}}$ and $s_{6\mathrm{m}}$ has been used since δ_1 equals 6 months.

by (50) and (54). Here, we have chosen the solution in (54) with $\Phi_{11} \equiv 0$, which seems to be a good choice in the context of our data, see Fig. 3.

Measurement and Transition Equations

Using the notation introduced above and following the lines of [Chatterjee, 2005], the Kalman filter measurement equation is given by

$$\tilde{P}_k \;=\; \tilde{A} + \tilde{B}\,\hat{s}_k + \epsilon_k \tag{74}$$

with $\mathbb{R}^M$-vectors

$$\tilde{P}_k \;:=\; \begin{pmatrix} \ln P_1^S(t_k, t_k+\tau_1) \\ \vdots \\ \ln P_1^S(t_k, t_k+\tau_M) \end{pmatrix}, \quad \tilde{A} \;:=\; \begin{pmatrix} A_{11}(\tau_1) \\ \vdots \\ A_{11}(\tau_M) \end{pmatrix}, \quad \tilde{B} \;:=\; \begin{pmatrix} B_{11}(\tau_1) \\ \vdots \\ B_{11}(\tau_M) \end{pmatrix}, \tag{75}$$

and (independent) measurement errors $\epsilon_k \sim \mathcal{N}(0, Q)$, where $Q \in \mathbb{R}^{M\times M}$ is diagonal with components $q_j, 1 \le j \le M$. Here, $\hat{s}_k$ denotes the filtered factor process s_1 at time step t_k. Note that A_{11} and B_{11} depend on the parameters κ_1, θ_1 and η_1.

Based on the conditional expectation and variance of the CIR process in (42), the Kalman filter state transition equation is given by

$$\hat{s}_k \;=\; \mathbb{E}[\,\hat{s}_k \,|\, \hat{s}_{k-1}\,] + u_k \;=\; \frac{\theta_1}{\kappa_1}(1 - e^{-\kappa_1 \Delta t}) + e^{-\kappa_1 \Delta t}\,\hat{s}_{k-1} + u_k, \tag{76}$$

where $u_k \sim \mathcal{N}(0, h_k^2)$ is independent of the measurement error ϵ_k and has variance

$$h_k^2 \;=\; \mathbb{V}[\,\hat{s}_k \,|\, \hat{s}_{k-1}\,] \;=\; \frac{\theta_1\,\eta_1^2}{2\,\kappa_1^2}(1 - e^{-\kappa_1 \Delta t})^2 + \frac{\eta_1^2}{\kappa_1}(e^{-\kappa_1 \Delta t} - e^{-2\,\kappa_1\,\Delta t})\,\hat{s}_{k-1}. \tag{77}$$

We refer the interested reader to [Chatterjee, 2005] for these and the following results.

Prediction and Update Steps

The Kalman filter prediction $\hat{s}_{k|k-1}$ of $\hat{s}_k$ based on the previous state $\hat{s}_{k-1}$ is given by

$$\hat{s}_{k|k-1} \;=\; \mathbb{E}[\,\hat{s}_k \,|\, \hat{s}_{k-1}\,] \;=\; \frac{\theta_1}{\kappa_1}(1 - e^{-\kappa_1 \Delta t}) + e^{-\kappa_1 \Delta t}\,\hat{s}_{k-1}. \tag{78}$$

Similarly, we set

$$\hat{q}_{k|k-1} \;=\; \mathbb{E}[\,(\hat{s}_k - \hat{s}_{k|k-1})^2 \,|\, \hat{s}_{k-1}\,] \;=\; e^{-2\kappa_1 \Delta t} q_{k-1} + h_k^2 \tag{79}$$

for the prediction covariance.

The forecast error is then given by

$$v_k \;=\; \tilde{P}_k - \tilde{A} - \tilde{B}\,\hat{s}_{k|k-1} \tag{80}$$

with covariance

$$F_k \;=\; \hat{q}_{k|k-1}\,\tilde{B}\,\tilde{B}^\top + Q. \tag{81}$$

The new state $\hat{s}_k$ and covariance $\hat{q}_k$ are then given by the update steps

$$\hat{s}_k \;=\; \hat{s}_{k|k-1} \;+\; \hat{q}_{k|k-1}\tilde{B}^\top F_k^{-1}\,v_k \quad \text{and} \quad \hat{q}_k \;=\; (1 - \hat{q}_{k|k-1}\tilde{B}^\top F_k^{-1}\,\tilde{B})\hat{q}_{k|k-1}. \tag{82}$$

Parameter Estimation

The parameter estimation is carried out using a quasi-MLE approach by maximizing the log-likelihood

$$\log L(\tilde{P}_1, \ldots, \tilde{P}_K \,|\, \kappa_1, \theta_1, \eta_1, Q) \;=\; -\frac{MK}{2}\log 2\pi \;-\; \frac{1}{2}\sum_{k=1}^{K}\Big(\log|F_k| + v_k^\top F_k^{-1} v_k\Big)\hat{q}_{k|k-1}. \tag{83}$$

Note that the forecast error v_k and its covariance F_k depend on the process parameters κ_1, θ_1 and η_1 (and the measurement error variances Q) and the filtered factor time series $\{\hat{s}_k\}_{k=1}^K$. However, the filtered factor time series can itself be considered a function of κ_1, θ_1 and η_1 (and Q).

Numerical results under $\mathbb{P}$

For the MLE-estimation, parameters were constrained by the lower and upper bounds

$$10^{-4} \;\leq\; \kappa_1, \theta_1, \eta_1 \;\leq\; 10^0. \tag{84}$$

The estimation procedure was run 500 times using MATLAB's internal implementation of the sequential quadratic programming approach with random starting parameter sets $(\kappa_1, \theta_1, \eta_1)$ satisfying the bounds above and with the measurement error variance set to $Q \equiv 10^{-8}$.

Optimal parameter sets cluster around the accumulation point $(0.2665, 0.0011, 0.9922)$, whose corresponding filtered spread factor time series $\hat{s}$ and implied log-spread time series are shown in Fig. 3. The parameter combination above implies a theoretical long-term mean of 0.0041 and a standard deviation of 0.0020 for the spread factor process s_1 (compared to empirical estimates of 0.0111 and 0.0071, respectively, from the filtered spread process $\hat{s}$). The parameters implied by the Kalman filter $(\kappa_1, \theta_1, \eta_1) = (0.2665, 0.0011, 0.9922)$ are comparable to the parameters implied by the market $(\kappa_1, \theta_1, \eta_1) = (0.2115, 0.0004, 0.9938)$ (cf. Section 5) for the considered date. The deviation is due to the historical backward-looking character of the Kalman filter implied parameters vs. the forward-looking feature of the market implied parameters.

7 Conclusion

In this paper, we concentrate upon a model with multiplicative relative spreads, inspired by the HJM and affine factor approaches, which allows for positive and ordered spreads. This model leads to tractable pricing formulas for typical interest-rate derivatives such as caplets. One of the main contributions of this paper is that we focus upon a joint modeling under $\mathbb{P}$ and $\mathbb{Q}$ and consider estimation methods under both measures. This model is easy to implement under the historical measure as well as to calibrate with respect to interest-rate derivatives. Numerical results are included.
An interesting topic for future research would be to develop a general term-structure framework taking stochastic discontinuities explicitly into account, which are for example related to monetary policymeetings of the ECB, see e.g. [Fontana et al., 2020]. Another interesting topic for future research would be to model a regime-switching multi-curve model with hidden Markov processes modelling different scenarios of the market.

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A The Multi-Curve HJM Framework: Results and Proofs

Using the notation introduced in the multi-curve HJM framework of section 2.2, we start by stating the $\mathbb{P}$-dynamics of the OIS- and XIBOR-bonds.

Theorem A.1 *The $\mathbb{P}$-dynamics of the OIS- and XIBOR-bonds are determined by*

$$dP_i(t,T) \;=\; P_i(t,T)\left(\left(r_i(t) - m_i(t,T) + \frac{1}{2}\|v_i(t,T)\|^2\right) dt \;-\; v_i(t,T)\, dW^{\mathbb{P}}(t)\right) \tag{85}$$

for $i = 0,\dots,N$, where $r_i(t) = f_i(t,t)$.

We next recall the $\mathbb{Q}$-dynamics of the OIS-bonds under the well-known drift condition (22).

Theorem A.2 *The $\mathbb{Q}$-dynamics of the OIS-bonds are given by*

$$dP_0(t,T) \;=\; P_0(t,T)\left(r_0(t)dt \;-\; v_0(t,T)\, dW^{\mathbb{Q}}(t)\right) \tag{86}$$

under the HJM drift condition (22)*, namely*

$$m_0(t,T) - \frac{1}{2}\|v_0(t,T)\|^2 \;=\; v_0(t,T)\,\gamma(t).$$

Here it is used that the HJM drift condition (22) is equivalent to the following condition in terms of the coefficients $\mu_0(t,T)$ and $\sigma_0(t,T)$, and this by a differentiation with respect to T:

$$\mu_0(t,T) - \sigma_0(t,T)\, v_0(t,T)^\top \;=\; \sigma_0(t,T)\,\gamma(t). \tag{87}$$

We now give a short proof of Theorem 2.1.

Proof of Theorem 2.1 [$\mathbb{Q}^{T_i}$-martingales]
(i) From (3), (85), (22) and Itô's formula, the dynamics of the OIS forward prices $P_i^D(t,T)$ are easily determined under the different probabilities as follows

$$\begin{aligned}
dP_i^D(t,T) \;&=\left(\frac{P_0(t,T)}{P_0(t,T_i)}\right)\Big[\left((v_0(t,T_i) - v_0(t,T))\,\gamma(t) + \|v_0(t,T_i)\|^2 - v_0(t,T)\,v_0(t,T_i)^\top\right) dt \\
&\qquad + \left(v_0(t,T_i) - v_0(t,T)\right) dW^{\mathbb{P}}(t)\,\Big] \\
&= P_i^D(t,T)\,\Big[\,\left(\|v_0(t,T_i)\|^2 - v_0(t,T)\,v_0(t,T_i)^\top\right) dt \\
&\qquad + \left(v_0(t,T_i) - v_0(t,T)\right) dW^{\mathbb{Q}}(t)\,\Big] \\
&= P_i^D(t,T)\,\left(v_0(t,T_i) - v_0(t,T)\right) dW^{\mathbb{Q}^{T_i}}(t),
\end{aligned}$$

where the last line confirms that the OIS forward prices $P_i^D(t,T)$ are $\mathbb{Q}^{T_i}$-martingales.
(ii) Similarly, for the XIBOR forward prices $P_i^L(t,T)$, we obtain

$$
\begin{aligned}
dP_i^L(t,T) \;&= \left(\frac{P_i(t,T)}{P_i(t,T_i)}\right)\Bigg[\, \Big((m_i(t,T_i)-m_i(t,T)) + \|v_i(t,T_i)\|^2 + \frac{1}{2}\|v_i(t,T)\|^2 - \frac{1}{2}\|v_i(t,T_i)\|^2 \\
&\qquad - v_i(t,T)\, v_i(t,T_i)^\top\Big)\, dt + \Big(v_i(t,T_i) - v_i(t,T)\Big)\, dW^{\mathbb{P}}(t)\,\Bigg] \\
&= P_i^L(t,T)\,\Bigg[\, \Big((m_i(t,T_i)-m_i(t,T)) + \frac{1}{2}\|v_i(t,T_i)-v_i(t,T)\|^2 \\
&\qquad - (v_i(t,T_i)-v_i(t,T))(\gamma(t) + v_0(t,T_i)^\top)\Big)\, dt \\
&\qquad + \Big(v_i(t,T_i)-v_i(t,T)\Big)\, dW^{\mathbb{Q}^{T_i}}(t)\,\Bigg]
\end{aligned}
\tag{88}
$$

for $i = 1,\ldots,N$ from Itô's formula. The dynamics (26) then follow under the XIBOR drift conditions (25). ■

We now concentrate upon the derivations of the useful Adjusted XIBOR drift conditions in Proposition 2.2.

Proof of Proposition 2.2 [Adjusted XIBOR drift condition]
From the well-known HJM OIS drift condition (22), it is easy to show that

$$
\begin{aligned}
&m_0(t,T_i) - m_0(t,T) \\
&\quad= \frac{1}{2}\|v_0(t,T_i)\|^2 + v_0(t,T_i)\,\gamma(t) - \frac{1}{2}\|v_0(t,T)\|^2 - v_0(t,T)\,\gamma(t) \\
&\quad= -\frac{1}{2}\|v_0(t,T_i)-v_0(t,T)\|^2 + (v_0(t,T_i)-v_0(t,T))(\gamma(t)+v_0(t,T_i)^\top),
\end{aligned}
\tag{89}
$$

which holds for all T_i and essentially reflects that the XIBOR drift condition (25) also holds for the OIS rates ($i=0$).

Using the notations introduced in (27)-(28), the XIBOR drift condition (25) can further be rewritten as

$$
\begin{aligned}
&\bar{m}_i(t,T_i) - \bar{m}_i(t,T) \\
&\quad= (m_i(t,T_i)-m_i(t,T)) - (m_0(t,T_i)-m_0(t,T)) \\
&\quad= -\frac{1}{2}\|v_i(t,T_i)-v_i(t,T)\|^2 \;+\; (v_i(t,T_i)-v_i(t,T))(\gamma(t)+v_0(t,T_i)^\top) \\
&\qquad + \frac{1}{2}\|v_0(t,T_i)-v_0(t,T)\|^2 - (v_0(t,T_i)-v_0(t,T))(\gamma(t)+v_0(t,T_i)^\top) \\
&\quad= -\frac{1}{2}\|\bar{v}_i(t,T_i)-\bar{v}_i(t,T)\|^2 + (\bar{v}_i(t,T_i)-\bar{v}_i(t,T))(\gamma(t)+v_0(t,T_i)^\top).
\end{aligned}
\tag{90}
$$

■

We next concentrate upon the dynamics of the XIBOR spreads $P_i^S(t,T)$.

Proof of Theorem 2.3 [$\mathbb{Q}^T$-martingale]

From (24), (26) and Itô's formula, the dynamics of the XIBOR spreads $P_i^S(t,T)$ in (6) are given by

$$\begin{aligned} dP_i^S(t,T) &= P_i^S(t,T) \Big[\Big(\|v_0(t,T_i) - v_0(t,T)\|^2 - (v_0(t,T_i) - v_0(t,T))(v_i(t,T_i) - v_i(t,T))^\top \Big) \, dt \\ &\qquad + \Big((v_i(t,T_i) - v_i(t,T)) - (v_0(t,T_i) - v_0(t,T)) \Big) \, dW^{\mathbb{Q}^{T_i}}(t) \Big] \\ &= P_i^S(t,T) \, (\bar{v}_i(t,T_i) - \bar{v}_i(t,T)) \, dW^{\mathbb{Q}^T}(t) \end{aligned} \tag{91}$$

for $i = 1, \ldots, N$, where in the last equality both the XIBOR drift condition (25) as well as the definitions of $\mathbb{Q}^T$ and $\mathbb{Q}^{T_i}$ are used. ■

Finally, we prove that the dynamics of the relative XIBOR spreads $P_{i/i-1}^S(t,T)$ are martingales under the measure $\mathbb{Q}_{i/i-1}^T$.

Proof of Theorem 2.4 [$\mathbb{Q}_{i/i-1}^T$-martingale]
The relative XIBOR spreads $P_{i/i-1}^S(t,T)$ in (7) evolve according to

$$\begin{aligned} dP_{i/i-1}^S(t,T) &= P_{i/i-1}^S(t,T) \Big[\Big(\|\bar{v}_{i-1}(t,T_{i-1}) - \bar{v}_{i-1}(t,T)\|^2 \\ &\qquad - (\bar{v}_i(t,T_i) - \bar{v}_i(t,T))(\bar{v}_{i-1}(t,T_{i-1}) - \bar{v}_{i-1}(t,T))^\top \Big) \, dt \\ &\qquad + \Big((\bar{v}_i(t,T_i) - \bar{v}_i(t,T)) - (\bar{v}_{i-1}(t,T_{i-1}) - \bar{v}_{i-1}(t,T)) \Big) \, dW^{\mathbb{Q}^T}(t) \Big] \end{aligned} \tag{92}$$

for $i = 1, \ldots, N$. The proof now follows immediately from the definition of the Radon-Nikodym derivative in (31) and the theorem of Girsanov; and in particular from the fact that $W^{\mathbb{Q}_{i/i-1}^T}$ defined by (33) are Brownian motions under $\mathbb{Q}_{i/i-1}^T$. ■

B Proof of the Caplet price in Theorem 4.2

We recall that in the Hull-White model, the zero-coupon prices are given by (see e.g. [Zagst, 2002] Lemma 4.24 on p. 138)

$$P_0(t,T) = \exp\left(A_0(t,T) + B_0(t,T) \, r(t)\right) \tag{93}$$

with

$$B_0(t,T) = \frac{-1}{\kappa_0}(1 - e^{-\kappa_0(T-t)}) \tag{94}$$

and

$$A_0(t,T) = \ln\left(\frac{P_0(0,T)}{P_0(0,t)}\right) - B_0(t,T) f_0(0,t) - \frac{1}{2} B_0(t,T)^2 \frac{\eta_0^2}{2\kappa_0}(1 - e^{-2\kappa_0 t}). \tag{95}$$

By using the change of measures in (45) and (23), the dynamics of the (OIS) short-rate r under $\mathbb{Q}^{T_i}$ are given by

$$dr(t) \; = \; (\tilde{\theta}_0(t) - \kappa_0 \, r(t)) \, dt \; + \; \eta_0 \, dW_0^{\mathbb{Q}^{T_i}}(t), \tag{96}$$

with $\tilde{\theta}_0(t) = \theta_0(t) - \frac{\eta_0^2}{\kappa_0}(1 - e^{-\kappa_0(T_i - t)})$, where we recall that $\theta_0(t)$ is well-known from calibrating the initial yield curve. Therefore, from (3) and (93), we have that

$$P_i^D(t,T) \; = \frac{P_0(t,T)}{P_0(t,T_i)} = \exp\left(A_0(t,T) - A_0(t,T_i) + (B_0(t,T) - B_0(t,T_i)) \, r(t)\right) \tag{97}$$

with the (OIS) short-rate r given under $\mathbb{Q}^{T_i}$ by (96), and $A_0(t,T)$ and $B_0(t,T)$ as in (94) and (95).

We further know from (57) that the XIBOR spreads P_i^S have the affine form

$$P_i^S(t,T) \;=\; \exp\left(\sum_{j=1}^{i}\int_T^{T_i} f_{j/j-1}(t,u)\,du\right) \;=\; \exp\left(\sum_{j=1}^{i} A_{ij}(t,T) + B_{ij}(t,T)\,s_j(t)\right) \tag{98}$$

with $A_{ij}(t,T)$ as in (54) and $B_{ij}(t,T)$ as in (50), with the function $C_{ij}(T)$ chosen as $C_{ij}(T) = \frac{1}{e^{-\kappa_j T} - e^{-\kappa_j T_i}}$. We further recall that for $j = 1, \dots, N$, the processes s_j are defined under $\mathbb{Q}^{T_i}$ (as well as under $\mathbb{P}$) as

$$ds_j(t) \;=\; (\theta_j - \kappa_j\,s_j(t))\,dt \;+\; \eta_j\sqrt{s_j(t)}\,dW_j^{\mathbb{Q}^{T_i}}(t), \tag{99}$$

with positive constants η_j, κ_j and θ_j.

In order to calculate the price of a caplet in (61), we use the well-known Carr-Madan formula (see [Carr and Madan, 1999]). Therefore we write the underlying in (61) as an exponential, namely

$$P_i^D(T,T)P_i^S(T,T) = e^{X_i(T)}$$

with

$$\begin{aligned} X_i(T) \;&=\; A_0(T,T) - A_0(T,T_i) + (B_0(T,T) - B_0(T,T_i))\,r(T) \\ &\quad + \sum_{j=1}^{i} (A_{ij}(T,T) + B_{ij}(T,T)\,s_j(T)). \end{aligned} \tag{100}$$

It remains to focus on the calculation of the generator function $\Upsilon_{0,T,l}(v) = \mathbb{E}_{\mathbb{Q}^{T_l}}\left(e^{vX_l(T)}\right)$ with $v \in \mathbb{C}$. Hereto, we will make use of the independence between the processes r and s_l for $l = 1, \dots, N$:

$$\mathbb{E}_{\mathbb{Q}^{T_l}}\left[e^{vX_l(T)}\right] = g_l(v,T)\,\mathbb{E}_{\mathbb{Q}^{T_l}}\left[\exp(v(B_0(T,T) - B_0(T,T_l))\,r(T)\right]\prod_{j=1}^{l}\mathbb{E}_{\mathbb{Q}^{T_l}}\left[\exp(vB_{lj}(T,T)\,s_j(T))\right] \tag{101}$$

with $g_l(v,T) = \exp\left(v(A_0(T,T) - A_0(T,T_l) + \sum_{j=1}^{l} A_{lj}(T,T))\right)$. Since

$$r(T) = r_0e^{-\kappa_0 T} + \; e^{-\kappa_0 T}\int_0^T \tilde{\theta}_0(s)e^{\kappa_0 s}ds + \eta_0 e^{-\kappa_0 T}\int_0^T e^{\kappa_0 s}dW_0^{\mathbb{Q}^{T_l}}(s)$$

follows a normal law, the first factor follows from the fact that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^{T_l}}\left[\exp(v(B_0(T,T) - B_0(T,T_l))\,r(T)\right] = \\ &\quad \exp(\mathbb{E}_{\mathbb{Q}^{T_l}}\left[v(B_0(T,T) - B_0(T,T_l))\,r(T)\right] + \frac{1}{2}\mathrm{Var}_{\mathbb{Q}^{T_l}}\left[v(B_0(T,T) - B_0(T,T_l))\,r(T)\right]) \end{aligned} \tag{102}$$

Noticing from (94) that $B_0(T,T) = 0$, it easily follows that

$$\mathbb{E}_{\mathbb{Q}^{T_l}}\left[v(B_0(T,T) - B_0(T,T_l))\,r(T)\right] = -vB_0(T,T_l)\left(r_0e^{-\kappa_0 T} + \int_0^T \tilde{\theta}_0(s)e^{-\kappa_0(T-s)}ds\right)$$

and

$$\mathrm{Var}_{\mathbb{Q}^{T_l}}\left[v(B_0(T,T) - B_0(T,T_l))\,r(T)\right] = v^2B_0(T,T_l)^2\eta_0^2\frac{1 - e^{-2\kappa_0 T}}{2\kappa_0}. \tag{103}$$

Each of the following factors

$$\mathbb{E}_{\mathbb{Q}^{T_l}}\left[\exp(vB_{lj}(T,T)\ s_j(T))\right]$$

follows from e.g. using Lemma 4.1 of [de Kort and Vellekoop, 2017]. Indeed, this Lemma leads to the following equality for $v \in \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}^{T_l}}\left[\exp(vB_{lj}(T,T)\ s_j(T))\Big|\ \mathcal{F}_t\right] = e^{-\bar{A}_{lj}(t,T)-\bar{B}_{lj}(t,T)\ s_j(t)}, \tag{104}$$

with the functions $\bar{A}_{lj}(t,T)$ and $\bar{B}_{lj}(t,T)$ defined by the PDE's

$$\begin{aligned} \frac{\partial \bar{B}_{lj}(t,T)}{\partial t} &= \kappa_j \bar{B}_{lj}(t,T) + \frac{1}{2}\eta_j^2(\bar{B}_{lj}(t,T))^2, \qquad 0 \le t \le T \\ \bar{B}_{lj}(T,T) &= -vB_{lj}(t,T) \end{aligned} \tag{105}$$

and

$$\begin{aligned} \frac{\partial \bar{A}_{lj}(t,T)}{\partial t} &= -\theta_j \bar{B}_{lj}(t,T), \qquad 0 \le t \le T \\ \bar{A}_{lj}(T,T) &= 0 \end{aligned} \tag{106}$$

and by noticing that these solutions $\bar{A}_{lj}(t,T)$ and $\partial\bar{B}_{lj}(t,T)$ are bounded functions. By solving the Riccati equations, one easily finds

$$\bar{B}_{lj}(t,T) = \frac{2\kappa_j}{\eta_j^2}\left(\frac{1}{1-\bar{C}_{lj}e^{\kappa_j t}} - 1\right)$$

with

$$\bar{C}_{lj} = \frac{e^{-\kappa_j T} vB_{lj}(T,T)\frac{\eta_j^2}{2}}{vB_{lj}(T,T)\frac{\eta_j^2}{2} - \kappa_j}$$

and

$$\begin{aligned} \bar{A}_{lj}(t,T) &= \theta_j \int_t^T \bar{B}_{lj}(s,T)\,ds \\ &= \frac{2\theta_j}{\eta_j^2}\ln\left(\frac{1-\bar{C}_{lj}e^{\kappa_j t}}{1-\bar{C}_{lj}e^{\kappa_j T}}\right). \end{aligned} \tag{107}$$

Next, the result in (104) can be generalized to $v \in \mathbb{C}$ by techniques as in Chapter 10 of [Filipovic, 2009]. Finally, the by now standard approach of [Carr and Madan, 1999] leads to the result in (62). ■

C Proof of the Option formula on XIBOR forward prices in Theorem 4.2

Using risk-neutral pricing and equation (6), the price of this option equals

$$\begin{aligned} C(t,U,T,i) &= P_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{P_0(U)}\left(P_0(U,T_i)P_i^L(U,T) - KP_0(U,T_{i-1})P_{i-1}^L(U,T)\right)_+ \Big|\ \mathcal{F}_t\right], \\ &= P_0(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{P_0(U)}\left(P_0(U,T)P_i^S(U,T) - KP_0(U,T)P_{i-1}^S(U,T)\right)_+ \Big|\ \mathcal{F}_t\right], \\ &= P_0(t,T)\,\mathbb{E}_{\mathbb{Q}^T}\left[\left(P_i^S(U,T) - KP_{i-1}^S(U,T)\right)_+ \Big|\ \mathcal{F}_t\right], \end{aligned} \tag{108}$$

by the definition of the forward-neutral probability measure $\mathbb{Q}^T$. In order to perform a change-of-measure to the probability measure $\mathbb{Q}^T_{i/i-1}$ defined by (31) and noticing from (8) that for all $t \leq T$

$$P_i^S(t,T) = P_{i/i-1}^S(t,T)P_{i-1}^S(t,T), \tag{109}$$

this can be easily rewritten as

$$\begin{aligned} C(t,U,T,i) &= P_0(t,T)P_{i-1}^S(t,T)\,\mathbb{E}_{\mathbb{Q}^T}\left[\frac{P_{i-1}^S(U,T)}{P_{i-1}^S(t,T)}\left(P_{i/i-1}^S(U,T)-K\right)_+ \,\Big|\, \mathcal{F}_t\right], \\ &= P_0(t,T)P_{i-1}^S(t,T)\,\mathbb{E}_{\mathbb{Q}^T_{i/i-1}}\left[\left(P_{i/i-1}^S(U,T)-K\right)_+ \,\Big|\, \mathcal{F}_t\right], \end{aligned} \tag{110}$$

where we used the Radon-Nikodym derivative in (31) which defines the probability $\mathbb{Q}^T_{i/i-1}$. ■