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Dynamic Factor Models: a Genealogy

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Matteo Barigozzi and Marc Hallin

Abstract Dynamic factor models have been developed out of the need of analyzing and forecasting time series in increasingly high dimensions. While mathematical statisticians faced with inference problems in high-dimensional observation spaces were focusing on the so-called *spiked-model-asymptotics*, econometricians adopted an entirely and considerably more effective asymptotic approach, rooted in the factor models originally considered in psychometrics. The so-called *dynamic factor model* methods, in two decades, has grown into a wide and successful body of techniques that are widely used in central banks, financial institutions, economic and statistical institutes. The objective of this chapter is not an extensive survey of the topic but a sketch of its historical growth, with emphasis on the various assumptions and interpretations, and a family tree of its main variants.

1 Factor models and the analysis of high-dimensional time series

With the fast-pace development of computing facilities, high-dimensional datasets are increasingly available, posing a genuine challenge to statisticians and econometricians. Faced with this situation and the need to analyze such datasets, new asymptotic scenarios and methods had to be developed. Mathematical statisticians mostly focused on the so-called spiked models (see, for instance, Johnstone (2001), Onatski et al. (2013, 2014)), which leads to beautiful mathematical results such as the phase

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transition phenomenon, the Marčenko-Pastur, and the Tracy-Widom laws but also, due to the "fixed-sized needle in a growing haystack" nature of their asymptotic scenario (see Hallin (2023) for details), somewhat limited practical consequences. More realistic asymptotics, of the "growing needle in a growing haystack" type, simultaneously were adopted by econometricians, which led to the by far more successful dynamic factor methods.

The objective of this paper is a historical account of the emergence of this dynamic factor approach to the analysis of high-dimensional time series, with emphasis on the assumptions, their interpretation, and the "genealogic" aspects of its development. The bibliography is unavoidably limited, and even sketchy; it unevitably but involuntarily overlooks a number of relevant contributions and reflects personal, hence biased, views for which we apologize.

2 Remote ancestry: Spearman (1904) and the psychometric roots of factor models

The family tree of factor models is rooted into early-century psychometrics, and it is usually admitted that the concept first appears more than a century ago in Spearman (1904). Spearman proposes factor analysis in order to account for the dependencies between several variables related with cognitive abilities measured on given individuals. The result was a two-factor theory in which cognitive performance was explained by two unobservable "factors": *general ability* and a second one which later on was dropped as non-significant. The concept of IQ or intelligence quotient usually refers to this general mental ability factor.

The objective of factor models, thus, is to account for cross-sectional dependencies—more specifically, cross-sectional covariances or correlations. Spearman's exposition does not match the mathematical standards of present-day psychometrics or statistics, and more precise mathematical descriptions of factor models only came somewhat later. Classical references are, among others, Hotelling (1933a,b), Bartlett (1937, 1938), Jöreskog (1969); see also Lawley and Maxwell (1971) for a classical textbook exposition, and Jöreskog (2007) for a historical account.

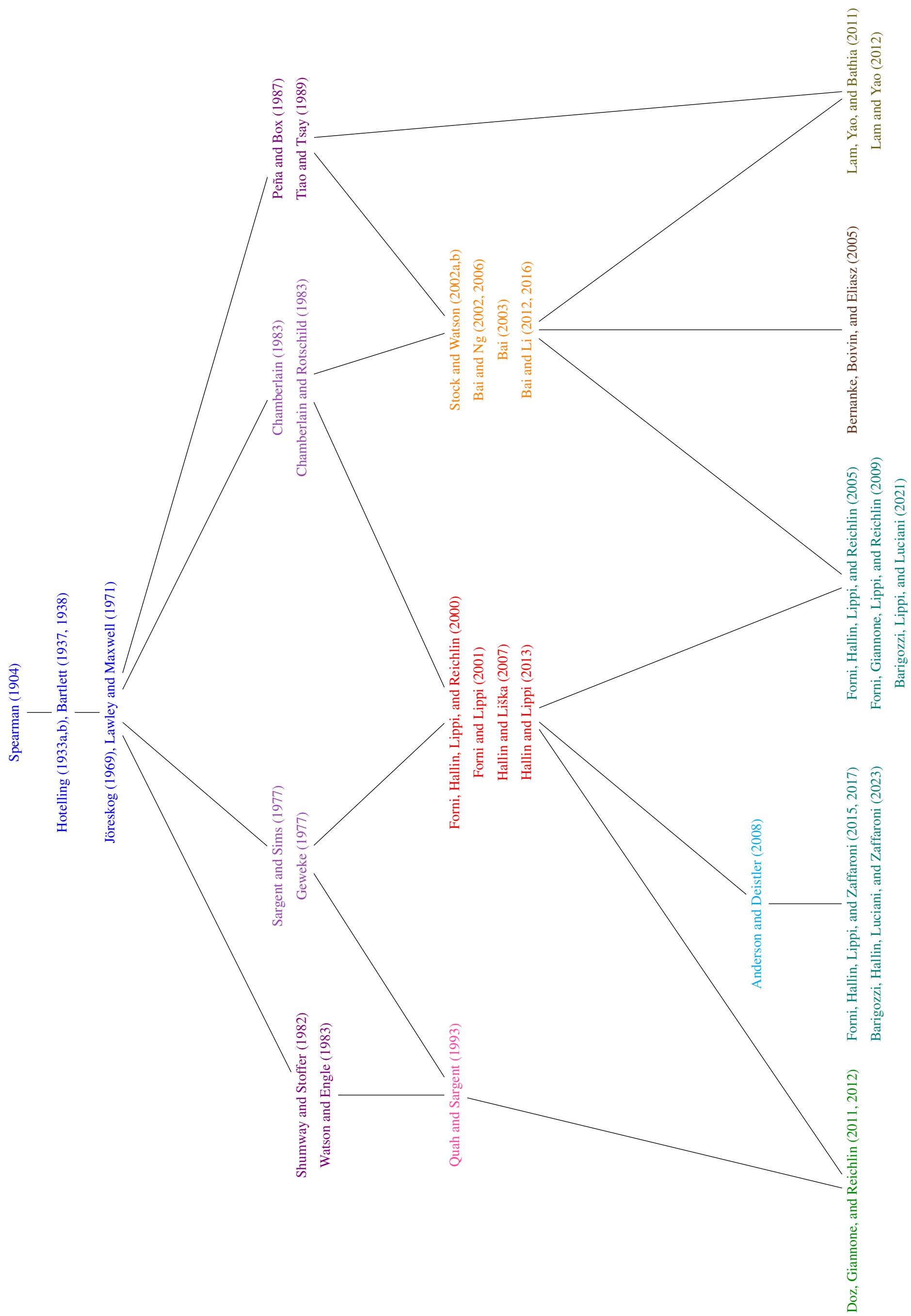
In modern language, a factor model with r factors (it is expected that $r \ll n$, where n is the number of components of the observed variable X) is a statistical model characterized by an equation of the form

$$\mathbf{X}_t = \chi_t + \xi_t := \mathbf{B}\mathbf{f}_t + \epsilon_t, \quad t = 1, \dots T, \tag{1}$$

where

(a) $\mathbf{X}_1, \dots, \mathbf{X}_T$ is an i.i.d. sample of n-dimensional observations $\mathbf{X}_t = (X_{1t}, \dots, X_{nt})'$ with (for ease of exposition and without any loss of generality) zero-mean, strictly positive variance, and finite second-order moments;

(b) $\mathbf{B} = (B_{ik})_{1\leq i\leq n,\, 1\leq k\leq r}$ is an $n \times r$ matrix of scalar *loadings*;


Spearman (1904)
Hotelling (1933a,b), Bartlett (1937, 1938)
Jöreskog (1969), Lawley and Maxwell (1971)
Peña and Box (1987)
Tiao and Tsay (1989)
Chamberlain (1983)
Chamberlain and Rotschild (1983)
Sargent and Sims (1977)
Geweke (1977)
Shumway and Stoffer (1982)
Watson and Engle (1983)
Stock and Watson (2002a,b)
Bai and Ng (2002, 2006)
Bai (2003)
Bai and Li (2012, 2016)
Forni, Hallin, Lippi, and Reichlin (2000)
Forni and Lippi (2001)
Hallin and Liška (2007)
Hallin and Lippi (2013)
Quah and Sargent (1993)
Anderson and Deistler (2008)
Lam, Yao, and Bathia (2011)
Lam and Yao (2012)
Bernanke, Boivin, and Eliasz (2005)
Forni, Hallin, Lippi, and Reichlin (2005)
Forni, Giannone, Lippi, and Reichlin (2009)
Barigozzi, Lippi, and Luciani (2021)
Forni, Hallin, Lippi, and Zaffaroni (2015, 2017)
Barigozzi, Hallin, Luciani, and Zaffaroni (2023)
Doz, Giannone, and Reichlin (2011, 2012)

(c) $\mathbf{f}_t = (f_{1t}, \dots, f_{rt})'$ is an i.i.d. process of latent (unobservable) r-dimensional variables, the *(common) factors* with zero-mean and unit variance;
(d) $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \dots, \epsilon_{nt})'$, called the *idiosyncratic component*, is an i.i.d. process of n-dimensional zero-mean variables with finite diagonal covariance matrix.

This equation decomposes the observation X_{it} into an unosbservable *common component* χ_{it} and an unobservable *idiosyncratic component* $\xi_{it} = \epsilon_{it}$. In order to identify this decomposition, it is assumed, moreover, that the following orthogonality conditions hold:

(i) $\mathrm{E}[f_{kt} f_{\ell t}] = 0,\ 1 \le k \ne \ell \le r,\ \ 1 \le t \le T;$
(ii) $\mathrm{E}[f_{kt} \epsilon_{jt'}] = 0,\ 1 \le k \le r,\ 1 \le j \le n,\ 1 \le t,\ t' \le T;$

In this statistical model, the space spanned by the loadings (that is, by the columns of $\mathbf{B}$) is identified, while the loadings $\mathbf{B}$ themselves and the factors $\mathbf{f}_t$ and the loadings $\mathbf{B}$ are only identified up to pre- and post-multiplication by an arbitrary $r \times r$ orthogonal matrix $\mathbf{O}$ and its inverse $\mathbf{O}'$: indeed, $\mathbf{Bf_t} = \mathbf{BO'Of_t}$ for any such matrix. This indeterminacy, however, should not be interpreted as a weakness—quite on the contrary, it provides a quite precious flexibility in the choice of interpretable factors.

Usual estimation methods are based on Gaussian maximum likelihood, and consistency (up to an orthogonal transformation) of the estimated loadings $\hat{\mathbf{B}}^{(T)}$ is achieved for fixed n as $T \to \infty$; we refer to Anderson and Rubin (1956) and Amemiya, et al. (1987) for a comprehensive coverage of the subject.

This consistency of $\hat{\mathbf{B}}^{(T)}$ as $T \to \infty$, however, does not allow for a consistent recovery of the factors $\mathbf{f}_t$. The latter have to be retrieved, for each t, as the linear projections $\hat{\mathbf{f}}_t^{(T)} := (\hat{\mathbf{B}}^{(T)\prime} \hat{\mathbf{B}}^{(T)})^{-1} \hat{\mathbf{B}}^{(T)\prime} \mathbf{X}_t$ of $\mathbf{X}_t$ onto the estimated loadings; now,

$$
\begin{aligned}
(\hat{\mathbf{B}}^{(T)\prime} \hat{\mathbf{B}}^{(T)})^{-1} \hat{\mathbf{B}}^{(T)\prime} \mathbf{X}_t &= (\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}'\mathbf{B}\mathbf{f}_t + (\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}' \boldsymbol{\epsilon}_t + o_{\mathrm{P}}(1) \\
&= \mathbf{f}_t + (\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}' \boldsymbol{\epsilon}_t + o_{\mathrm{P}}(1)
\end{aligned}
$$

as $T \to \infty$, where $(\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}' \boldsymbol{\epsilon}_t$, which does not depend on T, is not $o_{\mathrm{P}}(1)$ as $T \to \infty$. On the other hand, in view of Assumption (d), if we let the dimension n of $\mathbf{X}_t$ tend to infinity with, for instance, bounded (uniformly in n) diagonal idiosyncratic covariance matrix elements, $(\mathbf{B}'\mathbf{B})^{-1} \mathbf{B}' \boldsymbol{\epsilon}_t$ is $o_{\mathrm{P}}(1)$ as $n \to \infty$. This is the first sign that high-dimensional asymptotics are ideal in the context of factor models—the "blessing of dimensionality" phenomenon that will play a major behind-the-scenes role in the subsequent developments of factor models.

While the factors, in (1), are accounting for all cross-sectional covariances, they do not necessarily account for variances, since idiosyncratic variances can be quite high. Dropping the idiosyncratic component in (1) sometimes is interpreted as a dimension reduction technique, which may be misleading and quite dangerous in the presence of large idiosyncratic variances. This is in sharp contrast with Principal Component Analysis (PCA, introduced by Pearson (1901), three years before Spearman's first paper on factor models), where dropping the last eigenvalues and eigenvectors has a small impact on the total variance of the observation.

Note also that Equation (1), along with conditions (a)–(d) and (i)–(ii), constitutes a *statistical model*—that is, imposes on $\mathbf{X}_t$ restrictions that the distribution of a typical random vector $\mathbf{X}_t$ does not satisfy. Before performing a factor analysis on $\mathbf{X}_t$, thus, one should be cautious and check whether $\mathbf{X}_t$ satisfies the model assumptions. When it does, (1) may or may not be interpreted as describing a data-generating process of the "signal plus noise" type, with the idiosyncratic component playing the role of noise. This, again, is in sharp contrast with PCA, which is "model-free": provided that its second-order moment are finite, indeed, *any* $\mathbf{X}_t$ admits a decomposition into principal components. Tipping and Bishop (1999) showed that if we also assume homoskedasticity of idiosyncratic components then PCA estimates are equivalent to Gaussian maximum likelihood ones. The use of PCA to estimate factor models was first proposed by Hotelling (1933a,b), and then almost forgotten for quite some time, see Section 4.

As we shall see, all factor models are based, as Spearman's original one in (1), on a decomposition of the observation X_{it} into the sum $\chi_{it} + \xi_{it}$ of a common and an idiosyncratic component; they only differ by the various conditions ((a)–(d) and (i)–(ii)) imposed on this decomposition, which characterize various notions of "commonness" and "idiosyncrasy."

3 The pathbreaking generation: Geweke (1977), Sargent and Sims (1977), Chamberlain (1983), Chamberlain and Rothschild (1983)

In the late 1970s, it appeared that traditional econometric time series models—typically, VAR and VARMA models—were defeated by the curse of dimensionality and the increasingly high dimensions of the econometric series to be analyzed and predicted. A simple n-dimensional VAR(1) model, for instance, involves no less than $n(3n+1)/2$ parameters (n^2 autoregressive coefficients, n innovation variances, and $n(n-1)/2$ innovation covariances). This means 610 parameters for $n = 20$, 15,050 for $n = 100$, and 375,250 for $n = 1000$! A solution had to be found, able to deal with both the high-dimensional aspect of the data and their time series nature. The first steps towards that solution, based on serial extensions of the traditional factor model of Section 2, were taken in four pathbreaking papers: Geweke (1977), Sargent and Sims (1977), Chamberlain (1983), and Chamberlain and Rothschild (1983). The publication dates (running between 1977 and 1983) are mainly due to editorial hazards and do not reflect any significant precedence.

3.1 Geweke (1977) and dynamic loadings

Geweke (1977), shortly followed by Sargent and Sims (1977), first understood that, if factor models were to be used in standard econometric problems, some of the conditions imposed in Section 2 were to be relaxed. To begin with, the time-series nature of econometric data cannot be ignored: the i.i.d-ness assumptions in (a) and (c) cannot be maintained. Second—and this is an extremely innovative idea—the value f_{kt} of a factor at time t may be loaded by the observation with some lag: instead of contemporaneous loadings via a loading matrix $\mathbf{B}$ (call them *static loadings)*, Geweke considers *dynamic loadings* via *loading filters* $\mathbf{B}(L)$ where $\mathbf{B}(L) = \sum_{\nu=0}^{\infty}\mathbf{B}_\nu L^\nu$ is an $n\times r$ matrix of one-sided filters with square-summable entries. Here L, as usual, stands for the *lag operator*.

Geweke's is a statistical model characterized by an equation of the form

$$\mathbf{X}_t = \boldsymbol{\chi}_t + \boldsymbol{\xi}_t := \mathbf{B}(L)\mathbf{f}_t + \boldsymbol{\epsilon}_t = \sum_{\nu=0}^{\infty}\mathbf{B}_\nu \mathbf{f}_{k,t-\nu} + \boldsymbol{\epsilon}_t, \quad t = 1,\ldots T, \tag{2}$$

where

(a′) $\mathbf{X}_1,\ldots,\mathbf{X}_T$ is the finite realization an observed n-dimensional second-order stationary process $\mathbf{X} = (X_{1t},\ldots,X_{nt})'$ with (for ease of exposition and without any real loss of generality) zero-mean, strictly positive variance, and finite second-order moments;

(b′) $\mathbf{B}(L) := \left(\sum_{\nu=0}^{\infty} B_{ik\nu}L^\nu\right)_{1\le i\le n,\, 1\le k\le r}$ is an $n\times r$ matrix of *loading filters* with square-summable coefficients for any i and k;

(c′) $\mathbf{f}_t = (f_{1t},\ldots,f_{rt})'$ is a second-order stationary latent r-dimensional process of *factors* with $\mathrm{E}[f_{kt}] = 0$ and $\mathrm{E}[f_{kt}^2] = 1$ for $k = 1,\ldots,r$ and $t\in\mathbb{Z}$;

(d′) $\boldsymbol{\epsilon}_t = (\epsilon_{1t},\ldots,\epsilon_{nt})'$, called the *idiosyncratic component*, is an n-dimensional second-order stationary white noise process with finite diagonal covariance matrix.

It is assumed, moreover, that the following orthogonality conditions hold:

(i′) $\mathrm{E}[f_{kt}f_{\ell t'}] = 0,\ 1\le k\neq \ell \le r,\ 1\le t,t'\le T$;

(ii) $\mathrm{E}[f_{kt}\epsilon_{jt'}] = 0,\ 1\le k\le r,\ 1\le j\le n,\ 1\le t,\ t'\le T$.

Equation (2) is an extension of the classical equation (1); Assumptions (a′)–(d′), clearly, are relaxations of (a)–(d); condition (i′) is reinforcing condition (i) by requiring orthogonality of the factors at all leads and lags; condition (ii) remains unchanged. These conditions define an *exact dynamic* factor model.

Sargent and Sims (1977), under the name *unobservable index model*, and Geweke and Singleton (1981) provide an equivalent frequency-domain description (which we do not reproduce here) under the additional assumption, of course, of the existence of a spectrum for $\mathbf{f}_t$ and $\boldsymbol{\epsilon}_t$, hence for $\mathbf{X}_t$. Thanks to the assumption (d′) that the idiosyncratic processes are mutually orthogonal white noises (their lagged cross-covariances all are zero), the exact dynamic factor model is fully identified (again, up to an orthogonal transformation of the factors) for fixed n; see Geweke and Singleton (1981).

Estimation is typically performed by means of spectral Gaussian maximum likelihood, see Sargent and Sims (1977) and recent work by Fiorentini et al. (2018). As in the classical model , consistency as $T \to \infty$ with n fixed is possible only for the estimated loadings, not for the factors themselves.

3.2 Chamberlain (1983) and the approximate factor model

In Geweke's approach, the dimension n of the observation is fixed. His assumption (d$'$) of mutually orthogonal white noise idiosyncratic components, moreover, is quite unlikely to hold in practical situations—all the more so when n is large: it requires, indeed, that *all* cross-covariances be accounted for by the r factors. On the other hand, removing this condition (d$'$) results in an unidentifiable model where the factor-driven and the idiosyncratic components χ_t and $\boldsymbol{\xi}_t$ cannot be separated as $T \to \infty$. Finally, the high-dimensional nature of the observations, in a fixed-n approach, is not fully taken into account and the "blessing of dimensionality," moreover, is not exploited. This motivated Chamberlain (1983) and Chamberlain and Rothschild (1983) to consider an asymptotic scheme under which both n and T tend to infinity—a double-asymptotics approach that has, since then, become standard in high-dimensional statistics. The $(n, T(n))$ path along which this double-asymptotics scheme is achieved, i.e., the relative magnitude of n and T, in most identification and consistency results, plays no asymptotic role, though: in particular, n can be larger than T.

The model developed by Chamberlain (1983) and by Chamberlain and Rothschild (1983) comes back to the classical equation

$$\mathbf{X}_t^{(n)} = \chi_t^{(n)} + \xi_t^{(n)} := \mathbf{B}^{(n)}\mathbf{f}_t + \epsilon_t^{(n)} \quad t = 1, \ldots T, \;\; n \in \mathbb{N}. \tag{3}$$

This equation coincides with the traditional static-loading factor model of Equation (1), with an important difference, though: superscripts $^{(n)}$ have been added to $\mathbf{X}_t$, χ_t, ξ_t, $\mathbf{B}$, and ϵ_t in order to emphasize the fact that they are n-dimensional, with n tending to infinity just as T does. Chamberlain's assumptions, moreover, are quite mild: while (a$'$) and (c$'$) are borrowed from Geweke and (b) from the traditional model, Chamberlain relaxes Geweke's strong diagonality assumption (d$'$) into

(d$''$) $\epsilon_t^{(n)} = (\epsilon_{1t}, \ldots, \epsilon_{nt})'$, called the *idiosyncratic component*, is an n-dimensional second-order stationary process with finite covariance matrix $\boldsymbol{\Sigma}_\epsilon^{(n)}$.

This covariance matrix $\boldsymbol{\Sigma}_\epsilon^{(n)}$ needs not be diagonal.

As for the orthogonality conditions, Geweke's conditions (i$'$) and (ii$'$) are kept unchanged, but we add two new conditions:

(iii) denoting by $\lambda_{\chi;1}^{(n)} \geq \lambda_{\chi;2}^{(n)} \geq \ldots \geq \lambda_{\chi;r}^{(n)}$ the eigenvalues of the $n \times n$ rank r covariance matrix $\boldsymbol{\Sigma}_\chi^{(n)}$ of $\chi_t^{(n)}$, $\lim_{n\to\infty} \lambda_{\chi;r}^{(n)} = \infty$;

(iv) denoting by $\lambda^{(n)}_{\epsilon;1} \geq \lambda^{(n)}_{\epsilon;2} \geq \ldots \geq \lambda^{(n)}_{\epsilon;n}$ the eigenvalues of the $n \times n$ full-rank covariance matrix $\boldsymbol{\Sigma}^{(n)}_{\epsilon}$ of $\boldsymbol{\epsilon}^{(n)}_t$, $\lim_{n\to\infty} \lambda^{(n)}_{\epsilon;1} < \infty$.

These assumptions define an *approximate static factor model*. A similar setting was independently proposed by Connor and Korajczyk (1986).

A price is to be paid, however, for relaxing the diagonality assumption on $\boldsymbol{\Sigma}^{(n)}_{\epsilon}$: the model is no longer identifiable for fixed n and hence, still for fixed n, is no longer consistently estimable as $T \to \infty$. If, however, the cross-sectional dimension n tends to infinity, identifiability resurfaces under asymptotic form. Indeed, if $\mathbf{X}^{(n)}_t$ satisfies Equation (3) and the assumptions (a′), (b), (c′), (d″), (i)′, (ii″), (iii), and (iv) of the approximate static factor model, it follows from (iii) and (iv) and a straightforward application of Weyl's inequality that there exists a finite $r \in \mathbb{N}$ independent of n such that, denoting by $\lambda^{(n)}_{\mathbf{X};1} \geq \lambda^{(n)}_{\mathbf{X};2} \geq \ldots \geq \lambda^{(n)}_{\mathbf{X};n}$ the eigenvalues of the $n \times n$ covariance matrix $\boldsymbol{\Sigma}^{(n)}_{\mathbf{X}}$ of $\mathbf{X}^{(n)}_t$,

$$\lim_{n\to\infty} \lambda^{(n)}_{\mathbf{X};r} = \infty \quad \text{and} \quad \lim_{n\to\infty} \lambda^{(n)}_{\mathbf{X};r+1} < \infty.$$

Usual estimation approaches are PCA and (pseudo-)Gaussian maximum likelihood. Since we now assume that both n and T tend to infinity, we can consistently estimate both the loadings and the factors (up to orthogonal transformations, as usual). Hence, it is possible to consistently estimate also the common components: letting $n \to \infty$ wards off the curse of dimensionality problem— the previously mentioned *"blessing of dimensionality"* phenomenon.

4 Laying the modern foundations: Stock and Watson (2002), Bai (2003), Forni, Hallin, Lippi, and Reichlin (2000)

The powerful ideas of Geweke, Sargent and Sims, Chamberlain, and Rothschild were not fully exploited until, in the early 2000's, they were picked up, almost simultaneously, by three groups of econometricians working independently of each other. While the models developed by Stock and Watson (2002a,b) and Bai (2003) belong to the lineage of Chamberlain and Rothschild, with double asymptotics and static loadings, the model proposed by Forni et al. (2000) and Forni and Lippi (2001) is combining, under the name *Generalized* or *General Dynamic Factor Model* the attractive features of Geweke (1977) (dynamic loadings via filters rather than matrices) with those of Chamberlain (1983) (an approximate factor model where both n and T tend to infinity). The publication dates (running between 2000 and 2003) are due to refereeing hazards and do not reflect any significant precedence.

4.1 Static loadings and the approximate factor model: Stock and Watson (2002), Bai (2003), and some others

The factor model considered by Stock and Watson (2002a,b) and Bai (2003) (and Bai and Ng (2002) in their paper on the identification of the number of factors: see Section 5) is essentially the same approximate factor model as in Chamberlain (1983) and Chamberlain and Rothschild (1983)—with some minor variations in the assumptions. These papers provide a rigorous treatment of the asymptotic properties of the PCA-based estimators of the model, and show, as expected, that if both n and T tend to infinity, consistency (up to orthogonal transformations, as usual) is achieved for the loadings and the factors. Typically, once factors are extracted via PCA from an n-dimensional (n large) time series $\mathbf{X}_t$, they are used in a second step to predict a given target variables. This approach, in general, offers sizeable improvements over univariate or small-nd forecasting models, see, e.g., Stock and Watson (2002a,b) for empirical evidence.

These papers had a major impact on the econometric literature and largely contributed to the dissemination and the development of contemporary factor model methods. Further notable developments along the same lines are the study of *factor augmented prediction* (Bai and Ng 2006) and the high-dimensional extension of classical results on pseudo-Gaussian maximum likelihood estimation (Bai and Li 2012, 2016).

Predating Stock and Watson (2002a,b) and Bai (2003), one also should mention two groups of earlier contributions which considered extensions of the exact factor model apt to capture specific aspects of the observed time series:

(I) Engle and Watson (1981), Shumway and Stoffer (1982), Watson and Engle (1983), and Quah and Sargent (1993) adopted a "state-space approach"' where a dynamic equation for the factors, e.g., a VAR specification, is added to the static factor model, specifying a parametric structure for the factors' autocorrelations;

(II) Peña and Box (1987) and Tiao and Tsay (1989) revisited the exact static factor model in a time series context, thus assuming the idiosyncratic components to be a second-order stationary white noise process.

Approach (I) was extended to the high-dimensional setting $n \to \infty$ by Doz et al. (2011, 2012) who considered the use of the Kalman filter combined with Gaussian maximum likelihood estimation via the Expectation Maximization algorithm. This approach is among of the most frequently used in macroeconomic policy analysis; it is employed for now-casting (Giannone et al. (2008)) and for building indicators of economic activity (Barigozzi and Luciani (2021)). See also Poncela et al. (2021), for a survey.

Approach (II) was extended to the high-dimensional setting $n \to \infty$ by Lam et al., (2011) and Lam and Yao (2012) who still consider principal-component-based estimation but based on a sum of autocovariances—under an assumption of white noise idiosyncratic components which, however, is unlikely to hold in practice.

4.2 Dynamic loadings and the General Dynamic Factor Model: Forni, Lippi, Hallin, and Reichlin (2000)

The General or *Generalized* Factor Model (henceforth GDFM) was proposed by Forni et al. (2000) and Forni and Lippi (2001). It is combining the dynamic loadings ideas of Geweke (1977) and Sargent and Sims (1977) with the double asymptotics ($n, T \to \infty$) of Chamberlain (1983) and Chamberlain and Rothschild (1983). Dynamic loadings allow for capturing the lagged impacts of the common factors driving the common component

The following presentation is inspired from the time-domain exposition of Hallin and Lippi (2013), which avoids the spectral-domain approach originally used by Forni and Lippi (2001) to derive the of the GDFM. Its spirit also slightly differs from that of Lippi et al. (2023).

In this approach the observation, an $n \times T$ panel, is the finite realization, for $1 \le i \le n$ and $1 \le t \le T$ (n and T large), of a double-indexed stochastic process $\mathbf{X} := \{X_{it} | i \in \mathbb{N},\ t \in \mathbb{Z}\}$, that is, a collection of n observed time series of length T, related to n individuals or "cross-sectional items" or, equivalently, one single time series in dimension n. We denote by $\mathbf{X}_t^{(n)}$ the n-dimensional vector $(X_{1t}^{(n)}, \dots, X_{nt}^{(n)})'$, by $\mathbf{X}_t$ the fixed-t collection $\{X_{it} | i \in \mathbb{N}\}$, and by $\mathbf{X}^{(n)}$ the n-dimensional process $\{X_{it} | 1 \le i \le n,\ t \in \mathbb{Z}\}$. It is assumed throughout that $\mathbf{X}$ is second-order time-stationary, i.e., for all values of i, i', i'', t, and k, the variances $\mathrm{Var}(X_{it})$ and covariances $\mathrm{Cov}(X_{i't} X_{i'',t-k})$ exist, are finite, and do not depend on t. For simplicity, we also assume that all X_{it}'s have zero-mean centered and, in order to avoid trivialities, are non-degenerate: $\mathrm{E}[X_{it}] = 0$ and $0 < \mathrm{E}[X_{it}^2] < \infty$ for all $i \in \mathbb{N}$ and $t \in \mathbb{Z}$.

Under these assumptions, denote by $\mathcal{H}^{\mathbf{X}}$ the Hilbert space spanned by $\mathbf{X}$, equipped with the L_2 covariance scalar product, that is, the set of all L_2-convergent linear combinations of X_{it}'s and limits of L_2-convergent sequences thereof. Similarly, denote by $\mathcal{H}_t^{\mathbf{X}}$, $\mathcal{H}^{\mathbf{X}^{(n)}}$, and $\mathcal{H}_t^{\mathbf{X}^{(n)}}$ the subspaces of $\mathcal{H}^{\mathbf{X}}$ spanned , respectively, by $\{X_{is} |\ i \in \mathbb{N},\ s \le t\}$, by $\{X_{is} |\ 1 \le i \le n,\ t \in \mathbb{Z}\}$, and by $\{X_{is} |\ 1 \le i \le n,\ s \le t\}$. Let $\eta_0 := \sum_{i=1}^{\infty} \sum_{s=-\infty}^{\infty} a_{is} X_{is} \in \mathcal{H}^{\mathbf{X}}$. Then,

$$\eta_t := \sum_{i=1}^{\infty} \sum_{s=-\infty}^{\infty} a_{i,s+t} X_{i,s+t} \in \mathcal{H}^{\mathbf{X}} \quad \text{for all } t \in \mathbb{Z},$$

and we say that the process $\boldsymbol{\eta} := \{\eta_t | t \in \mathbb{Z}\}$ belongs to $\mathcal{H}^{\mathbf{X}}$.

The main idea in Hallin et al. (2000) and Forni and Lippi (2001) consists in decomposing $\mathcal{H}^{\mathbf{X}}$ into two mutually orthogonal subspaces $\mathcal{H}_{\mathrm{com}}^{\mathbf{X}}$ and $\mathcal{H}_{\mathrm{idio}}^{\mathbf{X}} := \left(\mathcal{H}_{\mathrm{com}}^{\mathbf{X}}\right)^{\perp}$ where $\mathcal{H}_{\mathrm{com}}^{\mathbf{X}}$ denotes the subspace spanned by the limits of all sequences of standardized linear combinations $\sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} a_{ik} X_{i,t-k}$ of the past, present, and future values of X_{it}'s with squared coefficients summing up to one exhibiting exploding variances as $n \to \infty$.

More precisely, call *common* any random variable ζ in $\mathcal{H}^{\mathbf{X}}$ with variance $0 < \sigma^2_\zeta$ such that ζ/σ_ζ is the limit in quadratic mean of a sequence $w^{(n)}_{\mathbf{X}}/(\mathrm{Var}(w^{(n)}_{\mathbf{X}}))^{1/2}$ of standardized elements of $\mathcal{H}^{\mathbf{X}}$ of the form

$$w^{(n)}_{\mathbf{X}} := \sum_{i=1}^{n}\sum_{k=-\infty}^{\infty} a^{(n)}_{ik} X_{i,t-k} \quad \text{with} \quad \sum_{i=1}^{n}\sum_{k=-\infty}^{\infty} (a^{(n)}_{ik})^2 = 1$$

such that $\lim_{n\to\infty} \mathrm{Var}(w^{(n)}_{\mathbf{X}}) = \infty$.

Define the Hilbert space $\mathcal{H}^{\mathbf{X}}_{\mathrm{com}}$ spanned by the collection of all common variables in $\mathcal{H}^{\mathbf{X}}$ and its orthogonal complement (with respect to $\mathcal{H}^{\mathbf{X}}$) $\mathcal{H}^{\mathbf{X}}_{\mathrm{idio}} := \left(\mathcal{H}^{\mathbf{X}}_{\mathrm{com}}\right)^{\perp}$ as $\mathbf{X}$'s *common* and *idiosyncratic spaces*, respectively. A process $\mathbf{X}$ is called *purely common* if $\mathcal{H}^{\mathbf{X}} = \mathcal{H}^{\mathbf{X}}_{\mathrm{com}}$ (hence, $\mathcal{H}^{\mathbf{X}}_{\mathrm{idio}} = \{0\}$), *purely idiosyncratic* if $\mathcal{H}^{\mathbf{X}} = \mathcal{H}^{\mathbf{X}}_{\mathrm{idio}}$ (hence, $\mathcal{H}^{\mathbf{X}}_{\mathrm{com}} = \{0\}$).

Projecting each X_{it} onto $\mathcal{H}^{\mathbf{X}}_{\mathrm{com}}$ and its orthogonal complement $\mathcal{H}^{\mathbf{X}}_{\mathrm{idio}}$ yields the factor model decomposition

$$X_{it} = \chi_{it} + \xi_{it}, \quad i \in \mathbb{N}, \;\; t \in \mathbb{Z} \tag{4}$$

of X_{it} into a *common component* χ_{it} and an *idiosyncratic* component ξ_{it}, respectively: call (4) the *General Dynamic Factor (GDFM) representation* of $\mathbf{X}$.

Contrary to the factor model decompositions (1), (2), and (3) previously considered, the GDFM decomposition (4) is *endogenous*; it always exists, and does not impose any restriction (beyond second-order stationarity) on the data-generating process of $\mathbf{X}$. In that sense, (4) is not a statistical *model*, but a representation result. Whether it constitutes the description of a data-generating process or not is relevant to the analysis. This representation result nature of (4) was first emphasized in Forni and Lippi (2001) where, however, a frequency domain approach is adopted.

So far, indeed, no assumption has been imposed on the second-order stationary process $\mathbf{X}$. Adding the requirement that, for any $n \in \mathbb{N}$, $\mathbf{X}^{(n)}$ admits a *spectral density matrix* $\theta \mapsto \mathbf{\Sigma}^{(n)}(\theta)$, $\theta \in (-\pi, \pi]$ with eigenvalues $\lambda^{(n)}_{X;1}(\theta) \geq \lambda^{(n)}_{X;2}(\theta) \geq \ldots \geq \lambda^{(n)}_{X;n}(\theta)$ such that

$$\lim_{n\to\infty} \lambda^{(n)}_{X;q}(\theta) = \infty \quad \text{and} \quad \lim_{n\to\infty} \lambda^{(n)}_{X;q+1}(\theta) < \infty, \quad \theta\text{-a.e. in } (-\pi, \pi],$$

for some finite $q \in \mathbb{N}$ independent of n, it can be shown (see Hallin and Lippi (2013)) that all $\{\chi_{it} | t \in \mathbb{Z}\}$'s are driven by a q-dimensional orthonormal white noise process $\{\mathbf{u}_t = (u_{1t}, \ldots, u_{qt})' | t \in \mathbb{Z}\}$ of *common shocks*. The GDFM decomposition (4), in that case, takes the form

$$X_{it} = \chi_{it} + \xi_{it} =: \sum_{i=1}^{q} \mathbf{B}_i(L)\mathbf{u}_t + \xi_{it} \quad i \in \mathbb{N}, \;\; t \in \mathbb{Z} \tag{5}$$

for some collection $\mathbf{B}_i(L) := \left(B_{i1}(L), \ldots, B_{iq}(L)\right)$ of one-sided linear $1 \times q$ square-summable filters $B_{ij}(L) := \sum_{k=0}^{\infty} B_{ijk} L^k$, $i \in \mathbb{N}$, $j = 1, \ldots, q$. We refer to Section 3.3

of Hallin and Lippi (2013) for the relation between the spectral density matrices $\mathbf{\Sigma}^{(n)}(\theta)$ and the filters $\mathbf{B}_i(L)$.

It follows from the above results that the GDFM (loading filters and factors) is asymptotically identified as $n \to \infty$. Forni et al. (2000) show that the common and idiosyncratic components χ_{it} and ξ_{it} can be consistently estimated, as $n, T \to \infty$, via dynamic (spectral) PCA, a technique introduced by Brillinger (2001) which, unfortunately, involves two-sided filters, hence performs poorly at the ends of the observation period—making it unsuitable in the context of prediction roblems. Forni et al. (2017) show that also the loadings and the factors can be consistently estimated, as $n, T \to \infty$, via a multi-step approach based on an equivalent autoregressive representation (not presented here) derived from the results by Anderson and Deistler (2008) and Forni et al. (2015) on singular stochastic processes. See also the recent results by Barigozzi et al. (2023). This latter approach only involves one-sided filters, and allows for constructing GDFM-based forecasts. Forni et al. (2018) show that such forecasts improve over the Stock and Watson (2002a) ones based on the static factor model.

4.3 Restricted General Dynamic Factor Model: Forni, Hallin, Lippi, and Reichlin (2005) and Forni, Giannone, Lippi, and Reichlin (2009)

Forni et al. (2005) consider a restricted version of the GDFM where it is assumed that the infinite singular moving average representation for the common component in (5) is in fact finite with maximum lag equal to s, say. In this case, the GDFM can be written as a model with static loadings and an $r = q(s+1)$-dimensional common factor process $\mathbf{F}_t = (\mathbf{u}_t', \dots, \mathbf{u}_{t-s}')'$. Note that under this model we must have $q \le r$, a finding often supported by data, see, e.g, the evidence in D'Agostino and Giannone (2012). The lagged values $\mathbf{u}_{t-1}, \dots, \mathbf{u}_{t-s}$ of the "original factors," moreover, should satisfy the pervasiveness conditions leading to condition (iii) of Chamberlain's *approximate static* factor model.

This *restricted General Dynamic Factor Model* is also often called the *approximate dynamic* factor model where, however, "dynamic" refers to the nature of the factors rather than to their loadings, which are static. Estimation is typically by dynamic PCA to recover the common component spectral density matrix plus classical (static) PCA on the recovered common component covariance matrix. This approach is particularly successful in forecasting and the construction of coincident indicators of economic activity, as, e.g, the EuroCoin indicator by Altissimo et al. (2010).

Forni et al. (2009) further assume that the factors follow a VAR process. It then becomes immediately clear that such a VAR has to be singular as soon as $s > 0$, i.e., the innovations must have dimension $q < r$. For more details, see the survey by Stock and Watson (2016). Clearly, this approach is almost equivalent to the state-space approach described in Section 4.1, with the only non-trivial difference that now the VAR for the factors is singular. Estimation is typically by PCA plus classical estimation of a VAR on the resulting estimators of the latent factors $\mathbf{F}_t$. Applications are in the macroeconometrics field of impulse response analysis, where the common shocks $\mathbf{u}_t$ (or an orthogonal transformation thereof) are identified as sources of economic fluctuations, see, e.g., Bernanke et al. (2005).

5 Identifying the number of factors

Irrespective of the choice of a factor model and the estimation method adopted, identifying the number r of factors in the static loadings case and/ the number q of common shocks in the GDFM is a crucial preliminary step. A number of methods have been considered.

The first one, based on information criteria, was proposed by Bai and Ng (2002) to determine r in the approximate static model, followed by Hallin and Liška (2007) to determine q in the GDFM. The latter work also proposes a tuned penalty version of the information criterion method. Back to the static model, Alessi et al. (2010) improve on Bai and Ng by combining their criterion with the same tuning idea. In a restricted GDFM setting, Amengual and Watson (2007) are adapting the Bai and Ng procedure to determining q, while Bai and Ng (2007) propose a way to jointly estimate r and q.

Another strand of methods is based on the empirical distribution of the eigenvalues of the covariance matrix of the observations. Ahn and Horestein (2007) propose an "eigenvalue ratio" and a "growth ratio" criterion. The eigenvalue ratio criterion consists in selecting as the number r of factors the number $\hat{r}$ that maximizes the ratio between the kth and the $(k+1)$th eigenvalues arranged in decreasing order of magnitude—a variant of the classical (and often decried) *scree test* method searching for a "clear break" in the spectrum. The growth ratio criterion proceeds similarly, now with the growth rates of the idiosyncratic variances associated with the fitting of k factors. More recently, Avarucci et al. (2022) developed dynamic counterparts, based on the eigenvalues of the spectral density matrix of the observations, of the eigenvalue ratio and growth ratio estimators of Ahn and Horenstein (2013). Finally, Onatski (2010) and Trapani (2018) consider the behavior of the difference between the the kth and the $(k+1)$th eigenvalues, showing that this difference, for $k = r$ diverges to infinity while it converges to zero for any k larger than r. Onatksi (2009) considers instead the asymptotic behavior of the eigenvalues of the spectral density matrix of the observations in order to determine q.

Popular factor number estimators, however, often suffer from the lack of significant eigengap in empirical eigenvalues and tend to over-estimate r due, for example, to the existence of non-pervasive factors affecting only a subset of the series, or the presence of moderate cross-sectional correlations in the idiosyncratic components. Barigozzi and Cho (2020) show how such overestimation can compromise the consistency of the principal component estimator in the approximate static factor model. They also propose a remedy involving a modified principal component estimator based on a rescaling of the sample eigenvector entries; this modified estimator is shown to be robust against the overestimation of r.

6 " ... as the stars of the heaven and as the sand which is upon the seashore"[1]

The foundational contributions described in Section 4 have triggered a veritable explosion of further papers on the subject, some of them refining or extending, some others developing related problems, along with countless applications in a variety of fields—much beyond econometrics and finance. Dynamic factor models, indeed, have emerged as a successful and widely used tool for analyzing the information contained in observed high-dimensional time-series (large panels of time series data), thereby obtaining now-casts and short-term forecasts of economic activity, financial volatility, inflation, etc. Factor models are used also in environmental and climatic sciences (Marotta and Mumtaz 2023), in health and biomedical studies (Peracchi and Rossetti 2022), and even, back to the origins of the method, in psychology (Molenaar and Ram 2009).

A Google search on "Dynamic Factor Model" brings no less than 435 million entries—*as many "as the stars of the heaven and as the sand which is upon the seashore!"* Below is a short personal, obviously highly incomplete, and unavoidably biased list of some of the uncountable offsprings of Stock and Watson (2002), Bai (2003), and Forni et al. (2000); we regroup them by subject.

Factor models and volatilities. Factor models as described in the previous sections only are dealing with unconditional covariances. In many applications—certainly so in finance—conditional covariances and volatilities are at least as important. Therefore, many papers have been devoted to volatilities and their forecasts in factor models. The basic and natural idea is a decomposition of volatilities into common and idiosyncratic. Some authors (Ng et al. 1992; Harvey et al. 1992; Connor et al. 2006; Sentana et al. 2008; Fan et al. 2015) are considering the volatility of the common components as the common volatility, neglecting as idiosyncratic the volatility of the idiosyncratic components. A different point of view is adopted in Barigozzi and Hallin (2016, 2017a, 2020) and Trucìos et al. (2022), where it is argued that the volatility of an idiosyncratic component, for instance, may well be exposed to common volatility shocks, and the volatility of a common component be affected by an idiosyncratic volatility shock, so that none of them should be neglected in the forecast exercise. This is used, e.g., in Hallin and Trucìos (2022), to produce forecasts of Value-at-Risk and Expected Shortfall of large portfolios.

Factor models and robustness. Factor model methods remain second-order ones: being based on second- order dependence structures,they are bound to be

[1] Genesis 22:17, King James version.

sensitive to the possible presence of outliers. Many outlier detection procedures are available in the time series literature, most of them restricted to univariate series. Relatively little attention has been given to robustness issues in the context of high-dimensional time series and factor models, though, with a handful of references such as Kristensen (2014) and Baragona and Battaglia (2007), who show that both the traditional (static) PCA methods and the more general dynamic PCA methods yield biased estimates in the presence of outliers; more recent contributions include Fan et al. (2018, 2019), Trucìos et al. (2019), Alonso et al. (2020), and He et al. (2022, 2023). The robustness of the Forni et al. (2015, 2017) approach is investigated in Trucìos et al. (2021).

Factor models with blocks and hierarchical factor models. A panel often decomposes into subpanels or blocks, according, for instance, to geographic criteria. Such features are treated via a (finite-n, large T) hierarchical static factor model by Kose et al. (2003) and (large n and T) Moench et al. (2013), in full generality and a (large n and T) GDFM by Hallin and Liška (2011), Hallin et al. (2011), and Barigozzi et al. (2018b).

Locally stationary factor models. In practice, time series observed over a long period of time T seldom are stationary; the evolution over time of the data-generating process, however, often can be considered to be smooth. This, in the univariate case, has motivated the development, initiated by Dahlhaus (1997), of the so-called *locally stationary* approach where it is assumed that the second-order structure is evolving slowly over time. Barigozzi et al. (2021a) consider the estimation of a time-varying version of the GDFM in which the factors are loaded via time-varying filters. A slightly different time-varying version of the GDFM, inspired by Forni et al. (2000), had been previously studied by Eichler et al. (2011) which, however, is entirely based on Brillinger's dynamic principal component analysis, hence suffers from the main drawback of dynamic principal components, which resorts to two-sided filters to recover the space spanned by the factors. Such two-sided filtering makes the Eichler et al. (2011) approach unsuitable for forecasting and impulse response analysis. This two-sidedness issue, in Barigozzi et al. (2021a), is taken care of via the one-sided approach developed in Forni et al. (2015, 2017). An approximate static version with time-varying loading matrices is considered in Hafner et al. (2011).

Integrated factors. Bai and Ng (2004) and Barigozzi et al. (2021b) extend the static factor model to allow for the presence of unit roots, both in the factors and in the idiosyncratic processes. While Bai and Ng (2004) develop a testing procedure for the presence of unit roots, and Barigozzi et al (2020, 2021b) assume the joint presence of unit roots jointly and deterministic trending components and study principal-component-based estimation as well as the estimation of a vector error correction model for the factors. This allows for computing impulse response functions in presence of permanent and transitory shocks. It is important to notice that a realistic factor model must allow for the idiosyncratic component to be integrated too—unless we assume the n observed series to be cointegrated with cointegration rank $(n-r)$, where r is the number of factors which, in this case, are also common trends (see Bai (2004) and Barigozzi and Trapani (2022)).

Functional factor models. Gao et al. (2019) consider the problem of forecasting high-dimensional functional time series via a heuristic two-stage approach combining a truncated-PCA dimension reduction and separate scalar factor-model analyses of the resulting (scalar) panels, conducted via an eigendecomposition of the long-run covariance operator, as opposed to the lag-zero covariance operator considered in static factor models; the truncation of the PCA decomposition, moreover, potentially may lead to a dramatic loss of information. In Gao et al. (2021), the same authors propose a factor-augmented version of their approach. Tang et al. (2021) propose a functional factor model allowing both factors and loadings to be functional, Guo et al. (2021) a functional factor model with functional factors and scalar loadings. In both cases, the model is considered as the description of a very specific data-generating process (no representation result), hence requires being checked; all the component functional time series, moreover, have to take values in the same Hilbert space, which is somewhat restrictive. Hallin et al. (2023); Tavakoli et al. (2023), on the other hand, extend to the functional case the approximate static model of Chamberlain, and establish a representation result; the component time series may take values in different Hilbert spaces, including the real line.

Factors plus networks. Among other possible dimension reduction techniques, sparse regressions, as lasso or ridge, are the main competitors of factor analysis which favors instead a dense modelling of the data. A crucial question to be asked in empirical analysis then is whether a given dataset is sparse or dense. It has been shown empirically that most economic time series datasets have a dense structure, so factor analysis should be preferred as it is likely to deliver better forecasts (De Mol et al. 2008). On the other hand, especially in financial data, once we control for common factors, there is evidence of non-negligible dependencies left in the idiosyncratic components (Barigozzi and Hallin 2017b). The theoretical properties of a factor plus sparse approach are studied by Fan et al. (2023) in the static loadings framework and by Barigozzi et al. (2023) in the GDFM framework. Idiosyncratic components there are modelled as a sparse VAR and the estimated coefficients, which are sparse matrices, have often been given a network interpretation in which edges represent non-zero conditional correlations.

Factors and breaks. The presence of structural breaks or change-points represents another important cause of deviation from the assumption of stationarity. Many papers have investigated this problem from an off-line perspective in the static factor model approach, proposing various tests for the presence of breaks and, possibly, estimators for their location: see Breitung and Eickmeier (2011), Chen et al. (2014), Han and Inoue (2014), Corradi and Swanson (2014), Yamamoto and Tanaka (2015), Cheng et al. (2016), Baltagi et al. (2017), Bai et al. (2017), Ma and Su (2018), Duan et al (2023). The case of the restricted dynamic factor model has been studied by Barigozzi et al. (2018a) who not only allow for changes in the loadings, but also in the number of factors, and in the autocorrelation function of the factors, as well as change-points in the idiosyncratic second-order structure. A similar approach but for the GDFM setting is in Cho et al. (2023). Finally, the issue of on-line, i.e., sequential, testing for change-points in factor models which, despite of its importance for practice, has not received much attention, is considered in Barigozzi and Trapani (2020).

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