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Model-based Fault Detection and Isolation for a Class of Discrete-Time Systems

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ABSTRACT

During the last decades, robust model-based fault detection and isolation (FDI) has received increasing interest in the literature. It is a fact that FDI methods can provide fault indicators for predictive maintenance and, consequently, preventing significant damages to the system components. Following this route, this doctorate thesis develops model-based fault detection and isolation techniques for linear discrete-time systems as well as for linear discrete-time descriptor systems. The proposed approach is based on the design of observer-based residual generators in order to detect and isolate actuator and/or sensor faults. Besides, a reference model is considered to describe the desired behavior from faults to residual, which allows for including requirements on fault decoupling as well as performance issues regarding robustness related to the optimization problem. Therefore, the residual generator is designed such that the system-filter connection approximately follows the fault to residual behavior provided by the reference model. A multi-objective optimization is designed involving different system criteria, such as H_∞/H_- and *peak-norm*/H_- , which depends on the class of inputs of the system (respectively, ℓ_2 or ℓ_∞ signals). The design conditions based on modified Lyapunov dissipation inequalities are cast in terms of Linear Matrix Inequalities (LMI) constraints. In order to illustrate the theoretical developments, numerical examples are provided as well as the application of the method to a simulated lithium-ion battery pack. There are four main contributions for this thesis. The design of FDI techniques integrated with a triangular reference model structure allows the fault detection and isolation when the faults do not occur simultaneously. Besides, the residual generator design conditions described in terms of LMI constraints made possible to consider mixed performance specifications and different classes of systems in a unified mathematical formalism. From a model transformation, the proposed method can also be applied to discrete-time descriptor systems. Finally, the proposed solution in this thesis can be successfully considered for practical-oriented applications, such as Li-ion battery packs.
Keywords: Fault detection and isolation. Linear Matrix Inequalities. Reference model. Robustness.

LIST OF FIGURES

ABBREVIATIONS

FD	Fault diagnosis.
FDI	Fault, detection and isolation.
LMI	Linear matrix inequality.
LTI	Linear time invariant.
SDP	Semidefinite programming.
LDTS	Linear discrete-time system.
LDTDS	Linear discrete-time descriptor system.

LIST OF SYMBOLS

$\mathbb{N}$	Set of Natural numbers.
$\mathbb{Z}$	Set of Integer numbers.
$\mathbb{Z}_{\geq}$	Set of nonnegative Integer numbers.
$\mathbb{R}$	Set of Real numbers.
$\mathbb{R}_{\geq}$	Set of nonnegative Real numbers.
$\mathbb{R}_{>}$	Set of positive Real numbers.
$\mathbb{R}^n$	Set of n dimensional real vectors.
$\mathbb{R}^{m\times n}$	Set of m-by-n real matrices.
$\mathbb{S}^n$	The space of real symmetric matrices in $\mathbb{R}^{n\times n}$, that is $\mathbb{S}^n = \{M \in \mathbb{R}^{n\times n} : M = M^T\}$.
I_n	Identity matrix of size $n \times n$.
$0_{m\times n}$	Matrix of zeros of size $m \times n$.
0_n	Matrix of zeros of size $n \times n$.
A^T	Transpose of a real matrix A.
A^{-1}	Inverse of a square full rank real matrix A.
$A > 0$	The symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite.
$A \geq 0$	The symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite.
$\mathrm{He}\{A\}$	Hermitian operator for a square matrix A, i.e., $\mathrm{He}\{A\} = A + A^T$.
$\mathrm{diag}\{\cdot\}$	Block-diagonal matrix.
$\star$	The transpose of the blocks outside the main diagonal block for a symmetric block matrix.
$\|f\|_2$	The 2-norm of a real valued vector $f(k)$, i.e., $\|f\|_2 = \sqrt{f(k)^T f(k)}$.
ℓ_2	The space of square summable vector sequences with norm $\|f\|_{\ell_2} = \sqrt{\sum_{k=0}^{\infty} f(k)^T f(k)}$.
ℓ_∞	The space of bounded vector sequences with norm $\|f\|_{\ell_\infty} = \sup_k \sqrt{f(k)^T f(k)}$.
f^+	The one-step-ahead shift operator of a sequence $f(k)$, i.e., $f(k+1)$.
$\|\mathcal{G}_{uy}\|_\infty$	The H_∞-norm of a matrix transfer function from u to y.
$\|\mathcal{G}_{uy}\|_{peak}$	The *peak*-norm (or H_{peak}) of a matrix transfer function from u to y.
$\|\mathcal{G}_{uy}\|_{-}$	The smallest singular value of a matrix transfer function from u to y, often refereed as the H_{-} index.

CONTENTS

1 CONTEXT AND MOTIVATION

For safety-critical systems such as energy storage systems, robotic systems or nuclear reactors, the occurrence of faults can have serious consequences in terms of human life, environmental impact and economic loss (CHEN, Ji; PATTON, R. J., 1999). There are many examples in terms of the effects caused by a fault that can be highlighted such as:

- the Boeing 737 max crashes in November 2010, originally due to sensor malfunction and the lack of exploitation of sensor redundancy;
- the disaster that occurred on 26^{th} April 1986 at Chernobyl nuclear power plant, Ukraine. The accident had a huge impact on the workers and local residents, causing health problems to thousands of people due to radiation exposure;
- the blaze on board a parked Boeing 787 Dreamliner at Heathrow Airport in July 2013, which was caused by the lithium battery-powered emergency locator transmitter of the plane. Due to the fact that its location is of difficult access, the passengers would not instantly notice the fire in the case of the aircraft being in flight which would eventually lead to an enormous tragedy;
- the explosion of an unmanned Ariane 5 rocket in French Guiana on 4^{th} June 1996, which happened less than one minute after its launch. A software error in the flight control system, more specifically in the inertial reference system, was the cause of the failure, resulting in an economic loss of $7 billion due to the costs regarding its development;
- overheating and burning of certain Samsung mobile phones, in particular the Note 7, due to faults occurring in their batteries. The main causes were insufficient insulation material within the battery, and lack of room to safely accommodate the electrodes, which could lead to a short circuit in case of improper manufacturing, and consequently a fire. After replacement of many mobile phones, which ended up having the same problem, the company permanently stopped the production of the Note 7.

While in most of the cases the occurrence of a fault cannot be prevented, analyses carried out subsequently reveal that the outcome caused by a fault could be avoided or, at least, minimized (KANEV, 2004). Hence, a process monitoring is fundamental in order to decrease the probability of having catastrophic events as those summarized above. In addition, a system capable of detecting and localizing faults is essential in order to prevent significant damages to other system components.

Another issue related to fault detection and isolation is the maintenance. Maintenance can be classified in three main categories: reactive, proactive and predictive. The first one is defined by the repair or the replacement of the equipment only when it stops working. This strategy minimizes the number of maintenance operations during its lifetime. However, this type of scheme makes the system to be unpredictable, and it can increase the maintenance costs in case of a bigger failure of the system caused by adopting the reactive maintenance strategy (SWANSON, 2001). A solution for this problem is to adopt a proactive maintenance, which is based on preventive actions with the view of monitoring the equipment and making minor repairs in order to prevent failure. According to (SWANSON, 2001), this kind of strategy can increase equipment lifetime, however the proactive maintenance requires interruption of production in order to perform the repair and, consequently, delaying the work. Hence, the predictive maintenance focuses on monitoring the equipment condition and its operation efficiency in order to provide enough information to maximize the interval between repairs as well as to minimize the occurrence and costs of equipment failures. This maintenance method is able to improve production effectiveness along with product quality (MOBLEY, 2002). The monitoring system that provides regular information on the state of health of the process is also called a fault diagnosis (FD) system.

As an example, wind turbines are often located in remote areas and hard-to-access structures, increasing the cost of operation and maintenance of the system. One way of dealing with wind turbines is the reactive maintenance, where the turbines work until a fault occurs, however this method considerably decreases the safety of the system. On the other hand, FD techniques can provide fault indicators for predictive maintenance, consequently avoiding faults that could occur due to the lack of maintenance in the wind turbines (LU et al., 2009). In this way, FD can help reducing maintenance costs by allowing the planning of maintenance operations in due time and hence avoiding costly unexpected plant shutdown and costly repair actions due to failure of some plant parts. In addition, a system capable of detecting and localizing faults is essential in order to prevent significant damages to other system components due to fault propagation.

Let us now provide some background on fault diagnosis in order to position the work and the contributions of this thesis.

1.1 BACKGROUND ON FAULT DIAGNOSIS AND PROBLEM STATEMENT

Modern control systems are becoming more complex and more automated since the demand for reliability, efficiency and operating safety in real processes is continuously growing (CHEN, Ji; PATTON, R. J., 1999). In a reliable system, the individual components, such as actuators, sensors and controllers, have a high probability of working properly over a specified time period. However, even in these conditions, we cannot avoid faults (DING, S. X., 2008). A fault can be defined as an unexpected change in the system parameters so its behavior does no longer correspond to the intended purpose. According to (BLANKE et al., 2006), from a physical viewpoint, a fault may represent an internal event in the system, a wrong control action or measurement, an error in the system design (the error is detected only in specific operation conditions, where the performance of the system is considerably reduced), among others. In terms of its general location, a fault can be classified into three types (BLANKE et al., 2006):

- plant faults – the dynamical properties of the system are changed;
- actuator faults – the influence of the manipulated input on the system is modified; and
- sensor faults – the measurement from the sensors may present inconsistency.

This work will be focused on the latter two types of faults, that is, actuator and sensor faults.

Process monitoring is important considering that, even if the occurrence of a fault may not induce a failure or a breakdown, it can change the performance of the system components and, consequently, cause the loss of system functionality (NOURA et al., 2009). The field of Fault Diagnosis (FD) has attracted recurring interest during the last decades, both in a research context and in the application on real processes, in order to ensure a safe and reliable system operation (ZHANG, Z.; JAIMOUKHA, 2014; JAIMOUKHA et al., 2006). Fault Diagnosis systems comprehend three main levels (BLANKE et al., 2006; NOBREGA, E. G. et al., 2000):

- fault detection: determines whether a fault has occurred or not;
- fault isolation: determines the location of the fault, that is, which component is faulty;
- fault identification: estimates the magnitude of the fault.

In most cases, FD systems are referred to as Fault Detection and Isolation (FDI) systems since fault identification is not necessarily required (MONTEIRO, 2015). In the present work, the FD system to be studied will embrace only the detection and isolation levels.

Regarding the way faults are modeled, a distinction is made between *additive faults*, and *multiplicative faults*. The first ones correspond to an unknown input in the system model, like a sensor bias. On the other hand, multiplicative faults are associated to changes in some model parameters which multiply the system input or state (BLANKE et al., 2006; KANEV, 2004).

Fault diagnosis systems rely on different approaches broadly classified into data-driven and model based methods (ZHONG, M. et al., 2018). The data-driven approaches address the FDI problem by analyzing historical records and online data. They rely on large data bases and learning methods to determine the system state of health. Broadly speaking they consist in determining a mapping between some data features and the different health states (healthy mode, fault 1 mode, fault 2 mode, ...). As faults have to be distinguished from changes in operating conditions, sufficient data must be available to cover all operating modes. When a process model is available it can be exploited to filter out operating mode changes automatically. Hence less data are required to design and tune model-based FDI systems. Besides, model-based methods are able to mathematically describe the process dynamics while keeping a physical meaning (often quite important on practical applications). Furthermore, there exist several mature results available in control theory providing powerful tools for model-based fault diagnosis.

Most model-based methods rely on the concept of analytical redundancy (BLANKE et al., 2006; GERTLER, 1991). The latter amounts to check whether the mathematical model of the system is consistent with the actual system behavior characterized by the recorded measurements (BLANKE et al., 2006). A discrepancy is an indication of a fault, disturbances, noises or modeling errors (GERTLER, 1997). According to (FRISK; NIELSEN, 2006), a central concept in FDI is the *residual*, which is a signal that verifies the consistency of the system at every time t based on the difference:

$$r(t) = f(y(t)) - f(\hat{y}(t)) \ , \tag{1}$$

where r is the residual vector, $f(y)$ is a function of the measured output vector of the system, and $f(\hat{y})$ represents its estimation. In the fault-free case, the residual signal is zero (or close to zero), on the other hand, when a fault occurs the residual significantly (BLANKE et al., 2006) departs from zero. The residual generation can be accomplished for systems described in continuous- and discrete-time. This work will concentrate on discrete-time models.

According to (MONTEIRO, 2015), there are three main model-based approaches for residual generation: parity relations, parameter estimation, and observer-based methods. The first method checks the consistency between the data recorded on the process over a finite moving horizon and an input/output model of this process over that horizon; for more information, see (GERTLER, 1997). The parameter estimation approach looks for changes in the parameters of the system dynamics via a system identification strategy (MONTEIRO, 2015). Lastly, the observer-based method is based on a mathematical model of the system and the development of an observer in order to estimate the states of interest (ZHANG, Z.; JAIMOUKHA, 2014). It is important to emphasize that the parity relations and the observer-based methods are designed to deal with additive faults primarily, while the parameter estimation method focuses on multiplicative faults.

Systems are generally influenced by unknown and unpredictable disturbances and noises. Moreover, there always exists a mismatch between the real process and its mathematical model (FRISK; NIELSEN, 2006). Hence, the main goal in the design of a residual generator is to increase the residual sensitivity to faults and simultaneously minimize the influence of non-fault signals such as disturbances and model uncertainties (NOBREGA, Euripedes G. et al., 2005). Since it is difficult to achieve a perfect disturbance decoupling, H_∞ techniques have been widely investigated in order to transform the decoupling problem into a sensitivity optimization problem (JAIMOUKHA et al., 2006). The solution to the FDI problem can be formulated based on different H_∞ methods such as Ricatti equation based techniques (MARCOS et al., 2005) and Linear Matrix Inequalities (LMIs) (NOBREGA, E. G. et al., 2000), (WANG, J. L. et al., 2007), (ZHONG, Maiying et al., 2003). When compared with Ricatti equation based approaches, LMI methods have some advantages, for instance, they are less restrictive in the design conditions, and the solutions for robust problems are less conservative (NOBREGA, E. G. et al., 2000). In addition, it is relatively easy to incorporate additional constraints into the formulation of LMIs (WANG, J. L. et al., 2007).

This present work aims to develop FDI schemes for discrete-time systems targeting the application in energy storage systems based on lithium-ion battery packs. More specifically, this thesis presents a novel robust observer-based FDI design technique for linear discrete-time systems as well as for a class of discrete-time descriptor systems, for which the design conditions are cast in terms of a convex optimization problem subject to LMI constraints. The results to be presented in this document consider mixed sensitivity specifications (either H_∞/H_- or H_{peak}/H_-) guaranteeing sensitivity to fault detection and isolation while mitigating the effects of unknown inputs.

1.2 PROPOSED APPROACH

The focus will be on observer-based residual generation because of the flexibility such schemes provide when accounting for multiple criteria as indicated above. Fault detection and isolation requires the definition of a coding set that characterizes how faults will affect the set of residuals. This information is typically specified in a so-called incidence matrix as illustrated in Table 1. Three residual and three faults are considered in the example. The crosses correspond to significantly non zero entries while the zero entries indicate faults that should affect the corresponding residual as little as possible. Once that matrix is defined the design of residual generator can be performed by stating a fault detection problem associated to each row of the incidence matrix. There is however no systematic way to choose the incidence matrix or to possibly optimize its structure; see, e.g., (MASSOUMNIA et al., 1989), (GERTLER, 1997) and (PATTON, R.; CHEN, J., 1997).

Table 1 – Incidence matrix.
Font: Own authorship.

	f_1	f_2	f_3
r_1	0	X	X
r_2	X	0	X
r_3	X	X	0

Fault isolation can be enforced in a more direct way by designing a filter that estimates the fault vector from the measured process input and output. However, this amounts to inverting the plant model to recover the fault input, which is a non-causal operation and hence cannot be achieved in practice. Therefore, to state a realistic problem, a reference model is introduced. According to (FRISK; NIELSEN, 2006), the reference model allows for including requirements on fault decoupling as well as performance issues regarding the transfer function from faults to the residual. This means that the use of a reference model contributes for a consistent optimization problem, ensuring good robustness properties and fulfillment of the structural constraints for fault isolation purposes. Hence, in this work, the residual reference model will be considered to describe the desired behavior from faults to residual of the residual generator with respect to faults as illustrated in Fig. 1, where $u(k)$ is the measured input, $w(k)$

is the disturbance input, $f(k)$ is the fault vector, $y(k)$ is the measured output, $r(k)$ represents the residual vector, $\breve{r}(k)$ is the desired residual, and $e_r(k)$ stands for a mismatch between $r(k)$ and $\breve{r}(k)$.

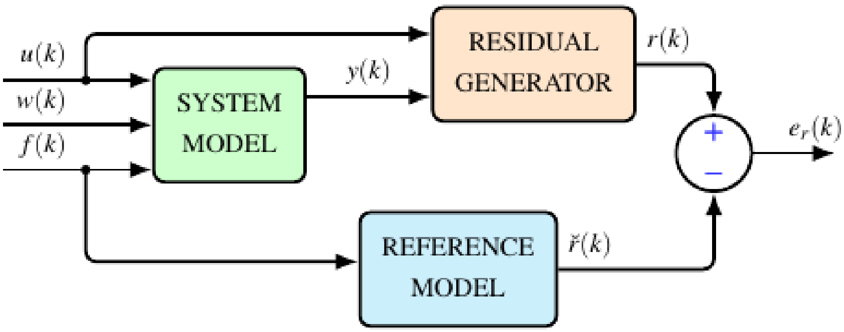


Figure 1 – Proposed setting for residual generation design.
Font: Own authorship.

In the setting shown in Fig. 1, the residual generator (i.e., an observer-based filter) should be designed such that the system-filter connection approximately follows the fault to residual behavior provided by the reference model. In order to properly establish tractable design conditions, measures of the input-to-output mappings (i.e., from w to r and from f and e_r) such as the H_∞ norm should be considered. In addition, if it is desirable to have a degree of freedom when defining the residual reference model a measure of the sensitivity from f to $\breve{r}$ will play an important role when designing the FDI scheme. Hence, a multi-objective optimization involving different system norms will be necessary (LI, Z. et al., 2011). For instance, in specialized literature, the H_- index (i.e., the minimum singular value of the matrix transfer function from fault to residual) is applied to measure the worst-case sensitivity from f to r, while the H_∞ norm is considered as a worst-case robustness measure (WANG, J. L. et al., 2007). If the residual generator can be designed in such a way that the series arrangement of the process and the residual generator has a behavior close to a diagonal reference model, then each residual is only sensitive to a single fault and simultaneous faults can be detected and isolated. However, this situation requires that the number of measurements is larger than or equal to the number of faults and disturbances. This condition is in general not fulfilled. In particular, if faults can affect each actuator and each sensor of a monitored process, a diagonal reference model cannot be considered. That is why other structures need to be considered for the reference model. The only condition to impose for fault isolation is that each column of the model transfer matrix is different from the others in terms of zero and non-zero components. In this way, each residual is expected to react only to certain faults and not to others, and the pattern of zero and non-zero residuals allows for fault isolation. To account for this observation while keeping mathematically tractable problems, a triangular reference model is chosen in the present work.

In this study, which applies a residual reference model approach with one degree of freedom (precisely, the reference model output gain), multi criteria constraints are considered to ensure: (*i*) low sensitivity to disturbances; (*ii*) low sensitivity from f to e_r (to yield an approximate model following); and (*iii*) a minimum sensitivity from f to $\check{r}$. In particular, the design conditions are established in terms of either H_∞/H_- or *peak*-norm/H_- mixed specifications depending on the class of inputs (respectively, ℓ_2 or ℓ_∞ signals). Moreover, the design conditions based on modified Lyapunov dissipation inequalities are cast in terms of LMI constraints which are numerically solved utilizing well-established SDP solvers.

In order to illustrate the theoretical developments, numerical examples are provided as well as the application of the method to a simulated lithium-ion battery pack. Particularly for the latter case study, the implementation of FDI systems based on state observers is important due to the fact that the estimation of system states that cannot be measured results in increased safety and reliability. When considering single a battery cell, for instance, only voltage, current and surface temperature can be measured using standard sensors, while the internal states of the battery, such as the state of charge (SOC) and critical surface concentration (CSC), cannot be measured (COUTO, 2018). In fact, the CSC directly affects the battery performance. For this reason, the measurement of the surface temperature (as well as current and cell voltage) is not sufficient and it is fundamental to estimate the battery internal state in order to increase the system reliability (ZHANG, C. et al., 2016).

1.3 THESIS STRUCTURE

The remaining of this thesis is divided into six chapters describing the proposed solution to address the FDI problem for linear discrete-time systems considering either a standard or descriptor state-space representation. Chapter 2.1 presents the state of the art of several approaches which are somehow related to the results covered by this doctorate thesis. Some key linear algebra results related to (LMI-based) robust control theory are presented in Chapter 3, which will be instrumental to derive the main results of this doctorate thesis. In Chapter 4, mixed H_∞/H_- and *peak-norm*/H_- reference model-based FDI methods are designed for linear discrete-time systems and they are applied to numerical examples to certify the effectiveness of the proposed techniques. Following the same route, Chapter 5 proposes two reference model-based FDI approaches targeted to linear discrete-time descriptor systems along with a numerical example demonstrating the efficiency of the proposed results. In Chapter 6, the performance of the FDI methodology proposed in the latter chapter is assessed considering a linear approximate model of a Li-Ion battery pack (precisely, two battery cells in series). The conclusions, contributions and future research of this doctorate thesis are discussed in Chapter 7. This thesis also contains four appendices. The first ones, Appendices A and B, present the proofs of the main theorems proposed throughout this work, while Appendices C and D summarize some proposed results on observer design for a class of nonlinear discrete-time systems, which can be in the future adopted to develop FDI systems for nonlinear systems.

2 STATE OF THE ART

The topic of FDI has been studied both in a research context and industry applications, in particular, in those looking for safety and reliability for the solution of control schemes (see (PATTON, R. J., 1994), (WANG, J. L. et al., 2007), (ZHONG, Maiying et al., 2003) and references therein). In (ISERMANN, 2005), for instance, an introduction for FDI is given and some results from different case studies are presented, such as a cabin pressure outflow valve actuator of a passenger aircraft, monitoring of the lateral driving behavior of passenger cars, and combustion engines. In a theoretical point of view, survey papers such as (HWANG et al., 2010) present an overview of recent works on FDI, for instance, methods to generate and test residuals for faults, as well as techniques to improve the robustness against unknown disturbances, noise, and model uncertainties. In (PATTON, R. J., 1994), the state of the art for robust fault diagnosis is reviewed with the main focus on residual generation and fault isolation. The paper (ISERMANN, 1997) introduces a general survey of supervision, fault detection and diagnosis approaches, such as parameter estimation, observer-based and parity equations, focusing on model-based fault detection methods.

Model-based FDI techniques focused in residual generation have been the object of numerous research works. In (GERTLER, 1997), the design of parity relations for fault detection and isolation is described, including decoupling from disturbance and some types of model uncertainties. A new kind of parity relation to solve the optimization problem for residual generation is proposed by (YE et al., 2004), which results showed a satisfactory and suitable fault detection when compared with other parity relation based approaches existing in the literature. Regarding the parameter estimation approach, (ISERMANN, 1993) provides an overview on fault diagnosis methods applied to machines, for example, a motor, a drive chain and a working process or load. However, the most widespread FDI approach is based on the use of observers to generate residual signals (see (CHEN, Ji; PATTON, R. J., 1999), (PATTON, R. J., 1994), and references therein).

In recent years, the application of observer techniques has been studied to deal with FDI issues. For instance, in (YANG et al., 2015) the analysis and design of observer-based fault detection for nonlinear systems is addressed. In addition, a scheme is proposed for a class of nonlinear systems, as well as different cases of application are discussed. The paper (MEYNEN et al., 2020) proposes a FDI strategy applied to distributed and decentralized systems. The main idea is to design a robust observer-based fault detection and isolation, where the last one was performed separately in order to decrease computational time. Simulation results on a three tanks system illustrated the effectiveness of the approach by uniquely detecting and isolating each fault. Differently, (HAES ALHELOU et al., 2020) proposes robust FDI schemes based on unknown input observer method applied to isolate faulty sensors on a micro-grid system, which simulation results show the effectiveness of the technique by considering cases of simultaneous faults. Other approaches can be referred for observer-based FDI systems, such as (COMMAULT et al., 2000) and references therein.

Following the trend in control system theory, H_∞ optimization methods have been applied due to the issue of robustness of the FDI schemes. An H_∞ Ricatti-based FDI technique is applied to a nonlinear system (the Boeing 747-100/200) in (MARCOS et al., 2005). Five H_∞ filters are designed at five different equilibrium points using the same interconnection and weights in order to study possible strategies to cover the entire flight envelope, where disturbances and model uncertainties are taken into account to provide robustness to the FDI scheme. Results show that the H_∞ FDI filters are capable of detecting and isolating faults for a determined neighborhood of the equilibrium points, on the other hand, the authors concluded that the filter performance and robustness properties can be improved by an adequate selection of the filter inputs, for instance.

Even though Ricatti-based techniques have been adopted for the design of H_∞ FDI filters, methods based on LMIs have proven to be very efficient to deal with robustness issues. In (NOBREGA, E. G. et al., 2000), an H_∞ FDI filtering approach is proposed to be designed for both the continuous-time and the discrete-time. The problem is cast in terms of an H_∞ LMI-based formulation with the view to minimize the worst-case estimation error over all bounded energy generalized disturbance (which includes disturbances and faults). A simulation example of a spring-mass-damper model is used to demonstrate the applicability of the method. Also considering an LMI-based formulation, the design of a bank of nonlinear H_∞ observers for sensor FDI is studied in (MATTEI et al., 2005). Taking into account that the system may have significant nonlinearities, the number of observers is determined to be the same as the number of faults due to the fact that faults may occur on different sensors (which makes the fault isolation hard to achieve in this case). A numerical example is presented in order to show the effectiveness of the proposed method in a large operation region by using a multi-model approach. This method ensures a low number of false alarms as well as detecting and isolating most of the sensor faults.

Since FDI systems seek for minimizing unknown input effects while the sensitivity to faults is maximized in the residual vector, a single H_∞ norm is not sufficient. For this reason, multi-objective approaches, such as H_-/H_∞, have been developed recently. In (WANG, J. L. et al., 2007), the H_- index and multi-objective H_-/H_∞ fault detection observer design problems are addressed in terms of LMI constraints, where both the infinite frequency range case and the finite frequency range case are considered. The proposed methods provide only an indirect approach to the H_- index and H_-/H_∞ observer problems for strictly proper systems, letting the development of a more direct approach for further research. In (LIANG et al., 2021), the focus is on FDI for a network of discrete-time multi-agent systems with disturbances. A bank of unknown input observers is proposed, where the optimization problem is cast in terms of LMI constraints, in order to guarantee sensitivity to faults while robust against non-decoupled unknown inputs. Another example is given by (HENRY, David, 2008), where a study is carried out to compare two model-based H_∞/H_- FDI filter schemes. The main idea is to apply the proposed approaches in the Microscope satellite thrusters in order to detect and isolate faults despite the presence of disturbances and noises. Simulation results are presented with the view to show that the faults are successfully detected and isolated.

Additionally, observer-based FDI approaches considering different system criteria have been applied to several classes of systems recently. In (SCHONS et al., 2020b), a robust filter is designed considering a mixed H_{-}/H_{∞} fault sensitivity and disturbance measure. The paper considers a triangular reference model structure with partially fixed dynamics with the view of approximating the behavior from faults to residual despite the presence of disturbances. The filter design is cast in terms of LMI constraints. Other works can be mentioned regarding the application of H_{-}/H_{∞} schemes, such as (NOBREGA, Euripedes G. et al., 2005), (AOUAOUDA et al., 2013), (JAIMOUKHA et al., 2006), (LIU, Nike; ZHOU, Kemin, 2007) and references therein.

Considering a class of nonlinear Lipschitz systems with unknown inputs, (KAMEL et al., 2009) proposes two approaches based on unknown input observer design to study the FDI problem. Both approaches are cast in terms of LMI constraints, where the first one makes use of residual techniques dedicated to FDI of actuator faults while the second approach is concerned with detecting and estimating an additive fault. In (PERTEW et al., 2005), where the design of an unknown input observer for nonlinear Lipschitz systems is addressed, the proposed method is equivalent to an H_{∞} optimization problem considering the same sufficient conditions as well as satisfying all the assumptions. Considering this Lipschitz observer design, the problem of sensor fault diagnosis for the class of nonlinear Lipschitz systems is studied in (PERTEW et al., 2007). The main idea is to build an observer based on an LMI design procedure to maximize the faults effect on the estimation error of the observer.

The H_{∞} observers design has also been extended for Lipschitz nonlinear descriptor systems in several research works. For instance, (ABBASZADEH, Masoud; MARQUEZ, Horacio J., 2012) proposes a robust LMI-based H_{∞} filtering method for continuous-time Lipschitz descriptor systems in the presence of norm-bounded uncertainties and disturbances. Simulation results show that the proposed H_{∞} filter guarantees the robustness against Lipschitz nonlinear uncertainties as well as asymptotic stability of the estimation error dynamics. An observer design in the LMI framework is also considered for discrete-time descriptor systems in (WANG, Zhenhua et al., 2012), which addresses the problem for the linear and nonlinear cases. In (SAHEREH et al., 2017), both continuous-time and discrete-time descriptor systems are considered for the design of H_{∞} filters in strict LMI formulations. The problem of designing a robust H_{-}/H_{∞} fault detection filter for a class of Lipschitz nonlinear descriptor systems is addressed in (BOULKROUNE et al., 2013). The idea is to minimize the effect of unknown inputs on the residual vector by using the H_{∞} norm while the sensitivity to faults is ensuring by the H_{-} index, in the LMI framework.

The main concern of FDI schemes is to derive a solution for the residuals such that the system is sensitive to faults while remaining insensitive to unknown inputs, such as disturbances, noises, and model uncertainties. Besides, the pattern of the residual upon occurrence of each fault should be different in order to ensure fault isolation. The design of robust FDI approaches using a reference model of the difference between the inputs and the residuals has been the object of several research works. For instance, (JAIMOUKHA et al., 2006) design an FDI observer such that the transfer matrix from faults to the residual is as close as possible to a diagonal transfer matrix, while minimizing the effects of disturbances in the residual vector. In order to illustrate the application of its algorithm, a jet engine example is considered, where results show that the proposed scheme can achieve fault isolation and provide disturbance decoupling. In (MAZARS et al., 2008), a reference model is designed for robust FDI residual generation using the H_∞ norm as a measure for the fault isolation and disturbance rejection performances. The solution is given by solving an LMI optimization problem. A numerical example considering a jet engine state-space model is given to illustrate the effectiveness of the proposed method. Other research methods can be referred for FDI approaches including a reference model, such as (FRISK; NIELSEN, 2006) and (ABDALLA et al., 2001).

2.1 A CONCISE REVIEW ON OBSERVER-BASED METHODS

Observer-based methods consider the output estimation error (or some function of this error) as residual. This error is built as the difference between the measured system output and its estimate provided by a state observer. Examples of such state observers are: Luenberger observer (REINELT; LUNDQUIST, 2005), Kalman filter (HEREDIA; OLLERO, 2009), unknown input observers (CHEN, W.; SAIF, 2006), H_∞ observers (MARCOS et al., 2005), among others.

As already pointed out in Section 1, even though there is an equivalence between the parity relations and the observer-based methods due to the fact that, under some hypothesis, observer-based approaches can be particularized to parity space regarding the detection and isolation of actuator and sensor faults, there are advantages related to the application of observer-based methods (MAGNI; MOUYON, 1994). Although the parity space approach is able to handle disturbances in the system with the addition of a filter to this end, the design of observer-based residual generators can be constrained in such a way that attenuation of disturbances and the rejection of unknown inputs is included in their design, making easy to handle noise in the system as well as multiple faults if the number of measurements is sufficient. From the implementation point of view for residual generation, parity relations approaches make use of a non-recursive form considering a finite sliding window. On the other hand, observer-based methods represents a recursive form corresponding most often to an infinite time horizon (DING, S. X., 2008), (CHOW; WILLSKY, 1984). The applicability of observer-based approaches is also large such as its implementation to nonlinear systems considering particular classes of nonlinearities (CHEN, Ji; PATTON, R. J., 1999).

For detecting a fault, a single residual is sufficient, however, the isolation of faults is only achieved with a set of residuals (GERTLER, 1991). Observer-based approaches can be used in order to design two different classes of residual sets, such as:

- fixed direction residuals: the residual vector has a fault specific direction in response to the occurrence of a single fault (GERTLER, 1991);
- structured residuals: only a specific subset of residuals is sensitive to a subset of faults while staying insensitive to the remaining faults (FRISK; NIELSEN, 2006), (PATTON, R. J., 1994).

The focus of this work is fault detection and isolation for linear discrete-time systems subject to additive (actuator and sensor) faults and disturbances. The main goal is to generate residuals that achieve a suitable trade-off between sensitivity to the faults and insensitivity to disturbances. Following this route, the H_- index is usually introduced to describe the worst-case sensitivity of the residual to the fault, as it is presented in (HOU; PATTON, R., 1996) and further developed in (LIU, J. et al., 2005), (LI, X.; LIU, H., 2013). On the other hand, the H_∞ norm is used for evaluating the (worst-case) disturbance effect on the residual. Solutions to the FDI problem based on H_∞ techniques have been proposed in numerous papers. For instance, in (JAIMOUKHA et al., 2006) the design of an H_∞ filter is developed such that the transfer matrix from faults to residual approximates a given diagonal transfer matrix including the minimization of disturbance effects. The authors in (MARCOS et al., 2005) develop H_∞ Ricatti-based FDI filters applied to a linearized model of an aircraft. The efficiency of these approaches has lead to the design of H_-/H_∞ fault detection observers or filters (DING, S. et al., 2000), (HOU; PATTON, R., 1996). Linear Matrix Inequality (LMI) approaches have also been proposed to solve multi-objectives optimization problems involving H_- and H_∞ type criteria (HENRY, D.; ZOLGHADRI, 2005), (WANG, J. L. et al., 2007), (JEE et al., 2012), and the present work also follows this route.

In the most line of research above mentioned, fault isolation is not considered explicitly. Even though residuals can be determined by solving successively fault detection problems in order to enforce the structure (MASSOUMNIA et al., 1989),(GERTLER, 1997),(PATTON, R.; CHEN, J., 1997), there is no systematic way to choose this structure. Fault isolation can be enforced in a more direct way by designing a filter that estimates the fault vector. This problem has been addressed, e.g., in (STOUSTRUP; NIEMANN, 2002), (MANGOUBI, 1998). A series of multi-objective FDI problems were considered using a reference model and taking into account the effect of disturbances and/or modelling uncertainties on the FDI filter output. The solutions were devised either in the form of observer-based residual generators (LI, Z.; JAIMOUKHA, 2009), (JAIMOUKHA et al., 2006) or residual generators with a general linear filter structure (CASAVOLA et al., 2005), (MARCOS et al., 2005), (MAZARS et al., 2008), (NOBREGA, E. G. et al., 2000). In most of the cases, the reference model is given a priori, often in the form of a diagonal transfer matrix. However, the determination of a reference model for fault detection has been addressed in (FRISK; NIELSEN, 2006) and (ZHONG, Maiying et al., 2003). In (MAZARS et al., 2008), the determination of the residual generator as well as the reference model is sought for FDI. The problem is recast in terms of quadratic matrix inequalities (QMIs) which cannot be solved exactly. Hence, an upper bound on the solution is achieved by linearizing the QMIs, leading to an algorithm cast in terms of a set of LMIs. This gives a valid solution for any structure of the reference model, even though the authors focus on a diagonal structure. Yet, this is restrictive as such a structure cannot be enforced as soon as the number of faults is larger than the number of measured outputs. The latter situation occurs frequently: it suffices to consider possible occurrence of faults on all the actuator and the sensors for a given plant (KINNAERT; PENG, 1995).

This doctorate thesis follows the same line of thought as presented in (MAZARS et al., 2008). However, in this work, a discrete-time setting is considered. Moreover, a triangular reference model structure with fixed dynamics is designed, which lends itself to an LMI approach. Therefore, the focus of the present work is on the automated determination of the residual generator structure with a view to fault detection and isolation.

3 INSTRUMENTAL TOOLS

This chapter introduces some results from control and systems theory of LTI discrete-times systems which will be instrumental to derive FDI methods for LTI discrete-time systems. In the sequel, the main mathematical tool (i.e., linear matrix inequality constraints) for numerically solving FDI design conditions is presented. Next, the two concepts of system norms (i.e., H_∞ and *peak*-norm) and a sensitivity index (i.e., H_- to be considered in this doctorate thesis are presented in order to ensure some performance specifications for FDI systems. This chapter ends by introducing LMI conditions for the computation of H_∞ and *peak* norms and the H_- sensitivity index.

3.1 LINEAR MATRIX INEQUALITIES

Linear matrix inequalities (LMIs) have been widely recognized and accepted nowadays. A basic definition for LMIs is that they are matrix inequalities that are linear in the matrix variables (YU; DUAN, 2013).

LMIs may arise in a wide diversity of mathematical problems. In control systems area, one of the most well-known LMIs is the continuous-time Lyapunov matrix inequality, which is described by

$$\mathcal{L}(P) = A^T P + PA + Q < 0, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{S}^n$, with $Q > 0$, are given matrices, and $P \in \mathbb{S}^n$ is a positive definite matrix to be determined. In (2), the notation $\mathcal{L}(P) < 0$ states that all eigenvalues of $\mathcal{L}(P)$ are negative or, equivalently, that $\mathcal{L}(P)$ is negative definite.

Now, the general form of an LMI will be defined. To this end, let $Q \in \mathbb{S}^n$, $G_i \in \mathbb{R}^{n\times n}$, and $R, H_i \in \mathbb{R}^{m\times n}$, $i = 1, 2, \dots, p$, be given and consider the following symmetric matrix:

$$\mathcal{M}(X) = R^T X + X^T R + Q + \sum_{i=1}^{p}(H_i^T X G_i + G_i^T X^T H_i), \tag{3}$$

with $X \in \mathbb{R}^{m\times n}$. Notice that $\mathcal{M}(X)$ is linear with respect to X. Hence, the general form of an LMI in X can be defined as

$$\mathcal{M}(X) < 0, \tag{4}$$

with X to be determined.

In particular, this doctorate thesis is concerned with linear discrete-time systems and the discrete-time Lyapunov matrix inequality is given:

$$A^T PA - P + Q < 0, \tag{5}$$

with A and Q being given and $P > 0$ to be determined, which can be viewed as a particular form of (4).

The analytical expression of the solution of an LMI is almost impossible to be obtained for more than 2 scalar decision variables. However, there exist several computational packages dedicated to derive a numerical solution to LMIs. Normally, LMI *solvers* work with scalar decision variables and there are applicatives (called *parsers*) which translate the general form in (3) to specific representations utilized by *solvers*. In particular, the solution of LMIs considered in this doctorate thesis were obtained considering the *parser* YALMIP (LÖFBERG, 2004) and *solver* MOSEK (APS, 2019) both running over MATLAB.

3.2 SOME USEFUL LINEAR ALGEBRA TOOLS

Typically, control related problems are not directly described in terms of LMI constraints. In this sense, there exist linear algebra tools that are used to overcome this issue by rewriting the problem of interest in terms of LMIs.

In the following, three results will be presented which are instrumental to derive LMI constraints in this doctorate thesis. The first one is the Schur's complement which will be largely utilized to turn a matrix inequality linear with respect to decision variables (YU; DUAN, 2013).

Lemma 1 *(Schur's complement) Let $X \in \mathbb{S}^{(n+m)}$ and consider the following partition:*

$$X = \begin{bmatrix} X_{11} & X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}, \quad X_{11} \in \mathbb{S}^n, \; X_{21} \in \mathbb{R}^{m \times n}, \; X_{22} \in \mathbb{S}^m.$$

Hence, the following statements hold:

1. *If X_{11} is nonsingular, then $X > 0$ if and only if*

$$X_{11} > 0, \quad X_{22} - X_{21} X_{11}^{-1} X_{21}^T > 0.$$

2. *If X_{22} is nonsingular, then $X > 0$ if and only if*

$$X_{22} > 0, \quad X_{11} - X_{21}^T X_{22}^{-1} X_{21} > 0.$$

Proof 1 *Consider the first statement of Lemma 1 and let the following matrix:*

$$T = \begin{bmatrix} I_n & 0 \\ -X_{21} X_{11}^{-1} & I_m \end{bmatrix}.$$

Now, pre- and post-multiplying $X > 0$ by T and its transpose, respectively, yields

$$\begin{bmatrix} I_n & 0 \\ -X_{21} X_{11}^{-1} & I_m \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} I_n & -X_{11}^{-1} X_{21}^T \\ 0 & I_m \end{bmatrix} > 0$$

$$\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} - X_{21} X_{11}^{-1} X_{21}^T \end{bmatrix} > 0$$

Therefore, statement 1 holds. The second statement of Lemma 1 can be similarly derived, which completes the proof.

Subsequently, the following lemma is largely utilized in control and optimization problems. This is an important LMI-based tool, since it makes it possible to deal with LMIs coupled to equality constraints (OLIVEIRA; SKELTON, 2001; KUSSABA et al., 2015; BRIAT, 2015).

Lemma 2 *(Finsler) Let $y \in \mathbb{R}^n$, $Q \in \mathbb{S}^n$ and $B \in \mathbb{R}^{m\times n}$, with $\text{rank}(B) = q$, $q \leq n$, and let $B^\perp$ denote a basis for the null space of B. Then, the following statements hold:*

(*i*) $y^T Q y > 0$, *for all* $By = 0$, $y \neq 0$;

(*ii*) $\left(B^\perp\right)^T Q B^\perp < 0$;

(*iii*) *There exists* $\mu \in \mathbb{R}$ *such that* $Q - \mu B^T B < 0$;

(*iv*) *There exists* $K \in \mathbb{R}^{n\times m}$ *such that* $Q + KB + B^T K^T < 0$.

The complete proof of Lemma 2 can be found in (OLIVEIRA; SKELTON, 2001). Notice that statement (*i*) is a constrained quadratic form, where the vector y is constrained to be in the null space of B, which can be parameterized by the vector $w = B^\perp y$, with $w \in \mathbb{R}^q$. Then, if the statement (*ii*) is pre- and post-multiplied by w^T and w, respectively, the statement (*i*) is recovered. In addition, if statements (*iii*) and (*iv*) are pre- and post-multiplied by y^T and y, hence $y^T Q y > 0$ for all y satisfying $By = 0$.

To end this section, the “S-Procedure", which allows to concatenate several scalar inequality constraints into a single one, is presented in the following (GHAOUI et al., 1994).

Lemma 3 *(S-Procedure) Let $\zeta \in \mathbb{R}^n$ and $F_i \in \mathbb{S}^n$, $i = 0, 1, \ldots, m$. Then, the following condition:*

$$\zeta^T F_0 \zeta \geq 0, \ \forall\ \zeta \ \text{ such that } \ \zeta^T F_i \zeta \geq 0, \ \forall\ i = 1, \ldots, m, \tag{6}$$

holds if there exist positive scalars $\tau_1, \ldots, \tau_m$ such that

$$F_0 - \sum_{i=1}^{m} \tau_i F_i \geq 0. \tag{7}$$

Proof 2 *Pre- and post-multiplying (7) by ζ^T and ζ, respectively, leads to*

$$\zeta^T F_0 \zeta \geq \sum_{i=1}^{m} \tau_i \zeta^T F_0 \geq 0,$$

since $\zeta^T F_i \zeta \geq 0$ and $\tau_i > 0$, which completes the proof.

Despite of its potential conservatism, the S-Procedure is a very useful tool in LMI-based robust analysis and control (YU; DUAN, 2013). When $m = 1$, the matrix inequality in (7) is a necessary and sufficient condition for the constrained quadratic form in (6); see, e.g., (GHAOUI et al., 1994).

3.3 SYSTEM INDEXES AND NORMS

In many control problems, it is of interest to measure the sensitivity of the system output with respect to input signals. For instance, for FDI systems, the residuals should have low (large) sensitivity to disturbance (fault) inputs. There are several ways to quantify the input-to-output gain of LTI systems which depend on the class of input signals (e.g., stochastic, ℓ_2 or ℓ_∞ signals) or the application itself (e.g., low or large sensitivity). When the considered input-to-output gain measure also implies that the system is stable, this measure is called a *system norm*. Otherwise, the input-to-output gain measure is only referred as a performance index.

Originally, most of system norms (and indexes) are defined in terms of the system matrix transfer function. However, this doctorate thesis considers the time-domain counterparts of H_∞ and *peak* norms as well as the H_- index of discrete-time LTI systems, which are more indicated to derive LMI-based conditions.

Before introducing the system norms (and index) to be considered in this doctorate thesis, let the following minimal realization of an LTI system

$$G_{uy} : \begin{cases} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{cases} \tag{8}$$

where $x \in \mathbb{R}^{n_x}$ is the state; $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input; $y \in \mathbb{R}^{n_y}$ is the output; A, B, C and D are real matrices having appropriate dimensions with A being Schur stable; and $\mathcal{U}$ represents the class of input signals (which will be defined later in this section).

Definition 1 *(H_∞-norm of systems) Consider the system in (8) with $\mathcal{U}$ representing the (n_u-dimensional) space of square summable vector sequences. Then, the H_∞-norm of system (8), or simply $\|G_{uy}\|_\infty$, is defined as*

$$\|G_{uy}\|_\infty := \sup_{u \neq 0} \frac{\|y\|_{\ell_2}}{\|u\|_{\ell_2}} \tag{9}$$

Definition 2 *(Peak-norm of systems) Consider the system in (8) and let*

$$\mathcal{U} := \left\{ u \in \mathbb{R}^{n_u} : u^T u \leq 1,\ u \neq 0 \right\} \tag{10}$$

Then, the peak-norm of system (8), or simply $\|G_{uy}\|_{peak}$, is defined as

$$\|G_{uy}\|_{peak} := \sup_{u \in \mathcal{U}} \|y\|_{\ell_\infty} \tag{11}$$

Either H_∞ or *peak*-norm will be considered in this doctorate thesis to measure the worst case system sensitivity to disturbances depending on the class of input signals. Notice that both system norms will be finite (i.e., well defined) when the system output is bounded, which is ensured if system (8) is asymptotically stable.

Definition 3 *(H_- index of systems) Consider the system in (8) with $\mathcal{U}$ representing the (n_u-dimensional) space of square summable vector sequences. Then, the H_--index of system (8), or simply $\|G_{uy}\|_-$, is defined as*

$$\|G_{uy}\|_- := \inf_{u\neq 0,\ x(0)=0} \frac{\|y\|_{\ell_2}}{\|u\|_{\ell_2}} \tag{12}$$

In contrast to H_∞ and *peak*-norm, the H_- index does not necessarily implies that system (8) is stable and for this reason is not a system norm (DING, S. X., 2008). In addition, since H_- index is utilized to measure the sensitivity of y with respect to u, the class of input signals will not play an important role. For instance, to handle amplitude bounded signals, it can be considered an ℓ_2 signal in a finite horizon.

Remark 1 *Let $G_{uy}(z)$ be matrix transfer function of system (8), i.e.,*

$$G_{uy}(z) = C(zI - A)^{-1}B + D.$$

Then, the H_∞ norm and H_- index are respectively equal to the largest and smallest singular values of $G_{uy}(e^{j\omega})$ over $\omega \in [0, 2\pi)$ (LIU, N.; ZHOU, K., 2007).

3.4 DETERMINING BOUNDS ON SYSTEM INDEXES

Bounds on the performance indexes $\|G_{uy}\|_\infty$, $\|G_{uy}\|_{peak}$ and $\|G_{uy}\|_-$ can be computed utilizing modified Lyapunov inequalities. For instance, let

$$V(k) = x(k)^T P x(k), \quad P \in \mathbb{S}^{n_x}, \quad P > 0, \tag{13}$$

and the following dissipation inequality:

$$\Delta V(k) + y(k)^T y(k) - \gamma u(k)^T u(k) < 0 \tag{14}$$

where γ is a positive scalar and

$$\Delta V(k) := V(k+1) - V(k). \tag{15}$$

Assuming that $u \in \ell_2$, notice that system G_{uy} in (8) will be asymptotically stable if (14) holds, since $u(k)$ will eventually vanishes to zero implying that $\Delta V(k) < 0$. In addition, for zero initial conditions, summing up (14) from $k = 0$ to $k \to \infty$ yields:

$$\begin{aligned} \sum_{k=0}^{\infty} \Delta V(k) + \sum_{k=0}^{\infty} y(k)^T y(k) &< \gamma \sum_{k=0}^{\infty} u(k)^T u(k) \\ V(\infty) - V(0) + \|y\|_{\ell_2}^2 &< \gamma \|u\|_{\ell_2}^2 \\ \|y\|_{\ell_2}^2 &< \gamma \|u\|_{\ell_2}^2 \end{aligned} \tag{16}$$

since $V(\infty) = V(0) = 0$. That is, γ is an upper-bound on $\|G_{uy}\|_\infty^2$.

The next result, known as the bounded real lemma (GHAOUI et al., 1994), provides an LMI-based constraint to determines the stability of system (8) while guaranteeing an upper bound γ on $\|G_{uy}\|_\infty^2$.

Lemma 4 *(Bounded Real Lemma:) Let γ be a given positive scalar. Then, the system defined in (8) is asymptotically stable and $\|G_{uy}\|_{\infty}^{2} \leq \gamma$ if there exists a $P \in \mathbb{S}^{n_x}$ such that $P > 0$ and*

$$\begin{bmatrix} A^T PA - P & A^T PB & C^T \\ * & B^T PB - \gamma I_{n_u} & D^T \\ * & * & -I_{n_y} \end{bmatrix} < 0. \tag{17}$$

Proof 3 *Firstly, suppose that (17) holds for some $P > 0$. Hence, notice that the block (1,1) of (17) implies that $A^T PA - P < 0$ which implies that the unforced system is asymptotically stable. Next, applying the Schur's complement to (17) yields*

$$\begin{bmatrix} A^T PA - P & A^T PB \\ * & B^T PB - \gamma I_{n_u} \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0.$$

Thus, pre- and post multiplying the above matrix inequality by $\begin{bmatrix} x^T & u^T \end{bmatrix}$ and its transpose leads to:

$$\begin{aligned} x^T(A^T PA - P + C^T C)x + 2x^T(A^T PB + C^T D)u + u^T B^T PBu - \gamma u^T u < 0 \\ x^T A^T PAx + 2x^T(A^T PB)u + u^T B^T PBu - x^T Px + y^T y - \gamma u^T u < 0 \\ (x^+)^T Px^+ - x^T Px + y^T y - \gamma u^T u < 0 \\ \Delta V(k) + y^T y - \gamma u^T u < 0, \end{aligned}$$

which completes the proof taking (14), (15) and (16) into account.

When system (8) is subject to non-vanishing input signals, one can consider the notion of input-to-state stability which basically guarantees that $\|x\|_{\ell_\infty}$ is bounded for bounded inputs in the ℓ_∞ sense. Before introducing a Lyapunov inequality ensuring the system input-to-state stability, consider the following definition adapted from (DE SOUZA et al., 2015) to our context.

Definition 4 *(Input-to-state stability) The system in (8) is said to be input-to-state stable (or simply ISS) if there exist a class $\mathcal{KL}$-function $a_1(\|x_0\|, k)$, a class $\mathcal{K}$-function $a_2(\|u(k)\|)$ and a positive scalar ρ such that the following holds*

$$\|x(k)\| \leq a_1(\|x(0)\|, k) + a_2(\|u(k)\|), \ \forall\, k \geq 0, \ \|x(0)\| \leq \rho, \ \|u(k)\| \leq 1. \tag{18}$$

Remark 2 *Let $s \in \mathbb{R}_{\geq}$, $t \in \mathbb{R}_{\geq}$ and $\rho \in \mathbb{R}_{>}$. A function $a(s)$ is said to be a class $\mathcal{K}$-function if it is continuous, strictly increasing and $a(0) = 0$. A function $b(s, t)$ is said to be a class $\mathcal{KL}$-function if it is a class $\mathcal{K}$ function for each fixed $t \geq 0$ and decreasing with respect to t with $b(s, t) \to 0$ as $t \to 0$.*

The following result will be instrumental to assess the input-to-state stability of a discrete-time system, which is a version of the Lyapunov characterization of ISS as proposed in (Z. P. JIANG; Y. WANG, 2001).

Lemma 5 *(Lyapunov characterization of ISS) Consider the system in (8) with $\mathcal{U}$ as defined in (10). The system is ISS if there exists a continuous function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq}$ and positive scalars ϵ_1, ϵ_2, ϵ_3 and ϵ_4 such that the following holds*

$$\epsilon_1 \|x\|^2 \leq V(k) \leq \epsilon_2 \|x\|^2 \tag{19}$$

$$\Delta V(k) \leq -\epsilon_3 \|x\|^2 + \epsilon_4 \|u\|^2 \tag{20}$$

for all $x \in \mathbb{R}^{n_x}$ and $u \in \mathcal{U}$.

Besides the characterization of ISS for discrete-time system, it is often of interest to estimate a state trajectory bounding set – sometimes referred in the literature as the system reachable set (GHAOUI et al., 1994). In other words, determine a set $\mathcal{X}$ such that $x(k)$, driven by $x(0) \in \mathcal{X}$ and $u(k) \in \mathcal{U}$, belongs to $\mathcal{X}$ for all $k > 0$. To this end, let $V(k)$ as defined in (13) and

$$\mathcal{X} := \{x \in \mathbb{R}^{n_x} : V(k) \leq 1\}, \tag{21}$$

and consider the following Lyapunov-like inequality:

$$\Delta V(k) \leq \tau\big(u^T u - V(k)\big), \quad x(k) \in \mathcal{X}, \quad u \in \mathcal{U}, \quad k \geq 0, \tag{22}$$

where $\tau \in (0, 1)$.

Following the reasoning established in (DE SOUZA et al., 2015), the condition in (22) implies that:

$$V(k_1 + 1) \leq (1 - \tau) V(k_1) + \tau u(k_1)^T u(k_1) \leq 1 - \tau + \tau = 1 \;\Rightarrow\; x(k_1 + 1) \in \mathcal{X}, \tag{23}$$

for any $k_1 \geq 0$, since $V(k_1) \leq 1$.

Next, in order to compute a bound on the system *peak*-norm, notice that the following constrained inequality

$$\gamma - y(k)^T y(k) \geq 0, \quad \forall\, (x, u) : x(k)^T P x(k) \leq 1,\; u(k)^T u(k) \leq 1, \tag{24}$$

implies $\|y(k)\|^2_{\ell_\infty} \leq \gamma$ assuming that (22) holds.

Hence, by applying the S-Procedure, (24) holds if there exist positive scalars β_1 and β_2 such that $\gamma - \beta_1 - \beta_2 \geq 0$ and:

$$\beta_1 u(k)^T u(k) + \beta_2 x(k)^T P x(k) - y(k)^T y(k) \geq 0. \tag{25}$$

The above developments can be summarized in the following result.

Lemma 6 *(Guaranteed* peak-*norm) Let* γ, τ, β_1 *and* β_2 *be given positive scalars such that* $\tau \in (0,1)$ *and* $\gamma - \beta_1 - \beta_2 \geq 0$*. Then, the system defined in* (8) *is ISS and* $\|G_{uy}\|^2_{peak} \leq \gamma$ *if there exists a* $P \in \mathbb{S}^{n_x}$ *satisfying the following LMIs:*

$$\begin{bmatrix} \beta_2 P & 0 & C^T \\ * & \beta_1 I_{n_u} & D^T \\ * & * & I_{n_y} \end{bmatrix} > 0, \tag{26}$$

$$\begin{bmatrix} A^T PA - (1-\tau)P & A^T PB \\ * & B^T PB - \tau I_{n_u} \end{bmatrix} < 0. \tag{27}$$

Moreover, for any $x(0) \in \mathcal{X}$ *and* $u(k) \in \mathcal{U}$*, the state trajectory* $x(k) \in \mathcal{X}$ *for all* $k \geq 0$*, with* $\mathcal{X}$ *being as defined in* (21)*.*

Proof 4 *Assume there is a solution* $P = P^T$ *to* (26) *and* (27)*. First, notice from the* $(1,1)$ *block of* (26) *that* $P > 0$*, since* $\beta_2 > 0$*.*

Next, pre- post-multiplying (27) *by* $\begin{bmatrix} x^T & u^T \end{bmatrix}$ *and its transpose, respectively, yields*

$$\Delta V(k) \leq -\tau V(k) + \tau \|u(k)\|^2 \leq -\tau\lambda \|x(k)\|^2 + \tau \|u(k)\|^2 \tag{28}$$

where λ *is the largest eigenvalue of* P*. Then, by the virtue of Lemma* 5*, the system is ISS.*

Now, applying the Schur's complement to (26)*, and pre- and post-multiplying the resulting matrix inequality by* $\begin{bmatrix} x^T & u^T \end{bmatrix}$ *and its transpose, respectively, leads to* (25)*. Hence, it follows that* $\|G_{uy}\|^2_{peak} \leq \gamma$*, since* $\|y(k)\|^2 \leq \gamma$ *for all* $k \geq 0$*.*

The proof is completed by noting that (28) *implies that* $\mathcal{X}$ *is a positively invariant set from the fact that* (23) *holds for all* $k \geq 0$*.*

Before ending this chapter, the LMI characterization of the H_- index has been proposed in (LI, X.; LIU, H., 2013). For completeness, the latter result, which establishes a necessary and sufficient condition to ensure a bound on $\|G_{uy}\|_-$, is presented in the following.

Lemma 7 *(Guaranteed system sensitivity) Assume that system* (8) *is asymptotically stable for* $u(k) \equiv 0$*. Let* γ *be a positive scalar. Then,* $\|G_{uy}\|_- > \gamma$ *if and only if there exists* $P \in \mathbb{S}^{n_x}$ *such that*

$$\begin{bmatrix} A^T PA - P + C^T C & A^T PB + C^T D \\ * & B^T PB + D^T D - \gamma^2 I_{n_u} \end{bmatrix} > 0 \tag{29}$$

Proof 5 *Pre- and post multiplying* (29) *by* $\begin{bmatrix} x^T & u^T \end{bmatrix}$ *and its transpose, respectively, yields*

$$\begin{aligned} (Ax+Bu)^T P(Ax+Bu) - x^T Px + (Cx+Du)^T(Cx+Du) - \gamma^2 u^T u > 0 \\ V(k+1) - V(k) + y^T y - \gamma^2 u^T u > 0 \end{aligned} \tag{30}$$

where $V(k) = x(k)^T Px(k)$*.*

Next, summing (30) up from $k = 0$ to $k \to \infty$, it follows that

$$\|y\|_{\ell_2}^2 - \gamma^2 \|u\|_{\ell_2}^2 > -V(\infty) \tag{31}$$

for zero initial conditions.

Assuming that the system is internally stable and $u \in \ell_2$, the condition in (31) implies that

$$\|y\|_{\ell_2} > \gamma \|u\|_{\ell_2},$$

which completes the proof.

4 REFERENCE MODEL-BASED FDI FOR LDTS

This chapter presents a reference model-based solution for the FDI problem applied to linear discrete-time systems. The chapter is divided in two solutions: while the first one is based on the $\mathcal{H}_\infty$ norm, the second approach takes into account the Peak norm. The goal is to guarantee fault detection and isolation taking into account the desired behavior given by a reference model despite the presence of disturbances.The filter design is cast as a convex optimization problem involving a set of LMI constraints. Hence, to this end, firstly the class of systems considered here is defined in Section 4.1, followed by the proposed observer-like filter, the residual generator, the reference model and the augmented system. Subsequently, Section 4.2 presents the design of an $\mathcal{H}_\infty$ reference model-based FDI for linear discrete-time systems, whereas Section 4.3 demonstrates the Peak norm solution for the same class of systems. The effectiveness of the proposed designs is illustrated by numerical examples.

4.1 PROBLEM OF INTEREST

Consider the following class of linear discrete-time systems:

$$x(k+1) = Ax(k) + B_u u(k) + B_w w(k) + B_f f(k), \tag{32a}$$

$$y(k) = Cx(k) + D_u u(k) + D_w w(k) + D_f f(k), \tag{32b}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is a measured input, $w(k) \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is an exogenous input, $y(k) \in \mathbb{R}^{n_y}$ is the measured output, $f(k) \in \mathcal{F} \subset \mathbb{R}^{n_f}$ is a vector containing actuator and/or sensor faults, and $A, B_u, B_w, B_f, C, D_u, D_w, D_f$ are given real matrices with appropriate dimensions. The class of signals defined by the sets $\mathcal{W}$ and $\mathcal{F}$ will be later specified in this chapter. The following conditions are assumed with respect to system (32):

A1 The number of measured inputs and outputs is equal to n, i.e., $n_y = n_u = n$.

A2 The number of faults is equal or smaller than the number of measurements, i.e., $n_f \leq 2n$.

Assumption **A1** imposes, for simplicity, that the system input-to-ouput map is square, whereas assumption **A2** establishes that the number of the system faults is smaller or equal to twice the number of the system inputs to ensure the well-posedness of the FDI problem.

Associate to system (32), consider the following observer-like filter:

$$\hat{x}(k+1) = A\hat{x}(k) + B_u u(k) + L(y(k) - \hat{y}(k)), \tag{33a}$$

$$\hat{y}(k) = C\hat{x}(k) + D_u u(k), \tag{33b}$$

where $\hat{x}(k) \in \mathbb{R}^{n_x}$ and $\hat{y}(k) \in \mathbb{R}^{n_y}$ are estimations of the state and output vectors, respectively, with the observer gain matrix $L \in \mathbb{R}^{n_x \times n_y}$ to be designed.

In addition, let

$$\tilde{x}(k) := x(k) - \hat{x}(k), \quad \tilde{x}(k) \in \mathbb{R}^{n_x}, \tag{34}$$

be the estimation error vector and consider the following residual generator:

$$\tilde{x}(k+1) = (A - LC)\tilde{x}(k) + (B_w - LD_w)w(k) + (B_f - LD_f)f(k), \tag{35a}$$

$$r(k) = QC_r(y(k) - \hat{y}(k)), \tag{35b}$$

where $r(k) \in \mathbb{R}^{n_f}$ is the residual vector, with $C_r \in \mathbb{R}^{n_f \times n}$ and $Q \in \mathbb{R}^{n_f \times n_f}$ to be designed such that the residual is sensitive to faults regardless the presence of exogenous disturbances.

In order to enforce sensitivity to faults occurring in system (32), consider the following reference model:

$$\breve{x}(k+1) = \breve{A}\breve{x}(k) + \breve{B}f(k), \tag{36a}$$

$$\breve{r}(k) = Q(\breve{C}\breve{x}(k) + \breve{D}f(k)), \tag{36b}$$

where $\breve{x}(k) \in \mathbb{R}^{n_q}$ and $\breve{r}(k) \in \mathbb{R}^{n_f}$ are the reference model state and residual, respectively, and $\breve{A}$, $\breve{B}$, $\breve{C}$ and $\breve{D}$ are given matrices with appropriate dimensions with $\breve{A}$ being Schur stable. It should be emphasized that the definition of matrices $\breve{A}$, $\breve{B}$, $\breve{C}$ and $\breve{D}$ must provide the desired behavior from faults to residuals in terms of the $\mathcal{H}_{-}$ index.

Next, consider the residual generator in (33), with (35b), the reference model in (36), and let the following augmented vector:

$$\bar{x} = \begin{bmatrix} \tilde{x} \\ \breve{x} \end{bmatrix}, \quad \bar{x}(k) \in \mathbb{R}^{n_a}, \quad n_a = n_x + n_q. \tag{37}$$

Thus, the following augmented system can be defined

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}f(k), \tag{38a}$$

$$e_r(k) = Q(\bar{C}\bar{x}(k) + \bar{D}f(k)), \tag{38b}$$

where $e_r(k) \in \mathbb{R}^{n_f}$ is the residual error as defined below

$$e_r(k) := r(k) - \breve{r}(k), \tag{39}$$

and the matrices $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ are given by

$$\bar{A} = \begin{bmatrix} A - LC & 0 \\ 0 & \breve{A} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_f - LD_f \\ \breve{B} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_r C & -\breve{C} \end{bmatrix}, \quad \bar{D} = C_r D_f - \breve{D}. \tag{40}$$

In the following sections, two different LMI approaches are devised for designing the residual generator (i.e., the matrices L, Q and C_r) which will depend on the class of disturbance and fault signals.

4.2 H_∞ APPROACH

Consider the class of linear discrete-time systems defined in (32), with **A1** and **A2**, and assume that the class of disturbance and fault are given by:

A3 $\mathcal{W} = \left\{ w \in \mathbb{R}^{n_w} : \sum_{k=0}^{\infty} w^T w = \|w\|_{\ell_2}^2 \leq c_w \right\}$

A4 $\mathcal{F} = \left\{ f \in \mathbb{R}^{n_f} : \sum_{k=0}^{\infty} f^T f = \|f\|_{\ell_2}^2 \leq c_f \right\}$

where c_w and c_f are given positive scalars defining bounds on the energy of $w(k)$ and $f(k)$, respectively. Notice that c_w and c_f can be normalized to 1 without loss of generality by appropriately rescaling the matrices B_w, D_w, B_f and D_f.

The main objective in this section is to design an $\mathcal{H}_-/\mathcal{H}_\infty$ residual generator as defined by (33) and (35b) such that the behavior from faults to residual is as close as possible to the behavior given by the reference model (36), while mitigating the effects of disturbance signals. In this context, the problem of concern is to design the residual generator free matrices (i.e., L, Q and C_r) such that the following holds:

I $\|\mathcal{G}_{wr}\|_\infty^2 \leq \gamma_w$,

II $\|\mathcal{G}_{fr} - \mathcal{G}_{f\breve{r}}\|_\infty^2 \leq \gamma_f$, and

III $\|\mathcal{G}_{f\breve{r}}\|_-^2 \geq \gamma_c$,

where $\mathcal{G}_{wr}$, $\mathcal{G}_{fr}$ and $\mathcal{G}_{f\breve{r}}$ represent the transfer functions from disturbance to residual, from fault to residual and from fault to reference residual, respectively, and γ_c, γ_w and γ_f are positive scalars defining the residual generator performance.

4.2.1 Determining a bound on $\|\mathcal{G}_{wr}\|_\infty$

In order to derive a solution to the filter design problem, let

$$V_1(\tilde{x}) = \tilde{x}^T P_1 \tilde{x}, \quad P_1 \in \mathbb{R}^{n_x \times n_x}, \quad P_1 > 0, \tag{41}$$

be a Lyapunov function candidate for the estimation error dynamics, as defined in (34), and consider the following inequality:

$$\Delta V_1(\tilde{x}) + r^T r - \gamma_w w^T w < 0, \tag{42}$$

where

$$\Delta V_1(\tilde{x}) = V_1(\tilde{x}(k+1)) - V_1(\tilde{x}(k)) \tag{43}$$

Assuming that the error dynamics is stable, i.e., $\Delta V_1(\tilde{x}) < 0$, $\forall\ \tilde{x} \neq 0$, and summing the inequality in (42) from $k = 0$ to ∞ yields:

$$\|r\|_{\ell_2}^2 - \gamma_w \|w\|_{\ell_2}^2 < V_1(\tilde{x}(0))$$

The above inequality for zero initial conditions implies that $\|\mathcal{G}_{wr}\|_\infty^2 \leq \gamma_w$ from the fact that:

$$\|\mathcal{G}_{wr}\|_\infty := \sup_{w \neq 0} \frac{\|r\|_{\ell_2}}{\|w\|_{\ell_2}}$$

Next, considering the residual generator in (35) and disregarding fault signals, the following is obtained from the inequality in (42):

$$\Big((A-LC)\tilde{x}+(B_w-LD_w)w\Big)^T P_1\Big((A-LC)\tilde{x}+(B_w-LD_w)w\Big)-\tilde{x}^T P_1\tilde{x}$$
$$+(C_rC\tilde{x}+C_rD_ww)^T Q^T Q(C_rC\tilde{x}+C_rD_ww)-\gamma_w w^T w<0,$$

which can be cast as follows:

$$\begin{bmatrix}\tilde{x}\\ w\end{bmatrix}^T\left\{\begin{bmatrix}-P_1 & 0\\ 0 & -\gamma_w I_{n_w}\end{bmatrix}+\begin{bmatrix}(A-LC)^T\\ (B_w-LD_w)^T\end{bmatrix}P_1\begin{bmatrix}(A-LC) & (B_w-LD_w)\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix}(C_rC)^T\\ (C_rD_w)^T\end{bmatrix}Q^TQ\begin{bmatrix}(C_rC) & (C_rD_w)\end{bmatrix}\right\}\begin{bmatrix}\tilde{x}\\ w\end{bmatrix}<0.$$

By applying the Schur complement (Lemma 1), the following condition is obtained:

$$\begin{bmatrix}-P_1 & 0 & (A-LC)^T & C^TC_r^T\\ 0 & -\gamma_w I_{n_w} & (B_w-LD_w)^T & D_w^TC_r^T\\ (A-LC) & (B_w-LD_w) & -P_1^{-1} & 0\\ C_rC & C_rD_w & 0 & -\bar{Q}\end{bmatrix}<0, \tag{44}$$

where

$$\bar{Q}=(Q^TQ)^{-1}. \tag{45}$$

Hence, pre- and post-multiplying (44) by

$$\text{diag}\{I_{n_x}, I_{n_w}, K, I_{n_f}\}$$

and its transpose yields:

$$\begin{bmatrix}-P_1 & 0 & (KA-L_KC)^T & C^TC_r^T\\ 0 & -\gamma_w I_{n_w} & (KB_w-L_KD_w)^T & D_w^TC_r^T\\ (KA-L_KC) & (KB_w-L_KD_w) & -KP_1^{-1}K^T & 0\\ C_rC & C_rD_w & 0 & -\bar{Q}\end{bmatrix}<0, \tag{46}$$

where $K\in\mathbb{R}^{n_x\times n_x}$ is a nonsingular free matrix and:

$$L_K=KL. \tag{47}$$

Now, notice from

$$(K-P_1)P_1^{-1}(K-P_1)^T\geq 0$$

that the following holds:

$$P_1-K-K^T\geq -KP_1^{-1}K^T, \tag{48}$$

for any nonsingular matrix K.

Hence, the following sufficient condition for ensuring that $\|\mathcal{G}_{wr}\|_\infty^2 \leq \gamma_w$ is derived from (46):

$$\begin{bmatrix} -P_1 & * & * & * \\ 0 & -\gamma_w I_{n_w} & * & * \\ (KA - L_K C) & (KB_w - L_K D_w) & P_1 - K - K^T & * \\ C_r C & C_r D_w & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{49}$$

where $(*)$ stands for the transpose of the blocks outside the main diagonal blocks.

4.2.2 Determining a bound on $\|\mathcal{G}_{fe_r}\|_\infty$

Consider the augmented system in Eq. (38). Thus, in order to determine a bound on the ℓ_2-gain from f to e_r, that is:

$$\|\mathcal{G}_{fe_r}\|_\infty := \|\mathcal{G}_{fr} - \mathcal{G}_{f\breve{r}}\|_\infty, \tag{50}$$

let

$$V_2(\bar{x}) = \bar{x}^T P_2 \bar{x}, \quad P_2 \in \mathbb{R}^{n_a \times n_a}, \quad n_a := n_x + n_q, \quad P_2 > 0, \tag{51}$$

and consider the following inequality:

$$\Delta V_2(\bar{x}) + e_r^T e_r - \gamma_f f^T f < 0, \tag{52}$$

with

$$\Delta V_2(\bar{x}) = V_2(\bar{x}(k+1)) - V_2(\bar{x}(k)) \tag{53}$$

with $\bar{x}$ representing the augmented state as defined in (37).

Next, consider the following partition of the matrix P_2

$$P_2 = \begin{bmatrix} P_{21} & P_{22}^T \\ P_{22} & P_{23} \end{bmatrix}, \tag{54}$$

where $P_{21} = P_{21}^T \in \mathbb{R}^{n_x \times n_x}$, $P_{22} \in \mathbb{R}^{n_q \times n_x}$, and $P_{23} = P_{23}^T \in \mathbb{R}^{n_q \times n_q}$, and let

$$K_2 = \begin{bmatrix} K & K_a \\ MK & K_b \end{bmatrix}, \tag{55}$$

with $K_a \in \mathbb{R}^{n_x \times n_q}$ and $K_b \in \mathbb{R}^{n_q \times n_q}$ being free matrices (to be designed), and $M \in \mathbb{R}^{n_q \times n_x}$ being a given real matrix.

Hence, in light of (54) and (55), the inequality in (52) can be cast as follows:

$$(\bar{A}\bar{x} + \bar{B}f)^T P_2 (\bar{A}\bar{x} + \bar{B}f) + (\bar{C}\bar{x} + \bar{D}f)^T Q^T Q (\bar{C}\bar{x} + \bar{D}f) - \bar{x}^T P_2 \bar{x} - \gamma_f f^T f < 0,$$

or, equivalently:

$$\begin{bmatrix} \bar{x} \\ f \end{bmatrix}^T \left\{ \begin{bmatrix} -P_2 & 0 \\ 0 & -\gamma_f I_{n_f} \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} P_2 \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} Q^T Q \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \right\} \begin{bmatrix} \bar{x} \\ f \end{bmatrix} < 0. \tag{56}$$

By applying the Schur complement, the above inequality is satisfied, if and only if the following matrix inequality holds:

$$\begin{bmatrix} -P_2 & 0 & \bar{A}^T & \bar{C}^T \\ 0 & -\gamma_f I_{n_f} & \bar{B}^T & \bar{D}^T \\ \bar{A} & \bar{B} & -P_2^{-1} & 0 \\ \bar{C} & \bar{D} & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{57}$$

with $\bar{Q}$ being as defined in (45).

Thus, pre- and post-multiplying Eq. (57) by

$$\text{diag}\{I_{n_a}, I_{n_f}, K_2, I_{n_f}\}$$

and its transpose yields:

$$\begin{bmatrix} -P_{21} & * & * & * & * & * \\ -P_{22} & -P_{23} & * & * & * & * \\ 0 & 0 & -\gamma_f I_{n_f} & * & * & * \\ \omega_{41} & K_a\breve{A} & \omega_{43} & \omega_{44} & * & * \\ \omega_{51} & K_b\breve{A} & \omega_{53} & \omega_{54} & \omega_{55} & * \\ C_r C & -\breve{C} & \omega_{63} & 0 & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{58}$$

from the fact that $P_2 - K_2 - K_2^T \geq -K_2 P_2^{-1} K_2^T$, where K_2 is as defined in (55) and

$$\begin{aligned} \omega_{41} &= KA - L_K C, & \omega_{43} &= KB_f - L_K D_f + K_a\breve{B}, \\ \omega_{44} &= P_{21} - K - K^T, & \omega_{51} &= MKA - ML_K C, \\ \omega_{53} &= MKB_f - ML_K D_f + K_b\breve{B}, & \omega_{54} &= P_{22} - MK - K_a^T, \\ \omega_{55} &= P_{23} - K_b - K_b^T, & \omega_{63} &= C_r D_f - \breve{D}. \end{aligned} \tag{59}$$

4.2.3 Determining a bound on $\|\mathcal{G}_{f\breve{r}}\|_-$

In view of Lemma 7 and the reference dynamics in (36), $\|\mathcal{G}_{f\breve{r}}\|_-^2$ is larger or equal to γ_c if and only if the following holds:

$$\begin{bmatrix} \breve{A}P_3\breve{A}^T - P_3 + \breve{B}\breve{B}^T & * \\ Q\breve{C}P_3\breve{A}^T + Q\breve{D}\breve{B}^T & Q\breve{C}P_3\breve{C}^T Q^T + Q\breve{D}\breve{D}^T Q^T - \gamma_c I_q \end{bmatrix} > 0, \tag{60}$$

where $P_3 = P_3^T \in \mathbb{R}^{n_q \times n_q}$ is to be designed.

Thus, pre- and post-multiplying (60) by

$$\text{diag}\{I_{n_q}, Q^{-1}\}$$

and its transpose leads to the following matrix inequality

$$\begin{bmatrix} \breve{A}P_3\breve{A}^T - P_3 + \breve{B}\breve{B}^T & * \\ \breve{C}P_3\breve{A}^T + \breve{D}\breve{B}^T & \breve{C}P_3\breve{C}^T + \breve{D}\breve{D}^T - \gamma_c\bar{Q} \end{bmatrix} > 0, \tag{61}$$

where $\bar{Q}$ is as defined in (45).

4.2.4 Residual Design

In view of the developments presented in Subsections 4.2.1, 4.2.2 and 4.2.3, the following theorem proposes residual design conditions in terms of LMI constraints to ensure that the residual generator performance specifications I, II and III hold.

Theorem 1 *Consider the residual generator in* (35)*, the reference model in* (36) *and the augmented system defined in* (38)*. Let* $\breve{A}$*,* $\breve{B}$*,* $\breve{C}$*,* $\breve{D}$*,* M *and* γ_c *be given. Suppose there exist symmetric matrices* P_1*,* P_{21}*,* P_{23}*,* P_3 *and* $\bar{Q}$*, free matrices* L_K*,* K*,* K_a*,* K_b*,* C_r *and* P_{22}*, and positive scalars* γ_f *and* γ_w *such that the LMIs in* (87)*,* (58) *and* (61) *hold. Then, the residual generator as defined in* (33) *and* (35)*, with*

$$L = K^{-1}L_K \quad \text{and} \quad Q = \bar{Q}^{-1/2},$$

ensures that:

(i) the unforced residual generator is asymptotically stable;

(ii) $\|\mathcal{G}_{wr}\|_\infty^2 \leq \gamma_w$*;*

(iii) $\|\mathcal{G}_{fe_r}\|_\infty^2 \leq \gamma_f$*; and*

(iv) $\|\mathcal{G}_{f\breve{r}}\|_-^2 \geq \gamma_c$*.*

The proof of the above theorem follows in the Appendix A.

4.2.5 Computational Issues

Consider the LMI constraints defined in (87), (58) and (61) which are associated to the specifications I, II and III, respectively. For a given minimum sensitivity to faults γ_c, an optimized residual generator may be derived in the sense of minimizing the mismatch between $\mathcal{G}_{fr}$ and $\mathcal{G}_{f\breve{r}}$ by means of the following optimization problem:

$$\min_{P_1,\ldots,Q,\gamma_w,\gamma_f} \gamma_f \text{ subject to } \begin{cases} (87), (58), (61) \\ \text{and } \gamma_w \leq \bar{\gamma}_w, \end{cases} \tag{62}$$

where $\bar{\gamma}_w$ represents an admissible fault sensitivity to exogenous disturbances in the H_∞ sense.

However, the above optimization problem can be quite conservative when the ℓ_2-gain from one fault f_i to the residual error e_r is relatively large compared to the others, for some f_j, $j \neq i$, $j \in \{1,\ldots,n_f\}$. To overcome this problem, the performance specification II may be relaxed to

$$\|\mathcal{G}_{f_ir} - \mathcal{G}_{f_i\breve{r}}\|_\infty^2 \leq \gamma_{f_i}, \quad i = 1,\ldots,n_f, \tag{63}$$

where $\gamma_{f_1}, \ldots, \gamma_{f_{n_f}}$ are positive scalars to be optimized and f_i stands for the i-th element of the fault input vector. The constraints in (63) can be accomplished by modifying the LMI in (57) (Subsection 4.2.2) to the following:

$$\begin{bmatrix} -P_{21} & * & * & * & * & * \\ -P_{22} & -P_{23} & * & * & * & * \\ 0 & 0 & -\Gamma_f & * & * & * \\ \omega_{41} & K_a\breve{A} & \omega_{43} & \omega_{44} & * & * \\ \omega_{51} & K_b\breve{A} & \omega_{53} & \omega_{54} & \omega_{55} & * \\ C_rC & -\breve{C} & \omega_{63} & 0 & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{64}$$

where

$$\Gamma_f = \mathrm{diag}\{\gamma_{f_1}, \ldots, \gamma_{f_{n_f}}\}.$$

Hence, an optimized solution can be obtained by means of the following convex optimization problem:

$$\min_{P_1,\ldots,\bar{Q},\gamma_w,\Gamma_f} \sum_{i=1}^{n_f} \gamma_{f_i} \text{ subject to } \begin{cases} (44), (61), (64) \\ \text{and } \gamma_w \leq \bar{\gamma}_w \end{cases} \tag{65}$$

Notice from (65) that specification II is satisfied with $\gamma_f = \max\{\gamma_{f_1}, \ldots, \gamma_{f_{n_f}}\}$.

4.2.6 Numerical example

In order to illustrate the H_∞ approach, consider a three tanks system as shown in Fig. 2. This system consists of three identical cylindrical tanks with transverse area S, connected to each other by cylindrical tubes with transverse area S_n and equal flow coefficients named μ_{13} and μ_{32}. The output in Tank 2, which represents the nominal flow of the system, has the same area S_n, but a different flow coefficient, labeled as μ_{20}. The actuators valves, represented by Valve 1 and Valve 2, which flow rates are defined by q_1 and q_2, respectively, are responsible for supplying Tanks 1 and 2. In this example, a disturbance input $w = \begin{bmatrix} w_1{}^T & w_2{}^T \end{bmatrix}^T$ is considered, where w_1 represents the flow of the pump and w_2 represents an energy bounded noise on Tank 1 level sensor. Moreover, notice from physical reasoning that the tank levels (x_i, with $i = 1, 2, 3$) and the valve flow rates (q_j, with $j = 1, 2$) are bounded by $x_{i_{\max}}$, $i = 1, 2, 3$, and $q_{j_{\max}}$, with $j = 1, 2$, respectively.

Consider the following numerical parameters for the system:

$$\begin{aligned} &S = 0.0154 \text{ m}^2, && S_n = 5 \times 10^{-5} \text{ m}^2, \\ &x_{i_{max}} = 0.62 \text{ m}, && q_{j_{max}} = 1.2 \times 10^{-4} \text{ m}^3\text{s}^{-1}, \\ &\mu_{13} = \mu_{32} = 0.5, && \mu_{20} = 0.675, \end{aligned} \tag{66}$$

as well as the state equilibrium points:

$$x_{s_1} = 0.6115 \text{ m}, \quad x_{s_2} = 0.4252 \text{ m}, \quad x_{s_3} = 0.5118 \text{ m}. \tag{67}$$

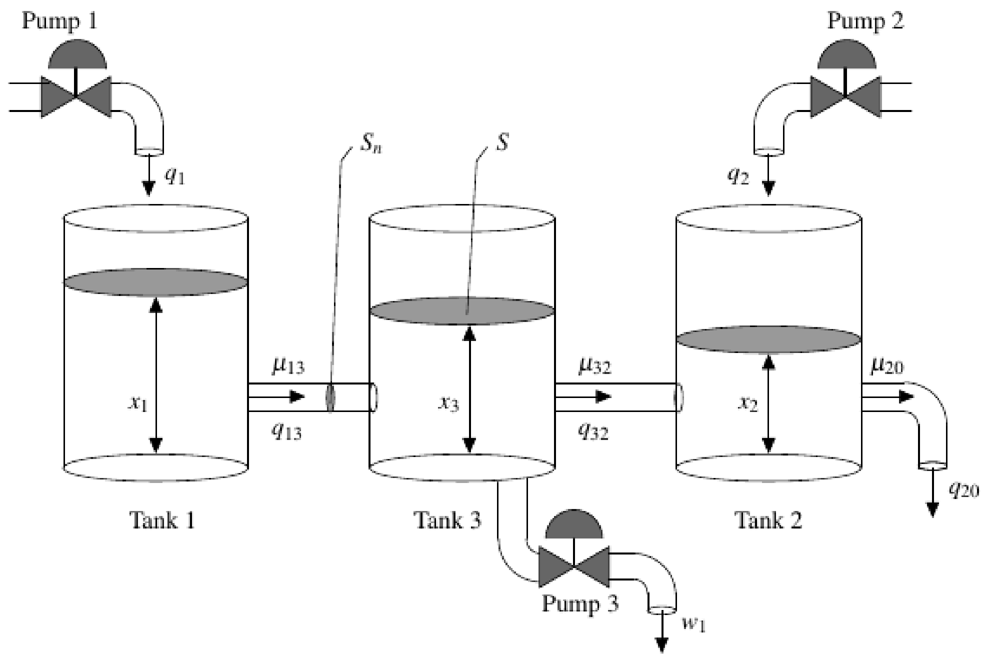


Figure 2 – Three tanks system.
Font: Own authorship.

The inputs q_1 and q_2 of the linearized model are normalized to belong to the range $q_i \in [0, 1]$, for $i = 1, 2$, with the following operating condition:

$$q_1 = 0.2916, \quad q_2 = 0.5417. \tag{68}$$

In this example, additive faults on Valve 1 and on the sensors measuring the level in Tanks 1 and 3 are considered in vector f. Hence, the linear approximate model of the three tanks system can be written as in (32), with

$$A = \begin{bmatrix} 0.9670 & 0.0006 & 0.0324 \\ 0.0006 & 0.9433 & 0.0344 \\ 0.0324 & 0.0344 & 0.9328 \end{bmatrix}, \quad B_u = 10^{-3} \begin{bmatrix} 22.9864 & 0.0047 \\ 0.0047 & 22.7053 \\ 0.3855 & 0.4106 \end{bmatrix},$$
$$B_w = 10^{-3} \begin{bmatrix} -0.0161 & 0 \\ -0.0171 & 0 \\ -0.9407 & 0 \end{bmatrix}, \quad B_f = 10^{-3} \begin{bmatrix} 22.9864 & 0 & 0 \\ 0.00047 & 0 & 0 \\ 0.3855 & 0 & 0 \end{bmatrix}, \tag{69}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix}, \quad D_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In view of the above setting, the system static gain from faults to measurements is given by:

$$K_{sys} = \begin{bmatrix} 2.369 & 1 & 0 \\ 1.684 & 0 & 1 \end{bmatrix}. \tag{70}$$

Notice that the dynamics behavior in low frequencies from faults to measurements in (70) has an appropriate structure for fault detection and isolation as explained later in this section.

Therefore, the reference model in (36) is chosen to be a copy of the fault to measurement dynamics, given by $\breve{A}$ and $\breve{B}$. Besides, to keep a triangular structure for fault detection and isolation purposes, matrices $\breve{C}$ and $\breve{D}$ are deduced from C and D_f. This means:

$$\breve{A} = A, \quad \breve{B} = B_f, \quad \breve{C} = \begin{bmatrix} C \\ 0_{1\times 3} \end{bmatrix}, \quad \breve{D} = \begin{bmatrix} D_f \\ 0_{1\times 3} \end{bmatrix}. \tag{71}$$

In (71), a third row of zeros has been included in $\breve{C}$ and $\breve{D}$ to ensure the same number of residuals and faults. Hence, the static gain of the reference model is:

$$K_{ref} = \begin{bmatrix} K_{sys} \\ 0_{1\times 3} \end{bmatrix} = \begin{bmatrix} 2.369 & 1 & 0 \\ 1.684 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{72}$$

Evaluating K_{sys} in (70), notice that the first two rows are sufficient to isolate the three faults provided that they do not occur simultaneously. This means that one can consider only the first two significant rows of K_{ref} in (72) for FDI evaluation purposes. Hence, taking into account (72), a structure regarding the behavior from faults to residual can be enforced such that:

- in the case of an occurrence of an actuator fault (f_a), both residuals r_1 and r_2 will be influenced by it;
- if a fault occurs in sensor 1 (f_{s_1}), only residual r_1 will be affected by this fault;
- similarly, only residual r_2 is influenced by the occurrence of sensor fault f_{s_2}.

Firstly, in this example, the performance specification II is considered and hence problem (62) is solved. In order to obtain a solution to the optimization problem, it is needed to define *a priori* the matrix M. In this example, we started by defining $M = \upsilon \begin{bmatrix} I_{n_q} & 0_{n_q\times(n_x-n_q)} \end{bmatrix}$ with $\upsilon = 1$, and then the parameter υ was tuned to optimize the solution. Therefore, the following parameter values are considered:

$$\gamma_C = 0.25, \quad M = 10^{-1} \cdot \begin{bmatrix} I_{n_q} & 0_{n_q\times(n_x-n_q)} \end{bmatrix}, \quad \bar{\gamma}_W = 0.0125, \tag{73}$$

where a griding was applied in the optimization problem for determining γ_C and $\bar{\gamma}_W$, and the optimization problem leads to

$$L = 10^{-3} \begin{bmatrix} 6.967 & -5.222 \\ 0.452 & 27.01 \\ 1.276 & 169.22 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.249 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 9754.95 \end{bmatrix}, \quad C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{74}$$

$$\gamma_W = 0.0125, \quad \gamma_f = 0.7993.$$

The static gain from faults to residuals, considering only the first two significant rows, is given by:

$$K_{fr} = \begin{bmatrix} 0.1651 & 0.1982 & -0.1808 \\ 0.0226 & -0.0087 & 0.0256 \end{bmatrix}. \tag{75}$$

Notice that the obtained solution, even in the absence of disturbance, did not yield a diagonal structure for the residual generator. Even though γ_f shown in (74) can ensure a satisfactory performance for specification II, it is not possible to guarantee fault isolation since a diagonal (or, at least, triangular) structure for the static gain in (75) is not achieved. To overcome this problem, the performance specification II is relaxed to $\|\mathcal{G}_{f_i r} - \mathcal{G}_{f_i \breve{r}}\| \leq \gamma_{f_i}$, for $i = 1, 2, 3$, as represented by the optimization problem in (65). Therefore, in this case, by defining

$$\gamma_C = 0.2, \quad M = 10 \begin{bmatrix} I_{n_q} & 0_{n_q \times (n_x - n_q)} \end{bmatrix}, \quad \bar{\gamma}_W = 0.0215,$$

and applying the optimization problem in (65) yields the following results:

$$L = 10^{-4} \begin{bmatrix} -0.183 & -0.08 \\ -0.144 & -0.053 \\ 0.333 & 0.142 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.2127 & 0 & 0 \\ -0.042 & 0.232 & 0 \\ 0 & 0 & 3121.85 \end{bmatrix}, \quad C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{76}$$

$$\gamma_W = 0.0215, \quad \gamma_{f_1} = 6.3944, \quad \gamma_{f_2} = 0.0013, \quad \gamma_{f_3} = 0.0011.$$

Analysing the results presented given in (74) and (76), notice that despite an apparent large mismatch between $\mathcal{G}_{fr}$ and $\mathcal{G}_{f\breve{r}}$ in Eq. (76) (since $\|\mathcal{G}_{fe_r}\|_\infty^2 \leq 6.3944$), fault isolation can be guaranteed from the resulting $\mathcal{G}_{fr}$, as demonstrated by the derived static gain:

$$K_{fr} = \begin{bmatrix} 0.5040 & 0.2127 & 0 \\ 0.2894 & -0.0431 & 0.2324 \end{bmatrix}. \tag{77}$$

Comparing the above gain matrix with the static gain structure of the reference model in (72), fault detection and isolation are guaranteed. This indicates that optimization problem (65) leads to less conservative results for FDI purposes when the magnitude of fault inputs are relatively different as shown in the following numerical experiments.

In order to verify the behavior of the system under influence of faults and disturbances, a simulation study has been carried out. In this example, the following control variation Δq_j, for $j = 1, 2$, and disturbance signals w_j, $j = 1, 2$, are defined:

- $\Delta q_1 = \Delta q_2 = 0.05$;
- $w_1 = 0.01 \sin(10^{-5} k + \pi/2)$;
- w_2 is a white noise with power equal to 0.0001.

The fault signals applied to the three tanks system are defined in order to represent 10% of each signal of interest. While the actuator fault magnitude was defined considering the maximum admissible value regarding q_i, the sensor faults values take into account the equilibrium points in (67). Hence, the actuator fault, defined as f_a, and the sensor faults, described by f_{s_1} and f_{s_2}, have the following magnitudes:

$$f_a = 0.1, \qquad f_{s_1} = 0.06115, \qquad f_{s_2} = 0.05118. \tag{78}$$

To evaluate the performance of the designed observer concerning the FDI problem, Fig. 3 presents the residual response considering the influence of fault, control and disturbance inputs. This means that the disturbance and control inputs are applied to the system during the entire time slot. On the other hand, for FDI evaluation purposes, each fault signal is applied to the system one at a time. Fig. 3a is referred to the case where the actuator fault happens at instant $k = 60$, where $f_{s_1} = f_{s_2} = 0$. On the other hand, Fig. 3b and Fig. 3c are related to each one of the sensor faults, which happen at instants $k = 70$ and $k = 80$, respectively. In this case, $f_a = 0$.

From these results, one can notice the satisfactory performance achieved by the proposed filter despite relatively large exogenous disturbances. Besides, fault detection and isolation are guaranteed.

4.3 PEAK NORM APPROACH

Consider the class of linear discrete-time systems in (32) satisfying **A1** and **A2** as defined in Section 4.1. In addition, assume that the disturbance and fault signals belong the to the following sets:

A5 $\mathcal{W} = \left\{ w \in \mathbb{R}^{n_w} : w^T w = \|w\|_2^2 \leq 1 \right\}$

A6 $\mathcal{F} = \left\{ f \in \mathbb{R}^{n_f} : f^T f = \|f\|_2^2 \leq 1 \right\}$

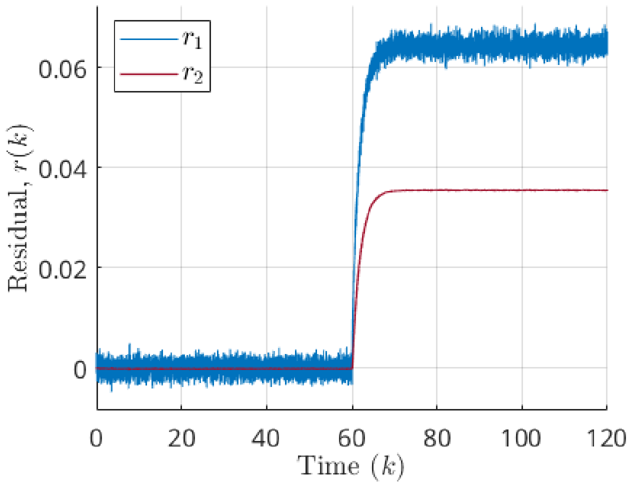
Assumptions **A5** and **A6** imply that both disturbance and fault signals are amplitude bounded. Notice that the sets $\mathcal{W}$ and $\mathcal{F}$ are normalized to 1 (one) which can be set without loss of generality by appropriately rescaling the matrices B_w, D_w, B_f and D_f.

Hence, considering the *peak*-norm concept for LTI discrete-time systems, the main objective of this section is to design a robust residual generator based on equations (33) and (35), such that the behavior from faults to residual is as close as possible to the one given by the reference model in (36). Therefore, the problem of concern in this case consists in determining the matrices L and Q of the filter in (33) and (35) such that:

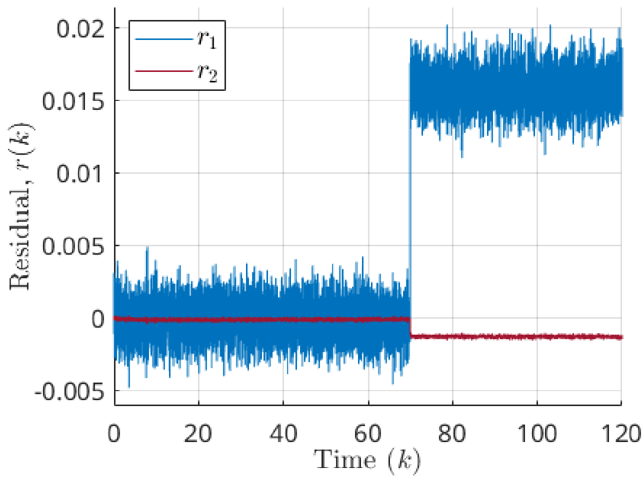
P-I $\|\mathcal{G}_{wr}\|_{peak}^2 \leq \gamma_w$,

P-II $\|\mathcal{G}_{fr} - \mathcal{G}_{f\breve{r}}\|_{peak}^2 \leq \gamma_f$,

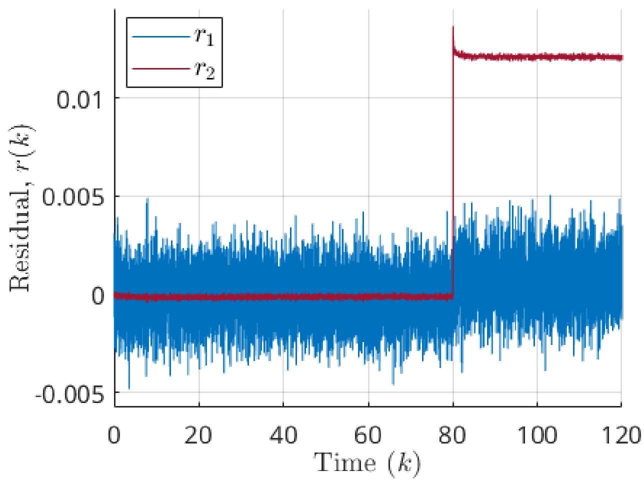
P-III $\|\mathcal{G}_{f\breve{r}}\|_{-}^2 \geq \gamma_c$;



(a) Residual response to the actuator fault.



(b) Residual response to the sensor fault 1.



(c) Residual response to the sensor fault 2.

Figure 3 – Residual response to fault, control and disturbance inputs.
Font: Own authorship.

where

$$\|\mathcal{G}_{wr}\|_{peak} = \sup_{w \in \mathcal{W}} \|r\|_{\ell_\infty}$$

is the *peak*-norm from w to r, with $\|\mathcal{G}_{fr} - \mathcal{G}_{f\check{r}}\|_{peak}$ being defined similarly, and $\|\mathcal{G}_{f\check{r}}\|_{-}$ is the smallest singular value of $\mathcal{G}_{f\check{r}}$.

Contrasting with the H_∞ approach, the matrix C_r of the residual generator in (35b) is constrained to be given in order to obtain a convex approach for the residual design problem in the *peak*-norm setting.

4.3.1 Determination of a bound on $\|\mathcal{G}_{wr}\|_{peak}$

Consider the Lyapunov function candidate in (41), the estimation error in (34), and let the following inequality:

$$\Delta V_1(\tilde{x}) \leq \tau_1(w^T w - V_1(\tilde{x})), \tag{79}$$

with τ_1 being a scalar belonging to the interval $(0, 1)$.

If the condition in (79) is satisfied, then the estimation error system in (35) is input-to-sate stable (ISS); see, e.g., (SONTAG; WANG, Y., 1996) and (JIANG; WANG, Y., 2001). In addition, to provide a bound γ_w on $\|\mathcal{G}_{wr}\|_{peak}$, the following must be guaranteed:

$$\gamma_w - r^T r \geq 0, \quad \forall\, (\tilde{x}, w) \quad : \quad \tilde{x}^T P_1 \tilde{x} \leq 1, \quad w^T w \leq 1. \tag{80}$$

Hence, by applying the S-Procedure, if there exists positive scalars α_1 and β_1 such that

$$\alpha_1 w^T w + \beta_1 \tilde{x}^T P_1 \tilde{x} - r^T r \geq 0 \tag{81}$$

holds, then the inequality in (80) is satisfied with

$$\gamma_w = \alpha_1 + \beta_1.$$

Furthermore, in order to obtain a tractable solution, the equation in (81) can be cast as follows:

$$\rho_1 w^T w + \tilde{x}^T P_1 \tilde{x} - \eta_1 r^T r \eta_1 \geq 0, \tag{82}$$

where

$$\gamma_w = \frac{\rho_1 + 1}{\eta_1^2}, \quad \rho_1 = \frac{\alpha_1}{\beta_1}, \quad \eta_1 = \frac{1}{\sqrt{\beta_1}},$$

which is equivalent to:

$$\begin{bmatrix} \tilde{x} \\ w \end{bmatrix}^T \left(\begin{bmatrix} P_1 & 0 \\ 0 & \rho_1 I_{n_w} \end{bmatrix} + \eta_1 \begin{bmatrix} C^T C_r^T \\ D_w^T C_r^T \end{bmatrix} Q^T Q \begin{bmatrix} C^T C_r^T \\ D_w^T C_r^T \end{bmatrix}^T \eta_1 \right) \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} \geq 0. \tag{83}$$

Hence, from the Schur complement, the following LMI is a necessary and sufficient condition for ensuring the boundedness of $\|\mathcal{G}_{wr}\|_{peak}$:

$$\begin{bmatrix} P_1 & * & * \\ 0 & \rho_1 I_{n_w} & * \\ \eta_1 C_r C & \eta_1 C_r D_w & \bar{Q} \end{bmatrix} > 0, \tag{84}$$

where $\bar{Q} = (Q^T Q)^{-1}$.

Next, notice that (79) can be written in the following form:

$$\Big((A-LC)\tilde{x}+(B_w-LD_w)w\Big)^T P_1 \Big((A-LC)\tilde{x}+(B_w-LD_w)w\Big) -(1-\tau_1)\tilde{x}^T P_1 \tilde{x}-\tau_1 w^T w \leq 0,$$

which can be cast as follows:

$$\begin{bmatrix} \tilde{x} \\ w \end{bmatrix}^T \left\{ \begin{bmatrix} -(1-\tau_1)P_1 & 0 \\ 0 & -\tau_1 I_{n_w} \end{bmatrix} + \begin{bmatrix} (A-LC)^T \\ (B_w-LD_w)^T \end{bmatrix} P_1 \begin{bmatrix} (A-LC) & (B_w-LD_w) \end{bmatrix} \right\} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} < 0.$$

By applying the Schur complement (1), the following condition is obtained:

$$\begin{bmatrix} -(1-\tau_1)P_1 & 0 & (A-LC)^T \\ 0 & -\tau_1 I_{n_w} & (B_w-LD_w)^T \\ (A-LC) & (B_w-LD_w) & -P_1^{-1} \end{bmatrix} < 0, \tag{85}$$

Hence, pre- and post-multiplying (85) by

$$\mathrm{diag}\{I_{n_x}, I_{n_w}, K\}$$

and its transpose yields:

$$\begin{bmatrix} -(1-\tau_1)P_1 & 0 & (KA-L_K C)^T \\ 0 & -\tau_1 I_{n_w} & (KB_w-L_K D_w)^T \\ (KA-L_K C) & (KB_w-L_K D_w) & -KP_1^{-1}K^T \end{bmatrix} < 0, \tag{86}$$

where $L_K = KL$.

Now, considering the condition (48), the following sufficient condition for ensuring that $\|\mathcal{G}_{wr}\|^2_{peak} \leq \gamma_w$ is derived:

$$\begin{bmatrix} -(1-\tau_1)P_1 & * & * \\ 0 & -\tau_1 I_{n_w} & * \\ (KA-L_K C) & (KB_w-L_K D_w) & P_1-K-K^T \end{bmatrix} < 0. \tag{87}$$

4.3.2 Determination of a bound on $\|\mathcal{G}_{fe_r}\|_{peak}$

Consider the augmented system in Eq. (38). Thus, in order to determine a bound on

$$\|\mathcal{G}_{fe_r}\|_{peak} := \|\mathcal{G}_{fr} - \mathcal{G}_{f\check{r}}\|_{peak}, \tag{88}$$

let

$$V_2(\bar{x}) = \bar{x}^T P_2 \bar{x}, \quad P_2 \in \mathbb{R}^{n_a \times n_a}, \quad n_a := n_x + n_q, \quad P_2 > 0, \tag{89}$$

be a Lyapunov function candidate and consider the following inequality:

$$\Delta V_2(\bar{x}) \leq \tau_2(f^T f - V_2(\bar{x})), \quad \tau_2 \in (0,1), \tag{90}$$

with $\Delta V_2(\bar{x}) = V_2(\bar{x}(k+1)) - V_2(\bar{x}(k))$.

Notice that the inequality in (90) can be represented in the following form:

$$(\bar{x}^+)^T P_2 \bar{x}^+ - (1-\tau_2)\bar{x}^T P_2 \bar{x} - \tau_2 f^T f \leq 0, \quad \forall\, (\bar{x}^+, \bar{x}, f) \;:\; -\bar{x}^+ + \bar{A}\bar{x} + \bar{B}_f = 0, \tag{91}$$

or, equivalently,

$$(\bar{A}\bar{x} + \bar{B}_f)^T P_2 (\bar{A}\bar{x} + \bar{B}_f) - (1-\tau_2)\bar{x}^T P_2 \bar{x} - \tau_2 f^T f \leq 0. \tag{92}$$

This means that:

$$\begin{bmatrix} \bar{x} \\ f \end{bmatrix}^T \left(\begin{bmatrix} -(1-\tau_2)P_2 & * \\ 0 & -\tau_2 I_{n_f} \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} P_2 \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} \right) \begin{bmatrix} \bar{x} \\ f \end{bmatrix} \leq 0,$$

By applying the Schur complement, the above inequality is satisfied, if and only if the following matrix inequality holds:

$$\begin{bmatrix} -(1-\tau_2)P_2 & 0 & \bar{A}^T \\ 0 & -\tau_2 I_{n_f} & \bar{B}^T \\ \bar{A} & \bar{B} & -P_2^{-1} \end{bmatrix} < 0. \tag{93}$$

Therefore, considering the matrix partitions of P_2 and K_2 as defined in (54) and (55), respectively, and the procedure developed in Section 4.2.2, the following matrix inequality is obtained:

$$\begin{bmatrix} \omega_{p11} & \omega_{p12} & \omega_{p13} & \omega_{p14} & \omega_{p15} \\ * & \omega_{p22} & \omega_{p23} & \omega_{p24} & \omega_{p25} \\ * & * & \omega_{p33} & \omega_{p34} & 0 \\ * & * & * & \omega_{p44} & 0 \\ * & * & * & * & \omega_{p55} \end{bmatrix} < 0, \tag{94}$$

where

$$\omega_{p11} = P_{21} - K - K^T, \qquad \omega_{p12} = P_{22}^T - K_a - K^T M^T,$$
$$\omega_{p13} = KA - L_K C, \qquad \omega_{p14} = K_a \breve{A},$$
$$\omega_{p15} = KB_f - L_K D_f + K_a \breve{B}, \qquad \omega_{p22} = P_{23} - K_b - K_b{}^T,$$
$$\omega_{p23} = MKA - ML_K C, \qquad \omega_{p24} = K_b \breve{A},$$
$$\omega_{p25} = MKB_f - ML_K D_f + K_b \breve{B}, \qquad \omega_{p33} = -(1-\tau_2)P_{21},$$
$$\omega_{p34} = -(1-\tau_2)P_{22}^T, \qquad \omega_{p44} = -(1-\tau_2)P_{23},$$
$$\omega_{p55} = -\tau_2 I_{n_f},$$

with L_K as defined in (47).

Next, a bound γ_f on $\|\mathcal{G}_{fe_r}\|_{peak}$ is guaranteed if the following is satisfied:

$$\gamma_f - e_r{}^T e_r \geq 0, \quad \forall\, (\bar{x}, f) \;:\; \bar{x}^T P_2 \bar{x} \leq 1, \quad f^T f \leq 1. \tag{95}$$

Hence, by applying the S-Procedure, the following sufficient condition for the inequality in (95) to hold is obtained:

$$\bar{x}^T \beta_2 P_2 \bar{x} + \alpha_2 f^T f - e_r{}^T e_r \geq 0, \tag{96}$$

with $\gamma_f = \alpha_2 + \beta_2$, which can be cast as follows:

$$\begin{bmatrix} \bar{x} \\ f \end{bmatrix}^T \left(\begin{bmatrix} P_2 & 0 \\ 0 & \rho_2 I_{n_f} \end{bmatrix} - \begin{bmatrix} (Q\bar{C})^T \\ (Q\bar{D})^T \end{bmatrix} \eta_2^2 \begin{bmatrix} Q\bar{C} & Q\bar{D} \end{bmatrix} \right) \begin{bmatrix} \bar{x} \\ f \end{bmatrix} \geq 0. \tag{97}$$

where

$$\rho_2 = \frac{\alpha_2}{\beta_2}, \quad \eta_2 = \frac{1}{\sqrt{\beta_2}}, \quad \gamma_f = \frac{1+\rho_2}{\eta_2^2}.$$

Notice that the following matrix inequality (derived from the Schur complement) is a necessary and sufficient condition for (97) hold:

$$\begin{bmatrix} P_2 & * & * \\ 0 & \rho_2 I_{n_f} & * \\ \eta_2 Q\bar{C} & \eta_2 Q\bar{D} & I_{n_f} \end{bmatrix} \geq 0. \tag{98}$$

Thus, pre- and post-multiplying (98) by

$$\mathrm{diag}\{I_{n_a}, I_{n_y}, Q^{-1}\}$$

and its transpose, leads to the following LMI:

$$\begin{bmatrix} P_{21} & * & * & * \\ P_{22}^T & P_{23} & * & * \\ 0 & 0 & \rho_2 I_{n_f} & * \\ \eta_2 C_r C & -\eta_2 \breve{C} & \eta_2 C_r D_f - \eta_2 \breve{D} & \bar{Q} \end{bmatrix} > 0, \tag{99}$$

where P_{21}, P_{22}, P_{23}, $\bar{Q}$ and η_2 are to be designed.

4.3.3 Residual Design

The following LMI-based theorem is established with a view to ensure that the residual generator is input-to-state stable while ensuring the performance specifications defined in P-I, P-II and P-III.

Theorem 2 *Consider the residual generator in Eq.* (35)*, the reference model in Eq.* (36) *and the augmented system in Eq.* (38)*. Let* $\breve{A}$, $\breve{B}$, $\breve{C}$, $\breve{D}$, M, C_r, γ_C, τ_1 *and* τ_2 *be given, with* $\tau_i \in (0,1)$, $i = 1,2$*. Suppose there exist symmetric matrices* P_1, P_{21}, P_{23}, P_3 *and* $\bar{Q}$*, free matrices* L_K, K, K_a, K_b*, and* P_{22}*, and positive scalars* ρ_1, ρ_2, η_1 *and* η_2 *such that the LMIs in* (**??**), (84), (94), (99) *and* (61) *are satisfied. Then, the following statements hold:*

i the estimation error system (35) *is ISS;*

ii $\|\mathcal{G}_{wr}\|_{peak} \leq \frac{\sqrt{1+\rho_1}}{\eta_1}$*;*

iii $\|\mathcal{G}_{fe_r}\|_{peak} \leq \frac{\sqrt{1+\rho_2}}{\eta_2}$*; and*

iv $\|\mathcal{G}_{f\breve{r}}\|_{-} \geq \gamma_c$*,*

with $L = K^{-1}L_K$ *and* $Q = \bar{Q}^{-1/2}$*.*

The proof of the above theorem follows in the Appendix B.

4.3.4 Computational Issues

Similarly to the procedure proposed in Section 4.2.5, one may apply Theorem 2, for instance, to minimize $\|\mathcal{G}_{wr}\|_{peak}$ and $\|\mathcal{G}_{fe_r}\|_{peak}$. However, the bounds γ_w and γ_f are non-convex with respect to ρ_1, η_1, ρ_2 and η_2 by noting that:

$$\gamma_w = \frac{1+\rho_1}{\eta_1^2} \quad \text{and} \quad \gamma_f = \frac{1+\rho_2}{\eta_2^2}$$

To overcome the above problem, the following solution is adopted in this thesis:

$$\min_{P_1,\ldots,Q,\rho_1,\rho_2,\eta_1,\eta_2} (\rho_1+\rho_2-\eta_1-\eta_2) \ \text{ subject to } \begin{cases} (84), (94), \\ (99), (61) \end{cases} \tag{100}$$

for a given bound γ_c on $\|\mathcal{G}_{f\breve{r}}\|_{-}$.

The optimization problem in (100) approximately minimizes $\gamma = \gamma_w + \gamma_f$. If needed, one can alternatively consider an weighted version $\gamma(\kappa) = (1-\kappa)(\rho_1-\eta_1)+\kappa(\rho_2-\eta_2)$, where $\kappa \in (0,1)$ is a parameter to be tuned by the designer.

4.3.5 Numerical example

To illustrate the *Peak*-norm approach, the same numerical example introduced in Section 4.2.6, consisting of the three tanks system depicted in Figure 2, is considered to evaluate the residual generator performance.

Firstly, for convenience, the matrix transfer function from the faults f to the output y is recalled:

$$\mathcal{G}_{sys} = \begin{bmatrix} g_{11} & 1 & 0 \\ g_{21} & 0 & 1 \end{bmatrix}, \tag{101}$$

where

$$g_{11} = \frac{0.02299z^2 - 0.04311z + 0.02019}{z^3 - 2.843z^2 - 2.692z - 0.8488},$$

$$g_{21} = \frac{0.0003855z^2 + 8.835 \cdot 10^{-6}z - 0.0003509}{z^3 - 2.843z^2 + 2.692z - 0.8488}.$$

It is worth remembering that the dynamics described in (101) has an appropriate structure for FDI purposes by analyzing the system static gain given in (70). Furthermore, as discussed in Section 4.2.6, $\mathcal{G}_{sys}$ can be straightforwardly applied to detect and isolate the faults assuming that they do not occur simultaneously. Then, in order to compare the results achieved by means of the H_∞ and *peak* norms, the reference model used here is the same one given in (71).

In order to apply the optimization problem defined in (100), consider the following numerica values:

$$\gamma_c = 0.125, \quad \tau_1 = 0.001, \quad \tau_2 = 0.006,$$
$$M = -0.9 \cdot \begin{bmatrix} I_{n_q} & 0_{n_q \times (n_x - n_q)} \end{bmatrix}, \quad C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{102}$$

The above parameters were chosen such that the LMI conditions are feasible and $\gamma_w + \gamma_f$ is approximately minimized. In particular:

- A fine griding was applied for designing τ_1 and τ_2, with $(\tau_1, \tau_2) \in (0,1) \times (0,1)$;
- The matrices C_r and M are set to be $C_r = \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}^T$ and $M = \vartheta \cdot \begin{bmatrix} I_{n_q} & 0_{n_q \times (n_x - n_q)} \end{bmatrix}$, where ϑ is a given parameter; and
- The parameters ϑ and γ_c were iteratively optimized using a coarse griding.

From the above setting, the optimization problem in (100) led to the following results:

$$Q = \begin{bmatrix} 0.125 & 0 & 0 \\ -0.0002 & 0.1252 & 0 \\ 0 & 0 & 182.393 \end{bmatrix}, \quad L = 10^{-3} \begin{bmatrix} 0.7658 & 0.2161 \\ 0.0248 & -0.0216 \\ 0.1247 & -0.1295 \end{bmatrix},$$
$$\rho_1 = 0.05, \quad \rho_2 = 0.0027, \quad \eta_1 = 24.77, \quad \eta_2 = 20.4,$$
$$\gamma_w = 0.0414, \quad \gamma_f = 0.0024.$$

The derived static gain with respect to the behavior from faults to residual is

$$K_{fr} = \begin{bmatrix} 0.2702 & 0.1150 & -0.0013 \\ 0.1918 & -0.0077 & 0.1246 \end{bmatrix}, \tag{103}$$

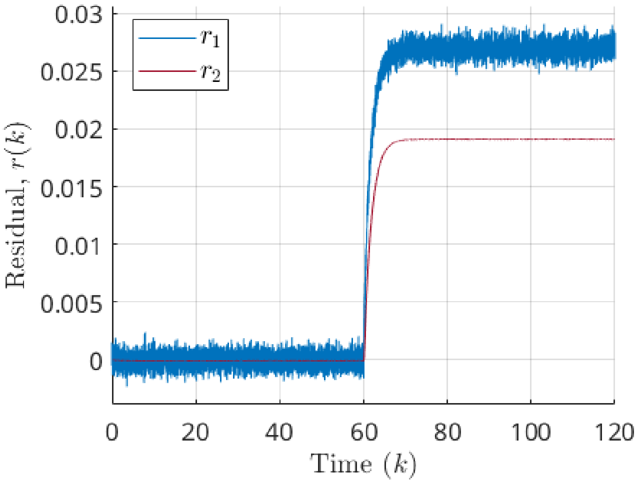
which indicates that that fault detection and isolation can be achieved despite the presence of persistent exogenous disturbances.

A simulation has been carried out in order to evaluate the performance of the designed residual generator for FDI purposes. Thus, the same control variation and disturbance signals as in Section 4.2.6 were considered as well as the fault signals defined in (78). Figure 4 shows the residual response to the application of control, disturbance and fault inputs. The control and disturbance signals are applied to the system during the whole time slot, while the fault signals are applied one at a time in order to evaluate the FDI performance. In particular, Figure 4a depicted the result related to the application of the actuator fault at instant $k = 60$ (with $f_{s_1} = f_{s_2} = 0$), while Figures 4b and 4c display the residual signal considering sensor faults f_{s_1} (at instant $k = 70$ with $f_{s2} = 0$ and $f_a = 0$) and f_{s_2} (at instant $k = 80$ with $f_{s1} = 0$ and $f_a = 0$), respectively.

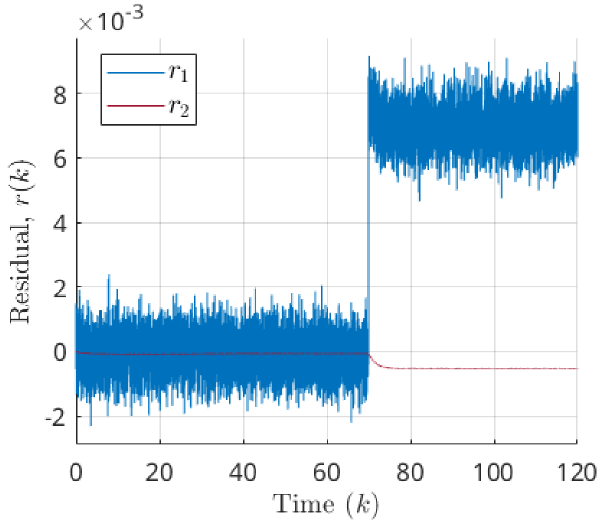
Notice in Figure 4a that both residuals were affected by the actuator fault f_a as expected. In Figure 4b, the isolation of the sensor fault f_{s_1} is guaranteed taking into account the residual r_1 response compared to the small influence of residual r_2. Similarly, in Figure 4c, the residual r_2 response is sufficiently large to indicate that the sensor f_{s_2} is faulty despite a noisy measurement. From these results, it should be emphasized the good performance achieved by the proposed residual generator based on the *peak*-norm approach.

4.4 CHAPTER CONCLUSION

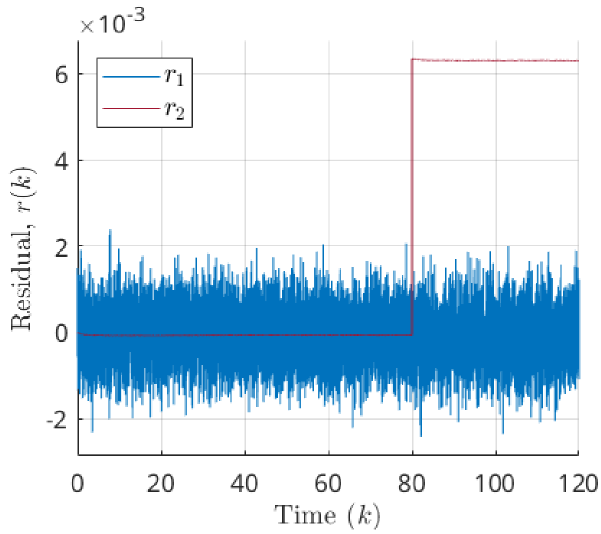
In this chapter, two different approaches (described in Sections 4.2 and 4.3) were proposed for designing a residual generator for FDI purposes based on an observer-like filter and a reference model representing the desired fault detection and isolation behavior. Specifically, convex optimization problems in terms of LMI constraints were devised to design the residual free parameters. Hence, considering the numerical example introduced in Section 4.2.6, the H_∞ approach was applied to ensure the asymptotic stability of the estimation error while guaranteeing the fault detection and isolation performance. In addition, the same numerical example is considered to illustrate the residual generator design considering the *peak*-norm approach which ensures the input-to-state stability of the estimation error dynamics while guaranteeing desired performance specifications. The results clearly indicated the effectiveness of the proposed methods, the *peak*-norm approach despite the large numbers of tuning parameters seems to be less sensible to large magnitude differences of fault inputs. Future research will be focused on designing FDI methods for a class of nonlinear discrete-time systems by means of H_∞ and *peak*-norm approaches. Preliminary results on the state estimation of nonlinear discrete-time systems assuming persistent disturbance signals are presented in the Appendix C.



(a) Residual response to the actuator fault.



(b) Residual response to sensor fault 1.



(c) Residual response to sensor fault 2.

Figure 4 – Residual response to fault, control and disturbance inputs.
Font: Own authorship.

5 REFERENCE MODEL-BASED FDI FOR LDTDS

This chapter presents a robust reference model-based solution for the FDI problem applied to a class of linear discrete-time descriptor systems based on either the H_∞- or *peak*-norm settings. Besides ordinary differential equations that appear in a traditional state-space representation, descriptor models also include algebraic equations, which are notably encountered when modeling electric systems as Kirchhoff's laws must be fulfilled. Therefore, similarly to Chapter 4, the goal is to ensure fault detection and isolation taking into account the desired behavior given by a reference model despite the presence of disturbances. The observer-like filter is designed in terms of LMI constraints considering a standard state-space realization of the descriptor model. Therefore, the class of descriptor systems and the standard state-space realization to be considered in this chapter are defined in Section 5.1, followed by the proposed filter, the residual generator, the reference model and the augmented system. Next, Section 5.2 presents the design of an H_∞ reference model-based FDI for linear discrete-time descriptor systems while Section 5.3 introduces the *peak*-norm counterpart. The effectiveness of the proposed design is illustrated by a numerical example in Section 5.4.

5.1 PROBLEM OF INTEREST

Consider the following class of regular and causal discrete-time descriptor systems:

$$Ex(k+1) = Ax(k) + B_u u(k) + B_w w(k) + B_f f(k), \tag{104a}$$

$$y(k) = Cx(k) + D_u u(k) + D_w w(k) + D_f f(k), \tag{104b}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector of the descriptor systems, $u(k) \in \mathbb{R}^{n_u}$ is the measured input, $w(k) \in \mathcal{W} \subset \mathbb{R}^{n_w}$ is the exogenous input, $y(k) \in \mathbb{R}^{n_y}$ is the measured output, $f(k) \in \mathcal{F} \subset \mathbb{R}^{n_f}$ is a vector containing actuator and/or sensor faults, and $E, A, B_u, B_w, B_f, C, D_u, D_w, D_f$ are given real matrices with appropriate dimensions. The class of signals defined by the sets $\mathcal{W}$ and $\mathcal{F}$ will be later specified in this chapter.

The following conditions are assumed with respect to the system defined in (104):

A1 The total number of input and output measurements is equal to n, i.e., $n = n_u + n_y$.

A2 The number of faults is equal or smaller than the number of measurements, i.e., $n_f \leq 2n$.

A3 $\text{rank}\{E\} = n_e$, with $n_e < n_x$.

A4 $\text{rank}\left\{\begin{bmatrix} E \\ C \end{bmatrix}\right\} = n_x$.

A5 $\text{rank}\left\{\begin{bmatrix} (zE - A) \\ C \end{bmatrix}\right\} = n_x, \; \forall\, z \in \mathbb{C} : 1 \leq |z| < \infty$.

Similarly to Chapter 4, assumptions **A1** and **A2** ensure the the well-posedness of the FDI problem. In addition, assumptions **A3**-**A5** are somehow standard in the literature of descriptor systems (DAROUACH, 2009; DAROUACH et al., 2017). In particular, **A3** is to state that the system defined in (104) is a descriptor system, whereas **A4** and **A5** imply respectively that the fast dynamics of the descriptor system is observable and its slow dynamics is detectable (DAI, 1989; BELOV et al., 2018).

From **A4**, notice that there exist matrices $T \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_x \times n_y}$ such that the following holds:

$$TE + RC = X\Phi = I_{n_x}, \tag{105}$$

where

$$X = \begin{bmatrix} T & R \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} E \\ C \end{bmatrix}. \tag{106}$$

Furthermore, any solution X to the equality in (105) can be parameterized as follows (BEN-ISRAEL; GREVILLE, 2003):

$$X = \Psi + ZS \tag{107}$$

where $\Psi \in \mathbb{R}^{n_x \times (n_x + n_y)}$ is the Moore-Penrose pseudoinverse of Φ, $Z \in \mathbb{R}^{n_x \times (n_x + n_y)}$ is an arbitrary matrix and

$$S = I_{n_x + n_y} - \Phi\Psi. \tag{108}$$

Since the matrix Z is a free matrix, without loss of generality, it can be assumed that there always exists a nonsingular matrix T satisfying (105), as discussed, e.g., in (KOENIG, 2006) and (RAO; MITRA, 1972).

Next, post-multiplying (105) by $x(k+1)$ leads to:

$$\begin{aligned} x(k+1) &= TEx(k+1) + RCx(k+1) \\ &= TEx(k+1) + R\big[y(k+1) - D_u u(k+1) - D_w w(k+1) - D_f f(k+1)\big]. \end{aligned} \tag{109}$$

Hence, replacing (104) into (109), the descriptor system can be represented in the following standard state-space form:

$$\begin{aligned} x(k+1) &= TAx(k) + TB_u u(k) + TB_w w(k) + TB_f f(k) + Ry(k+1) \\ &\qquad - RD_u u(k+1) - RD_w w(k+1) - RD_f f(k+1) \end{aligned} \tag{110a}$$

$$y(k) = Cx(k) + D_u u(k) + D_w w(k) + D_f f(k). \tag{110b}$$

Now, let the following notation:

$$\Delta f(k) = f(k+1) - f(k), \qquad f_a(k) = \begin{bmatrix} f(k) \\ \Delta f(k) \end{bmatrix}, \tag{111}$$

$$\Delta w(k) = w(k+1) - w(k), \qquad w_a(k) = \begin{bmatrix} w(k) \\ \Delta w(k) \end{bmatrix}.$$

Then, the system in (110) can be rewritten as:

$$x(k+1) = TAx(k) + TB_u u(k) + Ry(k+1) - RD_u u(k+1) + B_{w_a} w_a(k) + B_{f_a} f_a(k) \quad \text{(112a)}$$

$$y(k) = Cx(k) + D_u u(k) + D_{w_a} w_a(k) + D_{f_a} f_a(k), \quad \text{(112b)}$$

where

$$B_{w_a} = \begin{bmatrix} (TB_w - RD_w) & -RD_w \end{bmatrix}, \qquad B_{f_a} = \begin{bmatrix} (TB_f - RD_f) & -RD_f \end{bmatrix},$$
$$D_{w_a} = \begin{bmatrix} D_w & 0 \end{bmatrix}, \qquad D_{f_a} = \begin{bmatrix} D_f & 0 \end{bmatrix}. \quad \text{(113)}$$

Associate to the system representation in (112), consider the following observer-like filter:

$$\hat{x}(k+1) = TA\hat{x}(k) + TB_u u(k) - RD_u u(k+1) + Ry(k+1) + L\big(y(k) - \hat{y}(k)\big), \quad \text{(114a)}$$

$$\hat{y}(k) = C\hat{x}(k) + D_u u(k), \quad \text{(114b)}$$

where $\hat{x} \in \mathbb{R}^{n_x}$ and $\hat{y} \in \mathbb{R}^{n_y}$ are estimates of x and y, respectively, with $T \in \mathbb{R}^{n_x \times n_x}$, $R \in \mathbb{R}^{n_x \times n_y}$ and $L \in \mathbb{R}^{n_x \times n_y}$ to be designed.

Now, let

$$\tilde{x}(k) = x(k) - \hat{x}(k) \quad \text{(115)}$$

be the estimation error. Then, the error dynamics can be cast as follows:

$$\tilde{x}(k+1) = \big(TA - LC\big)\tilde{x}(k) + \big(B_{w_a} - LD_{w_a}\big)w_a(k) + \big(B_{f_a} - LD_{f_a}\big)f_a(k) \quad \text{(116)}$$

In light of the above developments, the proposed residual generator for the fault, detection and isolation of the class of discrete-time systems defined in (104) is summarized by the following equations:

$$\hat{x}(k+1) = \big(TA - LC\big)\hat{x}(k) + (TB_u - LD_u)u(k) - RD_u u(k+1) + Ry(k+1) + Ly(k) \quad \text{(117a)}$$

$$\hat{y}(k) = C\hat{x}(k) + D_u u(k), \quad \text{(117b)}$$

$$r(k) = QC_r\big(y(k) - \hat{y}(k)\big) \quad \text{(117c)}$$

where $C_r \in \mathbb{R}^{n_f \times n_y}$, $L \in \mathbb{R}^{n_x \times n_y}$ and $Q \in \mathbb{R}^{n_f \times n_f}$ are to be designed.

In addition, with the view of enforcing residual sensitivity to faults, consider the following fault-to-residual reference model:

$$\mathcal{G}_{f_a\breve{r}} : \begin{cases} \breve{x}(k+1) &= \breve{A}\breve{x}(k) + \breve{B}f_a(k) \\ \breve{r}(k) &= Q\big(\breve{C}\breve{x}(k) + \breve{D}f_a(k)\big) \end{cases} \quad \text{(118)}$$

where $\breve{x}(k) \in \mathbb{R}^{n_q}$ and $\breve{r}(k) \in \mathbb{R}^{n_f}$ are the reference model state and residual, respectively, and $\breve{A}$, $\breve{B}$, $\breve{C}$ and $\breve{D}$ are given matrices with appropriate dimensions with $\breve{A}$ being Schur stable. Similarly to the reference model considered in Chapter 4, it should be emphasized that the choice of matrices $\breve{A}$, $\breve{B}$, $\breve{C}$ and $\breve{D}$ must provide the desired behavior from faults to residuals taking into account the augmented fault vector as defined in (111).

Before ending this section, the following auxiliary systems will be instrumental for designing the residual generator in (117) considering either the H_∞ or *peak*-norm settings.

$$\mathcal{G}_{w_a r} : \begin{cases} \tilde{x}(k+1) &= \big(TA - LC\big)\tilde{x}(k) + \big(B_{w_a} - LD_{w_a}\big)w_a(k) \\ r(k) &= QC_r\big(C\tilde{x}(k) + D_{w_a}w_a(k)\big) \end{cases} \tag{119}$$

$$\mathcal{G}_{f_a e_r} : \begin{cases} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}f_a(k) \\ e_r(k) &= Q\big(\bar{C}\bar{x}(k) + \bar{D}f_a(k)\big) \end{cases} \tag{120}$$

where

$$e_r(k) := r(k) - \breve{r}(k) \tag{121}$$

is the mismatch between r and $\breve{r}$ and

$$\bar{x} = \begin{bmatrix} \tilde{x} \\ \breve{x} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} TA - LC & 0 \\ 0 & \breve{A} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \big(B_{f_a} - LD_{f_a}\big) \\ \breve{B} \end{bmatrix}, \tag{122}$$
$$\bar{C} = \begin{bmatrix} C_rC & -\breve{C} \end{bmatrix}, \quad \bar{D} = C_rD_{f_a} - \breve{D}.$$

5.2 H_∞-NORM APPROACH

The problem of concern in this section is to design the residual generator as defined in (117) assuming that $\mathcal{W}$ and $\mathcal{F}$ are ℓ_2 signals and considering a mixed H_∞/H_- performance such that the behavior from faults to residual is is approximated by the reference model defined in (118). In other words, the matrices T, R, L, Q and C_r of the residual generator in (117) should be designed such that the error dynamics in (116) is Schur stable and the following performance requirements hold:

(i) $\|\mathcal{G}_{w_a r}\|_\infty^2 \leq \gamma_1$,

(ii) $\|\mathcal{G}_{f_a e_r}\|_\infty^2 \leq \gamma_2$,

(iii) $\|\mathcal{G}_{f_a \breve{r}}\|_-^2 \geq \gamma_3$,

where $\mathcal{G}_{w_a r}$, $\mathcal{G}_{f_a e_r}$ and $\mathcal{G}_{f_a \breve{r}}$ are as defined in (119), (120) and (118), respectively, and γ_1, γ_2 and γ_3 are positive scalars defining the residual generator performance.

Notice that the observer dynamics as well as the performance criteria $\|\mathcal{G}_{w_a r}\|_\infty$, $\|\mathcal{G}_{f_a e_r}\|_\infty$ and $\|\mathcal{G}_{f_a \breve{r}}\|_-$ depend on the choice of the matrices T and R satisfying the equality constraint in (105). Thus, in view of the parametrization given in (107), one can exploit the degree of freedom introduced by the matrix Z when designing the residual generator in order to improve the overall performance of the FDI scheme.

Before introducing the main result of this section, consider the following parametrization of the matrices T and R which will be instrumental to devise a solution for the FDI problem for LDTDS in a mixed H_∞/H_- setting:

$$T = W_1 + Z_1 S_{11} + Z_2 S_{21}, \quad R = W_2 + Z_1 S_{12} + Z_2 S_{22}, \tag{123}$$

where

$$\Psi = \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \tag{124}$$

with Ψ and S being as defined in (107) and (108), respectively, and the matrices $W_1 \in \mathbb{R}^{n_x \times n_x}$, $W_2 \in \mathbb{R}^{n_x \times n_y}$, $Z_1 \in \mathbb{R}^{n_x \times n_x}$, $Z_2 \in \mathbb{R}^{n_x \times n_y}$, $S_{11} \in \mathbb{R}^{n_x \times n_x}$, $S_{12} \in \mathbb{R}^{n_x \times n_y}$, $S_{21} \in \mathbb{R}^{n_y \times n_x}$ and $S_{22} \in \mathbb{R}^{n_y \times n_y}$ being obtained by appropriately partitioning the matrices Ψ, Z and S, respectively.

5.2.1 Determining a bound on $\|\mathcal{G}_{w_a r}\|_\infty$

Let $V_1(\tilde{x}) = \tilde{x}^T P_1 \tilde{x}$, $P_1 > 0$, be a Lyapunov function candidate for the system $\mathcal{G}_{w_a r}$ as defined in (119) and consider the following H_∞ inequality

$$\Delta V_1(\tilde{x}) + r^T r - \gamma_1 {w_a}^T w_a < 0, \tag{125}$$

where $\Delta V_1(\tilde{x}) = V(\tilde{x}(k+1)) - V(\tilde{x})$.

Substituting the system dynamics and the output equation given in (119) into the above inequality yields

$$\begin{aligned}\left\{\left(TA - LC\right)\tilde{x} + \left(B_{w_a} - LD_{w_a}\right)w_a\right\}^T P_1 \left\{\left(TA - LC\right)\tilde{x} + \left(B_{w_a} - LD_{w_a}\right)w_a\right\} - \tilde{x}^T P_1 \tilde{x} \\ + (C_r C\tilde{x} + C_r D_{w_a} w_a)^T Q^T Q (C_r C\tilde{x} + C_r D_{w_a} w_a) - \gamma_1 {w_a}^T w_a < 0,\end{aligned} \tag{126}$$

which can be cast as follows:

$$\begin{aligned}\begin{bmatrix} \tilde{x} \\ w_a \end{bmatrix}^T \left\{ \begin{bmatrix} -P_1 & 0 \\ 0 & -\gamma_1 I_{2n_w} \end{bmatrix} + \begin{bmatrix} \left(TA - LC\right)^T \\ \left(B_{w_a} - LD_{w_a}\right)^T \end{bmatrix} P_1 \begin{bmatrix} \left(TA - LC\right) & \left(B_{w_a} - LD_{w_a}\right) \end{bmatrix} \right. \\ \left. + \begin{bmatrix} (C_r C)^T \\ (C_r D_{w_a})^T \end{bmatrix} Q^T Q \begin{bmatrix} (C_r C) & (C_r D_{w_a}) \end{bmatrix} \right\} \begin{bmatrix} \tilde{x} \\ w_a \end{bmatrix} < 0.\end{aligned} \tag{127}$$

From the Schur's complement, the above inequality is satisfied if the following matrix inequality holds:

$$\begin{bmatrix} -P_1 & 0 & \left(TA - LC\right)^T & C^T {C_r}^T \\ 0 & -\gamma_1 I_{2n_w} & \left(B_{w_a} - LD_{w_a}\right)^T & {D_{w_a}}^T {C_r}^T \\ TA - LC & B_{w_a} - LD_{w_a} & -P_1^{-1} & 0 \\ C_r C & C_r D_{w_a} & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{128}$$

where $\bar{Q} = (Q^T Q)^{-1}$.

Next, pre- and post-multiplying (128) by

$$\text{diag}\{I_{n_x}, I_{2n_w}, K, I_{n_f}\}$$

and its transpose leads to:

$$\begin{bmatrix} -P_1 & 0 & \left(T_K A - L_K C\right)^T & C^T {C_r}^T \\ 0 & -\gamma_1 I_{2n_w} & \Omega^T & {D_{w_a}}^T {C_r}^T \\ T_K A - L_K C & \Omega & -K P_1^{-1} K^T & 0 \\ C_r C & C_r D_{w_a} & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{129}$$

where $K \in \mathbb{R}^{n_x \times n_x}$ is a free nonsingular matrix and:

$$T_K = KT = KW_1 + Z_{1K} S_{11} + Z_{2K} S_{21}, \qquad L_K = KL, \qquad Z_{1K} = KZ_1, \qquad Z_{2K} = KZ_2,$$
$$\Omega = \left[(T_K B_w - R_K D_w) \quad -R_K D_w \right], \qquad R_K = KR = KW_2 + Z_{1K} S_{12} + Z_{2K} S_{22}. \tag{130}$$

Then, by recalling the condition in (48), the following LMI ensures that $\|\mathcal{G}_{w_a r}\|_\infty \leq \gamma_1$:

$$\begin{bmatrix} -P_1 & * & * & * \\ 0 & -\gamma_1 I_{2n_w} & * & * \\ T_K A - L_K C & \Omega & P_1 - K - K^T & * \\ C_r C & C_r D_{w_a} & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{131}$$

where P_1, K, L_K, Z_{1K}, Z_{2K}, C_r and $\bar{Q}$ are the decision variables.

5.2.2 Determining a bound on $\|\mathcal{G}_{f_a e_r}\|_\infty$

Let

$$V_2(\bar{x}) = \bar{x}^T P_2 \bar{x}, \quad P_2 \in \mathbb{R}^{n_a \times n_a}, \quad n_a := n_x + n_q, \quad P_2 > 0, \tag{132}$$

be a Lyapunov function candidate for the system defined in (120) and consider the following inequality:

$$\Delta V_2(\bar{x}) + {e_r}^T e_r - \gamma_2 {f_a}^T f_a < 0, \tag{133}$$

which implies for zero initial conditions that $\|\mathcal{G}_{f_a e_r}\|_\infty \leq \gamma_2$.

Thus, similarly to the development in (126) and taking into account the definition of the augmented dynamics in (120), it follows that:

$$(\bar{A}\bar{x} + \bar{B} f_a)^T P_2 (\bar{A}\bar{x} + \bar{B} f_a) + (\bar{C}\bar{x} + \bar{D} f_a)^T Q^T Q (\bar{C}\bar{x} + \bar{D} f_a) - \bar{x}^T P_2 \bar{x} - \gamma_2 {f_a}^T f_a < 0, \tag{134}$$

which can be cast as follows:

$$\begin{bmatrix} \bar{x} \\ f_a \end{bmatrix}^T \left\{ \begin{bmatrix} -P_2 & 0 \\ 0 & -\gamma_2 I_{2n_f} \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} P_2 \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} Q^T Q \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \right\} \begin{bmatrix} \bar{x} \\ f_a \end{bmatrix} < 0.$$

From the Schur's complement, the above inequality holds if the following matrix inequality is satisfied:

$$\begin{bmatrix} -P_2 & 0 & \bar{A}^T & \bar{C}^T \\ 0 & -\gamma_2 I_{2n_f} & \bar{B}^T & \bar{D}^T \\ \bar{A} & \bar{B} & -P_2^{-1} & 0 \\ \bar{C} & \bar{D} & 0 & -\bar{Q} \end{bmatrix} < 0. \tag{135}$$

Now, let the following matrix

$$\Lambda = \text{diag}\{I_{n_a}, I_{2n_f}, K_2, I_{n_f}\}, \tag{136}$$

where K_2 is as defined in (55), and consider the following partition of the matrix P_2

$$P_2 = \begin{bmatrix} P_{21} & * \\ P_{22} & P_{23} \end{bmatrix}, \tag{137}$$

with $P_{21} = P_{21}^T \in \mathbb{R}^{n_x \times n_x}$, $P_{22} \in \mathbb{R}^{n_q \times n_x}$ and $P_{23} = P_{23}^T \in \mathbb{R}^{n_q \times n_q}$.

Next, by pre- and post-multiplying (135) by Λ and Λ^T, respectively, it follows that:

$$\begin{bmatrix} -P_{21} & * & * & * & * & * \\ -P_{22} & -P_{23} & * & * & * & * \\ 0 & 0 & -\gamma_2 I_{2n_f} & * & * & * \\ \varpi_{41} & K_a\breve{A} & \varpi_{43} & \varpi_{44} & * & * \\ \varpi_{51} & K_b\breve{A} & \varpi_{53} & \varpi_{54} & \varpi_{55} & * \\ C_r C & -\breve{C} & \varpi_{63} & 0 & 0 & -\bar{Q} \end{bmatrix} < 0 \tag{138}$$

ensures that $\|\mathcal{G}_{f_a e_r}\|_\infty \leq \gamma_2$ holds from the fact that

$$P_2 - K_2 - K_2^T \geq -K_2 P_2^{-1} K_2^T,$$

where

$$\varpi_{41} = T_K A - L_K C, \qquad \varpi_{43} = \Upsilon_a + K_a\breve{B},$$

$$\varpi_{44} = P_{21} - K - K^T, \qquad \varpi_{53} = M\Upsilon_a + K_b\breve{B},$$

$$\varpi_{51} = MT_K A - ML_K C, \qquad \varpi_{54} = P_{22} - MK - K_a^T,$$

$$\varpi_{55} = P_{23} - K_b - K_b^T, \qquad \varpi_{63} = C_r D_{f_a} - \breve{D},$$

$$\Upsilon_a = \begin{bmatrix} T_K B_f - R_K D_f - L_K D_{f_a} & -R_K D_f \end{bmatrix},$$

with P_{21}, P_{22}, P_{23}, T_K, R_K, L_K, $\bar{Q}$, C_r, K, K_a and K_b to be designed.

5.2.3 H_∞ Residual Design

In view of the developments presented in Sections 5.2.1 and 5.2.2, and by recalling that $\|\mathcal{G}_{f_a \breve{r}}\|_-^2 \geq \gamma_3$ is satisfied if (61) holds with γ_c replaced by γ_3, the following Theorem proposes an LMI-based solution to the H_∞ residual generator problem for descriptor systems.

Theorem 3 *Consider the class of linear discrete-time descriptor systems in (104), the residual generator in (117), the reference model in (118) and the augmented system in (120). Let $\breve{A}$, $\breve{B}$, $\breve{C}$, $\breve{D}$, M and γ_3 be given. Suppose there exist symmetric matrices P_1, P_{21}, P_{23}, P_3 and $\bar{Q}$, free matrices L_K, K, Z_{1K}, Z_{2K}, K_a, K_b, C_r and P_{22}, and positive scalars γ_1 and γ_2 such that the LMIs in (131) and (138), and the following*

$$\begin{bmatrix} \breve{A}P_3\breve{A}^T - P_3 + \breve{B}\breve{B}^T & * \\ \breve{C}P_3\breve{A}^T + \breve{D}\breve{B}^T & \breve{C}P_3\breve{C}^T + \breve{D}\breve{D}^T - \gamma_3\bar{Q} \end{bmatrix} > 0, \tag{139}$$

$$H_e\left\{KW_1 + Z_{1K}S_{11} + Z_{2K}S_{21}\right\} > 0, \tag{140}$$

hold. Then, the residual generator defined in (117) considering C_r and

$$L = K^{-1}L_K, \qquad T = W_1 + K^{-1}\left(Z_{1k}S_{11} + Z_{2k}S_{21}\right), \tag{141}$$
$$Q = \bar{Q}^{-1/2}, \qquad R = W_2 + K^{-1}\left(Z_{1k}S_{12} + Z_{2k}S_{22}\right),$$

ensures that:

i) the unforced error dynamics is asymptotically stable;

ii) $\|\mathcal{G}_{w_a r}\|_\infty^2 \leq \gamma_1$;

iii) $\|\mathcal{G}_{f_a e_r}\|_\infty^2 \leq \gamma_2$; and

iv) $\|\mathcal{G}_{f_a \breve{r}}\|_-^2 \geq \gamma_3$.

Remark 3 *The LMI constraint in (140) has been considered to ensure that the matrix*

$$T_K = KW_1 + Z_{1K}S_{11} + Z_{2K}S_{21}$$

and consequently $T = K^{-1}T_K$ are nonsingular, since K is full rank.

Remark 4 *The same strategies developed in Section 4.2.5 can be directly applied to Theorem 3 in order to optimize the residual generator performance in the H_∞ sense.*

5.3 *PEAK*-NORM APPROACH

In this section, the residual generator synthesis for discrete-time descriptor systems is addressed considering a mixed H_-/*peak*-norm design criterion.To this end, assume that the augmented disturbance and fault input vectors of the standard system representation given in (112) satisfy the following

A6 $w_a \in \mathcal{W}_a$, with $\mathcal{W}_a := \{w_a \in \mathbb{R}^{2n_w} : w_a^T w_a \leq 1\}$;

A7 $f_a \in \mathcal{F}_a$, with $\mathcal{F}_a := \{f_a \in \mathbb{R}^{2n_f} : f_a^T f_a \leq 1\}$.

The above assumptions do not imply loss of generality, since one can appropriately re-scale the matrices B_{w_a}, D_{w_a}, B_{f_a} and D_{f_a}.

Hence, in view of the above setting, the problem of concern in this section can be stated as follows:

- *For given matrices M and C_r, determine the matrices T, R, L and Q of the residual generator defined in (117) such that the error dynamics is ISS and following performance requirements are satisfied:*

  (P-I) $\|\mathcal{G}_{w_a r}\|_{peak}^2 \leq \gamma_1$;

  (P-II) $\|\mathcal{G}_{f_a e_r}\|_{peak}^2 \leq \gamma_2$;

  (P-III) $\|\mathcal{G}_{f_a \breve{r}}\|_-^2 \geq \gamma_3$;

  where

  $$\|\mathcal{G}_{w_a r}\|_{peak} = \sup_{w_a \in \mathcal{W}_a} \|r\|_{\ell_\infty}, \quad \|\mathcal{G}_{f_a e_r}\|_{peak} = \sup_{f_a \in \mathcal{F}_a} \|e_r\|_{\ell_\infty}$$

  are the peak-*norms from w_a to r and f_a to e_r, respectively, and $\|\mathcal{G}_{f_a \breve{r}}\|_-$ is the smallest singular value of $\mathcal{G}_{f_a \breve{r}}$.*

Notice in contrast with the H_∞ approach that the matrix C_r will be constrained to be constant in order to derive numerically tractable design conditions.

5.3.1 Determining a bound on $\|\mathcal{G}_{w_a r}\|_{peak}$

Consider the following Lyapunov function candidate

$$V_1(\tilde{x}) = \tilde{x}^T P_1 \tilde{x}, \quad P_1 > 0, \quad P_1 \in \mathbb{R}^{n_x \times n_x}. \tag{142}$$

Hence, the system in (119) is ISS if the following inequality holds (JIANG; WANG, Y., 2001):

$$\Delta V_1(\tilde{x}) \leq \tau_1 \left(w_a^T w_a - V_1(\tilde{x}) \right), \quad \tau_1 \in (0,1). \tag{143}$$

Taking into account the representation of $\mathcal{G}_{w_a r}$ in (119), the above expression can be cast as follows:

$$(\tilde{x}^+)^T P_1 \tilde{x}^+ - (1-\tau_1)\tilde{x}^T P_1 \tilde{x} - \tau_1 w_a^T w_a \leq 0,$$
$$\forall\, (\tilde{x}^+, \tilde{x}, w_a) : -\tilde{x}^+ + (A - LC)\tilde{x} + (B_{w_a} - LD_{w_a})w_a = 0,$$

or, equivalently, as

$$\left((TA - LC)\tilde{x} + (B_{w_a} - LD_{w_a})w_a\right)^T P_1 \left((TA - LC)\tilde{x} + (B_{w_a} - LD_{w_a})w_a\right)$$
$$- (1 - \tau_1)\tilde{x}^T P_1 \tilde{x} - \tau_1 {w_a}^T w_a \leq 0,$$

which can be cast as follows:

$$\begin{bmatrix}\tilde{x}\\ w_a\end{bmatrix}^T\left\{\begin{bmatrix}-(1-\tau_1)P_1 & 0\\ 0 & -\tau_1 I_{n_{w_a}}\end{bmatrix}+\begin{bmatrix}(TA-LC)^T\\ (B_{w_a}-LD_{w_a})^T\end{bmatrix}P_1\begin{bmatrix}(TA-LC) & (B_{w_a}-LD_{w_a})\end{bmatrix}\right\}\begin{bmatrix}\tilde{x}\\ w_a\end{bmatrix}<0.$$

By applying the Schur complement (Lemma 1), the following condition is obtained:

$$\begin{bmatrix}-(1-\tau_1)P_1 & 0 & (TA-LC)^T\\ 0 & -\tau_1 I_{n_{w_a}} & (B_{w_a}-LD_{w_a})^T\\ (TA-LC) & (B_{w_a}-LD_{w_a}) & -P_1^{-1}\end{bmatrix}<0, \tag{144}$$

Hence, pre- and post-multiplying (85) by

$$\mathrm{diag}\{I_{n_x}, I_{n_{w_a}}, K\}$$

and its transpose yields:

$$\begin{bmatrix}-(1-\tau_1)P_1 & 0 & (T_KA-L_KC)^T\\ 0 & -\tau_1 I_{n_{w_a}} & (T_KB_{w_a}-L_KD_{w_a})^T\\ (T_KA-L_KC) & (T_KB_{w_a}-L_KD_{w_a}) & -KP_1^{-1}K^T\end{bmatrix}<0, \tag{145}$$

where $L_K = KL$ and $T_K = TK$. Considering the same developments in Section 5.2.1, the following sufficient condition for (143) to hold is obtained:

$$\begin{bmatrix}P_1-K-K^T & KA-L_KC & \Omega_1\\ * & -(1-\tau_1)P_1 & 0\\ * & * & -\tau_1 I_{2n_w}\end{bmatrix}<0, \tag{146}$$

where $K\in\mathbb{R}^{n_x\times n_x}$ is a free nonsingular matrix to be determined and

$$\Omega_1 = \begin{bmatrix}(T_K-R_KD_w-L_KD_w) & -R_KD_w\end{bmatrix}. \tag{147}$$

Now, assuming that (143) holds, a bound γ_1 on $\|\mathcal{G}_{w_ar}\|^2_{peak}$ should satisfy the following constrained inequality:

$$\gamma_1 - r^Tr \geq 0,\ \forall\ (\tilde{x}, w_a) : \tilde{x}^TP_1\tilde{x}\leq 1,\ w_a^Tw_a\leq 1, \tag{148}$$

which by applying the *S-Procedure* yields

$$\alpha_1 w_a^Tw_a + \beta_1\tilde{x}^TP_1\tilde{x} - r^Tr \geq 0, \tag{149}$$

with α_1 and β_1 being positive scalars to be designed and

$$\gamma_1 = \alpha_1 + \beta_1.$$

Next, let

$$\rho_1 = \frac{\alpha_1}{\beta_1} \quad\text{and}\quad \eta_1 = \frac{1}{\sqrt{\beta_1}}.$$

Hence, the expression in (149) can be cast by means of

$$\begin{bmatrix}\tilde{x}\\ w_a\end{bmatrix}^T\left(\begin{bmatrix}P_1 & *\\ 0 & \rho_1 I_{2n_w}\end{bmatrix}-\eta_1\begin{bmatrix}C^T C_r^T\\ D_{w_a}^T C_r^T\end{bmatrix}Q^T Q\begin{bmatrix}C_r C & C_r D_{w_a}\end{bmatrix}\eta_1\right)\begin{bmatrix}\tilde{x}\\ w_a\end{bmatrix}\geq 0,$$

which is satisfied if the following matrix inequality holds (derived from the Schur's complement):

$$\begin{bmatrix}P_1 & * & *\\ 0 & \rho_1 I_{2n_w} & *\\ \eta_1 C_r C & \eta_1 C_r D_{w_a} & \bar{Q}\end{bmatrix}>0, \tag{150}$$

where

$$\bar{Q}=(Q^T Q)^{-1} \quad \text{and} \quad \gamma_1=\frac{1+\rho_1}{\eta_1^2}.$$

5.3.2 Determining a bound on $\|\mathcal{G}_{f_a e_r}\|_{peak}$

Consider the following Lyapunov function candidate

$$V_2(\bar{x})=\bar{x}^T P_2\bar{x}, \quad P_2>0, \quad P_2\in\mathbb{R}^{n_a\times n_a}, \tag{151}$$

the system $\mathcal{G}_{f_a e_r}$ in (120), and the following dissipation inequality:

$$\Delta V_2(\bar{x})\leq\tau_2\left(f_a^T f_a - V_2(\bar{x})\right), \quad \tau_2\in(0,1). \tag{152}$$

Taking into account the representation of $\mathcal{G}_{f_a e_r}$, the latter expression can be written as follows:

$$(\bar{x}^+)^T P_2\bar{x}^+ - (1-\tau_2)\bar{x}^T P_1\bar{x} - \tau_2 f_a^T f_a \leq 0, \ \forall\ (\bar{x}^+,\bar{x},f_a)\ :\ -\bar{x}^+ + \bar{A}\bar{x} + \bar{B}_{f_a} = 0$$

or, equivalently,

$$(\bar{A}\bar{x}+\bar{B}_{f_a})^T P_2(\bar{A}\bar{x}+\bar{B}_{f_a}) - (1-\tau_2)\bar{x}^T P_2\bar{x} - \tau_2 f_a^T f_a\leq 0. \tag{153}$$

This means that:

$$\begin{bmatrix}\bar{x}\\ f_a\end{bmatrix}^T\left(\begin{bmatrix}-(1-\tau_2)P_2 & *\\ 0 & -\tau_2 I_{n_{f_a}}\end{bmatrix}+\begin{bmatrix}\bar{A}^T\\ \bar{B}_{f_a}^T\end{bmatrix}P_2\begin{bmatrix}\bar{A} & \bar{B}_{f_a}\end{bmatrix}\right)\begin{bmatrix}\bar{x}\\ f_a\end{bmatrix}\leq 0.$$

By applying the Schur complement, the above inequality is satisfied, if and only if the following matrix inequality holds:

$$\begin{bmatrix}-(1-\tau_2)P_2 & 0 & \bar{A}^T\\ 0 & -\tau_2 I_{n_{f_a}} & \bar{B}_{f_a}^T\\ \bar{A} & \bar{B}_{f_a} & -P_2^{-1}\end{bmatrix}<0. \tag{154}$$

Hence, considering the matrix partitions of P_2 and K_2 as defined in (54) and (55), respectively, and the procedure developed in Section 5.2.2, the following matrix inequality is obtained:

$$\begin{bmatrix}P_2-K_2-K_2^T & K_2\bar{A} & K_2\bar{B}_{f_a}\\ * & -(1-\tau_2)P_2 & 0\\ * & * & -\tau_2 I_{2n_{f_a}}\end{bmatrix}<0, \tag{155}$$

where $K_2 \in \mathbb{R}^{n_a \times n_a}$.

In order to obtain a numerical and tractable condition, the multiplier K_2 will be constrained to be as follows:

$$K_2 = \begin{bmatrix} K & K_a \\ MK & K_b \end{bmatrix} \tag{156}$$

with $M \in \mathbb{R}^{n_q \times n_x}$ being a given matrix and $K \in \mathbb{R}^{n_x \times n_x}$, $K_a \in \mathbb{R}^{n_x \times n_q}$ and $K_b \in \mathbb{R}^{n_q \times n_q}$ to be determined such that K_2 is nonsingular.

Then, considering the partition of P_2 as in (137), the definition of the matrices $\bar{A}$, and $\bar{B}$ in (122), the definition of K_2 in (156), the following sufficient condition for (152) is obtained:

$$\begin{bmatrix} P_{21} - K - K^T & \Omega_{12} & \Omega_{13} & K_a\breve{A} & \Omega_{15} \\ * & P_{23} - K_b - K_b^T & \Omega_{23} & K_b\breve{A} & \Omega_{25} \\ * & * & -(1-\tau_2)P_{21} & -(1-\tau_2)P_{22}^T & 0 \\ * & * & * & -(1-\tau_2)P_{23} & 0 \\ * & * & * & * & -\tau_2 I_{2n_f} \end{bmatrix} < 0 \tag{157}$$

where

$$\Omega_{12} = P_{22}^T - K_a - K^T M^T, \quad \Omega_{13} = T_K A - L_K C, \quad \Omega_{23} = M\Big(T_K A - L_K C\Big), \tag{158}$$
$$\Omega_{15} = \Big[(T_K B_f - R_K D_f - L_K D_f) \quad -R_K D_f\Big] + K_a\breve{B},$$
$$\Omega_{25} = M\Big[(T_K B_f - R_K D_f - L_K D_f) \quad -R_K D_f\Big] + K_b\breve{B}.$$

Now, assume that the dissipation inequality in (152) holds. Then, a bound γ_2 on $\|\mathcal{G}_{f_a e_r}\|_{peak}^2$ satisfies the following:

$$\gamma_2 - e_r^T e_r \geq 0, \ \forall\ (\bar{x}, f_a) : \bar{x}^T P_2 \bar{x} \leq 1, \ f_a^T f_a \leq 1,$$

which by applying the *S-Procedure* yields

$$\alpha_2 f_a^T f_a + \beta_2 \bar{x}^T P_2 \bar{x} - e_r^T e_r \geq 0, \tag{159}$$

with α_2 and β_2 being positive scalars to be designed and

$$\gamma_2 = \alpha_2 + \beta_2.$$

Next, let

$$\rho_2 = \frac{\alpha_2}{\beta_2} \quad \text{and} \quad \eta_2 = \frac{1}{\sqrt{\beta_2}}.$$

In light of the above definition, the inequality in (159) can be written in the following form

$$\begin{bmatrix} \bar{x} \\ f_a \end{bmatrix}^T \left(\begin{bmatrix} P_2 & * \\ 0 & \rho_2 I_{2n_f} \end{bmatrix} - \eta_2 \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} Q^T Q \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \eta_2 \right) \begin{bmatrix} \bar{x} \\ f_a \end{bmatrix} \geq 0,$$

which is satisfied if the following matrix inequality holds:

$$\begin{bmatrix} P_{21} & * & * & * \\ P_{22} & P_{23} & * & * \\ 0 & 0 & \rho_2 I_{2n_f} & * \\ \eta_2 C_r C & -\eta_2 \breve{C} & \eta_2(C_r D_{f_a} - \breve{D}) & \bar{Q} \end{bmatrix} > 0, \tag{160}$$

where $\bar{Q} = (Q^T Q)^{-1}$.

5.3.3 *Peak*-norm Residual Design

Notice that $\|\mathcal{G}_{f_a\breve{r}}\|_-^2 \geq \gamma_3$ is satisfied if (61) holds with γ_c replaced by γ_3. Then, by considering the latter developments, the following result proposes an LMI-based solution to the residual generator problem for descriptor systems in terms of the *peak*-norm approach.

Theorem 4 *Consider the class of linear discrete-time descriptor systems in* (104)*, the residual generator in* (117)*, the reference model in* (118) *and the augmented system in* (120)*. Let* $\breve{A}$, $\breve{B}$, $\breve{C}$, $\breve{D}$, C_r, M, γ_3 *and* $\tau_i \in (0,1)$, $i = 1,2$, *be given. Suppose there exist symmetric matrices* P_1, P_{21}, P_{23}, P_3 *and* $\bar{Q}$, *free matrices* L_K, K, Z_{1K}, Z_{2K}, K_a, K_b, *and* P_{22}, *and positive scalars* ρ_i, η_i, *and* γ_i, $i = 1,2$, *such that the LMIs in* (139), (140), (146), (150), (157) *and* (160) *hold. Then, the residual generator defined in* (117) *with*

$$L = K^{-1} L_K, \qquad T = W_1 + K^{-1}\left(Z_{1k} S_{11} + Z_{2k} S_{21}\right),$$

$$Q = \bar{Q}^{-1/2}, \qquad R = W_2 + K^{-1}\left(Z_{1k} S_{12} + Z_{2k} S_{22}\right),$$

ensures that:

i) the error dynamics is ISS;

ii) $\|\mathcal{G}_{w_a r}\|_{peak}^2 \leq \gamma_1$*;*

iii) $\|\mathcal{G}_{f_a e_r}\|_{peak}^2 \leq \gamma_2$*; and*

iv) $\|\mathcal{G}_{f_a\breve{r}}\|_-^2 \geq \gamma_3$*;*

where

$$\gamma_1 = \frac{1+\rho_1}{\eta_1^2} \quad \textit{and} \quad \gamma_2 = \frac{1+\rho_2}{\eta_2^2}.$$

Similarly to the procedure proposed in Section 4.3.4, Theorem 4 can be applied to obtain an optimized solution. For instance, upper-bounds on $\|\mathcal{G}_{wr}\|_{peak}$ and $\|\mathcal{G}_{fe_r}\|_{peak}$ can be minimized for a given bound γ_c on $\|\mathcal{G}_{f\breve{r}}\|_-$ by means of the following optimization problem:

$$\min_{P_1,\ldots,Q,\rho_1,\rho_2,\eta_1,\eta_2} \gamma(\kappa) \ \text{ subject to } \begin{cases} (139), (140), (146), (150), (157), (160), \\ \text{with } \ \gamma(\kappa) = (1-\kappa)(\rho_1 - \eta_1) + \kappa(\rho_2 - \eta_2), \end{cases} \tag{161}$$

where $\kappa \in (0,1)$ is a parameter to be tuned by the designer.

5.4 NUMERICAL EXAMPLE

In order to illustrate the proposed residual generator design for linear discrete-time descriptor systems, a numerical example representing two cascade subsystems with an equality constraint is presented in the following considering the H_∞-norm approach. To this end, consider the following matrices defining the descriptor system described in (104):

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0.99 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.99 & -10^{-10} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 10^{-2} & 0 & 0 & -1 \end{bmatrix}, \ B_u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ B_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$B_f = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 10^{-2} & 0 & 10^{-2} & 0 \\ 0 & 0 & 0 & 10^{-2} & 1 \end{bmatrix}, \ D_u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ D_f = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix}, \ D_w = 10^{-3} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and let

$$Y(z) = G_{uwf}(z) \begin{bmatrix} U(z) & W(z) & F(z) \end{bmatrix}^T, \tag{162}$$

where $G_{uwf}(z)$ is a transfer function matrix, and $Y(z)$, $U(z)$, $W(z)$ and $F(z)$ are the $\mathcal{Z}$ transform from the output, the control input, the disturbance and the fault, respectively. Therefore, the static gain can be defined as $K_{uwf} = G_{uwf}(z = 1)$, which is given by:

$$K_{uwf} = \begin{bmatrix} 4505 & 0.001 & -4505 & 0 \\ 9016 & 0.001 & -9016 & 1 \end{bmatrix}. \tag{163}$$

The reference model matrices in Eq. (118) are chosen in order to guarantee an appropriate structure to fault detection and isolation with regards to the dynamics from faults to measurements (i.e., either a diagonal or triangular matrix transfer function) and considering a stable dynamic behavior as close as possible to the dynamics of the original system, which for this particular example yields:

$$\breve{A} = \begin{bmatrix} 0.99 & -10^{-5} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0.99 & -10^{-4} \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ \breve{B} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\breve{C} = \begin{bmatrix} 0 & 0.01 & 0 & 0.01 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \ \breve{D} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

Now, let

$$\breve{R}_q(z) = G_{f_a \breve{r}_q}(z) F_a(z), \quad G_{f_a \breve{r}_q}(z) = \breve{C}(zI_{n_q} - \breve{A})^{-1}\breve{B} + \breve{D},$$

with $\breve{r}_q = Q^{-1}\breve{r}$ representing the desired residual without the gain Q, and $\breve{R}_q(z)$ and $F_a(z)$ denoting the $\mathcal{Z}$ transform of $\breve{r}_q$ and f_a, respectively. Thus, the reference model static gain can be defined as:

$$K_{ref} = G_{f_a\breve{r}_q}(1) = \begin{bmatrix} -1102.0004 & 0 & 0 & 0 \\ -101 & 1 & 0 & 0 \end{bmatrix}. \tag{164}$$

Hence, considering K_{ref} in Eq. (164) for FDI evaluation purposes, a triangular structure with respect to the behavior from faults to residual can be enforced such that:

- if an actuator fault (f_a) occurs, both residuals r_1 and r_2 will be influenced by it;
- if sensor 1 (f_{s_1}) is affected by a fault, only residual r_1 will respond to this fault;
- in the same way, in the occurrence of sensor fault f_{s_2}, only residual r_2 is affected by it.

Therefore, considering the parameters

$$\gamma_3 = 2 \cdot 10^{-7}, \quad \bar{\gamma}_1 = 0.5, \quad M = -I_{n_q \times n_x}, \tag{165}$$

the following optimization problem is applied

$$\min_{P_1,\ldots,\bar{Q},\gamma_1,\gamma_2} \gamma_2 \quad \text{subject to} \quad \begin{cases} \bar{\gamma}_1 - \gamma_1 \geq 0, \\ (131),\ (138)\text{-}(140) \end{cases}$$

leading to

$$L = \begin{bmatrix} -0.0008 & 0.0005 \\ 0.3050 & -0.0592 \\ 0.0025 & 0.0004 \\ -0.0970 & -0.0413 \\ 0.4737 & -0.5694 \end{bmatrix}, \quad Q = \begin{bmatrix} 2.2568 & 0 \\ 0.4579 & 90.3323 \end{bmatrix}, \tag{166}$$

$$C_r = \begin{bmatrix} 1.0099 & -0.0234 \\ 0.0625 & 0.6312 \end{bmatrix}, \quad \gamma_1 = 0.4650, \quad \gamma_2 = 0.7817.$$

The static gain from faults to residuals is given by:

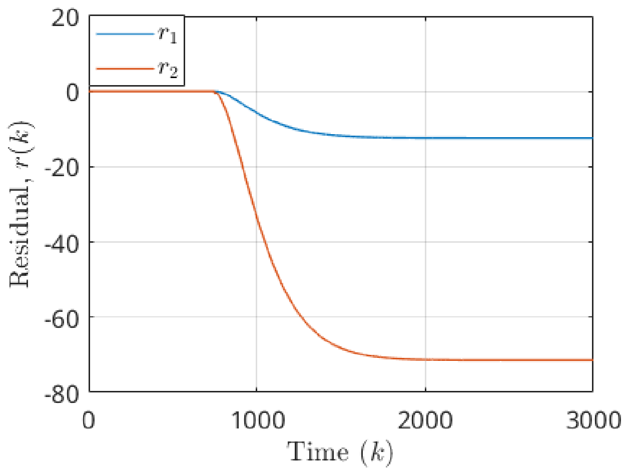
$$K_{fr} = \begin{bmatrix} 0.0134 & -0.0125 & 0.0541 & 0.1375 \\ 0.1184 & -1.6250 & 0.0469 & -1.8 \end{bmatrix}, \tag{167}$$

which demonstrates that fault detection and isolation can be guaranteed depending on the defined threshold. In this case, the chosen threshold is equal to 0.1. This means that the proposed optimization problem can ensure that the design conditions (i), (ii) and (iii) are satisfied.

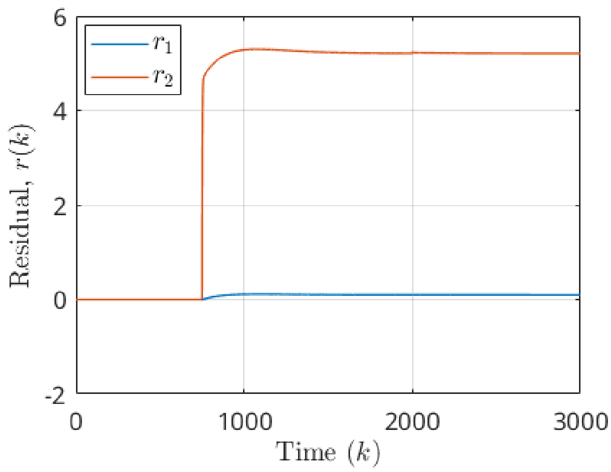
In order to analyze the behavior of the system under influence of faults and disturbances, a simulation has been developed. In this example, a control variation ($\Delta u_j(k)$, for $j = 1, 2$) is considered, with upper and lower values. Besides, the disturbance ($w(k)$) and the fault signals (actuator fault $f_1(k)$ and sensor fault $f_2(k)$) are defined:

- Δu_1 = -0.5, Δu_2 = 0.5;
- $w(k)$ = 1;
- $f_1(k)$ = 0.01, $f_2(k)$ = 1.

The control input is applied to the system during the entire time slot, whereas the disturbance signal occurs from time k = 665 s. Moreover, for FDI evaluation purposes, each fault signal is applied to the system one at a time. To evaluate the performance of the designed observer concerning the FDI problem, Figure 5a is referred to the case where the actuator fault happens at time k = 249, with $f_2(k)$ = 0. On the other hand, Figure 5b is related to occurrence of the sensor fault, which happen at k = 249. In this case, $f_1(k)$ = 0.



(a) Residual response to the actuator fault.



(b) Residual response to the sensor fault.

Figure 5 – Residual response to fault, control and disturbance inputs.
Font: Own authorship.

From these results, the satisfactory performance achieved by the proposed filter despite the presence of exogenous disturbances is clear. Taking into account a triangular structure with respect to the fault-residual static gain, in the occasion of an actuator fault (Figure 5a), both residuals (r_1 and r_2) are influenced by it, while the occurrence of a sensor fault (Figure 5b) only affects one residual (r_2). Therefore, fault detection and isolation are guaranteed.

5.5 CHAPTER CONCLUSION

In this chapter, two LMI-based approaches were proposed for designing residual generators for a class of linear discrete-time descriptor systems considering either a mixed H_-/H_∞ or $H_-/peak$-norm observer-like filter, which approximates the solution given by a reference model with the view to achieve fault detection and isolation. In order to obtain optimized solutions, the filter parameters are designed via *quasi*-convex optimization problems in terms of LMI constraints. A numerical example representing two cascade subsystems with an equality constraint (presented in Section 5.4) demonstrated the potentials of the proposed approach for the fault detection and isolation of descriptor systems. In particular, the derived residual generator (considering a mixed H_-/H_∞ design criterion) achieved a excellent FDI features. Future research will concentrate on designing residual generators for a class of nonlinear discrete-time descriptor systems.

6 APPLICATION TO LI-ION BATTERY PACKS

This chapter presents the application of a reference model-based FDI technique for a linear approximated model of Li-Ion battery packs. To this end, firstly, a nonlinear Li-ion battery cell model is introduced in Section 6.1, followed by the model of two batteries connected in series in Section 6.2. Next, Section 6.3 shows the linear approximation of the battery pack model around a prescribed operating condition. Due to reasons that will be explained later in this chapter, the linear approximate model of a battery pack is represented in terms of a class of linear discrete-time descriptor systems. Hence, Chapter 5 will be used as a basis for the residual generator design applied to Li-ion battery packs based on the H_∞-norm setting. The effectiveness of the proposed design for this application is illustrated by numerical examples. It is important to emphasize that the studied nonlinear descriptor model in this chapter is based on the unpublished work of (COUTO et al., 2021).

6.1 LI-ION BATTERY MODEL

A battery cell can be divided in three regions: two porous electrodes and a separator. The electrode regions are composed by two phases: a solid and an electrolyte. On the other hand, the separator region only has electrolyte phase. Since lithium is the element responsible for storing energy in a battery, when a battery is empty, all its lithium content is in the solid phase of the positive electrode. Moreover, when the battery is being charged, the lithium diffuses from the bulk of the positive electrode to the surface where it undergoes an oxidation reaction. The resulting lithium ions then travel within the electrolyte towards the negative electrode. Therefore, lithium ions go from the positive electrode up to the negative electrode by crossing the separator in the way. Once the negative electrode contains all the available lithium, the charging process is finished, while the discharging process is done when all the available lithium is on the positive electrode side (COUTO et al., 2021; COUTO, 2018).

The operation of a lithium-ion battery cell can be properly described by reduced-order electrochemical models, such as the so-called equivalent-hydraulic model (EHM). For more details on the EHM representation, please refer to (MILOCCO et al., 2014; COUTO et al., 2021). Figure 6 presents a schematic representation of a battery cell across the cell thickness, the solid-phase diffusion process as an EHM, and also the heat generated by the charging and discharging of the Li-Ion battery. A lithium-ion battery cell operation can be represented by a two tanks system. In this case, the tank system is illustrated through a single-particle analogy with the internal tank as the bulk concentration and the external one as the critical surface concentration (CSC). Additionally, the combination of these two gives rise to the state-of-charge (SOC) of the battery.

The valve coefficient g regulates the transfer of material entering into the particle u from the surface to the bulk. Although two EHM models can be used to describe the battery diffusion processes exemplified in Figure 6 (one for each electrode), a single EHM associated to

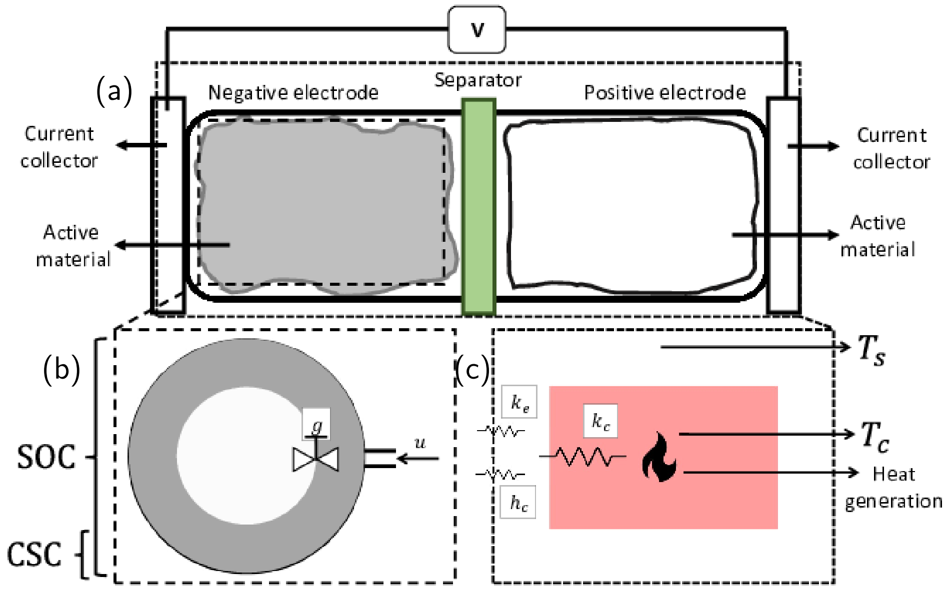


Figure 6 – Schematic representation of a battery cell and its model. (a) Longitudinal scheme of a cell, (b) electrode solid-phase diffusion process as an EHM, and (c) thermal model representation.
Font: (COUTO et al., 2021).

the diffusion-limiting electrode is often enough to describe the operation of a battery under mild conditions (i.e. current magnitudes below 1C[1]). Unlike other models, such as reduced-order EChMs (electrochemical models) and ECMs (equivalent circuit models), the EHM provides a simple model (only two states) while keeping its physical relevance. The two most important diffusion states are the concentration of lithium in the negative electrode, denoted as SOC, and in its surface, denoted as CSC.

Apart from lithium diffusion, thermal phenomena inside batteries are also fundamental in order to prevent safety hazards such as thermal runaway (BANDHAUER et al., 2011). It is important to emphasize that only the surface temperature (T_S) of the battery can be measured, although the heat builds up inside the battery. Therefore, in order to carefully monitor the thermal state of the battery, a model-based estimation is performed. In particular, a second-order thermal model (LIN et al., 2014) is used, where the temperature states include core (T_C) and surface (T_S) temperatures.

The nonlinear dynamic model that results from extending the EHM with thermal dynamics is denoted as TEHM, and it takes the following form in discrete-time (COUTO, 2018; COUTO; KINNAERT, 2018):

$$x(k+1) = f(u(k), x(k), v(k)) + b(T_\infty), \tag{168}$$

where T_∞ is the ambient temperature, the input $u(k) \in \mathbb{R}$ is the applied current, and $v(k) \in \mathbb{R}$ is the terminal voltage. The state vector consists of two components:

- a solid-phase diffusion component $x_S(k) = [\mathrm{SOC}(k),\ \mathrm{CSC}(k)]^T$;

[1] C-rate: normalization of the battery current in [A] with respect to the battery nominal capacity in [Ah].

- and a thermal component $x_T(k) = [T_c(k),\ T_s(k)]^T$ with core and surface temperatures, respectively.

This means that $x(k) = [x_s(k),\ x_T(k)]^T \in \mathbb{R}^4$. The nonlinear function f can also be split as $f = [f_s,\ f_T]^T$ in accordance with the state vector, where $f_s : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^2$ takes the form

$$f_s(u(k), x(k)) = \begin{bmatrix} \mathrm{SOC}(k) + \tau_s \gamma u(k) \\ \tau_s \frac{g(T_c(k))}{\beta(1-\beta)} \mathrm{SOC}(k) + \left(1 - \tau_s \frac{g(T_c(k))}{\beta(1-\beta)}\right) \mathrm{CSC}(k) + \tau_s \frac{\gamma}{1-\beta} u(k) \end{bmatrix}, \quad (169)$$

where τ_s is the sampling time, g is the valve coefficient, β represents the tank cross-section area for a given number of tanks, and γ is a given constant. Moreover, $f_T : \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^2$ is given by (LIN et al., 2014)

$$f_T(u(k), x(k), v(k)) = \begin{bmatrix} \left(1 - \tau_s \frac{k_c}{C_{pc}}\right) T_c(k) + \tau_s \frac{k_c}{C_{pc}} T_s(k) + \frac{\tau_s}{C_{pc}} \left(U_b^+(\mathrm{SOC}(k)) - U_b^-(\mathrm{SOC}(k)) - v(k)\right) u(k) \\ \tau_s \frac{k_c}{C_{ps}} T_c(k) + \left(1 - \tau_s \frac{k_c + h_c}{C_{ps}}\right) T_s(k), \end{bmatrix}, \quad (170)$$

with k_c being the thermal conductivity, h_c is the heat transfer coefficient, C_{pc} and C_{ps} are the core and surface specific heat capacity, and $b(T_\infty) = \tau_s \frac{h_c}{C_{ps}} T_\infty$ (COUTO et al., 2016). It is important to point out that the discretization procedure was determined by using the Euler approach. In Figure 6, the thermal process is represented, emphasizing the internal and surface temperatures, T_c and T_s, respectively, and how heat fluxes are governed by conductivities k_c or k_e and convection coefficient h_c.

For more details on the definition of the the different diffusion, thermal and electrochemical model variables, please refer to (COUTO et al., 2021). The function g is temperature-dependent, and it follows Arrhenius law:

$$g(T_c(k)) = g_{\mathrm{ref}} \exp\left(\frac{E_\Phi}{R_g}\left(\frac{1}{T_{\mathrm{ref}}} - \frac{1}{T_c(k)}\right)\right). \quad (171)$$

The model functions $U_b^\pm : \mathbb{R} \to \mathbb{R}$ are the equilibrium potentials $U^\pm$ evaluated at the bulk SOC for the positive and negative electrodes. For more details on the EHM and the thermal model, please refer to (COUTO et al., 2019; COUTO; KINNAERT, 2018).

The diffusion and thermal TEHM in Eq. (168) is complemented with an electrochemical output equation given by

$$v(k) = h(u(k), x(k)), \quad (172)$$

where the output $v \in \mathbb{R}$ is the voltage response of the battery and the nonlinear function $h : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ is represented by

$$h(u(k), x(k)) = \eta_s^+(u(k), x(k)) - \eta_s^-(u(k), x(k)) + U_s^+(\mathrm{SOC}(k)) - U_s^-(\mathrm{CSC}(k)) - R_f u(k). \quad (173)$$

The model functions $\eta_s^{\pm} : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ are the surface over potentials given by

$$\eta_s^{\pm}(u(k), x(k)) = \mp \frac{2R_g}{F} T_c(k) \sinh^{-1}\left(\frac{1}{i_0^{\pm}(x(k))} u(k) \right). \tag{174}$$

The exchange current density $i_0^{\pm} : \mathbb{R}^4 \to \mathbb{R}$ is given by

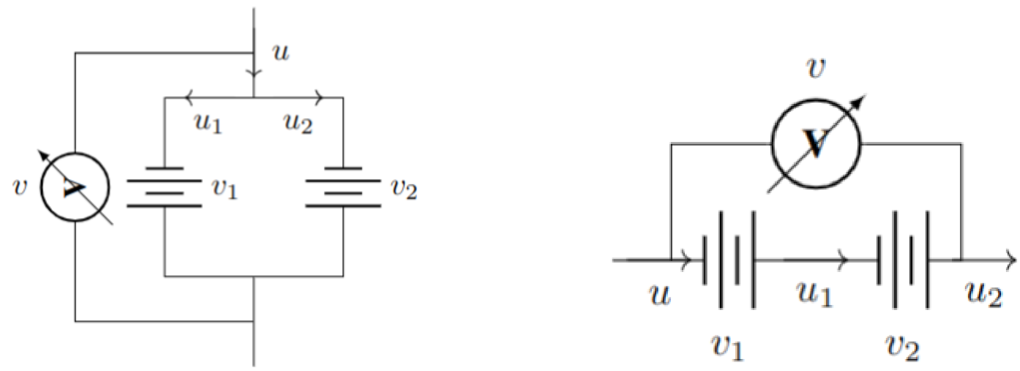
$$i_0^{\pm}(x(k)) = 2L^{\pm} a_s^{\pm} k_n^{\pm}(T_c(k)) c_{s,\max}^{\pm} \sqrt{c_e^0 z^{\pm}(k) \left(\sqrt{1 - z^{\pm}(k)} \right)}, \tag{175}$$

where $z^+(k) = \rho SOC(k) + \sigma$ for the positive electrode and $z^-(k) = CSC(k)$ for the negative electrode. The model functions $U_s^{\pm}$ are the equilibrium potentials evaluated at the surface CSC for the positive and negative electrodes. The model parameter R_f is a film resistance. Similarly as with the diffusion-related parameter g, the kinetic-related parameters $k_n^{\pm}$ depend on the temperature according to Eq. (171), with $k_n^{\pm}(T_c(k))$.

Remark 5 *The TEHM (Eq.(168),(172)) presented in this section is associated to a single battery cell. When considering two-cell arrangements, a TEHM is needed for each battery cell, and each cell TEHM is denoted with a subscript $i \in \{1, 2\}$ in parenthesis. For instance, the input current and the equilibrium potentials for the i-th battery cell will be denoted as $u_{(i)}$ and $U_{s,(i)}^{\pm}$, respectively. A two-cell arrangement modeling is presented in the next section.*

6.2 LI-ION BATTERY SERIES PACK MODEL

In order to deliver the energy/power required by a given application, lithium-ion battery cells have to be connected in series/parallel arrangements (COUTO et al., 2021). Therefore, the single cell TEHM designed in Section 6.1 has to be adapted to include cell interconnections as well as thermal and electrical couplings. In this section, two battery cells are considered and it is assumed that these cells are deployed side by side. Figure 7 illustrates both the parallel and series electrical topologies, respectively.



(a) Parallel configuration (b) Series configuration

Figure 7 – Possible parallel/series topologies for two electrically connected batteries. Font: (COUTO et al., 2021).

The first characteristic to be taken into account is that the cells are thermally coupled since they exchange heat due to their physical proximity. Hence, from energy balance, the state space representation in (168) for each i-th battery cell has to be suitably modified to accommodate heat exchange. This is achieved by substituting the thermal dynamics in (168), namely the last two equations, by the following model:

$$x_{T,i}(k+1) = [f_{T_c,i}(u_i(k), x_i(k), v_i(k)), f_{T_s,i}(x_i(k), x_j(k))]^T + b(T_\infty), \tag{176}$$

where $x_{T,i}(k) = [T_{c,i}(k),\ T_{s,i}(k)]^T \in \mathbb{R}^2$, the subscript j represents the j-th neighboring cell, the function $f_{T_c,i} : \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ corresponds to the first component in (170) and the function $f_{T_s,i} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ is given by

$$f_{T_s,i}(x_i(k), x_j(k)) = \tau_s \frac{k_{c,i}}{C_{ps,i}} T_{c,i}(k) + \left(1 - \tau_s \frac{k_{a,i}}{C_{ps,i}}\right) T_{s,i}(k) + \tau_s \frac{k_{e,j}}{C_{ps,j}} T_{s,j}(k), \tag{177}$$

where $k_{a,i} = k_{c,i} + k_{e,i} + h_{c,i}$.

Furthermore, the electrical interconnection of the cells depends on the network topology. In this work, the application will be focused on the series configuration. Therefore, the Kirchhoff's laws of current and voltage for the series configuration, illustrated in Figure 7(b), is represented as follows:

$$u(k) = u_1(k) = u_2(k), \tag{178}$$

$$v(k) = v_1(k) + v_2(k). \tag{179}$$

In practice, battery cells are not identical due to their different operating conditions resulting from e.g. differences in impedance or thermal environments. Hence, a variable Δ is introduced to represent battery mismatches. In the series configuration case, the voltage $v(k)$ can be written in terms of the voltage of cell 1, i.e., $v_1(k)$, as follows

$$v(k) = (2 + \Delta) v_1(k). \tag{180}$$

6.3 LINEAR APPROXIMATE MODEL

The resulting electrochemical model for two battery cells in a series topology is obtained by:

i) grouping a TEHM (Eq. (168),(172)) for each i-th, $i \in \{1, 2\}$, with a thermal model of the form of Eq. (176) for each cell;

ii) electrically interconnecting the cells through the algebraic equations (178),(180).

This model naturally renders itself into a descriptor-type of system. Hence, in order to design a linear state observer, one can approximate the resulting nonlinear descriptor system by a linear approximate representation around a nominal operating condition by using the Taylor series development limited to a first order. Here, we consider the operating condition of 0.45 for the SOC and -1 A for the input current. As a result, the approximate model takes the form given by

$$\begin{cases} E\xi(k+1) = A\xi(k) + B_q q(k) + B_w w(k), \\ \quad\quad y(k) = C\xi(k) + D_q q(k) + D_w w(k). \end{cases} \tag{181}$$

Furthermore, with the intention of designing an H_∞ fault detection and isolation approach for a class of discrete-time descriptor systems, the model in Eq. (181) can be rewritten as follows:

$$\begin{cases} E\xi(k+1) = A\xi(k) + B_q q(k) + B_w w(k) + B_f f(k), \\ \quad\quad y(k) = C\xi(k) + D_q q(k) + D_w w(k) + D_f f(k). \end{cases} \tag{182}$$

In this example, the SOC and the CSC of each battery of the battery pack composes the state $x(k)$, which means:

$$x_1(k) = [\mathrm{SOC}_1(k), \quad \mathrm{CSC}_1(k)]^T, \tag{183}$$

$$x_2(k) = [\mathrm{SOC}_2(k), \quad \mathrm{CSC}_2(k)]^T. \tag{184}$$

Therefore, the state vector $\xi(k) \in \mathbb{R}^{n_\xi}$, $n_\xi = 6$, is defined as

$$\xi(k) = \left[x_1(k)^T, \quad x_2(k)^T, \quad u_1(k), \quad v_1(k)\right]^T, \tag{185}$$

the measurable electrical variables $u(k)$ and $v(k)$ are grouped in $q(k) \in \mathbb{R}^2$ as

$$q(k) = \left[u(k), \quad v(k)\right]^T, \tag{186}$$

and the mismatch variables $\Delta u_1(k)$ and $\Delta v_1(k)$ representing disturbed current and voltage are included in $w(k) \in \mathbb{R}^{n_w}$, $n_w = 2$, as

$$w(k) = \left[\Delta u_1(k), \quad \Delta v_1(k)\right]^T. \tag{187}$$

The output signal $y(k)$ is defined as follows:

$$y(k) = \left[v(k), \quad v(k), \quad u(k), \quad v(k)\right]^T \tag{188}$$

The matrix E is given by

$$E = \mathrm{d}iag(I_4, 0_2) \in \mathbb{R}^{6\times 6}, \tag{189}$$

and the matrices $A \in \mathbb{R}^{6\times 6}$, $B_q \in \mathbb{R}^{6\times 2}$, $B_w \in \mathbb{R}^{6\times 2}$, $B_f \in \mathbb{R}^{6\times 2}$, $C \in \mathbb{R}^{4\times 6}$, $D_q \in \mathbb{R}^{4\times 2}$, $D_w \in \mathbb{R}^{4\times 2}$ and $D_f \in \mathbb{R}^{4\times 2}$ are the corresponding Jacobian matrices of f_ξ and h_ξ, yielding the following representation:

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & -8.81\cdot 10^{-6} & 0 \\ 12.24\cdot 10^{-3} & 0.987 & 0 & 0 & -29.24\cdot 10^{-6} & 0 \\ 0 & 0 & 1.0 & 0 & -6.46\cdot 10^{-6} & 0 \\ 0 & 0 & 12.24\cdot 10^{-3} & 0.987 & -21.44\cdot 10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};\ B_q=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.5 & 0 \\ 0 & -0.5 \end{bmatrix};$$

$$B_w=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix};\ B_f=10^{-6}\cdot\begin{bmatrix} 8.81 & 0 \\ 29.24 & 0 \\ 6.46 & 0 \\ 21.44 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};\ D_q=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};\ D_w=\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix};\ D_f=\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix};$$

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0.645 & 0.1246 & 3.0051 & 0.1248 & -0.0085 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3.0051 & 0.1248 & -0.0036 & 1 \end{bmatrix}. \tag{190}$$

In this example, we considered an actuator fault on the SOC of one of the batteries as well as a sensor fault in the output voltage $v(k)$.

For simplicity, in this chapter, the ambient temperature is assumed to be measured and thus its effect on the residual generator dynamics can be neglected. For more details, please refer to (COUTO et al., 2021).

6.4 H$_\infty$ FDI RESULTS

In order to achieve fault detection and isolation (FDI) for the battery pack problem, a robust reference model design based on the H_∞–norm approach (Section 5.2) is developed. Firstly, we consider the linear approximate model of two batteries in series assuming that the input current and total voltage sensor are subject to faults. Then, the matrices $\breve{A}$, $\breve{B}$, $\breve{C}$ and $\breve{D}$ of the reference model defined in (118) are chosen in order to have a similar dynamics to the system dynamics and to guarantee an appropriate structure for the fault detection and isolation from the augmented fault vector f_a to the reference model residual $\breve{r}$ yielding:

$$\breve{A}=\begin{bmatrix} 0.99 & -0.001 & 0 & 0 \\ 0.99 & 0.99 & 0 & 0 \\ 0 & 0 & 0.09 & 0.001 \\ 0 & 0 & 0.0001 & 0.01 \end{bmatrix},\ \breve{B}=\begin{bmatrix} 0.99 & 0 & 0.99 & 0 \\ 0 & -0.99 & 0 & -0.99 \\ -0.99 & 0 & -0.99 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\breve{C}=10^{-3}\cdot\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0.1 \end{bmatrix},\qquad \breve{D}=\begin{bmatrix} 2 & 0 & 30 & 0 \\ 0 & 2 & 0 & 60 \end{bmatrix}, \tag{191}$$

which results in the following reference model static gain

$$K_{ref} = \begin{bmatrix} 2.8992 & -0.0091 & 30.8992 & -0.0091 \\ 0 & 2 & 0 & 60 \end{bmatrix}. \tag{192}$$

Hence, for FDI evaluation purposes, an approximate diagonal structure with respect to the behavior from the fault vector $f = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$ to the residual vector $r = \begin{bmatrix} r_1 & r_2 \end{bmatrix}^T$ can be enforced leading to the following analysis:

- if the actuator fault f_1 occurs, only residual r_1 will be influenced by it; and
- if the total voltage sensor is affected by a failure f_2, only residual r_2 will respond to this fault.

Next, in order to design the residual generator by means of Theorem 3, we set

$$\gamma_3 = 0.2, \qquad M = 0_{4\times 6}, \tag{193}$$

and apply the following optimization problem:

$$\min_{P_1,\ldots,\bar{Q},\gamma_1,\gamma_2} \quad \gamma_1 + \gamma_2 \quad \text{subject to} \;\; (131),\ (138)\text{-}(140) \text{ and } Z_x \geq 0,$$

where

$$Z_x = \begin{bmatrix} 8 \cdot 10^2 \cdot I_{n_x} & {Z_{1K}}^T \\ Z_{1K} & I_{n_x} \end{bmatrix},$$

leading to

$$L = \begin{bmatrix} -1.2643 & -6.0045 & -0.045 & 1.851 \\ 24.5915 & 124.022 & 0.8648 & -47.1217 \\ -0.2398 & -0.0498 & 0.0005 & 0.2368 \\ -0.003 & 0.0054 & 0.0001 & 0.0028 \\ -1.3458 \cdot 10^{-9} & 14.73 \cdot 10^{-9} & -7.23 \cdot 10^{-9} & 3.4235 \cdot 10^{-9} \\ -0.0625 & 0.3227 & -0.4276 \cdot 10^{-3} & -0.888 \end{bmatrix}, \tag{194}$$

$$C_r = \begin{bmatrix} -0.0412 & -0.0802 & -0.002 & -1.6672 \\ -3.7662 & 0.0034 & -0.0052 & 0.1836 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.1884 & 0 \\ -2.258 \cdot 10^4 & 2.258 \cdot 10^4 \end{bmatrix},$$

$$\gamma_1 = 0.0913, \quad \gamma_2 = 1.5674.$$

The constraint $Z_x \geq 0$ has been added to the optimization to improve the conditioning of matrices T and R defined in (141), since it implies $\|Z_{1K}\|^2 \leq 8 \times 10^2$.

The resulting static gain from the fault vector f_a to the residual r is given by:

$$K_{fr} = \begin{bmatrix} -2.8992 & -1.4713 & -30.8992 & 3.3358 \\ 0.0002 & -5.7492 & 0 & -49.8862 \end{bmatrix}, \tag{195}$$

which demonstrates (despite the large value of γ_2) that fault detection and isolation can be achieved by considering of the following analysis:

(a) if the actuator fault f_1 occurs, only residual r_1 will be influenced by it; and

(b) if the total voltage sensor is affected by a failure f_2, both residuals will be affected.

In other words, the proposed optimization problem (Theorem 3) is able to guarantee that the design conditions (i), (ii) and (iii) are fulfilled. To evaluate the behavior of the residual generator when the system is under influence of faults, a simulation has been carried out. In this example, the input signal variation $q(k)$ (i.e., $u(k)$ and $v(k)$), the disturbance ($\Delta u_1(k)$ and $\Delta v_1(k)$) and the fault signals (actuator fault $f_1(k)$ and sensor fault $f_2(k)$) are defined as follows:

- $\Delta u_1 = \Delta v_1 = 0.01$ A;
- $u(k) = -200$ A;
- $v(k) = 50$ V;
- $f_1(k) = -500$ A;
- $f_2(k) = 0.02$ V;

In the following results, the input as well as the disturbance signal are applied to the system during the entire time slot. Moreover, for FDI evaluation purposes, it is considered that each fault signal occurs one at a time. Figure 8a refers to the case where the actuator fault happens at instant $k = 125$, with $f_2(k) = 0$. On the other hand, Figure 8b is related to occurrence of the sensor fault at instant $k = 125$ with $f_1(k) = 0$.

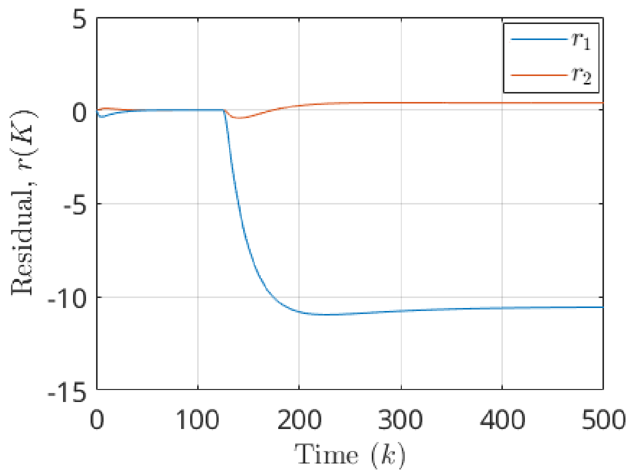
The results shown in Figures 5a and 5b clearly demonstrated the satisfactory performance achieved by the proposed residual generator despite the presence of step-like exogenous disturbances thanks to the triangular structure from the fault f to the residual r static gain. Notice that in the occasion of an actuator fault (Figure 5a), only residual r_1 is influenced by it, while the occurrence of a sensor fault (Figure 5b) affects both residuals (r_1 and r_2).

6.5 NONLINEAR MODEL SIMULATION

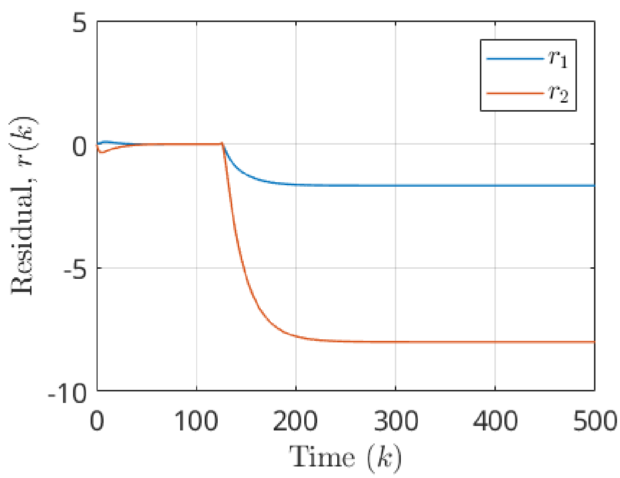
In this section, the effectiveness of the proposed residual general design approach is evaluated considering the nonlinear model defined in (168)-(175). For comparison purposes, the reference model, design and simulation conditions are the same as the ones presented in Section 6.4.

In the simulations, the input signal variation $q(k)$ (i.e., $u(k)$ and $v(k)$), the disturbance ($\Delta u_1(k)$ and $\Delta v_1(k)$) and the fault signals (actuator fault $f_1(k)$ and sensor fault $f_2(k)$) are as defined below:

- $\Delta u_1 = \Delta v_1 = 0.01$ A;
- $u(k) = -200$ A;



(a) Residual response to the actuator fault.



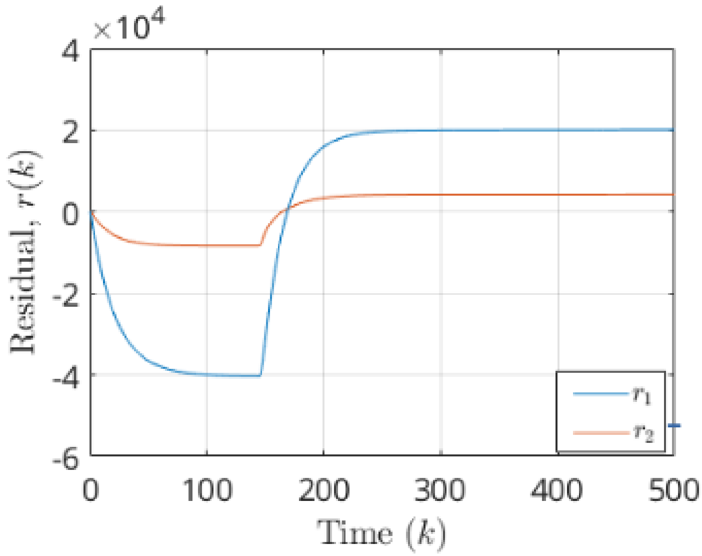
(b) Residual response to the sensor fault.

Figure 8 – Residual response to fault, control and disturbance inputs.
Font: Own authorship.

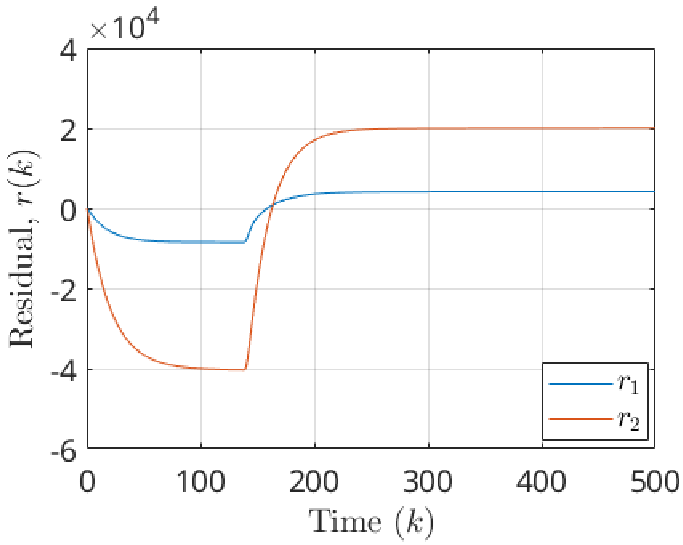
- $v(k) = 50$ V;
- $f_1(k) = -50$ A;
- $f_2(k) = 0.02$ V;

In addition, the input and the disturbance signals are applied to the system during the entire time slot and it is considered that each fault signal occurs one at a time. Figure 9a refers to the case where the actuator fault happens, i.e., $f_2(k) = 0$, while Figure 9b is related to occurrence of the sensor fault, i.e., $f_1(k) = 0$.

Despite of performance degradation compared to the results shown in Section 6.4, the results depicted in Figures 9a and 9b demonstrated (assuming suitable thresholds) that is still possible to detect the faults considering the proposed (linear) residual generator and the nonlinear model of the battery pack. Notice that in the occasion of an actuator fault (Figure 9a), residual r_1 is more influenced by it, while the occurrence of a sensor fault (Figure 9b)



(a) Residual response to the actuator fault.



(b) Residual response to the sensor fault.

Figure 9 – Residual response to fault, control and disturbance inputs.
Font: Own authorship.

affects more substantially residual r_2. It is important to emphasize that the residual generator provides satisfactory FDI results when dealing with relatively small magnitudes of the fault signals. Large inputs and fault signals may lead to unbounded residuals as noted in some simulations.

6.6 CHAPTER CONCLUSION

In this chapter, the LMI-based H_∞ approach was applied to design an FDI scheme for two Li-Ion batteries in the series configuration in which the dynamical model was embedded in the class of linear discrete-time descriptor systems introduced in Chapter 5. The resulting observer-like filter approximated the diagonal solution imposed by the reference model achieving fault detection and isolation. The simulations demonstrated the potentials of the proposed approach for the fault detection and isolation of (linear) descriptor systems while for the nonlinear case the FDI performance depends on the input signals magnitude. Future research will be concentrate on designing nonlinear residual generators considering the series and parallel configurations of battery-packs.

7 CONCLUDING REMARKS

In this thesis, we aimed at designing robust observer-based FDI techniques applied to discrete-time systems as well as discrete-time descriptor systems. The achieved results considered mixed sensitivity specifications (either H_∞/H_- or H_{peak}/H_-) for guaranteeing sensitivity to faults while attaining low sensitivity to disturbance and control inputs. To achieve this goal, the thesis contributions are highlighted in Section 7.1 while the list of publications (directly or indirectly) related to this thesis is reported in Section 7.2. Future lines of research are discussed in Section 7.3.

7.1 THESIS CONTRIBUTIONS

In order to achieve the thesis objectives, four main categories of contributions can be distinguished, which are presented in the following:

- **Reference model**: the design of FDI techniques integrated with a triangular reference model structure allows the fault detection and isolation when the faults do not occur simultaneously. The proposed reference model has a degree of freedom (given by the matrix Q) which facilitates the achievement of a desired sensitivity to faults. Also, one may consider a situation where there exist more faults than the number of measurements (i.e., when $n_f = 2n + 1$). This usually occurs when all sensors and actuators can be subject to faults (SCHONS et al., 2020a, 2020b) as shown in Chapters 4, 5 and 6.
- **Linear Matrix Inequality Framework**: the proposed residual generator design conditions described in terms of LMI constraints made possible to consider mixed performance specifications and different classes of systems in a unified mathematical formalism.
- **Descriptor models**: the main characteristic of a descriptor structure is the inclusion of algebraic constraints in the state and/or output equations, which is particularly interesting when modeling electric or mechanical systems; see, e.g., (BELOV et al., 2018). The proposed LMI conditions for descriptor systems do not impose any additional conservatism to the residual generator design when compared to the results derived for standard state space representations (thanks to the model transformation given in Chapter 5).
- **Application to the FDI problem of two Li-Ion batteries connected in series**: In Chapter 6, the H_∞/H_- result established in Chapter 5 is employed to the FDI problem of two batteries in series, showing that the proposed solution can be successfully considered for practical-oriented applications.

7.2 LIST OF PUBLICATIONS

In the following, the publications directly and indirectly connected to the scope of this thesis are presented. In particular, the publications **P3** and **P4** will be instrumental to the extension of proposed results to nonlinear systems as discussed in Section 7.3.

P1 D. Coutinho, S. Schons, M. Kinnaert and C.E. de Souza, "H_∞ Filter Design for a Class of Discrete-time Nonlinear Descriptor Systems". In *2019 IEEE 58th Conference on Decision and Control (CDC)*. Nice, France, pp. 623-628.

Main contribution: as this work was motivated by state observation of lithium-ion battery packs, nonlinearities in both state and output equations were considered, as well as the inclusion of a nonlinearity in the matrix coefficient of the one-step-ahead state vector.

P2 L.D. Couto, S. Schons, D. Coutinho and M. Kinnaert, "A Descriptor Modelling Approach for the Observer Design of Interconnected Li-ion Batteries Using Limited Measurements". In *2019 15th Advanced Control and Diagnosis (ACD)*. Bologna, Italy.

Main contribution: a framework that allows for state estimation of series or parallel arrangements of battery cells is presented. Besides, a robust LMI-based descriptor observer that minimizes the peak norm between current or voltage discrepancies and a given performance variable is designed.

P3 D. Coutinho, S. Schons, L.D. Couto and M. Kinnaert, "Robust observer design for discrete-time locally one-sided Lipschitz systems". *European Journal of Control*, 53:43-51, 2020.

Main contribution: the design of a robust state observer for one-sided Lipschitz nonlinear discrete-time systems with nonlinear state and output equations, minimizing an upper-bound on the ℓ_∞-induced system norm from the disturbance input to the error system performance output.

P4 S. Schons, D. Coutinho and M. Kinnaert, "H_-/H_∞ Reference Model-based Fault Detection and Isolation for Discrete-time Systems". In *2020 28th Mediterranean Conference on Control and Automation (MED)*. Saint-Raphaël, France, pp. 272-277.

Main contribution: an H_-/H_∞ FDI is designed considering a triangular reference model structure with fixed dynamics. The filter design is cast in terms of LMI constraints.

P5 L.D. Couto, S. Schons, D. Coutinho and M. Kinnaert, "Observer design for the series interconnection of Li-ion battery cells subject to reduced voltage information". In *Proceedings of the ASME Dynamic Systems and Control Conference*. Pittsburgh, PA, USA, 2020.

Main contribution: The lithium-ion battery pack model is formulated as a descriptor system subject to limited voltage and temperature measurements. The proposed state observer is able to estimate the state of each individual cell, including state of charge and inner temperature, as well as the unmeasured voltages.

P6 S. Schons, D. Coutinho and M. Kinnaert. "Reference Model-based Fault Detection and Isolation for Discrete-time Systems Subject to Persistent Disturbances". In *2020 23th Automatic Brazilian Congress*. Porto Alegre, Brazil.

Main contribution: a robust H_{-}/*peak*-norm filter is designed in order to guarantee fault detection and isolation for linear discrete-time systems. This paper considers a triangular reference model structure with partially fixed dynamics, which allows one to consider a situation where more faults than the number of measured system outputs are present. Finally, the problem is cast in terms of LMI constraints.

P7 D. Coutinho, C. E. de Souza, M. Kinnaert and S. Schons, "Robust observer design for a class of discrete-time nonlinear singular systems with persistent disturbances". *International Journal of Adaptive Control and Signal Processing*, 35:51–68, 2021.

Main contribution: by designing a state observer, an upper bound is minimized on a peak-to-peak performance index relating the disturbances/parametric uncertainties and a given observer performance signal while guaranteeing regional input-to-state stability of the estimation error dynamics. Besides, an estimate of the reachable set for the estimation error system is provided.

7.3 FUTURE RESEARCH

This work has covered state estimation and FDI techniques applied to linear discrete-time systems. Additionally, the state estimation problem of nonlinear (possibly uncertain) discrete-time systems has also been the subject of research as reported in publications **P3** and **P7** in the latter Section, which are summarized in Appendices C and D for completeness. In particular, (COUTINHO et al., 2020) proposed LMI-based conditions to the design of robust nonlinear observers for standard discrete-time one-sided Lipschitz systems, while (COUTINHO et al., 2019) addressed the problem of H_∞ filtering for nonlinear discrete-time descriptor systems locally satisfying Lipschitz-like conditions.

Therefore, following the same route of results presented in Chapters 4 and 5, future research will be focused on deriving FDI designed methods to (uncertain) nonlinear systems. A key issue to be better studied is how to define the reference model, since a linear reference model might be unsuitable or lead to conservative results.

Regarding the application on battery-packs, the extension of the work to a battery pack made of an arbitrary number of cells in series/parallel arrangement and handling actuator, sensor and internal faults simultaneously would be the ultimate goal. Therefore, future research will focus on reach this objective with a minimal number of sensors or with an instrumentation having the lowest cost.

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APPENDIX A – PROOF OF THEOREM 1

This appendix presents the proof of Theorem 1, where an H_-/H_∞ approach is designed with the view to guarantee fault detection and isolation for a class of linear discrete-time systems. Therefore, firstly, consider the LMI in (87) (Subsection 4.2.1). Notice from the block $(3,3)$ that the inequality

$$K + K^T - P_1 > 0$$

is obtained. Since $P_1 > 0$, it follows that K is full rank. In addition, taking the inequality in (48) into account, the following holds:

$$\begin{bmatrix} -P_1 & * & * & * \\ 0 & -\gamma_w I_{n_w} & * & * \\ KA - L_k C & KB_w - L_k D_w & -KP_1^{-1}K & * \\ C_r C & C_r D_w & 0 & -\bar{Q} \end{bmatrix} < 0, \tag{196}$$

Next, pre- and post-multiplying the above matrix inequality by

$$\text{diag}\{I_{n_x}, I_{n_w}, P_1 K^{-1}, I_{n_f}\}$$

and its transpose, respectively, and then by applying the Schur's complement leads to:

$$\begin{bmatrix} -P_1 & 0 \\ 0 & -\gamma_w I_{n_w} \end{bmatrix} + \begin{bmatrix} (A - LC)^T \\ (B_w - LD_w)^T \end{bmatrix} P_1 \begin{bmatrix} (A - LC) & (B_w - LD_w) \end{bmatrix} \\ + \begin{bmatrix} (C_r C)^T \\ (C_r D_w)^T \end{bmatrix} Q^T Q \begin{bmatrix} C_r C & C_r D_w \end{bmatrix} < 0 \tag{197}$$

Hence, pre- and post-multiplying (197) by $[\tilde{x}^{\,T} \;\; w^T]^T$ and its transpose yields the inequality given in (42) and thus the unforced system in (35) is asymptotically stable and the condition *(ii)* of Theorem 1 is satisfied, which means that $\|\mathcal{G}_{wr}\|_\infty^2 \leq \gamma_w$.

Now, suppose that the LMI in (57) (Subsection 4.2.2) is satisfied. Then, the following condition can be obtained:

$$\Omega < 0, \;\; \Omega = \begin{bmatrix} P_{21} - K - K^T & * \\ P_{22} - MK - K_a^T & P_{23} - K_b^T - K_b^T \end{bmatrix}, \tag{198}$$

or equivalently $K_2 + K_2^T - P_2 > 0$, where K_2 and P_2 are as defined in (54). Hence, from the fact that $P_2 > 0$, it follows that K_2 is full rank. Taking the fact that

$$P_2 - K_2 - K_2^T \geq -K_2 P_2^{-1} K_2^T,$$

it turns out that the LMI in (57), with the block Ω as defined in (198) being replaced by

$$-K_2 P_2^{-1} K_2^T,$$

holds. Thus, pre- and post-multiplying the resulting matrix inequality by

$$\text{diag}\{I_{n_a}, I_{n_f}, P_2 K_2^{-1}, I_{n_f}\}$$

and its transpose, respectively, yields:

$$\begin{bmatrix} -P_2 & 0 \\ 0 & -\gamma_f I_m \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} P_2 \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} Q^T Q \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} < 0 \tag{199}$$

from the Schur's complement. Further, pre- and post-multiplying (199) by $[\bar{x}^T \ \ f^T]^T$ and its transpose, respectively, leads to the inequality in (133). Hence, it follows from the bounded real lemma that the condition (*iii*) of Theorem 1 is satisfied, i.e., $\|\mathcal{G}_{fe_r}\|_\infty^2 \leq \gamma_f$.

Finally, the condition (*iv*) of Theorem 1, i.e., $\|\mathcal{G}_{f\breve{r}}\|_-^2 \geq \gamma_c$, follows straightforwardly from the LMI in (61) (Subsection 4.2.3) and (LI, X.; LIU, H., 2013), which completes the proof.

APPENDIX B – PROOF OF THEOREM 2

The proof of Theorem 2 is presented in this appendix, where a mixed $H_{-}/peak\text{-}norm$ approach is considered to design a residual based observer for the fault detection and isolation of linear discrete-time systems.

Firstly, consider the LMI in (**??**) (Subsection 4.3.1). From the block $(1,1)$, it follows that

$$K + K^T - P_1 > 0,$$

which implies that K is full rank since $P_1 > 0$. In addition, by noticing that

$$(K - P_1)P_1^{-1}(K - P_1)^T \geq 0,$$

the following holds:

$$P_1 - K - K^T \geq -KP_1^{-1}K^T, \tag{200}$$

for any nonsingular matrix K. As a result, the following inequality holds:

$$\begin{bmatrix} -KP_1^{-1}K & KA - L_k C & B_W - L_k D_W \\ * & -(1-\tau_1)P_1 & 0 \\ * & * & -\tau_1 I_{n_w} \end{bmatrix} < 0. \tag{201}$$

Next, pre- and post-multiplying the above by

$$\text{diag}\{P_1 K^{-1}, I_{n_x}, I_{n_w}\}$$

and its transpose, respectively, and then applying the Schur's complement leads to:

$$\begin{bmatrix} -P_1 & KA - L_k C \\ * & -(1-\tau_1)P_1 \end{bmatrix} - \begin{bmatrix} (B_W - L_k D_W)^T \\ 0 \end{bmatrix} \tau_1 I_{n_w} \begin{bmatrix} (B_W - L_k D_W) & 0 \end{bmatrix} < 0. \tag{202}$$

Thus, by pre- and post-multiplying (202) by $[\tilde{x}^T \;\; w^T]^T$ and its transpose, yields the inequality in (79) (Subsection 4.3.1). Hence, the condition

$$\|\mathcal{G}_{wr}\|_{peak} \leq \frac{\sqrt{1+\rho_1}}{\eta_1} \tag{203}$$

is satisfied and the error system is locally ISS stable.

Now, suppose that the LMI in (94) (Subsection 4.3.2) is satisfied. Then, the following condition can be obtained:

$$\Omega < 0, \quad \Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ * & \omega_{22} \end{bmatrix}, \tag{204}$$

or equivalently $K_2 + K_2^T - P_2 > 0$, where K_2 and P_2 are as defined in (54) (Subsection 4.2.2). Hence, from the fact that $P_2 > 0$, it follows that K_2 is full rank. Accounting for the fact that

$$P_2 - K_2 - K_2^T \geq -K_2 P_2^{-1} K_2^T,$$

it turns out that the LMI in (94) (Subsection 4.3.2), with the block Ω as defined in (204) being replaced by

$$-K_2 P_2^{-1} K_2^T,$$

holds. Thus, pre- and post-multiplying the resulting matrix inequality by

$$\mathrm{diag}\{P_2 K_2^{-1}, I_{n_a}, I_{n_f}\}$$

and its transpose, respectively, yields

$$\begin{bmatrix} -P_2 & 0 \\ 0 & -(1-\tau_2)P_{21} \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \end{bmatrix} P_2 \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} - \begin{bmatrix} \bar{C}^T \\ \bar{D}^T \end{bmatrix} \tau_2 I_{n_f} \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} < 0 \tag{205}$$

from the Schur's complement. Further, pre- and post-multiplying (205) by $[\bar{x}^T \;\; f^T]^T$ and its transpose, respectively, leads to the inequality in (90) (Subsection 4.3.2). Hence, it follows that

$$\|\mathcal{G}_{fe_r}\|_{peak} \leq \frac{\sqrt{1+\rho_2}}{\eta_2} \tag{206}$$

holds.

Finally, $\|\mathcal{G}_{f\breve{r}}\|_{-}^2 \geq \gamma_c$ follows straightforwardly from the LMI in (61) (Subsection 4.2.3) and (LI, X.; LIU, H., 2013), which completes the proof.

APPENDIX C – STATE ESTIMATION OF NONLINEAR DISCRETE-TIME SYSTEMS

In this appendix, a robust nonlinear observer design technique is proposed for a class of nonlinear one-sided Lipschitz discrete-time systems, where both state and output equations are nonlinear functions of the state, inputs and parameter uncertainties. This result was published as a journal paper and is summarized in this appendix. The reader may follow (COUTINHO et al., 2020) for the main results proofs and further details.

C.1 PROBLEM STATEMENT

Consider the following class of discrete-time systems:

$$x(k+1) = Ax(k) + \sum_{i=1}^{n} F_i \phi_i(x,u,w,\theta) + B_u u(k) + B_w w(k), \tag{207a}$$

$$y(k) = Cx(k) + \sum_{i=1}^{n} H_i \phi_i(x,u,w,\theta) + D_u u(k) + D_w w(k), \tag{207b}$$

where $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the vector of uncertain (possibly time-varying) parameters, $\phi_i : \mathcal{X} \times \mathcal{U} \times \mathcal{V} \times \Theta \to \mathbb{R}^{n_i}$, $i = 1, \ldots, n$, are nonlinear vector functions satisfying the conditions for uniqueness and existence of a solution to (207), and $A, F_1, \ldots, F_n, B_u, B_v, C, H_1, \ldots, H_n, D_u$ and D_v are given real matrices with appropriate dimensions. $\mathcal{X}, \mathcal{U}, \mathcal{V}$ and Θ are compact domains and, for simplicity of presentation, the domains $\mathcal{X}, \mathcal{V}$ and Θ are assumed to contain their origins.

Moreover, consider the following assumptions with respect to system (207):

A1 The pair (A, C) is detectable.

A2 $\phi_i(0,0,0,\theta) = 0$, $i = 1, \ldots, n$, for all $\theta \in \Theta$.

A3 There exist matrices $M_i \in \mathbb{R}^{n_x \times n_i}$, $Q_i = Q_i^T \in \mathbb{R}^{n_x \times n_x}$, $\bar{Q}_i = \bar{Q}_i^T \in \mathbb{R}^{n_v \times n_v}$, $\breve{Q}_i = \breve{Q}_i^T \in \mathbb{R}^{n_\theta \times n_\theta}$, $R_i = R_i^T \in \mathbb{R}^{n_x \times n_x}$, $\bar{R}_i = \bar{R}_i^T \in \mathbb{R}^{n_v \times n_v}$, $\breve{R}_i = \breve{R}_i^T \in \mathbb{R}^{n_\theta \times n_\theta}$, and real scalars s_i, $i = 1, \ldots, n$, such that the following holds for all $x, \hat{x} \in \mathcal{X}$, $u \in \mathcal{U}$, $w \in \mathcal{W}$, $\theta \in \Theta$ and $i = 1, \ldots, n$:

$$\left(\phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0)\right)^T M_i^T (x - \hat{x}) \leq \|Q_i^{1/2}(x-\hat{x})\|^2 + \|\bar{Q}_i^{1/2} w\|^2 + \|\breve{Q}_i^{1/2}\theta\|^2, \tag{208}$$

$$\|\phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0)\|^2 \leq \|R_i^{1/2}(x-\hat{x})\|^2 + \|\bar{R}_i^{1/2} w\|^2 + \|\breve{R}_i^{1/2}\theta\|^2 + s_i \left(\phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0)\right)^T M_i^T (x - \hat{x}). \tag{209}$$

A4 There exist positive scalars m_w and m_θ such that

$$w \in \mathcal{W} := \{w \in \mathbb{R}^{n_w} : w^T w \leq m_w\} \tag{210}$$

$$\theta \in \Theta := \{\theta \in \mathbb{R}^{n_\theta} : \theta^T \theta \leq m_\theta\} \tag{211}$$

Remark 6 *The inequalities in (208) and (209) are respectively local versions of the one-sided Lipschitz condition and quadratic inner bound property (for more details, see (ABBASZADEH, M.; MARQUEZ, H. J., 2010)), which provide a tighter upper bound on* $\|\phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0)\|$ *when compared to a standard locally Lipschitz condition. This point of view can be demonstrated by combining (208) and (209), which leads to:*

$$\begin{aligned}\|\phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0)\|^2 \leq\ & \|R_i^{1/2}(x-\hat{x})\|^2 + s_i\|Q_i^{1/2}(x-\hat{x})\|^2 \\ & + \|\bar{R}_i^{1/2}w\|^2 + s_i\|\bar{Q}_i^{1/2}w\|^2 \\ & + \|\breve{R}_i^{1/2}\theta\|^2 + s_i\|\breve{Q}_i^{1/2}\theta\|^2 \qquad (212) \\ \leq\ & \|S_i^{1/2}(x-\hat{x})\|^2 + \|\bar{S}_i^{1/2}w\|^2 + \|\breve{S}_i^{1/2}\theta\|^2 \qquad (213)\end{aligned}$$

where $S_i = R_i + |s_i|Q_i$, $\bar{S}_i = \bar{R}_i + |s_i|\bar{Q}_i$, $\breve{S}_i = \breve{R}_i + |s_i|\breve{Q}_i$. *Therefore, the combination of one-sided Lipschitz condition with quadratic boundedness, given by the right-hand side of (212), leads to a tighter upper bound on* $\|\phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0)\|$ *than the one provided by the standard Lipschitz condition given in (213). This fact makes one-sided Lipschitz conditions potentially less conservative than the traditional Lipschitz formulations.*

Thus, the idea is to design a nonlinear observer for the system in (207) which ensures that the estimation error system is locally input-to-state stable (ISS) while attenuating the effect of the disturbances and parameter uncertainty on the error.

In order to obtain an estimate $\hat{x}$ of x, consider the following nonlinear observer:

$$\hat{x}(k+1) = A\hat{x}(k) + \sum_{i=1}^{n} F_i\hat{\phi}_i(k) + B_u u(k) + L(y(k) - \hat{y}(k)), \qquad (214a)$$

$$\hat{y}(k) = C\hat{x}(k) + \sum_{i=1}^{n} H_i\hat{\phi}_i(k) + D_u u(k), \qquad (214b)$$

where $L \in \mathbb{R}^{n_x \times n_y}$ is to be designed and

$$\hat{\phi}_i(k) = \phi_i(\hat{x},u,0,0), \quad i = 1,\ldots,n. \qquad (215)$$

Furthermore, consider the estimation error defined in (34) (Section 4.1). Besides, let $\widetilde{\mathcal{X}}$ be the set induced by

$$\widetilde{\mathcal{X}} := \left\{\tilde{x} \in \mathbb{R}^{n_x} : \tilde{x} = x - \hat{x}, \quad x, \hat{x} \in \mathcal{X}\right\}. \qquad (216)$$

Then, the estimation error dynamics is given by

$$\tilde{x}(k+1) = (A - LC)\tilde{x}(k) + \sum_{i=1}^{n}(F_i - LH_i)\tilde{\phi}_i(k) + (B_w - LD_w)w, \quad \tilde{x}_0 = x(0) - \hat{x}(0), \quad (217)$$

where $\tilde{x} \in \widetilde{\mathcal{X}}$ and

$$\tilde{\phi}_i = \phi_i(x,u,w,\theta) - \phi_i(\hat{x},u,0,0), \quad i = 1,\ldots,n. \qquad (218)$$

In order to obtain a numerically tractable solution for the observer in (214) locally ensuring the ISS stability of the estimation error system while guaranteeing a peak-to-peak performance with respect to exogenous signals and parameter uncertainty, consider the following augmented disturbance input:

$$w_a(k) = \frac{1}{\sqrt{m}}\begin{bmatrix} w \\ \theta \end{bmatrix}, \quad m = m_w + m_\theta, \quad w_a \in \mathbb{R}^{n_{w_a}}, \quad n_{w_a} = n_w + n_\theta, \tag{219}$$

and the following performance output

$$e(k) = C_e\tilde{x}(k) + D_{w_a}w_a(k), \tag{220}$$

where $e \in \mathbb{R}^{n_e}$ and C_e, D_{w_a} are given real matrices with appropriate dimensions. From (210), (211) and (219), one can notice that the augmented disturbance vector w_a belongs to the following set:

$$\mathcal{W}_a := \{w_a \in \mathbb{R}^{n_{w_a}} : w_a^T w_a \leq 1\}. \tag{221}$$

Thus, considering the following system with the output given by (220)

$$\tilde{x}(k+1) = (A - LC)\tilde{x}(k) + \sum_{i=1}^{n}(F_i - LH_i)\tilde{\phi}_i(k) + B_{w_a}w_a(k), \tag{222}$$

with

$$B_{w_a} = \begin{bmatrix}(B_w - LD_w) & 0\end{bmatrix},$$

the peak norm of the error dynamics can be defined as follows:

$$\|\mathcal{G}_{w_a e}\|_{\text{peak}} = \sup\,\{\|e(k)\|_\infty,\ w_a \in \mathcal{W}_a,\ k \geq 0,\ \tilde{x}_0 = 0\}. \tag{223}$$

The notion of input-to-state stability used in this work consider the following definition which is consistent with the definition of ISS systems introduced in (Z. P. JIANG; Y. WANG, 2001).

Definition 5 *The equilibrium point $\tilde{x}(k) = 0$ of system in (222) is said to be locally ISS, if there exist sets $\mathcal{R} \subset \widetilde{\mathcal{X}}$ and $\mathcal{R}_0 \subseteq \mathcal{R}$ such that, for any $\tilde{x}(0) \in \mathcal{R}_0$ and $w_a \in \mathcal{W}_a$, $\tilde{x}(k)$ remains confined to $\mathcal{R}$, for all $k \geq 0$. Moreover, if $w_a(k)$ vanishes as $k \to \infty$ then $\tilde{x}(k) \overset{k\to\infty}{\longrightarrow} 0$.*

Hence, the following problems will be addressed in next section:

- For a given set $\mathcal{R}_0$ (called the set of admissible initial conditions), determine the gain L such that the error system in (222) is locally ISS;

- For a given performance output as in (220), determine the gain L such that the error system in (222) is locally ISS while minimizing an upper bound γ on $\|\mathcal{G}_{we}\|_{\text{peak}}$.

C.2 OBSERVER DESIGN

Consider the system defined in (222), the assumptions **A1**-**A4**, and the sets $\widetilde{\mathcal{X}}$ and $\mathcal{W}_a$ as defined in (216) and (221), respectively. Suppose there exist a Lyapunov function

$$V(\tilde{x}) = \tilde{x}^T P \tilde{x}, \quad P > 0, \tag{224}$$

and a real scalar $\tau \in (0, 1)$ such that the following conditions hold

$$\Delta V(\tilde{x}) \leq \tau({w_a}^T w_a - V(\tilde{x})) \quad \text{and} \quad \mathcal{R} \subset \widetilde{\mathcal{X}}, \tag{225}$$

for all $\tilde{x} \in \widetilde{\mathcal{X}}$, $w_a \in \mathcal{W}_a$ and $k \geq 0$. Then, the origin of system (222) is locally ISS and the region

$$\mathcal{R} := \left\{\tilde{x} \in \mathbb{R}^{n_x} : V(\tilde{x}) = \tilde{x}^T P \tilde{x} \leq 1\right\} \tag{226}$$

is a positively invariant set. Moreover, if there exists an integer k_0 such that $w_a(k) \equiv 0$, for all $k \geq k_0$, then $\tilde{x}(k) \to 0$ as $k \to \infty$.

It turns out that the conditions in (225) provide an estimate $\mathcal{R}$ of the error system reachable set as well as a bound on the norm of $\tilde{x}(k)$ as detailed below:

$$\underline{\lambda}\tilde{x}^T\tilde{x} \leq V(\tilde{x}) \leq 1 \;\Rightarrow\; \|\tilde{x}(k)\| \leq \sqrt{\frac{1}{\underline{\lambda}}}, \quad \forall\, k \geq 0,$$

where $\underline{\lambda}$ is the smallest eigenvalue of P.

Hence, with the view to derive an upper bound γ on $\|\mathcal{G}\|_{\text{peak}}$, consider the following constrained inequality:

$$e^T e \leq \gamma^2, \quad \forall\, (\tilde{x}, w_a) : \tilde{x} \in \mathcal{R}, \quad w_a \in \mathcal{W}_a. \tag{227}$$

Applying the S-Procedure, the following inequality is a sufficient condition for (227) to hold:

$$\beta_1 V(\tilde{x}) + \beta_2 {w_a}^T w_a - e^T e \geq 0, \quad \gamma = \sqrt{\beta_1 + \beta_2}, \tag{228}$$

with β_1 and β_2 being positive scalars to be determined.

Next, taking into account the estimation error, conditions (208) and (209), for $i = 1, \ldots, n$, can be recast as follows:

$$(M_i\tilde{\phi}_i)^T\tilde{x} \leq \|Q_i^{1/2}\tilde{x}\|^2 + \|J_i^{1/2}w_a\|^2 \tag{229}$$

$$\tilde{\phi}^T\tilde{\phi} \leq \|R_i^{1/2}\tilde{x}\|^2 + \|T_i^{1/2}w_a\|^2 + s_i(M_i\tilde{\phi})^T\tilde{x} \tag{230}$$

where

$$J_i = m \times \text{diag}\{\bar{Q}_i,\ \breve{Q}_i\} \quad \text{and} \quad T_i = m \times \text{diag}\{\bar{R}_i,\ \breve{R}_i\}.$$

Besides, to derive numerically tractable solutions, the state error domain $\widetilde{\mathcal{X}}$ is constrained to be the following polytopic set containing the origin:

$$\widetilde{\mathcal{X}} := \left\{\tilde{x} \in \mathbb{R}^{n_x} : {a_j}^T\tilde{x} \leq 1,\ j = 1, \ldots, n_f\right\}, \tag{231}$$

where $a_1, \ldots, a_{n_f} \in \mathbb{R}^{n_x}$ define the n_f faces of $\widetilde{\mathcal{X}}$.

Therefore, the following theorem can be established:

Theorem 5 *Consider the error system in (222) satisfying the conditions given in (229) and (230). Let $\widetilde{\mathcal{X}}$ and τ be respectively a given polytopic set as defined in (231) and a given scalar belonging to the interval $(0,1)$. Suppose there exist matrices $P = P^T$, G, K and positive scalars μ_i and ρ_i, $i = 1, \dots, n$, such that the following holds:*

$$\begin{bmatrix} 1 & a_j^T \\ a_j & P \end{bmatrix} > 0, \; j = 1, \dots, n_f, \tag{232}$$

$$\begin{bmatrix} \Omega_P + \Omega_\mu + \Omega_\rho & \Upsilon^T \\ \Upsilon & P - G - G^T \end{bmatrix} < 0, \tag{233}$$

where

$$\Omega_P = \mathrm{diag}\left\{(\tau - 1)P, 0_{n_1}, \dots, 0_{n_n}, -\tau I_{n_{wa}}\right\},$$

$$\Omega_\mu = \begin{bmatrix} \sum_{i=1}^n \mu_i Q_i & \star & \cdots & \star & 0 \\ -0.5\mu_1 M_1^T & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.5\mu_n M_n^T & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sum_{i=1}^n \mu_i J_i \end{bmatrix}, \tag{234}$$

$$\Omega_\rho = \begin{bmatrix} \sum_{i=1}^n \rho_i R_i & \star & \cdots & \star & 0 \\ 0.5\rho_1 s_1 M_1^T & -\rho_1 I_{n_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.5\rho_n s_n M_n^T & 0 & \cdots & -\rho_n I_{n_{wa}} & 0 \\ 0 & 0 & \cdots & 0 & \sum_{i=1}^n \rho_i T_i \end{bmatrix},$$

$$\Upsilon = \left[(GA - KC) \;\; (GF_1 - KH_1) \;\; \cdots \;\; (GF_n - KH_n) \;\; \left[\sqrt{m}(GB_v - KD_v) \;\; 0\right]\right].$$

Then, the equilibrium point $\tilde{x} = 0$ of the error system in (222), with $L = G^{-1}K$, is locally ISS and the error trajectory $\tilde{x}(k)$ driven by any $\tilde{x}(0) \in \mathcal{R}$ and $w_a \in \mathcal{W}_a$ remains bounded to $\mathcal{R}$, for all $k \geq 0$, where $\mathcal{R}$ is as defined in (226).

Theorem 5 can be applied in several ways. For instance, an optimized estimate $\mathcal{R}$ of the error system reachable set can be determined for a given set of admissible initial conditions. Hence, assume that

$$\mathcal{R}_0 := \left\{\tilde{x} \in \mathbb{R}^{n_x} : \tilde{x}^T P_0 \tilde{x} \leq 1\right\}, \; \mathcal{R}_0 \subseteq \mathcal{R}, \tag{235}$$

is the set of admissible initial conditions, where $P_0 > 0$ is a given matrix defining the size and shape of $\mathcal{R}_0$, and let λ be a positive scalar such that

$$P - \lambda I_{n_x} \geq 0. \tag{236}$$

Hence, the following optimization problem provides an optimized estimate $\mathcal{R}$ of the error system reachable set

$$\max_{P, G, \dots, \rho_n, \lambda, \tau} \lambda : P_0 - P \geq 0, \text{ subject to (232), (233) and (236).} \tag{237}$$

Notice in the above optimization problem that the condition in (233) is not jointly convex with respect to τ and P. To overcome this problem, a bisection algorithm can be applied taking into account that $\tau \in (0,1)$. In addition, it turns out that λ^{-1} is an upper-bound of $\|\tilde{x}(k)\|_\infty^2$, by noting that

$$\lambda \tilde{x}^T \tilde{x} \leq \tilde{x}^T P \tilde{x} \leq 1 \quad \Rightarrow \quad \|\tilde{x}(k)\|^2 \leq \frac{1}{\lambda}, \quad \forall\, k \geq 0. \tag{238}$$

In other words, the optimization problem in (237) provides an observer which minimizes $\sup \|\tilde{x}(k)\|_\infty$ while providing an optimized estimate of the error system reachable set ensuring the local input-to-state stability of the error system origin.

Next, if it is of interest to design the observer such that $\|\mathcal{G}\|_{\text{peak}}$ is minimized consider the following result.

Theorem 6 *Consider the performance output as defined in (220) and the error system in (222), satisfying (229) and (230). Let $\widetilde{\mathcal{X}}$, $\mathcal{R}_0$ and τ be respectively a given polytopic set as in (231), a given set of admissible initial errors as in (235) and a given scalar belonging to the interval $(0,1)$. Suppose there exist matrices $P = P^T$, G, K and positive scalars $\lambda, \beta, \beta_1, \mu_i$ and ρ_i, $i = 1,\ldots,n$ such that the conditions (232), (233), (236) and the following hold:*

$$P_0 - P \geq 0, \tag{239}$$

$$\begin{bmatrix} P & 0 & C_e{}^T \\ 0 & \beta I_{n_{w_a}} & D_{w_a}{}^T \\ C_e & D_{w_a} & \beta_1 I_{n_e} \end{bmatrix} \geq 0. \tag{240}$$

Then, the following holds for the error system in (222) with $L = G^{-1}K$:

- *(i) for any $\tilde{x}(0) \in \mathcal{R}_0$ and $w_a \in \mathcal{W}_a$, $\tilde{x}(k) \in \mathcal{R}$ for all $k \geq 0$, where $\mathcal{R}$ is as defined in (226); and*
- *(ii) $\|\mathcal{G}\|_{\text{peak}}^2 \leq \beta_1(1+\beta)$.*

In order to derive an observer as in (214), which guarantees that the error trajectory is confined to the set $\mathcal{R}$, for all $k \geq 0$, while minimizing an upper bound γ on $\|\mathcal{G}\|_{\text{peak}}$, the following optimization problem is proposed:

$$\min_{P,G,\ldots,\rho_n,\beta,\beta_1,\tau} (\beta + \beta_1): \text{ subject to (268), (269), (239) and (240).} \tag{241}$$

Similarly to the optimization problem proposed in (237), a bisection algorithm can be applied over $\tau \in (0,1)$ to obtain an optimized solution. Notice that the above optimization approximately minimizes the value of $\gamma = \beta_1 + \beta_1\beta$ in order to simplify a numerical solution.

C.3 ILLUSTRATIVE EXAMPLE

To demonstrate the effectiveness of the proposed state estimator, consider the following nonlinear discrete-time oscillator subject to a multiplicative noise measurement:

$$\begin{cases} x_1(k+1) &= 1.9x_1(k) + x_2(k) - x_1(k)^3 + u(k) \\ x_2(k+1) &= 0.5x_2(k) \\ y(k) &= x_1(k) + 0.1x_1(k)v(k) \end{cases} \tag{242}$$

where $v(k)$ is such that $v(k)^2 \leq 1$, $\forall\ k \geq 0$. With respect to (242), it is assumed that

$$u(k) = -2y(k) + y(k)^3,$$

which guarantees the local stability of the closed-loop system for all $x(k)$ belonging to the following set:

$$\mathcal{X} = \{x \in \mathbb{R}^2 : |x_i| \leq 1,\ i = 1, 2\}. \tag{243}$$

Hence, by defining

$$\phi_1(k) = -x_1(k)^3 \quad \text{and} \quad \phi_2 = x_1(k)v(k),$$

the following bounds can be obtained for $\tilde{\phi}_1(k)$ and $\tilde{\phi}_2(k)$

i $\tilde{\phi}_1 M_1^T \tilde{x} \leq 0$, $\tilde{\phi}_1^2 \leq s_1 \tilde{\phi}_1 M_1^T \tilde{x}$; and

ii $\tilde{\phi}_2^2 \leq \bar{R}_2 v^2$,

where $M_1 = [\ 1\ \ 0\ ]^T$, $s_1 = -1$ and $\bar{R}_2 = 1$.

In this example, an observer is derived for the system defined in (242), which minimizes $||\mathcal{G}||_{\text{peak}}$ assuming the following output performance

$$e(k) = x_1(k) - \hat{x}_1(k) \tag{244}$$

by means of optimization problem proposed in (241). Then, by setting

$$\mathcal{R}_0 = \{\tilde{x} \in \mathbb{R}^2 : 3\,\tilde{x}^T \tilde{x} \leq 1\},$$

the following results were obtained:

$$P = \begin{bmatrix} 1.1050 & -0.3563 \\ -0.3563 & 2.9330 \end{bmatrix}, \quad L = \begin{bmatrix} 1.5215 \\ 0.0607 \end{bmatrix}, \quad \tau = 0.08, \quad \gamma = 0.9704.$$

Figure 10 shows the estimation $\mathcal{R}$ of the reachable set and the set of admissible initial errors $\mathcal{R}_0$ as well as the phase portrait of the error system considering the following assumptions:

- Initial conditions $x(0) = [-0.10\ \ 0.56\ ]^T$ and $\hat{x}(0) = [\ 0\ \ 0\ ]^T$; and

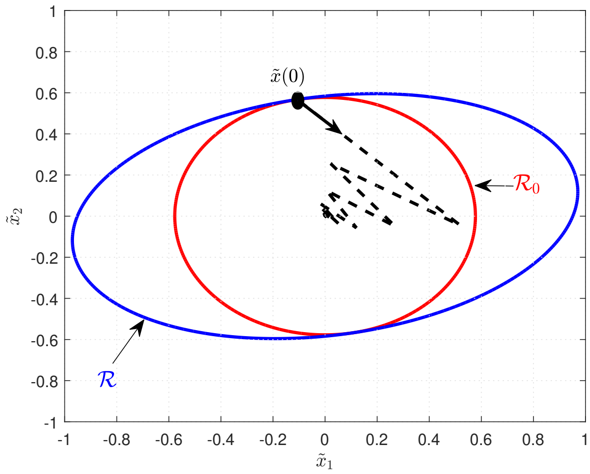


Figure 10 – A bounded error trajectory (dashed line) with representation of the sets $\mathcal{R}$ and $\mathcal{R}_0$.

- An uniformly distributed pseudo random sequence $v(k)$ such that $v^2(k) \leq 1$, for all $k \geq 0$.

This result demonstrates the effectiveness of the proposed observer, which designed conditions ensure that the error trajectory is locally bounded while providing an estimate of the error system reachable set with a guaranteed peak-to-peak performance. A comparative analysis between the result presented in Figure 10 and the method proposed by (BENALLOUCH et al., 2012) is made in (COUTINHO et al., 2020).

APPENDIX D – STATE ESTIMATION OF NONLINEAR DISCRETE-TIME DESCRIPTOR SYSTEMS

This chapter presents the summary results from the published paper (COUTINHO et al., 2019). Here, a convex technique is proposed to the design of standard H_∞ nonlinear observer-like filters for a class of nonlinear discrete-time descriptor systems. Consider the following specifications with respect to the system of interest:

- nonlinearities in the state and the output equations; and
- for the sake of generality, a nonlinearity in the matrix coefficient of the one-step-ahead state vector is included.

Therefore, consider the following class of nonlinear discrete-time descriptor systems:

$$Ex(k+1) = Ax(k) + (1+\delta)B_u u(k) + B_v v(k) + F\phi(x(k+1), x(k), u(k)), \tag{245a}$$
$$y(k) = Cx(k) + (1+\delta)D_u u(k) + D_v v(k) + H\psi(x(k), u(k)), \tag{245b}$$
$$x(0) = x_0, \tag{245c}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measurement output, $v \in \mathcal{V} \subset \ell_2^{n_v}$ is the disturbance input, $\phi \in \mathbb{R}^{n_\phi}$ and $\psi \in \mathbb{R}^{n_\psi}$ are known nonlinear functions and E, F and H are known constant real matrices with appropriate dimensions, with $\ell_2^{n_v}$ denoting the space of square summable n_v-dimensional vector sequences over $[0, \infty)$ with norm $\|(\cdot)\|_2 = \sqrt{(\cdot)^T(\cdot)}$. The additional terms involving the real scalar δ are introduced here to account for a possible model uncertainty with respect to the input $u(k)$. Besides the assumptions **A1**-**A3** in Chapter 5 with respect to the class of linear discrete-time descriptor systems in (104), the following conditions are assumed with respect to system (245):

A4 There exist compact sets $\mathcal{R} \subseteq \mathcal{X}$ and $\mathcal{V}$ such that for all $k \geq 0$, it holds that $x(k) \in \mathcal{R}$ for any $x_0 \in \mathcal{R}$, $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

A5 There exist real matrices M, N and Y with appropriate dimensions such that:

$$\|\phi(\xi_1, \zeta_1, u) - \phi(\xi_2, \zeta_2, u)\| \leq \|M(\xi_1 - \xi_2) + N(\zeta_1 - \zeta_2)\|, \tag{246a}$$
$$\|\psi(\zeta_1, u) - \psi(\zeta_2, u)\| \leq \|Y(\zeta_1 - \zeta_2)\|, \tag{246b}$$

for all ξ_1, ξ_2, ζ_1, $\zeta_2 \in \mathcal{X}$ and $u \in \mathcal{U}$.

Assumption **A4** guarantees that the system state trajectory is bounded in the domain of interest, and **A5** stands for a matrix version of standard local Lipschitz conditions. Inspired by (WANG, Z. et al., 2012), notice from **A2** that there exist matrices $T \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_x \times n_y}$ such that

$$TE + RC = I_{n_x}. \tag{247}$$

From now on, the dependence on k of the discrete-time representation in (245) will be often omitted to simplify the notation. Then, taking (247) into account, the system (245) can be cast as follows:

$$\begin{cases} x^+ = TAx + (1+\delta)TB_u u + TF\phi(x^+, x, u) + TB_v v + Ry^+ - (1+\delta)RD_u u^+ \\ \qquad\qquad\qquad\qquad\qquad\qquad - RH\psi(x^+, u^+) - RD_v v^+, \\ y \;= Cx + (1+\delta)D_u u + H\psi(x, u) + D_v v. \end{cases} \tag{248}$$

In view of the standard state-space representation (248) of the system defined in (245), the following state observer is proposed:

$$\begin{cases} \hat{x}^+ &= TA\hat{x} + TB_u u + TF\widehat{\phi} + Ry^+ + L(y - \hat{y}) - RD_u u^+ - RH\widehat{\psi}^+, \\ \hat{y} &= C\hat{x} + D_u u + H\widehat{\psi}, \\ \hat{x}(0) &= 0, \end{cases} \tag{249}$$

where $\widehat{\phi} := \phi(\hat{x}^+, \hat{x}, u)$, $\widehat{\psi} := \psi(\hat{x}, u)$, $\hat{x} \in \mathbb{R}^{n_x}$ is the estimate of x, and $L \in \mathbb{R}^{n_x \times n_y}$, $T \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_x \times n_y}$ are matrices to be determined.

Now, let

$$\tilde{x} := x - \hat{x} \tag{250}$$

be the estimation error vector. It is assumed that $\tilde{x} \in \widetilde{\mathcal{X}}$ with $\widetilde{\mathcal{X}}$ being a compact set containing $\tilde{x} = 0$. Then, taking (248)-(250) into account, the dynamics of $\tilde{x}$ can be cast as follows:

$$\begin{cases} \tilde{x}^+ &= (TA - LC)\tilde{x} + TF\widetilde{\phi} - RH\widetilde{\psi}^+ - LH\widetilde{\psi} + \tilde{B}w, \\ \tilde{x}(0) &= x_0, \end{cases} \tag{251}$$

where

$$\begin{aligned} \widetilde{\phi} &= \phi(x^+, x, u) - \phi(\hat{x}^+, \hat{x}, u), \\ \widetilde{\psi} &= \psi(x, u) - \psi(\hat{x}, u), \\ \widetilde{\psi}^+ &= \psi(x^+, u^+) - \psi(\hat{x}^+, u^+), \\ \tilde{B} &= \begin{bmatrix} -\delta RD_u & \delta(TB_u - LD_u) & -RD_v & (TB_v - LD_v) \end{bmatrix}, \\ w &= \begin{bmatrix} (u^+)^T & u^T & (v^+)^T & v^T \end{bmatrix}^T, \end{aligned} \tag{252}$$

with $w \in \mathbb{R}^{n_w}$ and $n_w = 2(n_u + n_v)$.

In order to obtain numerically tractable conditions for designing the gain L of the observer-based filter in (249), consider the following additional assumptions:

A6 $w \in \mathcal{W}(t, \mu)$, where

$$\mathcal{W}(t, \mu) := \left\{ w(k) \in \mathbb{R}^{n_w},\ k \in [0, t] : \sum_{i=0}^{t} w^T(i) w(i) \le \mu \right\}, \tag{253}$$

with t and μ being respectively a given positive integer and a given positive real scalar.

A7 $x_0 \in \mathcal{R}_0$, where

$$\mathcal{R}_0 := \left\{ x \in \mathbb{R}^{n_x} : x^T P_0 x \leq 1 \right\}, \tag{254}$$

with $P_0 > 0$ being a given real matrix defining the size and shape of $\mathcal{R}_0$.

Observe from assumption **A5** that the following holds with respect to the nonlinear functions appearing in the error system in (251):

$$\begin{aligned} \| \tilde{\phi} \| &\leq \| M\tilde{x}^+ + N\tilde{x} \|, \\ \| \tilde{\psi} \| &\leq \| Y\tilde{x} \|, \quad \| \tilde{\psi}^+ \| \leq \| Y\tilde{x}^+ \|, \end{aligned} \tag{255}$$

for all $\tilde{x}^+ \in \widetilde{\mathcal{X}}$ and $\tilde{x} \in \widetilde{\mathcal{X}}$.

Next, let

$$e = C_e\tilde{x} + D_\psi\tilde{\psi} + D_w w, \quad e \in \mathbb{R}^{n_e}, \tag{256}$$

where C_e, D_ψ and D_w are given real matrices with appropriate dimensions, be the performance output of the error system, and consider the following system norm definition:

$$\|\mathcal{G}_{we}\|_{\infty,[0,t]} := \sup_{\substack{0 \not\equiv w \in \mathcal{W}(t,\mu) \\ x_0 = 0}} \frac{\|e\|_{[2,[0,t]}}{\|w\|_{2,[0,t]}}, \tag{257}$$

where

$$\|(\cdot)\|_{2,[0,t]} = \sqrt{\textstyle\sum_{k=0}^{t} (\cdot)^T(\cdot)}\,.$$

Remark 7 *The performance output as defined in (256) is a function of the error system nonlinearities in order to cope with more general performance signals. Such a term arises, for instance, if it is of interest to consider the output estimation error $\tilde{y} := y - \hat{y}$ as the performance output.*

Therefore, the problem of concern is to determine the matrices L, T and R of the filter in (249) such that the error dynamics is bounded in a finite horizon $[0, t]$ while guaranteeing a prescribed (or optimized) upper-bound γ on $\|\mathcal{G}_{we}\|_{\infty,[0,t]}$.

Remark 8 *The proposed observer-based filter in (249) provides an estimate $\hat{x}(k)$ of $x(k)$ based on the measurements $\{y(i), u(i), i = 0, 1, \ldots, k\}$. For implementation purposes, $\hat{x}(k)$ can be obtained from the following recursive equations:*

$$\begin{aligned} f(k) = (TA - LC)\hat{x}(k-1) + (TB_u - LD_u)u(k-1) + Ry(k) + Ly(k-1) \\ - LH\hat{\psi}(k-1) - RD_u u(k), \end{aligned} \tag{258}$$

$$\hat{x}(k) - TF\hat{\phi}(k-1) + RH\hat{\psi}(k) = f(k), \tag{259}$$

where

$$\hat{\phi}(k-1) = \phi(\hat{x}(k), \hat{x}(k-1), u(k-1)), \tag{260}$$

$$\hat{\psi}(k) = \psi(\hat{x}(k), u(k)). \tag{261}$$

Notice that, since $\widehat{\phi}(k{-}1)$ *and* $\widehat{\psi}(k)$ *are nonlinear functions of* $\hat{x}(k)$*, Eq. (259) may be nonlinear with respect to* $\hat{x}(k)$*. Hence, to determine* $\hat{x}(k)$ *requires numerically solving a nonlinear equation at each time instant, which may not be convenient. However, if* $\phi(k-1)$ *and* $\psi(k)$ *do not depend on the system static state variables at time* k*, then (259) will be linear with respect to* $\hat{x}(k)$*.*

Therefore, in order to derive a solution to the filter design problem, let

$$V(\tilde{x}) = \tilde{x}^T P \tilde{x}, \quad P > 0, \tag{262}$$

be a Lyapunov function candidate for the estimation error system defined by (251) and (256), and consider the following dissipation inequality:

$$\Delta V(\tilde{x}(k)) + \gamma^{-1} e(k)^T e(k) - \gamma w(k)^T w(k) < 0, \tag{263}$$

where $\Delta V(\tilde{x}(k)) := V(\tilde{x}(k+1)) - V(\tilde{x}(k))$.

Supposing that the inequality in (263) holds for all $k \in [0, t]$, then summing the left-hand side of (263) from $k = 0$ to $k = t$ yields:

$$V(\tilde{x}(t)) + \gamma^{-1} \|e\|^2_{2,[0,t]} < V(x_0) + \gamma \|w\|^2_{2,[0,t]}. \tag{264}$$

Next, assuming that $w(k) \not\equiv 0$ over $[0, t]$, the above inequality implies that the following statements hold for any integer $k \in [1, t]$:

(*i*) if the matrix P_0 in (254) satisfies $P_0 \geq P$, then $V(\tilde{x}(k)) < 1 + \gamma\mu$, and thus $\tilde{x}$ is bounded on the time horizon $[0, t]$; and

(*ii*) $\|e\|^2_{2,[0,k]} < \gamma^2 \|w\|^2_{2,[0,k]}$ under the condition $x_0 = 0$, implying that $\|\mathcal{G}_{we}\|_{\infty,[0,t]} < \gamma$.

Before introducing the next result which proposes LMI based conditions to ensure that the above holds, consider the following polytopic inner approximation of $\mathcal{X}$ (i.e., $\mathcal{P} \subseteq \widetilde{\mathcal{X}}$):

$$\mathcal{P} := \left\{ p \in \mathbb{R}^{n_x} : |c_j^T p| \leq 1, \; j = 1, \ldots, m \right\}, \tag{265}$$

where $c_1, \ldots, c_m \in \mathbb{R}^{n_x}$ define the faces of $\mathcal{P}$.

Theorem 7 *Consider the error system in Eq. (251) under assumptions **A1-A6**. Let* $\mathcal{W}(t, \mu)$*,* $\mathcal{R}_0$ *and* $\mathcal{P}$*, as defined respectively in Eq. (253), Eq. (254) and Eq. (265), be given and* $\gamma > 0$ *be a given scalar. Suppose there exists real matrices* $P > 0$*,* K*,* T_K*,* L_K *and* R_K*, and positive scalars* λ*,* ρ *and* η *such that the following LMIs hold:*

$$P_0 - P \geq 0, \tag{266}$$

$$\begin{bmatrix} (1+\gamma\mu) & (1+\gamma\mu)c_j^T \\ (1+\gamma\mu)c_j & P \end{bmatrix} \geq 0,\ j = 1,\ldots,m, \tag{267}$$

$$\Omega = \left[\omega_{ij}\right]_{i,j=1,\ldots,10} < 0,\ \omega_{ij} = \omega_{ji}^T,\ \textit{for } i \neq j, \tag{268}$$

subject to

$$T_K E + R_K C - K = 0_{n_x}, \tag{269}$$

where the nonzero blocks ω_{ij} of matrix Ω are given by

$$\begin{aligned}
\omega_{11} &= P - K - K^T,\ \omega_{12} = T_K A - L_K C,\ \omega_{13} = T_K F,\ \omega_{14} = -R_K H,\ \omega_{15} = -L_K H, \\
\omega_{16} &= \tilde{B}_K,\ \omega_{18} = \lambda M^T, \omega_{19} = \rho Y^T,\ \omega_{22} = -P,\ \omega_{27} = C_e^T,\ \omega_{28} = \lambda N^T,\ \omega_{210} = \eta Y^T, \\
\omega_{33} &= -\lambda I_{n_\phi},\ \omega_{44} = -\rho I_{n_\psi},\ \omega_{55} = -\eta I_{n_\psi},\ \omega_{57} = D_\psi^T,\ \omega_{66} = -\gamma I_{n_w},\ \omega_{67} = D_w^T, \\
\omega_{77} &= -\gamma I_{n_e},\ \omega_{88} = -\lambda I_{n_\phi},\ \omega_{99} = -\rho I_{n_\psi},\ \omega_{1010} = -\eta I_{n_\phi},
\end{aligned}$$

with

$$\tilde{B}_K = \begin{bmatrix} -\delta R_K D_u & \delta(T_K B_u - L_K D_u) & -R_K D_v & (T_K B_v - L_K D_v) \end{bmatrix}.$$

Then, the state observer in Eq (249) with

$$L = K^{-1} L_K,\ T = K^{-1} T_K\ \textit{ and }\ R = K^{-1} R_K$$

ensures that the following holds for the estimation error system in (251) and (256):

1. *For any $x_0 \in \mathcal{R}_0$ and $w(k) \in \mathcal{W}(t,\mu)$, $\tilde{x}(k) \in \widetilde{\mathcal{R}}$ for all $k \in [0,t]$, where*

$$\widetilde{\mathcal{R}} := \left\{ p \in \mathbb{R}^{n_x} : p^T P p \leq 1 + \gamma\mu \right\}; \tag{270}$$

2. $\|\mathcal{G}_{we}\|_{\infty,[0,t]} < \gamma$.

The proof of Theorem 7 can be checked in (COUTINHO et al., 2019).

D.1 ILLUSTRATIVE EXAMPLE

In order to illustrate the effectiveness of the proposed technique, a numerical example is presented. Hence, consider the following nonlinear implicit system which has been adapted from the example in (BEIDAGHI et al., 2017) by replacing its uncertain parameter $\theta(k)$, with $|\theta(k)| \leq 1$, by the nonlinear function $\sin(x_1)$ and the addition of a disturbance signal v_2 in the measured output $y \in \mathbb{R}$:

$$\begin{bmatrix} \mathcal{F}(x^+, x, u, v) \\ \mathcal{F}_o(y, x, u, v) \end{bmatrix} = 0, \tag{271}$$

where $x = [x_1\ \ x_2\ \ x_3]^T \in \mathbb{R}^3$ is the state to be estimated based on y and the input $u \in \mathbb{R}$ which is corrupted by the exogenous disturbance v_1, and $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$, with

$$\mathcal{F}(\cdot) = \begin{bmatrix} \mathcal{F}_1^T(\cdot) & \mathcal{F}_2^T(\cdot) & \mathcal{F}_3^T(\cdot) \end{bmatrix}^T,$$

$$\mathcal{F}_o(\cdot) = -y + (0.5 + 2\cdot 10^{-4} s(x_1))x_1 + 0.3x_2 + 0.1v_2 + (0.2 - 4\cdot 10^{-4} s(x_1))x_3,$$

$$\mathcal{F}_1(\cdot) = -(1 + 0.01 s(x_1))x_1^+ + (0.51 + 2\cdot 10^{-3} s(x_1))x_1 + 0.32x_2 + (0.2 - 4\cdot 10^{-3} s(x_1))x_3 + 0.01(u + v_1),$$

$$\mathcal{F}_2(\cdot) = -(0.31 - 1\cdot 10^{-3} s(x_1))x_1 - 0.21x_2 + (0.15 - 2\cdot 10^{-3} s(x_1))x_3 + 0.2(u + v_1),$$

$$\mathcal{F}_3(\cdot) = (2\cdot 10^{-2} s(x_1) - 1)x_2^+ - 0.1x_1 - 0.1x_2 - 0.4x_3 - 0.1(u + v_1),$$

where $s(x_1) = \sin(x_1)$. It can be proved that the solution of $\mathcal{F}(x^+, x, 0, v) = 0$ is regular, causal and globally bounded.

Letting

$$\begin{cases} \phi(x^+, x)) = s(x_1)[\, x_1^+ \;\; x_2^+ \;\; x_1 \;\; x_3 \,]^T, \\ \psi(x) = s(x_1)[\, x_1 \;\; x_2 \,]^T, \end{cases} \tag{272}$$

the system in Eq. (271) can be written as in Eq. (245) with $\delta = 0$ and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\; A = \begin{bmatrix} 0.51 & 0.32 & 0.20 \\ -0.31 & -0.21 & 0.15 \\ -0.10 & -0.10 & -0.40 \end{bmatrix},\; B_u = \begin{bmatrix} 0.01 \\ 0.20 \\ -0.10 \end{bmatrix},\; B_v = \begin{bmatrix} B_u & 0_{3\times 1} \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5 \\ 0.3 \\ 2 \end{bmatrix}^T,\; F = 10^{-3} \begin{bmatrix} -10 & 0 & 2 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 20 & 0 & 0 \end{bmatrix},\; D_u = 0,\; H = 10^{-4}\begin{bmatrix} 2 & -4 \end{bmatrix},\; D_v = \begin{bmatrix} 0 & 0.1 \end{bmatrix}.$$

It is assumed that the performance output of the error system is defined by:

$$e(k) = C_e \tilde{x}(k) + D_\phi \tilde{\psi}(k), \tag{273}$$

where $\tilde{x}(k) = x(k) - \hat{x}(k)$, $\hat{x}(k) = [\, \hat{x}_1(k) \;\; \hat{x}_2(k) \;\; \hat{x}_3(k) \,]^T$, $\tilde{\psi}(k) = \psi(x)(k) - \psi(\hat{x})(k)$ and

$$C_e = \begin{bmatrix} 0.5 & 0 & 0.1 \end{bmatrix},\quad D_\psi = 10^{-4} \cdot \begin{bmatrix} 4 & -8 \end{bmatrix}.$$

In order to evaluate the performance of the proposed observer design technique with respect to exogenous disturbances, it is considered in this example that:

$$\mathcal{X} = \left\{ x \in \mathbb{R}^3 : |x_1| \le 0.5, |x_2| \le 1, |x_3| \le 2 \right\},$$

$$\mathcal{U} = \{ u \in \mathbb{R} : |u| \le 1 \} \quad \text{and} \quad \widetilde{\mathcal{X}} \equiv \mathcal{X}.$$

Considering (272) and since $|s(\alpha)| \le 1$, $\forall\, \alpha \in \mathbb{R}$, it follows that the assumption **A5** is satisfied with the following matrices:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\; N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Assuming $x_0 = 0$, $\mathcal{P} \equiv \mathcal{X}$ and $\mu = 2$, the following optimization problem is solved in order to minimize an upper-bound γ on $\|\mathcal{G}_{we}\|_{\infty,[0,t]}$

$$\min_{\substack{P,K,T_K,R_K,\\ L_K,\lambda,\rho,\eta,\gamma}} \gamma \text{ subject to (267), (268) and (269),}$$

leading to the following results:

$$P = \begin{bmatrix} 88.576 & 35.240 & -5.404 \\ 35.240 & 20.359 & -0.597 \\ -5.404 & -0.597 & 0.965 \end{bmatrix},\ R = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix},\ T = \begin{bmatrix} 1.000 & -0.0518 & 0.000 \\ 0.000 & 0.5040 & 1.000 \\ -2.500 & -0.6250 & -1.500 \end{bmatrix},$$

$$L = \begin{bmatrix} 0.9116 \\ -1.4705 \\ -0.0808 \end{bmatrix}, \gamma = 7.1774 \cdot 10^{-3},\ \lambda = 0.1255,\ \rho = 0.7074,\ \eta = 0.0277.$$

For illustrative purposes, Fig. 11a shows the performance output $e(k)$, whereas Fig. 11b and 11c respectively display the state errors $\tilde{x}_1(k)$ and $\tilde{x}_3(k)$ (it is noticed that $\tilde{x}_2(k)$ is quite similar to $\tilde{x}_1(k)$) considering $x_0 = \begin{bmatrix} 0.075 & 0.1 & 0.2 \end{bmatrix}^T$,

$$u(k) = 0.5\sin(0.02\pi k),\ v_1(k) = 0.25\sin(0.2\pi k),$$

and with $v_2(k)$ being a uniformly distributed random sequence over the interval $[-1, 1]$. Notice the excellent performance achieved by the proposed filter despite relatively large exogenous disturbances. This filter has been implemented considering the following recursive equations:

1. $f(k) = (TA - LC)\hat{x}(k-1) + TF_b\hat{\phi}_b(k-1) + TB_u u(k-1) + Ry(k) + Ly(k-1) - LH\hat{\psi}(k-1) - RD_u u(k)$;

2. $\hat{x}_1(k) = (1 + 0.01\sin(\hat{x}_1(k-1))^{-1} f_1(k)$;

3. $\hat{x}_2(k) = (1 - 0.02\sin(\hat{x}_1(k-1))^{-1} f_2(k)$;

4. $\hat{x}_3(k) = f_3(k) - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\left(RH\hat{\psi}(k) - TF_a\hat{\phi}_a(k-1)\right)$;

where $f(k) = \begin{bmatrix} f_1(k) & f_2(k) & f_3(k) \end{bmatrix}^T$ and

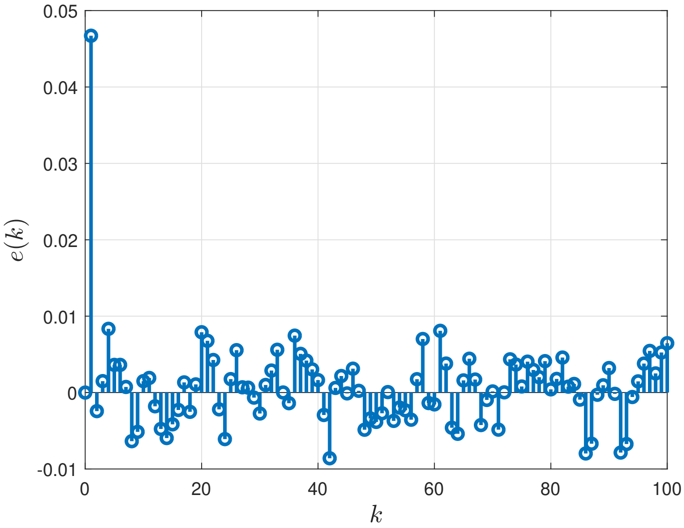
$$\hat{\psi}(k) = s(\hat{x}_1(k))\begin{bmatrix} \hat{x}_1(k) & \hat{x}_2(k) \end{bmatrix}^T,\quad \begin{bmatrix} \hat{\phi}_a(k-1) \\ --- \\ \hat{\phi}_b(k-1) \end{bmatrix} = \hat{\phi}(k-1),\ \begin{bmatrix} TF_a & | & TF_b \end{bmatrix} = TF,$$

$$\hat{\phi}_a(k-1) = s(\hat{x}_1(k-1))\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix},\quad \hat{\phi}_b(k-1) = s(\hat{x}_1(k-1))\begin{bmatrix} \hat{x}_1(k-1) \\ \hat{x}_3(k-1) \end{bmatrix}.$$

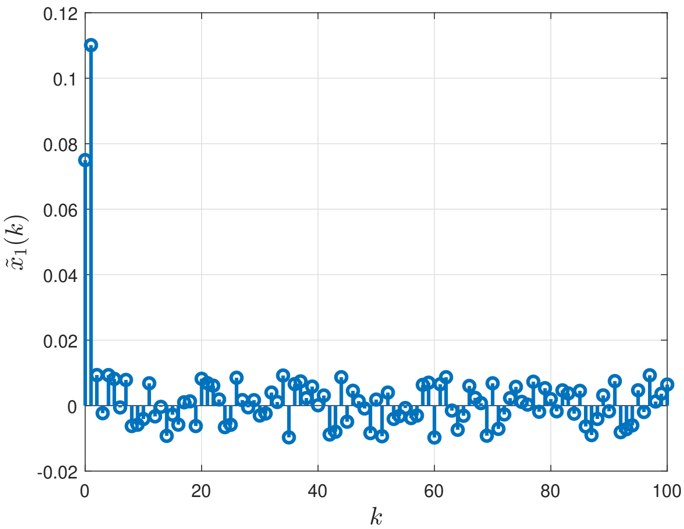
Noting that $\widehat{\phi}_a(k-1)$ and $\widehat{\psi}(k)$ do not depend on the estimate $\hat{x}_3(k)$ of the static state variable $x_3(k)$, as pointed out in Remark 8, it turns out that the estimates $\hat{x}_i(k)$, $i=1,2,3$, are computed solving only linear equations. Indeed, observe that $\hat{x}_3(k)$ as above is given by:

$$\hat{x}_3(k) = f_3(k) - 0.001s(\hat{x}_1(k))\hat{x}_1(k) + 0.002s(\hat{x}_1(k))\hat{x}_2(k) \\ + 0.025s(\hat{x}_1(k-1))\hat{x}_1(k) - 0.03s(\hat{x}_1(k-1))\hat{x}_2(k).$$

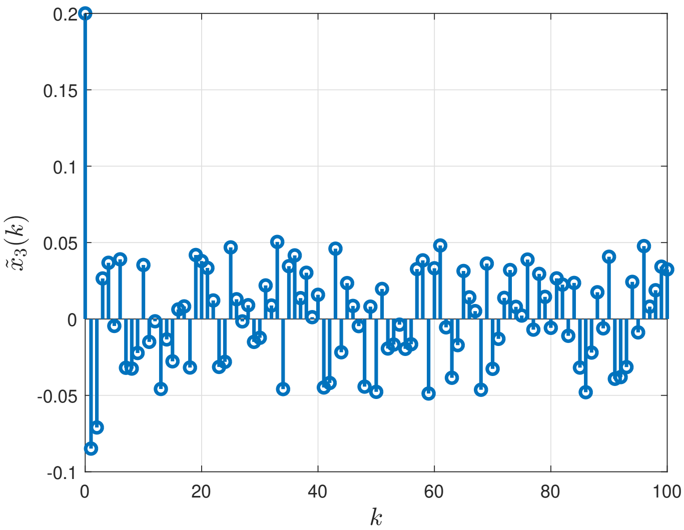
In general, the filter implementation requires numerically solving, at each time instant, a system of algebraic nonlinear equations with respect to the state estimate. However, when the model nonlinearities only involve the system dynamic states and the control input, the above mentioned system of equations becomes linear and the filter lends itself to a straightforward implementation.



(a) Estimation error performance output.



(b) Estimation error $\tilde{x}_1(k)$.



(c) Estimation error $\tilde{x}_3(k)$.

Figure 11 – Estimation error.
Font: Own authorship.