ECARES

Semiparametrically Efficient Tests of Multivariate Independence Using Center-Outward Quadrant, Spearman, and Kendall Statistics

Hongjian Shi
Department of Mathematics, TUM School of Computation,
Information and Technology, Technical University of Munich

Mathias Drton
Department of Mathematics, TUM School of Computation,
Information and Technology, Technical University of Munich

Marc Hallin
ECARES, Department of Mathematics,
Université libre de Bruxelles

Fang Han
Department of Statistics,
University of Washington, Seattle,

January 2023

ECARES working paper 2023-03

ECARES
ULB - CP 114/04
50, F.D. Roosevelt Ave., B-1050 Brussels BELGIUM
www.ecares.org

Semiparametrically Efficient Tests of Multivariate Independence Using Center-Outward Quadrant, Spearman, and Kendall Statistics

Hongjian Shi[*], Mathias Drton[†], Marc Hallin[‡] and Fang Han[§]

Abstract

Defining multivariate generalizations of the classical univariate ranks has been a long-standing open problem in statistics. Optimal transport has been shown to offer a solution in which multivariate ranks are obtained by transporting data points to a grid that approximates a uniform reference measure (Chernozhukov et al., 2017; Hallin, 2017; Hallin et al., 2021). We take up this new perspective to develop and study multivariate analogues of the sign covariance/quadrant statistic, Kendall's tau, and Spearman's rho. The resulting tests of multivariate independence are genuinely distribution-free, hence uniformly valid irrespective of the actual (absolutely continuous) distributions of the observations. Our results provide asymptotic distribution theory for these new test statistics, with asymptotic approximations to critical values to be used for testing independence as well as a power analysis of the resulting tests. This includes a multivariate elliptical Chernoff–Savage property, which guarantees that, under ellipticity, our nonparametric tests of independence enjoy an asymptotic relative efficiency of one or larger with respect to the classical Gaussian procedures.

1 Introduction

Testing independence is a fundamental problem in statistical inference and has numerous important applications in areas such as graphical modeling, causal inference, or inference in genomics. For broader applications, nonparametric methods are useful since they can detect non-linear dependencies without imposing moment conditions. However, in terms of statistical efficiency, nonparametric methods were initially deemed inferior to parametric methods from well approximating models. This belief was overturned in the seminal works of Hodges and Lehmann (1956) and Chernoff and Savage (1958) who proved that nonparametric methods based on univariate ranks not only perform well, but in fact can be more efficient than their parametric competitors. In this paper, we consider the problem of testing multivariate independence and show that also in higher dimension there are genuinely distribution-free and, thus, uniformly valid tests that enjoy this efficiency phenomenon.

[*]Department of Mathematics, TUM School of Computation, Information and Technology, Technical University of Munich, 85748 Garching bei München, Germany; e-mail: hongjian.shi@tum.de

[†]Department of Mathematics, TUM School of Computation, Information and Technology, Technical University of Munich, 85748 Garching bei München, Germany; e-mail: mathias.drton@tum.de

[‡]ECARES and Department of Mathematics, Université Libre de Bruxelles, Brussels, Belgium; email: mhallin@ulb.ac.be

[§]Department of Statistics, University of Washington, Seattle, WA 98195, USA; e-mail: fanghan@uw.edu

Specifically, we leverage the recently introduced concept of center-outward ranks and signs in order to devise the tests, and we work with generalized versions of so-called Konijn alternatives to assess local powers. The tests we devise can be deemed a safe and powerful replacement for Wilks' classical test, even under Gaussian distributions.

1.1 Testing multivariate independence

The problem of testing for independence between two random variables with unspecified densities has been among the very first applications of rank-based methods in statistical inference. Spearman's correlation coefficient was proposed in the early 1900s (Spearman, 1904), and Kendall's rank correlation goes back to Kendall (1938); both well before Wilcoxon (1945) gave his rank sum and signed rank tests for location.

The multivariate version of the same problem—testing independence between two random vectors with unspecified densities—is significantly harder, crucially due to the difficulty of defining a multivariate counterpart to univariate ranks. The first attempt to provide a rank-based alternative to the Gaussian likelihood ratio method of Wilks (1935) was developed in Chapter 8 of Puri and Sen (1971) and, for almost thirty years, has remained the only rank-based approach to the problem. The proposed tests, however, are based on componentwise rankings and are not distribution-free—unless, of course, both vectors have dimension one, in which case we are back to the traditional context of bivariate independence (e.g., Chapter III.6 of Hájek and Šidák, 1967). This issue persists in more recent work, e.g., that of Lin (2017), Weihs et al. (2018), and Moon and Chen (2022).

Alternatives to the Puri and Sen tests have appeared with the developments of multivariate concepts of signs and ranks. Based on Randles (1989)'s concept of *interdirections*, Gieser (1993) and Gieser and Randles (1997) proposed a sign test extending the univariate quadrant test (Blomqvist, 1950). Taskinen et al. (2005) also proposed a sign test, based on the so-called standardized *spatial signs*; their test is asymptotically equivalent to the one of Gieser and Randles under elliptical symmetry assumptions. *Spatial ranks* are introduced, along with the spatial signs, in Taskinen et al. (2005), where multivariate analogs of Spearman's *rho* and Kendall's *tau* are considered; the Spearman tests (involving Wilcoxon scores) are extended, in Taskinen et al. (2004), to the case of arbitrary square-integrable score functions, which includes van der Waerden (normal) score tests.

All these tests are enjoying, under elliptical symmetry, many of the attractive properties of their traditional univariate counterparts. In particular, under the assumptions of elliptical symmetry, they are asymptotically distribution-free. Local powers are obtained in Taskinen et al. (2004) against elliptical extensions of the so-called *Konijn alternatives* (Konijn, 1956). Chernoff–Savage and Hodges–Lehmann results (Chernoff and Savage, 1958; Hodges and Lehmann, 1956) have been established by Hallin and Paindaveine (2008), showing that the van der Waerden version of the Taskinen–Kankainen–Oja test uniformly dominates Wilks' optimal Gaussian procedure (Wilks, 1935), which entails the Pitman-nonadmissibility of the latter as a pseudo-Gaussian test.

We note here that the above work does provide test statistics that are asymptotically distribution-free in subclasses such as elliptical distributions. However, from the perspective we take here, such subclasses are too restrictive; the assumption of elliptical symmetry, in particular, is extremely strong, and unlikely to hold in most applications. Moreover, there is a crucial difference between finite-sample and asymptotic distribution-freeness. Indeed, one should be wary that a sequence of tests $\psi^{(n)}$ with asymptotic size $\lim_{n\to\infty} \mathrm{E}_{\mathrm{P}}[\psi^{(n)}] = \alpha$ under any element P in a class $\mathcal{P}$ of distributions does not necessarily have asymptotic size α under unspecified $\mathrm{P} \in \mathcal{P}$: the convergence of $\mathrm{E}_{\mathrm{P}}[\psi^{(n)}]$ to α, indeed, typically is not uniform over $\mathcal{P}$, so that, in general, $\lim_{n\to\infty} \sup_{\mathrm{P}\in\mathcal{P}} \mathrm{E}_{\mathrm{P}}[\psi^{(n)}] \neq \alpha$. Genuinely distribution-free tests $\phi^{(n)}$, where $\mathrm{E}_{\mathrm{P}}[\psi^{(n)}]$ does not depend on P, do not suffer that problem, and this is why finite-sample distribution-freeness is a fundamental property.

Palliating these limitations of the existing procedures by defining genuinely distribution-free—now over the class of all absolutely continuous distributions—multivariate extensions of the quadrant, Spearman, and Kendall tests, based on a "maximal distribution-free" concept of multivariate ranks and signs, is thus highly desirable. It is the objective of this paper.

1.2 Center-outward signs and ranks

For dimension $d > 1$, the real space $\mathbb{R}^d$ lacks a canonical ordering. As a result, the problem of defining, in dimension $d > 1$, concepts of signs and ranks enjoying the properties that make the traditional ranks so successful in univariate statistical inference has been an open problem for more than half a century. One of the most important properties is the exact distribution-freeness (for i.i.d. samples from absolutely continuous distributions). In an important new development involving optimal transport, the concept of center-outward ranks and signs was proposed recently by Chernozhukov et al. (2017), Hallin (2017), and Hallin et al. (2021) and enjoys a property of "maximal distribution-freeness", contrary to earlier concepts put forth — marginal ranks (Puri and Sen, 1971), spatial ranks (Oja, 2010), depth-based ranks (Liu and Singh, 1993; Zuo and He, 2006), and Mahalanobis ranks and signs (Hallin and Paindaveine, 2002a,b).

While this paper is focused on how center-outward ranks and signs naturally allow us to define distribution-free multivariate versions of the popular quadrant, Spearman, and Kendall tests, other types of tests have been considered in the literature. Center-outward ranks and signs have been used recently by Shi et al. (2022a) in the construction of distribution-free versions of distance covariance tests for multivariate independence, and a general framework for using center-outward ranks and signs in order to design distribution-free tests of multivariate independence that are consistent has been developed in Shi et al. (2022b). Multivariate ranks (based on measure transportation to the unit cube rather than the unit ball) have been used similarly in Ghosal and Sen (2022) and Deb and Sen (2022).

Center-outward ranks and signs also have been used successfully in other statistical problems: rank tests and R-estimation for VARMA models (Hallin et al., 2022b, 2023), rank tests for multiple-output regression and MANOVA (Hallin et al., 2022a), two-sample goodness-of-fit tests (Deb and Sen, 2022; Deb et al., 2021; Hallin and Mordant, 2021), and multiple-output quantile regression, linear (Carlier et al., 2016, 2017) and nonparametric (del Barrio et al., 2022).

1.3 Motivating examples

The importance and advantages of center-outward sign- and rank-based tests of independence are illustrated with the following real data and simulation examples.

1.3.1 Stock market data

The considered data, collected from Yahoo! Finance (finance.yahoo.com), contains prices for stocks that are in the Standard & Poor's 100 (S&P 100) index for every year between 2003 and 2012. We will analyze the seasonal log returns of daily adjusted closing prices.

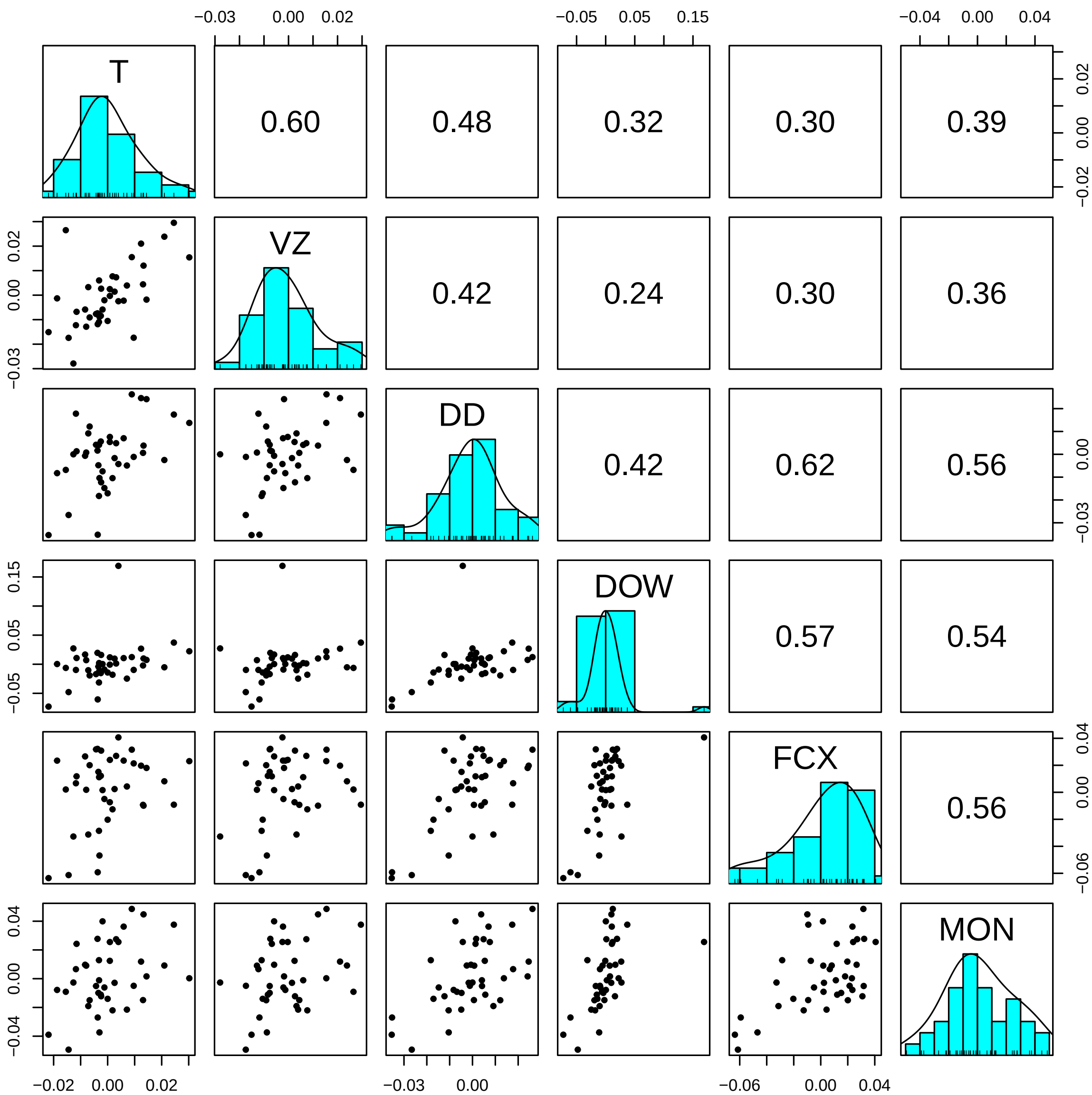

Figure 1: Univariate histograms (on the diagonal), bivariate scatter plots (below the diagonal), and the Pearson correlation coefficients (above the diagonal) for the seasonal log returns of stocks "AT&T Inc [T]", "Verizon Communications [VZ]," "Du Pont (E.I.) [DD]," "Dow Chemical [DOW]," "Freeport-McMoran Cp & Gld [FCX]," and "Monsanto Co. [MON]."

The S&P 100 stocks are classified into ten sectors by Global Industry Classification Standard (GICS). To illustrate the advantages of the proposed tests, we focus on detecting dependence between two sectors in S&P 100: (1) Telecommunication, including stocks "AT&T Inc [T]" and "Verizon Communications [VZ]"; and (2) Materials, including stocks "Du Pont (E.I.) [DD]", "Dow Chemical [DOW]", "Freeport-McMoran Cp & Gld [FCX]", and "Monsanto Co. [MON]."

Figure 1 presents univariate histograms and bivariate scatterplots for the seasonal log returns of the six stocks mentioned above. We notice that all univariate variables are skewed and/or heavy-tailed. We also notice that the (linear) relationship between any stock in (T, VZ) and any stock in (DD, DOW, FCX, MON) is not obvious, and the corresponding Pearson correlation is relatively small.

We apply the four rank-based tests proposed in Section 3.3 to the seasonal log returns of (T, VZ) coupled with either (DD, DOW), or (DD, FCX), or (DD, MON), or (DOW, FCX), or (DOW, MON), or (FCX, MON). The tests are denoted $\psi^{(n)}_{\text{sign}}$ (sign test, denoted as "sign" in tables and figures), $\psi^{(n)}_{\text{Spearman}}$ (Spearman test, denoted as "Spe"), $\psi^{(n)}_{\text{Kendall}}$ (Kendall test, denoted as "Ken"), and $\psi^{(n)}_{J\text{vdW}}$ (van der Waerden or normal score test, denoted as "vdW"). They are based on the matrices defined in (3.1)–(3.4), respectively. As a benchmark, we also include the Gaussian likelihood ratio test (LRT). The p-values for the tests are reported in Table 1. One observes that the proposed tests perform no worse than the LRT in the first three columns, while in the last three columns, the proposed tests yield strong evidence against independence while the LRT is not able to reject the null at the significance level of 0.05.

Table 1: P-values based on four proposed tests as well as LRT for the independence between US stock adjusted closing prices between 2003 and 2012.

		(DD, DOW)	(DD, FCX)	(DD, MON)	(DOW, FCX)	(DOW, MON)	(FCX, MON)
sign		0.002	0.050	0.027	0.013	0.008	0.023
Spe		< 0.001	0.005	0.004	0.007	0.001	0.011
Ken	(T,VZ)	0.002	0.013	0.016	0.011	0.002	0.036
vdW		< 0.001	0.012	0.011	0.006	< 0.001	0.010
LRT		0.011	0.016	0.009	0.174	0.072	0.074

Rank-based sign, Spearman, Kendall, and van der Waerden tests, thus, are able to detect dependencies that the traditional LRT cannot.

1.3.2 Finite-sample size of likelihood ratio tests

The LRT, moreover, can suffer from severe size inflation when the observations are not Gaussian. While this is not surprising, it is worth to briefly illustrate inflated test sizes in the context of simulated data of various dimensions.

Example 1.1. The data are a sample of independent copies of the random vector $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')'$ where $\mathbf{X}_1 = \exp(\mathbf{Z}_1)$ (values in $\mathbb{R}^{d_1}$), $\mathbf{X}_2 = \exp(\mathbf{Z}_2)$ (values in $\mathbb{R}^{d_2}$), and $(\mathbf{Z}_1, \mathbf{Z}_2) \sim N(\mathbf{0}, \sigma^2\mathbf{I}_{d_1+d_2})$. The empirical sizes for the LRT, based on $10,000$ simulations at the nominal significance level of 0.05 with a sample size $n = 100$, dimension $d_1 = d_2 \in \{2, 3, 5, 7\}$ and $\sigma \in \{0.5, 1.0, 1.5, 2.0\}$, are reported in Table 2. Note that σ, a standard error for the Gaussian $\mathbf{Z}_1$ and $\mathbf{Z}_2$, is a skewness parameter for the lognormals $\mathbf{X}_1$ and $\mathbf{X}_2$. As one might expect, the size grows for increasing skewness. Similar effects result from increasing the dimensions d_1 and d_2.

Table 2: Sizes of LRT for log-normal distributions at nominal significance level 0.05 for various dimensions d_1 and d_2 and various values of the skewness parameter σ.

	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 1.5$	$\sigma = 2.0$
$(d_1, d_2) = (2, 2)$	0.059	0.079	0.101	0.102
$(d_1, d_2) = (3, 3)$	0.067	0.101	0.134	0.155
$(d_1, d_2) = (5, 5)$	0.095	0.154	0.211	0.243
$(d_1, d_2) = (7, 7)$	0.133	0.197	0.272	0.317

1.4 Outline of the paper

The paper is organized as follows. Section 2 briefly reviews the notion of center-outward ranks and signs, and Section 3 introduces our tests of multivariate independence based on center-outward ranks and signs. In Section 4, we establish an elliptical Chernoff–Savage property for our center-outward test based on van der Waerden scores, which uniformly dominates, against Konijn alternatives, Wilks' test for multivariate independence. We also derive an analog of Hodges and Lehmann (1956)'s result for the problem under study. Numerical studies are provided in Section 5. The paper ends with a short conclusion in Section 6. All the proofs are relegated to Section 7.

2 Center-outward distribution functions, ranks, and signs

2.1 Definitions

Denoting by $\mathbb{S}_d$ and $\mathcal{S}_{d-1}$, respectively, the open unit ball and the unit hypersphere in $\mathbb{R}^d$, let U_d stand for the spherical[1] uniform distribution over $\mathbb{S}_d$. Let $\mathcal{P}_d$ be the class of Lebesgue-absolutely continuous distributions over $\mathbb{R}^d$. For any P in $\mathcal{P}_d$, the main result in McCann (1995) then implies the existence of a P-a.s. unique gradient $\nabla\phi$ of a convex (and lower semi-continuous) function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla\phi$ pushes P forward to U_d, i.e., $\nabla\phi(\mathbf{Z}) \sim \mathrm{U}_d$ under $\mathbf{Z} \sim \mathrm{P}$. Call *center-outward distribution function* of P any version $\mathbf{F}_\pm$ of this a.e. unique gradient.

Turning to sample versions, denote by $\mathbf{Z}^{(n)} := \big(\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}\big)$, $n \in \mathbb{N}$ a triangular array of i.i.d. d-dimensional random vectors with distribution P. The *empirical center-outward distribution function* $\mathbf{F}_\pm^{(n)}$, associated with $\mathbf{Z}^{(n)}$, maps the n-tuple $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}$ to a "regular" grid $\mathfrak{G}_n$ of the unit ball $\mathbb{S}_d$.

This regular grid $\mathfrak{G}_n$ is expected to approximate the spherical uniform distribution U_d over the unit ball $\mathbb{S}_d$. The only mathematical requirement needed for the asymptotic results below is the

[1] Namely, U_d is the spherical distribution with uniform (over $[0, 1]$) radial density—equivalently, the product of a uniform over the distances to the origin and a uniform over the unit sphere $\mathcal{S}_{d-1}$. For $d = 1$, U_1 coincides with the Lebesgue uniform over $(-1, 1)$.

weak convergence, as $n \to \infty$, of the uniform discrete distribution over $\mathfrak{G}_n$ to the spherical uniform distribution U_d. A spherical uniform i.i.d. sample of n points over $\mathbb{S}_d$ (almost surely) satisfies such a requirement. Since the spherical uniform is highly symmetric, further symmetries can be imposed on $\mathfrak{G}_n$, though, which only can improve the convergence to U_d. In the sequel, we throughout assume that the grids $\mathfrak{G}_n$ are symmetric with respect to the origin, i.e., $\mathbf{u} \in \mathfrak{G}_n$ implies $-\mathbf{u} \in \mathfrak{G}_n$, which considerably simplifies formulas. More symmetry, however, can be assumed: see Section 2.2 below. It should be insisted, however, that such symmetry assumptions do not restrict the generality of the results since the construction of the grid is entirely under control.

The empirical counterpart $\mathbf{F}_{\pm}^{(n)}$ of $\mathbf{F}_{\pm}$ is defined as the bijective mapping from $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}$ to the grid $\mathfrak{G}_n$ that minimizes $\sum_{i=1}^n \big\|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)}) - \mathbf{Z}_i^{(n)}\big\|^2$. That mapping is unique with probability one; in practice, it is obtained via a simple optimal assignment (pairing) algorithm—a linear program; see Shi et al. (2022a, Section 5.1) for a review and references therein. Call (rescaled) *center-outward rank* of $\mathbf{Z}_i^{(n)}$ the modulus

$$R_{i;\pm}^{(n)} \coloneqq \big\|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)})\big\|, \quad i = 1, \ldots, n \tag{2.1}$$

and *center-outward sign* of $\mathbf{Z}_i^{(n)}$ the unit vector

$$\mathbf{S}_{i;\pm}^{(n)} \coloneqq \mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)}) / \big\|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)})\big\| \quad \text{for } \mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)}) \neq \mathbf{0}; \tag{2.2}$$

put $\mathbf{S}_{i;\pm}^{(n)} = \mathbf{0}$ for $\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)}) = \mathbf{0}$.

2.2 Grid selection

The way ranks and signs in (2.1) and (2.2) are constructed, thus, depends on the way the grid $\mathfrak{G}_n$ is selected. As already mentioned, the only requirement for asymptotic results (as in Proposition 2.1 below) is the weak convergence to the spherical uniform U_d of the empirical distribution, $\mathrm{U}_{\mathfrak{G}}^{(n)}$, say, over $\mathfrak{G}_n$. The closer $\mathrm{U}_{\mathfrak{G}}^{(n)}$ to U_d, the better. Imposing on $\mathfrak{G}_n$ some of the many symmetries of U_d only can improve the finite-sample performance of $\mathrm{U}_{\mathfrak{G}}^{(n)}$ as an approximation of U_d.

Since U_d is the product of a uniform over the distances to the origin and a uniform over the unit sphere $\mathcal{S}_{d-1}$, it is natural to select $\mathfrak{G}_n$ such that $\mathrm{U}_{\mathfrak{G}}^{(n)}$ similarly factorizes. This can be obtained as follows:

(a) first factorize n into $n = n_R n_S + n_0$, with $0 \le n_0 < \min(n_R, n_S)$;[2]

(b) next consider a "regular array" $\mathfrak{S}_{n_S} \coloneqq \{\mathbf{s}_1^{n_S}, \ldots, \mathbf{s}_{n_S}^{n_S}\}$ of n_S points on the sphere $\mathcal{S}_{d-1}$ (see Remark 2.1 below);

(c) construct the grid $\mathfrak{G}_n$ consisting in the collection of the $n_R n_S$ gridpoints of the form

$$\big(r/(n_R+1)\big)\mathbf{s}_s^{n_S}, \quad r = 1, \ldots, n_R, \; s = 1, \ldots, n_S,$$

along with (n_0 copies of) the origin in case $n_0 \neq 0$: in total $n - (n_0 - 1)$ or n distinct points, thus, according as $n_0 > 0$ or $n_0 = 0$.

Rather than the rescaled ranks obtained via such a grid, one then may prefer, as in Hallin et al. (2021, 2022a, 2023), the "non-rescaled" ones $(n_R + 1)R_{i;\pm}^{(n)}$, taking values $1, \ldots, n_R$ (and 0, in case $n_0 \neq 0$).

[2]Note that this implies that $n_0/n = o(1)$ as $n \to \infty$. See Mordant (2021, Chapter 7.4) for a discussion of the selection of n_R and n_S.

Remark 2.1. By "regular" array $\mathfrak{S}_{n_S}$ over $\mathcal{S}_{d-1}$ in (b) above, we mean "as regular as possible" an array $\mathfrak{S}_{n_S}$—in the sense, for example, of the *low-discrepancy sequences* of the type considered in numerical integration, Monte-Carlo methods, and experimental design.[3] The asymptotic results in Proposition 2.1 below only require the weak convergence, as $n_S \to \infty$, of the uniform discrete distribution over $\mathfrak{S}_{n_S}$ to the uniform distribution over $\mathcal{S}_{d-1}$. A uniform i.i.d. sample of points over $\mathcal{S}_{d-1}$ (almost surely) satisfies that requirement. However, one can easily construct arrays that are "more regular" than an i.i.d. one. In particular, in order for the grid $\mathfrak{G}_n$ described in (a)–(c) above to be symmetric with respect to the origin, one could select an even value of n_S and see that $\mathbf{s}_s^{n_S} \in \mathfrak{S}_n$ implies $-\mathbf{s}_s^{n_S} \in \mathfrak{S}_{n_S}$, so that $\sum_{s=1}^{n_S} \mathbf{s}_s^{n_S} = \mathbf{0}$.

Remark 2.2. Some desirable finite-sample properties, such as strict independence between the ranks and the signs, hold with grids constructed as in (a)–(c) above provided, however, that $n_0 = 0$ or 1. This is due to the fact that the mapping from the sample to the grid is no longer injective for $n_0 \geq 2$. This fact, which has no asymptotic consequences (since the number n_0 of tied values involved is $o(n)$ as $n \to \infty$), is easily taken care of by performing the following tie-breaking device in step (c) of the construction of $\mathfrak{G}_n$:

(i) randomly select n_0 directions $\mathbf{s}_1^0, \ldots, \mathbf{s}_{n_0}^0$ in $\mathfrak{S}_{n_S}$, then

(ii) replace the n_0 copies of the origin with the new gridpoints

$$[1/2(n_R+1)]\mathbf{s}_1^0, \ldots, [1/2(n_R+1)]\mathbf{s}_{n_0}^0. \tag{2.3}$$

This new grid (for simplicity, the same notation $\mathfrak{G}_n$ is used as for the original one) no longer has multiple points: the optimal pairing between the sample and the grid is bijective and finite-sample independence between the resulting ranks and signs.

The selection of $\mathfrak{G}_n$ is to be decided by the statistician and, as long as $\mathrm{U}_{\mathfrak{G}}^{(n)}$ converges weakly to U_d, has no impact on the applicability, nor the optimality properties (which are of an asymptotic nature) of our tests. Below, we assume a symmetric $\mathfrak{G}_n$ ($\mathbf{u} \in \mathfrak{G}_n$ implies $-\mathbf{u} \in \mathfrak{G}_n$), which considerably simplifies formulas without restricting the generality of our results.

2.3 Main properties

This section summarizes some of the main properties of the concepts defined in Section 2.1; further properties and the proofs can be found in Hallin et al. (2021), Hallin et al. (2022a), and Hallin (2022). In the following propositions, the grids $\mathfrak{G}_n$ used in the construction of the empirical center-outward distribution function $\mathbf{F}_\pm^{(n)}$ are only required to satisfy the minimal assumption of an empirical distribution converging weakly to U_d.

Proposition 2.1. *Let $\mathbf{F}_\pm$ denote the center-outward distribution function of $\mathrm{P} \in \mathcal{P}_d$. Then,*

(i) *$\mathbf{F}_\pm$ is a probability integral transformation of $\mathbb{R}^d$, that is, $\mathbf{F}_\pm(\mathbf{Z}) \sim \mathrm{U}_d$ if and only if $\mathbf{Z} \sim \mathrm{P}$; under $\mathbf{Z} \sim \mathrm{P}$, $\|\mathbf{F}_\pm(\mathbf{Z})\|$ is uniform over $[0,1)$, $\mathbf{F}_\pm(\mathbf{Z})/\|\mathbf{F}_\pm(\mathbf{Z})\|$ is uniform over the sphere $\mathcal{S}_{d-1}$, and they are mutually independent.*

Let $\mathbf{Z}_1^{(n)}, \ldots, \mathbf{Z}_n^{(n)}$ be i.i.d. with distribution $\mathrm{P} \in \mathcal{P}_d$ and empirical center-outward distribution function $\mathbf{F}_\pm^{(n)}$. Then, if there are no ties in the grid $\mathfrak{G}_n$,

[3] See also Hallin and Mordant (2021) for a spherical version of the so-called Halton sequences.

(ii) $\big(\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_1^{(n)}), \ldots, \mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_n^{(n)})\big)$ *is uniformly distributed over the* $n!$ *permutations of* $\mathfrak{G}_n$*;*

(iii) the n*-tuple* $\big(\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_1^{(n)}), \ldots, \mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_n^{(n)})\big)$ *is* strongly essentially maximal ancillary*;*[4]

(iv) (pointwise convergence) for any $i = 1, \ldots, n$*,*

$$\left\|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)}) - \mathbf{F}_{\pm}(\mathbf{Z}_i^{(n)})\right\| \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

Assume, moreover, that P *is in the so-called class* $\mathcal{P}_d^+ \subset \mathcal{P}_d$ *of distributions with nonvanishing densities—namely, the class of distributions with density* $f := \mathrm{dP}/\mathrm{d}\mu_d$ *(*μ_d *the* d*-dimensional Lebesgue measure) such that, for all* $D \in \mathbb{R}^+$*, there exist constants* $\lambda^-_{D;\mathrm{P}}$ *and* $\lambda^+_{D;\mathrm{P}}$ *satisfying*

$$0 < \lambda^-_{D;\mathrm{P}} \leq f(\mathbf{z}) \leq \lambda^+_{D;\mathrm{P}} < \infty \tag{2.4}$$

for all $\mathbf{z}$ *with* $\|\mathbf{z}\| \leq D$*. Then,*

(v) there exists a version of $\mathbf{F}_{\pm}$ *defining a homeomorphism between the punctured unit ball* $\mathbb{S}_d \backslash \{\mathbf{0}\}$ *and* $\mathbb{R}^d \setminus \mathbf{F}_{\pm}^{-1}(\{\mathbf{0}\})$*; that version has a continuous inverse* $\mathbf{Q}_{\pm}$ *(with domain* $\mathbb{S}_d \backslash \{\mathbf{0}\}$*), which naturally qualifies as* P*'s center-outward quantile function;*

(vi) (Glivenko–Cantelli)

$$\max_{1\leq i\leq n}\left\|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_i^{(n)}) - \mathbf{F}_{\pm}(\mathbf{Z}_i^{(n)})\right\| \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

The results (v) and (vi), credited to Figalli (2018), Hallin (2017), del Barrio et al. (2018), Hallin et al. (2021), can be extended (del Barrio et al., 2020) to a more general[5] class $\mathcal{P}_d^{\text{conv}}$ of absolutely continuous distributions, with density f supported on a open convex set of $\mathbb{R}^d$ but not necessarily the whole space, while the definition of $\mathbf{F}_{\pm}$ given in Hallin et al. (2021) aims at selecting, for each $\mathrm{P} \in \mathcal{P}_d$, a version of $\nabla\phi$ which, whenever $\mathrm{P} \in \mathcal{P}_d^{\text{conv}}$, is yielding that homeomorphism. For the sake of simplicity, since we are not interested in quantiles, we stick here to the P-a.s. unique definition given above for $\mathrm{P} \in \mathcal{P}_d$.

Center-outward distribution functions, ranks, and signs also inherit, from the invariance of squared Euclidean distances, elementary but quite remarkable invariance and equivariance properties under shifts, orthogonal transformations, and global rescaling (see Hallin et al. (2022a)). Denote by $\mathbf{F}_{\pm}^{\mathbf{Z}}$ the center-outward distribution function of $\mathbf{Z}$ and by $\mathbf{F}_{\pm}^{\mathbf{Z};\mathfrak{G}_n}$ the empirical center-outward distribution function of an i.i.d. sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ associated with a grid $\mathfrak{G}_n$.

Proposition 2.2. *Let* $\boldsymbol{\mu} \in \mathbb{R}^d$*,* $k \in \mathbb{R}^+$*, and denote by* $\mathbf{O}$ *a* $d \times d$ *orthogonal matrix. Then,*

(i) $\mathbf{F}_{\pm}^{\boldsymbol{\mu}+k\mathbf{OZ}}(\boldsymbol{\mu} + \mathbf{Oz}) = \mathbf{OF}_{\pm}^{\mathbf{Z}}(\mathbf{z})$*,* $\mathbf{z} \in \mathbb{R}^d$*;*

(ii) denoting by $\mathbf{F}_{\pm}^{\boldsymbol{\mu}+k\mathbf{OZ};\mathbf{O}\mathfrak{G}_n}$ *the empirical center-outward distribution function computed from the sample* $\boldsymbol{\mu} + k\mathbf{OZ}_1, \ldots, \boldsymbol{\mu} + k\mathbf{OZ}_n$ *and the grid* $\mathbf{O}\mathfrak{G}_n := \{\mathbf{Ou} : \mathbf{u} \in \mathfrak{G}_n\}$*,*

$$\mathbf{F}_{\pm}^{\boldsymbol{\mu}+k\mathbf{OZ};\mathbf{O}\mathfrak{G}_n}(\boldsymbol{\mu} + k\mathbf{OZ}_i) = \mathbf{OF}_{\pm}^{\mathbf{Z};\mathfrak{G}_n}(\mathbf{Z}_i), \quad i = 1, \ldots, n. \tag{2.5}$$

Note that the orthogonal transformations in Proposition 2.2 include the permutations of $\mathbf{Z}$'s components. Invariance with respect to such permutations is an essential requirement for hypothesis testing in multivariate analysis.

[4] See Section 2.4 and Appendices D.1 and D.2 of Hallin et al. (2021) for a precise definition and a proof of this essential property.

[5] Namely, $\mathcal{P}_d^+ \subsetneq \mathcal{P}_d^{\text{conv}} \subsetneq \mathcal{P}_d$.

3 Rank-based tests for multivariate independence

3.1 Center-outward test statistics for multivariate independence

In this section, we describe the test statistics we are proposing for testing independence between two random vectors. Consider a sample

$$(\mathbf{X}'_{11}, \mathbf{X}'_{21})', (\mathbf{X}'_{12}, \mathbf{X}'_{22})', \ldots, (\mathbf{X}'_{1n}, \mathbf{X}'_{2n})'$$

of n i.i.d. copies of some $(d_1+d_2)=d$-dimensional random vector $(\mathbf{X}'_1, \mathbf{X}'_2)'$ with Lebesgue-absolutely continuous joint distribution $\mathrm{P} \in \mathcal{P}_d$ and density f. We are interested in the null hypothesis under which $\mathbf{X}_1$ and $\mathbf{X}_2$, with unspecified marginal distributions $\mathrm{P}_1 \in \mathcal{P}_{d_1}$ (density f_1) and $\mathrm{P}_2 \in \mathcal{P}_{d_2}$ (density f_2), respectively, are mutually independent: f then factorizes into $f = f_1 f_2$.

For $k = 1, 2$ and $i = 1, 2, \ldots, n$, denote by $R^{(n)}_{ki;\pm}$ and $\mathbf{S}^{(n)}_{ki;\pm}$ the center-outward rank and the sign of $\mathbf{X}_{ki}$ computed from $\mathbf{X}_{k1}, \mathbf{X}_{k2}, \ldots, \mathbf{X}_{kn}$ and the grid $\mathfrak{G}_n$. Recall that we throughout assume that if $\mathbf{u} \in \mathfrak{G}_n$ then $-\mathbf{u} \in \mathfrak{G}_n$. This implies that $\sum_{i=1}^n \mathbf{S}^{(n)}_{ki;\pm} = \mathbf{0}$ and $\sum_{i=1}^n J_k\big(R^{(n)}_{ki;\pm}\big)\mathbf{S}^{(n)}_{ki;\pm} = \mathbf{0}$ for any *score function* $J_k : [0,1) \to \mathbb{R}$, $k = 1, 2$.

Consider now the $d_1 \times d_2$ matrices

$$\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}} := \frac{1}{n} \sum_{i=1}^n \mathbf{S}^{(n)}_{1i;\pm} \mathbf{S}^{(n)\prime}_{2i;\pm}, \tag{3.1}$$

$$\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Sperman}} := \frac{1}{n(n_R+1)^2} \sum_{i=1}^n R^{(n)}_{1i;\pm} R^{(n)}_{2i;\pm} \mathbf{S}^{(n)}_{1i;\pm} \mathbf{S}^{(n)\prime}_{2i;\pm}, \tag{3.2}$$

$$\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}} := \binom{n}{2}^{-1} \sum_{i<i'} \mathrm{sign}\Big[\Big(R^{(n)}_{1i;\pm}\mathbf{S}^{(n)}_{1i;\pm} - R^{(n)}_{1i';\pm}\mathbf{S}^{(n)}_{1i';\pm}\Big) \times \Big(R^{(n)}_{2i;\pm}\mathbf{S}^{(n)}_{2i;\pm} - R^{(n)}_{2i';\pm}\mathbf{S}^{(n)}_{2i';\pm}\Big)'\Big], \tag{3.3}$$

where $\mathrm{sign}\big[\mathbf{M}\big]$ stands for the matrix collecting the signs of the entries of a real matrix $\mathbf{M}$. More generally, let

$$\underset{\sim}{\mathbf{W}}^{(n)}_{J} := \frac{1}{n} \sum_{i=1}^n J_1\Big(R^{(n)}_{1i;\pm}\Big) J_2\Big(R^{(n)}_{2i;\pm}\Big) \mathbf{S}^{(n)}_{1i;\pm} \mathbf{S}^{(n)\prime}_{2i;\pm}, \tag{3.4}$$

where the *score functions* $J_k : [0,1) \to \mathbb{R}$, $k = 1, 2$ are square-integrable differences of two monotone increasing functions, with

$$0 < \sigma^2_{J_k} := \int_0^1 J_k^2(u)\mathrm{d}u < \infty. \tag{3.5}$$

This assumption on score functions will be made throughout the remainder of the paper. The matrices defined in (3.1)–(3.4) clearly constitute matrices of cross-covariance measurements based on center-outward ranks and signs (for $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, signs only). For $d_1 = 1 = d_2$, it is easily seen that $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Spearman}}$, and $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}}$, up to scaling constants, reduce to the classical quadrant, Spearman, and Kendall test statistics, while $\underset{\sim}{\mathbf{W}}^{(n)}_{J}$ yields a matrix-valued score-based extension of Spearman's correlation coefficient.

3.2 Asymptotic representation and asymptotic normality

Evidently, neither the ranks nor the signs are mutually independent: $n-1$ rank and sign pairs indeed determine the last pair. However, as we show now, each of the rank-based matrices defined in (3.1)–(3.4) has an asymptotic representation in terms of i.i.d. variables. More precisely, defining $\mathbf{S}_{ki;\pm}$ as $\mathbf{F}_{k;\pm}(\mathbf{X}_{ki})/\big\|\mathbf{F}_{k;\pm}(\mathbf{X}_{ki})\big\|$ if $\mathbf{F}_{k;\pm}(\mathbf{X}_{ki}) \neq \mathbf{0}$ and $\mathbf{0}$ otherwise for $k=1,2$, let

$$\mathbf{W}^{(n)}_{\text{sign}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{S}_{1i;\pm}\mathbf{S}'_{2i;\pm}, \tag{3.6}$$

$$\mathbf{W}^{(n)}_{\text{Spearman}} := \frac{1}{n}\sum_{i=1}^{n} \mathbf{F}_{1;\pm}(\mathbf{X}_{1i})\mathbf{F}'_{2;\pm}(\mathbf{X}_{2i}), \tag{3.7}$$

$$\mathbf{W}^{(n)}_{\text{Kendall}} := \binom{n}{2}^{-1}\sum_{i<i'} \text{sign}\Big[\Big(\mathbf{F}_{1;\pm}(\mathbf{X}_{1i}) - \mathbf{F}_{1;\pm}(\mathbf{X}_{1i'})\Big) \times \Big(\mathbf{F}_{2;\pm}(\mathbf{X}_{2i}) - \mathbf{F}_{2;\pm}(\mathbf{X}_{2i'})\Big)'\Big], \tag{3.8}$$

and

$$\mathbf{W}^{(n)}_{J} := \frac{1}{n}\sum_{i=1}^{n} J_1\Big(\big\|\mathbf{F}_{1;\pm}(\mathbf{X}_{1i})\big\|\Big) J_2\Big(\big\|\mathbf{F}_{2;\pm}(\mathbf{X}_{2i})\big\|\Big)\mathbf{S}_{1i;\pm}\mathbf{S}'_{2i;\pm}. \tag{3.9}$$

The following asymptotic representation results then hold under the null hypothesis of independence (hence, also under contiguous alternatives).

Lemma 3.1. *Under the null hypothesis of independence, as both n_R and n_S tend to infinity,* $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}} - \mathbf{W}^{(n)}_{\text{sign}}\big)$, $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Spearman}} - \mathbf{W}^{(n)}_{\text{Spearman}}\big)$, $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}} - \mathbf{W}^{(n)}_{\text{Kendall}}\big)$, *and* $\text{vec}\big(\underset{\sim}{\mathbf{W}}^{(n)}_{J} - \mathbf{W}^{(n)}_{J}\big)$ *are $o_{\text{q.m.}}(n^{-1/2})$. Here* $\text{vec}(\mathbf{A})$ *denotes* $(\mathbf{a}_1^\top, \dots, \mathbf{a}_m^\top)$ *for an $n \times m$ matrix* $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)$, *and* $X_n = o_{\text{q.m.}}(R_n)$ *means* $X_n = Y_n R_n$ *and* $\lim \text{E}[|Y_n|^2] \to 0$.

The asymptotic normality for $\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}})$, $\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Spearman}})$, $\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}})$, and $\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{J})$ follows immediately from the asymptotic representation results and the standard central-limit behavior of $\text{vec}(\mathbf{W}^{(n)}_{\text{sign}})$, $\text{vec}(\mathbf{W}^{(n)}_{\text{Spearman}})$, $\text{vec}(\mathbf{W}^{(n)}_{\text{Kendall}})$, and $\text{vec}(\mathbf{W}^{(n)}_{J})$.

Proposition 3.1. *Under the null hypothesis of independence, as both n_R and n_S tend to infinity,* $n^{1/2}\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}})$, $n^{1/2}\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Spearman}})$, $n^{1/2}\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}})$, *and* $n^{1/2}\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{J})$ *are asymptotically normal with mean vectors $\mathbf{0}_{d_1d_2}$ and covariance matrices*

$$\frac{1}{d_1d_2}\mathbf{I}_{d_1d_2}, \quad \frac{1}{9d_1d_2}\mathbf{I}_{d_1d_2}, \quad \frac{4}{9}\mathbf{I}_{d_1d_2}, \quad \textit{and} \quad \frac{\sigma^2_{J_1}\sigma^2_{J_2}}{d_1d_2}\mathbf{I}_{d_1d_2},$$

respectively.

3.3 Center-outward sign, Spearman, Kendall, and score tests

Associated with $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}$, $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Spearman}}$, $\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}}$, and $\underset{\sim}{\mathbf{W}}^{(n)}_{J}$ are the sign, Spearman, Kendall, and score test statistics

$$\underset{\sim}{T}^{(n)}_{\text{sign}} := nd_1d_2\big\|\underset{\sim}{\mathbf{W}}^{(n)}_{\text{sign}}\big\|^2_{\text{F}}, \qquad \underset{\sim}{T}^{(n)}_{\text{Spearman}} := 9nd_1d_2\big\|\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Spearman}}\big\|^2_{\text{F}},$$

$$\underset{\sim}{T}^{(n)}_{\text{Kendall}} := \frac{9n}{4} \|\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}}\|^2_{\text{F}}, \qquad \text{and} \qquad \underset{\sim}{T}^{(n)}_{J} := \frac{nd_1d_2}{\sigma^2_{J_1}\sigma^2_{J_2}} \|\underset{\sim}{\mathbf{W}}^{(n)}_{J}\|^2_{\text{F}},$$

respectively. Here, $\|\mathbf{M}\|_{\text{F}}$ is the Frobenius norm of a matrix $\mathbf{M}$, and $\sigma^2_{J_1}$, $\sigma^2_{J_2}$ are defined as in (3.5).

In view of the asymptotic normality results in Proposition 3.1, the tests (denoted respectively by $\psi^{(n)}_{\text{sign}}$, $\psi^{(n)}_{\text{Spearman}}$, $\psi^{(n)}_{\text{Kendall}}$, and $\psi^{(n)}_{J}$) rejecting the null hypothesis of independence whenever $\underset{\sim}{T}^{(n)}_{\text{sign}}$, $\underset{\sim}{T}^{(n)}_{\text{Spearman}}$, $\underset{\sim}{T}^{(n)}_{\text{Kendall}}$, or $\underset{\sim}{T}^{(n)}_{J}$ exceed the $(1-\alpha)$-quantile $\chi^2_{d_1d_2;1-\alpha}$ of a chi-square distribution with d_1d_2 degrees of freedom have asymptotic level α. These tests are, however, strictly distribution-free, and exact critical values can be computed or simulated as well. The tests based on $\underset{\sim}{T}^{(n)}_{\text{sign}}$, $\underset{\sim}{T}^{(n)}_{\text{Spearman}}$, and $\underset{\sim}{T}^{(n)}_{\text{Kendall}}$ are multivariate extensions of the traditional quadrant, Spearman, and Kendall tests, respectively, to which they reduce for $d_1 = 1 = d_2$.

4 Local asymptotic power

While there is only one way for two random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ to be independent, their mutual dependence can take many forms. The classical benchmark, in testing for bivariate independence, is a "local" form of an independent component analysis model that goes back to Konijn (1956). A multivariate extension of such alternatives has been considered also by Gieser and Randles (1997), Taskinen et al. (2003) and Hallin and Paindaveine (2008) in the elliptical context. We extend it further to more general, non-elliptical situations.

4.1 Generalized Konijn alternatives

Let $\mathbf{X}^* = (\mathbf{X}^{*\prime}_1, \mathbf{X}^{*\prime}_2)'$, where $\mathbf{X}^*_1$ and $\mathbf{X}^*_2$ are mutually independent random vectors, with absolutely continuous distributions P_1 over $\mathbb{R}^{d_1}$ and P_2 over $\mathbb{R}^{d_2}$ and densities f_1 and f_2, respectively. Then $\mathbf{X}^*$ has density $f = f_1f_2$ over $\mathbb{R}^d$. Consider

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} := \mathbf{M}_\delta \begin{pmatrix} \mathbf{X}^*_1 \\ \mathbf{X}^*_2 \end{pmatrix} := \begin{pmatrix} (1-\delta)\mathbf{I}_{d_1} & \delta\mathbf{M_1} \\ \delta\mathbf{M_2} & (1-\delta)\mathbf{I}_{d_2} \end{pmatrix} \begin{pmatrix} \mathbf{X}^*_1 \\ \mathbf{X}^*_2 \end{pmatrix} \tag{4.1}$$

where $\delta \in \mathbb{R}$ and $\mathbf{M}_1 \in \mathbb{R}^{d_1\times d_2}$, $\mathbf{M}_2 \in \mathbb{R}^{d_2\times d_1}$ are nonzero. For given P_1, P_2, $\mathbf{M}_1$, and $\mathbf{M}_2$, the distribution $\text{P}^{\mathbf{X}}$ of $\mathbf{X}$ belongs to a one-parameter family $\mathcal{P}^{\mathbf{X}} := \{\text{P}^{\mathbf{X}}_\delta \,|\, \delta \in \mathbb{R}\}$.

On f_1 and f_2, we make the following assumption (by $\mathbf{\Sigma}_k > 0$ we mean $\mathbf{\Sigma}_k$ positive definite).

Assumption 4.1.

(K1) The densities f_1 and f_2 are such that

$$\int_{\mathbb{R}^{d_k}} \mathbf{x} f_k(\mathbf{x}) \mathrm{d}\mathbf{x} = \mathbf{0} \quad \textit{and} \quad 0 < \int_{\mathbb{R}^{d_k}} \mathbf{x}\mathbf{x}' f_k(\mathbf{x}) \mathrm{d}\mathbf{x} =: \mathbf{\Sigma}_k < \infty, \quad k = 1, 2.$$

(K2) The functions $\mathbf{x}_k \mapsto (f_k(\mathbf{x}_k))^{1/2}$, $k = 1, 2$ admit quadratic mean partial derivatives[6]

$$D_\ell[(f_k)^{1/2}], \quad \ell = 1, \ldots, d_k, \; k = 1, 2.$$

(K3) Letting

$$\boldsymbol{\varphi} := (\boldsymbol{\varphi}'_1, \boldsymbol{\varphi}'_2)' := (\varphi_{1;1}, \ldots, \varphi_{1;d_1}, \varphi_{2;1}, \ldots, \varphi_{2;d_1})'$$

[6]Existence of quadratic mean partial derivatives is equivalent to quadratic mean differentiability; this was shown in Lind and Roussas (1972) and independently rediscovered by Garel and Hallin (1995, Lemma 2.1).

with

$$\varphi_{k;\ell} := -2D_\ell[(f_k)^{1/2}]/(f_k)^{1/2} \overset{a.e.}{=} -\partial_\ell f_k/f_k, \quad \ell = 1, \ldots, d_k, \ k = 1, 2,$$

it holds that, for $k = 1, 2$ and $\ell = 1, \ldots, d_k$, $0 < \int_{\mathbb{R}^{d_k}} \big(\varphi_{k;\ell}(\mathbf{x})\big)^2 < \infty$, and [7]

$$\mathcal{J}_k := \mathrm{Var}\left(\mathbf{X}_k^{*\prime}\boldsymbol{\varphi}_k(\mathbf{X}_k^*)\right) = \int_{\mathbb{R}^{d_k}} \left(\mathbf{x}'\boldsymbol{\varphi}_k(\mathbf{x}) - d_k\right)^2 f_k(\mathbf{x})\mathrm{d}\mathbf{x} < \infty.$$

It should be stressed that these assumptions are not to be imposed on the observations in order for our tests to be valid but only intend to provide an analytically convenient benchmark for the comparison of local power.

Under $\mathrm{P}_0^{\mathbf{X}}$, $\mathbf{X}_1 = \mathbf{X}_1^*$ and $\mathbf{X}_2 = \mathbf{X}_2^*$ are mutually independent; for $\delta \neq 0$, call $\mathrm{P}_\delta^{\mathbf{X}}$ a (generalized) *Konijn alternative* to $\mathrm{P}_0^{\mathbf{X}}$. Sequences of the form $\mathrm{P}_{n^{-1/2}\tau}^{\mathbf{X}}$ with $\tau \neq 0$, as we shall see, constitute local alternatives to the null hypothesis of independence in a sample of size n. More precisely, the following local asymptotic normality (LAN) property holds in the vicinity of $\delta = 0$. Let

$$\boldsymbol{\mathcal{I}}_k := \int_{\mathbb{R}^{d_k}} \boldsymbol{\varphi}(\mathbf{x})\boldsymbol{\varphi}'(\mathbf{x}) f_k(\mathbf{x})\mathrm{d}\mathbf{x} < \infty.$$

Lemma 4.1. *Let P_1 and P_2 satisfy Assumption 4.1. Then, denoting by $\mathbf{X}^{(n)} := (\mathbf{X}_1, \ldots, \mathbf{X}_n)$, $n \in \mathbb{N}$ a triangular array of n independent copies of $\mathbf{X} \sim \mathrm{P}_0^{\mathbf{X}}$, for given nonzero $\mathbf{M}_1$ and $\mathbf{M}_2$, the family $\mathcal{P}_{\mathbf{X}}$ of Konijn alternatives is LAN at $\delta = 0$ with root-n contiguity rate, central sequence*

$$\Delta^{(n)}(\mathbf{X}^{(n)}) := n^{-1/2} \sum_{i=1}^{n} \Big[\mathbf{X}_{1i}'\mathbf{M}_2'\boldsymbol{\varphi}_2(\mathbf{X}_{2i}) + \mathbf{X}_{2i}'\mathbf{M}_1'\boldsymbol{\varphi}_1(\mathbf{X}_{1i}) - \Big(\mathbf{X}_{1i}'\boldsymbol{\varphi}_1(\mathbf{X}_{1i}) - d_1\Big) - \Big(\mathbf{X}_{2i}'\boldsymbol{\varphi}_2(\mathbf{X}_{2i}) - d_2\Big)\Big] \tag{4.2}$$

and Fisher information

$$\gamma^2 := \mathcal{J}_1 + \mathcal{J}_2 + \mathrm{vec}'(\boldsymbol{\Sigma}_1)\,\mathrm{vec}\big(\mathbf{M}_2'\boldsymbol{\mathcal{I}}_2\mathbf{M}_2\big) + \mathrm{vec}'(\boldsymbol{\Sigma}_2)\,\mathrm{vec}\big(\mathbf{M}_1'\boldsymbol{\mathcal{I}}_1\mathbf{M}_1\big) + \mathrm{tr}(\mathbf{M}_1\mathbf{M}_2) + \mathrm{tr}(\mathbf{M}_2\mathbf{M}_1). \tag{4.3}$$

In other words, under $\mathrm{P}_0^{\mathbf{X}}$,

$$\Lambda^{(n)}(\mathbf{X}^{(n)}) := \log \frac{\mathrm{dP}_{n^{-1/2}\tau}^{\mathbf{X}}}{\mathrm{dP}_0^{\mathbf{X}}}(\mathbf{X}^{(n)}) = \tau\Delta^{(n)}(\mathbf{X}^{(n)}) - \frac{1}{2}\tau^2\gamma^2 + o_{\mathrm{P}}(1) \tag{4.4}$$

and $\Delta^{(n)}(\mathbf{X}^{(n)})$ is asymptotically normal, with mean zero and variance γ^2 as $n \to \infty$.

4.2 Limiting distributions and Pitman efficiencies

For the univariate two-sample location problem, Chernoff and Savage (1958) and Hodges and Lehmann (1956) established celebrated lower bounds for the asymptotic relative efficiency (ARE) of traditional normal-score (van der Waerden) and Wilcoxon rank tests with respect to the Gaussian procedure (the two-sample Student test). These results were extended by Hallin and Paindaveine (2002b) to Mahalanobis ranks-and-signs-based location tests under multivariate elliptical distributions, by Hallin and Paindaveine (2002a) to VARMA time-series models with elliptical innovations,

[7]Integration by parts yields $\int_{\mathbb{R}^{d_k}} \boldsymbol{\varphi}_k(\mathbf{x}) f_k(\mathbf{x})\mathrm{d}\mathbf{x} = \mathbf{0}$, $\int_{\mathbb{R}^{d_k}} \mathbf{x}'\boldsymbol{\varphi}_k(\mathbf{x}) f_k(\mathbf{x})\mathrm{d}\mathbf{x} = d_k$, and $\int_{\mathbb{R}^{d_k}} \mathbf{x}\boldsymbol{\varphi}_k(\mathbf{x})' f_k(\mathbf{x})\mathrm{d}\mathbf{x} = \mathbf{I}_{d_k}$, $k = 1, 2$; see also Garel and Hallin (1995, page 555).

by Paindaveine (2004) for the shape parameter of elliptical distributions. Elliptical Chernoff–Savage and Hodges–Lehmann bounds for the AREs (with respect to Hotelling) of measure-transportation-based center-outward rank and sign tests were first obtained by Deb et al. (2021) in the context of two-sample location problems. In this section, we aim at establishing elliptical Chernoff–Savage and Hodges–Lehmann results for the AREs of our center-outward rank tests based on van der Waerden and Wilcoxon scores, respectively, with respect to Wilks' test.

To this end, we first derive the limiting distributions of $\underset{\sim}{T}_J^{(n)}$ and $\underset{\sim}{T}_{\text{Kendall}}^{(n)}$ under the sequence of local Konijn alternatives $\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}$.

Theorem 4.1. *Let* P_1 *and* P_2 *satisfy Assumption 4.1. If the observations are* n *independent copies with distribution* $\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}$, *for given nonzero* $\mathbf{M}_1$ *and* $\mathbf{M}_2$, *then*

(i) the limiting distribution of the test statistic $\underset{\sim}{T}_J^{(n)}$ *is noncentral chi-square with* $d_1 d_2$ *degrees of freedom and noncentrality parameter*

$$\frac{\tau^2 d_1 d_2}{\sigma^2_{J_1}\sigma^2_{J_2}}\Big\|\mathrm{E}_{H_0}\Big[\mathbf{J}_1(\mathbf{F}_{1;\pm}(\mathbf{X}_1))\mathbf{R}\mathbf{J}_2(\mathbf{F}_{2;\pm}(\mathbf{X}_2))'\Big]\Big\|^2_{\mathrm{F}},$$

where $\mathbf{R} := \mathbf{X}_1'\mathbf{M}_2'\boldsymbol{\varphi}_2(\mathbf{X}_2) + \mathbf{X}_2'\mathbf{M}_1'\boldsymbol{\varphi}_1(\mathbf{X}_1)$ *and*

$$\mathbf{J}_k(\mathbf{u}) := J_k(\|\mathbf{u}\|)\frac{\mathbf{u}}{\|\mathbf{u}\|}\mathbf{1}_{[\|\mathbf{u}\|\neq 0]}, \quad \mathbf{u}\in\mathbb{S}_d;$$

(ii) the limiting distribution of the Kendall test statistic $\underset{\sim}{T}_{\text{Kendall}}^{(n)}$ *is noncentral chi-square with* $d_1 d_2$ *degrees of freedom and noncentrality parameter*

$$9\tau^2\Big\|\mathrm{E}_{H_0}\Big[\mathbf{F}^{\square}_{1;\pm}(\mathbf{X}_1)\mathbf{R}\mathbf{F}^{\square}_{2;\pm}(\mathbf{X}_2)'\Big]\Big\|^2_{\mathrm{F}},$$

where, denoting by F_{kj} *the cumulative distribution function of* $\big(\mathbf{F}_{k;\pm}(\mathbf{X}_k)\big)_j$,

$$\big(\mathbf{F}^{\square}_{k;\pm}(\mathbf{X}_k)\big)_j := 2F_{kj}\Big(\big(\mathbf{F}_{k;\pm}(\mathbf{X}_k)\big)_j\Big) - 1, \qquad j=1,\ldots,d, \quad k=1,2.$$

Wilks' (log) likelihood ratio test statistic is $T^{(n)}_{\text{Wilks}} := n\log V^{(n)}$ with

$$V^{(n)} := \frac{\det(\mathbf{S}_1^{(n)})\det(\mathbf{S}_2^{(n)})}{\det(\mathbf{S}^{(n)})},$$

where $\det(\cdot)$ is the determinant of a matrix, $\mathbf{S}_k^{(n)}$ is the sample covariance matrix of $\mathbf{X}_{k1},\ldots,\mathbf{X}_{kn}$, $k=1,2$, and $\mathbf{S}^{(n)}$ is the sample covariance matrix of $(\mathbf{X}_{11}',\mathbf{X}_{21}')',\ldots,(\mathbf{X}_{1n}',\mathbf{X}_{2n}')'$. Under $\mathrm{P}^{\mathbf{X}}_0$ (the null hypothesis of independence), $T^{(n)}_{\text{Wilks}}$ is asymptoticaly chi-square with $d_1 d_2$ degrees of freedom. Wilk's tests $\psi^{(n)}_{\text{Wilks}}$ rejects the null hypothesis at whenever $T^{(n)}_{\text{Wilks}}$ exceeds the chi-square quantile of order $(1-\alpha)$ and has asymptotic size α. Suppose that the assumptions in Theorem 4.1 hold. Then the limiting distribution of Wilks' statistic under $\mathrm{P}^{\mathbf{X}}_{n^{-1/2}\tau}$ is noncentral chi-square, with $d_1 d_2$ degrees of freedom and noncentrality parameter

$$\tau^2\Big\|\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\Sigma}_2^{-1/2} + \boldsymbol{\Sigma}_1^{-1/2}\mathbf{M}_1\boldsymbol{\Sigma}_2^{1/2}\Big\|^2_{\mathrm{F}};$$

see, e.g., page 919 of Taskinen et al. (2005).

Now we are ready to compute the asymptotic relative efficiencies, within the family $\mathcal{P}^{\mathbf{X}}$ of Konijn alternatives characterized by elliptical P_1 and P_2 (with a slight abuse of language, call it an *elliptical* Konijn family), of our center-outward rank tests with respect to Wilks' likelihood ratio test.

Proposition 4.1. *Let* P_1 *and* P_2 *be elliptically symmetric distributions, that is, have densities of the form*

$$f_k(\mathbf{x}_k) \propto (\det(\mathbf{\Sigma}_k))^{-1/2}\phi_k\Big(\sqrt{\mathbf{x}_k'\mathbf{\Sigma}_k^{-1}\mathbf{x}_k}\Big), \quad k = 1, 2,$$

satisfying Assumption 4.1. Then, the Pitman asymptotic relative efficiency (ARE), within the elliptical family $\mathcal{P}^{\mathbf{X}}$ *of Konijn alternatives, of the center-outward test* $\psi_J^{(n)}$ *based on the score functions* J_k, $k = 1, 2$ *with respect to Wilks' test* $\psi_{\mathrm{Wilks}}^{(n)}$ *is*

$$\mathrm{ARE}(\psi_J^{(n)}, \psi_{\mathrm{Wilks}}^{(n)}) = \frac{\Big\|D_1C_2\mathbf{\Sigma}_1^{1/2}\mathbf{M}_2'\mathbf{\Sigma}_2^{-1/2} + D_2C_1\mathbf{\Sigma}_1^{-1/2}\mathbf{M}_1\mathbf{\Sigma}_2^{1/2}\Big\|_{\mathrm{F}}^2}{d_1d_2\sigma_{J_1}^2\sigma_{J_2}^2\Big\|\mathbf{\Sigma}_1^{1/2}\mathbf{M}_2'\mathbf{\Sigma}_2^{-1/2} + \mathbf{\Sigma}_1^{-1/2}\mathbf{M}_1\mathbf{\Sigma}_2^{1/2}\Big\|_{\mathrm{F}}^2}, \tag{4.5}$$

where

$$C_k \equiv C_k(J_k, \phi_k) := \mathrm{E}[J_k^{-1}(U)\rho_k(\tilde{F}_k^{-1}(U))],$$
$$D_k \equiv D_k(J_k, \phi_k) := \mathrm{E}[J_k^{-1}(U)\tilde{F}_k^{-1}(U))],$$

$\rho_k := -\phi_k'/\phi_k$, $\tilde{F}_k$ *denotes the cumulative distribution function of* $\|\mathbf{Y}_k\|$ *with* $\mathbf{Y}_k := \mathbf{\Sigma}_k^{-1/2}\mathbf{X}_k$, *and* U *is a random variable uniformly distributed over* $(0, 1)$. *In particular, if* $\mathbf{\Sigma}_1\mathbf{M}_2' = \mathbf{M}_1\mathbf{\Sigma}_2$, *we have*

(i) $\mathrm{ARE}(\psi_{J^{\mathrm{vdW}}}^{(n)}, \psi_{\mathrm{Wilks}}^{(n)}) \geq 1$, *where* J_k^{vdW}, $k = 1, 2$ *are the van der Waerden or Gaussian score functions* $J_k^{\mathrm{vdW}}(u) := \big(F_{\chi^2_{d_k}}^{-1}(u)\big)^{1/2}$ *with* $F_{\chi^2_d}$ *the* χ^2_d *cumulative distribution function;*

(ii) $\mathrm{ARE}(\psi_{J^{\mathrm{W}}}^{(n)}, \psi_{\mathrm{Wilks}}^{(n)}) \geq \Omega(d_1, d_2) \geq 9/16$, *where* J_k^{W}, $k = 1, 2$ *are the Wilcoxon score functions defined as* $J_k^{\mathrm{W}}(u) := u$

$$\Omega(d_1, d_2) := \frac{9(2c_{d_1}^2 + d_1 - 1)^2(2c_{d_2}^2 + d_2 - 1)^2}{1024d_1d_2c_{d_1}^2c_{d_2}^2}, \tag{4.6}$$
$$c_d := \inf\Big\{x > 0 \;\Big|\; \Big(\sqrt{x}B_{\sqrt{2d-1}/2}(x)\Big)' = 0\Big\},$$
$$B_a(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + a + 1)}\Big(\frac{x}{2}\Big)^{2m+a}.$$

Note that the ARE values in (4.5) depend on the underlying covariance structure ($\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$) of $\mathbf{X}_1$ and $\mathbf{X}_2$; a similar fact was already pointed out by Gieser (1993). Most the existing literature (e.g. Gieser (1993), Gieser and Randles (1997), Taskinen et al. (2003, 2004), Taskinen et al. (2005), Hallin and Paindaveine (2008) and Deb et al. (2021)) therefore focuses on the spherically symmetric case ($\mathbf{\Sigma}_1 = \mathbf{I}_{d_1}$, $\mathbf{\Sigma}_2 = \mathbf{I}_{d_2}$). We are filling this gap by providing the explicit ARE value for general $\mathbf{\Sigma}_k$'s. The claim (i) shows the Pitman non-admissibility against elliptic Konijn alternatives of Wilks' test, which is uniformly dominated by our center-outward test with van der Waerden scores. This is comparable with Theorem 4.1 in Deb et al. (2021). Claim (ii) is a multivariate extension of Hodges and Lehmann (1956)'s result; the infimum of $\Omega(d_1, d_2)$, $9/16 \approx 0.562$, is achieved for $d_1, d_2 \to \infty$. Table 3 gives numerical values of $\Omega(d_1, d_2)$ for fixed $d_1, d_2 \leq 10$.

Table 3: Some numerical values of the lower bound (4.6) for the Pitman asymptotic relative efficiency, against elliptical Konijn alternatives, of the center-outward Wilcoxon score test with respect to Wilks', for dimensions $d_1, d_2 \leq 10$.

d_1 \ d_2	1	2	3	4	5	6	7	8	9	10
1	0.856	0.884	0.867	0.850	0.837	0.826	0.817	0.809	0.803	0.797
2	0.884	0.913	0.895	0.878	0.864	0.853	0.844	0.836	0.829	0.823
3	0.867	0.895	0.878	0.861	0.847	0.836	0.827	0.819	0.813	0.807
4	0.850	0.878	0.861	0.845	0.831	0.820	0.811	0.804	0.797	0.792
5	0.837	0.864	0.847	0.831	0.818	0.807	0.799	0.791	0.785	0.779
6	0.826	0.853	0.836	0.820	0.807	0.797	0.788	0.781	0.775	0.769
7	0.817	0.844	0.827	0.811	0.799	0.788	0.779	0.772	0.766	0.761
8	0.809	0.836	0.819	0.804	0.791	0.781	0.772	0.765	0.759	0.754
9	0.803	0.829	0.813	0.797	0.785	0.775	0.766	0.759	0.753	0.748
10	0.797	0.823	0.807	0.792	0.779	0.769	0.761	0.754	0.748	0.742

5 Numerical experiments

We now present numerical experiments that aim to illustrate the theory presented and to study the performance of the proposed tests outside the setting of elliptical distributions. We compare the empirical performance of the following tests:

(i) the center-outward sign test $\psi^{(n)}_{\text{sign}}$,

(ii) the center-outward Spearman test $\psi^{(n)}_{\text{Spearman}}$,

(iii) the center-outward Kendall test $\psi^{(n)}_{\text{Kendall}}$,

(iv) the center-outward van der Waerden score test $\psi^{(n)}_{J\text{vdW}}$,

(v) the center-outward Wilcoxon score version of the distance covariance test (Shi et al., 2022a),

(vi) the center-outward van der Waerden score version of the distance covariance test (Shi et al., 2022b), and

(vii) Wilks' likelihood ratio test in the Gaussian model $\psi^{(n)}_{\text{Wilks}}$.

5.1 Simulation results

Example 5.1. The data are generated as a sample of n independent copies of the (d_1+d_2)-dimensional random vector $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2')'$ under model (4.1) with $d_1 = d_2 = d_0$, $\delta = \delta^{(n)} = n^{-1/2}\tau$, and $\mathbf{M}_1 = \mathbf{M}_2' = \mathbf{I}_{d_0}$. The following null distributions were considered.

Gaussian and nearly Gaussian:

(a) $\mathbf{X}_1^*, \mathbf{X}_2^* \overset{\text{i.i.d.}}{\sim} N(\mathbf{0}, \mathbf{I}_{d_0})$, standard Gaussian in dimension d_0;

(b) $\mathbf{X}_1^*, \mathbf{X}_2^* \overset{\text{i.i.d.}}{\sim} 0.5N(\mathbf{0}, \mathbf{I}_{d_0}) + 0.5N(\mathbf{1}, \mathbf{I}_{d_0})$ (a non-elliptical multiivariate mixture of Gaussians).

Heavy-tailed:

(c) $\mathbf{X}_1^*, \mathbf{X}_2^* \overset{\text{i.i.d.}}{\sim} t_3(\mathbf{0}, \mathbf{I}_{d_0})$, where $t_\nu(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} \in \mathbb{R}^p$, $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$ denotes the distribution with density

$$\frac{\Gamma\left[(\nu+p)/2\right]}{\Gamma(\nu/2)\nu^{p/2}\pi^{p/2}\left|\boldsymbol{\Sigma}\right|^{1/2}}\left[1+\frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-(\nu+p)/2}$$

(an elliptical distribution with finite second-order moments);

(d) $\mathbf{X}_1^*, \mathbf{X}_2^* \overset{\text{i.i.d.}}{\sim} t_2(\mathbf{0}, \mathbf{I}_{d_0})$ (an elliptical distribution with infinite second-order moments);

(e) $\mathbf{X}_1^*, \mathbf{X}_2^* \overset{\text{i.i.d.}}{\sim} t_1(\mathbf{0}, \mathbf{I}_{d_0})$ (an elliptical distribution with infinite second-order moments);

(f) $\mathbf{X}_{1i}^*, \mathbf{X}_{2i}^* \overset{\text{i.i.d.}}{\sim} t_3(0,1)$ for $i = 1, \dots, d_0$ (a heavy-tailed non-elliptical distribution invariant under coordinate permutations);

(g) $\mathbf{X}_{1i}^*, \mathbf{X}_{2i}^* \overset{\text{i.i.d.}}{\sim} t_2(0,1)$ for $i = 1, \dots, d_0$ (a heavy-tailed non-elliptical distribution invariant under coordinate permutations);

(h) $\mathbf{X}_{1i}^*, \mathbf{X}_{2i}^* \overset{\text{i.i.d.}}{\sim} t_1(0,1)$ for $i = 1, \dots, d_0$ (a heavy-tailed non-elliptical distribution invariant under coordinate permutations).

Skewed:

(i) $\mathbf{X}_{1i}^*, \mathbf{X}_{2i}^* \overset{\text{i.i.d.}}{\sim} \chi_2^2$ for $i = 1, \dots, d_0$ (a skewed non-elliptical distribution);

(j) $\mathbf{X}_{1i}^*, \mathbf{X}_{2i}^* \overset{\text{i.i.d.}}{\sim} \chi_1^2$ for $i = 1, \dots, d_0$ (a skewed non-elliptical distribution).

Tables 4–13 report empirical powers (rejection frequencies) of the seven tests, based on $1,000$ simulations with nominal significance level $\alpha = 0.05$, dimensions $d_1 = d_2 \in \{2, 3, 5, 7\}$, and sample size $n \in \{432, 864, 1728\}$. The parameter τ takes values in $\{0, 0.2, 0.4, 0.6, 0.8\}$.

Overall, the center-outward van der Waerden score test has the best power except when the true underlying distribution is Gaussian (case (a)) or nearly Gaussian (case (b)), in which case Wilks, quite unsurprisingly, prevails. The performances of the center-outward Spearman and van der Waerden score tests, and the two distance covariance tests are comparable, while the center-outward sign and Kendall tests perform unsatisfactorily in some cases, which is not unexpected since the center-outward ranks and signs are not efficiently used.

Tables 4 and 6 confirm Proposition 4.1, that under ellipticity, the center-outward van der Waerden score test uniformly dominates Wilks' optimal Gaussian procedure, and the center-outward Spearman test has Pitman asymptotic relative efficiency no smaller than 0.77 with respect to Wilks' test when $d_1, d_2 \leq 7$ (see Table 3). In the Gaussian case (a) (see Table 4), the center-outward van der Waerden score test is second best out of all competing tests, and in the multivariate t_3 distribution case (c), the center-outward van der Waerden score test is best when the sample size is sufficiently large.

Table 5 shows the performance of the seven competing tests when the distributions are the normal mixtures, which are non-elliptical but nearly Gaussian. Table 5 suggests that, in this case, Wilks' test has the best finite-sample performance compared to all the center-outward sign- and rank-based tests, with the center-outward van der Waerden score test matching similar limiting performance

Although cases (d) and (e) are not covered by Proposition 4.1 due to infinite second-order moments, Tables 7 and 8 indicate that the center-outward van der Waerden score test remains best among the competing tests, while Wilks's test suffers from severe size inflation due to heavy-tailedness.

Tables 9–11 show the performance of the seven competing tests when the distributions are non-elliptical and heavy-tailed. We notice that the center-outward van der Waerden score test is the best one out of all competing tests while Wilks's test suffers from severe size inflation due to heavy-tailedness, which is similar to the multivariate t-distribution cases (c)–(e) (see Tables 6–8). Comparing Table 8 (representing the multivariate t_1-distribution case (e)) and Table 11 (representing the marginal t_1-distribution case (h)), the major difference is that the center-outward Kendall test performs unsatisfactorily in the latter case as the dimension increases.

Tables 12–13 show the performance of the seven competing tests when the distributions are non-elliptical and skewed. We notice that when the sample size is large enough, the center-outward Spearman and van der Waerden score tests, and the two distance covariance tests perform better than the center-outward sign and Kendall tests, and all center-outward sign- and rank-based tests are much more powerful than Wilks' test. The only exception occurs when the dimension is relatively large and the sample size is small, which can be attributed to the rate of convergence of empirical center-outward ranks and signs, a rate that fastly decreases as the dimension increases.

6 Conclusion

Optimal transport provides an entirely new approach to rank-based statistical inference in dimension $d \geq 2$. The new multivariate ranks retain many of the favorable properties one is used to from the classical univariate ranks. Here, we demonstrate how the new multivariate ranks can be used for a definition of multivariate versions of popular rank correlations such as Kendall's tau or Spearman's rho. We show how the new multivariate rank correlations yield fully distribution-free, yet powerful and computationally efficient tests of independence. A highlight of our results is the fact that the use of van der Waerden scores allows one to design a nonparametric test whose asymptotic efficiency under arbitrary elliptical densities never drops below that of Wilks' test against the so-called Konijn alternatives—not even under a Gaussian model.

Table 4: Empirical powers of the seven competing tests in Example 5.1(a). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.053	0.057	0.088	0.160	0.263	0.050	0.055	0.086	0.139	0.263	0.045	0.073	0.088	0.149	0.268
	Spe	0.051	0.064	0.132	0.233	0.411	0.053	0.067	0.107	0.202	0.393	0.044	0.063	0.119	0.215	0.394
	Ken	0.047	0.066	0.135	0.232	0.388	0.055	0.072	0.105	0.201	0.372	0.053	0.065	0.106	0.206	0.387
	vdW	0.055	0.063	0.129	0.229	0.402	0.051	0.066	0.110	0.207	0.394	0.044	0.063	0.119	0.217	0.404
	dCov	0.045	0.075	0.113	0.217	0.385	0.053	0.067	0.103	0.194	0.372	0.051	0.063	0.111	0.199	0.359
	dCvdW	0.051	0.071	0.112	0.221	0.386	0.054	0.070	0.103	0.195	0.376	0.048	0.068	0.107	0.200	0.368
	LRT	0.055	0.077	0.145	0.259	0.469	0.052	0.073	0.129	0.243	0.427	0.047	0.066	0.118	0.221	0.427
$(d_1, d_2) = (3, 3)$	sign	0.040	0.065	0.096	0.153	0.270	0.045	0.068	0.111	0.163	0.290	0.054	0.059	0.095	0.158	0.284
	Spe	0.050	0.075	0.108	0.192	0.351	0.050	0.060	0.122	0.210	0.362	0.045	0.067	0.126	0.208	0.386
	Ken	0.045	0.077	0.109	0.193	0.349	0.047	0.058	0.119	0.215	0.374	0.051	0.066	0.122	0.217	0.369
	vdW	0.048	0.071	0.108	0.189	0.368	0.053	0.064	0.120	0.221	0.395	0.046	0.067	0.128	0.219	0.401
	dCov	0.045	0.063	0.110	0.183	0.330	0.047	0.063	0.106	0.201	0.353	0.054	0.067	0.117	0.200	0.363
	dCvdW	0.041	0.068	0.113	0.184	0.347	0.045	0.067	0.115	0.207	0.371	0.052	0.063	0.120	0.209	0.378
	LRT	0.049	0.081	0.142	0.271	0.489	0.043	0.071	0.145	0.266	0.483	0.044	0.071	0.139	0.263	0.456
$(d_1, d_2) = (5, 5)$	sign	0.049	0.065	0.073	0.119	0.225	0.050	0.060	0.079	0.140	0.243	0.051	0.058	0.087	0.153	0.278
	Spe	0.059	0.063	0.104	0.149	0.249	0.055	0.061	0.087	0.143	0.264	0.053	0.058	0.100	0.166	0.291
	Ken	0.053	0.060	0.098	0.145	0.255	0.052	0.065	0.089	0.142	0.266	0.052	0.061	0.095	0.174	0.310
	vdW	0.049	0.063	0.097	0.153	0.268	0.051	0.057	0.085	0.163	0.290	0.053	0.059	0.101	0.178	0.336
	dCov	0.048	0.063	0.089	0.129	0.229	0.055	0.065	0.085	0.138	0.259	0.051	0.060	0.103	0.159	0.289
	dCvdW	0.051	0.066	0.086	0.133	0.266	0.053	0.065	0.081	0.155	0.284	0.049	0.057	0.099	0.176	0.328
	LRT	0.060	0.086	0.145	0.309	0.582	0.061	0.079	0.138	0.278	0.527	0.045	0.070	0.127	0.245	0.525
$(d_1, d_2) = (7, 7)$	sign	0.050	0.057	0.063	0.109	0.165	0.045	0.049	0.080	0.114	0.181	0.041	0.056	0.085	0.126	0.211
	Spe	0.059	0.087	0.101	0.120	0.164	0.055	0.065	0.091	0.134	0.197	0.059	0.060	0.090	0.139	0.225
	Ken	0.054	0.063	0.084	0.116	0.166	0.044	0.061	0.081	0.136	0.180	0.047	0.060	0.087	0.132	0.223
	vdW	0.049	0.065	0.083	0.118	0.189	0.041	0.056	0.090	0.135	0.217	0.048	0.055	0.092	0.156	0.259
	dCov	0.044	0.061	0.065	0.081	0.139	0.033	0.059	0.062	0.103	0.158	0.039	0.051	0.075	0.122	0.196
	dCvdW	0.049	0.057	0.076	0.113	0.185	0.043	0.049	0.083	0.123	0.199	0.041	0.055	0.085	0.143	0.249
	LRT	0.077	0.109	0.166	0.347	0.617	0.056	0.071	0.145	0.298	0.572	0.055	0.067	0.132	0.285	0.523

Table 5: Empirical powers of the seven competing tests in Example 5.1(b). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.043	0.061	0.088	0.131	0.259	0.050	0.052	0.093	0.145	0.247	0.049	0.058	0.095	0.161	0.262
	Spe	0.049	0.063	0.118	0.213	0.373	0.045	0.068	0.109	0.217	0.389	0.055	0.069	0.129	0.230	0.403
	Ken	0.045	0.073	0.121	0.209	0.382	0.049	0.057	0.111	0.213	0.379	0.051	0.065	0.115	0.219	0.380
	vdW	0.041	0.065	0.119	0.218	0.384	0.043	0.066	0.113	0.222	0.394	0.049	0.067	0.121	0.235	0.418
	dCov	0.041	0.061	0.113	0.190	0.350	0.050	0.059	0.098	0.201	0.367	0.047	0.067	0.120	0.224	0.373
	dCvdW	0.043	0.061	0.113	0.185	0.355	0.050	0.057	0.109	0.203	0.374	0.046	0.064	0.127	0.219	0.376
	LRT	0.047	0.066	0.135	0.262	0.444	0.047	0.064	0.118	0.248	0.426	0.055	0.065	0.130	0.245	0.430
$(d_1, d_2) = (3, 3)$	sign	0.045	0.063	0.092	0.142	0.273	0.043	0.054	0.087	0.183	0.294	0.049	0.056	0.096	0.169	0.291
	Spe	0.055	0.079	0.118	0.186	0.344	0.042	0.069	0.099	0.217	0.369	0.049	0.075	0.129	0.212	0.382
	Ken	0.047	0.071	0.119	0.193	0.330	0.049	0.066	0.098	0.220	0.357	0.055	0.071	0.113	0.207	0.356
	vdW	0.050	0.077	0.116	0.195	0.355	0.045	0.066	0.105	0.219	0.373	0.048	0.077	0.133	0.214	0.383
	dCov	0.046	0.072	0.111	0.174	0.323	0.043	0.056	0.097	0.201	0.359	0.053	0.073	0.108	0.199	0.353
	dCvdW	0.043	0.075	0.106	0.189	0.328	0.040	0.057	0.093	0.204	0.363	0.050	0.067	0.111	0.200	0.365
	LRT	0.051	0.078	0.143	0.274	0.493	0.052	0.064	0.122	0.277	0.462	0.058	0.084	0.145	0.263	0.448
$(d_1, d_2) = (5, 5)$	sign	0.052	0.055	0.083	0.121	0.209	0.047	0.060	0.082	0.143	0.244	0.049	0.063	0.081	0.151	0.259
	Spe	0.052	0.063	0.096	0.135	0.233	0.055	0.067	0.086	0.159	0.283	0.046	0.067	0.093	0.167	0.291
	Ken	0.047	0.059	0.095	0.139	0.251	0.054	0.069	0.087	0.159	0.279	0.054	0.067	0.094	0.173	0.279
	vdW	0.045	0.055	0.091	0.134	0.253	0.049	0.065	0.091	0.163	0.308	0.051	0.071	0.097	0.185	0.315
	dCov	0.051	0.062	0.083	0.129	0.220	0.053	0.061	0.087	0.153	0.258	0.045	0.061	0.097	0.167	0.281
	dCvdW	0.045	0.061	0.085	0.146	0.251	0.055	0.060	0.095	0.169	0.295	0.049	0.063	0.092	0.177	0.299
	LRT	0.066	0.074	0.153	0.305	0.553	0.059	0.066	0.133	0.291	0.528	0.056	0.069	0.133	0.272	0.491
$(d_1, d_2) = (7, 7)$	sign	0.037	0.057	0.066	0.101	0.181	0.047	0.057	0.085	0.097	0.194	0.051	0.053	0.097	0.127	0.237
	Spe	0.065	0.072	0.083	0.123	0.182	0.055	0.076	0.091	0.109	0.209	0.057	0.070	0.085	0.129	0.227
	Ken	0.053	0.065	0.071	0.105	0.193	0.055	0.064	0.089	0.113	0.203	0.053	0.064	0.087	0.131	0.226
	vdW	0.047	0.063	0.077	0.109	0.197	0.051	0.062	0.085	0.113	0.225	0.054	0.057	0.095	0.139	0.265
	dCov	0.041	0.055	0.061	0.082	0.145	0.045	0.065	0.071	0.092	0.173	0.044	0.053	0.079	0.125	0.191
	dCvdW	0.041	0.060	0.071	0.103	0.192	0.050	0.062	0.089	0.108	0.215	0.051	0.058	0.093	0.134	0.263
	LRT	0.081	0.102	0.180	0.344	0.627	0.065	0.081	0.167	0.299	0.553	0.053	0.068	0.131	0.281	0.559

Table 6: Empirical powers of the seven competing tests in Example 5.1(c). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.042	0.065	0.107	0.235	0.382	0.046	0.074	0.110	0.237	0.418	0.044	0.062	0.123	0.227	0.408
	Spe	0.043	0.081	0.148	0.298	0.525	0.055	0.067	0.138	0.319	0.513	0.057	0.071	0.146	0.299	0.526
	Ken	0.049	0.076	0.161	0.321	0.543	0.054	0.077	0.143	0.316	0.531	0.043	0.065	0.156	0.309	0.520
	vdW	0.043	0.082	0.150	0.301	0.529	0.056	0.073	0.134	0.314	0.538	0.053	0.071	0.149	0.311	0.538
	dCov	0.044	0.073	0.145	0.299	0.512	0.058	0.071	0.137	0.301	0.505	0.053	0.073	0.151	0.289	0.487
	dCvdW	0.041	0.073	0.139	0.317	0.523	0.055	0.072	0.141	0.307	0.529	0.052	0.069	0.148	0.296	0.503
	LRT	0.058	0.086	0.143	0.275	0.485	0.057	0.069	0.133	0.252	0.464	0.057	0.067	0.143	0.238	0.415
$(d_1, d_2) = (3, 3)$	sign	0.062	0.073	0.105	0.241	0.425	0.056	0.060	0.133	0.260	0.466	0.043	0.063	0.119	0.264	0.479
	Spe	0.053	0.064	0.129	0.257	0.458	0.049	0.063	0.129	0.276	0.489	0.044	0.063	0.127	0.287	0.509
	Ken	0.059	0.068	0.139	0.273	0.490	0.056	0.066	0.138	0.281	0.523	0.046	0.059	0.132	0.291	0.515
	vdW	0.058	0.067	0.135	0.278	0.504	0.051	0.061	0.139	0.305	0.537	0.044	0.067	0.141	0.322	0.569
	dCov	0.045	0.070	0.115	0.250	0.463	0.053	0.061	0.121	0.273	0.497	0.045	0.059	0.129	0.274	0.505
	dCvdW	0.049	0.071	0.129	0.271	0.499	0.055	0.059	0.129	0.297	0.543	0.043	0.061	0.135	0.309	0.557
	LRT	0.075	0.087	0.156	0.295	0.529	0.063	0.076	0.135	0.257	0.487	0.064	0.082	0.133	0.253	0.471
$(d_1, d_2) = (5, 5)$	sign	0.047	0.061	0.093	0.175	0.352	0.050	0.061	0.093	0.223	0.431	0.049	0.063	0.107	0.249	0.441
	Spe	0.057	0.076	0.103	0.175	0.310	0.046	0.067	0.107	0.193	0.363	0.053	0.063	0.119	0.219	0.385
	Ken	0.050	0.068	0.108	0.191	0.340	0.051	0.063	0.107	0.214	0.407	0.053	0.068	0.137	0.234	0.421
	vdW	0.051	0.075	0.106	0.195	0.384	0.047	0.065	0.104	0.242	0.441	0.053	0.060	0.133	0.274	0.480
	dCov	0.055	0.062	0.089	0.167	0.314	0.056	0.063	0.106	0.206	0.375	0.049	0.060	0.125	0.231	0.390
	dCvdW	0.048	0.071	0.108	0.191	0.382	0.055	0.066	0.106	0.259	0.458	0.055	0.060	0.135	0.289	0.492
	LRT	0.102	0.115	0.159	0.331	0.595	0.084	0.103	0.147	0.307	0.547	0.071	0.088	0.153	0.283	0.513
$(d_1, d_2) = (7, 7)$	sign	0.047	0.059	0.082	0.134	0.245	0.046	0.053	0.075	0.145	0.287	0.041	0.057	0.093	0.184	0.341
	Spe	0.062	0.077	0.099	0.138	0.233	0.058	0.066	0.091	0.147	0.233	0.055	0.061	0.083	0.161	0.277
	Ken	0.061	0.068	0.087	0.141	0.249	0.053	0.061	0.089	0.148	0.276	0.047	0.072	0.089	0.165	0.306
	vdW	0.055	0.057	0.085	0.145	0.261	0.051	0.054	0.093	0.165	0.307	0.044	0.065	0.100	0.207	0.365
	dCov	0.043	0.058	0.063	0.113	0.173	0.042	0.053	0.081	0.119	0.204	0.043	0.059	0.071	0.147	0.261
	dCvdW	0.048	0.059	0.081	0.134	0.252	0.045	0.058	0.085	0.165	0.295	0.042	0.066	0.095	0.199	0.361
	LRT	0.121	0.146	0.212	0.359	0.626	0.098	0.119	0.198	0.323	0.559	0.088	0.121	0.167	0.303	0.521

Table 7: Empirical powers of the seven competing tests in Example 5.1(d). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.057	0.064	0.144	0.306	0.543	0.046	0.084	0.147	0.334	0.555	0.047	0.083	0.168	0.347	0.608
	Spe	0.063	0.081	0.190	0.404	0.673	0.047	0.084	0.205	0.445	0.695	0.047	0.075	0.206	0.427	0.711
	Ken	0.054	0.083	0.196	0.413	0.695	0.047	0.089	0.205	0.445	0.697	0.047	0.083	0.211	0.433	0.709
	vdW	0.059	0.077	0.199	0.415	0.695	0.048	0.087	0.208	0.469	0.738	0.049	0.081	0.216	0.471	0.740
	dCov	0.061	0.075	0.175	0.383	0.662	0.043	0.083	0.201	0.424	0.676	0.051	0.078	0.187	0.421	0.688
	dCvdW	0.057	0.076	0.187	0.391	0.685	0.047	0.085	0.203	0.438	0.697	0.045	0.085	0.203	0.443	0.709
	LRT	0.076	0.100	0.167	0.300	0.545	0.070	0.089	0.143	0.278	0.509	0.063	0.083	0.139	0.262	0.482
$(d_1, d_2) = (3, 3)$	sign	0.051	0.062	0.158	0.323	0.578	0.041	0.059	0.173	0.335	0.635	0.052	0.080	0.175	0.404	0.667
	Spe	0.054	0.065	0.167	0.367	0.631	0.052	0.082	0.185	0.387	0.708	0.051	0.073	0.192	0.409	0.673
	Ken	0.057	0.067	0.175	0.380	0.656	0.059	0.077	0.199	0.401	0.701	0.048	0.073	0.193	0.440	0.719
	vdW	0.051	0.072	0.182	0.391	0.680	0.056	0.075	0.203	0.439	0.747	0.054	0.076	0.215	0.489	0.757
	dCov	0.055	0.063	0.163	0.340	0.609	0.059	0.077	0.175	0.373	0.683	0.047	0.077	0.187	0.407	0.673
	dCvdW	0.049	0.064	0.175	0.369	0.656	0.059	0.069	0.201	0.414	0.723	0.044	0.079	0.214	0.458	0.744
	LRT	0.100	0.111	0.194	0.325	0.575	0.091	0.117	0.164	0.306	0.533	0.073	0.103	0.152	0.255	0.501
$(d_1, d_2) = (5, 5)$	sign	0.051	0.057	0.129	0.258	0.463	0.049	0.075	0.128	0.318	0.577	0.037	0.077	0.160	0.345	0.651
	Spe	0.065	0.077	0.127	0.239	0.432	0.057	0.084	0.135	0.279	0.548	0.050	0.069	0.152	0.312	0.583
	Ken	0.057	0.071	0.129	0.258	0.475	0.051	0.079	0.136	0.297	0.570	0.057	0.069	0.162	0.328	0.628
	vdW	0.055	0.071	0.140	0.275	0.530	0.047	0.083	0.146	0.339	0.649	0.046	0.077	0.186	0.400	0.727
	dCov	0.059	0.067	0.107	0.235	0.436	0.060	0.078	0.138	0.274	0.541	0.048	0.070	0.157	0.307	0.583
	dCvdW	0.048	0.063	0.131	0.280	0.517	0.053	0.081	0.154	0.335	0.638	0.045	0.077	0.187	0.375	0.709
	LRT	0.143	0.172	0.238	0.400	0.635	0.127	0.160	0.223	0.326	0.601	0.127	0.139	0.206	0.313	0.529
$(d_1, d_2) = (7, 7)$	sign	0.048	0.055	0.103	0.177	0.328	0.050	0.059	0.105	0.208	0.429	0.049	0.061	0.119	0.261	0.545
	Spe	0.067	0.079	0.119	0.183	0.333	0.062	0.073	0.105	0.208	0.365	0.050	0.075	0.121	0.242	0.448
	Ken	0.054	0.064	0.114	0.197	0.339	0.061	0.071	0.106	0.208	0.398	0.054	0.065	0.125	0.261	0.491
	vdW	0.056	0.063	0.113	0.203	0.384	0.058	0.063	0.111	0.238	0.463	0.047	0.073	0.135	0.313	0.592
	dCov	0.047	0.058	0.081	0.143	0.277	0.047	0.054	0.090	0.175	0.323	0.042	0.065	0.102	0.205	0.405
	dCvdW	0.050	0.054	0.107	0.187	0.368	0.051	0.059	0.115	0.229	0.451	0.045	0.064	0.126	0.305	0.572
	LRT	0.167	0.185	0.289	0.441	0.691	0.142	0.188	0.231	0.381	0.614	0.139	0.158	0.198	0.331	0.538

Table 8: Empirical powers of the seven competing tests in Example 5.1(e). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.051	0.142	0.445	0.737	0.912	0.047	0.179	0.517	0.819	0.942	0.042	0.204	0.578	0.864	0.965
	Spe	0.044	0.247	0.671	0.930	0.986	0.040	0.293	0.766	0.966	0.992	0.049	0.311	0.833	0.977	0.993
	Ken	0.051	0.234	0.643	0.922	0.984	0.049	0.277	0.740	0.948	0.990	0.051	0.285	0.794	0.969	0.992
	vdW	0.053	0.291	0.725	0.956	0.989	0.041	0.367	0.833	0.981	0.994	0.049	0.397	0.897	0.988	0.994
	dCov	0.053	0.217	0.615	0.902	0.982	0.050	0.242	0.713	0.941	0.988	0.041	0.273	0.783	0.971	0.991
	dCvdW	0.047	0.227	0.644	0.916	0.985	0.049	0.251	0.737	0.953	0.989	0.043	0.299	0.813	0.977	0.991
	LRT	0.075	0.265	0.433	0.645	0.822	0.053	0.242	0.424	0.594	0.817	0.045	0.215	0.397	0.581	0.805
$(d_1, d_2) = (3, 3)$	sign	0.058	0.134	0.459	0.755	0.946	0.051	0.183	0.549	0.875	0.976	0.061	0.217	0.636	0.927	0.981
	Spe	0.063	0.219	0.657	0.917	0.995	0.053	0.265	0.769	0.973	0.995	0.056	0.321	0.859	0.986	0.992
	Ken	0.060	0.211	0.619	0.905	0.992	0.050	0.244	0.714	0.967	0.994	0.061	0.277	0.807	0.977	0.990
	vdW	0.060	0.241	0.697	0.935	0.997	0.049	0.304	0.822	0.983	0.995	0.053	0.387	0.897	0.989	0.992
	dCov	0.049	0.179	0.583	0.895	0.993	0.055	0.229	0.709	0.956	0.993	0.055	0.278	0.801	0.977	0.991
	dCvdW	0.052	0.185	0.607	0.894	0.995	0.050	0.241	0.739	0.960	0.992	0.058	0.304	0.809	0.976	0.990
	LRT	0.131	0.323	0.541	0.713	0.881	0.103	0.283	0.485	0.664	0.868	0.089	0.253	0.473	0.661	0.837
$(d_1, d_2) = (5, 5)$	sign	0.054	0.104	0.328	0.619	0.873	0.042	0.131	0.431	0.824	0.963	0.049	0.174	0.554	0.881	0.975
	Spe	0.062	0.156	0.491	0.813	0.971	0.049	0.197	0.641	0.939	0.993	0.048	0.252	0.775	0.981	0.989
	Ken	0.055	0.130	0.448	0.779	0.951	0.051	0.177	0.560	0.916	0.985	0.053	0.204	0.695	0.953	0.979
	vdW	0.054	0.153	0.502	0.819	0.976	0.049	0.198	0.661	0.955	0.991	0.047	0.276	0.816	0.983	0.988
	dCov	0.059	0.125	0.415	0.753	0.955	0.053	0.170	0.566	0.916	0.985	0.048	0.202	0.704	0.968	0.984
	dCvdW	0.048	0.124	0.433	0.763	0.957	0.048	0.163	0.575	0.923	0.985	0.050	0.217	0.715	0.962	0.982
	LRT	0.187	0.408	0.636	0.811	0.944	0.155	0.369	0.577	0.757	0.919	0.133	0.340	0.541	0.736	0.891
$(d_1, d_2) = (7, 7)$	sign	0.049	0.085	0.222	0.421	0.723	0.046	0.104	0.311	0.641	0.898	0.044	0.136	0.405	0.797	0.955
	Spe	0.069	0.156	0.385	0.651	0.903	0.058	0.156	0.489	0.844	0.976	0.055	0.196	0.606	0.935	0.988
	Ken	0.060	0.126	0.347	0.580	0.859	0.055	0.134	0.411	0.777	0.956	0.053	0.163	0.511	0.904	0.983
	vdW	0.053	0.129	0.342	0.619	0.897	0.047	0.144	0.489	0.859	0.975	0.048	0.195	0.621	0.941	0.987
	dCov	0.047	0.095	0.270	0.521	0.833	0.052	0.111	0.372	0.759	0.961	0.046	0.146	0.515	0.889	0.979
	dCvdW	0.053	0.091	0.274	0.521	0.828	0.050	0.120	0.396	0.777	0.957	0.046	0.165	0.533	0.902	0.978
	LRT	0.247	0.488	0.692	0.861	0.955	0.218	0.448	0.654	0.818	0.931	0.148	0.401	0.623	0.777	0.914

Table 9: Empirical powers of the seven competing tests in Example 5.1(f). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.052	0.056	0.119	0.211	0.403	0.045	0.064	0.103	0.208	0.358	0.044	0.069	0.108	0.233	0.395
	Spe	0.053	0.068	0.151	0.292	0.505	0.045	0.073	0.151	0.277	0.505	0.042	0.069	0.144	0.300	0.521
	Ken	0.056	0.066	0.156	0.301	0.542	0.047	0.082	0.152	0.298	0.496	0.045	0.070	0.146	0.315	0.527
	vdW	0.051	0.062	0.154	0.303	0.523	0.045	0.073	0.158	0.288	0.523	0.045	0.069	0.145	0.311	0.529
	dCov	0.058	0.060	0.144	0.277	0.502	0.040	0.073	0.143	0.272	0.466	0.045	0.069	0.133	0.284	0.501
	dCvdW	0.057	0.065	0.149	0.282	0.521	0.041	0.075	0.144	0.279	0.485	0.046	0.071	0.137	0.299	0.517
	LRT	0.053	0.078	0.155	0.275	0.499	0.057	0.075	0.136	0.248	0.438	0.055	0.067	0.129	0.249	0.441
$(d_1, d_2) = (3, 3)$	sign	0.048	0.071	0.105	0.225	0.393	0.049	0.056	0.109	0.229	0.416	0.052	0.060	0.122	0.248	0.435
	Spe	0.055	0.071	0.131	0.256	0.452	0.049	0.071	0.131	0.265	0.461	0.051	0.064	0.131	0.292	0.501
	Ken	0.061	0.065	0.129	0.261	0.486	0.056	0.069	0.135	0.283	0.491	0.056	0.065	0.140	0.293	0.519
	vdW	0.052	0.070	0.129	0.270	0.471	0.046	0.072	0.137	0.286	0.489	0.049	0.063	0.145	0.319	0.534
	dCov	0.053	0.065	0.110	0.238	0.420	0.048	0.059	0.121	0.255	0.443	0.049	0.065	0.130	0.273	0.489
	dCvdW	0.058	0.073	0.115	0.253	0.462	0.041	0.061	0.129	0.273	0.483	0.055	0.062	0.137	0.300	0.526
	LRT	0.071	0.094	0.136	0.280	0.509	0.061	0.078	0.135	0.274	0.484	0.065	0.069	0.149	0.261	0.465
$(d_1, d_2) = (5, 5)$	sign	0.049	0.059	0.089	0.162	0.290	0.043	0.071	0.101	0.176	0.353	0.046	0.058	0.098	0.211	0.412
	Spe	0.054	0.072	0.103	0.171	0.281	0.042	0.067	0.099	0.179	0.353	0.047	0.066	0.104	0.217	0.391
	Ken	0.053	0.064	0.089	0.172	0.313	0.043	0.063	0.097	0.186	0.361	0.045	0.074	0.105	0.212	0.392
	vdW	0.049	0.067	0.107	0.195	0.325	0.040	0.069	0.107	0.211	0.408	0.042	0.069	0.113	0.245	0.465
	dCov	0.053	0.064	0.079	0.171	0.270	0.041	0.067	0.091	0.168	0.359	0.056	0.071	0.102	0.209	0.379
	dCvdW	0.045	0.062	0.092	0.185	0.315	0.041	0.078	0.106	0.201	0.389	0.049	0.071	0.109	0.243	0.455
	LRT	0.082	0.091	0.160	0.327	0.593	0.051	0.085	0.156	0.297	0.564	0.055	0.085	0.127	0.274	0.501
$(d_1, d_2) = (7, 7)$	sign	0.051	0.058	0.074	0.119	0.206	0.049	0.061	0.081	0.138	0.260	0.052	0.070	0.090	0.155	0.313
	Spe	0.063	0.071	0.097	0.125	0.216	0.047	0.068	0.105	0.143	0.236	0.053	0.066	0.091	0.151	0.263
	Ken	0.059	0.069	0.085	0.121	0.217	0.049	0.074	0.078	0.136	0.224	0.057	0.070	0.093	0.151	0.260
	vdW	0.057	0.058	0.093	0.129	0.239	0.046	0.067	0.091	0.147	0.281	0.051	0.067	0.102	0.166	0.335
	dCov	0.044	0.051	0.073	0.094	0.182	0.037	0.052	0.077	0.120	0.207	0.047	0.050	0.078	0.126	0.223
	dCvdW	0.049	0.059	0.081	0.124	0.219	0.043	0.067	0.084	0.144	0.278	0.055	0.065	0.106	0.167	0.325
	LRT	0.099	0.115	0.183	0.349	0.648	0.080	0.094	0.173	0.307	0.596	0.059	0.095	0.136	0.285	0.532

Table 10: Empirical powers of the seven competing tests in Example 5.1(g). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.052	0.069	0.137	0.313	0.524	0.045	0.081	0.145	0.309	0.515	0.055	0.066	0.151	0.308	0.537
	Spe	0.053	0.101	0.197	0.415	0.666	0.049	0.082	0.185	0.397	0.679	0.053	0.074	0.205	0.423	0.702
	Ken	0.056	0.099	0.185	0.434	0.695	0.041	0.082	0.195	0.421	0.676	0.053	0.071	0.202	0.425	0.686
	vdW	0.052	0.105	0.193	0.433	0.681	0.043	0.087	0.199	0.434	0.702	0.054	0.087	0.217	0.452	0.732
	dCov	0.057	0.091	0.181	0.404	0.641	0.048	0.075	0.165	0.385	0.648	0.055	0.080	0.183	0.404	0.661
	dCvdW	0.054	0.093	0.190	0.415	0.662	0.041	0.084	0.175	0.401	0.667	0.054	0.079	0.192	0.427	0.677
	LRT	0.070	0.097	0.171	0.306	0.547	0.062	0.087	0.140	0.295	0.491	0.057	0.082	0.139	0.267	0.452
$(d_1, d_2) = (3, 3)$	sign	0.053	0.070	0.148	0.292	0.532	0.053	0.067	0.151	0.337	0.585	0.055	0.079	0.159	0.335	0.628
	Spe	0.047	0.085	0.152	0.337	0.587	0.056	0.082	0.183	0.378	0.652	0.048	0.081	0.191	0.390	0.703
	Ken	0.049	0.088	0.169	0.336	0.603	0.055	0.078	0.180	0.359	0.655	0.051	0.079	0.183	0.403	0.711
	vdW	0.049	0.076	0.166	0.369	0.642	0.055	0.080	0.201	0.406	0.706	0.046	0.076	0.200	0.435	0.766
	dCov	0.046	0.073	0.159	0.312	0.571	0.047	0.078	0.173	0.351	0.625	0.056	0.080	0.177	0.367	0.687
	dCvdW	0.050	0.078	0.171	0.343	0.613	0.051	0.076	0.189	0.380	0.677	0.051	0.082	0.187	0.403	0.741
	LRT	0.085	0.093	0.183	0.330	0.575	0.089	0.105	0.150	0.296	0.543	0.087	0.093	0.140	0.281	0.512
$(d_1, d_2) = (5, 5)$	sign	0.055	0.067	0.120	0.225	0.373	0.043	0.066	0.121	0.277	0.502	0.053	0.071	0.128	0.283	0.569
	Spe	0.067	0.078	0.123	0.223	0.405	0.048	0.071	0.127	0.277	0.511	0.049	0.070	0.147	0.287	0.561
	Ken	0.055	0.060	0.119	0.227	0.389	0.052	0.067	0.119	0.266	0.471	0.055	0.071	0.131	0.287	0.533
	vdW	0.057	0.076	0.127	0.252	0.463	0.055	0.070	0.145	0.319	0.594	0.048	0.079	0.158	0.348	0.645
	dCov	0.049	0.071	0.121	0.212	0.389	0.045	0.067	0.115	0.265	0.487	0.055	0.075	0.143	0.276	0.548
	dCvdW	0.056	0.065	0.123	0.251	0.443	0.047	0.070	0.133	0.303	0.573	0.052	0.074	0.148	0.328	0.624
	LRT	0.123	0.146	0.237	0.380	0.632	0.113	0.127	0.199	0.341	0.580	0.107	0.135	0.175	0.279	0.534
$(d_1, d_2) = (7, 7)$	sign	0.048	0.052	0.101	0.155	0.273	0.051	0.061	0.094	0.179	0.369	0.042	0.056	0.115	0.226	0.448
	Spe	0.066	0.079	0.115	0.167	0.286	0.065	0.073	0.107	0.163	0.332	0.057	0.069	0.105	0.229	0.406
	Ken	0.059	0.071	0.092	0.165	0.232	0.058	0.066	0.098	0.160	0.303	0.043	0.065	0.097	0.204	0.377
	vdW	0.050	0.062	0.117	0.176	0.310	0.051	0.065	0.107	0.197	0.406	0.039	0.067	0.120	0.262	0.505
	dCov	0.051	0.056	0.078	0.123	0.224	0.046	0.063	0.090	0.141	0.279	0.050	0.065	0.100	0.185	0.372
	dCvdW	0.052	0.055	0.095	0.171	0.297	0.050	0.062	0.098	0.183	0.386	0.037	0.066	0.119	0.250	0.501
	LRT	0.151	0.173	0.255	0.421	0.687	0.139	0.161	0.226	0.381	0.593	0.125	0.149	0.195	0.327	0.539

Table 11: Empirical powers of the seven competing tests in Example 5.1(h). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.045	0.155	0.453	0.773	0.939	0.053	0.187	0.548	0.857	0.973	0.043	0.229	0.618	0.881	0.974
	Spe	0.054	0.257	0.711	0.945	0.993	0.050	0.344	0.828	0.975	0.993	0.042	0.375	0.880	0.983	0.992
	Ken	0.056	0.225	0.646	0.919	0.990	0.052	0.291	0.755	0.955	0.993	0.045	0.319	0.805	0.973	0.989
	vdW	0.048	0.293	0.753	0.962	0.994	0.053	0.404	0.875	0.988	0.995	0.039	0.456	0.922	0.988	0.993
	dCov	0.054	0.218	0.635	0.924	0.989	0.055	0.295	0.755	0.967	0.992	0.037	0.315	0.825	0.974	0.991
	dCvdW	0.049	0.227	0.665	0.932	0.992	0.051	0.311	0.785	0.972	0.992	0.040	0.338	0.851	0.979	0.991
	LRT	0.080	0.247	0.420	0.613	0.803	0.062	0.224	0.381	0.592	0.782	0.049	0.220	0.401	0.571	0.775
$(d_1, d_2) = (3, 3)$	sign	0.053	0.174	0.489	0.821	0.955	0.040	0.194	0.594	0.909	0.985	0.053	0.237	0.687	0.941	0.981
	Spe	0.053	0.287	0.719	0.952	0.993	0.051	0.325	0.824	0.984	0.994	0.055	0.396	0.908	0.987	0.989
	Ken	0.061	0.215	0.604	0.883	0.981	0.047	0.219	0.698	0.963	0.992	0.055	0.274	0.787	0.970	0.985
	vdW	0.053	0.297	0.754	0.963	0.995	0.045	0.352	0.860	0.987	0.994	0.055	0.447	0.926	0.987	0.989
	dCov	0.056	0.222	0.660	0.928	0.990	0.058	0.253	0.777	0.980	0.994	0.051	0.329	0.858	0.983	0.986
	dCvdW	0.053	0.246	0.676	0.934	0.991	0.050	0.266	0.783	0.979	0.994	0.049	0.342	0.865	0.984	0.986
	LRT	0.098	0.334	0.522	0.712	0.847	0.091	0.277	0.477	0.681	0.826	0.073	0.261	0.445	0.641	0.816
$(d_1, d_2) = (5, 5)$	sign	0.039	0.127	0.389	0.709	0.924	0.049	0.164	0.535	0.877	0.973	0.033	0.223	0.667	0.930	0.968
	Spe	0.067	0.197	0.572	0.868	0.970	0.049	0.247	0.732	0.968	0.989	0.065	0.335	0.869	0.979	0.985
	Ken	0.057	0.136	0.351	0.680	0.892	0.051	0.151	0.474	0.828	0.969	0.052	0.201	0.590	0.903	0.971
	vdW	0.056	0.197	0.556	0.884	0.977	0.049	0.241	0.762	0.976	0.989	0.050	0.355	0.881	0.978	0.983
	dCov	0.054	0.159	0.485	0.821	0.963	0.050	0.193	0.656	0.945	0.989	0.055	0.272	0.796	0.974	0.978
	dCvdW	0.044	0.151	0.493	0.832	0.967	0.046	0.205	0.673	0.953	0.987	0.045	0.290	0.797	0.969	0.976
	LRT	0.202	0.410	0.604	0.805	0.916	0.160	0.362	0.575	0.750	0.897	0.134	0.335	0.527	0.722	0.858
$(d_1, d_2) = (7, 7)$	sign	0.044	0.115	0.239	0.501	0.769	0.052	0.119	0.381	0.717	0.929	0.044	0.149	0.544	0.858	0.963
	Spe	0.072	0.171	0.399	0.727	0.910	0.068	0.193	0.561	0.884	0.976	0.056	0.233	0.743	0.965	0.983
	Ken	0.061	0.123	0.231	0.427	0.667	0.059	0.125	0.289	0.571	0.844	0.057	0.125	0.385	0.734	0.936
	vdW	0.053	0.153	0.371	0.706	0.916	0.052	0.176	0.563	0.882	0.980	0.047	0.231	0.747	0.961	0.980
	dCov	0.050	0.099	0.281	0.595	0.851	0.044	0.129	0.428	0.803	0.971	0.042	0.166	0.635	0.929	0.977
	dCvdW	0.046	0.130	0.298	0.623	0.853	0.047	0.144	0.485	0.833	0.968	0.043	0.189	0.669	0.927	0.973
	LRT	0.242	0.493	0.712	0.830	0.938	0.201	0.425	0.643	0.829	0.904	0.170	0.391	0.575	0.770	0.899

Table 12: Empirical powers of the seven competing tests in Example 5.1(i). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.049	0.070	0.113	0.263	0.435	0.049	0.064	0.132	0.244	0.417	0.051	0.058	0.126	0.240	0.433
	Spe	0.053	0.108	0.279	0.595	0.859	0.053	0.106	0.317	0.655	0.905	0.046	0.108	0.339	0.713	0.935
	Ken	0.053	0.095	0.213	0.458	0.737	0.055	0.097	0.218	0.459	0.732	0.043	0.079	0.216	0.485	0.756
	vdW	0.053	0.106	0.288	0.597	0.859	0.055	0.108	0.344	0.685	0.922	0.046	0.112	0.395	0.767	0.955
	dCov	0.053	0.093	0.240	0.547	0.812	0.059	0.097	0.274	0.564	0.857	0.043	0.088	0.278	0.625	0.888
	dCvdW	0.057	0.092	0.238	0.535	0.797	0.059	0.097	0.261	0.561	0.847	0.047	0.090	0.275	0.629	0.884
	LRT	0.059	0.075	0.143	0.263	0.450	0.059	0.080	0.135	0.240	0.422	0.055	0.068	0.137	0.263	0.444
$(d_1, d_2) = (3, 3)$	sign	0.037	0.071	0.123	0.280	0.508	0.048	0.067	0.161	0.300	0.530	0.045	0.073	0.150	0.331	0.588
	Spe	0.059	0.086	0.214	0.511	0.799	0.045	0.098	0.275	0.613	0.887	0.053	0.114	0.343	0.721	0.945
	Ken	0.045	0.082	0.152	0.392	0.678	0.044	0.088	0.196	0.408	0.692	0.045	0.085	0.208	0.437	0.764
	vdW	0.054	0.085	0.211	0.480	0.787	0.047	0.091	0.259	0.574	0.870	0.054	0.109	0.317	0.673	0.933
	dCov	0.050	0.080	0.202	0.480	0.779	0.045	0.095	0.241	0.559	0.860	0.047	0.100	0.290	0.633	0.924
	dCvdW	0.048	0.076	0.185	0.433	0.754	0.045	0.089	0.228	0.499	0.820	0.044	0.086	0.268	0.571	0.885
	LRT	0.046	0.077	0.132	0.259	0.485	0.053	0.072	0.143	0.251	0.447	0.046	0.068	0.138	0.273	0.447
$(d_1, d_2) = (5, 5)$	sign	0.053	0.059	0.111	0.204	0.411	0.037	0.067	0.119	0.275	0.499	0.039	0.057	0.141	0.299	0.569
	Spe	0.064	0.074	0.121	0.244	0.454	0.053	0.071	0.132	0.347	0.609	0.051	0.074	0.159	0.445	0.735
	Ken	0.054	0.069	0.125	0.210	0.420	0.053	0.065	0.112	0.285	0.495	0.044	0.065	0.145	0.307	0.572
	vdW	0.055	0.064	0.131	0.257	0.509	0.052	0.065	0.143	0.361	0.638	0.041	0.071	0.167	0.432	0.748
	dCov	0.055	0.067	0.136	0.245	0.451	0.054	0.066	0.130	0.341	0.594	0.048	0.061	0.165	0.439	0.713
	dCvdW	0.053	0.062	0.138	0.253	0.501	0.052	0.077	0.143	0.345	0.621	0.043	0.062	0.168	0.403	0.714
	LRT	0.051	0.079	0.161	0.295	0.545	0.053	0.093	0.139	0.268	0.514	0.051	0.073	0.121	0.293	0.501
$(d_1, d_2) = (7, 7)$	sign	0.037	0.071	0.072	0.144	0.261	0.043	0.064	0.095	0.171	0.327	0.045	0.062	0.111	0.215	0.447
	Spe	0.065	0.085	0.094	0.158	0.250	0.057	0.067	0.098	0.179	0.307	0.051	0.065	0.131	0.225	0.434
	Ken	0.053	0.071	0.086	0.154	0.251	0.053	0.062	0.085	0.169	0.322	0.051	0.059	0.125	0.195	0.402
	vdW	0.040	0.079	0.095	0.168	0.309	0.051	0.066	0.101	0.197	0.381	0.047	0.065	0.140	0.257	0.513
	dCov	0.039	0.067	0.068	0.131	0.225	0.047	0.055	0.079	0.163	0.307	0.042	0.065	0.106	0.193	0.402
	dCvdW	0.041	0.077	0.081	0.165	0.302	0.044	0.061	0.100	0.193	0.384	0.047	0.064	0.125	0.251	0.501
	LRT	0.070	0.107	0.169	0.375	0.627	0.059	0.080	0.147	0.307	0.559	0.045	0.081	0.141	0.283	0.539

Table 13: Empirical powers of the seven competing tests in Example 5.1(j). The rejection frequencies of (i) the center-outward sign test ("sign"), (ii) the center-outward Spearman test ("Spe"), (iii) the center-outward Kendall test ("Ken"), (iv) the center-outward van der Waerden score test ("vdW"), (v) the center-outward Wilcoxon score version of the distance covariance test ("dCov"), (vi) the center-outward van der Waerden score version of the distance covariance test ("dCvdW"), and (vii) the likelihood ratio test ("LRT") are based on 1,000 replicates.

	n	432					864					1728				
	τ	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
$(d_1, d_2) = (2, 2)$	sign	0.045	0.091	0.244	0.513	0.798	0.044	0.087	0.250	0.557	0.826	0.048	0.117	0.265	0.565	0.838
	Spe	0.046	0.338	0.831	0.989	1.000	0.053	0.448	0.946	0.999	1.000	0.059	0.572	0.989	1.000	1.000
	Ken	0.045	0.184	0.597	0.921	0.995	0.057	0.205	0.712	0.956	0.999	0.055	0.254	0.765	0.987	1.000
	vdW	0.043	0.325	0.810	0.991	1.000	0.058	0.447	0.943	0.999	1.000	0.058	0.587	0.987	1.000	1.000
	dCov	0.047	0.293	0.815	0.988	1.000	0.058	0.385	0.951	0.999	1.000	0.061	0.533	0.991	1.000	1.000
	dCvdW	0.043	0.262	0.765	0.979	1.000	0.057	0.344	0.919	0.999	1.000	0.057	0.502	0.986	1.000	1.000
	LRT	0.061	0.085	0.134	0.243	0.436	0.063	0.076	0.131	0.249	0.416	0.057	0.087	0.139	0.249	0.425
$(d_1, d_2) = (3, 3)$	sign	0.054	0.087	0.316	0.665	0.913	0.054	0.110	0.401	0.781	0.963	0.051	0.139	0.467	0.850	0.985
	Spe	0.061	0.185	0.649	0.957	0.996	0.057	0.317	0.875	0.995	1.000	0.047	0.473	0.981	1.000	1.000
	Ken	0.057	0.129	0.441	0.835	0.984	0.063	0.157	0.586	0.928	0.998	0.049	0.201	0.704	0.978	0.999
	vdW	0.053	0.165	0.601	0.942	0.993	0.059	0.286	0.826	0.992	1.000	0.051	0.395	0.949	1.000	1.000
	dCov	0.049	0.181	0.660	0.960	0.997	0.062	0.301	0.871	0.997	1.000	0.048	0.444	0.980	1.000	1.000
	dCvdW	0.043	0.155	0.567	0.933	0.995	0.061	0.237	0.795	0.992	1.000	0.049	0.331	0.925	0.999	1.000
	LRT	0.071	0.087	0.158	0.273	0.479	0.066	0.075	0.136	0.280	0.467	0.060	0.074	0.138	0.253	0.447
$(d_1, d_2) = (5, 5)$	sign	0.050	0.075	0.181	0.424	0.745	0.052	0.094	0.261	0.607	0.915	0.049	0.089	0.335	0.761	0.979
	Spe	0.057	0.078	0.206	0.421	0.735	0.051	0.105	0.319	0.669	0.939	0.055	0.119	0.477	0.910	0.997
	Ken	0.056	0.071	0.175	0.401	0.734	0.051	0.095	0.236	0.559	0.885	0.055	0.081	0.285	0.723	0.965
	vdW	0.055	0.082	0.212	0.501	0.831	0.052	0.110	0.341	0.717	0.970	0.051	0.127	0.471	0.912	0.999
	dCov	0.050	0.089	0.217	0.467	0.805	0.056	0.109	0.331	0.711	0.965	0.057	0.125	0.496	0.923	0.999
	dCvdW	0.051	0.090	0.229	0.523	0.847	0.051	0.105	0.340	0.725	0.973	0.055	0.106	0.459	0.893	0.999
	LRT	0.074	0.108	0.163	0.314	0.559	0.053	0.082	0.158	0.290	0.533	0.058	0.075	0.136	0.287	0.491
$(d_1, d_2) = (7, 7)$	sign	0.043	0.066	0.111	0.211	0.423	0.043	0.066	0.129	0.319	0.600	0.054	0.073	0.187	0.441	0.761
	Spe	0.054	0.075	0.128	0.195	0.348	0.060	0.072	0.122	0.280	0.518	0.055	0.074	0.161	0.420	0.727
	Ken	0.051	0.070	0.104	0.187	0.363	0.052	0.068	0.121	0.251	0.507	0.047	0.072	0.150	0.369	0.659
	vdW	0.045	0.086	0.120	0.241	0.463	0.050	0.071	0.140	0.357	0.647	0.049	0.081	0.194	0.522	0.835
	dCov	0.040	0.067	0.097	0.187	0.346	0.052	0.071	0.124	0.277	0.541	0.050	0.080	0.178	0.439	0.762
	dCvdW	0.049	0.066	0.118	0.235	0.465	0.047	0.071	0.141	0.361	0.646	0.047	0.073	0.197	0.528	0.841
	LRT	0.090	0.115	0.179	0.341	0.616	0.087	0.105	0.163	0.313	0.555	0.061	0.077	0.131	0.287	0.520

7 Proofs

Proof of Lemma 3.1. We only need to prove the result for $\text{vec}(\underset{\sim}{\mathbf{W}}_J^{(n)})$ (of which $\text{vec}(\underset{\sim}{\mathbf{W}}_{\text{sign}}^{(n)})$ and $\text{vec}(\underset{\sim}{\mathbf{W}}_{\text{Spearman}}^{(n)})$ are particular cases) and $\text{vec}(\underset{\sim}{\mathbf{W}}_{\text{Kendall}}^{(n)})$.

(a) Starting with $\text{vec}\underset{\sim}{\mathbf{W}}_J^{(n)}$, let

$$\mathbf{Y}_{ki}^{(n)} := J_k\Big(\frac{R_{ki;\pm}^{(n)}}{n_R+1}\Big)\mathbf{S}_{ki;\pm}^{(n)} \quad \text{and} \quad \mathbf{Y}_{ki} := J_k\Big(\big\|\mathbf{F}_{k;\pm}(\mathbf{X}_{ki})\big\|\Big)\mathbf{S}_{ki;\pm}, \quad k=1,2,$$

and rewrite $n^{1/2}\big(\underset{\sim}{\mathbf{W}}_J^{(n)} - \mathbf{W}_J^{(n)}\big)_{j\ell}$, $j=1,\dots,d_1$, $\ell=1,\dots,d_2$ as

$$n^{-1/2}\sum_{i=1}^{n}\Big(\big(\mathbf{Y}_{1i}^{(n)}\big)_j\big(\mathbf{Y}_{2i}^{(n)}\big)_\ell - \big(\mathbf{Y}_{1i}\big)_j\big(\mathbf{Y}_{2i}\big)_\ell\Big).$$

Let us show that

$$\mathrm{E}\Big[\Big\{n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}\big)_j\big(\mathbf{Y}_{2i}\big)_\ell\Big\}^2\Big], \tag{7.1}$$

$$\mathrm{E}\Big[\Big\{n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}^{(n)}\big)_j\big(\mathbf{Y}_{2i}^{(n)}\big)_\ell\Big\}^2\Big], \tag{7.2}$$

and

$$\mathrm{E}\Big[\Big\{n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}^{(n)}\big)_j\big(\mathbf{Y}_{2i}^{(n)}\big)_\ell\Big\}\Big\{n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}\big)_j\big(\mathbf{Y}_{2i}\big)_\ell\Big\}\Big] \tag{7.3}$$

tend to the same limit as n_R and n_S tend to infinity.

First consider (7.1). Due to the independence between $\big(\mathbf{Y}_{1i}\big)_j$ and $\big(\mathbf{Y}_{2i}\big)_\ell$, and the independence between $\big(\mathbf{Y}_{ki}\big)_j$ and $\big(\mathbf{Y}_{ki'}\big)_\ell$, $k=1,2$, $i\neq i'$, $j=1,\dots,d_1$, $\ell=1,\dots,d_2$, we have

$$\mathrm{E}\Big[\Big\{n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}\big)_j\big(\mathbf{Y}_{2i}\big)_\ell\Big\}^2\Big] = \mathrm{E}\big[\big(\mathbf{Y}_{11}\big)_j^2\big]\cdot\mathrm{E}\big[\big(\mathbf{Y}_{21}\big)_\ell^2\big].$$

Turning to (7.2), since $\big((\mathbf{Y}_{11}^{(n)})_j,\dots,(\mathbf{Y}_{1n}^{(n)})_j\big)$ and $\big((\mathbf{Y}_{21}^{(n)})_\ell,\dots,(\mathbf{Y}_{2n}^{(n)})_\ell\big)$ are independent for $j=1,\dots,d_1$, $\ell=1,\dots,d_2$, and in view of Proposition 2.1(ii), we have (see, for instance, Theorem 2 in Hoeffding (1951)),

$$\mathrm{E}\Big[\Big\{n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}^{(n)}\big)_j\big(\mathbf{Y}_{2i}^{(n)}\big)_\ell\Big\}^2\Big] = \frac{1}{n(n-1)}\sum_{\mathbf{u}_1\in\mathfrak{G}_n^{d_1}}\Big(\big(\mathbf{J}_1(\mathbf{u}_1)\big)_j\Big)^2\sum_{\mathbf{u}_2\in\mathfrak{G}_n^{d_2}}\Big(\big(\mathbf{J}_2(\mathbf{u}_2)\big)_\ell\Big)^2,$$

where the right-hand side, by the properties of the grid $\mathfrak{G}_n^{d_k}$ and the fact that score functions J_k, $k=1,2$ have bounded variation, tends to

$$\mathrm{E}\big[\big(\mathbf{J}_1(\mathbf{V}_1)\big)_j^2\big]\cdot\mathrm{E}\big[\big(\mathbf{J}_2(\mathbf{V}_2)\big)_\ell^2\big] = \mathrm{E}\big[\big(\mathbf{Y}_{11}\big)_j^2\big]\cdot\mathrm{E}\big[\big(\mathbf{Y}_{21}\big)_\ell^2\big]$$

with $\mathbf{V}_k\sim\mathrm{U}_{d_k}$, $k=1,2$.

Next, we obtain for (7.3):

$$\mathrm{E}\Big[n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}^{(n)}\big)_j\big(\mathbf{Y}_{2i}^{(n)}\big)_\ell\times n^{-1/2}\sum_{i=1}^{n}\big(\mathbf{Y}_{1i}\big)_j\big(\mathbf{Y}_{2i}\big)_\ell\Big]$$

$$= n^{-1}\Big\{ \sum_{i=1}^{n} \Big(\mathrm{E}\big[\big(\mathbf{Y}_{1i}^{(n)}\big)_j \big(\mathbf{Y}_{1i}\big)_j\big]\Big)\Big(\mathrm{E}\big[\big(\mathbf{Y}_{2i}^{(n)}\big)_\ell \big(\mathbf{Y}_{2i}\big)_\ell\big]\Big)$$
$$+ \sum_{i\neq i'} \Big(\mathrm{E}\big[\big(\mathbf{Y}_{1i'}^{(n)}\big)_j \big(\mathbf{Y}_{1i}\big)_j\big]\Big)\Big(\mathrm{E}\big[\big(\mathbf{Y}_{2i'}^{(n)}\big)_\ell \big(\mathbf{Y}_{2i}\big)_\ell\big]\Big)\Big\}. \tag{7.4}$$

Since, for $k = 1, 2$,

$$\mathrm{E}\big[\big(\mathbf{Y}_{ki}^{(n)}\big)_j \big(\mathbf{Y}_{ki}\big)_j\big] + \sum_{i':i'\neq i} \mathrm{E}\big[\big(\mathbf{Y}_{ki'}^{(n)}\big)_j \big(\mathbf{Y}_{ki}\big)_j\big] = \mathrm{E}\Big[\big(\mathbf{Y}_{ki}\big)_j \sum_{i'=1}^{n} \big(\mathbf{Y}_{ki'}^{(n)}\big)_j\Big]$$
$$= \mathrm{E}\Big[\big(\mathbf{Y}_{ki}\big)_j \sum_{\mathbf{u}\in\mathfrak{G}_n^{d_k}} \big(\mathbf{J}_k(\mathbf{u})\big)_j\Big] = \mathrm{E}\Big[\big(\mathbf{Y}_{ki}\big)_j\Big]\Big(\sum_{\mathbf{u}\in\mathfrak{G}_n^{d_k}} \mathbf{J}_k(\mathbf{u})\Big)_j = 0,$$

and since, by symmetry, for all $i \neq i'$,

$$\mathrm{E}\big[\big(\mathbf{Y}_{ki}^{(n)}\big)_j \big(\mathbf{Y}_{ki}\big)_j\big] = \mathrm{E}\big[\big(\mathbf{Y}_{k1}^{(n)}\big)_j \big(\mathbf{Y}_{k1}\big)_j\big],$$
$$\mathrm{E}\big[\big(\mathbf{Y}_{ki'}^{(n)}\big)_j \big(\mathbf{Y}_{ki}\big)_j\big] = \mathrm{E}\big[\big(\mathbf{Y}_{k2}^{(n)}\big)_j \big(\mathbf{Y}_{k1}\big)_j\big],$$

we deduce that the right-hand side of (7.4) equals

$$n^{-1}\Big\{ n\Big(\mathrm{E}\big[\big(\mathbf{Y}_{11}^{(n)}\big)_j \big(\mathbf{Y}_{11}\big)_j\big]\Big)\Big(\mathrm{E}\big[\big(\mathbf{Y}_{21}^{(n)}\big)_\ell \big(\mathbf{Y}_{21}\big)_\ell\big]\Big)$$
$$+ n(n-1)\Big((n-1)^{-1}\mathrm{E}\big[\big(\mathbf{Y}_{11}^{(n)}\big)_j \big(\mathbf{Y}_{11}\big)_j\big]\Big)\Big((n-1)^{-1}\mathrm{E}\big[\big(\mathbf{Y}_{21}^{(n)}\big)_\ell \big(\mathbf{Y}_{21}\big)_\ell\big]\Big)\Big\}$$
$$= \frac{n}{n-1}\mathrm{E}\big[\big(\mathbf{Y}_{11}^{(n)}\big)_j \big(\mathbf{Y}_{11}\big)_j\big] \cdot \mathrm{E}\big[\big(\mathbf{Y}_{21}^{(n)}\big)_\ell \big(\mathbf{Y}_{21}\big)_\ell\big]. \tag{7.5}$$

In view of the Proof of Theorem 3.1 in del Barrio et al. (2018) and the Proof of Proposition 3.3 in Hallin et al. (2021), we deduce

$$\mathbf{Y}_{k1}^{(n)} - \mathbf{Y}_{k1} \longrightarrow 0 \quad \text{a.s.}$$

while

$$\lim_{n\to\infty} \mathrm{E}\big[\|\mathbf{Y}_{k1}^{(n)}\|^2\big] = \lim_{n_R\to\infty} n_R^{-1} \sum_{r=1}^{n_R} J_k^2\Big(\frac{r}{n_R+1}\Big)$$
$$= \int_0^1 J_k^2(u)\mathrm{d}u = \mathrm{E}\big[\|\mathbf{Y}_{k1}\|^2\big] < \infty.$$

It then follows (see, e.g., part *(iv)* of Theorem 5.7 in Shorack (2017, Chap. 3)) that

$$\mathrm{E}\big[\|\mathbf{Y}_{k1}^{(n)} - \mathbf{Y}_{k1}\|^2\big] \to 0, \quad k = 1, 2 \ \text{ as } n_R, n_S \to \infty.$$

In particular,

$$\mathrm{E}\big[\big(\mathbf{Y}_{k1}^{(n)} - \mathbf{Y}_{k1}\big)_j^2\big] \to 0$$

and thus

$$\mathrm{E}\big[\big(\mathbf{Y}_{k1}^{(n)}\big)_j \big(\mathbf{Y}_{k1}\big)_j\big]$$
$$= 2^{-1}\Big(\mathrm{E}\big[\big(\mathbf{Y}_{k1}^{(n)}\big)_j^2\big] + \mathrm{E}\big[\big(\mathbf{Y}_{k1}\big)_j^2\big] - \mathrm{E}\big[\big(\mathbf{Y}_{k1}^{(n)} - \mathbf{Y}_{k1}\big)_j^2\big]\Big) \to \mathrm{E}\big[\big(\mathbf{Y}_{k1}\big)_j^2\big].$$

It follows that the right-hand side of (7.5) tends to $\mathrm{E}\big[\big(\mathbf{Y}_{11}\big)_j^2\big] \cdot \mathrm{E}\big[\big(\mathbf{Y}_{21}\big)_\ell^2\big]$, which concludes the proof of part (a) of the lemma.

(b) The case of the Kendall matrix $\text{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}})$ is slightly different, although the arguments in the proof are quite similar. We consider the Hájek projection of U-statistics (see, e.g., Proof of Theorem 7.1 in Hoeffding (1948)) for

$$\Big(\mathbf{W}^{(n)}_{\text{Kendall}}\Big)_{j\ell} = \binom{n}{2}^{-1} \sum_{i<i'} \text{sign}\Big(\big(\mathbf{F}_{1;\pm}(\mathbf{X}_{1i}) - \mathbf{F}_{1;\pm}(\mathbf{X}_{1i'})\big)_j\Big) \times \text{sign}\Big(\big(\mathbf{F}_{2;\pm}(\mathbf{X}_{2i}) - \mathbf{F}_{2;\pm}(\mathbf{X}_{2i'})\big)_\ell\Big).$$

It holds by Application 9(d) in Hoeffding (1948) that

$$\Big(\mathbf{W}^{(n)}_{\text{Kendall}}\Big)_{j\ell} = \frac{2}{n}\sum_{i=1}^{n} \Big\{2F_{1j}\Big(\big(\mathbf{F}_{1;\pm}(\mathbf{X}_{1i})\big)_j\Big) - 1\Big\} \times \Big\{2F_{2\ell}\Big(\big(\mathbf{F}_{2;\pm}(\mathbf{X}_{2i})\big)_\ell\Big) - 1\Big\} + o_{\text{q.m.}}(n^{-1/2}), \tag{7.6}$$

where F_{1j}, $F_{2\ell}$ denote the cumulative distribution functions of $\big(\mathbf{F}_{1;\pm}(\mathbf{X}_1)\big)_j$ and $\big(\mathbf{F}_{2;\pm}(\mathbf{X}_2)\big)_\ell$, respectively.

We also have the Hájek projection of combinatorial statistics (see, e.g., page 242 of Barbour and Eagleson (1986); also refer to Chapter II.3.1 of Hájek and Šidák (1967)) for

$$\Big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}}\Big)_{j\ell} = \binom{n}{2}^{-1} \sum_{i<i'} \text{sign}\Big(\big(\mathbf{F}^{(n)}_{1;\pm}(\mathbf{X}_{1i}) - \mathbf{F}^{(n)}_{1;\pm}(\mathbf{X}_{1i'})\big)_j\Big) \times \text{sign}\Big(\big(\mathbf{F}^{(n)}_{2;\pm}(\mathbf{X}_{2i}) - \mathbf{F}^{(n)}_{2;\pm}(\mathbf{X}_{2i'})\big)_\ell\Big),$$

which implies

$$\Big(\underset{\sim}{\mathbf{W}}^{(n)}_{\text{Kendall}}\Big)_{j\ell} = \frac{2}{n}\sum_{i=1}^{n} \Big\{2F^{(n)}_{1j;\text{mid}}\Big(\big(\mathbf{F}^{(n)}_{1;\pm}(\mathbf{X}_{1i})\big)_j\Big) - 1\Big\} \times \Big\{2F^{(n)}_{2\ell;\text{mid}}\Big(\big(\mathbf{F}^{(n)}_{2;\pm}(\mathbf{X}_{2i})\big)_\ell\Big) - 1\Big\} + o_{\text{q.m.}}(n^{-1/2}), \tag{7.7}$$

where $F^{(n)}_{1j;\text{mid}}$, $F^{(n)}_{2\ell;\text{mid}}$ denote the *mid-cumulative distribution functions* (Parzen, 2004) of $\big(\mathbf{F}^{(n)}_{1;\pm}(\mathbf{X}_1)\big)_j$ and $\big(\mathbf{F}^{(n)}_{2;\pm}(\mathbf{X}_2)\big)_\ell$, respectively. Here the *mid-cumulative distribution function* of a random variable X is defined as

$$F_{\text{mid}}(x) \coloneqq [\text{P}(X \le x) + \text{P}(X < x)]/2.$$

Finally, we can show that the difference between the right-hand sides of (7.7) and (7.6) is $o_{\text{q.m.}}(n^{-1/2})$ along the same lines as in the proof for part (a), which completes the proof. □

Proof of Lemma 4.1. It follows from the quadratic mean differentiability of $f_1^{1/2}$ and $f_2^{1/2}$ and the differentiability with respect to δ of $\mathbf{M}_\delta$ that, denoting by $[\mathbf{V}]_1$ and $[\mathbf{V}]_2$, respectively, the first d_1 and last d_2 components of a d-dimensional vector $\mathbf{V}$,

$$\delta \mapsto \Big(\frac{\mathrm{d}\text{P}^{\mathbf{X}}_\delta}{\mathrm{d}\mu_d}(\mathbf{x})\Big)^{1/2} = \Big(\big|\det\big(\mathbf{M}_\delta\big)\big|^{-1}\Big)^{1/2} f_1([\mathbf{M}_\delta^{-1}\mathbf{x}]_1) f_2([\mathbf{M}_\delta^{-1}\mathbf{x}]_2)$$

also is differentiable in quadratic mean. The quadratic expansion of the log-likelihood ratio

$$\log \frac{\mathrm{dP}^{\mathbf{X}}_{\delta+n^{-1/2}\tau}}{\mathrm{dP}^{\mathbf{X}}_{\delta}}(\mathbf{X}^{(n)})$$

follows (see, e.g., Theorem 12.2.3 (i) in Lehmann and Romano (2005)), yielding, at $\delta = 0$, the second-order asymptotic representation (4.4). The explicit forms (4.2) and (4.3) of the central sequence $\Delta^{(n)}(\mathbf{X}^{(n)})$ and the Fisher information γ^2 for $\delta = 0$ are obtained via elementary differentiation. The asymptotic normality result for $\Delta^{(n)}(\mathbf{X}^{(n)})$ follows from part (ii) of the same Theorem 12.2.3 in Lehmann and Romano (2005). □

Proof of Theorem 4.1. We only give the proof for $\underset{\sim}{T}^{(n)}_J$; the proof for $\underset{\sim}{T}^{(n)}_{\text{Kendall}}$ is similar and hence omitted. Applying the multivariate central limit theorem (Bhattacharya and Ranga Rao, 1986, Equation (18.24)) to the asymptotic form of $\Lambda^{(n)}(\mathbf{X}^{(n)})$ (see Lemma 3.1), we deduce that, under the null hypothesis,

$$\Big(n^{1/2}\mathrm{vec}(\mathbf{W}^{(n)}_J), \Lambda^{(n)}(\mathbf{X}^{(n)})\Big) \rightsquigarrow N_{d_1d_2+1}\left(\begin{pmatrix}\mathbf{0}_{d_1d_2}\\ -\frac{1}{2}\tau^2\gamma^2\end{pmatrix}, \begin{pmatrix}\sigma_J^2\mathbf{I}_{d_1d_2} & \tau\mathbf{v}\\ \tau\mathbf{v}' & \tau^2\gamma^2\end{pmatrix}\right)$$

where $\sigma_J^2 := \sigma_{J_1}^2\sigma_{J_2}^2/(d_1d_2)$,

$$\mathbf{v} := \mathrm{Cov}_{H_0}\Big[\mathrm{vec}\Big(\mathbf{J}_1(\mathbf{F}_{1;\pm}(\mathbf{X}_1))\mathbf{J}_2(\mathbf{F}_{2;\pm}(\mathbf{X}_2))'\Big),$$
$$\mathbf{X}_1'\mathbf{M}_2'\boldsymbol{\varphi}_2(\mathbf{X}_2) + \mathbf{X}_2'\mathbf{M}_1'\boldsymbol{\varphi}_1(\mathbf{X}_1) - \Big(\mathbf{X}_1'\boldsymbol{\varphi}_1(\mathbf{X}_1) - d_1\Big) - \Big(\mathbf{X}_2'\boldsymbol{\varphi}_2(\mathbf{X}_2) - d_2\Big)\Big],$$

and thus, by Lemma 4.1,

$$\Big(n^{1/2}\mathrm{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_J), \Lambda^{(n)}(\mathbf{X}^{(n)})\Big) \rightsquigarrow N_{d_1d_2+1}\left(\begin{pmatrix}\mathbf{0}_{d_1d_2}\\ -\frac{1}{2}\tau^2\gamma^2\end{pmatrix}, \begin{pmatrix}\sigma_J^2\mathbf{I}_{d_1d_2} & \tau\mathbf{v}\\ \tau\mathbf{v}' & \tau^2\gamma^2\end{pmatrix}\right).$$

Then we employ a corollary to Le Cam's third lemma (van der Vaart, 1998, Example 6.7) to obtain that, under the alternative sequences,

$$n^{1/2}\mathrm{vec}(\underset{\sim}{\mathbf{W}}^{(n)}_J) \rightsquigarrow N_{d_1d_2}(\tau\mathbf{v}, \sigma_J^2\mathbf{I}_{d_1d_2}).$$

The result follows. □

Proof of Proposition 4.1. Direct computation yields

$$\begin{aligned}
&\mathrm{E}_{H_0}\Big[\mathbf{J}_1(\mathbf{F}_{1;\pm}(\mathbf{X}_1))\Big(\mathbf{X}_1'\mathbf{M}_2'\boldsymbol{\varphi}_2(\mathbf{X}_2)\Big)\mathbf{J}_2(\mathbf{F}_{2;\pm}(\mathbf{X}_2))'\Big]\\
&= \mathrm{E}_{H_0}\Big[\mathbf{J}_1\Big(\frac{\mathbf{Y}_1}{\|\mathbf{Y}_1\|}\tilde{F}_1(\|\mathbf{Y}_1\|)\Big)\Big(\mathbf{Y}_1'\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\varphi}_2(\boldsymbol{\Sigma}_2^{1/2}\mathbf{Y}_2)\Big)\mathbf{J}_2\Big(\frac{\mathbf{Y}_2}{\|\mathbf{Y}_2\|}\tilde{F}_2(\|\mathbf{Y}_2\|)\Big)'\Big]\\
&= \mathrm{E}_{H_0}\Big[\frac{\mathbf{Y}_1}{\|\mathbf{Y}_1\|}J_1\Big(\tilde{F}_1(\|\mathbf{Y}_1\|)\Big)\Big(\mathbf{Y}_1'\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\varphi}_2(\boldsymbol{\Sigma}_2^{1/2}\mathbf{Y}_2)\Big)J_2\Big(\tilde{F}_2(\|\mathbf{Y}_2\|)\Big)\frac{\mathbf{Y}_2'}{\|\mathbf{Y}_2\|}\Big]\\
&= D_1C_2\boldsymbol{\Sigma}_1^{1/2}\mathbf{M}_2'\boldsymbol{\Sigma}_2^{-1/2}.
\end{aligned}$$

The first result then follows from Hannan (1956, Equation (5)). The next part follows from the proof of Propositions 1 and 2 in Hallin and Paindaveine (2008); see also Theorem 1 in Paindaveine (2004) and Proposition 7 in Hallin and Paindaveine (2002a). □

Acknowledgement

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 883818).

References

[1] Barbour, A. D. and Eagleson, G. K. (1986). Random association of symmetric arrays. *Stochastic Anal. Appl.*, 4(3):239–281.

[2] Bhattacharya, R. N. and Ranga Rao, R. (1986). *Normal Approximation and Asymptotic Expansions* (Rpt. ed.). Robert E. Krieger Publishing Co., Inc., Melbourne, FL.

[3] Blomqvist, N. (1950). On a measure of dependence between two random variables. *Ann. Math. Statist.*, 21(4):593–600.

[4] Carlier, G., Chernozhukov, V., and Galichon, A. (2016). Vector quantile regression: an optimal transport approach. *Ann. Statist.*, 44(3):1165–1192.

[5] Carlier, G., Chernozhukov, V., and Galichon, A. (2017). Vector quantile regression beyond the specified case. *J. Multivariate Anal.*, 161:96–102.

[6] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.*, 29(4):972–994.

[7] Chernozhukov, V., Galichon, A., Hallin, M., and Henry, M. (2017). Monge-Kantorovich depth, quantiles, ranks and signs. *Ann. Statist.*, 45(1):223–256.

[8] Deb, N., Bhattacharya, B. B., and Sen, B. (2021). Efficiency lower bounds for distribution-free Hotelling-type two-sample tests based on optimal transport. Available at arXiv:2104.01986v2.

[9] Deb, N. and Sen, B. (2022). Multivariate rank-based distribution-free nonparametric testing using measure transportation. *J. Amer. Statist. Assoc.* (in press).

[10] del Barrio, E., Cuesta-Albertos, J. A., Hallin, M., and Matrán, C. (2018). Smooth cyclically monotone interpolation and empirical center-outward distribution functions. Available at arXiv:1806.01238v1.

[11] del Barrio, E., González-Sanz, A., and Hallin, M. (2020). A note on the regularity of optimal-transport-based center-outward distribution and quantile functions. *J. Multivariate Anal.*, 180:104671, 13.

[12] del Barrio, E., González-Sanz, A., and Hallin, M. (2022). Nonparametric multiple-output center-outward quantile regression. Available at arXiv:2204.11756v2.

[13] Figalli, A. (2018). On the continuity of center-outward distribution and quantile functions. *Nonlinear Anal.*, 177(part B):413–421.

[14] Garel, B. and Hallin, M. (1995). Local asymptotic normality of multivariate ARMA processes with a linear trend. *Ann. Inst. Statist. Math.*, 47(3):551–579.

[15] Ghosal, P. and Sen, B. (2022). Multivariate ranks and quantiles using optimal transport: Consistency, rates and nonparametric testing. *Ann. Statist.*, 50(2):1012–1037.

[16] Gieser, P. W. (1993). *A new nonparametric test for independence between two sets of variates.* PhD thesis, University of Florida. Available at https://ufdc.ufl.edu/AA00003658/00001 and https://www.proquest.com/docview/304041219.

[17] Gieser, P. W. and Randles, R. H. (1997). A nonparametric test of independence between two vectors. *J. Amer. Statist. Assoc.*, 92(438):561–567.

[18] Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests.* Academic Press, New York-London; Academia Publishing House of the Czechoslovak Academy of Sciences, Prague.

[19] Hallin, M. (2017). On distribution and quantile functions, ranks and signs in $\mathbb{R}^d$: a measure transportation approach. Available at https://ideas.repec.org/p/eca/wpaper/2013-258262.html.

[20] Hallin, M. (2022). Measure transportation and statistical decision theory. *Annu. Rev. Stat. Appl.*, 9:401–424.

[21] Hallin, M., del Barrio, E., Cuesta-Albertos, J., and Matrán, C. (2021). Distribution and quantile functions, ranks and signs in dimension d: A measure transportation approach. *Ann. Statist.*, 49(2):1139–1165.

[22] Hallin, M., Hlubinka, D., and Hudecová, Š. (2022a). Efficient fully distribution-free center-outward rank tests for multiple-output regression and MANOVA. *J. Amer. Statist. Assoc.* (in press).

[23] Hallin, M., La Vecchia, D., and Liu, H. (2022b). Center-outward R-estimation for semiparametric VARMA models. *J. Amer. Statist. Assoc.*, 117(538):925–938.

[24] Hallin, M., La Vecchia, D., and Liu, H. (2023). Rank-based testing for semiparametric VAR models: a measure transportation approach. *Bernoulli*, 29(1):229–273.

[25] Hallin, M. and Mordant, G. (2021). On the finite-sample performance of measure transportation-based multivariate rank tests. Available at arXiv:2111.04705v2.

[26] Hallin, M. and Paindaveine, D. (2002a). Optimal procedures based on interdirections and pseudo-Mahalanobis ranks for testing multivariate elliptic white noise against ARMA dependence. *Bernoulli*, 8(6):787–815.

[27] Hallin, M. and Paindaveine, D. (2002b). Optimal tests for multivariate location based on interdirections and pseudo-Mahalanobis ranks. *Ann. Statist.*, 30(4):1103–1133.

[28] Hallin, M. and Paindaveine, D. (2008). Chernoff-Savage and Hodges-Lehmann results for Wilks' test of multivariate independence. In *Beyond parametrics in interdisciplinary research: Festschrift in honor of Professor Pranab K. Sen*, volume 1 of *Inst. Math. Stat. (IMS) Collect.*, pages 184–196. Inst. Math. Statist., Beachwood, OH.

[29] Hannan, E. J. (1956). The asymptotic powers of certain tests based on multiple correlations. *J. Roy. Statist. Soc. Ser. B*, 18(2):227–233.

[30] Hodges, Jr., J. L. and Lehmann, E. L. (1956). The efficiency of some nonparametric competitors of the t-test. *Ann. Math. Statist.*, 27(2):324–335.

[31] Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.*, 19(3):293–325.

[32] Hoeffding, W. (1951). A combinatorial central limit theorem. *Ann. Math. Statist.*, 22(4):558–566.

[33] Kendall, M. G. (1938). A new measure of rank correlation. *Biometrika*, 30(1/2):81–93.

[34] Konijn, H. S. (1956). On the power of certain tests for independence in bivariate populations. *Ann. Math. Statist.*, 27(2):300–323.

[35] Lehmann, E. L. and Romano, J. P. (2005). *Testing Statistical Hypotheses* (3rd ed.). Springer Texts in Statistics. Springer, New York.

[36] Lin, J. (2017). *Copula versions of RKHS-based and distance-based criteria*. PhD thesis, Pennsylvania State University. Available at https://etda.libraries.psu.edu/catalog/14485jul268.

[37] Lind, B. and Roussas, G. (1972). A remark on quadratic mean differentiability. *Ann. Math. Statist.*, 43(3):1030–1034.

[38] Liu, R. Y. and Singh, K. (1993). A quality index based on data depth and multivariate rank tests. *J. Amer. Statist. Assoc.*, 88(421):252–260.

[39] McCann, R. J. (1995). Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.*, 80(2):309–323.

[40] Moon, H. and Chen, K. (2022). Interpoint-ranking sign covariance for the test of independence. *Biometrika*, 109(1):165–179.

[41] Mordant, G. (2021). *Transporting probability measures: some contributions to statistical inference*. PhD thesis, UCL-Université Catholique de Louvain. Available at http://hdl.handle.net/2078.1/250201.

[42] Oja, H. (2010). *Multivariate Nonparametric Methods with R: An Approach Based on Spatial Signs and Ranks*, volume 199 of *Lecture Notes in Statistics*. Springer, New York.

[43] Paindaveine, D. (2004). A unified and elementary proof of serial and nonserial, univariate and multivariate, Chernoff-Savage results. *Stat. Methodol.*, 1(1-2):81–91.

[44] Parzen, E. (2004). Quantile probability and statistical data modeling. *Statist. Sci.*, 19(4):652–662.

[45] Puri, M. L. and Sen, P. K. (1971). *Nonparametric Methods in Multivariate Analysis*. John Wiley & Sons, Inc., New York-London-Sydney.

[46] Randles, R. H. (1989). A distribution-free multivariate sign test based on interdirections. *J. Amer. Statist. Assoc.*, 84(408):1045–1050.

[47] Shi, H., Drton, M., and Han, F. (2022a). Distribution-free consistent independence tests via center-outward ranks and signs. *J. Amer. Statist. Assoc.*, 117(537):395–410.

[48] Shi, H., Hallin, M., Drton, M., and Han, F. (2022b). On universally consistent and fully distribution-free rank tests of vector independence. *Ann. Statist.*, 50(4):1933–1959.

[49] Shorack, G. R. (2017). *Probability for Statisticians* (2nd ed.). Springer Texts in Statistics. Springer, Cham, Switzerland.

[50] Spearman, C. (1904). The proof and measurement of association between two things. *Amer. J. Psychol.*, 15(1):72–101.

[51] Taskinen, S., Kankainen, A., and Oja, H. (2003). Sign test of independence between two random vectors. *Statist. Probab. Lett.*, 62(1):9–21.

[52] Taskinen, S., Kankainen, A., and Oja, H. (2004). Rank scores tests of multivariate independence. In *Theory and applications of recent robust methods*, Stat. Ind. Technol., pages 329–341. Birkhäuser, Basel.

[53] Taskinen, S., Oja, H., and Randles, R. H. (2005). Multivariate nonparametric tests of independence. *J. Amer. Statist. Assoc.*, 100(471):916–925.

[54] van der Vaart, A. W. (1998). *Asymptotic Statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, United Kingdom.

[55] Weihs, L., Drton, M., and Meinshausen, N. (2018). Symmetric rank covariances: a generalized framework for nonparametric measures of dependence. *Biometrika*, 105(3):547–562.

[56] Wilcoxon, F. (1945). Individual comparisons by ranking methods. *Biometrics Bulletin*, 1(6):80–83.

[57] Wilks, S. S. (1935). On the independence of k sets of normally distributed statistical variables. *Econometrica*, 3(3):309–326.

[58] Zuo, Y. and He, X. (2006). On the limiting distributions of multivariate depth-based rank sum statistics and related tests. *Ann. Statist.*, 34(6):2879–2896.