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Jonathan Hoseana & Ryan Aziz

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The Generalized Sum of Remainders Map and Its Fixed Points

JONATHAN HOSEANA  
Parahyangan Catholic University  
Bandung, Indonesia  
j.hoseana@unpar.ac.id

RYAN AZIZ  
Universitas Prasetiya Mulya  
South Jakarta, Indonesia  
ryan.aziz@pmbs.ac.id

Over the last few decades, number theory has been used as the main tool for studying the iteration of some interesting maps. Among others (but not discussed here), there is Collatz’s famous $3n + 1$ map: a self-map on the set of natural numbers which halves the argument if it is even and computes three times the argument plus one if it is odd. A famous open conjecture states that every orbit of this map eventually reaches one. Similarly, mathematicians have studied Bulgarian solitaire: a self-map on the space of finite integer multisets, some of whose orbits are proved to be eventually constant [4, 5]. In the same spirit, in this paper, we address the iteration of another number-theoretic map which produces intriguing orbits.

Take a finite nonempty set of positive integers, say $K = \{1, 3, 4\}$. Starting with any positive integer, say 11, produce a new number by computing the sum of the remainders upon dividing this integer by every element of $K$. Since dividing 11 by 1, by 3, and by 4 leaves remainders 0, 2, and 3, respectively, our new number is $0 + 2 + 3 = 5$. Now, apply the same process into this new number. That is, divide it by every element of $K$ and compute the sum of the remainders. Doing this repeatedly, we obtain the following sequence: $(11, 5, 3, 3, 3, 3, \ldots)$. In this case, the sequence becomes constant. However, if we start with, for example, 8, we obtain the sequence $(8, 2, 4, 1, 2, 4, 1, \ldots)$ which turns out to be periodic of period 3. The question is: Starting with any positive integer, is there a way to predict the behavior of the resulting sequence?

More formally, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any finite nonempty set $K$ of positive integers, we define a map $\rho_K : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by

$$\rho_K(x) = \sum_{k \in K} (x \mod k)$$

for every $x \in \mathbb{N}_0$. We are interested in the dynamics of this map, that is, the behavior of the sequence $(x, \rho_K(x), \rho_K(\rho_K(x)), \ldots)$ for any given initial value $x \in \mathbb{N}_0$. This sequence is referred to as the orbit of $x$.

Some maps similar to $\rho_K$ are found in the literature. For instance, Spivey [6] discusses a map which computes the sum of the remainders upon dividing any positive integer by every positive integer not exceeding it. Therefore, in this map the set $K$ depends on the argument, that is, $K = \{1, 2, \ldots\}$. Boju and Funar [1, page 76] eliminate this dependence by using $K = \{1, 2, \ldots, m\}$, where $m$ is a fixed positive integer. Our map (1) can be viewed as a further generalization, which consists of replacing $K$ with any fixed finite nonempty set of positive integers. The dynamics of our map...
Therefore, 

\( \rho_K \), therefore, may also share some similarities with the maps found in the literature, and so our study of \( \rho_K \) might also be beneficial towards understanding them in the perspective of dynamical systems.

To illustrate a typical behavior of our map, let us now return to our choice \( K = \{1, 3, 4\} \). We can compute

\[
(\rho_K(x))_{x=0}^\infty = (0, 2, 4, 3, 1, 3, 2, 4, 2, 1, 3, 5, 0, 2, 4, 3, 1, 3, 2, 4, 2, 1, 3, 5, \ldots ) ,
\]

from which we make the following two important observations.

- Notice that \( \rho_K(x + 12) = \rho_K(x) \) for every \( x \in \mathbb{N}_0 \). That is, the sequence \( (\rho_K(x))_{x=0}^\infty \) is periodic of period \( 12 = \text{lcm} \{1, 3, 4\} \). Therefore, it suffices to restrict our attention to initial values in \( \{0, 1, \ldots, 11\} \).

- For every \( x \in \mathbb{N}_0 \), since \( x \mod 1 \in \{0\} \), \( x \mod 3 \in \{0, 1, 2\} \), and \( x \mod 4 \in \{0, 1, 2, 3\} \), we have

\[
\rho_K(x) = (x \mod 1) + (x \mod 3) + (x \mod 4) \in \{0, 1, 2, 3, 4, 5\}.
\]

Having computed the images of all initial values of interest, we can now give a complete description of all their orbits by constructing the directed graph shown in Figure 1, having vertex set \( \{0, 1, \ldots, 11\} \) and edge set

\[
\{(x, \rho_K(x)) : x \in \{0, 1, \ldots, 11\}\}.
\]

We call this graph the orbit graph of the map \( \rho_K \).

![Figure 1](image)

The graph shows the existence of three disjoint sets which form cycles, namely \( \{0\} \), \( \{1, 2, 4\} \), and \( \{3\} \). In other words, 0 and 3 are periodic points of prime period 1, i.e., fixed points, whereas 1, 2, and 4 are periodic points of prime period 3 forming a 3-cycle. Different choices of the set \( K \) could give periodic points of other periods, as shown in Figures 2 and 3.

For any finite \( K \subset \mathbb{N} \), it is clear that \( \rho_K \) takes only finitely many possible values, so it is obvious that every initial condition is eventually periodic. However, the existence of periodic points of various specified periods is not at all obvious. Given any finite nonempty set \( K \) of positive integers and any \( n \in \mathbb{N} \), one could question, for instance, whether the map \( \rho_K \) possesses periodic points of prime period \( n \).

We would agree that answering this question demands a thorough analysis of the number-theoretic properties of the set \( K \), as well as that of the initial value \( x \). This paper aims to begin this analysis for the simplest type of periodic points, namely fixed points.
The general setup

We briefly restate our general setup. For the precise definitions of dynamical and number-theoretic terminology used in this paper, we refer the reader to Devaney [3] and Burton [2], respectively.

Let $K \subset \mathbb{N}$ be finite and nonempty. Define the map $\rho_K : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ by the rule

$$\rho_K(x) = \sum_{k \in K} (x \mod k)$$

for every $x \in \mathbb{N}_0$. First, we prove the general versions of our two observations in the introduction.

**Proposition 1.** The sequence $(\rho_K(x))_{x=0}^{\infty}$ is periodic of period $\text{lcm } K$.

**Proof.** Notice that

$$\rho_K(x + \text{lcm } K) = \sum_{k \in K} [(x + \text{lcm } K) \mod k]$$

$$= \sum_{k \in K} (x \mod k) = \rho_K(x),$$

where we have used the fact that $k$ evenly divides $\text{lcm } K$ for every $k \in K$. ■

**Proposition 2.** For every nonnegative integer $x$,

$$\rho_K(x) \in \left\{0, 1, \ldots, \sum_{k \in K} k - |K| \right\}. $$

**Proof.** For every nonnegative integer $x$, we have $x \mod k \in \{0, 1, \ldots, k - 1\}$ for all $k \in K$, so $\rho_k(x) = \sum_{k \in K} (x \mod k)$ belongs to the required set. ■

These two propositions allow us to restrict the domain and the codomain of $\rho_K$, so that we can formally define it as follows:

**Definition 1.** Let $K \subset \mathbb{N}$ be finite and nonempty, and let

$$I_K := \{0, 1, \ldots, \text{lcm } K - 1\} \quad \text{and} \quad S_K := \left\{0, 1, \ldots, \sum_{k \in K} k - |K| \right\}. $$

The generalized sum of remainders map over $K$ is defined as the map $\rho_K : I_K \cup S_K \rightarrow S_K$ given by

$$\rho_K(x) = \sum_{k \in K} (x \mod k).$$

Looking at our previous example, one might be tempted to define $\rho_K$ as a map from $I_K$ to $S_K$. However, notice that we can have $I_K \subset S_K$ (see, for example, Figure 3), in which case the map $\rho_K$ is not well-defined for every element in $S_K \setminus I_K$. Hence, the domain of $\rho_K$ is set to be $I_K \cup S_K$. 

Fixed points

It is clear that 0 is a fixed point of $\rho_K$ for any finite nonempty $K \subset \mathbb{N}$. If $|K| = 1$, it is easy to see that every point in $I_K \cup S_K$ is fixed. Therefore, we only need to consider the case $|K| \geq 2$. We now give a sufficient condition for the existence of nonzero fixed points. We define the notation $L_K := \text{lcm}(K \setminus \{\max K\})$. That is, $L_K$ is the least common multiple of the numbers in the set $K \setminus \{\max K\}$.

**Theorem 1.** Let $|K| \geq 2$, and let $L_K < \max K$. Then every nonnegative multiple of $L_K$ less than $\max K$ is a fixed point of $\rho_K$.

**Proof.** Let $x < \max K$ be an arbitrary nonnegative multiple of $L_K$. Then $k \mid x$, which implies $x \mod k = 0$, for all $k \in K \setminus \{\max K\}$. Consequently,

$$
\rho_K(x) = \sum_{k \in K} (x \mod k)
= \sum_{k \in K \setminus \{\max K\}} (x \mod k) + [x \mod (\max K)]
= 0 + [x \mod (\max K)]
= x \mod (\max K) = x,
$$

since $x < \max K$. This means that $x$ is a fixed point of $\rho_K$. □

Theorem 1 tells us that under the given condition, every nonnegative multiple of $L_K$ less than $\max K$ is a fixed point of $\rho_K$. However, it does not guarantee that there is no other fixed point which is not a multiple of $L_K$. As an example, in the case of $K = \{3, 6, 7\}$, this theorem tells us that every nonnegative multiple of lcm $\{3, 6\} = 6$ less than $\max\{3, 6, 7\} = 7$ is a fixed point of $\rho_K$. These multiples are 0 and 6. However, one can easily verify that 11 is also a fixed point of $\rho_K$. The orbit graph of $\rho_K$ is displayed in Figure 2.

![Figure 2](image)

**Figure 2** The orbit graph of $\rho_K$, where $K = \{3, 6, 7\}$.

If the given condition is violated, then a nonzero fixed point may or may not exist. In some cases, it does. For instance, Boju and Funar’s map [1, page 76] with...
Proposition 3. Let \( |K| \geq 2 \), and let \( L_K > \max K \). Let \( x \) be a nonzero fixed point of \( \rho_K \) which is not a multiple of \( L_K \). Then

1. \( x \geq \max K \), and
2. \( x = \max K \) if and only if \( x \in K \cap I_K \setminus \{x\} \) and \( x \) is a fixed point of \( \rho_{K \setminus \{x\}} \).

Proof. To prove (1), suppose for a contradiction that \( x < \max K \). Then \( x \mod (\max K) = x \). Since \( x \) is a fixed point of \( \rho_K \), we have \( \rho_K(x) = x \), that is,

\[
\sum_{k \in K} (x \mod k) = x
\]

\[
\sum_{k \in K \setminus \{\max K\}} (x \mod k) + [x \mod (\max K)] = x
\]

\[
\sum_{k \in K \setminus \{\max K\}} (x \mod k) = 0.
\]

Since \( x \mod k \geq 0 \) for every \( k \in K \setminus \{\max K\} \), we must have \( x \mod k = 0 \) for every \( k \in K \setminus \{\max K\} \). This means that \( k | x \) for every \( k \in K \setminus \{\max K\} \). It follows that \( L_K | x \), which is a contradiction.

The necessity part of (2) is a direct consequence of (1) because \( x \in K \) and \( x \geq \max K \) implies \( x = \max K \). To prove the sufficiency, suppose \( x = \max K \). Clearly, \( x \in K \). Also, \( 0 < x \leq L_K - 1 \), so \( x \in I_K \setminus \{x\} \). Therefore, \( x \) belongs to the domain of \( \rho_{K \setminus \{x\}} \). Since \( x \) is a fixed point of \( \rho_K \), we have \( \rho_K(x) = x \), and therefore

\[
\rho_{K \setminus \{x\}}(x) = \sum_{k \in K \setminus \{x\}} (x \mod k) = \sum_{k \in K \setminus \{x\}} (x \mod k) + (x \mod x)
\]

\[
= \sum_{k \in K} (x \mod k) = \rho_K(x) = x.
\]

This proves that \( x \) is a fixed point of \( \rho_{K \setminus \{x\}} \). \( \blacksquare \)

In Figure 1, we can see that the trivial fixed point 0 appears as an isolated vertex. This prompts the following definition:

Definition 2. A fixed point of \( \rho_K \) is said to be isolated if it is not the image of any element of \( I_K \cup S_K \) other than itself.

In general, the trivial fixed point is not always isolated. For example, take \( K = \{1, 3, 9\} \). Here we have \( I_K = \{0, 1, \ldots, 8\} \) and \( S_K = \{0, 1, \ldots, 10\} \). The orbit graph of \( \rho_K \) in Figure 3 shows that \( \rho_K(9) = 0 \).

![Figure 3](image_url)
Proposition 4. The trivial fixed point of the map \( \rho_K \) is isolated if and only if \( S_K \subseteq I_K \).

Proof. First, suppose \( S_K \subseteq I_K \). Since for every \( x \in I_K \cup S_K = I_K \) we have \( x < \text{lcm } K \), it follows that \( \rho_K(x) = 0 \) if and only if \( x = 0 \). Conversely, suppose \( S_K \supset I_K \). Then \( \text{lcm } K \in S_K \subseteq I_K \cup S_K \), so we can compute \( \rho_K(\text{lcm } K) = 0 \), proving that the trivial fixed point is not isolated. \( \blacksquare \)

One might question whether nonzero isolated fixed points exist. It is easy to show that 1 is a nonzero isolated fixed point of \( \rho_K \), where \( K = \{1, 2\} \). However, determining a condition for their existence is not a straightforward task. Tables displaying the list of nonzero isolated fixed points for various choices of \( K \) appear in an appendix to the online version of this article.

Pairwise relatively prime set

We shall prove that in the special case in which the integers in \( K \) are pairwise relatively prime, the only fixed points of \( \rho_K \) are those given by Theorem 1 (as seen in the example in the introduction). Before doing so, we first prove the following lemma:

Lemma 1. Let \( k \in \mathbb{N} \), and let \( x_1, x_2, \ldots, x_k \in \mathbb{N} \) be pairwise distinct. Then
\[
x_1x_2 \ldots x_k \geq x_1 + x_2 + \ldots + x_k - k.
\]

Proof. First we prove that for any distinct \( a, b \in \mathbb{N} \) with \( a, b \geq 2 \) we have
\[
ab \geq a + b.
\]
Without loss of generality, let \( a \geq b \). Then \( ab \geq a \cdot 2 = a \cdot (1 + 1) = a + a \geq a + b \), as required. Now we prove inequality (2). Since it is trivially true for \( k = 1 \), let \( k \in \mathbb{N} \) with \( k \geq 2 \), and let \( x_1, x_2, \ldots, x_k \in \mathbb{N} \) be pairwise distinct. If \( x_i \geq 2 \) for every \( i \in \{1, 2, \ldots, k\} \), then applying inequality (3) repeatedly yields
\[
x_1x_2 \ldots x_{k-2}x_{k-1}x_k = (x_1x_2 \ldots x_{k-2}x_{k-1})x_k
\geq (x_1x_2 \ldots x_{k-2})x_{k-1} + x_k
\geq x_1x_2 \ldots x_{k-2} + x_{k-1} + x_k
\vdots
\geq x_1 + x_2 + \ldots + x_k
\geq x_1 + x_2 + \ldots + x_k - k,
\]
as desired. Otherwise, there exists \( i \in \{1, 2, \ldots, k\} \) such that \( x_i < 2 \), implying that \( x_i = 1 \). Then \( x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}, x_k \) are all \( \geq 2 \) since \( x_1, x_2, \ldots, x_k \) are pairwise distinct positive integers. Therefore,
\[
x_1x_2 \ldots x_k = x_1x_2 \ldots x_{i-1} \cdot 1 \cdot x_{i+1} \ldots x_{k-1}x_k
= x_1x_2 \ldots x_{i-1}x_{i+1} \ldots x_{k-1}x_k
\geq x_1 + x_2 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_{k-1} + x_k
= x_1 + x_2 + \ldots + x_{i-1} + 1 + x_{i+1} + \ldots + x_{k-1} + x_k - 1
= x_1 + x_2 + \ldots + x_{i-1} + x_i + x_{i+1} + \ldots + x_{k-1} + x_k - 1
\]
≥ x_1 + x_2 + \ldots + x_k - k,

where the first inequality follows from applying inequality (3) repeatedly, as above.

The following corollary of Lemma 1 is easily derived.

**Corollary 1.** For any finite nonempty subset $K$ of pairwise relatively prime positive integers, we have $|I_K| \geq |S_K| - 1$.

**Proof.** By Lemma 1, we have

$$|I_K| = \text{lcm} K = \prod_{k \in K} k \geq \sum_{k \in K} k - |K| = \left( \sum_{k \in K} k - |K| + 1 \right) - 1 = |S_K| - 1,$$

as desired.

We now state and prove our promised result.

**Theorem 2.** Let $K$ be a finite set of pairwise relatively prime positive integers with $|K| \geq 2$ and such that $\prod_{k \in K \setminus \{\max K\}} k < \max K$. Then the fixed points of $\rho_K$ are precisely the nonnegative multiples of $\prod_{k \in K \setminus \{\max K\}} k$ less than $\max K$.

**Proof.** Since in this case we have $L_K = \prod_{k \in K \setminus \{\max K\}} k$, then by Theorem 1, every nonnegative multiple of $\prod_{k \in K \setminus \{\max K\}} k$ less than $\max K$ is a fixed point of $\rho_K$. Now we prove the converse. Let $x$ be an arbitrary fixed point of $\rho_K$, then $x = \rho_K(x)$, that is, $x = \sum_{k \in K} (x \mod k)$. Equivalently,

$$x - \lfloor x \mod (\max K) \rfloor = \sum_{k \in K \setminus \{\max K\}} x \mod k.$$

Since the left-hand side is a multiple of $\max K$, then so is the right-hand side. But the right-hand side is a nonnegative integer not exceeding

$$\sum_{k \in K \setminus \{\max K\}} k - (|K| - 1) \leq \prod_{k \in K \setminus \{\max K\}} k < \max(K),$$

where the first inequality follows from Lemma 1. This forces $x - \lfloor x \mod (\max K) \rfloor = 0$, i.e.,

$$0 = \sum_{k \in K \setminus \{\max K\}} (x \mod k).$$

Now each summand on the right hand side is nonnegative, so they must all be zero, implying that every element of $K \setminus \{\max K\}$ divides $x$. It follows that $x$ is a multiple of $L_K = \prod_{k \in K \setminus \{\max K\}} k$, as desired.

In the special case where $|K| = 2$, the hypothesis of Theorem 2 is always satisfied, and therefore we can easily deduce the following corollary:

**Corollary 2.** Let $K$ be a set of two relatively prime positive integers. Then the map $\rho_K$ has exactly $\left\lceil \frac{\max(K)}{\min(K)} \right \rceil$ fixed points, specifically, all nonnegative multiples of $\min(K)$ less than $\max(K)$.

This corollary gives a complete description of fixed points of $\rho_K$ for $|K| = 2$. One can also see that if $1 \notin K$, then we have $I_{K \cup \{1\}} = I_K$, $S_{K \cup \{1\}} = S_K$, and indeed, $\rho_{K \cup \{1\}} = \rho_K$. Therefore, the above corollary also extends to the case in which $|K| = 3$, $1 \in K$, and $\gcd(K \setminus \{1\}) = 1$. 
Conclusion

We have introduced the generalized sum of remainders map $\rho_K$, given by (1). In the case $\text{lcm}(K \setminus \text{max}(K)) < \text{max}(K)$, we have established a family of fixed points (Theorem 1) and proved that these are the only fixed points if the elements of $K$ are pairwise relatively prime (Theorem 2). We have also derived some results in the opposite case (Proposition 3) as well as a necessary and sufficient condition for the trivial fixed point to be isolated (Proposition 4). The study can be continued by considering the following open questions:

1. Is there a necessary and sufficient condition for the existence of nonzero fixed points, and more generally, nonzero periodic points?
2. Similarly, is there a firm condition for a nonzero fixed point, and more generally, for a periodic orbit, to be isolated?
3. Does the number of iterations needed for an orbit to reach one of the periodic points depend regularly on its initial value?
4. Can we further generalise to the case where $K$ is a multiset, rather than a set?

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Appendix

The online version of this article contains an appendix with tables listing the fixed points of the map $\rho_K$ for every nonempty subset $K$ of {7, 12, 13, 17, 23, 25}.

REFERENCES


Summary. We generalize the sum of remainders map by replacing the set of summation indices with an arbitrary finite nonempty set of positive integers. The iteration of this map produces interesting dynamical behavior, which we explore. In particular, we discuss the existence of nonzero fixed points and some of their properties.

JONATHAN HOSEANA (MR Author ID: 1221226) is a lecturer in the Department of Mathematics, Parahyangan Catholic University, Bandung, Indonesia. He received a BSc from the same university, before an MSc and a PhD from Queen Mary University of London. Besides doing mathematics, he enjoys reading about Catholicism and, when in good health, traveling on airplanes and riding roller coasters.

RYAN AZIZ (MR Author ID: 1294379) is currently a faculty member in the Department of Business Mathematics, Universitas Prasetiya Mulya, Indonesia. He obtained his doctoral degree from Queen Mary, University of London. Although currently working on the theory of noncommutative geometry and quantum groups, he enjoys all kinds of mathematics. He also enjoys good books in his spare time.