FINITE ELEMENT ANALYSIS OF NONLINEAR MAGNETIC FIELDS

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Abstract—The basic equations of electromagnetism are written in the form of a quasi-harmonic equation. The application of the weighted residual process leads to a non-linear system of algebraic equations which is solved by a full Newton-Raphson procedure. The iteration scheme is developed and applied to numerical examples.

1. INTRODUCTION

Two dimensional magnetic fields can easily be reduced to a quasi-harmonic equation by introduction of the vector potential. Since the permeability depends on the induction, and thus on the unknown solution, the problem is non-linear and has to be solved iteratively.

Many papers have presented different numerical schemes for solving the non-linear field equation. C. W. Trowbridge[1] presents a survey of the historical development of computer programs using either finite difference or finite element formulations. A review of papers based on finite differences[2-5] or finite elements [6-15] is presented in the bibliography, some of them being analyzed in[1] and compared with respect to convergence properties, computer requirements, accuracy and agreement with experimental results.

The present paper presents a finite element formulation based on the use of isoparametric elements. Particular shapes of magnetisation curves are chosen and the resulting non-linear algebraic system is easily solved by applying the pure Newton-Raphson procedure. Numerical examples show how the computer times may be reduced by choosing an adequate initial solution and by introducing a small under-relaxation factor. Numerical results are presented for a magnetic self and for a turboalternator.

2. BASIC EQUATIONS

The Maxwell equations for magnetic field problems are

\[ \nabla \times \mathbf{H} = \mathbf{J}; \quad \mathbf{B} = \mu_s \mathbf{H}; \quad \text{div} \mathbf{B} = 0 \]  

where \( \mathbf{H} \) is the magnetic field vector, \( \mathbf{B} \) is the induction vector, \( \mu_s \) is the permeability, \( \mathbf{J} \) is the current density vector. Introducing the magnetic vector potential \( \mathbf{A} \) by

\[ \mathbf{B} = \nabla \times \mathbf{A}; \quad \text{div} \mathbf{A} = 0 \]  

equations (1) give

\[ \frac{\partial}{\partial x} \left[ \frac{1}{\mu_s} \frac{\partial A_j}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \frac{1}{\mu_s} \frac{\partial A_j}{\partial y} \right] + J = 0 \]  

since for two dimensional problems

\[ \mathbf{A} = A_k \mathbf{k}; \quad \mathbf{B} = B_i \mathbf{i} + B_j \mathbf{j} + B_k \mathbf{k}; \quad J = J \mathbf{k} \]  

\( i, j, k \) being the unit vectors in a right-handed cartesian reference frame.

The two dimensional domain to be studied is divided into finite elements on which the vector
potential is supposed to have a polynomial variation

$$A = \sum_i A_i N_i(x, y). \quad (n \text{ nodes per element}) \quad (5)$$

Applying the weighted residual process to equation (3), with the shape functions $N_i(x, y)$ as weighting functions[7], equation (3) becomes

$$\int_\Omega N_i(x,y) \left[ \frac{\partial}{\partial x} \frac{1}{\mu_s} \frac{\partial A}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\mu_s} \frac{\partial A}{\partial y} + J \right] d\Omega = 0 \quad (6a)$$

$$[K] \{\delta\} = \{f\} \quad (6b)$$

with

$$K_{ij} = \int_\Omega \frac{1}{\mu_s} \left[ \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] d\Omega \quad (7a)$$

$$f_i = \int_\Omega J \cdot N_i(x,y) d\Omega \quad (7b)$$

$$\delta_i = A_i \quad \text{(value of A at node i).} \quad (7c)$$

3. FINITE ELEMENT SOLUTION PROCEDURE

The computer program developed at the Stress Analysis Department of the University of Brussels uses quadratic isoparametric quadrilaterals (8 nodes) or triangles (6 nodes). Their well-known shape functions are given in[6].

Since the magnetic permeability depends on the induction, equation (6) is non-linear and has to be solved iteratively. The magnetisation curves introduced in the program are approximated by a set of straight lines. A tangential permeability, constant on each section, is defined by

$$\mu_t = \frac{dB}{dH} \quad (8)$$

while the permeability defined by equation (1) will be called "secant permeability".

For onedimensional problems, \(\{\delta\}\) has one component only) a graphical interpretation for solving equation (6) is represented on Fig. 1.

For multidimensional problems, the representation remains valid but must be extended to hyperplanes instead of straight lines. Also the form of the hyper-surface will be more complicated although still composed of hyperplanes.

Fig. 1. Graphical interpretation of equation 6.
3.3. The secant stiffness method*

The iteration procedure defines the new solution vector $\{\delta\}_{n+1}$ from $\{\delta\}_n$ by

$$\begin{align*}
[K]_n\{\delta\}_{n+1} &= \{f\} \\
\end{align*}$$

(9)

where $[K]_n$ is given by equation (7a). In many cases, this method has proved to be nonconvergent, resulting in a limit cycle (Fig. 2: the arrows show how the solution proceeds), if $[K]_n\{\delta\}_n$ and $[K]_{n+2}\{\delta\}_{n+2}$ are situated on the same hyperplane (here straight line).

3.2. The tangential stiffness method

Starting from an approximation $\{\delta\}_n$, a new solution vector $\{\delta\}_{n+1}$ is given by (Fig. 3)

$$\begin{align*}
[K_T]_n(\{\delta\}_{n+1} - \{\delta\}_n) &= \{f\} - [K]_n\{\delta\}_n \tag{10}
\end{align*}$$

with

$$\begin{align*}
[K_T]_n &= \int_\Omega \frac{1}{\mu_T} \left[ \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] d\Omega
\end{align*}$$

(11)

This procedure has proved to converge rapidly but (as for the secant stiffness procedure) a new system resolution must be performed at each iteration since the matrix $[K_T]_n$ of the linearized system is not constant. In order to try to reduce the computer times, a modified Newton–Raphson scheme may be applied, keeping the system matrix constant during several iterations, allowing partial system resolutions to be used (equation 12). This procedure needs however more steps to converge and in fact, is not more economical than the full Newton–Raphson method.

$$\begin{align*}
[K]_n(\{\delta\}_{n+1} - \{\delta\}_n) &= \{f\} - [K]_n\{\delta\}_n \tag{12}
\end{align*}$$

3.3. Convergence test

The solution procedure stops when

$$\begin{align*}
CN &= \sqrt{\sum_{i=1}^{\text{number of nodes}} \frac{(\delta_i^{n+1} - \delta_i^n)^2}{(\delta_i^n)^2}} < \varepsilon
\end{align*}$$

(13)

$CN$ is the convergence norm and $\delta^n$ is the $n$th iterate of the vector potential.

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*"Stiffness" by analogy with structural analysis properties.
4.1. Magnetic self

The geometry of the magnetic self is defined on Fig. 4. The finite element mesh takes into account the conditions of symmetry. The boundary conditions are

\[
\begin{align*}
A &= 0 \text{ on the exterior boundary,} \\
A &= 0 \text{ along the vertical symmetry axis,} \\
\frac{\partial A}{\partial n} &= 0 \text{ along the horizontal symmetry axis,}
\end{align*}
\]

and it is sufficient to consider a quarter of the geometry (drawn in heavy lines on Fig. 4).

The problem has been solved for two different values of the currents: 6000 A and 24000 A per conductor. If \( \varepsilon = 0.005 \) (i.e. 0.5%), the solution obtained is considered to be adequate and will be called the converged solution. The flux lines obtained after four iterations (24000 A case) may be compared with the converged solution on Figs. 5 and 6. The converged solution is obtained after 16 iterations when the procedure starts from \( A = 0 \) everywhere, but if the converged solution corresponding to 6000 A is used as initial value, the method converges only in 3 steps allowing computer time to be saved.
4.2. Turboalternator.

The geometry of the turboalternator is defined on Fig. 7. The currents have a constant density in the rotor windings, corresponding to the empty-load nominal conditions.

The finite element mesh is presented on Fig. 8, using 111 triangular elements, 231 quadrilaterals and 951 nodes.

Starting from $A = 0$ everywhere, a convergence level of 0.1% is reached after 9 iterations (full Newton–Raphson scheme). The flux lines are plotted on Fig. 9. An enlarged view is shown on Fig. 10, with the corresponding finite element mesh on Fig. 11. Some conclusions may be drawn from these plots:

(a) as expected, the flux is not uniformly distributed along the circumferential air-gap. A higher induction occurs opposite to the teeth and the flux tends to go from a stator tooth to a rotor tooth.

(b) nearly all the magnetic flux passes through the teeth. The higher the angle of the tooth with the vertical symmetry axis, the higher is the induction.
Fig. 8. Turboalternator mesh.

Fig. 9. Turboalternator flux lines.

Fig. 10. Turboalternator flux lines (enlarged view).

Fig. 11. Finite element mesh (enlarged view).
(c) however, the normality conditions of the flux lines with respect to boundaries between air (or copper) and iron seem not to be fulfilled. This is due to the computer plotting subroutine: the intersection points between the element boundaries and the flux curves are determined by use of quadratic interpolation formulae (equation 5) but these points are not connected with parabolic curves (as they should) but with straight lines easier to program in the plotting subroutine (Fig. 12).

Another source of errors is also the rather coarse mesh used around the teeth, but the core size of the CDC 6500-96 K computer of the University of Brussels did not allow a more refined mesh without increasing tremendously the input/output times.

The next figures show the evolution of the vector potential along the vertical axis of symmetry (Fig. 13) and along a circle (Fig. 14) at the mid-side of the stator teeth. The computed curves are compared with other numerical results obtained with use of the method presented by Winslow [3] modified by de La Vallée Poussin and Lion [5] in order to accelerate the convergence and summarized in [16]. The agreement is very good since the relative difference between the two solutions has an order of magnitude of 2%.

4.3. Accelerated schemes

4.3.1. Under-relaxation factor. A relaxation factor $\alpha$ has been introduced in equation (10) giving

$$[K_T]_n(\{\delta\}_{n-1} - \{\delta\}_n) = \{f\} - [K]_n\{\delta\}_n$$

(14)

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![Fig. 12. Flux line determination.](image)

![Fig. 13. Evolution of A along the vertical axis of symmetry ($x =$ results from [16]).](image)
and numerical experiments have been performed in order to determine which values of $\alpha$ give an accelerated convergence. One general conclusion appears from Fig. 15. For values of $\alpha$ greater than one (over-relaxation), the number of iterations for convergence is never smaller and nearly always far higher than for $\alpha = 1$. With under-relaxation values, the number of iterations may be reduced ($\alpha = 0.95$) or slightly increased. The numerical experiments show that a good range of values should be $(0.95-1.00)$ and that the conclusion may depend on the problem considered.

4.3.2. Initial solution. The choice of the initial solution to start the iterative procedure has a major influence on the rate of convergence. It has been shown in Section 4.1. that the use of the solution corresponding to other values of the currents could strongly reduce the computation effort. This remark should be kept in mind in the search of several working points of a given machine.

Fig. 14. Evolution of $\Lambda$ along the mean circumference ($x = \text{results from [16]}$).

![Graph showing the evolution of $\Lambda$ along the mean circumference](image)

\[
\{\delta\}_{n+1} = \{\delta\}_n + \alpha[\{\delta\}_{n+1} - \{\delta\}_n]
\] (15)

Fig. 15. Influence of the relaxation factor, for two different magnetic selfs.

![Graph showing the influence of the relaxation factor](image)
In connection with the particular choice of magnetisation curves composed of straight lines, the present Newton–Raphson procedure has shown to be rapidly convergent. The finite element method is well suited to solve geometrically complicated problems. In these cases, however, the data preparation is a formidable task since, in general, no mesh generation can be adapted to the geometrical complexity of the problem.

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