Asymptotic efficiency of some nonparametric tests for location on hyperspheres

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Abstract

In the present paper, we show that several classical nonparametric tests for multivariate location in the Euclidean case can be adapted to nonparametric tests for the location problem on hyperspheres. The tests we consider are spatial signed-rank tests for location on hyperspheres. We compute the asymptotic powers of the latter tests in the classical rotationally symmetric case. In particular, we show that the spatial signed-rank test uniformly dominates the spatial sign test and has performances that are extremely close to the asymptotically optimal test in the well-known von Mises-Fisher case. Monte-Carlo simulations confirm our asymptotic results.

Keywords: Directional statistics, location tests, spatial signed-ranks

1. Introduction

Directional data consist in observed directions often modeled as realizations of a random vector $\mathbf{X}$ taking values on the unit hypersphere $\mathcal{S}^{p-1} := \{\mathbf{v} \in \mathbb{R}^p : \mathbf{v}'\mathbf{v} = 1\}$ of $\mathbb{R}^p$. We refer the reader to the monographs Mardia and Jupp (2000), Jammalamadaka and SenGupta (2001) and Ley and Verdebout (2017) for an overview of this field. When dealing with directional data, a very common parameter of interest is the so-called mean direction parameter $\boldsymbol{\theta} := \mathrm{E}[\mathbf{X}]/\|\mathrm{E}[\mathbf{X}]\| \in \mathcal{S}^{p-1}$ for the problem at hand. The parameter $\boldsymbol{\theta}$, that can be viewed as a "north pole" for the problem under study, often plays the role of location parameter or symmetry center. For instance $\boldsymbol{\theta}$ is the symmetry center of several *rotationally symmetric distributions.* Absolutely continuous rotationally symmetric distributions have densities of the form $\mathbf{x} \to c_f f(\mathbf{x}'\boldsymbol{\theta})$, where f is a positive *angular function* and c_f is a normalizing constant. Below when the rotationally symmetric model is considered, we assume that f is such that $\boldsymbol{\theta} \in \mathcal{S}^{p-1}$ is well identified. The most important parametric model for directional data, that belongs to the class of rotationally symmetric distributions, is the von Mises-Fisher (vMF) model characterized by densities on $\mathcal{S}^{p-1}$ of the form $\mathbf{x} \to c_{p,\kappa} \exp(\kappa \mathbf{x}'\boldsymbol{\theta})$, where $\kappa > 0$ is a so-called concentration parameter that regulates the probability mass in the vicinity of $\boldsymbol{\theta}$ and $c_{p,\kappa}$ is a normalizing constant.

Inference on $\boldsymbol{\theta}$ within the rotationally symmetric framework has been the subject of many contributions. For the point estimation problem, among others, Ko (1992) and Ko and Chang (1993) studied M-estimators while Ley et al. (2013) provided one-step rank-based estimators. For the problem of testing $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ for some fixed unit vector $\boldsymbol{\theta}_0 \in \mathcal{S}^{p-1}$, Watson (1983), Hayakawa and Puri (1985) and Hayakawa (1990) studied the VMF likelihood ratio and score tests while Paindaveine and Verdebout (2015, 2017, 2020) studied various tests (including sign tests and tests based on (univariate) ranks and spatial signs) in different rotationally symmetric setups.

Spatial signed-rank tests for multivariate location problems involving Euclidean data have been proposed and studied in Möttönen and Oja (1995), Möttönen et al. (1997) and Hettmansperger et al. (1997). The latter tests enjoy many attractive features including nice invariance properties, validity robustness and nice asymptotic relative efficiency properties with respect to parametric procedures. In the present paper, we show how one can obtain nonparametric tests including spatial signed-rank tests for the one-sample hyperspherical location problem. We compute the asymptotic powers of our nonparametric tests under rotationally symmetry. We show that the spatial signed-rank tests performs essentially as well as the optimal von Mises test and dominates the spatial sign test in the von Mises case. The paper is organized as follows: in Section 2, we properly define the class of tests studied in the paper. In Section 3, we compare our tests through asymptotic relative efficiencies under rotational symmetry. Finally, in Section 4, we study the small-sample properties of our tests through Monte-Carlo experiments.

2. Asymptotically distribution-free test statistics

Consider the spherical location problem that consists in testing $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ for some fixed $\boldsymbol{\theta}_0 \in \mathcal{S}^{p-1}$. Local alternatives to $\mathcal{H}_0$ can be obtained by considering sequences of underlying distributions with location parameters $\boldsymbol{\theta}_n := \boldsymbol{\theta}_0 + n^{-1/2}\boldsymbol{\tau}_n$; where $\boldsymbol{\tau}_n$ is a bounded sequence such that $\boldsymbol{\theta}_n$ belongs to $\mathcal{S}^{p-1}$ for all n; that is $\|\boldsymbol{\theta}_n\| = 1$. Therefore, the sequence $\boldsymbol{\tau}_n$ must satisfy $1 = \|\boldsymbol{\theta}_n\|^2 = 1 + 2n^{-1/2}\boldsymbol{\tau}_n'\boldsymbol{\theta}_0 + n^{-1}\boldsymbol{\tau}_n'\boldsymbol{\tau}_n$, or equivalently $\boldsymbol{\tau}_n'\boldsymbol{\theta}_0 = -\frac{1}{2n^{1/2}}\boldsymbol{\tau}_n'\boldsymbol{\tau}_n$. It follows that the sequence $\boldsymbol{\tau}_n$ is such that $\boldsymbol{\tau}_n'\boldsymbol{\theta}_0 = o(1)$ as $n \to \infty$. In other words, if the underlying sequence of models is sufficiently regular in the sense that the perturbations $\boldsymbol{\theta}_0 + n^{-1/2}\boldsymbol{\tau}_n$ provide contiguous alternatives to $\mathcal{H}_0$, we can safely assume that $\boldsymbol{\tau}_n$ belongs to the orthogonal complement to $\boldsymbol{\theta}_0$ defined as $\mathcal{M}(\boldsymbol{\Gamma}_{\boldsymbol{\theta}_0})$, where $\mathcal{M}(\mathbf{A})$ denotes the span generated by the columns of a full column rank matrix $\mathbf{A}$, and $\boldsymbol{\Gamma}_{\boldsymbol{\theta}}$ is a $p \times (p-1)$ semi-orthogonal matrix such that $\boldsymbol{\Gamma}_{\boldsymbol{\theta}}'\boldsymbol{\Gamma}_{\boldsymbol{\theta}} = \mathbf{I}_{p-1}$ and $\boldsymbol{\Gamma}_{\boldsymbol{\theta}}\boldsymbol{\Gamma}_{\boldsymbol{\theta}}' = \mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}'$, $\boldsymbol{\theta} \in \mathcal{S}^{p-1}$.

It follows from the rationale above that if the i.i.d. directions $\mathbf{X}_1, \ldots, \mathbf{X}_n$

at hand have mean direction $\boldsymbol{\theta}$, a naturel way to test $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ is to test that the projections $\mathbf{Y}_1(\boldsymbol{\theta}_0) := \boldsymbol{\Gamma}'_{\boldsymbol{\theta}_0}\mathbf{X}_1, \ldots, \mathbf{Y}_n(\boldsymbol{\theta}_0) := \boldsymbol{\Gamma}'_{\boldsymbol{\theta}_0}\mathbf{X}_n$ of $\mathbf{X}_1, \ldots, \mathbf{X}_n$ onto $\mathcal{M}(\boldsymbol{\Gamma}_{\boldsymbol{\theta}_0})$ have expectation zero. As a result, it is very natural to consider test statistics based on quantities of the form

$$\mathbf{Q}_{\mathbf{J}^{(n)}} := n^{-1/2} \sum_{i=1}^{n} \mathbf{J}^{(n)}(\mathbf{Y}_i(\boldsymbol{\theta}_0)), \tag{2.1}$$

where the mapppings $\mathbf{J}^{(n)} : \mathbb{R}^{p-1} \to \mathbb{R}^{p-1}$ are such that under the null hypothesis $\mathrm{E}[\mathbf{J}^{(n)}(\mathbf{Y}_1(\boldsymbol{\theta}_0))] = \mathbf{0}$. If $\mathbf{J}^{(n)}$ and the underlying common distribution of the $\mathbf{X}_i$'s under the null hypothesis are such that, still under the null hypothesis, $\mathbf{Q}_{\mathbf{J}^{(n)}}$ converges weakly to a Gaussian vector with mean zero and full-rank covariance matrix $\boldsymbol{\Sigma}_{\mathbf{J}}$, letting $\hat{\boldsymbol{\Sigma}}_{\mathbf{J}}$ be a consistent estimator of $\boldsymbol{\Sigma}_{\mathbf{J}}$, a classical test for $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ then rejects the null hypothesis for large values of

$$\mathbf{Q}'_{\mathbf{J}^{(n)}} \hat{\boldsymbol{\Sigma}}_{\mathbf{J}}^{-1} \mathbf{Q}_{\mathbf{J}^{(n)}}. \tag{2.2}$$

Several classical tests for spherical location belong to the class of tests of the form (2.2):

(i) the classical Watson (1983) test under rotational symmetry studied in Paindaveine and Verdebout (2017, 2020) is obtained by taking $\mathbf{J}^{(n)}(\mathbf{Y}) = \mathbf{Y} =: \mathbf{J}^{(n)}_{\mathrm{W}}(\mathbf{Y})$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{J}} = \frac{(n^{-1}\sum_{i=1}^{n}(1-(\mathbf{X}_i'\boldsymbol{\theta}_0)^2))}{(p-1)}\mathbf{I}_{p-1}$ in (2.2). The resulting test $\phi^{(n)}_{\mathrm{W}}$ rejects the null hypothesis at the asymptotic level α when

$$W_n := \frac{(p-1)}{\sum_{i=1}^{n}(1-(\mathbf{X}_i'\boldsymbol{\theta}_0)^2)} \sum_{i,j=1}^{n} \mathbf{Y}_i'(\boldsymbol{\theta}_0)\mathbf{Y}_j(\boldsymbol{\theta}_0) > \chi^2_{p-1;1-\alpha}, \tag{2.3}$$

where $\chi^2_{k;\nu}$ denotes the quantile of order ν of the chi-square distribution with k degrees of freedom. Following the construction above, the Watson test can obviously be robustified by considering its nonparametric version

$\phi^{(n)}_{\mathrm{W,non}}$ that rejects the null hypothesis at the asymptotic level α when

$$\tilde{W}_n := \frac{1}{n} \sum_{i,j=1}^{n} \mathbf{Y}'_i(\boldsymbol{\theta}_0)\hat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{J}^{(n)}_{\mathrm{W}}} \mathbf{Y}_j(\boldsymbol{\theta}_0) > \chi^2_{p-1;1-\alpha},$$

where $\hat{\boldsymbol{\Sigma}}_{\mathbf{J}^{(n)}_{\mathrm{W}}} := n^{-1} \sum_{i=1}^{n} \mathbf{Y}_i(\boldsymbol{\theta}_0)\mathbf{Y}'_i(\boldsymbol{\theta}_0)$. The central limit theorem directly entails (since the $\mathbf{Y}_i(\boldsymbol{\theta}_0)$'s are bounded) that $\tilde{W}_n$ is asymptotically distribution-free under the null hypothesis. Clearly, the two tests $\phi^{(n)}_{\mathrm{W,non}}$ and $\phi^{(n)}_{\mathrm{W}}$ are asymptotically equivalent in the rotationally symmetric case.

(ii) the spatial sign test under rotational symmetry recently studied in García-Portugués et al. (2020) is obtained by taking $\mathbf{J}^{(n)}(\mathbf{Y}) = \mathbf{Y}/\|\mathbf{Y}\| =: \mathbf{J}^{(n)}_{\mathrm{sgn}}(\mathbf{Y})$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{J}} = \frac{1}{(p-1)}\mathbf{I}_{p-1}$ in (2.2). The resulting test $\phi^{(n)}_{\mathrm{sign}}$ rejects the null hypothesis at the asymptotic level α when

$$S_n := \frac{(p-1)}{n} \sum_{i,j=1}^{n} \frac{\mathbf{Y}'_i(\boldsymbol{\theta}_0)\mathbf{Y}_j(\boldsymbol{\theta}_0)}{\|\mathbf{Y}_i(\boldsymbol{\theta}_0)\|\|\mathbf{Y}_j(\boldsymbol{\theta}_0)\|} > \chi^2_{p-1;1-\alpha}.$$

The spatial sign test $\phi^{(n)}_{\mathrm{sign}}$ that is asymptotically valid under rotational symmetry only can also be turned into a nonparametric test $\phi^{(n)}_{\mathrm{sign,non}}$ that rejects the null hypothesis at the asymptotic level α when

$$\tilde{S}_n := \frac{1}{n} \sum_{i,j=1}^{n} \frac{\mathbf{Y}'_i(\boldsymbol{\theta}_0)\hat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{J}^{(n)}_{\mathrm{sign}}} \mathbf{Y}_j(\boldsymbol{\theta}_0)}{\|\mathbf{Y}_i(\boldsymbol{\theta}_0)\|\|\mathbf{Y}_j(\boldsymbol{\theta}_0)\|} > \chi^2_{p-1;1-\alpha},$$

where $\hat{\boldsymbol{\Sigma}}_{\mathbf{J}^{(n)}_{\mathrm{sign}}} := n^{-1} \sum_{i=1}^{n} \frac{\mathbf{Y}_i(\boldsymbol{\theta}_0)\mathbf{Y}'_i(\boldsymbol{\theta}_0)}{\|\mathbf{Y}_i(\boldsymbol{\theta}_0)\|^2}$. As for the Watson test above, $\phi^{(n)}_{\mathrm{sign,non}}$ and $\phi^{(n)}_{\mathrm{sign}}$ are asymptotically equivalent under rotational symmetry.

Following the original construction of spatial signed-rank tests of Möttönen et al. (1997), we consider the spatial signed-rank test for hyperspherical location obtained by taking

$$\mathbf{J}^{(n)}(\mathbf{Y}) = \frac{1}{2n}\sum_{j=1}^{n} \frac{\mathbf{Y} - \mathbf{Y}_j(\boldsymbol{\theta}_0)}{\|\mathbf{Y} - \mathbf{Y}_j(\boldsymbol{\theta}_0)\|} + \frac{\mathbf{Y} + \mathbf{Y}_j(\boldsymbol{\theta}_0)}{\|\mathbf{Y} + \mathbf{Y}_j(\boldsymbol{\theta}_0)\|} =: \mathbf{J}^{(n)}_{\text{rank}}(\mathbf{Y})$$

in (2.1). Letting $\hat{\boldsymbol{\Sigma}}_{\mathbf{J}^{(n)}_{\text{rank}}} := n^{-1}\sum_{i=1}^{n} \mathbf{J}^{(n)}_{\text{rank}}(\mathbf{Y}_i(\boldsymbol{\theta}_0))(\mathbf{J}^{(n)}_{\text{rank}}(\mathbf{Y}_i(\boldsymbol{\theta}_0)))'$, the non-parametric spatial signed-rank test $\phi^{(n)}_{\text{rank,non}}$ rejects the null hypothesis at the asymptotic level α when

$$R_n := \mathbf{Q}'_{\mathbf{J}^{(n)}_{\text{rank}}} \hat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{J}^{(n)}_{\text{rank}}} \mathbf{Q}_{\mathbf{J}^{(n)}_{\text{rank}}} > \chi^2_{p-1;1-\alpha}.$$

In the next Section, we study the asymptotic efficiency of $\phi^{(n)}_{\text{rank,non}}$ with respect to its competitors $\phi^{(n)}_{\text{W,non}}$ and $\phi^{(n)}_{\text{sign,non}}$ under rotational symmetry.

3. Asymptotic efficiency under rotational symmetry

As mentioned in the introduction inference for hyperspherical location has mainly been studied under the assumption of rotational symmetry. Let $\mathbf{X}$ be a random vector taking values in $\mathcal{S}^{p-1}$ and consider the following *tangent-normal* decomposition

$$\begin{aligned} \mathbf{X} &= (\mathbf{X}'\boldsymbol{\theta})\boldsymbol{\theta} + (\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}')\mathbf{X} \\ &= (\mathbf{X}'\boldsymbol{\theta})\boldsymbol{\theta} + \boldsymbol{\Gamma}_{\boldsymbol{\theta}}\mathbf{Y}(\boldsymbol{\theta}) = \mathbf{X}'\boldsymbol{\theta}\boldsymbol{\theta} + (1 - (\mathbf{X}'\boldsymbol{\theta})^2)^{1/2}\boldsymbol{\Gamma}_{\boldsymbol{\theta}}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}), \end{aligned} \tag{3.4}$$

where $\mathbf{Y}(\boldsymbol{\theta}) := \boldsymbol{\Gamma}'_{\boldsymbol{\theta}}\mathbf{X}$ and $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}) := \mathbf{Y}(\boldsymbol{\theta})/\|\mathbf{Y}(\boldsymbol{\theta})\|$ is a multivariate sign (unit) vector taking values on $\mathcal{S}^{p-2}$. If $\mathbf{X}$ has an absolutely continuous rotationally symmetric distribution with location $\boldsymbol{\theta}$ and angular function f, it is well-known that (see for instance Paindaveine and Verdebout (2017)) (i) the density of $\mathbf{X}'\boldsymbol{\theta}$ is of the form

$$u \to c_f f(u)(1-u^2)^{(p-3)/2}, \quad u \in (-1, 1), \tag{3.5}$$

for some normalizing constant c_f, (ii) the sign vector $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$ is uniformly distributed on $\mathcal{S}^{p-2}$ and (iii) $\mathbf{X}'\boldsymbol{\theta}$ and $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X})$ are independent. Note that the famous vMF distribution obtained by taking $f(u) = \exp(\kappa u)$ for some $\kappa > 0$ in (3.5) clearly belongs to the class of rotationally symmetric distributions. Writing $\mathrm{P}^{(n)}_{\boldsymbol{\theta},f}$ for the joint distribution of i.i.d. observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$ with a common rotationally symmetric distribution with angular function f and location parameter $\boldsymbol{\theta}$, we have the following result that describes the asymptotic behavior of the spatial signed-rank test under local alternatives.

Proposition 3.1. *Let $\boldsymbol{\tau}_n$ be a bounded sequence such that $\boldsymbol{\tau} := \lim_{n\to\infty} \boldsymbol{\tau}_n$ and $\boldsymbol{\theta}_0 + n^{-1/2}\boldsymbol{\tau}_n \in \mathcal{S}^{p-1}$ and assume that f is almost everywhere differentiable (with a.e. derivative f') such that ($\phi_f(u) := \frac{f'}{f}(u)$) $\mathrm{E}[\phi_f^2(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i^2(\boldsymbol{\theta}_0)] < \infty$, where $U_i(\boldsymbol{\theta}_0) := \sqrt{1-(\mathbf{X}_i'\boldsymbol{\theta}_0)^2}$. Then, as $n \to \infty$ under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0+n^{-1/2}\boldsymbol{\tau}_n,f}$, R_n is asymptotically chi-square with $p-1$ degrees of freedom and non-centrality parameter (expectations are taken under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0,f}$)*

$$\frac{\mathrm{E}^2[q(U_i(\boldsymbol{\theta}_0))\phi_f(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i(\boldsymbol{\theta}_0)]}{\mathrm{E}[q^2(U_i(\boldsymbol{\theta}_0))](p-1)}\boldsymbol{\tau}'(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0')\boldsymbol{\tau},$$

where $q(u) := \mathrm{E}[(u - Y_{p-1})/(Y_1^2 + \ldots + Y_{p-2}^2 + (u - Y_{p-1})^2)^{1/2}]$, with $(Y_1, \ldots, Y_{p-1})'$ having the same distribution as $\mathbf{Y}_i(\boldsymbol{\theta}_0)$ under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0,f}$.

See the supplementary material for a proof. It directly follows from Paindaveine and Verdebout (2017) that the Watson test statistic W_n (and therefore $\tilde{W}_n$) in (2.3) converges weakly under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0+n^{-1/2}\boldsymbol{\tau}_n,f}$ to a chi-square random variable with $p-1$ degrees of freedom and non-centrality parameter (expectations are taken under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0,f}$)

$$\frac{\mathrm{E}^2[\phi_f(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i^2(\boldsymbol{\theta}_0)]}{\mathrm{E}[U_i^2(\boldsymbol{\theta}_0)](p-1)}\boldsymbol{\tau}'(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0')\boldsymbol{\tau}.$$

It follows that the Asymptotic Relative Efficiency (ARE) of $\phi^{(n)}_{\text{rank,non}}$ with respect to $\phi^{(n)}_{\mathrm{W,non}}$ is given by

$$\mathrm{ARE}_f(\phi^{(n)}_{\mathrm{rank,non}}/\phi^{(n)}_{\mathrm{W,non}}) = \frac{\mathrm{E}^2[q(U_i(\boldsymbol{\theta}_0))\phi_f(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i(\boldsymbol{\theta}_0)]\mathrm{E}[U_i^2(\boldsymbol{\theta}_0)]}{\mathrm{E}[q^2(U_i(\boldsymbol{\theta}_0))]\mathrm{E}^2[\phi_f(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i^2(\boldsymbol{\theta}_0)]}.$$

As shown in Paindaveine and Verdebout (2017) the Watson test $\phi^{(n)}_{\mathrm{W}}$ (and therefore $\phi^{(n)}_{\mathrm{W,non}}$) is locally and asymptotically optimal in the vMF case $f(u) = \exp(\kappa u)$ with $\kappa > 0$ and $\phi_f(u) = \kappa$ for which the Cauchy-Schwarz inequality indeed directly yields

$$\mathrm{ARE}_{\mathrm{vMF}(\kappa)}(\phi^{(n)}_{\mathrm{rank,non}}/\phi^{(n)}_{\mathrm{W,non}}) = \frac{\mathrm{E}^2[q(U_i(\boldsymbol{\theta}_0))U_i(\boldsymbol{\theta}_0)]}{\mathrm{E}[q^2(U_i(\boldsymbol{\theta}_0))]\mathrm{E}[U_i^2(\boldsymbol{\theta}_0)]} \leq 1.$$

Although, the Watson test $\phi^{(n)}_{\mathrm{W,non}}$ asymptotically dominates the spatial signed-rank-based test in the von Mises case, the domination is extremely weak as shown in Figure 1 below. Working along the same lines as above, we can easily obtain that

$$\mathrm{ARE}_f(\phi^{(n)}_{\mathrm{rank,non}}/\phi^{(n)}_{\mathrm{sign,non}}) = \frac{\mathrm{E}^2[q(U_i(\boldsymbol{\theta}_0))\phi_f(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i(\boldsymbol{\theta}_0)]}{\mathrm{E}[q^2(U_i(\boldsymbol{\theta}_0))]\mathrm{E}^2[\phi_f(\mathbf{X}_i'\boldsymbol{\theta}_0)U_i(\boldsymbol{\theta}_0)]},$$

which in the von Mises case yields (using successively the Jensen and the Cauchy-Schwarz inequalities)

$$\mathrm{ARE}_{\mathrm{vMF}(\kappa)}(\phi^{(n)}_{\mathrm{rank,non}}/\phi^{(n)}_{\mathrm{sign,non}}) = \frac{\mathrm{E}^2[q(U_i(\boldsymbol{\theta}_0))U_i(\boldsymbol{\theta}_0)]}{\mathrm{E}[q^2(U_i(\boldsymbol{\theta}_0))]\mathrm{E}^2[U_i(\boldsymbol{\theta}_0)]} \geq 1,$$

so that the spatial signed-rank test uniformly (in κ) dominates the spatial sign-based test in the von Mises case. In Figure 1 below, we provide (approximated) values of $\mathrm{ARE}_{\mathrm{vMF}(\kappa)}(\phi^{(n)}_{\mathrm{rank,non}}/\phi^{(n)}_{\mathrm{W,non}})$ and $\mathrm{ARE}_{\mathrm{vMF}(\kappa)}(\phi^{(n)}_{\mathrm{rank,non}}/\phi^{(n)}_{\mathrm{sign,non}})$ for various values of κ. The approximations have been obtained by classical Monte-Carlo simulations (based on 5,000 replications) to compute the various expectations in the ARE expressions. Inspection of Figure 1 confirms the results obtained above but also shows that although the Watson test asymptotically dominates the spatial signed-rank test in the von Mises case, the domination is extremely weak. Actually, the spatial signed-rank test and the Watson test perform equally well for not too large concentrations. Note that $\kappa = 100$ and $\kappa = 1000$ provide highly concentrated data around the location. In the next Section, we compare the small-sample properties of the Waston test $\phi^{(n)}_{\mathrm{W,non}}$, the sign test $\phi^{(n)}_{\mathrm{sign,non}}$ and the spatial signed-rank test $\phi^{(n)}_{\mathrm{rank,non}}$ through Monte-Carlo simulations.

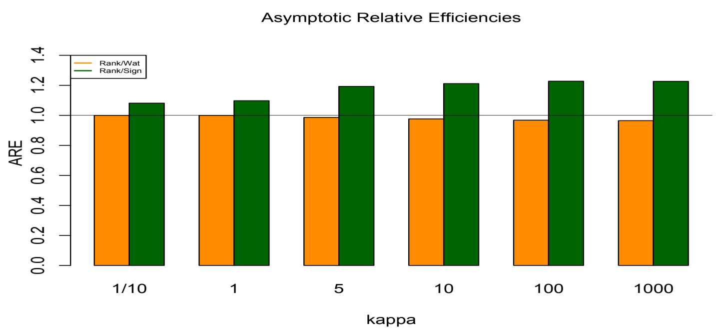

Figure 1: Asymptotic Relative Efficiencies $\mathrm{ARE}_{\mathrm{vMF}(\kappa)}(\phi^{(n)}_{\mathrm{rank}}/\phi^{(n)}_{\mathrm{W}})$ and $\mathrm{ARE}_{\mathrm{vMF}(\kappa)}(\phi^{(n)}_{\mathrm{rank}}/\phi^{(n)}_{\mathrm{sign}})$ for various values of κ.

4. Monte-Carlo comparison

In this section, we compare the finite sample behavior of the Waston tests $\phi^{(n)}_{\mathrm{W}}$ and $\phi^{(n)}_{\mathrm{W,non}}$, the sign tests $\phi^{(n)}_{\mathrm{sign}}$ and $\phi^{(n)}_{\mathrm{sign,non}}$ and the spatial signed-rank test $\phi^{(n)}_{\mathrm{rank,non}}$ under various sampling schemes. First we generated $N = 2,500$ mutually independent samples of i.i.d. trivariate ($k = 3$) random vectors

$$\mathbf{X}^{(\rho)}_j, \quad \rho = 1, \ldots, 4, \;\; j = 1, \ldots, (n =)100,$$

with location $\boldsymbol{\theta}_0 := \mathbf{e}_1 := (1,0,0)'$. The $\mathbf{X}_j^{(1)}$'s and the $\mathbf{X}_j^{(2)}$'s have a vMF distribution with location $\boldsymbol{\theta}_0$ and concentration 5 and 20 respectively, the $\mathbf{X}_j^{(3)}$'s and the $\mathbf{X}_j^{(4)}$'s have a tangent elliptical distribution with location $\boldsymbol{\theta}_0$, concentration 5 and shape matrix $\mathbf{V} = \mathrm{diag}(v, 1/v)$ respectively with $v = 4$ and $v = 10$. Tangent elliptical distributions recently studied in García-Portugués et al. (2020) are not rotationally symmetric; the distribution of the projection of the observations in the tangent space to the location is elliptical with a scatter controlled by the shape parameter $\mathbf{V}$. The value $\mathbf{V} = \mathbf{I}_{p-1}$ provides a spherical distribution for the projection and therefore rotational symmetry of the observation. In each scenario, we performed the five tests $\phi_{\mathrm{W}}^{(n)}$, $\phi_{\mathrm{W,non}}^{(n)}$, $\phi_{\mathrm{sign}}^{(n)}$, $\phi_{\mathrm{sign,non}}^{(n)}$ and $\phi_{\mathrm{rank,non}}^{(n)}$ at the asymptotic level .05 for the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. In Figure 2, we provide the empirical rejection frequencies (among the N replications) of the five tests. Inspection of Figure 2 reveals that all the tests are asymptotically valid in the von Mises case as expected while the non-parametric tests $\phi_{\mathrm{W,non}}^{(n)}$, $\phi_{\mathrm{sign,non}}^{(n)}$ and $\phi_{\mathrm{rank,non}}^{(n)}$ are the only test satisfying the nominal level constraint under tangent elliptical distributions.

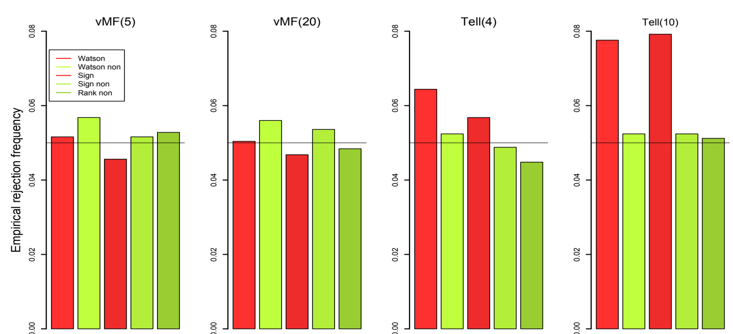

Figure 2: Empirical rejection frequencies of the Waston tests $\phi_{\mathrm{W}}^{(n)}$ and $\phi_{\mathrm{W,non}}^{(n)}$, the sign test $\phi_{\mathrm{sign}}^{(n)}$ and $\phi_{\mathrm{sign,non}}^{(n)}$ and the spatial signed-rank test $\phi_{\mathrm{rank,non}}^{(n)}$ performed at the nominal level .05 under the null hypothesis.

We performed a second simulation to compare the small-sample powers of

$\phi^{(n)}_{\mathrm{W,non}}$, $\phi^{(n)}_{\mathrm{sign,non}}$ and $\phi^{(n)}_{\mathrm{rank,non}}$. We generated $N = 2,500$ mutually independent samples of i.i.d. trivariate ($k = 3$) random vectors

$$\mathbf{X}^{(\rho)}_{\ell;j}, \quad \rho = 1, 2, 3,\ \ell = 0, \ldots, 3,\ j = 1, \ldots, (n =)100,$$

with location $\boldsymbol{\theta}_\ell := \mathbf{B}_\ell \mathbf{e}_1$, where

$$\mathbf{B}_\ell := \begin{pmatrix} \cos(\ell/50) & -\sin(\ell/50) & 0 \\ \sin(\ell/50) & \cos(\ell/50) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When $\ell = 0$, the matrix $\mathbf{B}_0$ is the 3-dimensional identity matrix so that in this case, the spherical location parameter is $\boldsymbol{\theta}_0 = \mathbf{e}_1$. The $\mathbf{X}^{(1)}_{\ell;j}$'s have a vMF distribution with location $\boldsymbol{\theta}_\ell$ and concentration 5, the $\mathbf{X}^{(2)}_{\ell;j}$'s have a tangent elliptical distribution with location $\boldsymbol{\theta}_\ell$, concentration 5 and shape parameter $\mathbf{V} = \mathrm{diag}(4, 1/4)$ while the $\mathbf{X}^{(3)}_{\ell;j}$'s have a rotationally symmetric distribution with location $\boldsymbol{\theta}_\ell$ and angular function $f(u) = \log(u + 2)$. In each scenario, we performed the three tests $\phi^{(n)}_{\mathrm{W,non}}$, $\phi^{(n)}_{\mathrm{sign,non}}$ and $\phi^{(n)}_{\mathrm{rank,non}}$ at the asymptotic level .05 for the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. The values $\ell = 1, \ldots, 3$ yield to distributions that are increasingly far from the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$. In Figure 3 we provide the empirical rejection frequencies of the three tests. Inspection of Figure 3 reveals that $\phi^{(n)}_{\mathrm{W,non}}$ and $\phi^{(n)}_{\mathrm{rank,non}}$ do behave almost similarly overall. Both strongly dominate the sign test outside of the von Mises case.

Acknowledgements

Thomas Verdebout's research is supported by the Program of Concerted Research Actions (ARC) of the Université Libre de Bruxelles and the Fonds Thelam from the Fondation Roi Baudouin.

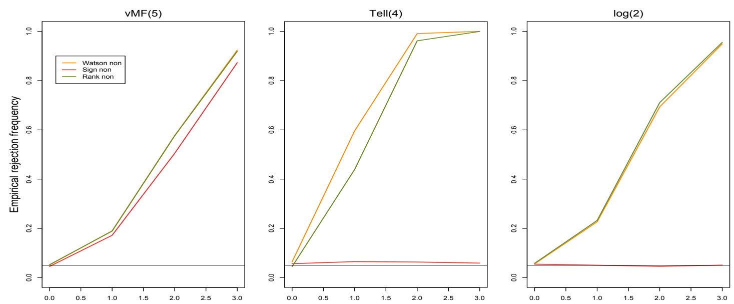

Figure 3: Empirical rejection frequencies of the Waston test $\phi^{(n)}_{\mathrm{W,non}}$, the sign test $\phi^{(n)}_{\mathrm{sign,non}}$ and the spatial signed-rank test $\phi^{(n)}_{\mathrm{rank,non}}$ performed at the nominal level .05

References

García-Portugués, E., Paindaveine, D., Verdebout, T., 2020. On optimal tests for rotational symmetry against new classes of hyperspherical distributions. Journal of the American Statistical Association 115, 1873–1887.

Hayakawa, T., 1990. On tests for the mean direction of the langevin distribution. Annals of the Institute of Statistical Mathematics 42, 359–373.

Hayakawa, T., Puri, M.L., 1985. Asymptotic expansions of the distributions of some test statistics. Annals of the Institute of Statistical Mathematics 37, 95–108.

Hettmansperger, T.P., Möttönen, J., Oja, H., 1997. Affine-invariant multivariate one-sample signed-rank tests. Journal of the American Statistical Association 92, 1591–1600.

Jammalamadaka, S.R., SenGupta, A., 2001. Topics in Circular Statistics. Series on Multivariate Analysis, World Scientific, Singapore.

Ko, D., 1992. Robust estimation of the concentration parameter of the von mises-fisher distribution. Annals of Statistics , 917–928.

Ko, D., Chang, T., 1993. Robust m-estimators on spheres. Journal of Multivariate Analysis 45, 104–136.

Ley, C., Swan, Y., Thiam, B., Verdebout, T., 2013. Optimal r-estimation of a spherical location. Statistica Sinica 23, 305–333.

Ley, C., Verdebout, T., 2017. Modern Directional Statistics. Chapman & Hall/CRC Interdisciplinary Statistics Series, CRC Press, Boca Raton.

Mardia, K.V., Jupp, P., 2000. Directional Statistics. Wiley, New York.

Möttönen, J., Oja, H., 1995. Multivariate spatial sign and rank methods. Journal of Nonparametric Statistics 5, 201–213.

Möttönen, J., Oja, H., Tienari, J., 1997. On the efficiency of multivariate spatial sign and rank tests. Annals of Statistics 25, 542–552.

Paindaveine, D., Verdebout, T., 2015. Optimal rank-based tests for the location parameter of a rotationally symmetric distribution on the hypersphere, in: Hallin, M., Mason, D., Pfeifer, D., Steinebach, J. (Eds.), Mathematical Statistics and Limit Theorems. Springer, Cham, pp. 249–269.

Paindaveine, D., Verdebout, T., 2017. Inference on the mode of weak directional signals: a Le Cam perspective on hypothesis testing near singularities. Annals of Statistics 45, 800–832.

Paindaveine, D., Verdebout, T., 2020. Inference for spherical location under high concentration. Annals of Statistics 48, 2982–2998.

Watson, G.S., 1983. Statistics on Spheres. Wiley, New York.