# AN INTERPOLATING FAMILY OF SIZE DISTRIBUTIONS


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Abstract. Power laws and power laws with exponential cut-off are two distinct families of size distributions. In the present paper, we propose a unified treatment of both families by building a family of distributions that interpolates between them, hence the name Interpolating Family (IF) of distributions. Our original construction, which relies on techniques from statistical physics, provides a connection for hitherto unrelated distributions like the Pareto and Weibull distributions, and sheds new light on them. The IF also contains several distributions that are neither of power law nor of power law with exponential cut-off type. We calculate quantile-based properties, moments and modes for the IF. This allows us to review known properties of famous size distributions and to provide in a single sweep these characteristics for various less known (and new) special cases of our Interpolating Family.

*Keywords:* Exponential Cut-off, Flexible Modeling, Pareto distribution, Power Law, Weibull distribution

## 1. Introduction

Size distributions are probability laws designed to model data that only take non-negative values or values above a certain threshold. Typical examples of such data are claim sizes in actuarial science, wind speeds in meteorology or lifetime data. Nonetheless, the spectrum of application areas is much broader and non-negative observations appear naturally in survival analysis (Lawless, 2003; Lee and Wang, 2003), environmental science (Marchenko and Genton, 2010), network traffic modeling (Mitzenmacher, 2004), reliability theory (Rausand and Høyland, 2004), economics (Eeckhout, 2004; Luttmer, 2007; Gabaix, 2016), seismology (Ley and Simone, 2020) and hydrology (Clarke, 2002), to cite but these. Parametric distributions are a simple and effective way to convey the information contained in those data. Given the range of distinct domains of application, it is not surprising that there exists a plethora of different size distributions and that the quest for the right size distribution in a given situation has developed into a very active research area over the past years (Dominicy and Sinner, 2017).

A very popular size distribution is the Pareto distribution, also called the Pareto type I distribution, with probability density function

$$x \mapsto \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, \quad x \in [x_0, \infty),$$

where $x_0 \geqslant 0$ is a location parameter and $\alpha > 0$ is a shape parameter known as the tail or Pareto index. Vilfredo Pareto used this law to model the distribution of income as well as the allocation of wealth among individuals (Pareto, 1964). Over the years, the Pareto law has further been applied to city size, file size distribution of internet traffic, the size of meteorites, or the size of sand particles (Reed and Jorgensen, 2004).

The Pareto distribution is a member of the power laws, which are typically of the form $x \mapsto kx^{-\alpha}$, with normalizing constant $k$ and power $\alpha > 0$. Power law distributions are employed in a vast range of situations, such as the modeling of the number of hits on web pages (Huberman and Adamic, 1999), income of top earners in areas of arts, sports and business (Rosen, 1981), and inequalities of income and wealth (Piketty and Zucman, 2014; Toda and Walsh, 2015). For further information and references, we refer the interested reader to Sornette (2003) and Mitzenmacher (2004).

A popular alternative to power laws are power laws with exponential cut-off, whose densities take the form $x \mapsto kx^{-\alpha}\mathrm{e}^{-\beta x}$ with normalizing constant $k$, power exponent $\alpha > 0$ and rate parameter $\beta > 0$. A power law with exponential cut-off behaves like a power law for small values of $x$, while its tail behavior is governed by a decreasing exponential. A famous representative of this class of distributions is the Weibull distribution with density

$$x \mapsto \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1} e^{-\left(\frac{x}{\sigma}\right)^{\alpha}}, \quad x \in [0, \infty),$$

where $\alpha > 0$ is a shape parameter regulating tail-weight and $\sigma > 0$ is a scale parameter. While the Weibull distribution already appeared in Rosin and Rammler (1933) to describe the distribution of particle size, it gained its prominence and name after Waloddi Weibull who showed in Weibull (1951) that the distribution can be successfully applied to seven very different case studies. Nowadays, the Weibull distribution is widely used in various domains such as life data analysis (Nelson, 2005), wind speed modeling (Manwell et al., 2009) and hydrology (Clarke, 2002). For an overview of various methods to estimate its parameters, we refer the reader to Gugliani et al. (2018).

Power laws and power laws with exponential cut-off are mostly studied apart from each other, due to their disparity. In the present paper, we shall build a bridge between these two classes of size distributions by proposing an over-arching family of distributions that interpolates between both classes, hence the name *Interpolating family of size distributions* (for simplicity, we shall from now on also refer to it as IF distribution). The corresponding density, which we shall discuss in detail later in the paper, is of the form

$$\mathrm{IF}(x;p,b,c,q,x_0) = \frac{|b|q}{c}\left(\frac{x-x_0}{c}\right)^{b-1} G_p(x)^{-q-1}\left(1-\frac{1}{p+1}G_p(x)^{-q}\right)^{p}, \tag{1}$$

for $x \in [x_0, \infty)$, and with $p \in [0,\infty], b \neq 0, c, q > 0, x_0 \geqslant 0$ and

$$G_p(x) = (p+1)^{-\frac{1}{q}} + \left(\frac{x-x_0}{c}\right)^{b}.$$

The roles of the various parameters will be described in Section 2.2. The alert reader will no doubt have recognized the densities of various size distributions included in (1), such as the Pareto and Weibull. The essence of the IF rests on its construction: we have used a technique from statistical mechanics (see Section 2) that allows us to interpolate between the Pareto and Weibull distributions, even more generally, between power laws and power laws with exponential cut-off. Thus, we are precisely finding a path from one end of the spectrum to the other, and this moreover in a constructive way. Besides providing a link between these *a priori* distinct families of size distributions, our proposal also permits to treat their properties such as moments, quantiles, modes in a unique way. Thus, by studying these properties of the IF, we are reviewing existing characteristics for certain famous distributions and at the same time we are uncovering these properties for less studied size distributions.

The remainder of the paper is organized as follows. In Section 2 we describe the Interpolating Family of size distributions, explain how it interpolates between power laws and power laws with exponential cut-off and elucidate the role of each of the five parameters. In Section 3 we summarize some of the special cases of the IF. We then provide quantile-based properties in Section 4, moment-based results in Section 5, and mode-related results in Section 6. Final comments are stated in Section 7, while technical derivations are presented in the Appendix.

## 2. The Interpolating Family: construction and parameter interpretation

In this section we present in detail our original construction leading to the Interpolating Family of size distributions. Section 2.2 expounds on the role of each of the five parameters.

### 2.1. Construction of the family.

As announced in the Introduction, our goal is to build a size distribution which incorporates both power laws and power laws with exponential cut-off. To show that (1) indeed satisfies this requirement, we start by writing up power law distributions and power law distributions with exponential cut-off in a unified language.

#### 2.1.1. *Power laws.*

The probability density function (pdf) of a typical power law distribution corresponds to

$$x \mapsto q(1+x)^{-q-1}, \quad x \in [0, \infty),$$

where the tail behavior is governed by the shape parameter $q > 0$. To get a more flexible distribution, one may add various parameters, such as a scale parameter $c > 0$, a location parameter $x_0 \geqslant 0$ and/or a shape parameter $b > 0$, leading to

$$x \mapsto \frac{bq}{c}\left(\frac{x-x_0}{c}\right)^{b-1}\left(1+\left(\frac{x-x_0}{c}\right)^b\right)^{-q-1}, \quad x \in [x_0, \infty).$$

Alternatively, in terms of the function $G_0(x) = 1 + \left(\frac{x-x_0}{c}\right)^b$, the pdf can be written under the form

$$f_0(x) = q\ g_0(x)G_0(x)^{-q-1}, \quad x \in [x_0, \infty),$$

where $g_0(x) = \frac{\mathrm{d}}{\mathrm{d}x}G_0(x) = \frac{b}{c}\left(\frac{x-x_0}{c}\right)^{b-1}$. We point out that the function $G_0(x)$ has been chosen such that the following boundary conditions are satisfied: $G_0(x_0) = 1$ and $\lim_{x\to\infty} G_0(x) = \infty$.

2.1.2. *Power laws with exponential cut-off.* The pdf of a typical power law distribution with exponential cut-off reads

$$x \mapsto qx^{-q-1}e^{-x^{-q}}, \quad x \in [0, \infty).$$

The shape parameter $q > 0$ still controls the tail behavior and, just as for power laws, we may increase the flexibility of the model by adding scale, location and shape parameters to get

$$x \mapsto \frac{bq}{c}\left(\frac{x-x_0}{c}\right)^{-bq-1} e^{-\left(\frac{x-x_0}{c}\right)^{-bq}}, \quad x \in [x_0, \infty).$$

Alternatively, we may write the pdf in terms of the function $G_\infty(x) = \left(\frac{x-x_0}{c}\right)^b$ as

$$f_\infty(x) = q\ g_\infty(x)G_\infty(x)^{-q-1}e^{-G_\infty(x)^{-q}}, \quad x \in [x_0, \infty),$$

where $g_\infty(x) = \frac{\mathrm{d}}{\mathrm{d}x}G_\infty(x) = \frac{b}{c}\left(\frac{x-x_0}{c}\right)^{b-1}$. Note that the function $G_\infty(x)$ has been chosen such that $G_\infty(x_0) = 0$ and $\lim_{x\to\infty} G_\infty(x) = \infty$.

2.1.3. *Interpolating Family.* If we want a highly flexible distribution including both power laws and power laws with exponential cut-off, we need a way to build densities interpolating between $f_0(x)$ and $f_\infty(x)$. To this end, we introduce a mild variant of the one-parameter deformation of the exponential function popularized in the seminal paper Tsallis (1988) in the context of non-extensive statistical mechanics. A more detailed account can be found in the review paper Tsallis (2002).

For any $p \in [0, \infty]$, we define the $p$-exponential[1] by

$$e_p(x) = \left(1 - \frac{1}{p+1}x\right)^p, \quad x \in [0, p+1].$$

The extreme cases $p = 0$ and $p \to \infty$ respectively correspond to $e_0(x) = 1$ over $[0, 1]$ and $e_\infty(x) = e^{-x}$ over $[0, \infty)$. With this in mind, it is natural to consider densities of the type

$$f_p(x) = q \; g_p(x) G_p(x)^{-q-1} e_p\left(G_p(x)^{-q}\right), \quad x \in [x_0, \infty), \tag{2}$$

with $g_p(x) = \frac{\mathrm{d}}{\mathrm{d}x} G_p(x)$, where we have not defined the function $G_p(x)$ yet. Just as $e_p(x)$ interpolates between 1 and $e^{-x}$, the mapping $G_p$ should also vary between $G_0$ and $G_\infty$. Hence, with the parameters $c > 0$, $b > 0$ and $x_0 \geqslant 0$ bearing the same interpretation as before, the map $G_p$ could be chosen as $G_p(x) = k + \left(\frac{x - x_0}{c}\right)^b$ for some constant $k$. A quick calculation shows that $k = (p+1)^{-\frac{1}{q}}$ is the right choice for $f_p(x)$ to integrate to one over its domain. Consequently

$$G_p(x) = (p+1)^{-\frac{1}{q}} + \left(\frac{x - x_0}{c}\right)^b, \quad x \in [x_0, \infty).$$

[1]The classical $q$-exponential defined by Tsallis (1988) has the form $\tilde{e}_q(x) = (1 + (1-q)x)^{\frac{1}{1-q}}$. We slightly modified the deformation path in order to simplify the calculations.

Since $G_p(x)^{-q}$ with $q > 0$ maps $[x_0, \infty)$ onto $[0, p+1]$, the function $e_p(G_p(x)^{-q})$ is well-defined. The pointwise convergence of the resulting density $f_p$ to $f_0$ as $p$ tends to zero (respectively $f_p$ to $f_\infty$ as $p \to \infty$) can be shown by straightforward limit calculations which we omit here.

The density (2) now almost corresponds to the density announced in the Introduction. Relaxing the condition $b > 0$ into $b \in \mathbb{R}_0$, we finally end up with

$$\mathrm{IF}(x; p, b, c, q, x_0) = \mathrm{sign}(b) q \, g_p(x) G_p(x)^{-q-1} e_p\left(G_p(x)^{-q}\right), \quad x \in [x_0, \infty). \tag{3}$$

The relaxation on $b$ only entails a minor change in the normalizing constant, which remains extremely simple. We call IF the *Interpolating Family* of size distributions as it interpolates between power laws and power laws with exponential cut-off. The density depends on five parameters $p, b, c, q$ and $x_0$, which we will discuss in more detail in the next section.

2.2. **Interpretation of the parameters.** For the sake of illustration, we provide density plots of the IF distribution in Figure 1. Except for the parameter we are varying, all the parameters remain fixed to $p = 1$, $b = 1$, $c = 200$, $q = 2$ and $x_0 = 0$.

Figure 1 provides a visual inspection of the roles endorsed by the five parameters: $x_0 \geqslant 0$ is a location parameter (smaller than or equal to the lowest value of the data), $c > 0$ a scale parameter, $q > 0$ a tail-weight parameter, and $b \in \mathbb{R}_0$ a shape parameter regulating the skewness. By changing the sign of $b$ in IF, one gets the Inverse-IF distribution (such as, for instance, the Rayleigh and Inverse Rayleigh

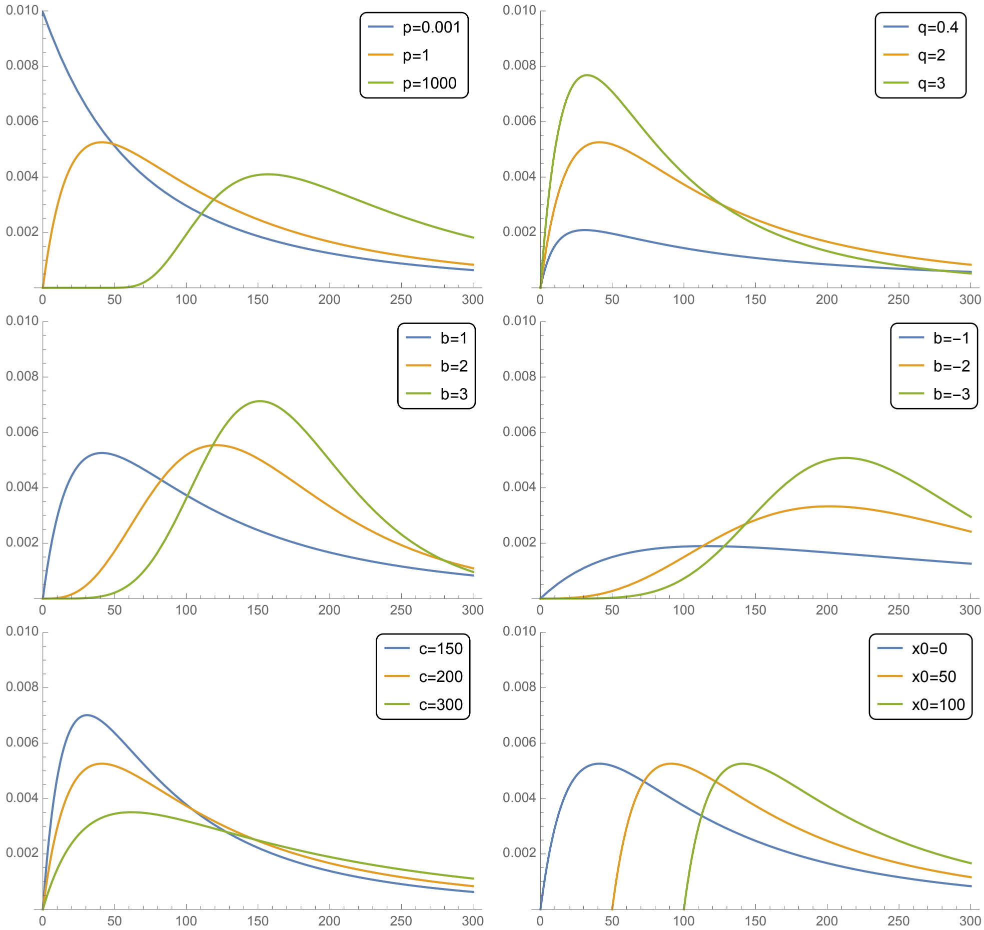


FIGURE 1. Density plots of the IF distribution where, except for the parameter we vary in each plot, the parameters remain fixed to $p = 1$, $b = 1$, $c = 200$, $q = 2$ and $x_0 = 0$.

distribution, see below). A crucial role is played by $p \in [0, \infty]$ as it enables us to interpolate between power laws and power laws with exponential cut-off. We therefore name it *interpolation parameter*.

## 3. Special cases and three main IF subfamilies

One major appeal of the IF distribution is that it contains a plethora of well-known size distributions as special cases. For a clearer structure, we define three four-parameter subfamilies:

- the IF1 distribution where $p = 0$,

- the IF2 distribution where $p \to \infty$,
- the IF3 distribution where $p \in (0, \infty)$ and $b = 1$.

Of course, there remain several other parameter combinations in the Interpolating Family, and perhaps in the future other interesting subfamilies will be given special attention. We also by no means claim to be exhaustive in the special cases cited here, and the reader may well find other known distributions that are special cases of the IF but not mentioned here.

**The IF1 distribution.** In the power law limit $p = 0$, the pdf of the resulting four-parameter family of distributions, called *Interpolating Family of the first kind (IF1)*, is given by

$$f_0(x) = \mathrm{sign}(b) q \, g_0(x) G_0(x)^{-q-1} = \frac{|b|q}{c} \left(\frac{x - x_0}{c}\right)^{b-1} \left(1 + \left(\frac{x - x_0}{c}\right)^b\right)^{-q-1},$$

where $x \in [x_0, \infty)$. Special cases of the IF1 distribution are, in decreasing order of the number of parameters, the Lindsay–Burr type III distribution ($b < 0$), the Pareto type IV distribution ($b > 0$), the Dagum distribution ($b < 0$ and $x_0 = 0$), the Pareto type II distribution ($b = 1$), the Pareto type III distribution ($b > 0$ and $q = 1$), the Tadikamalla–Burr type XII distribution ($b > 0$ and $x_0 = 0$), the Pareto type I distribution ($b = 1$ and $c = x_0 > 0$), the Lomax distribution ($b = 1$ and $x_0 = 0$), the Burr type XII distribution ($b > 0, c = 1$ and $x_0 = 0$) and the Fisk distribution ($b > 0, q = 1$ and $x_0 = 0$).

**The IF2 distribution.** In the power law with exponential cut-off limit $p \to \infty$, the pdf of the resulting four-parameter family of distributions, called *Interpolating Family of the second kind (IF2)*, is given by

$$f_\infty(x) = \operatorname{sign}(b)q\, g_\infty(x)G_\infty(x)^{-q-1}e^{-G_\infty(x)^{-q}} = \frac{|b|q}{c}\left(\frac{x-x_0}{c}\right)^{-bq-1} e^{-\left(\frac{x-x_0}{c}\right)^{-bq}},$$

where $x \in [x_0, \infty)$. Special cases of the IF2 distribution are, in decreasing order of the number of parameters, the Weibull distribution ($b = -1$; if also $x_0 = 0$, we find the two-parameter Weibull distribution), the Fréchet distribution ($b = 1$; if also $x_0 = 0$, we find the two-parameter Fréchet distribution), the Gumbel type II distribution ($b = 1$ and $x_0 = 0$), the Rayleigh distribution ($b = -1, q = 2$ and $x_0 = 0$), the Inverse Rayleigh distribution ($b = 1, q = 2$ and $x_0 = 0$), the Exponential distribution ($b = -1, q = 1$ and $x_0 = 0$), and the Inverse Exponential distribution ($b = 1, q = 1$ and $x_0 = 0$).

**The IF3 distribution.** The *Interpolating Family of the third kind (IF3)* is characterized by $0 < p < \infty$ and $b = 1$, resulting in the pdf

$$f_{p,1}(x) = \frac{q}{c}\left((p+1)^{-\frac{1}{q}} + \frac{x-x_0}{c}\right)^{-q-1}\left(1 - \frac{1}{p+1}\left((p+1)^{-\frac{1}{q}} + \frac{x-x_0}{c}\right)^{-q}\right)^{p},$$

where $x \in [x_0, \infty)$. Special cases of the IF3 distribution are the Generalized Lomax distribution ($x_0 = 0$) and the Stoppa distribution ($x_0 = c(p+1)^{-\frac{1}{q}}$).

**Distribution tree.** A visual summary of the structure inherent to the IF distribution with its various special cases is given in Figure 2 below. Since the inverse of each distribution is obtained by switching the sign of the parameter $b$, we only give the tree for positive values of $b$.

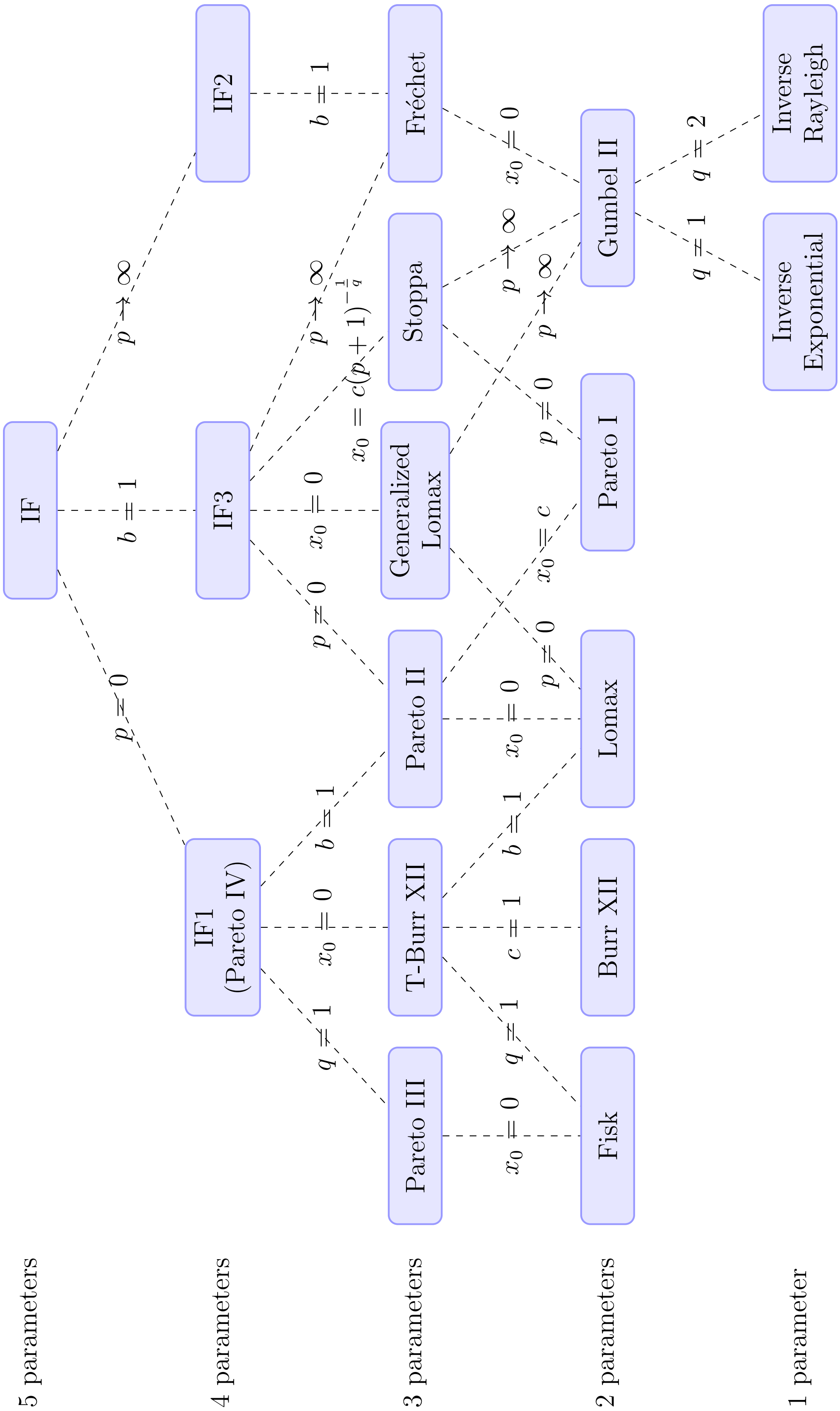


FIGURE 2. Distribution tree of the Interpolating Family of size distributions

## 4. Quantile-based properties

In this section we present and discuss quantile-based properties of the IF distribution. Since it contains so many special cases, the subsequent results provide in a single sweep those properties for the various distributions mentioned in Section 3.

4.1. **Cumulative distribution function, survival function and hazard function.** One major advantage of the IF distribution is that the cumulative distribution function (cdf) can be written under closed form:

$$F_p(x) = \begin{cases} \left(1 - \frac{1}{p+1} G_p(x)^{-q}\right)^{p+1} & \text{if } b > 0, \\ 1 - \left(1 - \frac{1}{p+1} G_p(x)^{-q}\right)^{p+1} & \text{if } b < 0. \end{cases} \tag{4}$$

Consequently, the survival or reliability function $S_p(x) = 1 - F_p(x)$ is extremely simple, too. The same holds true for the hazard function, defined as the ratio of the pdf and the survival function:

$$H_p(x) = \frac{f_p(x)}{S_p(x)} = \begin{cases} q\ g_p(x) G_p(x)^{-q-1} \frac{\left(1-\frac{1}{p+1} G_p(x)^{-q}\right)^{p}}{1-\left(1-\frac{1}{p+1} G_p(x)^{-q}\right)^{p+1}} & \text{if } b > 0, \\ -q\ g_p(x) G_p(x)^{-q-1} \frac{1}{\left(1-\frac{1}{p+1} G_p(x)^{-q}\right)} & \text{if } b < 0. \end{cases}$$

4.2. **Quantile function and median.** Very conveniently, the quantile function takes a nice form thanks to the simple expression of the cdf (4). Given the wide range of quantile-based statistical tools and methods such as QQ-plots, interquartile range or quantile regression, this is a very welcomed feature of the IF distribution. For $b > 0$, the quantile function is given by

$$Q_p^+(y) = F_p^{-1}(y) = \begin{cases} x_0 + c\left((1-y)^{-\frac{1}{q}} - 1\right)^{\frac{1}{b}} & \text{if } p = 0, \\ x_0 + c(p+1)^{-\frac{1}{bq}}\left(\left(1 - y^{\frac{1}{p+1}}\right)^{-\frac{1}{q}} - 1\right)^{\frac{1}{b}} & \text{if } 0 < p < \infty, \\ x_0 + c\left(\ln\left(\frac{1}{y}\right)\right)^{-\frac{1}{bq}} & \text{if } p \to \infty, \end{cases}$$

for $y \in [0,1]$. The expression for $b < 0$ is readily obtained via the relationship $Q_p^-(y) = Q_p^+(1-y)$, and we define the quantile function $Q_p(y)$ as $Q_p^+(y)$ if $b > 0$ and as $Q_p^-(y)$ if $b < 0$. The median is uniquely defined as

$$\text{Median} = \begin{cases} x_0 + c\left((2^{\frac{1}{q}} - 1\right)^{\frac{1}{b}} & \text{if } p = 0, \\ x_0 + c(p+1)^{-\frac{1}{bq}}\left(\left(1 - 2^{-\frac{1}{p+1}}\right)^{-\frac{1}{q}} - 1\right)^{\frac{1}{b}} & \text{if } 0 < p < \infty, \\ x_0 + c\left(\ln(2)\right)^{-\frac{1}{bq}} & \text{if } p \to \infty. \end{cases}$$

4.3. **Random variable generation.** The closed form of the quantile functions entails a straightforward random variable generation process from the IF. Indeed, it suffices to generate a random variable $U$ from a uniform distribution on the interval $[0,1]$, and then apply $Q_p$ to it. The resulting random variable $Q_p(U)$ follows the IF distribution. The simplicity of the procedure is particularly important for Monte Carlo simulation purposes and shows that all size distributions that are part of the Interpolating Family benefit from a straightforward random variable generation procedure.

## 5. Moments, mean and variance

We now provide the general moment expressions of the IF distribution. Particular focus shall be given to the mean and variance expressions, which we can write out explicitly in terms of the gamma and beta functions. We conclude the section with a table containing the mean expressions for the various size distributions mentioned in Section 3.

The $r^{\text{th}}$ moment of the IF distribution is given by

$$\mathbb{E}\left[X^r\right] = \int_{x_0}^{\infty} x^r \ \mathrm{IF}(x; p, b, c, q, x_0) \ \mathrm{d}x.$$

We will first treat the cases when $p < \infty$. Making the change of variables $y = (p+1)^{-\frac{1}{q}} + \left(\frac{x-x_0}{c}\right)^b$ and applying Newton's binomial theorem, we get

$$\mathbb{E}\left[X^r\right] = \sum_{i=0}^{r} \binom{r}{i} x_0^i c^{r-i} \underbrace{\int_{(p+1)^{-\frac{1}{q}}}^{\infty} q \ y^{-q-1} \left(y - (p+1)^{-\frac{1}{q}}\right)^{\frac{r-i}{b}} \left(1 - \frac{y^{-q}}{p+1}\right)^p \mathrm{d}y.}_{I(p,b,q)}$$

Note how the sign of $b$ vanishes during this change of variable. The following result establishes under which conditions the moments of the IF distribution exist and are finite.

**Proposition 1.** *The $r^{th}$ moment of the IF distribution for $p < \infty$ exists and is finite if and only if $b > 0$ and $r < bq$ or $b < 0$ and $r < -b$.*

The proof is provided in the Appendix. It is in principle possible to write out $I(p, b, q)$ as an infinite series of beta functions, but since this expression is rather intricate and needs to be worked out on a case-by-case basis just like $I(p, b, q)$, we refrain from doing so. However, in what follows, we compute the integral $I(p, b, q)$ for the four-parameter distributions IF1 and IF3 and obtain there more tractable expressions.

**Moments of the IF1 distribution.** If we plug in $p = 0$ and then set $z = \frac{1}{y}$ we get

$$I(0, b, q) = \int_1^\infty q\, y^{-q-1} (y-1)^{\frac{r-i}{b}} \, \mathrm{d}y = q \int_0^1 z^{q-1-\frac{r-i}{b}} (1-z)^{\frac{r-i}{b}} \, \mathrm{d}z.$$

If either $b > 0$ and $r < bq$ or else $b < 0$ and $r < -b$ (i.e., under the conditions identified in Proposition 1), then this can be written as

$$I(p, b, q) = qB(q - \tfrac{r-i}{b}, 1 + \tfrac{r-i}{b}),$$

where $B(\cdot, \cdot)$ stands for the beta function. The $r^{\text{th}}$ moment of the IF1 distribution is thus given by

$$\mathbb{E}\left[X^r\right] = \sum_{i=0}^{r} \binom{r}{i} x_0^i c^{r-i} qB(q - \tfrac{r-i}{b}, 1 + \tfrac{r-i}{b}) \quad \text{if} \begin{cases} b > 0 \text{ and } r < bq, \\ b < 0 \text{ and } r < -b, \end{cases}$$

otherwise the moment does not exist. The mean and variance of the IF1 distribution are then respectively given by

$$\mathbb{E}(X) = x_0 + cq\, B\left(q - \frac{1}{b}, 1 + \frac{1}{b}\right) \quad \text{if} \begin{cases} b > \frac{1}{q}, \\ b < -1, \end{cases} \tag{5}$$

and

$$\mathbb{V}(X) = c^2 \left[ q\, B\left(q - \frac{2}{b}, 1 + \frac{2}{b}\right) - \left( q\, B\left(q - \frac{1}{b}, 1 + \frac{1}{b}\right)\right)^2 \right] \quad \text{if} \begin{cases} b > \frac{2}{q}, \\ b < -2. \end{cases}$$

Note that sometimes it can be convenient to rewrite the mean (5) under the form

$$\mathbb{E}(X) = x_0 + c \frac{\Gamma\left(q - \frac{1}{b}\right) \Gamma\left(1 + \frac{1}{b}\right)}{\Gamma(q)}$$

with $\Gamma(\cdot)$ the gamma function. Special cases of the mean expressions can be found in Table 1.

**Moments of the IF3 distribution.** If $b = 1$, for $r < q$ we make the change of variables $z = \frac{y^{-q}}{p+1}$ and get

$$\begin{aligned} I(p,1,q) &= (p+1)^{1-\frac{r-i}{q}} \int_0^1 \left(z^{-\frac{1}{q}} - 1\right)^{r-i} (1-z)^p \, \mathrm{d}z \\ &= (p+1)^{1-\frac{r-i}{q}} \sum_{k=0}^{r-i} \tbinom{r-i}{k} (-1)^k \int_0^1 z^{-\frac{1}{q}(r-i-k)} (1-z)^p \, \mathrm{d}z \\ &= (p+1)^{1-\frac{r-i}{q}} \sum_{k=0}^{r-i} \tbinom{r-i}{k} (-1)^k B\left(1 - \tfrac{1}{q}(r-i-k), p+1\right). \end{aligned}$$

These manipulations are possible since the integral is finite under $r < q$. Consequently, the $r^{\text{th}}$ moment of the IF3 distribution is given by

$$\mathbb{E}\left[X^r\right] = \sum_{i=0}^{r} \tbinom{r}{i} x_0^i c^{r-i} \ (p+1)^{1-\frac{r-i}{q}} \sum_{k=0}^{r-i} \tbinom{r-i}{k} (-1)^k B\left(1 - \tfrac{1}{q}(r-i-k), p+1\right).$$

The associated mean and variance are

$$\mathbb{E}(X) = x_0 + c(p+1)^{1-\frac{1}{q}} \left( B\left(1 - \frac{1}{q}, p+1\right) - \frac{1}{p+1} \right) \quad \text{if } 1 < q,$$

and

$$\begin{aligned} \mathbb{V}(X) &= c^2 (p+1)^{1-\frac{2}{q}} \left[ B\left(1 - \frac{2}{q}, p+1\right) - 2B\left(1 - \frac{1}{q}, p+1\right) + \frac{1}{p+1} \right] \\ &\quad - c^2 (p+1)^{2-\frac{2}{q}} \left[ B\left(1 - \frac{1}{q}, p+1\right) - \frac{1}{p+1} \right]^2 \quad \text{if } 2 < q. \end{aligned}$$

The special cases of the mean expressions for the Generalized Lomax and Stoppa distributions can be found in Table 1.

Let us now consider the case $p = \infty$.

**Moments of the IF2 distribution.** The $r$-th moment is calculated as

$$\mathbb{E}\left[X^r\right] = \int_{x_0}^{\infty} x^r \ \frac{|b|q}{c} \left(\frac{x - x_0}{c}\right)^{-bq-1} e^{-\left(\frac{x-x_0}{c}\right)^{-bq}} \, \mathrm{d}x.$$

In this case the finite moments conditions are more easily seen and require no formal statement under the form of a proposition. When $b < 0$, all moments exist, while for $b > 0$ we can see that the integrand behaves like $x^{r-bq-1}$ for large values of $x$, implying existence of thr $r$-th moment iff $r < bq$. Under these conditions, the change of variables $y = \left(\frac{x-x_0}{c}\right)^{-bq}$ combined with Newton's binomial theorem implies that the integral is equal to

$$\sum_{i=0}^{r} \binom{r}{i} x_0^i c^{r-i} \int_0^\infty y^{-\frac{r-i}{bq}} e^{-y} \, \mathrm{d}y = \sum_{i=0}^{r} \binom{r}{i} x_0^i c^{r-i} \Gamma\left(1 - \frac{r-i}{bq}\right)$$

by definition of the gamma function. The $r^{\text{th}}$ moment of the IF2 distribution is therefore given by

$$\mathbb{E}\left[X^r\right] = \sum_{i=0}^{r} \tbinom{r}{i} x_0^i c^{r-i} \, \Gamma\left(1 - \tfrac{r-i}{bq}\right) \quad \text{if} \begin{cases} b > 0 \text{ and } r < bq, \\ b < 0, \end{cases}$$

otherwise the moment does not exist. The corresponding mean and variance take on the expressions

$$\mathbb{E}(X) = x_0 + c\,\Gamma\left(1 - \frac{1}{bq}\right) \quad \text{if} \begin{cases} b > \frac{1}{q}, \\ b < 0, \end{cases}$$

and

$$\mathbb{V}(X) = c^2 \left[\Gamma\left(1 - \frac{2}{bq}\right) - \left(\Gamma\left(1 - \frac{1}{bq}\right)\right)^2\right] \quad \text{if} \begin{cases} b > \frac{2}{q}, \\ b < 0. \end{cases}$$

Special cases of the mean expressions can be found in Table 1.

## 6. Unimodality and location of the mode

Determining the mode of a distribution is an important issue, which we tackle in this section. We study the derivative of $x \mapsto f_p(x)$, with

| Distribution name | # par. | Parameters $(p, b, c, q, x_0)$ | Mean $\mathbb{E}[X]$ | Constraint |
|---|---|---|---|---|
| Pareto IV | 4 | $(0, \frac{1}{\gamma} > 0, c, q, x_0)$ | $x_0 + cqB(q-\gamma, 1+\gamma)$ | $\gamma < q$ |
| Lindsay–Burr III | 4 | $(0, b < 0, c, q, x_0)$ | $x_0 + cqB\left(q - \frac{1}{b}, 1 + \frac{1}{b}\right)$ | $b < -1$ |
| Dagum | 3 | $(0, b < 0, c, q, 0)$ | $cqB\left(q - \frac{1}{b}, 1 + \frac{1}{b}\right)$ | $b < -1$ |
| Pareto II | 3 | $(0, 1, c, q, x_0)$ | $x_0 + \frac{c}{q-1}$ | $q > 1$ |
| Pareto III | 3 | $(0, \frac{1}{\gamma} > 0, c, 1, x_0)$ | $x_0 + c\Gamma(1-\gamma)\Gamma(1+\gamma)$ | $\gamma < 1$ |
| Tadikamalla–Burr XII | 3 | $(0, b > 0, c, q, 0)$ | $cqB\left(q - \frac{1}{b}, 1 + \frac{1}{b}\right)$ | $bq > 1$ |
| Fisk | 2 | $(0, b > 0, c, 1, 0)$ | $c\Gamma\left(1 - \frac{1}{b}\right)\Gamma\left(1 + \frac{1}{b}\right)$ | $b > 1$ |
| Lomax | 2 | $(0, 1, c, q, 0)$ | $\frac{c}{q-1}$ | $q > 1$ |
| Pareto I | 2 | $(0, 1, x_0, q, x_0)$ | $\frac{q}{q-1}x_0$ | $q > 1$ |
| Burr XII | 2 | $(0, b > 0, 1, q, 0)$ | $qB\left(q - \frac{1}{b}, 1 + \frac{1}{b}\right)$ | $bq > 1$ |
| Weibull | 3 | $(\infty, -1, c, q, x_0)$ | $x_0 + c\Gamma\left(1 + \frac{1}{q}\right)$ | |
| Fréchet | 3 | $(\infty, 1, c, q, x_0)$ | $x_0 + c\Gamma\left(1 - \frac{1}{q}\right)$ | $q > 1$ |
| Gumbel II | 2 | $(\infty, 1, c, q, 0)$ | $c\Gamma\left(1 - \frac{1}{q}\right)$ | $q > 1$ |
| Rayleigh | 1 | $(\infty, -1, c, 2, 0)$ | $\frac{c}{2}\sqrt{\pi}$ | |
| Inverse Rayleigh | 1 | $(\infty, 1, c, 2, 0)$ | $c\sqrt{\pi}$ | |
| Exponential | 1 | $(\infty, -1, c, 1, 0)$ | $c$ | |
| Inverse Exponential | 1 | $(\infty, 1, c, 1, 0)$ | Not defined | Violated |
| Generalized Lomax | 3 | $(m-1, 1, c, q, 0)$ | $c\, m^{1-\frac{1}{q}}\left(B\left(1 - \frac{1}{q}, m\right) - \frac{1}{m}\right)$ | $q > 1$ |
| Stoppa | 3 | $(m-1, 1, c, q, cm^{-\frac{1}{q}})$ | $c\, m^{1-\frac{1}{q}}\ B\left(1 - \frac{1}{q}, m\right)$ | $q > 1$ |

TABLE 1. Expressions for the mean

particular emphasis on the three main subfamilies IF1, IF2 and IF3 described in Section 3. As we show in the Appendix, the derivative of the pdf vanishes either at the boundary $x = x_0$ of the domain or at

$$x = x_0 + c(p+1)^{-\frac{1}{bq}}\left(t^{-\frac{1}{q}} - 1\right)^{\frac{1}{b}},$$

where $t$ is solution of the almost cyclic equation

$$(b-1)t^{-\frac{1}{q}}(1-t) - b(q+1)(t^{-\frac{1}{q}} - 1)(1-t) + pbq(t^{-\frac{1}{q}} - 1)t = 0. \quad (6)$$

This allows us to draw the following conclusions regarding the modality of the IF distribution.

- The mode of the IF1 distribution ($p = 0$) is given by

$$\begin{cases} x_0 & \text{if } b = -\frac{1}{q} \text{ or } b = 1, \\ x_0 + c\left(\frac{b-1}{bq+1}\right)^{\frac{1}{b}} & \text{if } b < -\frac{1}{q} \text{ or } b > 1, \end{cases}$$

whereas in the remaining cases, i.e. $b \in ]-\frac{1}{q}; 1[$, there is a vertical asymptote at $x = x_0$. We plot in Figure 3 a contour plot of the mode of the IF1 distribution.

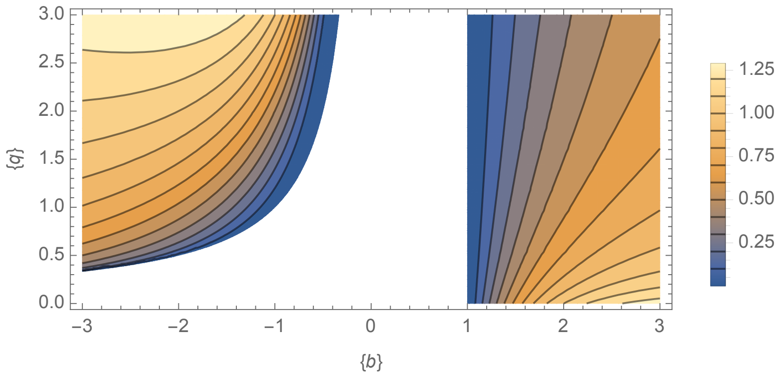


FIGURE 3. Contour plot of the mode of the IF1 distribution as a function of the parameters, with $c = 1$ and $x_0 = 0$.

- The mode of the IF2 distribution ($p \to \infty$) is given by

$$\begin{cases} x_0 & \text{if } b = -\frac{1}{q}, \\ x_0 + c\left(\frac{bq}{bq+1}\right)^{\frac{1}{bq}} & \text{if } b < -\frac{1}{q} \text{ or } b > 0, \end{cases}$$

whereas in the remaining cases, i.e. $b \in ]-\frac{1}{q}, 0]$, there is a vertical asymptote at $x = x_0$. Figure 4 shows a contour plot of the mode of the IF2 distribution.

- The mode of the IF3 distribution ($0 < p < \infty$ and $b = 1$) is given by

$$x_0 + c(p+1)^{-\frac{1}{q}}\left(\left(\frac{q+1}{(p+1)q+1}\right)^{-\frac{1}{q}} - 1\right).$$

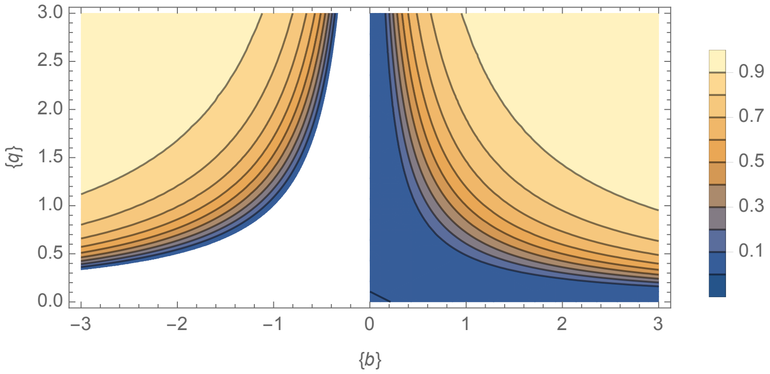


FIGURE 4. Contour plot of the mode of the IF2 distribution as a function of the parameters, with $c = 1$ and $x_0 = 0$.

A contour plot of the mode of the IF3 distribution can be seen in Figure 5.

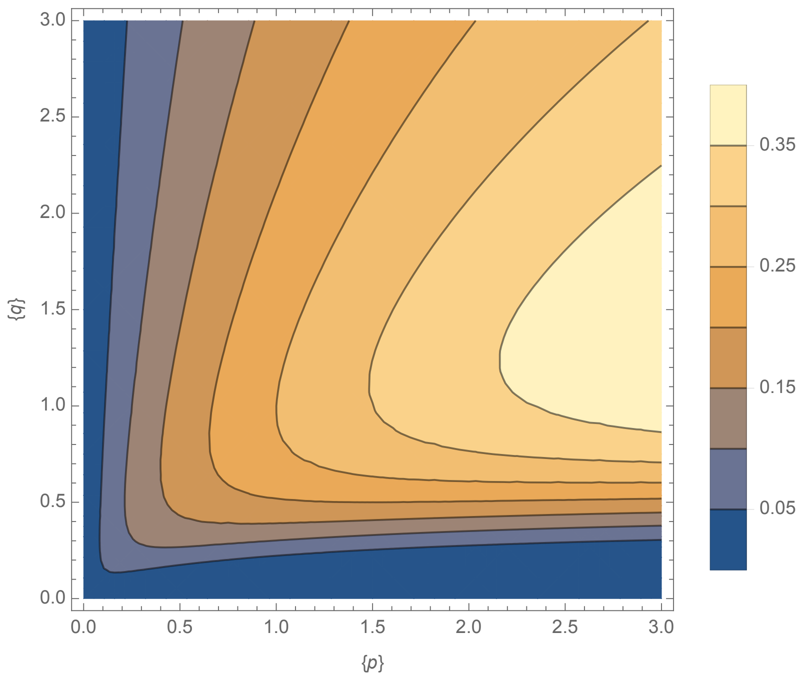


FIGURE 5. Contour plot of the mode of the IF3 distribution as a function of the parameters, with $c = 1$ and $x_0 = 0$.

For calculation details, see the Appendix. We note that all three subfamilies are unimodal, which is coherent with the related special cases from the literature. Moreover, we have derived the exact expressions of the modes. This unimodality is a very attractive feature from an interpretability point of view: bimodal or multimodal distributions are arguably best modeled as a mixture of unimodal distributions. It is therefore not surprising that many new distributions are built with the target of being unimodal; for modern examples, see e.g. Jones (2014); Kato and Jones (2015); Fujisawa and Abe (2015).

## 7. Final remarks

We have built in this paper an overarching family of size distributions, the Interpolating Family of distributions, and shown how it indeed interpolates between power laws and power laws with exponential cut-off. This sheds interesting new light on these highly distinct types of size distributions, and we hope that our construction inspired from statistical physics will stimulate researchers to search for bridges between other apparently unrelated distributions. Understanding the links between probability laws and grouping them into classes with similar properties has become particularly important nowadays, given the plethora of new distributions. We refer the interested reader to the review papers Jones (2015) and Babić et al. (2019) for further information about classifying flexible distributions for univariate respectively multivariate data, and to Ley et al. (2021) for an overview and discussion of advantages and limitations of flexible models.

Finally, we recall that the aim of this research was to treat power laws and power laws with exponential cut-off in a unified way, and to develop general properties for the IF distribution. We purposely did not provide inferential procedures for the full 5-parameter IF distribution since we have noticed that distinct combinations of the five parameters lead to nearly identical shapes of the density, and consequently maximum likelihood estimation may be ill-defined. We therefore recommend to restrict to subfamilies for inferential purposes[2], or to consider interesting new special cases of the IF distribution. Our results readily yield the theoretical basis for such future research.

## Appendix

**Proof of Proposition 1.** We start by performing the change of variables $y^{-q}/(p+1) = t$ inside the integral $I(p,b,q)$, yielding

$$(p+1)^{1-\frac{r-i}{qb}} \int_0^1 \left(t^{-1/q}-1\right)^{\frac{r-i}{b}} (1-t)^p \, dt.$$

Since $p < \infty$ and $t \in [0,1]$, the factor $(1-t)^p$ is always bounded by 1 and hence causes no problem. Concentrating our attention on $\left(t^{-1/q}-1\right)^{\frac{r-i}{b}}$, we need to distinguish two cases:

- $b > 0$: The finiteness of the integral depends on $\left(t^{-1/q}-1\right)^{\frac{r-i}{b}}$ when $t$ approaches 0, in which case the expression behaves like $t^{\frac{i-r}{qb}}$ and hence is finite iff $\frac{i-r}{qb} > -1$, which is equivalent to $r - i < qb$. Since this needs to hold for every $i \in [0,r]$, we conclude that the $r$-th moment is finite iff $r < bq$.
- $b < 0$: The finiteness of the integral depends on $\left(t^{-1/q}-1\right)^{\frac{r-i}{b}}$ when $t$ approaches 1, in which case the expression behaves like $(1-t)^{\frac{r-i}{b}}$ and hence is finite iff $\frac{r-i}{b} > -1$, which is equivalent to $r-i < -b$. Since this needs to hold for every $i \in [0,r]$, we conclude that the $r$-th moment is finite iff $r < -b$.

This concludes the proof. □

[2]Given the simple form of the cumulative distribution function, special cases of the IF are tailor-made for dealing with censored data.

**Mode calculation.** The derivative of the pdf (3) vanishes if and only if

$$\begin{aligned} 0 = \quad & (b-1)\left(\tfrac{x-x_0}{c}\right)^{b-2}\left((p+1)^{-\frac{1}{q}}+\left(\tfrac{x-x_0}{c}\right)^b\right)^{-q-1}\left(1-\tfrac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+\left(\tfrac{x-x_0}{c}\right)^b\right)^{-q}\right)^p \\ & -b(q+1)\left(\tfrac{x-x_0}{c}\right)^{2b-2}\left((p+1)^{-\frac{1}{q}}+\left(\tfrac{x-x_0}{c}\right)^b\right)^{-q-2}\left(1-\tfrac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+\left(\tfrac{x-x_0}{c}\right)^b\right)^{-q}\right)^p \\ & +\tfrac{pbq}{p+1}\left(\tfrac{x-x_0}{c}\right)^{2b-2}\left((p+1)^{-\frac{1}{q}}+\left(\tfrac{x-x_0}{c}\right)^b\right)^{-2q-2}\left(1-\tfrac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+\left(\tfrac{x-x_0}{c}\right)^b\right)^{-q}\right)^{p-1}. \end{aligned}$$

If we set $y = \frac{x-x_0}{c}$, then the above holds true if either

$$y^{b-2}\left((p+1)^{-\frac{1}{q}}+y^b\right)^{-q-2}\left(1-\frac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+y^b\right)^{-q}\right)^{p-1} = 0$$

or

$$\begin{aligned} 0 = (b-1)\left((p+1)^{-\frac{1}{q}}+y^b\right)\left(1-\frac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+y^b\right)^{-q}\right) \quad (7) \\ -b(q+1)y^b\left(1-\frac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+y^b\right)^{-q}\right)+\frac{pbq}{p+1}y^b\left((p+1)^{-\frac{1}{q}}+y^b\right)^{-q}. \end{aligned}$$

The only solution that the first equation can possibly admit is $y = 0$. This corresponds to $x = x_0$, i.e. to the boundary of the domain. On the other hand, equation (7) may admit interior solutions. We separate the analysis of equation (7) in two parts: first the case $p$ finite from which we deduce the mode of the IF1 and IF3 subfamilies and second the case $p \to \infty$ which gives the mode of the IF2 subfamily. If $p$ is finite, then we set

$$t = \frac{1}{p+1}\left((p+1)^{-\frac{1}{q}}+y^b\right)^{-q}$$

and equation (7) simplifies to the almost cyclic equation (6):

$$(b-1)t^{-\frac{1}{q}}(1-t) - b(q+1)\left(t^{-\frac{1}{q}}-1\right)(1-t) + pbq\left(t^{-\frac{1}{q}}-1\right)t = 0.$$

Solving this equation in all generality is possible numerically but we will restrict ourselves to show how to get closed-form solutions for the two subfamilies IF1 and IF3. For the IF1 distribution ($p = 0$), equation (6) further simplifies to

$$(b-1)t^{-\frac{1}{q}}(1-t) - b(q+1)\left(t^{-\frac{1}{q}} - 1\right)(1-t) = 0.$$

While we recover the boundary solution $x = x_0$ if $b > 0$, we also find an interior solution $x = x_0 + c\left(\frac{b-1}{bq+1}\right)^{\frac{1}{b}}$ if either $b < -\frac{1}{q}$ or $b > 1$. Repeating the procedure with the second derivative of the pdf (3), a straightforward but tedious calculation shows that the interior solution thus found indeed corresponds to a maximum and that the mode occurs on the boundary $x = x_0$ if either $b = -\frac{1}{q}$ or $b = 1$.

For the IF3 distribution ($0 < p < \infty$ and $b = 1$), equation (6) further simplifies to

$$-(q+1)\left(t^{-\frac{1}{q}} - 1\right)(1-t) + pq\left(t^{-\frac{1}{q}} - 1\right)t = 0.$$

This equation admits two solutions: the boundary solution $x = x_0$ and the interior solution $x = x_0 + c(p+1)^{-\frac{1}{q}}\left(\left(\frac{q+1}{(p+1)q+1}\right)^{-\frac{1}{q}} - 1\right)$. One can then check that the latter corresponds to the mode of the IF3 distribution and that this mode, and thus the interior solution, moves towards the boundary as $p$ and $q$ tend to zero.

On the other hand, for the IF2 distribution ($p \to \infty$), equation (7) simplifies to

$$0 = (b-1)y^b - b(q+1)y^b + bqy^{b-bq}.$$

We deduce that the derivative of the pdf of the IF2 vanishes either at the boundary $x = x_0$ if $b > 0$ or at the interior point $x = x_0 + c\left(\frac{bq}{bq+1}\right)^{\frac{1}{bq}}$ if $b < -\frac{1}{q}$ or $b > 0$. Similarly as for the IF1, tedious second derivative calculations reveal that the interior solution always corresponds to a maximum and that the mode occurs on the boundary if either $b = -\frac{1}{q}$ or $b = 0$.

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