EIGENVALUE FLUCTUATIONS FOR RANDOM ELLIPTIC OPERATORS IN HOMOGENIZATION REGIME

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ABSTRACT. This work is devoted to the asymptotic behavior of eigenvalues of an elliptic operator with rapidly oscillating random coefficients on a bounded domain with Dirichlet boundary conditions. A sharp convergence rate is obtained for eigenvalues towards those of the homogenized problem, as well as a quantitative two-scale expansion result for eigenfunctions. Next, a quantitative central limit theorem is established for fluctuations of isolated eigenvalues; more precisely, a pathwise characterization of eigenvalue fluctuations is obtained in terms of the so-called homogenization commutator, in parallel with the recent fluctuation theory for the solution operator.

1. Introduction

Let $\boldsymbol{a}$ be a stationary and ergodic random coefficient field on $\mathbb{R}^d$ with symmetric values in $\mathbb{R}^{d\times d}$, with the following boundedness and uniform ellipticity properties, for some deterministic constant $\nu > 0$,

$$|\boldsymbol{a}(x)e| \le |e|, \qquad e\cdot \boldsymbol{a}(x)e \ge \nu |e|^2, \qquad \text{almost surely,} \quad \text{for all } x,e\in\mathbb{R}^d, \tag{1.1}$$

and denote by $(\Omega,\mathbb{P})$ the underlying probability space. In the sequel, we further assume that $\boldsymbol{a}$ satisfies some strong mixing condition, and our main results focus for simplicity on a Gaussian model, see Section 2.1. Given a bounded $C^{1,1}$ domain $U\subset\mathbb{R}^d$, we consider the sequence of rescaled operators $-\nabla\cdot\boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla$ on $H^1_0(U)$. We consider their eigenvalues $\{\lambda^k_\varepsilon\}_{k\ge1}$, listed in increasing order and repeated according to multiplicity, and we choose corresponding orthonormal eigenfunctions $\{g^k_\varepsilon\}_{k\ge1}\subset H^1_0(U)$,

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g^k_\varepsilon \;=\; \lambda^k_\varepsilon g^k_\varepsilon \quad \text{in } U, \qquad \|g^k_\varepsilon\|_{\mathrm{L}^2(U)}=1. \tag{1.2}$$

As is well-known, see e.g. [25, Section 11], the eigenvalues $\{\lambda^k_\varepsilon\}_{k\ge1}$ converge almost surely to the corresponding eigenvalues $\{\bar\lambda^k\}_{k\ge1}$ of the homogenized operator $-\nabla\cdot\bar{\boldsymbol{a}}\nabla$,

$$-\nabla\cdot\bar{\boldsymbol{a}}\nabla\bar g^k \;=\; \bar\lambda^k\bar g^k \quad \text{in } U, \qquad \|\bar g^k\|_{\mathrm{L}^2(U)}=1, \tag{1.3}$$

where the effective coefficient $\bar{\boldsymbol{a}}\in\mathbb{R}^{d\times d}$ is defined in each direction e_α, $1\le\alpha\le d$, by

$$\bar{\boldsymbol{a}}e_\alpha = \mathbb{E}\left[\boldsymbol{a}(\nabla\varphi_\alpha+e_\alpha)\right],$$

in terms of the so-called corrector gradient $\nabla\varphi_\alpha$, which is defined as the unique almost sure gradient solution in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)^d$ of the corrector equation

$$-\nabla\cdot\boldsymbol{a}(\nabla\varphi_\alpha+e_\alpha)=0, \qquad \text{in } \mathbb{R}^d,$$

such that $\nabla\varphi_\alpha$ is a stationary field with vanishing expectation and bounded second moment. In addition, normalized eigenfunctions are known to converge weakly in $H^1_0(U)$ to the corresponding eigenfunctions of the homogenized operator (up to taking linear combinations in case of multiple eigenvalues), see e.g. [25, Section 11]. In the present contribution, we further establish sharp convergence rates and we analyze random fluctuations. More precisely, our results are twofold:

(i) We prove an optimal convergence rate for eigenvalues and provide a two-scale description of eigenfunctions. Note that this also applies to the case of multiple eigenvalues. Such results were only known previously in the easier periodic setting [26, 27].

(ii) We characterize joint fluctuations of *simple* eigenvalues in form of a quantitative central limit theorem, thus answering a question raised by Biskup, Fukushima, and König in [6, Section 2.2]. More precisely, we unravel the pathwise structure of fluctuations: in the spirit of our recent work with Gloria and Otto [15] (see also the related heuristics by Armstrong, Gu, and Mourrat in [23]), we show that fluctuations of λ_ε^k are pointwise close to fluctuations of $\int_U \Xi^\circ_{\alpha\beta}(\frac{\cdot}{\varepsilon})\partial_\alpha \bar g^k \partial_\beta \bar g^k$,[1] in terms of the so-called standard homogenization commutator

$$\Xi^\circ_{\alpha\beta} := e_\beta \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})(\nabla \varphi_\alpha + e_\alpha). \tag{1.4}$$

In other words, while we found in [15, 14] that Ξ° governs fluctuations of the solution operator, we show in the present contribution that this quantity also governs eigenvalue fluctuations. This pathwise relation then reduces the characterization of eigenvalue fluctuations to the scaling limit of Ξ°, which we already studied extensively in [15, 16, 11]. In particular, under suitable strong mixing conditions, eigenvalue fluctuations are Gaussian.

These different results make heavy use of refined tools from the recent quantitative theory of stochastic homogenization as developed in [1, 21, 20, 16].

We briefly explain how our fluctuation result relates to the spectral statistics conjecture for random operators. Rescaling the eigenvalue relation (1.2), eigenvalues of the operator $-\nabla \cdot \boldsymbol{a} \nabla$ on the dilated domain $\frac{1}{\varepsilon}U$ coincide with $\{\varepsilon^2 \lambda_\varepsilon^k\}_{k\ge 1}$, and we consider the large-volume limit $\varepsilon \downarrow 0$. In this contribution, we show that the first eigenvalues have joint Gaussian fluctuations, in the sense that the vector $\varepsilon^{-d/2}(\lambda_\varepsilon^1 - \mathbb{E}[\lambda_\varepsilon^1], \ldots, \lambda_\varepsilon^n - \mathbb{E}[\lambda_\varepsilon^n])$ is asymptotically Gaussian for any fixed n. This Gaussian fluctuation result at the bottom of the spectrum is new and should be compared to the conjecture that eigenvalues have local Poisson statistics in spectral regions where localization holds (in particular, at edges of the spectrum other than its bottom) and have random matrix GOE statistics in the bulk of regions where delocalization holds. Rigorous results on Poisson statistics in the localized regime were pioneered by Minami [28] for the Anderson model, and we refer to [18, 24, 9] and references therein for recent developments, but to our knowledge the problem still remains open for the divergence-form operator $-\nabla \cdot \boldsymbol{a} \nabla$ apart from the 1D case covered in [31]. Rigorous results on GOE statistics in the delocalized regime are only known in the simplified setting of random band matrix models [8, 7].

The article is organized as follows. Precise assumptions and main results are stated in Section 2. We focus for simplicity on a Gaussian model for the coefficient field $\boldsymbol{a}$, in which case Malliavin calculus is available and simplifies the analysis. In Section 3, we recall some useful tools from the quantitative theory of stochastic homogenization, including corrector estimates and large-scale regularity theory, and we recall notations from Malliavin calculus. Proofs of the main results are postponed to Section 4.

[1]Throughout, we use Einstein's convention of summation on repeated indices, here on $1 \le \alpha, \beta \le d$.

Notation.

- We denote by $C \geq 1$ any constant that only depends on the dimension d, on the ellipticity constant ν in (1.1), on the domain U, and on $\|a_0\|_{W^{2,\infty}}$ and $\int_{\mathbb{R}^d}[\mathcal{C}_0]_\infty$ in (2.1) and (2.2) below. We use the notation $\lesssim$ (resp. $\gtrsim$) for $\leq C\times$ (resp. $\geq \frac{1}{C}\times$) up to such a multiplicative constant C. We write $\simeq$ when both $\lesssim$ and $\gtrsim$ hold. We add subscripts to $C, \lesssim, \gtrsim, \simeq$ to indicate dependence on other parameters.
- We denote by $B_r(x)$ the ball of radius r centered at x in $\mathbb{R}^d$, and we write for shortness $B_r := B_r(0)$, $B(x) := B_1(x)$, and $B := B_1(0)$.
- For a function g and an exponent $1 \leq p < \infty$, we write $[g]_p(x) := (\fint_{B(x)} |g|^p)^{1/p}$ for the local moving L^p-averages, and similarly $[g]_\infty(x) := \sup_{B(x)} |g|$. For averages at the scale ε, we write $[g]_{p;\varepsilon}(x) := (\fint_{B_\varepsilon(x)} |g|^p)^{1/p}$.

2. Main results

2.1. **Assumptions.** Let $U \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ domain. For the random coefficient field $\boldsymbol{a}$, we focus on a Gaussian model: more precisely, we set

$$\boldsymbol{a}(x) := a_0(G(x)), \tag{2.1}$$

where $a_0 \in C^2_b(\mathbb{R}^\kappa)^{d\times d}$ is such that the boundedness and uniform ellipticity requirements (1.1) are pointwise satisfied, and where $G : \mathbb{R}^d \times \Omega \to \mathbb{R}^\kappa$ is an $\mathbb{R}^\kappa$-valued centered stationary Gaussian random field on $\mathbb{R}^d$ with covariance function $\mathcal{C} : \mathbb{R}^d \to \mathbb{R}^{\kappa\times\kappa}$, constructed on a probability space $(\Omega, \mathbb{P})$. In addition, we assume that G has integrable correlations in the following sense: starting from the representation

$$G_i = \mathcal{C}_{0;ij} * \xi_j,$$

where ξ is an $\mathbb{R}^\kappa$-valued Gaussian white noise on $\mathbb{R}^d$ and where the kernel $\mathcal{C}_0 : \mathbb{R}^d \to \mathbb{R}^{\kappa\times\kappa}$ satisfies $\mathcal{C}_{0;il} * \mathcal{C}_{0;lj} = \mathcal{C}_{ij}$, we assume that $\mathcal{C}_0$ satisfies the integrability condition

$$\int_{\mathbb{R}^d} [\mathcal{C}_0]_\infty < \infty. \tag{2.2}$$

In particular, this entails that the covariance function $\mathcal{C}$ itself satisfies the same integrability condition $\int_{\mathbb{R}^d}[\mathcal{C}]_\infty < \infty$. Moreover, $\mathcal{C}$ is necessarily continuous, so that G and $\boldsymbol{a}$ are stochastically continuous and jointly measurable on $\mathbb{R}^d \times \Omega$.

Remark 2.1 (Relaxation of assumptions)**.** This Gaussian model (2.1)–(2.2) allows to exploit Malliavin calculus techniques, which strongly simplify the analysis. Our approach can be repeated mutatis mutandis in a corresponding Poisson model or in the iid discrete setting, using corresponding stochastic calculus techniques, e.g. [30, 10]. It can be further adapted to the case of a degraded stochastic calculus in form of multiscale variance inequalities as we introduced in [12, 13] with Gloria, which are available for a much wider class of mixing coefficient fields. The general case of an α-mixing coefficient field is however much more demanding: we believe that it can be treated using the recent techniques of [1, 22], but we do not pursue in that direction here. Finally, the integrability condition (2.2) is easily relaxed: the Gaussian model with non-integrable correlations can be treated similarly but would yield different scalings as in [20, 14, 11].

2.2. **Convergence rate for eigenvalues and eigenfunctions.** The following result provides a sharp convergence rate for eigenvalues, as well as a quantitative two-scale expansion for corresponding eigenfunctions. Note that the statement also covers multiple eigenvalues; in case of a simple eigenvalue $\bar\lambda^k$, the projection $\bar\pi^k[g_\varepsilon^k]$ is reduced to $\bar g^k$. The square root in the convergence rates $(\varepsilon\mu_d(\frac1\varepsilon))^{1/2}$ is due to boundary layers: for corresponding eigenvalue problems on a box with periodic boundary conditions, boundary issues disappear and a direct inspection of the proof would yield the optimal rate $\varepsilon\mu_d(\frac1\varepsilon)$.

Theorem 2.2. *For all $k\ge1$, denoting by $\bar\pi^k[\cdot]$ the orthogonal projection of $\mathrm{L}^2(U)$ onto the (possibly multidimensional) eigenspace associated with $\bar\lambda^k$, we have for all $q<\infty$,*

$$\|\lambda_\varepsilon^k-\bar\lambda^k\|_{\mathrm{L}^q(\Omega)} \lesssim_{k,q} (\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}, \tag{2.3}$$

$$\|g_\varepsilon^k-\bar\pi^k[g_\varepsilon^k]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))} \lesssim_{k,q} (\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}, \tag{2.4}$$

$$\|\nabla g_\varepsilon^k-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar\pi^k[g_\varepsilon^k]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))} \lesssim_{k,q} (\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}, \tag{2.5}$$

in terms of

$$\mu_d(r) := \left\{\begin{array}{lll} 1 & : & d>2,\\ \log(2+r)^{\frac12} & : & d=2,\\ (1+r)^{\frac12} & : & d=1.\end{array}\right. \tag{2.6}$$

2.3. **Eigenvalue fluctuations.** The following result shows that eigenvalue fluctuations are governed to leading order by fluctuations of the so-called standard homogenization commutator (1.4). Combined with the scaling limit for the latter in [15, 16, 11], this yields a full characterization of eigenvalue fluctuations together with a convergence rate. Note that this result is restricted to *simple* eigenvalues. As in Theorem 2.2, the convergence rates $(\varepsilon\mu_d(\frac1\varepsilon))^{1/2}$ can be replaced by $\varepsilon\mu_d(\frac1\varepsilon)$ in case of corresponding eigenvalue problems on a periodic box.

Theorem 2.3. *For all $k\ge1$ such that $\bar\lambda^k$ is simple, we have for all $q<\infty$,*

$$\varepsilon^{-\frac d2}\Big\|\lambda_\varepsilon^k-\mathbb{E}[\lambda_\varepsilon^k]-\int_U\Xi^\circ_{\alpha\beta}(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\partial_\beta\bar g^k\Big\|_{\mathrm{L}^q(\Omega)}\lesssim_{k,q}(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}, \tag{2.7}$$

where we recall that the standard homogenization commutator Ξ° is defined in (1.4), and where μ_d is given by (2.6). Combined with the known scaling limit for Ξ°, cf. [15, 16, 11], *this yields for all $k_1,\ldots,k_n\ge1$ such that $\bar\lambda^{k_1},\ldots,\bar\lambda^{k_n}$ are simple,*

$$W_2\bigg(\varepsilon^{-\frac d2}\Big(\big(\lambda_\varepsilon^{k_1}-\mathbb{E}[\lambda_\varepsilon^{k_1}]\big),\ldots,\big(\lambda_\varepsilon^{k_n}-\mathbb{E}[\lambda_\varepsilon^{k_n}]\big)\Big)\,;\,\mathcal{N}_{k_1,\ldots,k_n}\bigg)\lesssim_{k_1,\ldots,k_n}(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12},$$

where $W_2(\cdot;\cdot)$ denotes the 2-Wasserstein distance and where $\mathcal{N}_{k_1,\ldots,k_n}$ stands for the n-dimensional centered Gaussian vector with covariance

$$\mathbb{E}\left[(\mathcal{N}_{k_1,\ldots,k_n})_i(\mathcal{N}_{k_1,\ldots,k_n})_j\right] = \int_{\mathbb{R}^d}(\nabla\bar g^{k_i}\otimes\nabla\bar g^{k_i}):\mathcal{Q}\,(\nabla\bar g^{k_j}\otimes\nabla\bar g^{k_j}),$$

where the 4th-order tensor $\mathcal{Q}\in\mathbb{R}^{d\times d\times d\times d}$ is given by the following Green–Kubo formula, for any cut-off function $\chi\in C^\infty_c(\mathbb{R}^d)$ with $\chi(0)=1$,

$$\mathcal{Q}_{\alpha'\beta'\alpha\beta} := \lim_{L\uparrow\infty}\int_{\mathbb{R}^d}\chi(\tfrac1Lx)\,\mathrm{Cov}\left[\Xi^\circ_{\alpha'\beta'}(0);\Xi^\circ_{\alpha\beta}(x)\right]dx. \tag{2.8}$$

Remark 2.4. As shown in [15, 16, 11], although the covariance function of the homogenization commutator Ξ° is only borderline integrable,

$$\big|\mathrm{Cov}\left[\Xi^\circ_{\alpha'\beta'}(0);\Xi^\circ_{\alpha\beta}(x)\right]\big| \lesssim (1+|x|)^{-d},$$

the limit (2.8) indeed exists and the convergence holds with rate $O(L^{-1}\mu_d(L))$. Alternatively, in terms of Malliavin calculus, the effective tensor $\mathcal{Q}$ can be expressed as

$$\mathcal{Q}_{\alpha'\beta'\alpha\beta} := \int_{\mathbb{R}^d} \mathcal{C}_{ij}(y)\,\mathbb{E}\Big[\big((\nabla\varphi_{\beta'}+e_{\beta'})\cdot\partial_i a_0(G)(\nabla\varphi_{\alpha'}+e_{\alpha'})\big)(0) \\ \times(\mathcal{L}+1)^{-1}\big((\nabla\varphi_\beta+e_\beta)\cdot\partial_j a_0(G)(\nabla\varphi_\alpha+e_\alpha)\big)(y)\Big]\,dy,$$

where $\mathcal{L}$ is the Ornstein–Uhlenbeck operator associated with the Malliavin calculus with respect to the underlying Gaussian field G, cf. (3.4) below.

3. Main tools

In this section, we recall useful tools both from the quantitative theory of stochastic homogenization, including corrector estimates and large-scale regularity theory, and from Malliavin calculus.

3.1. **Tools from quantitative homogenization theory.** The following result recalls the definition of correctors and flux corrector, e.g. [21, Lemma 1], which are key to describe fine oscillations of the solution operator. Note that the flux corrector σ_α is defined as a vector potential for the flux $q_\alpha = \boldsymbol{a}(\nabla\varphi_\alpha+e_\alpha)-\bar{\boldsymbol{a}}e_\alpha$, cf. (3.2), and the defining equation (3.1) amounts to choosing the Coulomb gauge.

Lemma 3.1 (Correctors; [21])**.** *For all $1\le\alpha\le d$, there exists a unique solution φ_α to the following infinite-volume corrector problem:*

- *Almost surely, φ_α belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies in the weak sense*
$$-\nabla\cdot\boldsymbol{a}(\nabla\varphi_\alpha+e_\alpha) = 0, \qquad \text{in } \mathbb{R}^d.$$
- *The gradient field $\nabla\varphi_\alpha$ is stationary, has vanishing expectation, and has bounded second moment, and φ_α satisfies the anchoring condition $\fint_B\varphi_\alpha=0$ almost surely.*

In addition, there exists a unique random 2-tensor field $\sigma_\alpha=\{\sigma_{\alpha\beta\gamma}\}_{1\le\beta,\gamma\le d}$ that satisfies the following infinite-volume problem:

- *For all $1\le\beta,\gamma\le d$, almost surely, $\sigma_{\alpha\beta\gamma}$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies in the weak sense*
$$-\triangle\sigma_{\alpha\beta\gamma} = \partial_\beta(q_\alpha)_\gamma-\partial_\gamma(q_\alpha)_\beta, \qquad \text{in } \mathbb{R}^d, \tag{3.1}$$
in terms of the flux $q_\alpha := \boldsymbol{a}(\nabla\varphi_\alpha+e_\alpha)-\bar{\boldsymbol{a}}e_\alpha$.
- *The gradient field $\nabla\sigma_\alpha$ is stationary, has vanishing expectation, and has bounded second moment, and σ_α satisfies the anchoring condition $\fint_B\sigma_\alpha=0$ almost surely.*

In particular, this definition entails

$$\nabla\cdot\sigma_\alpha=q_\alpha, \qquad \sigma_{\alpha\beta\gamma}=-\sigma_{\alpha\gamma\beta}. \tag{3.2}$$

Next, in the present Gaussian setting (2.1)–(2.2), we have the following moment bounds on corrector gradients, as well as optimal estimates on the sublinearity of correctors, see [1, 20]. In dimension $d>2$, these estimates ensure that correctors φ,σ can be chosen themselves as stationary fields.

Theorem 3.2 (Corrector estimates; [1, 20]). *For all $q<\infty$,*

$$\|[\nabla\varphi]_2\|_{\mathrm{L}^q(\Omega)}+\|[\nabla\sigma]_2\|_{\mathrm{L}^q(\Omega)}\lesssim_q 1,$$

and for all $x\in\mathbb{R}^d$,

$$\|[\varphi]_2(x)\|_{\mathrm{L}^q(\Omega)}+\|[\sigma]_2(x)\|_{\mathrm{L}^q(\Omega)}\lesssim_q \mu_d(|x|),$$

where we recall that μ_d is given by (2.6). In addition, the following Meyers type improvement holds: there exists a constant $C_0\simeq 1$ such that for all $2\le p\le 2+\frac{1}{C_0}$ the local quadratic averages $[\cdot]_2$ in the above estimates can be replaced by $[\cdot]_p$.

A key insight in quantitative stochastic homogenization theory is the idea of large-scale regularity, which started with Avellaneda and Lin [5] in the periodic setting, then with Armstrong and Smart [3] in the random setting, and was fully developed in recent years in [2, 1, 21]: due to homogenization, the heterogeneous elliptic operator $-\nabla\cdot\boldsymbol{a}\nabla$ can be expected to inherit the same regularity properties as its homogenized version $-\nabla\cdot\bar{\boldsymbol{a}}\nabla$ on large scales. For our purpose in this work, we focus on large-scale L^p-regularity and we appeal to a convenient annealed version that we established in [16, Theorem 6.1] with Otto. More precisely, while in [16] only interior L^p-regularity was established, the following is further stated to hold globally on any bounded domain with Dirichlet boundary conditions: the proof follows as in [16, Section 6] up to replacing the use of large-scale interior Lipschitz regularity by corresponding global regularity as developed in [1, Section 3.5] (see also [17]).

Theorem 3.3 (Annealed L^p-regularity; [16, 1]). *Let $D\subset\mathbb{R}^d$ be a bounded $C^{1,\gamma}$ domain for some $\gamma>0$. For all $0<\varepsilon\le 1$ and $h\in C^\infty_c(D;\mathrm{L}^\infty(\Omega)^d)$, if $u_{\varepsilon;h}$ is almost surely the unique solution in $H^1_0(D)$ of*

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla u_{\varepsilon;h}=\nabla\cdot h,\qquad \text{in } U,$$

then there holds for all $1<p,q<\infty$ and $\delta>0$,

$$\|[\nabla u_{\varepsilon;h}]_{2;\varepsilon}\|_{\mathrm{L}^p(D;\mathrm{L}^q(\Omega))}\lesssim_{D,p,q,\delta}\|[h]_{2;\varepsilon}\|_{\mathrm{L}^p(D;\mathrm{L}^{q+\delta}(\Omega))}.$$

3.2. **Tools from Malliavin calculus.** We recall some classical notation and tools from Malliavin calculus; we refer e.g. to [29] for details. We set

$$\mathcal{G}(\zeta):=\int_{\mathbb{R}^d}G\cdot\zeta,\qquad\text{for all }\zeta\in C^\infty_c(\mathbb{R}^d)^\kappa,$$

which are jointly Gaussian random variables with covariance

$$\mathrm{Cov}\left[\mathcal{G}(\zeta);\mathcal{G}(\zeta')\right]=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathcal{C}_{ij}(x-y)\,\zeta_i(x)\zeta'_j(y)\,dxdy,\qquad \zeta,\zeta'\in C^\infty_c(\mathbb{R}^d)^\kappa.$$

Defining $\mathfrak{H}$ as the closure of $C^\infty_c(\mathbb{R}^d)^\kappa$ for this (semi)norm,

$$\|\zeta\|_{\mathfrak{H}}^2:=\langle\zeta,\zeta\rangle_{\mathfrak{H}},\qquad \langle\zeta,\zeta'\rangle_{\mathfrak{H}}:=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathcal{C}_{ij}(x-y)\,\zeta_i(x)\zeta'_j(y)\,dxdy,$$

we may extend by density the definition of $\mathcal{G}(\zeta)\in\mathrm{L}^2(\Omega)$ to all $\zeta\in\mathfrak{H}$. The space $\mathfrak{H}$ (up to taking the quotient with respect to the kernel of $\|\cdot\|_{\mathfrak{H}}$) is a separable Hilbert space and embeds isometrically into $\mathrm{L}^2(\Omega)$ via $\zeta\mapsto\mathcal{G}(\zeta)$. In view of the integrability condition (2.2), the norm of $\mathfrak{H}$ is bounded by

$$\|\zeta\|_{\mathfrak{H}}\lesssim\|[\zeta]_1\|_{\mathrm{L}^2(\mathbb{R}^d)}.\tag{3.3}$$

Without loss of generality we can assume that the probability space is endowed with the σ-algebra generated by the Gaussian field G, so that the linear subspace

$$\mathcal{S}(\Omega) := \Big\{ g(\mathcal{G}(\zeta_1), \ldots, \mathcal{G}(\zeta_n)) \,:\, n \in \mathbb{N},\, g \in C_c^\infty(\mathbb{R}^n),\, \zeta_1, \ldots, \zeta_n \in \mathfrak{H} \Big\}$$

is dense in $\mathrm{L}^2(\Omega)$. We may thus define operators on this simpler subspace $\mathcal{S}(\Omega)$ before extending them by density to $\mathrm{L}^2(\Omega)$. For a random variable $X \in \mathcal{S}(\Omega)$, say $X = g(\mathcal{G}(\zeta_1), \ldots, \mathcal{G}(\zeta_n))$, we define its Malliavin derivative $DX \in \mathrm{L}^2(\Omega; \mathfrak{H})$ as

$$DX := \sum_{i=1}^n \zeta_i \, (\partial_i g)(\mathcal{G}(\zeta_1), \ldots, \mathcal{G}(\zeta_n)).$$

We can check that this operator $D : \mathcal{S}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega; \mathfrak{H})$ is closable, and we still denote by D its closure. Next, we define the divergence operator D^* as the adjoint of D, and we construct the so-called Ornstein–Uhlenbeck operator

$$\mathcal{L} := D^* D, \tag{3.4}$$

which is well-defined as an essentially self-adjoint nonnegative operator on $\mathcal{S}(\Omega) \subset \mathrm{L}^2(\Omega)$. With this notation, we may now state the following useful classical result; a short proof and relevant references can be found e.g. in [16, Proposition 4.1].

Proposition 3.4 (Malliavin–Poincaré inequality)**.** *For all $X \in \mathcal{S}(\Omega)$ and $q < \infty$,*

$$\|X - \mathbb{E}[X]\|_{\mathrm{L}^{2q}(\Omega)} \lesssim q^{\frac12} \|DX\|_{\mathrm{L}^{2q}(\Omega;\mathfrak{H})}.$$

4. Proof of main results

This section is devoted to the proof of our main results. After a few preliminary estimates, Theorems 2.2 and 2.3 are established in Sections 4.2 and 4.3, respectively. Note that the $C^{1,1}$ regularity of the domain U is only used to have $W^{2,\infty-}$ estimates on homogenized eigenfunctions, cf. Lemma 4.2 below, while all other arguments only require $C^{1,\gamma}$ regularity for some $\gamma > 0$.

4.1. **Preliminary estimates.** The following lemma provides uniform bounds on eigenvalues and eigenfunctions. Uniform bounds on gradients of eigenfunctions, cf. (4.3), are based on large-scale regularity theory.

Lemma 4.1. *For all $k \geq 1$, we have almost surely,*

$$\lambda_\varepsilon^k \simeq |k|^2, \tag{4.1}$$

$$|g_\varepsilon^k| \lesssim_k 1, \tag{4.2}$$

and for all $1 < p, q < \infty$,

$$\|[\nabla g_\varepsilon^k]_{2;\varepsilon}\|_{\mathrm{L}^p(U;\mathrm{L}^q(\Omega))} \lesssim_{k,p,q} 1. \tag{4.3}$$

Proof. We split the proof into two steps.

Step 1. Proof of deterministic estimates (4.1) and (4.2).
The first estimate (4.1) follows from a spectral comparison argument based on the uniform ellipticity condition (1.1). We turn to the proof of (4.2) and we appeal to a similar reproducing kernel trick as in [6]: the eigenvalue relation (1.2) yields for all $k \geq 1$ and $t \geq 0$,

$$g_\varepsilon^k = e^{\lambda_\varepsilon^k t} P_\varepsilon^t g_\varepsilon^k,$$

in terms of the Dirichlet semigroup $P_\varepsilon^t := e^{t\nabla\cdot\boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla}$ on U. Noting that the latter is bounded by the corresponding whole-space semigroup, and appealing to the Nash–Aronson estimates [4] (see also [25, Appendix A]), we deduce almost surely

$$|g_\varepsilon^k| \lesssim e^{\lambda_\varepsilon^k t} \int_U t^{-\frac{d}{2}} \exp(-\tfrac{\nu}{8t}|\cdot - y|^2)\, |g_\varepsilon^k(y)|\, dy.$$

Choosing $t = 1$, using (4.1), and recalling that g_ε^k is normalized, we conclude

$$|g_\varepsilon^k| \lesssim_k 1.$$

Step 2. Proof of (4.3).
Considering the unique solution $h_\varepsilon^k \in H_0^1(U)$ of the Laplace equation

$$\triangle h_\varepsilon^k = \lambda_\varepsilon^k g_\varepsilon^k \qquad \text{in } U, \tag{4.4}$$

we can rewrite the eigenvalue relation (1.2) as

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k = \nabla\cdot(\nabla h_\varepsilon^k) \qquad \text{in } U.$$

Appealing to annealed L^p-regularity in form of Theorem 3.3, we deduce for all $1 < p, q < \infty$ and $\delta > 0$,

$$\|[\nabla g_\varepsilon^k]_{2;\varepsilon}\|_{\mathrm{L}^p(U;\mathrm{L}^q(\Omega))} \lesssim_{p,q,\delta} \|[\nabla h_\varepsilon^k]_{2;\varepsilon}\|_{\mathrm{L}^p(U;\mathrm{L}^{q+\delta}(\Omega))} \lesssim \|\nabla h_\varepsilon^k\|_{\mathrm{L}^\infty(\Omega;\mathrm{L}^\infty(U))}.$$

Schauder regularity theory applied to equation (4.4) yields almost surely

$$\|\nabla h_\varepsilon^k\|_{\mathrm{L}^\infty(U)} \lesssim \lambda_\varepsilon^k \|g_\varepsilon^k\|_{\mathrm{L}^\infty(U)},$$

and the conclusion (4.3) then follows from (4.1)–(4.2). □

The following lemma concerns the regularity of eigenfunctions of the homogenized operator. The proof follows from global L^p-regularity theory in $C^{1,1}$ domains, e.g. [19, Theorem 9.13], applied to the eigenvalue relation (1.3).

Lemma 4.2. *For all $k \ge 1$, we have for all $1 < p < \infty$,*

$$\|\bar g^k\|_{W^{2,p}(U)} \lesssim_{k,p} 1.$$

4.2. **Convergence of eigenvalues and eigenfunctions.** This section is devoted to the proof of Theorem 2.2. We start with the following estimate on the fluctuation scaling of eigenvalues. This is often used in the sequel as a concentration result.

Lemma 4.3 (Fluctuation scaling)**.** *For all $k \ge 1$ and $q < \infty$,*

$$\|\lambda_\varepsilon^k - \mathbb{E}[\lambda_\varepsilon^k]\|_{\mathrm{L}^q(\Omega)} \lesssim_{k,q} \varepsilon^{\frac{d}{2}}.$$

Proof. In terms of Malliavin calculus, cf. Proposition 3.4, centered moments can be estimated by

$$\|\lambda_\varepsilon^k - \mathbb{E}[\lambda_\varepsilon^k]\|_{\mathrm{L}^q(\Omega)} \lesssim_q \|D\lambda_\varepsilon^k\|_{\mathrm{L}^q(\Omega;\mathfrak{H})}. \tag{4.5}$$

Starting from identity

$$\lambda_\varepsilon^k = \int_U \nabla g_\varepsilon^k \cdot \boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k,$$

the Malliavin derivative can be written as

$$D\lambda_\varepsilon^k = \int_U \nabla g_\varepsilon^k \cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k + 2\int_U \nabla D g_\varepsilon^k \cdot \boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k.$$

Since the eigenvalue relation and the normalization of g_ε^k ensure that the second right-hand side term is

$$\int_U \nabla Dg_\varepsilon^k \cdot \boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k \,=\, \lambda_\varepsilon^k \int_U g_\varepsilon^k Dg_\varepsilon^k \,=\, \tfrac12 \lambda_\varepsilon^k D\|g_\varepsilon^k\|_{\mathrm{L}^2(U)} \,=\, 0,$$

we deduce

$$D\lambda_\varepsilon^k \,=\, \int_U \nabla g_\varepsilon^k \cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k. \tag{4.6}$$

The definition (2.1) of $\boldsymbol{a}$ yields for any test function $\psi \in C_b(U)$,

$$\int_U \psi\, D_z\boldsymbol{a}(\tfrac{\cdot}{\varepsilon}) \,=\, \varepsilon^d \psi(\varepsilon z)\,\partial a_0(G(z)), \tag{4.7}$$

so that the above becomes

$$D\lambda_\varepsilon^k \,=\, \varepsilon^d \nabla g_\varepsilon^k(\varepsilon\cdot)\cdot \partial a_0(G(\cdot))\nabla g_\varepsilon^k(\varepsilon\cdot).$$

Inserting this into (4.5), and using the integrability condition (2.2) in form of (3.3), we obtain after rescaling,

$$\|\lambda_\varepsilon^k - \mathbb{E}[\lambda_\varepsilon^k]\|_{\mathrm{L}^q(\Omega)} \,\lesssim_q\, \varepsilon^{\frac d2}\, \|[\nabla g_\varepsilon^k]_{2;\varepsilon}\|^2_{\mathrm{L}^{2q}(\Omega;\mathrm{L}^4(U))}, \tag{4.8}$$

and the conclusion follows from (4.3). □

Next, we establish the following convergence result for eigenvalues and eigenspaces. The proof is based on using two-scale expansion to show that homogenized eigenvalues $\{\bar\lambda^k\}_k$ are approximate eigenvalues for the heterogeneous problem (1.2). Recall that the weight μ_d is defined in (2.6).

Lemma 4.4 (Convergence of eigenvalues and eigenspaces)**.**

(i) For all $k \ge 1$ and $q < \infty$, we have

$$\|\lambda_\varepsilon^k - \bar\lambda^k\|_{\mathrm{L}^q(\Omega)} \lesssim_{k,q} (\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}.$$

(ii) For all $k \ge 1$ and $q < \infty$, we have

$$\|g_\varepsilon^k - \bar\pi^k[g_\varepsilon^k]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))} \,\lesssim_{k,q}\, (\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12},$$

where we recall that $\bar\pi^k[\cdot]$ stands for the orthogonal projection of $\mathrm{L}^2(U)$ onto the eigenspace associated with $\bar\lambda^k$.

Proof. We split the proof into three steps.

Step 1. Construction of approximate eigenvalues.
Given a parameter $\rho \in [\varepsilon, 1]$ to be later optimized (depending on ε), set

$$U_\rho := \{x \in U : \mathrm{dist}(x, \partial U) > \rho\}, \qquad \partial_\rho U := U \setminus U_\rho,$$

choose a cut-off function $\eta_\rho \in C_c^\infty(U)$ such that

$$\eta_\rho|_{U_\rho} = 1, \qquad 0 \le \eta_\rho \le 1, \qquad |\nabla\eta_\rho| \lesssim \tfrac1\rho, \tag{4.9}$$

and consider the following truncated two-scale expansion,

$$h_{\varepsilon,\rho}^k \,:=\, \bar g^k + \varepsilon\eta_\rho\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha \bar g^k \,\in\, H_0^1(U). \tag{4.10}$$

The eigenvalue relation for $\bar g^k$ yields

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla h_{\varepsilon,\rho}^k \,=\, \bar\lambda^k \bar g^k - \nabla\cdot\Big((\boldsymbol{a}-\bar{\boldsymbol{a}})(\tfrac{\cdot}{\varepsilon})\nabla\bar g^k + \boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla\big(\varepsilon\eta_\rho\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\big)\Big). \tag{4.11}$$

Let us reformulate the right-hand side. Expanding the gradient and inserting the definition of the flux corrector σ, cf. (3.2), we can rewrite

$$\begin{aligned}&(\boldsymbol{a}-\bar{\boldsymbol{a}})(\tfrac{\cdot}{\varepsilon})\nabla\bar g^k+\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla\big(\varepsilon\eta_\rho\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\big)\\ &\quad=\ (1-\eta_\rho)(\boldsymbol{a}-\bar{\boldsymbol{a}})(\tfrac{\cdot}{\varepsilon})\nabla\bar g^k+\varepsilon(\boldsymbol{a}\varphi_\alpha)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\rho\partial_\alpha\bar g^k)+\big(\boldsymbol{a}(\nabla\varphi_\alpha+e_\alpha)-\bar{\boldsymbol{a}}e_\alpha\big)(\tfrac{\cdot}{\varepsilon})\eta_\rho\partial_\alpha\bar g^k\\ &\quad=\ (1-\eta_\rho)(\boldsymbol{a}-\bar{\boldsymbol{a}})(\tfrac{\cdot}{\varepsilon})\nabla\bar g^k+\varepsilon(\boldsymbol{a}\varphi_\alpha)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\rho\partial_\alpha\bar g^k)+(\nabla\cdot\sigma_\alpha)(\tfrac{\cdot}{\varepsilon})\eta_\rho\partial_\alpha\bar g^k.\end{aligned}$$

and we note that Leibniz' rule and the skew-symmetry of σ yield for the last right-hand side term,

$$\nabla\cdot\Big((\nabla\cdot\sigma_\alpha)(\tfrac{\cdot}{\varepsilon})\eta_\rho\partial_\alpha\bar g^k\Big)\ =\ (\nabla\cdot\sigma_\alpha)(\tfrac{\cdot}{\varepsilon})\cdot\nabla(\eta_\rho\partial_\alpha\bar g^k)\ =\ -\nabla\cdot\Big(\sigma_\alpha(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\rho\partial_\alpha\bar g^k)\Big).$$

Inserting these identities into (4.11), we are led to the following approximate eigenvalue relation,

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla h^k_{\varepsilon,\rho}\ =\ \bar\lambda^k h^k_{\varepsilon,\rho}-\bar\lambda^k s^k_{\varepsilon,\rho}-\nabla\cdot r^k_{\varepsilon,\rho},\tag{4.12}$$

where the remainder terms $r^k_{\varepsilon,\rho},s^k_{\varepsilon,\rho}$ are defined by

$$\begin{aligned} r^k_{\varepsilon,\rho}\ &:=\ (1-\eta_\rho)(\boldsymbol{a}-\bar{\boldsymbol{a}})(\tfrac{\cdot}{\varepsilon})\nabla\bar g^k+\varepsilon(\boldsymbol{a}\varphi_\alpha-\sigma_\alpha)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\rho\partial_\alpha\bar g^k),\\ s^k_{\varepsilon,\rho}\ &:=\ h^k_{\varepsilon,\rho}-\bar g^k\ =\ \varepsilon\eta_\rho\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k.\end{aligned}\tag{4.13}$$

We now estimate these remainders. As $1-\eta_\rho$ and $\nabla\eta_\rho$ are supported in $\partial_\rho U$, cf. (4.9), we can estimate by Hölder's inequality, for all $p>2$,

$$\begin{aligned}\|r^k_{\varepsilon,\rho}\|_{\mathrm{L}^2(U)}\ \lesssim\ |\partial_\rho U|^{\frac12}\|\nabla\bar g^k\|_{\mathrm{L}^\infty(U)}+\rho^{-1}\varepsilon\|(\varphi,\sigma)(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2(\partial_\rho U)}\|\nabla\bar g^k\|_{\mathrm{L}^\infty(U)}\\ +\varepsilon\|(\varphi,\sigma)(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^p(U)}\|\nabla^2\bar g\|_{\mathrm{L}^{\frac{2p}{p-2}}(U)},\end{aligned}$$

and thus, taking the $\mathrm{L}^q(\Omega)$ norm of both sides, appealing to the corrector estimates of Theorem 3.2 for $p>2$ close enough to 2, using that $\partial_\rho U$ has volume $|\partial_\rho U|\simeq\rho$, and appealing to Lemma 4.2 for regularity of $\bar g^k$, we deduce for all $q<\infty$,

$$\|r^k_{\varepsilon,\rho}\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\ \lesssim_{k,q}\ \rho^{\frac12}+\rho^{-\frac12}\varepsilon\mu_d(\tfrac1\varepsilon),\tag{4.14}$$

and similarly,

$$\|s^k_{\varepsilon,\rho}\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\ \lesssim_{k,q}\ \varepsilon\mu_d(\tfrac1\varepsilon).\tag{4.15}$$

Step 2. Convergence of eigenspaces: given $\delta\in(0,1]$, denoting by $\pi^k_{\varepsilon,\delta}[\cdot]$ the orthogonal projection of $\mathrm{L}^2(U)$ onto $\mathrm{span}\{g^j_\varepsilon:|\lambda^j_\varepsilon-\bar\lambda^k|\le\delta\}$, we show for all $q<\infty$,

$$\|\bar g^k-\pi^k_{\varepsilon,\delta}[\bar g^k]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\ \lesssim_{k,q}\ \delta^{-1}(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}.\tag{4.16}$$

We claim that it suffices to establish the following result: given $\rho\in[\varepsilon,1]$, in terms of the truncated two-scale expansion $h^k_{\varepsilon,\rho}$, cf. (4.10), we have for all $q<\infty$,

$$\|h^k_{\varepsilon,\rho}-\pi^k_{\varepsilon,\delta}[h^k_{\varepsilon,\rho}]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\ \lesssim_{k,q}\ \delta^{-1}\big(\rho^{\frac12}+\rho^{-\frac12}\varepsilon\mu_d(\tfrac1\varepsilon)\big).\tag{4.17}$$

Indeed, as $\bar g^k=h^k_{\varepsilon,\rho}-s^k_{\varepsilon,\rho}$, with $s^k_{\varepsilon,\rho}$ estimated in (4.15), the latter entails

$$\|\bar g^k-\pi^k_{\varepsilon,\delta}[\bar g^k]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\ \lesssim_{k,q}\ \delta^{-1}\big(\rho^{\frac12}+\rho^{-\frac12}\varepsilon\mu_d(\tfrac1\varepsilon)\big),$$

and the claim (4.16) follows after optimizing with respect to $\rho\in[\varepsilon,1]$, which amounts to choosing $\rho=\varepsilon\mu_d(\frac1\varepsilon)$.

We turn to the proof of (4.17). For that purpose, starting point is the following spectral decomposition,

$$h^k_{\varepsilon,\rho}-\pi^k_{\varepsilon,\delta}[h^k_{\varepsilon,\rho}]\,=\,\sum_{j:|\lambda^j_\varepsilon-\bar\lambda^k|>\delta}\Big(\int_U h^k_{\varepsilon,\rho}g^j_\varepsilon\Big)g^j_\varepsilon. \tag{4.18}$$

In order to estimate the norm of the right-hand side, we note that projecting the approximate eigenvalue relation (4.12) onto g^j_ε yields

$$(\lambda^j_\varepsilon-\bar\lambda^k)\int_U h^k_{\varepsilon,\rho}g^j_\varepsilon\,=\,-\bar\lambda^k\int_U s^k_{\varepsilon,\rho}g^j_\varepsilon+\int_U r^k_{\varepsilon,\rho}\cdot\nabla g^j_\varepsilon,\qquad\text{for all } j. \tag{4.19}$$

Inserting this into (4.18) and using that $\{g^j_\varepsilon\}_j$ is an orthonormal basis of $\mathrm{L}^2(U)$, we infer

$$\|h^k_{\varepsilon,\rho}-\pi^k_{\varepsilon,\delta}[h^k_{\varepsilon,\rho}]\|^2_{\mathrm{L}^2(U)}\,\le\,\sum_{j:|\lambda^j_\varepsilon-\bar\lambda^k|>\delta}\frac{2}{(\lambda^j_\varepsilon-\bar\lambda^k)^2}\bigg(\Big|\bar\lambda^k\int_U s^k_{\varepsilon,\rho}g^j_\varepsilon\Big|^2+\Big|\int_U r^k_{\varepsilon,\rho}\cdot\nabla g^j_\varepsilon\Big|^2\bigg).$$

In terms of the solution $v^k_{\varepsilon,\rho}\in H^1_0(U)$ of the auxiliary problem

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla v^k_{\varepsilon,\rho}\,=\,\nabla\cdot r^k_{\varepsilon,\rho}\qquad\text{in } U, \tag{4.20}$$

we can write

$$\int_U r^k_{\varepsilon,\rho}\cdot\nabla g^j_\varepsilon\,=\,-\int_U\nabla v^k_{\varepsilon,\rho}\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g^j_\varepsilon\,=\,-\lambda^j_\varepsilon\int_U v^k_{\varepsilon,\rho}g^j_\varepsilon,$$

so that the above becomes

$$\|h^k_{\varepsilon,\rho}-\pi^k_{\varepsilon,\delta}[h^k_{\varepsilon,\rho}]\|^2_{\mathrm{L}^2(U)}\,\le\,\sum_{j:|\lambda^j_\varepsilon-\bar\lambda^k|>\delta}\frac{2((\lambda^j_\varepsilon)^2+(\bar\lambda^k)^2)}{(\lambda^j_\varepsilon-\bar\lambda^k)^2}\bigg(\Big|\int_U s^k_{\varepsilon,\rho}g^j_\varepsilon\Big|^2+\Big|\int_U v^k_{\varepsilon,\rho}g^j_\varepsilon\Big|^2\bigg).$$

Noting that for $|\lambda^j_\varepsilon-\bar\lambda^k|>\delta$ we have

$$\frac{2((\lambda^j_\varepsilon)^2+(\bar\lambda^k)^2)}{(\lambda^j_\varepsilon-\bar\lambda^k)^2}\,\le\,4+\frac{6(\bar\lambda^k)^2}{(\lambda^j_\varepsilon-\bar\lambda^k)^2}\,\lesssim_k\,\delta^{-2},$$

and using again that $\{g^j_\varepsilon\}_j$ is an orthonormal basis of $\mathrm{L}^2(U)$, we are led to

$$\|h^k_{\varepsilon,\rho}-\pi^k_{\varepsilon,\delta}[h^k_{\varepsilon,\rho}]\|_{\mathrm{L}^2(U)}\,\lesssim_k\,\delta^{-1}\big(\|s^k_{\varepsilon,\rho}\|_{\mathrm{L}^2(U)}+\|v^k_{\varepsilon,\rho}\|_{\mathrm{L}^2(U)}\big).$$

Poincaré's inequality combined with an energy estimate for (4.20) yields

$$\|v^k_{\varepsilon,\rho}\|_{\mathrm{L}^2(U)}\,\lesssim\,\|\nabla v^k_{\varepsilon,\rho}\|_{\mathrm{L}^2(U)}\,\lesssim\,\|r^k_{\varepsilon,\rho}\|_{\mathrm{L}^2(U)}.$$

Combining this with the above, together with the bounds (4.14)–(4.15) on the remainders $r^k_{\varepsilon,\rho}$, $s^k_{\varepsilon,\rho}$, the claim (4.17) follows.

Step 3. Conclusion.
Assume that $\bar\lambda^k$ has multiplicity $s\ge1$, with

$$\bar\lambda^{k-1}<\bar\lambda^k=\ldots=\bar\lambda^{k+s-1}<\bar\lambda^{k+s}.$$

Given $K>0$, choosing $\delta=K(\varepsilon\mu_d(\frac1\varepsilon))^{\frac12}$, the event $\sharp\{j:|\lambda^j_\varepsilon-\bar\lambda^k|\le\delta\}<s$ entails that the projection $\pi^k_{\varepsilon,\delta}[\cdot]$ defined in Step 2 is a projection onto a subspace of dimension $<s$. As a consequence, under this event, there exists a unit vector in $\operatorname{span}\{\bar g^k,\ldots,\bar g^{k+s-1}\}$ for which the projection vanishes, and therefore there must exist j with $k\le j\le k+s-1$ such

that $\|\bar g^j-\pi^k_{\varepsilon,\delta}[\bar g^j]\|_{\mathrm{L}^2(U)}\ge s^{-\frac12}$. Combining this observation with a union bound, Markov's inequality, and (4.16), we deduce for all $q<\infty$,

$$\begin{aligned}
&\mathbb{P}\Big[\sharp\big\{j:|\lambda^j_\varepsilon-\bar\lambda^k|\le K(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\big\}<s\Big]\\
&\qquad\le\quad \mathbb{P}\Big[\max_{k\le j\le k+s-1}\|\bar g^j-\pi^k_{\varepsilon,\delta}[\bar g^j]\|_{\mathrm{L}^2(U)}\ge s^{-\frac12}\Big]\\
&\qquad\le\quad s^{\frac q2}\sum_{j=k}^{k+s-1}\|\bar g^j-\pi^k_{\varepsilon,\delta}[\bar g^j]\|^q_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\\
&\qquad\lesssim_{k,s,q}\quad\big(\delta^{-1}(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\big)^q\;=\;K^{-q}.
\end{aligned}\tag{4.21}$$

Also recall that Lemma 4.3 ensures that the law of each eigenvalue is strongly concentrated: more precisely, using Markov's inequality and the fact that $\varepsilon^{d/2}\le\varepsilon\mu_d(\frac1\varepsilon)$, it gives for all $j\ge1$ and $q<\infty$,

$$\mathbb{P}\Big[|\lambda^j_\varepsilon-\mathbb{E}[\lambda^j_\varepsilon]|>K(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\Big]\;\lesssim_{j,q}\;K^{-q}.\tag{4.22}$$

In view of (4.1), there exists a deterministic constant $c_k>0$ and a deterministic index set I_k such that almost surely, for all $\varepsilon>0$,

$$\inf_j|\lambda^j_\varepsilon-\bar\lambda^k|\le c_k,\qquad\{j:|\lambda^j_\varepsilon-\bar\lambda^k|\le c_k\}\subset I_k,\qquad\sharp I_k\lesssim_k 1.\tag{4.23}$$

By a union bound, provided $2K(\varepsilon\mu_d(\frac1\varepsilon))^{\frac12}\le c_k$, we may then write

$$\begin{aligned}
&\mathbb{P}\Big[\sharp\big\{j:|\mathbb{E}[\lambda^j_\varepsilon]-\bar\lambda^k|\le 2K(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\big\}<s\Big]\\
&\qquad\le\mathbb{P}\Big[\sharp\big\{j:|\lambda^j_\varepsilon-\bar\lambda^k|\le K(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\big\}<s\Big]+\sum_{j\in I_k}\mathbb{P}\Big[|\lambda^j_\varepsilon-\mathbb{E}[\lambda^j_\varepsilon]|>K(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\Big],
\end{aligned}$$

and thus, inserting (4.21) and (4.22), we deduce for all $K>0$ and $q<\infty$,

$$\mathbb{P}\Big[\sharp\big\{j:|\mathbb{E}[\lambda^j_\varepsilon]-\bar\lambda^k|\le 2K(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\big\}<s\Big]\;\lesssim_{k,s,q}\;K^{-q}.$$

As the left-hand side is the probability of a deterministic event, while the right-hand side can be made <1 by choosing $K\simeq_{k,s,q}1$ large enough, we infer

$$\sharp\big\{j:|\mathbb{E}[\lambda^j_\varepsilon]-\bar\lambda^k|\lesssim_{k,s,q}(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}\big\}\ge s.$$

Since it is already known that the set $\{\lambda^j_\varepsilon\}_j$ converges almost surely to $\{\bar\lambda^j\}_j$ with multiplicities, see e.g. [25, Section 11], we can deduce

$$\max_{k\le j\le k+s-1}|\mathbb{E}[\lambda^j_\varepsilon]-\bar\lambda^k|\;\lesssim_{k,s,q}\;(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}.$$

Appealing again to the concentration result of Lemma 4.3, this yields part (i) of the statement.

It remains to establish part (ii). First note that a straightforward linear algebraic argument yields

$$\max_{k\le j\le k+s-1}\|g^j_\varepsilon-\bar\pi^k[g^j_\varepsilon]\|_{\mathrm{L}^2(U)}\;\lesssim_s\;\max_{k\le j\le k+s-1}\|\bar g^j-\pi^k_\varepsilon[\bar g^j]\|_{\mathrm{L}^2(U)},\tag{4.24}$$

where $\pi_\varepsilon^k[\cdot]$ stands for the orthogonal projection of $\mathrm{L}^2(U)$ onto $\mathrm{span}\{g_\varepsilon^k,\ldots,g_\varepsilon^{k+s-1}\}$. Now choosing

$$\delta_k \,:=\, \big(\tfrac12|\bar\lambda^k-\bar\lambda^{k-1}|\big)\wedge\big(\tfrac12|\bar\lambda^k-\bar\lambda^{k+s}|\big)\wedge c_k \quad>0,$$

where we recall that c_k is chosen in (4.23), we note that this projection π_ε^k coincides with $\pi_{\varepsilon,\delta_k}^k$ unless there is some $k\le j\le k+s-1$ with $|\lambda_\varepsilon^j-\bar\lambda^k|>\delta_k$, or unless there is some $j<k$ or some $j>k+s-1$ with $|\lambda_\varepsilon^j-\bar\lambda^k|\le\delta_k$. In view of the choice of δ_k, this actually means that the projection π_ε^k coincides with $\pi_{\varepsilon,\delta_k}^k$ unless there is some $j\in I_k$ with $|\lambda_\varepsilon^j-\bar\lambda^j|>\delta_k$. By conditioning and Markov's inequality, we may then estimate

$$\begin{aligned}\|\bar g^j-\pi_\varepsilon^k[\bar g^j]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))} &\le \|\bar g^j-\pi_{\varepsilon,\delta_k}^k[\bar g^j]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}+2\Big(\sum_{j\in I_k}\mathbb{P}\big[|\lambda_\varepsilon^j-\bar\lambda^j|>\delta_k\big]\Big)^{\frac1q}\\ &\lesssim_k \|\bar g^j-\pi_{\varepsilon,\delta_k}^k[\bar g^j]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}+\sum_{j\in I_k}\|\lambda_\varepsilon^j-\bar\lambda^j\|_{\mathrm{L}^q(\Omega)}.\end{aligned}$$

Combining this with (4.16) and with part (i) of the statement, and inserting the result into (4.24), the conclusion (ii) follows. ☐

It remains to establish the corrector result stating the accuracy of the two-scale expansion of eigenfunctions.

Lemma 4.5 (Corrector result)**.** *For all $k\ge1$ and $q<\infty$, we have*

$$\|\nabla g_\varepsilon^k-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar\pi^k[g_\varepsilon^k]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\;\lesssim_{k,q}\;(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}. \tag{4.25}$$

In addition, the following Meyers type improvement holds: there exists a constant $C_0\simeq1$ such that for all $2\le p\le 2+\frac{1}{C_0}$ the $\mathrm{L}^2(U)$ norm can be replaced by an $\mathrm{L}^p(U)$ norm in (4.25), at the price of replacing the rate $(\varepsilon\mu_d(\frac1\varepsilon))^{1/2}$ by $(\varepsilon\mu_d(\frac1\varepsilon))^{1/p}$.

Proof. Assume that $\bar\lambda^k$ has multiplicity $s\ge1$, with

$$\bar\lambda^{k-1}<\bar\lambda^k=\ldots=\bar\lambda^{k+s-1}<\bar\lambda^{k+s}.$$

Given a parameter $\rho\in[\varepsilon,1]$, we choose a cut-off function η_ρ as in (4.9). For all j with $k\le j\le k+s-1$, we consider the following truncated two-scale expansion for g_ε^j, taking into account the multiplicity of the homogenized eigenspace,

$$\tilde h_{\varepsilon,\rho}^j\;:=\;\bar\pi^k[g_\varepsilon^j]+\varepsilon\eta_\rho\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar\pi^k[g_\varepsilon^j]\quad\in\;H_0^1(U).$$

Comparing with (4.10), this means $\tilde h_{\varepsilon,\rho}^j=\sum_{l=k}^{k+s-1}(\int_U g_\varepsilon^j\bar g^l)h_{\varepsilon,\rho}^l$. Starting point is the approximate eigenvalue relation (4.12), which we reorganize as

$$-\nabla\cdot\boldsymbol a(\tfrac{\cdot}{\varepsilon})\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\;=\;(\lambda_\varepsilon^j-\bar\lambda^k)\bar\pi^k[g_\varepsilon^j]+\lambda_\varepsilon^j(g_\varepsilon^j-\bar\pi^k[g_\varepsilon^j])+\nabla\cdot\tilde r_{\varepsilon,\rho}^j, \tag{4.26}$$

where the remainder $\tilde r_{\varepsilon,\rho}^j$ is given by

$$\tilde r_{\varepsilon,\rho}^j\;:=\;(1-\eta_\rho)(\boldsymbol a-\bar{\boldsymbol a})(\tfrac{\cdot}{\varepsilon})\nabla\bar\pi^k[g_\varepsilon^j]+\varepsilon(\boldsymbol a\varphi_\alpha-\sigma_\alpha)(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\rho\partial_\alpha\bar\pi^k[g_\varepsilon^j]).$$

We split the proof into two steps.

Step 1. Proof of (4.25).
Testing equation (4.26) with $g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j\in H_0^1(U)$ itself, using Poincaré's inequality and (4.1), we find

$$\int_U|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)|^2\;\lesssim_j\;|\lambda_\varepsilon^j-\bar\lambda^k|^2+\int_U|g_\varepsilon^j-\bar\pi^k[g_\varepsilon^j]|^2+\int_U|\tilde r_{\varepsilon,\rho}^j|^2.$$

Taking the $\mathrm{L}^q(\Omega)$ norm of both sides, appealing to Lemma 4.4, and estimating the last two right-hand side term as in (4.14), we deduce

$$\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\,\lesssim_j\,(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}+\rho^{\frac12}+\rho^{-\frac12}\varepsilon\mu_d(\tfrac1\varepsilon).$$

Now decomposing

$$\nabla\tilde h_{\varepsilon,\rho}^j\,=\,(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar\pi^k[g_\varepsilon^j]-(1-\eta_\rho)\nabla\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar\pi^k[g_\varepsilon^j]+\varepsilon\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\nabla(\eta_\rho\partial_\alpha\bar\pi^k[g_\varepsilon^j]),\quad(4.27)$$

similar estimates yield

$$\|\nabla g_\varepsilon^j-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar\pi^k[g_\varepsilon^j]\|_{\mathrm{L}^q(\Omega;\mathrm{L}^2(U))}\,\lesssim_{k,q}\,(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}+\rho^{\frac12}+\rho^{-\frac12}\varepsilon\mu_d(\tfrac1\varepsilon).$$

The conclusion (4.25) follows after optimizing with respect to $\rho\in[\varepsilon,1]$, which amounts to choosing $\rho=\varepsilon\mu_d(\frac1\varepsilon)$.

Step 2. Meyers improvement.
Rewriting the approximate eigenvalue relation (4.26) as

$$-\triangle(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\,=\,\nabla\cdot\Big((\tfrac{2}{1+\nu}\boldsymbol a(\tfrac{\cdot}{\varepsilon})-\mathrm{Id})\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\Big)+\tfrac{2}{1+\nu}\Big((\lambda_\varepsilon^j-\bar\lambda^k)\bar\pi^k[g_\varepsilon^j]+\lambda_\varepsilon^j(g_\varepsilon^j-\bar\pi^k[g_\varepsilon^j])+\nabla\cdot\tilde r_{\varepsilon,\rho}^j\Big),$$

the standard L^p regularity theory for the Laplace equation in U yields for all $1<p<\infty$,

$$\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^p(U)}\,\le\,K(p)\|(\tfrac{2}{1+\nu}\boldsymbol a(\tfrac{\cdot}{\varepsilon})-\mathrm{Id})\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^p(U)}+2K(p)\Big(|\lambda_\varepsilon^j-\bar\lambda^k|\|\bar\pi^k[g_\varepsilon^j]\|_{W^{-1,p}(U)}+\lambda_\varepsilon^j\|g_\varepsilon^j-\bar\pi^k[g_\varepsilon^j]\|_{W^{-1,p}(U)}+\|\tilde r_{\varepsilon,\rho}^j\|_{\mathrm{L}^p(U)}\Big),$$

where by interpolation the multiplicative constants satisfy

$$\lim_{p\to2}K(p)=K(2)=1.\qquad(4.28)$$

The uniform ellipticity condition (1.1) yields

$$|\tfrac{2}{1+\nu}\boldsymbol a(\tfrac{\cdot}{\varepsilon})-\mathrm{Id}\,|\le\tfrac{1-\nu}{1+\nu},$$

and thus, also appealing to the Sobolev inequality and to (4.1), we get for all $2\le p\le\frac{2d}{d-2}$,

$$\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^p(U)}\,\le\,K(p)\tfrac{1-\nu}{1+\nu}\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^p(U)}+C_{j,p}\Big(|\lambda_\varepsilon^j-\bar\lambda^k|+\|g_\varepsilon^j-\bar\pi^k[g_\varepsilon^j]\|_{\mathrm{L}^2(U)}+\|\tilde r_{\varepsilon,\rho}^j\|_{\mathrm{L}^p(U)}\Big).$$

Taking the $\mathrm{L}^q(\Omega)$ norm of both sides of this estimate, appealing to Lemma 4.4, and estimating the last right-hand side term as in (4.14), we deduce

$$\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^q(\Omega;\mathrm{L}^p(U))}\,\le\,K(p)\tfrac{1-\nu}{1+\nu}\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^q(\Omega;\mathrm{L}^p(U))}+C_{j,p,q}\Big((\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}+\rho^{\frac1p}+\rho^{\frac1p-1}\varepsilon\mu_d(\tfrac1\varepsilon)\Big).\quad(4.29)$$

Recalling (4.28) and $\frac{1-\nu}{1+\nu}<1$, we can choose $C_0\simeq1$ such that

$$K(p)\tfrac{1-\nu}{1+\nu}\,\le\,(\tfrac{1-\nu}{1+\nu})^{\frac12}\,<1\qquad\text{provided }|p-2|\le\tfrac{1}{C_0}.$$

This allows to absorb the first right-hand side term in (4.29): for all $2\le p\le2+\frac1{C_0}$,

$$\|\nabla(g_\varepsilon^j-\tilde h_{\varepsilon,\rho}^j)\|_{\mathrm{L}^q(\Omega;\mathrm{L}^p(U))}\,\lesssim_{j,p,q}\,(\varepsilon\mu_d(\tfrac1\varepsilon))^{\frac12}+\rho^{\frac1p}+\rho^{\frac1p-1}\varepsilon\mu_d(\tfrac1\varepsilon).$$

Further decomposing $\nabla \tilde{h}^j_{\varepsilon,\rho}$ as in (4.27), and optimizing with respect to ρ, the Meyers improvement of (4.25) follows. □

4.3. **Eigenvalue fluctuations.** This section is devoted to the proof of Theorem 2.3. While the fluctuation scaling is already captured in Lemma 4.3, we now turn to the characterization of leading-order fluctuations and their pathwise description (2.7).

Proof of Theorem 2.3. Let $k \geq 1$ be fixed such that the eigenvalue $\bar\lambda^k$ is simple. The eigenvalue relations for g^k_ε and $\bar g^k$ yield

$$(\lambda^k_\varepsilon - \bar\lambda^k) \int_U g^k_\varepsilon \, \bar g^k \;=\; \int_U \nabla \bar g^k \cdot (\boldsymbol{a}(\tfrac{\cdot}{\varepsilon}) - \bar{\boldsymbol{a}}) \nabla g^k_\varepsilon,$$

or alternatively,

$$\lambda^k_\varepsilon - \bar\lambda^k \;=\; \int_U \nabla \bar g^k \cdot (\boldsymbol{a}(\tfrac{\cdot}{\varepsilon}) - \bar{\boldsymbol{a}}) \nabla g^k_\varepsilon \;+\; (\lambda^k_\varepsilon - \bar\lambda^k)\Big(1 - \int_U g^k_\varepsilon \bar g^k\Big), \tag{4.30}$$

where the first right-hand side term involves the so-called "homogenization commutator" of the eigenfunction g^k_ε, that is, the vector field $(\boldsymbol{a}(\tfrac{\cdot}{\varepsilon}) - \bar{\boldsymbol{a}})\nabla g^k_\varepsilon$, in the terminology of [15]. Taking inspiration from the fluctuation theory for the solution operator in [15], we expect that the homogenization commutator can be replaced by its two-scale expansion, and we are led to postulating the following approximation,

$$\lambda^k_\varepsilon - \mathbb{E}[\lambda^k_\varepsilon] \sim \int_U \Xi^\circ_{\alpha\beta}(\tfrac{\cdot}{\varepsilon}) \partial_\alpha \bar g^k \partial_\beta \bar g^k,$$

where we recall the definition of the standard homogenization commutator, cf. (1.4),

$$\Xi^\circ_{\alpha\beta} \;=\; e_\beta \cdot (\boldsymbol{a} - \bar{\boldsymbol{a}})(\nabla \varphi_\alpha + e_\alpha).$$

It remains to estimate the approximation error. For that purpose, in view of (4.30), we can write for all $q < \infty$,

$$\Big\| \lambda^k_\varepsilon - \mathbb{E}[\lambda^k_\varepsilon] - \int_U \Xi^\circ_{\alpha\beta}(\tfrac{\cdot}{\varepsilon}) \partial_\alpha \bar g^k \partial_\beta \bar g^k \Big\|_{\mathrm{L}^q(\Omega)} \;=\; \|E^k_\varepsilon - \mathbb{E}[E^k_\varepsilon]\|_{\mathrm{L}^q(\Omega)},$$

where we have set for abbreviation

$$E^k_\varepsilon \;:=\; \int_U \nabla \bar g^k \cdot (\boldsymbol{a}(\tfrac{\cdot}{\varepsilon}) - \bar{\boldsymbol{a}}) \Big(\nabla g^k_\varepsilon - (\nabla\varphi_\alpha + e_\alpha)(\tfrac{\cdot}{\varepsilon}) \partial_\alpha \bar g^k\Big) + (\lambda^k_\varepsilon - \bar\lambda^k)\Big(1 - \int_U g^k_\varepsilon \bar g^k\Big). \tag{4.31}$$

Appealing to Malliavin calculus, cf. Proposition 3.4, we deduce for all $q < \infty$,

$$\Big\| \lambda^k_\varepsilon - \mathbb{E}[\lambda^k_\varepsilon] - \int_U \Xi^\circ_{\alpha\beta}(\tfrac{\cdot}{\varepsilon}) \partial_\alpha \bar g^k \partial_\beta \bar g^k \Big\|_{\mathrm{L}^q(\Omega)} \;\lesssim_q\; \|DE^k_\varepsilon\|_{\mathrm{L}^q(\Omega;\mathfrak{H})}. \tag{4.32}$$

To estimate the right-hand side, we first proceed to a suitable computation of the Malliavin derivative DE^k_ε, and we split the proof into two steps.

Step 1. Proof of

$$\begin{aligned} DE^k_\varepsilon \;=\; & \int_U \Big(\nabla g^k_\varepsilon + (\nabla \phi_\beta + e_\beta)(\tfrac{\cdot}{\varepsilon}) \partial_\beta \bar g^k\Big) \cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon}) \Big(\nabla g^k_\varepsilon - (\nabla \varphi_\alpha + e_\alpha)(\tfrac{\cdot}{\varepsilon}) \partial_\alpha \bar g^k\Big) \\ & - \varepsilon \int_U \nabla(\partial_\alpha \bar g^k \partial_\beta \bar g^k) \cdot \Big((\boldsymbol{a}\varphi_\beta + \sigma_\beta)(\tfrac{\cdot}{\varepsilon}) \nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon}) + (\varphi_\beta D\boldsymbol{a})(\tfrac{\cdot}{\varepsilon})(\nabla \varphi_\alpha + e_\alpha)(\tfrac{\cdot}{\varepsilon}) \Big). \end{aligned} \tag{4.33}$$

By definition (4.31) of E_ε^k, its Malliavin derivative can be decomposed as

$$DE_\varepsilon^k = \int_U \nabla\bar g^k\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\Big(\nabla g_\varepsilon^k-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\Big)$$
$$+\int_U \nabla\bar g^k\cdot(\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})-\bar{\boldsymbol{a}})\Big(\nabla Dg_\varepsilon^k-\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\Big)+D\bigg((\lambda_\varepsilon^k-\bar\lambda^k)\Big(1-\int_U g_\varepsilon^k\bar g^k\Big)\bigg). \qquad (4.34)$$

We start by reformulating the last right-hand side term,

$$D\bigg((\lambda_\varepsilon^k-\bar\lambda^k)\Big(1-\int_U g_\varepsilon^k\bar g^k\Big)\bigg) = -(\lambda_\varepsilon^k-\bar\lambda^k)\int_U\bar g^kDg_\varepsilon^k+(D\lambda_\varepsilon^k)\Big(1-\int_U g_\varepsilon^k\bar g^k\Big). \qquad (4.35)$$

We further reformulate the first right-hand side term in this identity. Taking the Malliavin derivative of the eigenvalue relation for g_ε^k, we find

$$(-\lambda_\varepsilon^k-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla)Dg_\varepsilon^k = \nabla\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k+(D\lambda_\varepsilon^k)g_\varepsilon^k,$$

hence, testing this relation with $\bar g^k$ and using the eigenvalue relation for $\bar g^k$,

$$-(\lambda_\varepsilon^k-\bar\lambda^k)\int_U\bar g^kDg_\varepsilon^k-(D\lambda_\varepsilon^k)\int_U g_\varepsilon^k\bar g^k$$
$$= -\int_U\nabla\bar g^k\cdot(\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})-\bar{\boldsymbol{a}})\nabla Dg_\varepsilon^k-\int_U\nabla\bar g^k\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k.$$

Combining this with (4.34) and (4.35), we deduce after straightforward simplifications,

$$DE_\varepsilon^k = \int_U \nabla\bar g^k\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\Big(\nabla g_\varepsilon^k-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\Big)$$
$$-\int_U(\partial_\alpha\bar g^k)\nabla\bar g^k\cdot(\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})-\bar{\boldsymbol{a}})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})-\int_U\nabla\bar g^k\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k+D\lambda_\varepsilon^k,$$

or equivalently, further using (4.6),

$$DE_\varepsilon^k = \int_U \nabla\bar g^k\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\Big(\nabla g_\varepsilon^k-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\Big)$$
$$-\int_U(\partial_\alpha\bar g^k)\nabla\bar g^k\cdot(\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})-\bar{\boldsymbol{a}})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})+\int_U(\nabla g_\varepsilon^k-\nabla\bar g^k)\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla g_\varepsilon^k. \qquad (4.36)$$

Next, we further reformulate the second right-hand side term in this identity. In terms of the skew-symmetric flux corrector σ, cf. (3.2), integrating by parts, using Leibniz' rule, and noting that the Malliavin derivative of the corrector equation takes the form

$$-\nabla\cdot\boldsymbol{a}\nabla D\varphi_\beta = \nabla\cdot D\boldsymbol{a}(\nabla\varphi_\beta+e_\beta), \qquad (4.37)$$

we easily get

$$\int_U(\partial_\alpha\bar g^k)\nabla\bar g^k\cdot(\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})-\bar{\boldsymbol{a}})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})$$
$$= \int_U(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\,(-\boldsymbol{a}\nabla\varphi_\beta+\nabla\cdot\sigma_\beta)(\tfrac{\cdot}{\varepsilon})\cdot\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})$$
$$= \int_U(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\,(\nabla\varphi_\beta)(\tfrac{\cdot}{\varepsilon})\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})$$
$$+\,\varepsilon\int_U\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\cdot(\boldsymbol{a}\varphi_\beta+\sigma_\beta)(\tfrac{\cdot}{\varepsilon})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})$$

$$+\,\varepsilon\int_U \nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\cdot(\varphi_\beta D\boldsymbol{a})(\tfrac{\cdot}{\varepsilon})(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon}).$$

Inserting this into (4.36), and reorganizing the terms, the claim (4.33) follows.

Step 2. Conclusion.
In terms of the solution $v_{\varepsilon;\alpha}\in H^1_0(U)$ of the auxiliary problem

$$-\nabla\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla v_{\varepsilon;\alpha}\,=\,\nabla\cdot\Big((\boldsymbol{a}\varphi_\beta-\sigma_\beta)(\tfrac{\cdot}{\varepsilon})\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\Big),\qquad\text{in }U,\tag{4.38}$$

we can write

$$\int_U\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\cdot(\boldsymbol{a}\varphi_\beta+\sigma_\beta)(\tfrac{\cdot}{\varepsilon})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\,=\,-\int_U\nabla v_{\varepsilon;\alpha}\cdot\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon}),$$

and thus, in view of the Malliavin derivative of the corrector equation, cf. (4.37),

$$\int_U\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\cdot(\boldsymbol{a}\varphi_\beta+\sigma_\beta)(\tfrac{\cdot}{\varepsilon})\nabla D\varphi_\alpha(\tfrac{\cdot}{\varepsilon})\,=\,\int_U\nabla v_{\varepsilon;\alpha}\cdot D\boldsymbol{a}(\tfrac{\cdot}{\varepsilon})(\nabla\varphi_\alpha(\tfrac{\cdot}{\varepsilon})+e_\alpha).$$

Inserting this into (4.33), and recalling that the definition (2.1) of $\boldsymbol{a}$ yields (4.7), we are led to

$$\begin{aligned}DE^k_\varepsilon\,=\,&\varepsilon^d\Big(\nabla g^k_\varepsilon(\varepsilon\cdot)+(\nabla\phi_\beta+e_\beta)\partial_\beta\bar g^k(\varepsilon\cdot)\Big)\cdot\partial a_0(G)\Big(\nabla g^k_\varepsilon(\varepsilon\cdot)-(\nabla\varphi_\alpha+e_\alpha)\partial_\alpha\bar g^k(\varepsilon\cdot)\Big)\\&-\varepsilon^{d+1}\Big(\nabla v_{\varepsilon;\alpha}(\varepsilon\cdot)+\varphi_\beta\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)(\varepsilon\cdot)\Big)\cdot\partial a_0(G)(\nabla\varphi_\alpha+e_\alpha).\end{aligned}$$

Inserting this into (4.32), and using the integrability condition (2.2) in form of (3.3), we obtain after rescaling, for all $2<p,q<\infty$,

$$\begin{aligned}&\Big\|\lambda^k_\varepsilon-\mathbb{E}[\lambda^k_\varepsilon]-\int_U\Xi^\circ_{\alpha\beta}(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\partial_\beta\bar g^k\Big\|_{\mathrm{L}^q(\Omega)}\\&\lesssim_q\ \varepsilon^{\frac d2}\Big\|\big[\nabla g^k_\varepsilon+(\nabla\varphi_\beta+e_\beta)(\tfrac{\cdot}{\varepsilon})\partial_\beta\bar g^k\big]_{2;\varepsilon}\Big\|_{\mathrm{L}^{2q}(\Omega;\mathrm{L}^{\frac{2p}{p-2}}(U))}\\&\qquad\qquad\times\Big\|\big[\nabla g^k_\varepsilon-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\big]_{2;\varepsilon}\Big\|_{\mathrm{L}^{2q}(\Omega;\mathrm{L}^p(U))}\\&\quad+\varepsilon^{1+\frac d2}\|[\nabla\varphi_\alpha+e_\alpha]_2\|_{\mathrm{L}^{2q}(\Omega)}\Big(\|[\nabla v_{\varepsilon;\alpha}]_{2;\varepsilon}\|_{\mathrm{L}^2(U;\mathrm{L}^{2q}(\Omega))}\\&\qquad\qquad+\Big\|\big[\varphi_\beta(\tfrac{\cdot}{\varepsilon})\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\big]_{2;\varepsilon}\Big\|_{\mathrm{L}^2(U;\mathrm{L}^{2q}(\Omega))}\Big).\end{aligned}$$

Appealing to annealed L^p regularity in form of Theorem 3.3 for equation (4.38), we find

$$\|[\nabla v_{\varepsilon;\alpha}]_{2;\varepsilon}\|_{\mathrm{L}^2(U;\mathrm{L}^{2q}(\Omega))}\ \lesssim_{p,q}\ \Big\|\big[(\boldsymbol{a}\varphi_\beta-\sigma_\beta)(\tfrac{\cdot}{\varepsilon})\nabla(\partial_\alpha\bar g^k\partial_\beta\bar g^k)\big]_{2;\varepsilon}\Big\|_{\mathrm{L}^2(U;\mathrm{L}^{3q}(\Omega))}.$$

Further appealing to the corrector estimates of Theorem 3.2 and to the estimates of Lemmas 4.1 and 4.2 on $\lambda^k_\varepsilon,g^k_\varepsilon,\bar g^k$, we deduce for all $2<p,q<\infty$,

$$\begin{aligned}\varepsilon^{-\frac d2}\Big\|\lambda^k_\varepsilon-\mathbb{E}[\lambda^k_\varepsilon]-\int_U\Xi^\circ_{\alpha\beta}(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\partial_\beta\bar g^k\Big\|_{\mathrm{L}^q(\Omega)}\ &\lesssim_{k,p,q}\ \varepsilon\mu_d(\tfrac1\varepsilon)\\&\quad+\Big\|\big[\nabla g^k_\varepsilon-(\nabla\varphi_\alpha+e_\alpha)(\tfrac{\cdot}{\varepsilon})\partial_\alpha\bar g^k\big]_{2;\varepsilon}\Big\|_{\mathrm{L}^{2q}(\Omega;\mathrm{L}^p(U))}.\end{aligned}$$

The conclusion (2.7) follows from the Meyers improvement of the corrector result in Lemma 4.5 provided that $p>2$ is chosen close enough to 2. □

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