A Tractable Overlapping Generations Structure for Quantitative DSGE Models

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This paper develops a novel tractable overlapping generations (OLG) structure that is suitable for use in rich quantitative dynamic stochastic general equilibrium (DSGE) models. The OLG structure assumes that newborn agents receive a wealth transfer such that equilibrium consumption during the first period of life represents a *time-invariant* share of aggregate consumption. Under efficient risk sharing across contemporaneous cohorts, this implies that *aggregate* consumption obeys a (quasi-)Euler equation that is isomorphic to the Euler equation of an infinitely-lived representative agent. As a result, DSGE models, with the proposed OLG structure, can be solved as conveniently as standard DSGE models with infinitely-lived representative agents. The great tractability of the OLG structure here constitutes an important advantage over conventional OLG models, especially when agents are long-lived. While highly tractable, the present OLG structure maintains key predictions of standard OLG models, namely the possibility of low (even negative) real interest rates and of equilibrium indeterminacy.

JEL codes: E1,E3,C6
Keywords: overlapping generations; dynamic stochastic general equilibrium models; Euler equation; subjective discount factor; transversality condition; multiple equilibria.

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1. Introduction

This paper develops a novel tractable overlapping generations (OLG) structure, with finitely-lived agents, that is suitable for use in rich quantitative dynamic stochastic general equilibrium (DSGE) models of business cycle fluctuations. The OLG setup yields a (quasi-)Euler equation in terms of *aggregate* (economy-wide) consumption that is isomorphic to the Euler equation of an infinitely-lived representative agent (ILRA). This aggregation property implies that generic DSGE models, with the proposed OLG structure, can be solved as conveniently as ILRA DSGE models. Importantly, the subjective discount factor in the aggregate OLG Euler equation can differ from the subjective discount factor in ILRA models (the aggregate OLG discount factor may exceed unity). The OLG structure here can thus generate equilibria with low -- even negative -- real interest rates. Also, there are no transversality conditions (TVC) for aggregate asset stocks, in the OLG structure (by contrast to ILRA models). Therefore, multiple equilibria can exist.

In essence, this paper shows that one can transform a generic ILRA DSGE model into an OLG DSGE model (with the OLG structure presented here) just by dropping the ILRA model's TVC. The format of other aggregate equations remains unchanged in the OLG DSGE model, but the OLG structure allows greater freedom in calibrating the subjective discount factor.

The central assumption of the proposed OLG structure is that, at each date, newborn agents receive a wealth transfer (from older agents) that is set in such a manner that equilibrium consumption during the first period of life represents a *time-invariant* share of aggregate consumption. If consumption risk is efficiently shared among all *contemporaneous* age-groups, via complete insurance markets (as often postulated in DSGE models), then the assumed time-invariant consumption share of newborn agents implies that (under CRRA utility) <u>each</u> older age group's equilibrium consumption likewise represents a time-invariant, but age-dependent, share of aggregate consumption. This permits the aggregation of Euler equations across age-groups. That aggregation result holds irrespective of agents' life-span, and so the present OLG framework can easily handle agents with long life-spans.

To justify the central assumption (about transfers and the constant newborn consumption share), it can be noted that, in reality, there are large resource transfers to young individuals driven by altruism and social norms (e.g., parental support and bequests; tax-funded child benefits and education spending). Empirically, the per capita consumption of children and younger consumers closely tracks aggregate consumption; changes in the *relative* consumption of different age groups are dwarfed by changes in aggregate consumption.[1] This motivates the assumption that the equilibrium consumption share of newborn agents is time-invariant. In the OLG structure here, the transfer scheme to newborns, and the time-invariant newborn consumption share targeted by that scheme, are taken as an exogenous institutional/sociopolitical datum.

The tractability of the structure here constitutes an important advantage over standard OLG models, especially when agents are long-lived. OLG models are workhorses of dynamic macroeconomic theory, and they represent a key alternative to ILRA models. In OLG models, competitive equilibria may fail to be Pareto-optimal; OLG models may have multiple equilibria and imply low real interest rates.[2] For analytical convenience, much of the *theoretical* OLG literature assumes that individuals live only two (or three) periods. Such a setting is not suitable for studying economic fluctuations at a business cycle frequency. Given an empirical life expectancy of about 80 years (in advanced countries), a realistic OLG business cycle model for quarterly data would need to assume agents with a life-span of several hundred periods. Solving *stochastic* models with a large number of overlapping *heterogeneous* age cohorts is challenging.[3] This explains why the overwhelming majority of quantitative DSGE macro models (see survey by Wieland et al. (2016)) does not use OLG structures, but assumes an ILRA setting. By contrast, the time-invariance of age-specific consumption shares, in the novel OLG structure developed here, makes solving an OLG DSGE model simple.[4] It has to be stressed that, while

[1] For time series on consumption by age cohorts see, e.g., d'Albis et al. (2019) [data for France] and Saito (2001) [US, UK and Japan].

[2] The basic OLG framework was developed by Allais (1947) and Samuelson (1958). For overviews of OLG models see, e.g., Guesnerie and Woodford (1992), Weil (2008) and Niepelt (2019). During the last two decades, advanced countries' short-term real interest rates have been very low by historical standards; theoretical analyses of low rates often use OLG models (e.g., Carvalho et al. (2016) and Papetti (2021)).

[3] OLG models with long-lived heterogeneous agents can be solved more easily in a deterministic environment, i.e. under perfect foresight (e.g., Auerbach and Kotlikoff (1987), Mc Morrow and Roeger (2004)). However, the analysis of economic fluctuations requires models with stochastic shocks.

[4] In models with a Blanchard (1985) "perpetual youth" OLG structure (that postulates uncertain individual life-time, with a constant death probability per period), *approximate* aggregation across age-groups is possible (but more complicated), based on linearization of individual generation's optimality conditions (e.g., Devereux (2011), Albonico et al. (2021)); the resulting approximate OLG aggregate equations are not isomorphic to those of an ILRA model. By contrast, the novel OLG structure developed here permits very simple *exact* aggregation; thus, the structure here is suited for non-linear model solutions (also, here, the OLG aggregate equations have the same format at ILRA equations; see above). The presentation of the novel OLG structure in Sect. 2 postulates that life-time is non-random (and identical for all individuals); the key substantive results (implied by the key assumption of a constant new-born consumption share) continue to hold when a "perpetual youth" structure is used instead (exact aggregation remains possible).

much more tractable, the OLG structure here maintains key predictions of standard OLG models (possibility of low interest rates and equilibrium indeterminacy). I hope this paper will enable the development of rich OLG DSGE models.

This paper discusses Real Business Cycle (RBC) and New Keynesian (NK) models that embed the OLG structure developed here. Equilibria of the OLG RBC model are locally determinate, but globally indeterminate. The OLG NK model may be locally indeterminate when the aggregate subjective discount factor is greater than or equal to unity.

2. A tractable OLG structure

Assume an economy in which a measure 1 of agents is born each period. All agents live N<∞ periods, and thus the total population has measure N. I refer to agents who are in the *i*-th period of their life at date *t* as 'age-group *i*' at date *t*. All members of the same age-group are identical. Let $c_{i,t}$ and $L_{i,t}$ denote the consumption and labor hours of age-group *i* at *t*. The expected life-time utility of the generation born at date *t* is

$$E_t \sum_{s=1}^{N} \beta^{s-1} \{\log(c_{s,t+s-1}) - \tfrac{1}{1+1/\eta} (L_{s,t+s-1})^{1+1/\eta}\}, \qquad (1)$$

where β>0 and η>0 are the subjective discount factor and the (Frisch) labor supply elasticity, respectively. (The key substantive results discussed below continue to hold under non-unitary consumption risk aversion, and when the subjective discount factor is age dependent.) $\beta<1$ is not required here, as life-time utility is well-defined (bounded) even if $\beta\geq 1$, given the finite life span N. By contrast, $\beta<1$ has to hold for an infinitely-lived agent (under log-utility). Aggregate consumption at date t is $C_t = \sum_{i=1}^{N} c_{i,t}$.

2.1. Risk sharing

Assume that, at each date *t*, a complete set of one-period Arrow-Debreu insurance contracts is traded, so that, in equilibrium, consumption risk is efficiently shared across all agents who are alive at both *t* and *t*+1. Under the assumed CRRA (log) period utility, this implies that consumption growth between *t* and *t*+1 is equated across those contemporaneous agents, for all states of the world:

$$c_{i+1,t+1}/c_{i,t} = c_{2,t+1}/c_{1,t} \text{ for } i=2,..,N\text{-}1. \qquad (2)$$

Let $\lambda_{i,t} \equiv c_{i,t}/C_t \geq 0$ denote the share of age-group i's consumption in aggregate consumption, at t. (2) implies

$$\lambda_{i+1,t+1}/\lambda_{i,t} = \lambda_{2,t+1}/\lambda_{1,t} \text{ for } i=2,..,N\text{-}1. \tag{3}$$

(3) and the adding up constraint

$$\sum_{i=1}^{N} \lambda_{i,t+1} = 1 \tag{4}$$

define N-1 equations in the N consumption shares at date t+1. Assume that the equilibrium consumption share of age-group 1 is time-invariant: $\lambda_{1,t} = \lambda_1 \;\; \forall t$, which requires a suitable wealth transfer (endowment) from older age-groups to age-group 1. A discussion of transfer schemes is provided in the Appendix (as shown there, a constant consumption share of age-group 1 is achieved by allocating to that age group a time-invariant share of aggregate wealth). λ_1 is here treated as exogenous. The following analysis focuses on equilibria in which the (endogenous) consumption shares of age-groups i>1 too are time-invariant, $\lambda_{i,t} = \lambda_i$. Given λ_1, those steady state shares for i>1 are uniquely determined and obey

$$\lambda_{i+1} = (1-\lambda_1)\lambda_i/(1-\lambda_N) \text{ for } i=1,..,N\text{-}1.\text{[5]} \tag{5}$$

2.2. Aggregate Euler equation, subjective discount factor, no transversality condition

Only date t age-groups i=1,..,N-1 can hold assets between dates t and t+1. Let r_{t+1} be the rate of return of a traded asset, between t and t+1. The date t Euler equation of age-group i=1,..,N-1 for that asset is $E_t \rho_{t,t+1} \times (1+r_{t+1}) = 1$, where $\rho_{t,t+1} = \beta c_{i,t}/c_{i+1,t+1}$ is the intertemporal marginal rate of substitution (IMRS) between t and t+1. Full risk sharing implies that the IMRS is equated across those age-groups (see (2)). Thus,

$$\rho_{t,t+1} = \beta \sum_{i=1}^{N-1} c_{i,t} / \sum_{i=2}^{N} c_{i,t+1} = \beta(C_t - c_{N,t})/(C_{t+1} - c_{1,t+1}) = \beta[(1-\lambda_N)/(1-\lambda_1)] \cdot C_t/C_{t+1}.$$

Therefore,

$$E_t \tilde{\beta}(C_t/C_{t+1}) \times (1+r_{t+1}) = 1, \text{ with } \tilde{\beta} \equiv \beta \times (1-\lambda_N)/(1-\lambda_1). \tag{6}$$

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[5] Given λ_1, equations (3),(4) yield a unique solution for date t+1 consumption shares $\{\lambda_{i,t+1}\}$, as a function of the date t shares $\{\lambda_{i,t}\}$. That solution is: $\lambda_{i+1,t+1} = (1-\lambda_1)\lambda_{i,t}/(1-\lambda_{N,t})$ for i=1,..,N-1. When consumption shares for age-groups i>2 *differ* from steady state shares, at some date, then shares at later dates converge asymptotically to the steady state shares. Numerical experiments show that convergence is fast.

$\widetilde{\beta}$ is increasing in λ_1. $\lambda_1=1/N$ implies $\lambda_i=1/N$ for $i>1$, which entails $\widetilde{\beta}\equiv\beta$; hence, $\widetilde{\beta}>\beta$ holds when $\lambda_1>1/N$. [6]

(6) shows that the OLG model implies a (quasi-)Euler equation, in terms of <u>aggregate</u> consumption, that has the same form as the Euler equation of an infinitely-lived representative agent (ILRA). However, an important difference, relative to ILRA models, is that the subjective discount factor in the aggregate OLG Euler equation can exceed unity: $\widetilde{\beta}>1$ is possible.[7] Thus, the OLG economy can generate a low (even negative) steady state real interest rate, $r=(1-\widetilde{\beta})/\widetilde{\beta}$. Prominent estimates of Euler equations based on aggregate consumption data have reported fitted subjective discount factors that exceed unity (e.g. Eichenbaum et al. (1988)). The OLG structure here can account for that empirical finding.

In the OLG structure, each <u>individual</u> agent holds zero assets, at the end of her life (see Appendix), but the path of <u>aggregate</u> asset holdings is not constrained by a terminal condition. By contrast, in an ILRA economy, optimizing individual behavior requires that a transversality condition (TVC) for aggregate asset stocks holds.

3. Embedding the OLG structure in DSGE models

The above OLG structure can easily be built into a wide range of DSGE macro models. Assume that labor and the assets of different age-groups are homogeneous. Then the *static* equilibrium conditions, the budget constraints, and the laws of motion of individual asset holdings can be aggregated across age cohorts, which delivers equations in aggregate variables that are identical to corresponding equations in ILRA DSGE model.

The upshot is that one can transform an ILRA DSGE model into an OLG DSGE model (with the OLG structure developed here) by simply dropping the TVC. The format of the

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[6] Intuitively, a higher age-group 1 consumption share implies lower growth of *individual* consumption over the life-cycle, which must be accompanied by a lower steady state interest rate (and a higher $\widetilde{\beta}$). Note that (5) implies $\lambda_{i+1}=\lambda_1 Q^i$ for i=1,..,N-1, with $Q\equiv(1-\lambda_1)/(1-\lambda_N)$. (4) gives $\lambda_1\sum_{i=1}^{N}Q^{i-1}=1$. This equation allows to solve for Q and λ_N as functions of λ_1. Q is a decreasing function of λ_1, so $\widetilde{\beta}\equiv\beta/Q$ is increasing in λ_1. $\lambda_1=1/N$ implies Q=1, so that $\lambda_i=1/N$ for $i>1$.

[7]Note that $\widetilde{\beta}>1$ can hold even when $\beta<1$, provided $(1-\lambda_N)/(1-\lambda_1)$ is sufficiently large. Assume, e.g., N=320 quarters (80 year life-span), and β=0.99; then λ_1=1.0373% implies $\widetilde{\beta}$=1.005 (steady state real interest rate: -2% per annum).

aggregate Euler equation and of other aggregate equations remains unchanged, but the OLG structure allows greater freedom in calibrating the subjective discount factor.

I next discuss Real Business Cycle (RBC) and New Keynesian economies that embed the novel OLG structure.

3.1. OLG RBC model

I begin the analysis of the OLG RBC model by highlighting its aggregation properties. The RBC model assumes flexible prices and wages, and competitive goods and factor markets (e.g., King and Rebelo (1999)). There is an aggregate production function $Y_t=\theta_t(K_t)^{\alpha}(L_t)^{1-\alpha}$, $0<\alpha<1$, where Y_t and θ_t are aggregate output and exogenous total factor productivity (TFP), respectively. $L_t=\sum_{i=1}^{N}L_{i,t}$ and $K_t=\sum_{i=1}^{N}K_{i,t}$ are the aggregate labor input and the aggregate capital stock (at the beginning of period *t*), where $K_{i,t}$ is the capital stock owned by age-group *i*. Let the law of motion of aggregate capital be $K_{t+1}=(1-\delta)K_t+I_t$, where I_t is gross investment while $0\leq\delta\leq 1$ is the rate of capital depreciation. Under the OLG structure developed in this paper, the following (quasi-) Euler equation for capital holds (from (6)): $E_t\widetilde{\beta}(C_t/C_{t+1})\times(1+r_{K,t+1})=1$, where $r_{K,t+1}\equiv\alpha Y_{t+1}/K_{t+1}-\delta$ is the return on capital between dates *t* and *t*+1 (marginal product of capital net of depreciation). Each age-group *i* equates her marginal rate of substitution between consumption and leisure to the real wage rate (marginal product of labor) $w_t\equiv(1-\alpha)Y_t/L_t$; thus, $w_t/c_{i,t}=(L_{i,t})^{1/\eta}$. With time-invariant age-group-specific consumption shares, $c_{i,t}=\lambda_i C_t$, age-group *i's* labor supply is, thus: $L_{i,t}=(w_t/C_t)^{\eta}(\lambda_i)^{-\eta}$. Hence, aggregate labor is $L_t=(w_t/C_t)^{\eta}\sum_{i=1}^{N}(\lambda_i)^{-\eta}$. An ILRA model with life-time utility $E_t\sum_{s=1}^{\infty}\beta^{s-1}\{\log(C_{t+s-1})-\frac{\widetilde{\mu}}{1+1/\eta}(L_{t+s-1})^{1+1/\eta}\}$, where $0<\beta<1$ and $\widetilde{\mu}\equiv\{\sum_{i=1}^{N}(\lambda_i)^{-\eta}\}^{-1/\eta}$, induces the same aggregate labor supply equation as the OLG economy. Furthermore, aggregation of all age-groups' budget constraints delivers the ILRA resource constraint $Y_t=C_t+I_t$. [8]

[8] In the ILRA RBC model, the TVC for capital is: $\lim_{\tau\to\infty}\beta^{\tau}E_tK_{t+\tau+1}/C_{t+\tau}=0$. That condition does <u>not</u> hold in the OLG RBC model.

I now present numerical simulation results for the OLG RBC economy. I compare model predictions for $\widetilde{\beta}=0.99$ and for $\widetilde{\beta}=1.01$. Following King and Rebelo (1999), I set $\alpha=\frac{2}{3}, \delta=0.025, \eta=4$ and assume that log TFP is an AR(1) process with autocorrelation 0.979 and innovation standard deviation 0.72%. The steady state C/Y and K/Y ratios are 0.76 and 9.49 respectively when $\widetilde{\beta}=0.99$, compared to 0.45 and 22.08 when $\widetilde{\beta}=1.01$. Thus, the capital/output ratio is much higher (while the C/Y ratio is lower) when $\widetilde{\beta}=1.01$. I log-linearize the model around the deterministic steady state. For all values of $\widetilde{\beta}$ consistent with an interior steady state, the linearized model has a unique non-explosive (bounded) solution, i.e. the equilibrium is <u>locally</u> determinate.[9] The local solution (for aggregate variables) generated by an OLG RBC model with $\widetilde{\beta}<1$ is identical to that of an ILRA RBC model with $\beta=\widetilde{\beta}$.

The log-linearized equilibrium policy functions for capital and output are:

$$\widehat{K_{t+1}}=0.95\widehat{K_t}+0.11\widehat{\theta}_t, \quad \widehat{Y}_t=0.11\widehat{K}_t+1.50\widehat{\theta}_t \quad \text{when } \widetilde{\beta}=0.99;$$

$$\widehat{K_{t+1}}=0.98\widehat{K_t}+0.07\widehat{\theta}_t, \quad \widehat{Y}_t=0.18\widehat{K}_t+1.72\widehat{\theta}_t \quad \text{when } \widetilde{\beta}=1.01.$$

Here $\widehat{X}_t\equiv\log(X_t/X)$ is the relative deviation of variable X_t from steady state (X). Note that output is more sensitive to the level of capital and to TFP, when $\widetilde{\beta}=1.01$. The standard deviations of Hodrick-Prescott-filtered log output, consumption and investment are 1.41%. 0.58% and 4.24%, respectively, when $\widetilde{\beta}=0.99$, compared to 1.61%. 0.36% and 2.64%, when $\widetilde{\beta}=1.01$. Hence, the OLG RBC model with $\widetilde{\beta}=1.01$ implies that output is more volatile, while consumption and investment are less volatile than in a standard ILRA RBC model with $\widetilde{\beta}=0.99$.

Kollmann (2022) shows that, despite <u>local</u> determinacy, the OLG RBC model is <u>globally</u> indeterminate (existence of an infinity of stationary equilibria with bounded paths of consumption, hours, investment and output). This global indeterminacy reflects the absence of a transversality condition (TVC) for capital in the OLG model, and it obtains even when $\widetilde{\beta}<1$. (By contrast, in the ILRA RBC model, the TVC pins down a unique equilibrium.) Using a globally

[9] The OLG model has an interior steady state (with strictly positive consumption and capital) for $0<\widetilde{\beta}\leq 1.017$. For all values of $\widetilde{\beta}$ in that range, the number of eigenvalues of the linearized state-space form equals the number of non-predetermined variables, i.e. the Blanchard and Kahn (1980) condition for local determinacy is satisfied.

accurate non-linear model solution, Kollmann (2022) documents that beliefs-driven (sunspot) equilibria in an OLG RBC economy (no TVC) can capture key empirical business cycle stylized facts, even when there are no shocks to productivity (or to other fundamental driving forces).

3.2. An OLG New Keynesian model

In contrast to the OLG RBC economy, a New Keynesian (NK) sticky-prices economy, with the OLG structure developed here, may be <u>locally</u> indeterminate when $\widetilde{\beta}\geq 1$. We can transform the aggregate equations of a standard ILRA NK model into an OLG NK model by replacing the ILRA subjective discount factor $\beta<1$ by $\widetilde{\beta}$. Consider the following linearized three-equation textbook NK model (e.g., Galí (2015), Kollmann (2021a,b,c)):

$$\pi_t=\kappa\cdot x_t+\widetilde{\beta}E_t\pi_{t+1};\quad i_{t+1}=\gamma\pi_t;\quad E_t x_{t+1}=x_t+\{i_{t+1}-E_t\pi_{t+1}-r_{t+1}\}, \tag{7}$$

where π_t, x_t are inflation and the output gap at date t, respectively; i_{t+1}, r_{t+1} are the nominal interest rate, and the exogenous (stationary) natural real interest rate (i.e. the real interest rate in a flex-prices economy) between t and t+1. (Inflation and interest rates are expressed as deviations from steady state, while the output gap is defined as the relative deviation of output from its natural level.) The three equations listed in (7) are a Phillips equation, a monetary policy (Taylor) rule and an Euler equation, respectively. $\kappa\geq 0$ is the slope of the Phillips curve (κ is a decreasing function of the degree of price stickiness), while γ captures the response of the policy interest rate to inflation; the Taylor principle is assumed to hold: $\gamma>1$. When the OLG structure developed in this paper is assumed, the subjective discount factor appearing in the Phillips curve is the parameter $\widetilde{\beta}$ that may exceed unity (see (6)).[10] When $\widetilde{\beta}<1$, the NK economy has a unique non-explosive equilibrium, for any values $\kappa\geq 0$, $\gamma>1$. When $\widetilde{\beta}\geq 1$, by contrast, multiple non-explosive equilibria may exist. E.g., one sees immediately that, for $\kappa=0$, the processes $\pi_{t+1}=(1/\widetilde{\beta})\pi_t+\varepsilon_{t+1}$, $x_{t+1}=x_t+\{[\gamma-(1/\widetilde{\beta})]\pi_t-r_{t+1}\}+\eta_{t+1}$ satisfy (7), for *arbitrary* disturbances $\varepsilon_{t+1}, \eta_{t+1}$ of zero conditional mean ($E_t\varepsilon_{t+1}=E_t\eta_{t+1}=0$); these inflation and output gap processes are non-

[10] The Phillips equation can be derived from the linearized first-order condition of the profit maximization problem of a monopolistic firm facing convex price adjustment costs. At date t, that firm optimally trades off price adjustment costs at t and t+1. Assume that, under the OLG structure, this trade-off is evaluated using the intertemporal marginal rate of substitution of households alive at t and t+1. Then $\widetilde{\beta}$ is the relevant subjective discount factor in the monopolist's price setting equation, which implies that $\widetilde{\beta}$ appears in the Phillips equation.

explosive when $\widetilde{\beta}\geq 1$. This confirms the equilibrium indeterminacy. When $\kappa>0$, equilibrium indeterminacy arises for values of $\widetilde{\beta}$ strictly larger than unity.[11]

4. Conclusion

This paper develops a novel tractable overlapping generations (OLG) structure that is suitable for use in rich quantitative dynamic stochastic general equilibrium (DSGE) models. The OLG structure here yields a (quasi-)Euler equation in terms of *aggregate* consumption that is isomorphic to the Euler equation of an infinitely-lived representative agent (ILRA). This aggregation property implies that DSGE models, with the proposed OLG structure, can be solved as conveniently as ILRA DSGE models. While highly tractable, the structure presented here maintains key predictions of standard OLG models, namely the possibility of low real interest rates and of equilibrium indeterminacy. The central insight of this paper is that one can transform a generic ILRA DSGE model into an OLG DSGE model (with the OLG structure developed here) by simply dropping the ILRA model's transversality conditions. The format of other aggregate equations remains unchanged in the OLG model, but the OLG structure allows greater freedom in calibrating the subjective discount factor.

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[11] Combining equations (7) gives $E_t\pi_{t+2}=[(1+\kappa+\widetilde{\beta})/\widetilde{\beta}]E_t\pi_{t+1}-[(\gamma\kappa+1)/\widetilde{\beta}]\pi_t+(\kappa/\widetilde{\beta})r_{t+1}$. I verified numerically that, for empirically relevant values of κ and γ (0<κ≤0.5; 1<γ≤5), the difference equation has characteristic root(s) on or inside the unit circle, which implies solution indeterminacy, when $\widetilde{\beta}\geq 1+\gamma\kappa$.

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APPENDIX: Transfers in the OLG structure

For the OLG structure presented in Sect. 2, this Appendix discusses transfers from age-groups i=2,..,N to newborn agents (age-group 1) that support an equilibrium with a time-invariant newborn consumption share (λ_1). A variety of transfer schemes can be envisaged. The structure of suitable transfer schemes depends on the features of the underlying economy. As an illustration, I here analyze transfers in the RBC economy of Sect. 3.1.

Transfers to newborn agents: a simple example (OLG RBC economy)

I now present a simple lump sum transfer scheme that ensures that the consumption share of age-group 1 is $\lambda_1=1/N$, which implies $\lambda_i=1/N$ for i=2,..,N, so that consumption is equated across all age-groups (see Sect. 2.2). (A more general analysis of transfer schemes is provided below.) Denote the transfer made by age-group i >1 to age-group 1 by $T_{i,t}$. In an equilibrium with equal consumption (shares), hours worked are equated across age-groups, so that $L_{i,t}=L_t/N$. All age-groups thus have the same labor income, denoted $W_t\equiv(1-\alpha)Y_t/N$. As is common in OLG models, assume that all agents are born with zero capital holdings, $K_{1,t}=0$. As households can invest their wealth in capital or in a complete set of state-contingent bonds, the composition of portfolios is indeterminate. I here discuss transfers in a symmetric equilibrium in which all age-groups hold zero state-contingent bonds, and in which age-groups i=1,..,N-1 hold the same amount capital at the end of any period t: $K_{i+1,t+1}=K_{t+1}/(N-1)$; the oldest age-group, i=N, does not hold capital at the end of the period: $K_{N+1,t+1}=0$.

In this symmetric equilibrium, the age-groups' budget constraints at date t are:

For i=1: $\quad K_{t+1}/(N-1)+C_t/N = W_t+\sum_{i=2}^{N} T_{i,t}$;

For i=2,..,N-1: $\quad K_{t+1}/(N-1)+C_t/N = W_t - T_{i,t} + (1+r_{K,t})K_t/(N-1)$;

For i=N: $\quad C_t/N = W_t - T_{N,t} + (1+r_{K,t})K_t/(N-1)$,

with $r_{K,t}=\alpha Y_t/K_t-\delta$. Solving these budget constraints for transfers (using $C_t=Y_t-[K_{t+1}-(1-\delta)K_t]$) gives:

$$T_{i,t} = \{[1+r_{K,t}]K_t/(N-1)-K_{t+1}/(N-1)\}/N \text{ for } i=2,..,N\text{-}1 \text{ and } T_{N,t} = \{[1+r_{K,t}]K_t/(N-1)+K_{t+1}\}/N.$$

Thus, age-groups i=2,..,N-1 transfer (to age-group 1) a fraction 1/N of their respective capital income less investment. The transfer by age-group i=N amounts to a fraction 1/N of capital income, plus a fraction 1/N of the aggregate end-of-period capital stock. The total transfer received by age-group 1 is $\sum_{i=2}^{N} T_i = \{[1+r_{K,t}]K_t + K_{t+1}/(N-1)\}/N$. When N is large, the total transfer is close to the beginning-of-period capital stock plus capital income, per age group, $[1+r_{K,t}]K_t/N$.

More general analysis of wealth and transfers

I next provide a more general analysis of transfer schemes, allowing for arbitrary values of the targeted newborn consumption ratio $\lambda_1 > 0$, and highlighting the role of state-contingent bonds. Taking account of one-period state-contingent bonds, the date t budget constraint of age-group i=1,..,N is:

$$\int p_t(S_{t+1}) F_{i+1,t+1}(S_{t+1}) dS_{t+1} + K_{i+1,t+1} + c_{i,t} = F_{i,t} + K_{i,t}(1+r_{K,t}) + W_t - T_{i,t}, \tag{A.1}$$

where $p_t(S_{t+1})$ is the date t price of an insurance contract that pays one units of output in period t+1 if and only if state of the world S_{t+1} is realized at date t+1. $F_{i+1,t+1}(S_{t+1})$ is the number of claims of this type held by age-group i, at the end of period t. [12] For i>1, $T_{i,t}$ is the lump sum transfer made by age-group i>1 to age-group 1. $-T_{1,t} = \sum_{i=2}^{N} T_{i,t}$ is the total transfer received by age-group 1.

Age-group 1 has zero income from capital and state-contingent bonds: $K_{1,t}$=$F_{1,t}$=0. Agents who are in the last period of their life, at date t, cannot issue claims that oblige them to make future payments, and it is not in their interest to acquire claims to future payments or to invest in physical capital; thus, $K_{N+1,t+1}$=0 and $F_{N+1,t+1}(S_{t+1}) = 0 \;\; \forall S_{t+1}$.

Age-groups i=1,..,N-1 equate their probability-weighted intertemporal marginal rate of substitution to the prices of state-contingent claims: $p_t(S_{t+1}) = \pi_t(S_{t+1}) \beta c_{i,t}/c_{i+1,t+1}(S_{t+1}) \;\; \forall S_{t+1}$ where $\pi_t(S_{t+1})$ is the conditional probability (density) of state S_{t+1}, given date t information.

[12] $c_{i,t}, K_{i+1,t+1}, T_{i,t}, r_{K,t}, W_t$ and $F_{i,t}$ depend on the state of the world S_t. To simplify notation, I suppress the dependence of choice variables on the state of the world, unless confusion arises.

Financial market completeness implies thus the risk-sharing condition (2). Note that the price of state-contingent assets can be expressed as: $p_t(S_{t+1}) = \pi_t(S_{t+1})\rho_{t,t+1}$, with $\rho_{t,t+1}=\beta c_{i,t}/c_{i+1,t+1}$ for i=1,..,N-1. This allows to write the budget constraint (A.1) as $E_t\rho_{t,t+1}F_{i+1,t+1}+K_{i+1,t+1}+c_{i,t}= F_{i,t}+K_{i,t}(1+r_{K,t})+W_t-T_{i,t}$. Let $H_{i,t}=W_t+E_t\rho_{t,t+1}H_{i+1,t+1}$ and $Q_{i,t}=-T_{i,t}+E_t\rho_{t,t+1}Q_{i+1,t+1}$ denote the present value of age-group i's wage income and of her transfer income, respectively, with $H_{i+1,t+1}= Q_{i+1,t+1}=0$ for $i\geq$N. $H_{i,t}$ and $Q_{i,t}$ are age-group i's human capital and her 'transfer capital', respectively. Using these definitions, age-group i's budget constraint can be written as

$$E_t\rho_{t,t+1}A_{i+1,t+1} + c_{i,t} = A_{i,t} \text{ with } A_{i,t}\equiv F_{i,t}+H_{i,t}+Q_{i,t}+K_{i,t}(1+r_{K,t}) . \tag{A.2}$$

$A_{i,t}$ is age-group i's total wealth at the beginning of period t. Agents hold zero assets at the end of their life, and thus age-group N consumes her (beginning-of-period) assets: $c_{N,t} = A_{N,t}$.

Age-group i=1,..,N thus faces the present value budget constraint $A_{i,t}=E_t\sum_{s=0}^{N-i}\rho_{t,t+s}c_{i+s,t+s}$ where $\rho_{t,t}\equiv1$ while, for $s\geq1$, $\rho_{t,t+s}\equiv\prod_{\tau=0}^{\tau=s-1}\rho_{t+\tau,t+\tau+1}$. Given efficient risk sharing across contemporaneous age-groups (see (2)), $\rho_{t,t+s}=\beta^s c_{i,t}/c_{i+s,t+s}$ holds for $0<s\leq N$-i. Thus $A_{i,t}= c_{i,t}\sum_{s=0}^{N-i}\beta^s$ and hence

$$c_{i,t}=\phi_i \cdot A_{i,t}, \text{ with } \quad \phi_i\equiv(1-\beta)/(1-\beta^{N-i+1}) \quad \text{for } i=1,..,N.$$

Each period, age-group i consumes thus a fraction ϕ_i of her wealth. ϕ_i is age-group-specific, but time invariant. In an equilibrium with time-invariant age-group consumption shares, the period t wealth of age-group i equals thus $A_{i,t}=(\lambda_i/\phi_i)C_t$. The wealth share of age-group i is:

$$A_{i,t}/\sum_{s=1}^{N}A_{s,t} = (\lambda_i/\phi_i)/\sum_{s=1}^{N}(\lambda_s/\phi_s) \equiv \kappa_i .$$

Note that this wealth share is time-invariant. Thus, an equilibrium with time-invariant age-group-specific consumption shares exhibits time-invariant age-group wealth shares. The consumption share of age-group 1, λ_1, pins down the (steady state) consumption shares of older age-groups, i.e. λ_i is a function of λ_1: $\lambda_i=\Lambda_i(\lambda_1)$ for i=2,..,N. There is, hence, a unique mapping from λ_1 to κ_i, the wealth share of age-group i:

$$\kappa_i = (\Lambda_i(\lambda_1)/\phi_i)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s).$$

If the new-born age-group is allocated a wealth share $\kappa_1 = (\lambda_1/\phi_1)/\sum_{s=1}^{N}(\Lambda_s(\lambda_1)/\phi_s)$, then this sustains an equilibrium in which the consumption share of the new-born age-group is λ_1. A consumption allocation in which all age-groups have an identical consumption share $\lambda_i = 1/N$ is sustained by allocating to age-group 1 a wealth share $\kappa_1 = (1/\phi_1)/\sum_{s=1}^{N} 1/\phi_s$. As an example, assume that life lasts 80 years, i.e. N=320 quarters, and that the quarterly subjective discount factor is $\beta = 0.99$; then the consumption allocation with equal consumption shares $\lambda_i = 1/N = 0.3125\%$ requires an age-group 1 wealth share of $\kappa_1 = 0.4267\%$.

Age-group 1 holds zero initial bonds and zero initial capital. Thus, the initial wealth of age-group 1 is the sum of her human capital and of her 'transfer capital' (see (A.2)): $A_{1,t} \equiv H_{1,t} + Q_{1,t}$. Because $c_{1,t} = \phi_1 A_{1,t}$, a time-invariant age-group 1 consumption share λ_1 obtains when $A_{1,t} = (\lambda_1/\phi_1)C_t$ and $Q_{1,t} = (\lambda_1/\phi_1)C_t - H_{1,t}$ hold.