Curvature Instability in Passive Diffractive Resonators

M. Tlidi,[1] A. G. Vladimirov,[2] and Paul Mandel[1]

[1]*Optique Nonlinéaire Théorique, Université Libre de Bruxelles, Campus Plaine CP 231, 1050 Bruxelles, Belgium*

[2]*Physics Faculty, St. Petersburg State University, 198904 St. Petersburg, Russia*

(Received 25 September 2001; published 18 November 2002)

We study the stability of localized structures in a passive optical bistable system. We show that there is a critical value of the input field intensity above which localized structures are unstable with respect to a curvature instability. Beyond this instability boundary, a transition from the localized branch of solutions to stable hexagons is found.

DOI: 10.1103/PhysRevLett.89.233901 PACS numbers: 42.65.Sf, 42.60.Mi, 42.65.Pc, 89.75.Kd

Nonlinear cavities driven by an external field can display localized structures (LS), also called spatial solitons. They arise in the domain where two branches of stable solutions coexist, typically a branch of homogeneous steady states and either another branch of steady states [1] or a branch of spatially periodic solutions [2,3] arising from a Turing or spatial modulational instability [4]. In the degenerate optical parametric oscillator [5,6] the formation of LS is related to the phase indetermination in a bistable system and does not rely on a spatial modulational instability. LS consist of bright or dark pulses in the transverse profile of the intracavity field amplitude. If there is more than one LS, they can be either spatially independent and randomly distributed or clustered and forming spatial patterns in the transverse plane. They may also exhibit periodic or chaotic oscillations in time [7,8]. Reviews on the LS formation can be found in [9].

Experimental evidence of LS [10,11] has reinforced interest in the spatial confinement of light. There is a possibility that LS could be used as bits for information manipulation [12,13]. Therefore, the question of stability of a LS is central and sources of instabilities must be carefully scrutinized. In this Letter, we report on a new instability of two types of LS, circular spots and stripes, in a driven passive resonator described by a Lugiato-Lefever model with saturating nonlinearity. We show that there is a critical value of the input field intensity, above which these two LS are unstable with respect to a curvature instability. For appropriate values of the parameters, the LS exhibits an elliptical deformation and splitting. The circular LS, for instance, breaks down in two bright spots connected by a tube of lower intensity. A single localized stripe can also be destroyed by this curvature instability. When the instability takes place, either type of LS exhibits a transition toward the same stationary π-hexagon (or honeycomb) pattern which fills the transverse plane. The numerical results are supported by a qualitative analytical analysis. The curvature instability bears analogies with the morphological instability occurring during the solidification process in fluid dynamics [14]. This behavior also establishes a connection between the optical reaction-diffraction systems and the chemical reaction-diffusion systems [15] where a similar instability has been observed. Experimental observation of the curvature instability is also possible in a liquid crystal light valve with feedback [16]. Similar instabilities affecting the circular shape of LS [17] and a single stripe [18] have been reported for other nonlinear optical systems involving high dimensions. In this Letter we show that an equation of the complex Ginzburg-Landau–type with saturating nonlinearity is sufficient to display the curvature instability.

We consider a passive resonator with plane mirrors, filled by a resonant two-level medium without population inversion and driven by a coherent plane-wave injected signal. We assume that the medium relaxes much faster than the cavity field. In this regime, two types of LS may be generated, depending on the magnitude of the critical wave number associated with the modulational instability: circular localized bright spots [12] and a stationary localized single stripe [19]. Both types of LS can occur for the same parameter values [19]. In the mean field approximation, the space-time evolution of the intracavity field is described by the following dimensionless partial differential equation:

$$\frac{\partial E}{\partial t} = E_i + f(E, E^*) + i\mathcal{L}_\perp E, \tag{1}$$

$$f(E, E^*) = -(1 + i\theta)E - \frac{2C(1 - i\Delta)E}{1 + \Delta^2 + |E|^2}, \tag{2}$$

where E is the normalized slowly varying complex envelope of the electric field. The input field amplitude E_i is real and constant. The detuning parameters are $\theta = (\omega_c - \omega_e)/\kappa$ and $\Delta = (\omega_a - \omega_e)/\gamma_\perp$, where ω_c (ω_e, ω_a) are the cavity (external, atomic) frequency, κ and $\gamma_\perp$ are the cavity and the atomic polarization decay rates, respectively. The cooperativity parameter C determines the steady state bistability. The diffraction operator is $\mathcal{L}_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The homogeneous steady state solutions of Eq. (1) $\bar{I} = |\bar{E}|^2$ are given by the implicit equation $E_i^2 = \bar{I}[(1 + F)^2 + (\theta - \Delta F)^2]$ with $F = 2C/(1 + \Delta^2 + \bar{I})$. For simplicity, we will consider the limit of an absorptive bistability ($\Delta = 0$). As we shall see in the following, this assumption does not affect the generality of our analysis. The cavity intensity $\bar{I}$ as a function of the input intensity E_i^2 is bistable if $C > C_c$, where C_c is the real solution of $(C_c - 4)(1 + 2C_c^2) = 27\theta^2 C_c^2$. The corresponding critical intensity is $\bar{I}_c = (1 + 2C_c)/(C_c - 1)$. We consider the domain $\theta < 0$ ($\omega_c < \omega_e$) where $\bar{I}$ undergoes a modulational instability leading to the formation of stationary patterns which are periodic in space [3,20]. The instability domain is bounded by two thresholds $I_T^{\pm} = [C - 1 \pm \sqrt{C(C-4)}]/2$. The critical wave number corresponding to the maximum gain is $k_T = \sqrt{-\theta}$ at both bifurcation points.

We first consider the circular localized bright spots. They are stationary radially symmetric localized structures $E(x, y) = E_0(r)$. They derive from Eq. (1) with $\partial E/\partial t = 0$, $\mathcal{L}_\perp = \partial^2/\partial r^2 + (1/r)\partial/\partial r$, and the boundary condition $E_0(r) \to 0$ when $r \to \infty$. In order to calculate LS we use the following procedure. We first integrate Eq. (1) from $r = l \gg 1$ to $r = r_1 = O(1)$, with the initial condition $E_0 = (a + ib)K_0(\mu l)v_1 + (a - ib)K_0(\mu^* l)v_2$ where μ and $(v_1, v_2)^T$ are, respectively, the eigenvalue and eigenvector of the matrix

$$i\begin{pmatrix} \frac{\partial f}{\partial E} & \frac{\partial f}{\partial E^*} \\ -\frac{\partial f^*}{\partial E} & -\frac{\partial f^*}{\partial E^*} \end{pmatrix}_{E=E_0},$$

where f is given by Eq. (2) and K_0 is the modified Bessel function of the second kind. Second, we perform the integration from $r = r_0 = 10^{-8} \ll 1$ to $r = r_1$; the initial condition used is $E_0 = A + iB$ and $\partial E_0/\partial r = 0$. The unknown parameters a, b, A, and B are determined by matching the two solutions at $r = r_1$. This allows us to calculate the 2D LS amplitude as a function of the input field amplitude. To study the stability of 2D LS, we introduce the perturbations

$$E = E_0(r) + \delta E(r) e^{\lambda_m t} e^{im\varphi}.$$

We consider $m \neq 0$ since only these perturbations affect the shape of the bounding circle of the LS. Linearizing Eq. (1) with respect to δE leads to the characteristic equation for the eigenvalues λ_m

$$\lambda_m \delta E = i\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2} - \theta\right)\delta E - \delta E - \frac{2C\,\delta E}{1 + |E_0|^2} + \frac{2C(|E_0|^2 \delta E + E_0^2 \delta E^*)}{(1 + |E_0|^2)^2}.$$

A discretization of the differential operators in the interval $[r_0, l]$ is adopted to calculate numerically the eigenvalues λ_m. This leads to an eigenvalue problem for a real $2N \times 2N$ matrix. The calculation is performed for $N = 256$. The eigenvalue with the greatest real part is plotted in Fig. 1 for $m = 0, 1, 3$, and 4. It is clearly seen from this figure that in the domain $19 \leq E_i \leq 20.5$ only perturbations with the angular index $m = 2$ become unstable as the driving field E_i increases. This instability occurs at $E_i = 19.81$. Other perturbations of the 2D LS corresponding to $m = 0, 1, 3$, and 4 decay.

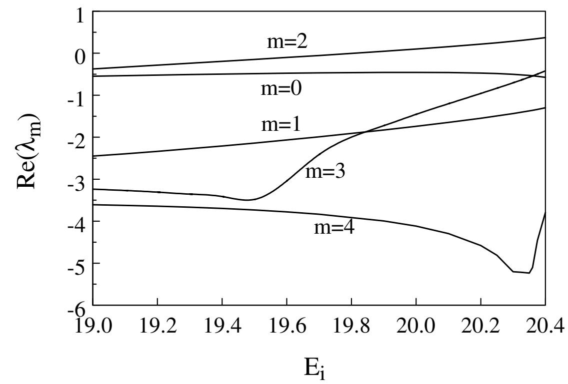

FIG. 1. Real part of the nonzero eigenvalues λ_m corresponding to the angular harmonics $m = 0, 1, 2, 3$, and 4 as a function of the input field amplitude. Parameters are $\theta = -2$ and $C = 20$.

To make explicit comparisons with these results, numerical simulations of the full model Eq. (1) have been made for the range of parameters where the system exhibits a pattern forming instability which affects only a portion of the upper S-shaped steady state intensity. We fix θ and C and let the input field amplitude be the control parameter. We have exclusively looked at periodic boundary conditions in both transverse directions. Let us consider an initial condition leading to stable circular localized structures for the input field $E_i = 18$ with $\theta = -2$ and $C = 20$. They are similar to the LS analyzed in [2,12]. With the same initial condition but a larger input field intensity, the circular shape evolves quickly to an ellipse [see Figs. 2(a1) and 2(a2)]. This bright spot becomes unstable with respect to radially asymmetric distortions. This curvature instability arises in the vicinity of the boundary layer connecting the LS and the stable homogeneous background. As time increases, the elliptical spot grows by elongation and splits into two bright spots connected by a tube of lower amplitude [see Figs. 2(a3)]. This transient process of deformation, elongation, and splitting continues until the system reaches a stable hexagonal branch. This leads to the filling of the plane by a periodic pattern as illustrated on the left side of Fig. 2. We stress that the processes of deformation, splitting, and transition to the hexagonal static attractor are not specific to the zero atomic detuning ($\Delta = 0$). They can occur for a nonzero detuning. As an example, for $\Delta = 0.5$, $\theta = -2$, and $C = 20$, the curvature instability occurs at $E_i = 20.36$.

Equation (1) also supports single localized stripes (SLS). We first calculate the SLS solution by using a procedure similar to that outlined for the bright spots. It allows us to plot the maximum amplitude $|E_0(x)|_{x=0}$ as

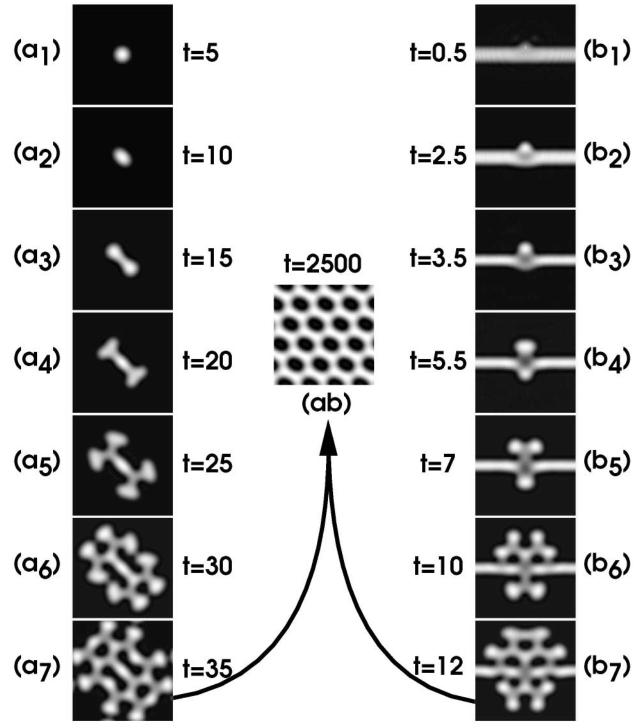

FIG. 2. Curvature instability affecting both circular and single stripe localized structures. Time sequence of the real part of the intracavity field. Maxima are plain white and the grid size is 256.

a function of input field amplitude (Fig. 3). To study the stability of these solutions we linearize Eq. (1) around the SLS solution $E_0(x)$. This yields a linear operator $\hat{L}$ which can be represented as $\hat{L} = \hat{L}_0 + \hat{\sigma}\partial^2/\partial y^2$. The linear operator $\hat{L}_0$ describes the stability of 1D LS $E(x) = E_0(x)$ and $\hat{\sigma} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Replacing $\partial^2/\partial y^2$ by $-k^2$ in $\hat{L}$, where k is the wave number of the perturbation along the y direction, we obtain the eigenvalue problem

$$\hat{L}(k)\vec{\psi}(k) = (\hat{L}_0 - k^2\hat{\sigma})\vec{\psi}(k) = \lambda(k)\vec{\psi}(k). \qquad (3)$$

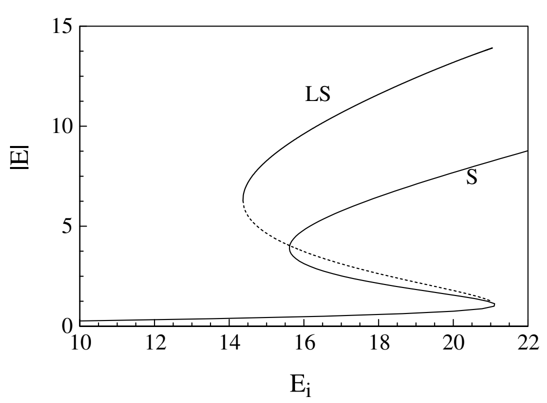

FIG. 3. 1D localized structures for $\theta = -2$ and $C = 20$. The branches of localized structures are labeled LS. The homogeneous response curve is labeled S.

The invariance of Eq. (1) with respect to the transformation $x \to x + c$ with arbitrary constant c implies that for $k = 0$ this equation has the eigenvalue $\lambda(0) = 0$ with eigenfunction $\vec{\psi}(0) = \vec{\psi}_0 = \partial_x[\mathrm{Re}E_0(x), \mathrm{Im}E_0(x)]^T$. It is the translational neutral mode of $\hat{L}_0$. To study the stability of the SLS, let us consider the eigenvalue $\lambda(k)$ which verifies $\lim_{k\to 0}\lambda(k) = 0$ and its eigenfunction $\lim_{k\to 0}\vec{\psi}(k) = \vec{\psi}_0$. The eigenvalue $\lambda(k)$ is negative (positive) for small nonzero k and therefore the SLS is stable (unstable) if $d^2\lambda(k)/dk^2|_{k=0}$ is negative (positive).

From Eq. (3), we get the following stability criterion for the SLS:

$$\left.\frac{d^2\lambda}{dk^2}\right|_{k=0} = -2\frac{\langle\vec{\psi}_0^\dagger, \hat{\sigma}\vec{\psi}_0\rangle}{\langle\vec{\psi}_0^\dagger, \vec{\psi}_0\rangle} < 0,$$

where $\vec{\psi}_0^\dagger$ is the eigenfunction of the adjoint operator $\hat{L}_0^\dagger$ with zero eigenvalue. For $\theta = -2$ and $C = 20$, which are the parameters chosen for all figures, the function $d^2\lambda/dk^2|_{k=0}$ vanishes at $E_i = E_{th} = 20.02$ and is monotonically increasing close to that point. This corresponds to the threshold associated with the curvature instability as shown in Fig. 4.

Numerical simulations of Eq. (1) have also been performed to check these analytical results. The initial condition consists of a single stripe along the x direction to which we add a pointlike perturbation along the y direction. The initial perturbation grows and the single stripe exhibits deformation and splitting along the y direction [see Figs. 2(b1), 2(b2), 2(b3), and 2(b4)]. Then a transition to π-hexagons occurs [see Figs. 2(b5), 2(b6), 2(b7), and 2(ab)]. The wavelength associated with the π-hexagon [Fig. 2(ab)] is essentially the same as that

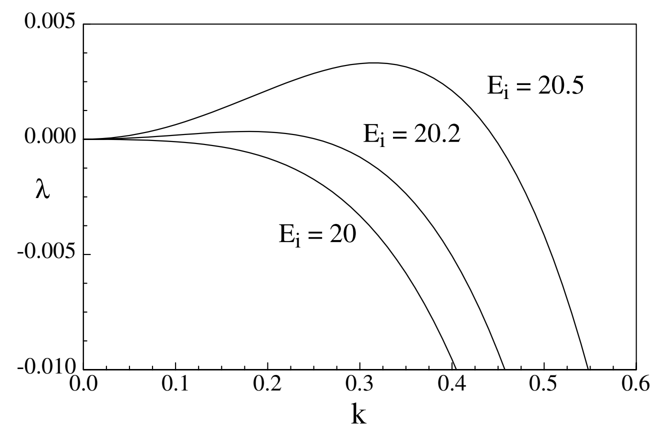

FIG. 4. Stability of 1D localized stripe along the x coordinate with respect to small perturbations acting on the y direction. For $k = 0$, the eigenvalue is always equal to zero. It corresponds to the neutral mode associated with the translational symmetry along the x axis. The single localized stripe is stable for $E_i = 20$, weakly unstable for $E_i = 20.2$, and unstable for $E_i = 20.5$. Parameters are $\theta = -2$ and $C = 20$.

calculated from the linear stability analysis of the homogeneous state: $\Lambda_T = 2\pi/k_T = 2\pi/\sqrt{-\theta} \approx 4.44$.

In summary, this analysis reveals the existence of curvature instability that prevents the stabilization of localized structures when the input field exceeds some threshold. These results are supported by analytical, though implicit, results. The results obtained from numerical simulations are in good agreement with the analytical results.

Fruitful discussions with R. Lefever are gratefully acknowledged. This research was supported in part by the Fonds National de la Recherche Scientifique (Belgium), by the InterUniversity Attraction Pole program of the Belgian government, and by the European Science Foundation. The financial support of Fonds Emile Defay is also acknowledged.

[1] W. Mc Laughlin, J.V. Moloney, and A.C. Newell, Phys. Rev. Lett. **51**, 75 (1983); N.N. Rosanov and G.V. Khodova, Opt. Sectrosc. **72**, 1394 (1992).

[2] M. Tlidi, P. Mandel, and R. Lefever, Phys. Rev. Lett. **73**, 640 (1994).

[3] M. Tlidi and P. Mandel, Chaos Solitons Fractals **4**, 1457 (1994); A.J. Scroggie, W.J. Firth, G.S. McDonald, M. Tlidi, R. Lefever, and L.A. Lugiato, Chaos Solitons Fractals **4**, 1323 (1994).

[4] L.A. Lugiato and R. Lefever, Phys. Rev. Lett. **58**, 2209 (1987).

[5] S. Trillo, M. Haelterman, and A.P. Sheppard, Opt. Lett. **22**, 970 (1997); K. Staliunas and V.J. Sánchez-Morcillo, Phys. Lett. A **241**, 28 (1998); M. Le Berre, E. Ressayre, and A. Tallet, J. Opt. B **1**, 153 (1999); G.L. Oppo, A.J. Scroggie, and W.J. Firth, J. Opt. B **1**, 133 (1999); M. Tlidi, M. Le Berre, E. Ressayre, A. Tallet, and L. Di Menza, Phys. Rev. A **61**, 043806 (2000).

[6] V.B. Taranenko, K. Staliunas, and C.O. Weiss, Phys. Rev. Lett. **81**, 2236 (1998).

[7] M. Le Berre *et al.*, Phys. Rev. A **56**, 3150 (1997); M. Tlidi, P. Mandel, and M. Haelterman, Phys. Rev. E **56**, 6524 (1997); M. Tlidi and P. Mandel, Phys. Rev. A **59**, R2575 (1999); D.V. Skryabin and W.J. Firth, Opt. Lett. **24**, 1056 (1999); D.V. Skryabin, Phys. Rev. E **60**, R3508 (1999).

[8] A.G. Vladimirov *et al.*, Quantum Electron. **27**, 949 (1997); **28**, 55 (1998); A.G. Vladimirov *et al.*, J. Opt. B **1**, 101 (1999).

[9] N.N. Rosanov, *Progress in Optics* (North-Holland, Amsterdam, 1996), Vol. 35; P. Mandel, *Theoretical Problems in Cavity Nonlinear Optics* (Cambridge University Press, Cambridge, 1997); L.A. Lugiato, M. Brambilla, and A. Gatti, *Optical Pattern Formation*, Advances in Atomic, Molecular and Optical Physics (Academic Press, Boston, 1998), Vol. 40, p. 229. F.T. Arecchi, S. Boccaletti, and P.L. Ramazza, Phys. Rep. **318**, 1 (1999).

[10] V.B. Taranenko, K. Staliunas, and C.O. Weiss, Phys. Rev. A **56**, 1582 (1997); G. Slekys, K. Staliunas, and C.O. Weiss, Opt. Commun. **149**, 113 (1998); C.O. Weiss *et al.*, Appl. Phys. B **68**, 151 (1999); M. Vaupel, A. Maître, and C. Fabre, Phys. Rev. Lett. **83**, 5278 (1999); B. Schäpers, M. Feldmann, T. Ackemann, and W. Lange, Phys. Rev. Lett. **85**, 748 (2000); V.B. Taranenko *et al.*, Phys. Rev. A **61**, 063818 (2000); V.B. Taranenko *et al.*, J. Opt. Soc. Am. B **19**, 684 (2002).

[11] P.L. Ramazza *et al.*, J. Opt. B **2**, 399 (2000); G. Slekys *et al.*, J. Opt. B **2**, 443 (2000).

[12] W.J. Firth and A.J. Scroggie, Phys. Rev. Lett. **76**, 1623 (1996); M. Brambilla, L.A. Lugiato, and M. Stefani, Europhys. Lett. **34**, 109 (1996).

[13] S. Longhi, Opt. Lett. **20**, 695 (1995); K. Staliunas and V.J. Sánchez-Morcillo, Opt. Commun. **139**, 306 (1996); S. Longhi, Phys. Scr. **56**, 611 (1997); B.A. Samson and M.A. Vorontsov, Phys. Rev. A **56**, 1621 (1997); M. Brambilla *et al.*, Phys. Rev. Lett. **79**, 2042 (1997); D. Michaelis *et al.*, Phys. Rev. A **56**, R3366 (1997); L. Spinelli *et al.*, Phys. Rev. A **58**, 2542 (1998); G. Tissoni *et al.*, J. Opt. Soc. Am. B **16**, 2083 (1999); **16** 2095 (1999); P. Lodahl, M. Bache, and M. Saffman, Phys. Rev. A **63**, 023815 (2001); R. Vilaseca *et al.*, Phys. Rev. Lett. **87**, 083902 (2001).

[14] W.W. Mullins and R.F. Sekerka, J. Appl. Phys. **34**, 323 (1963); **35**, 444 (1964).

[15] J.E. Pearson, Science **261**, 189 (1993); K.J. Lee *et al.*, Science **261**, 192 (1993); R. Goldstein *et al.*, Phys. Rev. E **53**, 3933 (1996); P.W. Davies *et al.*, J. Phys. Chem. A **102**, 8236 (1998).

[16] P.L. Ramazza (private communication).

[17] T. Maggipinto, M. Brambilla, G.K. Harkness, and W.J. Firth, Phys. Rev. E **62**, 8726 (2000).

[18] U. Peschel, D. Michaelis, C. Etrich, and F. Lederer, Phys. Rev. E **58**, R2745 (1998); R. Gallego, M. San Miguel, and R. Toral, Phys. Rev. E **61**, 2241 (2000).

[19] M. Tlidi and P. Mandel, Europhys. Lett. **44**, 449 (1998); M. Tlidi, P. Mandel, and R. Lefever, Phys. Rev. Lett. **81**, 979 (1998).

[20] W. Firth and A.J. Scroggie, Europhys. Lett. **26**, 521 (1994).