Body-centered cubic dissipative crystal formation in a dispersive and diffractive optical parametric oscillator

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We show that coupling diffraction and chromatic dispersion lead to body-centered cubic and hexagonally packed cylinders of dissipative optical crystals in a degenerate optical parametric oscillator. The stabilization of these crystals is a direct consequence of the interaction between the modulational and the quasi-neutral modes. © 2003 Optical Society of America

Frequency conversion by means of optical parametric conversion is a fundamental phenomenon for the generation of tunable coherent radiation in quadratic media. The optical parametric oscillator is a promising device in several applications, ranging from the generation of nonclassical states of light to low-noise measurements and detection to optical coherent information processing. On the one hand, in large-area devices the coupling of nonlinearity and diffraction triggers the spatial modulation instability, which leads to the spontaneous formation of two-dimensional (2D) periodic structures or spatial solitons

\[
\frac{\partial u_1}{\partial t} = -(1 - i \delta_1)u_1 + u_1^* u_2 + i \left( L_{\perp} \pm \frac{\delta_2}{\partial \tau^2} \right) u_1, \tag{1}
\]

\[
\frac{\partial u_2}{\partial t} = -\gamma \left[ (1 + i \delta_2)u_2 + u_1^2 - S \right] + i \left( a L_{\perp} \pm |\beta| \frac{\delta_2}{\partial \tau^2} \right) u_2, \tag{2}
\]

where \( u_{1,2} \) are the normalized slowly varying complex envelopes of the signal and the pump at frequencies \( \omega \) and \( 2\omega \), respectively. The time \( t \), which describes the slow evolution of the field envelopes, has been scaled so that the field decay rate is unity; \( \tau \) is the normalized time in a reference frame traveling at the group velocity of light in the \( \chi^{(2)} \) medium. The parameters \( \delta_{1,2} \) are the detuning at both frequencies, and \( \gamma \) is the ratio of the photon lifetimes at frequencies \( \omega \) and \( 2\omega \). Diffraction and dispersion appear when the operator \( a L_{\perp} \pm |\beta| \frac{\delta_2}{\partial \tau^2} \) acts on the Euclidean space \( \mathbf{r} = (x, y, \tau) \), where \( a \) and \( \beta \) are the ratios between the diffraction and dispersion coefficients of the pump and the signal fields, respectively. The phase-matching condition imposes that \( a = 1/2 \). The \( + \) or \( - \) sign in front of the second-order derivative with respect to \( \tau \) is the sign of the crystal dispersion.

Equations (1) and (2) admit two types of homogeneous steady states (HSSs): (i) the nonlasing state \( u_1 = 0 \) and \( u_2 = S/(1 + i \delta_2) \) and (ii) the lasing state \( u_1 \neq 0 \) satisfying \( S^2 = S_{\text{th}}^2 + 2(1 - \delta_1 \delta_2)I_1 + I_2^2 \) and \( I_2 = 1 + \delta_1^2 \), with \( S_{\text{th}} = \left[ (1 + \delta_1^2)(1 + \delta_2^2) \right]^{1/2} \) and \( I_{1,2} = |u_{1,2}|^2 \). The two solutions coincide at the lasing threshold \( S = S_{\text{th}} \). The nonlasing state undergoes a pattern-forming instability at \( S = M = (1 + \delta_2^2)^{1/2} \). At

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this bifurcation point the most unstable wave number is \( k_M = \sqrt{-\delta_1} \).

We focus on the situation that is usually realized experimentally, namely, the near-resonant signal regime that is obtained for a small detuning parameter \( \delta_3 \). In that regime the nonlinear dynamics of Eqs. (1) and (2) can be reduced to an order parameter equation for a real amplitude \( E = \Re(u_1) \), which is derived as a solvability condition through a standard weakly nonlinear analysis:

\[
\frac{\partial E}{\partial t} = \left( \frac{S^2}{1 + \delta_2} - 1 \right) E - \frac{2}{1 + \delta_2} E^3 - \left[ \delta_1 + \left( L_\perp \pm \frac{\partial^2}{\partial \tau^2} \right) \right]^2 E.
\] (3)

Equation (3) generalizes the Swift–Hohenberg model that has been derived in 2D diffraction systems. The HSSs of Eq. (3) are \( E^0 = 0 \) and \( E^\pm = [(S^2 - S_0^2)/2]^{1/2} \). These solutions coincide at the lasing threshold \( S = S_{th} \). We focus on the anomalous dispersion regime (positive dispersion). In that case the HSSs undergo 3D pattern-forming instabilities at \( S_{M1} = (1 + \delta_2)^{1/2} \) and \( S_{M2} = [(3\delta_2^2/2 + 1)(1 + \delta_2^2)]^{1/2} \). The unstable wave numbers at \( S_{M1,2} \) are the same and are given by \( k_M = \sqrt{-\delta_1} \). Near these bifurcation points the periodic solutions can be approximated by a linear superposition of pairs of wave vectors \( k_l \) with the same modulus \( |k_l| = k_M \) and the homogeneous quasi-neutral mode

\[
E(x, y, \tau, t) = E_0(t) + \sum_{l=1}^{N} [E_l(t) \exp(i(k_l \cdot r) + \text{c.c.})],
\] (4)

where c.c. denotes complex conjugate and \( r = (x, y, \tau) \). The 3D stripes and rhombic crystals are obtained for \( N = 1 \) and \( N = 2 \), respectively. The HPC corresponds to \( N = 3 \). The FCC and the quasi-periodic crystals are obtained for \( N = 4 \) and \( N = 5 \), respectively. The BCC crystals correspond to \( N = 6 \). Since we are interested in the 3D periodic structures, the quasi-periodic crystals (\( N = 5 \)) are not considered. The rhombic cells and the FCC structures are intrinsically unstable. Therefore they are not considered here.

To construct the 3D nonlinear solutions of Eq. (3), we use a truncated Fourier mode expansion of \( E \), including the homogeneous quasi-neutral mode. We adopt the method used in Ref. 23, which allows us to derive amplitude equations for 3D solutions corresponding to \( N = 1, 3, 6 \) and to assess their stability. The results of the nonlinear analysis are summarized in the 3D bifurcation diagram shown in Fig. 1, where we plot the maximum amplitude of the 3D structures corresponding to 3D stripes, HPCs, and BCC structures as a function of the input field amplitude. As can be seen, in the domain \( S_{M1} < S < S_H \) only 3D stripes are stable. The HPC structures become stable when \( S > S_H \) with \( S_H = [S_{th}^2 - 23\delta_2^2(1 + \delta_2^2)/74]^{1/2} \). From this expression we can see clearly that stable HPC structures appear before the lasing threshold, i.e., \( S < S_{th} \). As the input field amplitude is further increased, for \( S > S_B \), the BCC structures are stable. Note, however, that the HPCs and the BCC structures appear supercritically. This property contrasts strongly with those of previous studies on 3D crystal formation. The amplitude of the quasi-neutral modes corresponding to HPC and BCC is plotted in Fig. 2. They correspond to the average values of the field amplitude. The amplitude of the zero mode of 3D stripes coincides with the HSS \( E^0 = 0 \).

The analytical predictions obtained from the normal form analysis are confirmed and extended by numerical simulations of the full dynamical Eqs. (1) and (2). In agreement with the geometry of the ring resonator, we consider periodic boundary conditions in the three directions. Our results are illustrated in Fig. 3, where we can see, as predicted from the normal form...
Fig. 3. Isosurface corresponding to (a) the HPCs and (b) the BCC structures for the pump field amplitude. They are obtained by numerical simulations of the full model of Eqs. (1) and (2). Parameters are $\delta_1 = -\delta_2 = -0.5$, $\beta = 0.5$, and $\gamma = 1$. (a) $S = 1.22$. (b) $S = 1.3$. Mesh integration is $48 \times 48 \times 48$.

analysis, that the DOPO supports HPC and BCC structures, which are both stable.

In summary, we have described the formation of 3D dissipative crystals in a diffractive and dispersive degenerate optical parametric oscillator. We show that the existence and stability of the hexagonally packed cylinders and the body-centered cubic dissipative crystals strongly depend on the interaction between the 3D modulational and the quasi-neutral modes. Without this interaction these structures do not exist. Numerical simulations of the full model Eqs. (1) and (2) confirm the analytical predictions.

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