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Delta Conjectures and Theta refinements

Thesis presented by Anna Vanden Wyngaerd
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"Nothing is yours. It is to use. It is to share. If you will not share it, you cannot use it".

– Ursula K. Le Guin, *The Dispossessed*

I rather enjoyed writing this manuscript. I began in the spring of 2020, when the world was shook by the COVID-19 pandemic and millions were suddenly confined to their homes. For me, the writing of this thesis was a perfect quarantine occupation. This last phase of a four year undertaking felt almost contemplative, meditative.

Even though these pages were written in a literal quarantine, I was far from isolated during the years when its content was created. Many people contributed in one way or another to the successful completion of this project.

It is a simple truth that without Michele D'Adderio, I would not have written this thesis. I am not sure I would have even applied for the job if he had not convinced me. Once I got it, Michele was an extremely helpful, encouraging, available and patient supervisor. At the very beginning told me "writing a thesis is only a secondary effect of the actual goal, which is to turn you into an independent researcher". If this goal has not been achieved yet, it is through lack of effort on Michele's part. For all this, I thank him.

If Michele is my mathematical father figure, then Alessandro Iraci, a.k.a. Sasha, is my brother. It turns out that being stuck on a problem is more fun together than alone. I am glad that he endured the freezing cold of Belgian's warmer seasons to come work here.

I am grateful to the members of my jury and PhD committee – Dimitri Leemans, Joost Vercruysse, Jacopo Gandini and Anton Mellit – for taking the time to read and discuss my work.

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Introduction

The story of the Delta conjecture is one with three sides. It is a *combinatorial* formula for a certain *symmetric function*, that is intimately related to some *representation of the symmetric group*.

The ring of symmetric functions $\Lambda_{\mathbb{K}}$ over a field $\mathbb{K}$ is the ring of formal power series with coefficients in $\mathbb{K}$, of bounded degree in a countably infinite amount of variables, invariant by any permutation of those variables. This ring is naturally graded, i.e. $\Lambda_{\mathbb{K}} = \oplus_{n\in\mathbb{N}}\Lambda_{\mathbb{K}}^{(n)}$, where $\Lambda_{\mathbb{K}}^{(n)}$ is the subspace of homogeneous symmetric functions of degree n. The dimension of $\Lambda_{\mathbb{K}}^{(n)}$ is equal to the number of *partitions* λ of n, denoted $\lambda \vdash n$, which are vectors of positive integers whose entries are weakly decreasing and sum to n. There are many interesting bases of this space, indexed by partitions λ, like the elementary symmetric functions e_λ, the homogeneous symmetric functions h_λ, the power symmetric functions p_λ and the Schur function s s_λ.

Given the n-th symmetric group $\mathfrak{S}_n$, denote $\mathcal{C}(\mathfrak{S}_n)$ the space of its class functions (i.e. the space of functions $f : \mathfrak{S}_n \to \mathbb{C}$ that are constant on conjugacy classes). As two permutations are conjugate if and only if they have the same cycle type, there are exactly as many conjugacy classes of $\mathfrak{S}_n$ as there are partitions of n. For $\chi \in \mathcal{C}(\mathfrak{S}_n)$, denote by χ_λ the constant value of χ on the permutations of cycle type λ. The *Frobenius characteristic map* is

$$\begin{aligned} \mathcal{F} : \mathcal{C}(\mathfrak{S}_n) &\to \Lambda_{\mathbb{C}}^{(n)} \\ \chi &\mapsto \sum_{\lambda\vdash n} \frac{1}{z_\lambda}\chi_\lambda p_\lambda, \end{aligned}$$

where $z_\lambda \in \mathbb{N}$ is the size of the conjugacy class of cycle type λ. This map is an isomorphism of vector spaces (see [Sag01, Theorem 4.7.4]). It thus provides a remarkable correspondence between finite dimensional representations V of $\mathfrak{S}_n$ and symmetric functions. Indeed any such V is determined up to isomorphism by its character χ^V (see [Sag01, Corollary 1.9.4]), a class function of $\mathfrak{S}_n$, which bijectively corresponds to a symmetric function $\mathcal{F}(\chi^V)$. Frobenius showed, that under this correspondence, the characters of the irreducible representations of $\mathfrak{S}_n$ (of which there are as many as there are partitions of n) map exactly onto the Schur functions s_λ (see [Sag01, Theorem 4.6.4]). By Maschke's theorem (see [Sag01, 1.5.3]), all finite dimensional representations V of $\mathfrak{S}_n$ can be decomposed into a finite direct sum of irreducible representations. Since $\chi^{V\oplus W} = \chi^V + \chi^W$, this implies that χ is the character of a representation of $\mathfrak{S}_n$ if and only if $\mathcal{F}(\chi)$ is *Schur positive*, i.e. the coefficients of its expansion in the Schur basis are non-negative integers.

Thus, Frobenius provides a correspondence between finite dimensional representations of $\mathfrak{S}_n$ and Schur positive symmetric functions. The type of symmetric group representations V that interest us will be *bi-graded*, i.e. $V = \oplus_{i,j\in\mathbb{N}} V^{(i,j)}$ for some representations $V^{(i,j)}$ of $\mathfrak{S}_n$. The *bi-graded Frobenius characteristic* of the character[1] χ^V of such a representation is defined to be

$$\mathcal{F}_{q,t}(V) = F_{q,t}(\chi^V) \coloneqq \sum_{i,j\in\mathbb{N}} q^i t^j \mathcal{F}(\chi^{V^{(i,j)}}),$$

which is an element of $\Lambda^{(n)}_{\mathbb{C}(q,t)}$. The coefficients of its expansion in the Schur basis are elements of $\mathbb{N}[q,t]$, which gives an updated notion of Schur positivity in $\Lambda^{(n)}_{\mathbb{C}(q,t)}$.

Macdonald positivity and the $n!$ theorem

In 1988, Macdonald introduced a family of symmetric functions[2] (see [Mac88], [Mac95]), with coefficients in $\mathbb{Q}(q,t)$ that form a basis of $\Lambda_{\mathbb{Q}(q,t)}$. These polynomials have attracted a lot of attention over the years, for many reasons. First, they play a unifying role in symmetric function theory, as for suitable choices of q and t, Macdonald's polynomials specialise to a number of other important families of symmetric functions: Schur functions, Hall-Littlewood symmetric functions, Jack symmetric functions and zonal symmetric functions. Furthermore, there seem to be deep connections between Macdonald's theory and other fields including statistical physics, affine Hecke algebras, and Hilbert schemes (see for example [LV96], [Hai06] and [Hai99], respectively).

Also, Macdonald introduced a slightly modified version of his polynomials which he conjectured to be Schur positive. In [GH93], Garsia and Haiman defined a normalised version of Macdonald's modified polynomials, denoted H_λ in this text, and defined a family of bi-graded $\mathfrak{S}_n$-modules, $\mathcal{H}_\lambda$, the bi-graded Frobenius characteristic of which they conjectured to equal H_λ. Their conjecture resisted proof for more than a decade, during which time it was reduced to the question of showing that the dimension of $\mathcal{H}_\lambda$, for any λ, equals $n!$; this became known as the $n!$ conjecture. This conjecture finally became a theorem due to Haiman [Hai01], whose proof uses an algebraic theoretical approach, originally outlined by Procesi. With it the Schur positivity of the H_λ was established.

Garsia and Haiman's conjecture and the search for its proof revealed remarkable connections between Macdonald polynomial theory and representation theory of the symmetric group. For example, the same authors introduced the space of *diagonal harmonics* $\mathcal{DH}_n$, which essentially contains all the $\mathcal{H}_\lambda$ as subrings, and is defined as follows. Consider the ring $\mathcal{R}_n \coloneqq \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ and the *diagonal action* of $\mathfrak{S}_n$ defined as

$$\sigma \cdot f(x_1,\ldots,x_n,y_1,\ldots,y_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)},y_{\sigma(1)},\ldots,y_{\sigma(n)})$$

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[1] By slight abuse of terminology and notation, we will use "(bi-graded) Frobenius characteristic of a representation V" denoted $\mathcal{F}(V)$ or $\mathcal{F}_{q,t}(V)$ to mean the (bi-graded) Frobenius characteristic of its character, $\mathcal{F}(\chi^V)$ or $\mathcal{F}_{q,t}(\chi^V)$

[2] See [GR05] for a historical review of some notable results concerning Macdonald's polynomials.

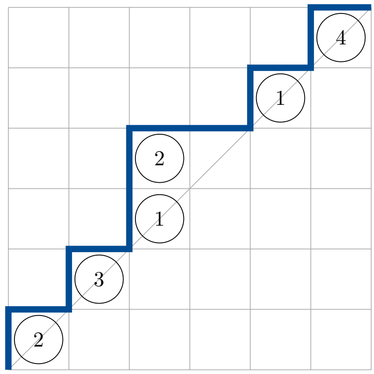


Figure 1: An element of $\mathsf{LD}(6)$.

for $f \in \mathcal{R}_n$ and $\sigma \in \mathfrak{S}_n$. Next set $\mathcal{I}_n$ to be the ideal of constant-free invariants with respect to this action. The diagonal harmonics are $\mathcal{DH}_n \coloneqq \mathcal{R}_n/\mathcal{I}_n$. The ring $\mathcal{R}_n$ is naturally bi-graded by the homogeneous bi-degree of the x and y variables. Since $\mathcal{I}_n$ is a homogeneous ideal, this bi-grading is inherited by $\mathcal{DH}_n$ and so we may construct its bi-graded Frobenius characteristic $\mathcal{F}_{q,t}(\mathcal{DH}_n)$, a Schur positive symmetric function. Bergeron an Garsia noticed that, up to simple multiplicative constant before each term, the Macdonald expansion of $\mathcal{F}_{q,t}(\mathcal{DH}_n)$ is very similar to the one of e_n. This inspired them to define the ∇ operator [BG99], as the linear operator satisfying $\nabla H_\lambda = T_\lambda H_\lambda$, for any partition λ, where T_λ is a simple constant in $\mathbb{N}[q,t]$ (see Definition 2.29). They conjectured that $\mathcal{F}_{q,t}(\mathcal{DH}_n) = \nabla e_n$, which Haiman showed to be a consequence his $n!$ theorem [Hai02].

The shuffle theorem

The third side of the story, the combinatorics, solidified when– building on Haiman's assertion that $\dim(\mathcal{DH}_n) = (n+1)^{n-1}$ [Hai02]– Haglund, Haiman, Loehr, Remmel and Ulyanov proposed a combinatorial formula for ∇e_n [HHL$^+$05] in terms of *labelled Dyck paths* of size (or the closely related *parking functions*, of which there are $(n+1)^{n-1}$). Their prediction became known as the *shuffle conjecture* [3] [4] and it reads

$$\nabla e_n = \sum_{P \in \mathsf{LD}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P;$$

where $\mathsf{LD}(n)$ is the set of labelled Dyck paths P of size n, dinv and area are statistics that encode some combinatorial information about such paths and x^P is a monomial naturally obtained from the labelling of P (see Chapter 4 for the precise definitions and Figure 1 for an illustration of a labelled Dyck path). Some special cases of this formula were already known at the time, most famously the q,t-Catalan positivity

[3] See Section 5.3 for an explanation of this term.
[4] See [vW20] for a very nice account of its history.

theorem, predicted by Haglund in [Hag03] and proved by himself and Garsia in [GH02]. Several years passed before Haglund, Morse and Zabrocki conjectured what they called a "compositional" refinement of the shuffle conjecture [HMZ12]. They introduced a family of symmetric functions C_α, indexed by *compositions* α of n (i.e. a vector of positive integers summing to n, denoted $\alpha \vDash n$) with the property that $\sum_{\alpha \vDash n} C_\alpha = e_n$. They then posited that ∇C_α equals the the same combinatorial formula as the shuffle conjecture above except that the sum is taken only over the paths with diagonal composition α (see Definition 4.16). It was this compositional formula that Carlsson and Mellit proved in [CM18], implying the shuffle theorem. Their proof is an impressive feat, introducing many new tools such as the Dyck path algebra and their raising and lowering operators. The publication of their paper marked the end of the very successful story of the shuffle theorem.

The Delta conjecture

While Carlsson and Mellit were working on their proof of the shuffle conjecture, Haglund, Remmel and Wilson formulated the Delta conjecture [HRW18]. The Delta operators, first introduced in [BGHT99], are are two families of closely related linear operators of $\Lambda_{\mathbb{Q}(q,t)}$ defined by

$$\Delta_f H_\lambda = f[B_\lambda] H_\lambda \qquad\qquad \Delta'_f H_\lambda = f[B_\lambda - 1] H_\lambda$$

for any $f \in \Lambda_{\mathbb{Q}(q,t)}$, where $f[B_\lambda]$ and $f[B_\lambda - 1]$ are some constants in $\mathbb{Q}(q,t)$ (see Section 1.5 and Definition 2.29). They generalise ∇ in the sense that on $\Lambda^{(n)}_{\mathbb{Q}(q,t)}$, we have $\Delta_{e_n} = \Delta'_{e_{n-1}} = \nabla$. The Delta conjecture is a pair of combinatorial formulas for the symmetric function $\Delta'_{e_{n-k-1}} e_n$, of the same general form as the shuffle theorem, except that the sum is over *decorated labelled Dyck paths* of size n with k *decorations* on *rises* (first formula) or *valleys* (second formula). These decorations have an impact on the area and dinv statistics (see Chapter 4 for the precise definitions). For $k = 0$, the Delta conjecture reduces to the shuffle theorem. The general case is still open today. In the Delta conjecture paper, the authors established the easy fact that the rise version of the combinatorics is a positive sum of *LLT-polynomials*, which were defined in [LLT97], and shown to be Schur positive in a preprint by Grojnowski and Haiman [GH07].

Theta operators

In [DIV20], we introduced a family of operators, Θ_k, (see Defintition 2.37) that satisfy $\Theta_k \nabla e_{n-k} = \Delta'_{e_{n-k-1}} e_n$ (see Theorem 3.36). There are many reasons why these operators are interesting, one of them being that $\Theta_k \nabla C_\alpha$, for α a composition of $n-k$, seems to be the appropriate symmetric function for a compositional version of the Delta conjecture (see Conjecture 5.10). This brings us one step closer to a potential generalisation of Carlsson and Mellit's proof to the Delta context.

For several years, there was no representation theoretic aspect to the Delta conjecture. That is, even though its truth would imply the Schur positivity of

$\Delta'_{e_{n-k-1}} e_n$ and thus we might construct a direct sum of irreducible representations of $\mathfrak{S}_n$ whose bi-graded Frobenius characteristic coincides with this symmetric function, there was no known "naturally occurring" module of which we could say the same. This changed when Zabrocki [Zab19] introduced the module of *super-diagonal coinvariants*. In [DIV20], we used Zabrocki's breakthrough to define a more general[5] module $\mathcal{M}_{n,2}$ which we describe here. Consider

$$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n, \eta_1, \dots, \eta_n]$$

where the θ and η variables are sets of n anti-commuting or *Grassmanian* variables[6]. As before, the diagonal action of $\mathfrak{S}_n$ permutes the 4 sets of variables simultaneously. Then $\mathcal{M}_{n,2}$ is the quotient of this ring by the constant-free invariants of this action. If $\mathcal{M}_{n,2}^{(k,l)}$ is the homogeneous subspace of degree k, l in the θ, η variables, respectively, then we conjectured that

$$\mathcal{F}(M_{n,2}^{(k,l)}) = \Theta_k \Theta_l \nabla e_{n-(k+l)}.$$

For $l = 0$, we recover Zabrocki's conjecture for $\Theta_k \nabla e_{n-k} = \Delta'_{e_{n-k-1}} e_n$. The fact that Zabrocki's conjecture seems to generalise so naturally using the Θ_k operators is another argument in favour of studying of these operators.

More Delta conjectures

The Delta conjecture described above is far from the only combinatorial formula for (seemingly) Schur positive symmetric functions that was formulated since and inspired by the shuffle theorem. For instance, the Delta conjecture paper [HRW18] containes another *generalised Delta conjecture*: a formula for $\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n$ in terms of *decorated partially labelled Dyck paths* (see Chapter 4).

What is more, developments remarkably similar to the shuffle and Delta conjecture, with combinatorics based on *square paths*[7] instead of Dyck paths and symmetric functions relating to p_n instead of e_n, started in 2007. In that year, Loehr and Warrington formulated their *square conjecture* [LW07], a formula for $\nabla (-1)^{n-1} p_n$ in terms of *labelled square paths* (see Chapter 4). It contained their *q, t-square theorem* [CL06] as a special case (which is to the square conjecture what the q, t-Catalan theorem is to the shuffle theorem). In [Ser17], Sergel proved that the shuffle theorem implies the square conjecture, which thus became a theorem. Her proof used a *schedule formula* (see section 7.1) for square paths, that allowed for an expression of the combinatorics of square paths in terms of Dyck paths. In [DIV19] and [IV20], we proposed a combinatorial formulas for

- $\frac{[n-k]_t}{[n]_t} (-1)^{n-1} \Delta_{e_{n-k}} p_n$ in terms of *rise decorated labelled square paths*,
- $\frac{[n-k]_q}{[n]_q} (-1)^{n-1} \Delta_{e_{n-k}} p_n$ in terms of *valley decorated labelled square paths*.

[5] Zabrocki's module $\mathcal{M}_{n,1}$ has only one set of Grassmanian variables

[6] We have $\theta_i \theta_j = -\theta_j \theta_i$, $\eta_i \eta_j = -\eta_j \eta_i$ for all $i, j \in \{1, \dots, n\}$, any other product of variables commutes.

[7] Square paths are lattice paths from $(0, 0)$ to (n, n) using unit north and east steps, ending with an east step.

- $(-1)^{n-1}\Theta_k \nabla p_{n-k}$ also in terms of *valley decorated labelled square paths*, be it a slightly modified version of them.

We refer to these formulas the *Delta square conjecture*. In the same papers, we formulate *generalised* versions of these conjectures: we predict that applying Δ_{h_m} to the symmetric function yields similar combinatorics but with *partial labellings*. In [IV20] we show, using the same strategy as Sergel for the square conjecture, that valley version of the generalised Delta conjecture implies the valley version of the generalised Delta square conjecture (modified version).

Content and organisation

This thesis focuses on the symmetric function and combinatorial aspects of the Delta and related conjectures.

- Chapter 1 and Chapter 2 set the stage by providing classical definitions and results related to symmetric function theory in general (Chapter 1) and Macdonald polynomials in particular (Chapter 2). Section 2.4 contains our definition of the Theta operators, which first appeared in [DIV20].
- Chapter 3 is dedicated to symmetric function identities. Sections 3.1 and 3.2 lay out relevant identities from the literature. Section 3.3 establishes a key summation formula which was originally proved in [DIV18]. Section 3.4 contains the proof of some key results concerning the Theta operators from [DIV20].
- Chapter 4 sets up the combinatorial definitions related to Dyck and square paths. Most of these are not original, except for some relating to our original conjectures (see the next chapter).
- Chapter 5 lists all the formulas (conjectural and otherwise) related to the Delta conjecture. Our contributions are: the generalised Delta square conjecture (both versions) and all the conjectures involving the Theta operators.
- In Chapter 6, we prove the *generalised shuffle theorem*, i.e. the interpretation of $\Delta_{h_m}\nabla e_n$ in terms of partially labelled Dyck paths. Actually we prove a refinement of this result called its *touching* refinement whose combinatorics specify how many times the Dyck paths touch the line $x = y$. We first put out this result in [DIV20].
- In Chapter 7 we show that the valley version of the touching generalised Delta conjecture implies our formula for $(-1)^{n-1}\Theta_k\nabla p_{n-k}$ in terms of partially labelled valley decorated square paths. We use the same general strategy that Sergel used to prove that the shuffle theorem implies the square theorem [Ser17]. As in Sergel's paper we formulate a schedule formula for our combinatorics, which was inspired by the one in [HS19]. Our formula however is for labelled paths and not for parking or *preference functions*, in other words we allow for repeated labels. So it also provides a new factorisation of all the previously discovered schedule formulas concerning square and Dyck paths. This conditional result, combined with the one in Chapter 6, gives

a formula for $\Delta_{h_m}(-1)^{n-1}p_n$, which we call the *generalised square theorem*. This chapter contains the results of [IV20].

- In Chapter 8 we extend the combinatorial framework of the proof of the compositional shuffle conjecture [CM18] to rise decorated Dyck paths. In particular, we prove an extension of the "main recursion" in their paper that relates decorated Dyck paths to the raising and lowering operators. In this way, we reduce the rise version of the compositional Delta conjecture to a conjectural identity of operators. The content of this Chapter appears in [DIV20].

Finally, we include some ideas for future research (page 121).

Chapter 1

Symmetric function theory: an introduction

We give an introduction to symmetric function theory, highly catered to the needs of this thesis. The main sources used are [Ber09], [Mac95],[Sag01],[Sta99], where the interested reader can find more details.

Consider a field $\mathbb{K}$ and let $X = (x_1, x_2, \dots)$ be an alphabet of a countably infinite amount of variables. Let $\mathbb{P}$ be the set of positive numbers (i.e. $\mathbb{N} \setminus \{0\}$) and $\mathbb{N}^{\mathbb{P}}$ the be functions $\mathbb{P} \to \mathbb{N}$, or equivalently the set of sequences $(\alpha_1, \alpha_2, \dots)$ with $\alpha_i \in \mathbb{N}$. The *support* of an integer sequence $\alpha \in \mathbb{N}^{\mathbb{P}}$ is the set of $i \in \mathbb{N}$ such that $\alpha_i \neq 0$. Set $x^\alpha := \prod_{i \in \mathbb{P}} x_i^{\alpha_i}$. Now define

$$\mathbb{K}[[X]] := \left\{ \sum_{\alpha \in \mathbb{N}^{\mathbb{P}}} c_\alpha x^\alpha \mid c_\alpha \in \mathbb{K}, \alpha \text{ has finite support} \right\}$$

the ring of formal power series with coefficients in $\mathbb{K}$ and variables X. Given a monomial $c_\alpha x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}} \in \mathbb{K}[[X]]$, its *degree* is $\alpha_{i_1} + \cdots + \alpha_{i_k}$. The degree of an arbitrary element of $\mathbb{K}[[X]]$ is the supremum of the degrees of its monomials, which might be infinite. We denote by $\mathbf{B}\mathbb{K}[[X]]$ the subring of $\mathbb{K}[[X]]$ of elements of finite (or bounded) degree.

Let $\mathfrak{S}_\infty$ be the group of bijections $\mathbb{P} \to \mathbb{P}$ with the composition operation. We define an action of $\mathfrak{S}_\infty$ on $\mathbf{B}\mathbb{K}[[X]]$ as the permutation of its variables: for $\sigma \in \mathfrak{S}_\infty$ and $f \in \mathbf{B}\mathbb{K}[[X]]$ define

$$\sigma \cdot f(x_1, x_2, \dots) := f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, \dots).$$

Definition 1.1. A *symmetric function* is an element $f \in \mathbf{B}\mathbb{K}[[X]]$ such that for all $\sigma \in \mathfrak{S}_\infty$, $\sigma \cdot f = f$. The set of symmetric functions with coefficients in $\mathbb{K}$ is denoted by $\Lambda_{\mathbb{K}}$.

In other words, a symmetric function is a polynomial series of bounded degree in an infinite number of variables, stable by any permutation of these variables.

Remark 1.2. The invariants of $\mathbf{B}\mathbb{K}[[X]]$ under $\mathfrak{S}_\infty$ (i.e. symmetric functions) are the same as the invariants under the subgroup of bijections $\mathbb{P} \to \mathbb{P}$ that fix all but a finite number of elements, denoted by $\mathfrak{S}_{(\infty)}$. Indeed, consider $\sum_{\alpha\in\mathbb{N}^{\mathbb{P}}} c_\alpha x^\alpha$ and element of $\mathbf{B}\mathbb{K}[[X]]$. It is invariant by $\mathfrak{S}_\infty$ if and only if $c_\alpha = c_{\sigma\cdot\alpha}$ for all $\sigma \in \mathfrak{S}_\infty$ where $\sigma\cdot(\alpha_1, \alpha_2, \ldots,) = (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots)$. Since the α have finite support, invariance by $\mathfrak{S}_{(\infty)}$ implies invariance by $\mathfrak{S}_\infty$. There is a natural isomorphism $\mathfrak{S}_{(\infty)} \simeq \cup_{n\in\mathbb{N}}\mathfrak{S}_n$ where an element in $\mathfrak{S}_n$ of the left hand side corresponds to the permutation of $\mathbb{P}$ fixing all integers strictly n.

The ring $\Lambda_{\mathbb{K}}$ has a natural grading.

Definition 1.3. An element $f \in \mathbf{B}\mathbb{K}[[X]]$ is called *homogeneous of degree* n if all its monomials have degree n. The set of symmetric functions with coefficients in $\mathbb{K}$, homogeneous of degree n is denoted $\Lambda_{\mathbb{K}}^{(n)}$.

It is clear that

$$\Lambda_{\mathbb{K}} = \bigoplus_{n\in\mathbb{N}} \Lambda_{\mathbb{K}}^{(n)}.$$

Definition 1.4. Given $k \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{P}^k$ such that $\lambda_1 \geq \cdots \geq \lambda_k$ we set

$$m_\lambda := \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k},$$

where the sum is over all $(i_1, \ldots, i_k) \in \mathbb{P}^k$ yielding distinct monomials $x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$. We call m_λ the *monomial symmetric function* associated to λ. We set $m_\emptyset = 1$.

Example. If $\lambda = (2, 1, 1)$ then

$$m_{2,1,1} = x_1^2 x_2 x_3 + x_2^2 x_1 x_3 + x_2^2 x_1 x_3 + x_1^2 x_2 x_4 + \cdots$$

It is not hard to see that the monomial symmetric functions are indeed symmetric. In fact, upon some reflection, one notices that the m_λ, indexed by $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{P}^k$ for some $k \in \mathbb{N}$ and such that $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$, are a linear basis for the vector space $\Lambda_{\mathbb{K}}^{(n)}$ over $\mathbb{K}$.

This suggests that the objects denoted here by λ, which are called *partitions*, will play a central role in the theory of symmetric functions. So we elaborate on these objects in the next section.

1.1 Partitions and tableaux

Definition 1.5. Let $n \in \mathbb{N}$. A *partition* of n is a vector $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integer entries such that $\lambda_1 \geq \cdots \geq \lambda_k$ and $|\lambda| := \sum_{i=1}^k \lambda_i = n$. We will use the notation $\lambda \vdash n$ for partitions of n.

The *parts* of λ are its components and its *length*, $\ell(\lambda)$ is its number of parts. The set of all partitions is denoted by Par and the set of partitions of n by $\mathsf{Par}(n)$. The size of the set $\mathsf{Par}(n)$ is referred to by $p(n)$.

The following is a related but distinct notion.

Definition 1.6. A *composition* of n is a vector $(\alpha_1, \ \ldots, \alpha_k)$ of positive integer entries such that $\sum_{i=1}^k \alpha_i = n$. We denote $\alpha \models n$ and set $\ell(\alpha) = k$ to be the *length* of the composition. The *size* n of a composition is denoted by $|\alpha|$.

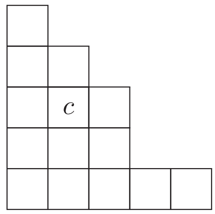


Figure 1.1: A Young diagram.

The set of partitions is a subset of the set of compositions.

Definition 1.7. Given α and β two composition, its *concatenation* is the composition $\alpha\beta := (\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \beta_1, \ldots, \beta_{\ell(\beta)})$.

Convention 1.8. There is exactly one composition of 0, the empty partition, denoted by $\lambda = \emptyset$.

Let us associate some pictures to these objects.

Definition 1.9. We associate a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ to the subset of $\mathbb{N} \times \mathbb{N}$

$$\begin{aligned} S_\lambda := &\{(0,0), \ldots, (\lambda_1 - 1, 0), (0,1), \ldots, (\lambda_2 - 1, 1), \\ &(0, k-1), \ldots, (\lambda_k - 1, k-1)\}. \end{aligned}$$

Now for each $(i,j) \in S_\lambda$, draw the square with vertices $(i,j), (i+1,j), (i,j+1)$ and $(i+1,j+1)$. We call such a square a *cell* or a square of λ, with coordinates (i,j). The resulting diagram is called the *Young diagram* of λ[1]. In other words the Young diagram of λ consisting of k rows of squares, where the i-th row from the bottom consists of λ_i squares and the rows are aligned to the left.

We will often identify a partition λ with the set S_λ or its Young diagram.

Example. The Young diagram of the partition $(5,3,3,2,1)$ is shown in Figure 1.1. The cell c has coordinates $(1,2)$.

Definition 1.10. Given the Young diagram of a partition λ, perform the orthogonal symmetry with respect to the line $x = y$. We obtain the Young diagram of the *conjugate partition* of λ, denoted by λ'.

Example. If $\lambda = (5,3,3,2,1)$ then Figure 1.2 is the Young diagram of its conjugate λ', so $\lambda' = (5,4,3,1,1)$.

Notation 1.11. We define a shorthand for some partitions that will turn up a lot.

- $(1^n) := \underbrace{(1, \ldots, 1)}_{n \text{ times}}$, which is called a *column partition* due to the shape of its Young diagram. For example, we draw the Young diagram of (1^3), see the left diagram of Figure 1.3 .

[1] We use the French convention, which uses of cartesian coordinates, as opposed to the English convention, which uses matrix-like coordinates.

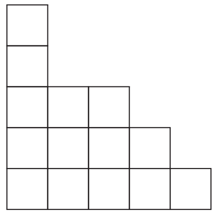


Figure 1.2: Conjugate of the diagram in Figure 1.1.

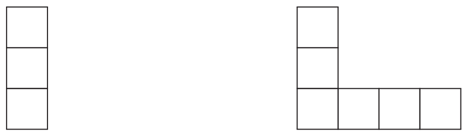


Figure 1.3: The row partition (1^3) (left) and hook partition $(4, 1^2)$ (right).

- $(n-k, 1^k) := (n-k, \underbrace{1, \ldots, 1}_{k \text{ times}})$, which is called a *hook shape* partition due to the shape of its Young diagram. For example, we draw here $(4, 1^2)$, see the right diagram of Figure 1.3. Be careful, when $k = n$ we get $(0, 1^n)$ which is *not a partition* and *not equal to* (1^n).

Partition constants

Given (a cell of) a partition, there are a number of constants in that will come up frequently.

Definition 1.12. Given a cell of the Young diagram of a partition λ we define its arm $a_\lambda(c)$, co-arm $a'_\lambda(c)$, leg[2] $l_\lambda(c)$ and co-leg $a'_\lambda(c)$ to be the number of cells that lie strictly to the east, west, north, and south of c, respectively.

Example. The cell labelled c in the partition $\lambda = (9, 8, 7, 7, 4, 4, 3, 2)$ in Figure 1.4 has $a_\lambda(c) = 4, a'_\lambda(c) = 2, l_\lambda(c) = 3$ and $l'_\lambda(c) = 3$.

Definitions 1.13. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition, then we set

- $n(\lambda) := \sum_{i=1}^{k} (i-1)\lambda_i$
- $m_i(\lambda) := |\{j \mid \lambda_j = i\}|$
- $z_\lambda := \prod_{i=1}^{k} i^{m_i(\lambda)} m_i(\lambda)!$.

Here are some observations about these quantities.

[2]This terminology makes sense for the upside down English convention.

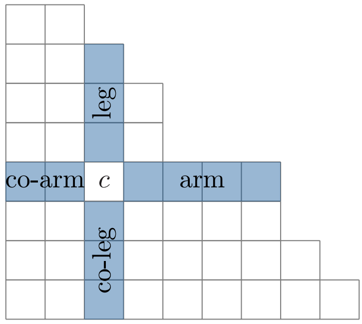


Figure 1.4: Limbs and co-limbs of a partition.

- The quantity z_λ is exactly the number of permutations of $\{1, \dots, n\}$ of cycle type λ.
- For all the λ_i cells in the i-th row of λ, the co-leg equals $i-1$. So we have

$$n(\lambda) = \sum_{c\in\lambda} l'_\lambda(c) = \sum_{c\in\lambda} l_\lambda(c), \tag{1.14}$$

  where the second equality is obtained from the obvious fact that for any column of λ the sum of the co-legs of its cells equals the sum of the legs of its cells.

Ordering partitions

Definition 1.15. We denote $\lambda \subseteq \mu$ if the Young diagram of λ is "contained" in the Young diagram of μ, in other words, $\lambda_i \leq \mu_i$ for all i. This defines a partial order on Par called the *containment order.*

Definition 1.16. We define the *dominance order* on $\mathsf{Par}(n)$ as follows: suppose λ, μ are partitions of n, then we denote $\lambda \preceq \mu$ if and only if for all $j \in \mathbb{P}$

$$\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$$

where we consider $\lambda_j = 0$ for $j > \ell(\lambda)$.

Remark 1.17. For any partition λ there are but a finite amount of partitions that are strictly smaller than λ, with respect to $\subseteq$ or $\preceq$. It follows that $(\mathsf{Par}, \subseteq)$ and $(\mathsf{Par}(n), \preceq)$ are *well founded*[3]posets and thus we may use inductive proofs of statements about these spaces.

We will need the following is an elementary fact about $\preceq$, see [Mac95, 1.11]

Proposition 1.18. *Suppose λ, μ are partition of n, then $\lambda \preceq \mu$ if and only if $\mu' \preceq \lambda'$.*

[3]A poset is said to be well founded if all of its subsets contain a minimal element

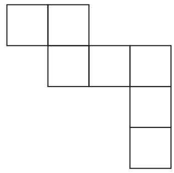


Figure 1.5: The Young diagram of the skew partition $(4,4,4,2)/(3,3,1)$.

Skew partitions

Definition 1.19. A *skew partition* λ/μ is a pair of partitions $\mu \subseteq \lambda$. Its *size* is $|\lambda| - |\mu|$.

The Young diagram $S_{\lambda/\mu}$ of a skew partition λ/μ is $S_\lambda \setminus S_\mu$. For example, we draw the diagram of $(4,4,4,2)/(3,3,1)$ in Figure 1.5. We identify a partition λ with the skew partition $\lambda/\emptyset$ so that the set of skew partitions contains the set of partitions.

Young tableaux

Related to partitions are the following objects.

Definition 1.20. Given a (skew) partition λ/μ, a *filling* of its Young diagram is a function $f : S_{\lambda/\mu} \to \mathbb{P}$, in other words for every cell c of λ a filling f picks a positive integer $f(c)$. For every cell c of $S_{\lambda/\mu}$ draw $f(c)$ inside the square corresponding to c. A filling is said to be

- *standard* if it is strictly increasing in rows (from left to right) and columns (from bottom to top).
- *semi-standard* if it is weakly increasing in rows and strictly increasing in columns.

A *Young tableau* or YT is a pair $(\lambda/\mu, f)$ where f is a filling of the Young diagram of λ/μ. A (semi)-standard Young tableau or (S)SYT is a YT whose filling is (semi)-standard.

The set of standard (respectively semi-standard) Young tableau with diagram, or *shape*, λ/μ is denoted $\mathsf{SYT}(\lambda/\mu)$ (respectively $\mathsf{SSYT}(\lambda/\mu)$).

Definition 1.21. The *content* of a YT is the vector of integers obtained by setting its i-th component to the number of i's in its filling.

Example. In Figure 1.6 are represented a standard Young tableau (left) and a semi-standard Young tableau (right). The content of the left tableau is $(0,2,0,2,0,3,2,3)$ and the content of the right one is $(1,1,3,0,2,2,0,2,1)$.

We introduce some very interesting numbers related to tableaux.

Definition 1.22. Given $\lambda, \mu \in \mathsf{Par}$ we define the *Kostka number*

$$K_{\lambda,\mu} = \#\text{ SSYT of shape } \lambda \text{ and content } \mu.$$

8			
6	8		
2	7	8	
	4	6	
	2	4	6	7

9			
8	8		
5	6	6	
	2	5	
	1	3	3	3

Figure 1.6: A standard Young tableau (left) and a semi-standard Young tableau.

4					
3	3	3			
2	2	2	2		
1	1	1	1	1	1

Figure 1.7: The only tableau of shape and content $(6, 4, 3, 1)$.

We make some observations about these numbers.

Proposition 1.23. *Let* $\lambda, \mu \in \mathsf{Par}$ *then*

(i) $K_{\lambda,\lambda} = 1$

(ii) $K_{\lambda,\mu} \neq 0$ *if and only if* $|\lambda| = |\mu|$ *and* $\mu \preceq \lambda$.

Proof. (i) The only tableau of shape *and* content λ is the tableau whose bottom row of cells are filled with 1's, the second to bottom row with 2's and so on. See Figure 1.7 for an illustration.

(ii) If $\lambda \prec \mu$ in the dominance order, there must exist a k such that $\mu_1+\cdots+\mu_k > \lambda_1 + \cdots + \lambda_k$. It follows that there must be a k in a row above the k-th row from the bottom of λ. This is in contradiction with the condition of strictly increasing labels in columns.

When $\mu \preceq \lambda$ then the filling of λ row by row from left to right bottom to top with μ_1 1's, μ_2 2's and so on, gives a SSYT of shape λ and filling μ.

□

1.2 Some standard bases and results

We turn our attention back to symmetric functions. We have already observed that $\{m_\lambda\}_{\lambda \vdash n}$ is a linear basis for the space $\Lambda_{\mathbb{K}}^{(n)}$, which is thus of dimension $p(n)$. There are several other notable basis.

Definition 1.24. The *n-th elementary symmetric function* is

$$e_n(X) := \sum_{1 \leq i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

for $n \in \mathbb{P}$ and $e_0 \coloneqq 1$.

Definition 1.25. The *n-th (complete) homogeneous symmetric function* is

$$h_n(X) \coloneqq \sum_{1 \le i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

for $n \in \mathbb{P}$ and $h_0 \coloneqq 1$.

This function is the sum of *all* monomials of degree n, hence the adjective "complete".

Definition 1.26. The *n-th power symmetric function* is

$$p_n(X) \coloneqq \sum_{i \in \mathbb{P}} x_i^n$$

for $n \in \mathbb{P}$ and $p_0 \coloneqq 1$.

Convention 1.27. We extend the definitions of these function to indices in $\mathbb{Z}$ by setting $e_n = h_n = p_n = 0$ for all $n < 0$. This convention will make for nicer formulas.

Example. For $n = 3$ we get

$$\begin{aligned}
e_3 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \cdots \\
h_3 &= x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_3^3 + \cdots \\
p_3 &= x_1^3 + x_2^3 + x_3^3 + x_4^3 + \cdots
\end{aligned}$$

We extend these definitions to partition indexes by applying the following multiplicative rule

$$\begin{aligned}
e_\lambda &\coloneqq e_{\lambda_1} \cdots e_{\lambda_k} \\
h_\lambda &\coloneqq h_{\lambda_1} \cdots h_{\lambda_k} \\
p_\lambda &\coloneqq p_{\lambda_1} \cdots p_{\lambda_k}
\end{aligned}$$

using the convention $e_\emptyset = h_\emptyset = p_\emptyset = 1$.

Let's introduce another, arguably the most important[4], family of symmetric functions.

Definition 1.28. Let $\lambda \in \mathsf{Par}$. The *Schur function* associated to λ is

$$s_\lambda(X) \coloneqq \sum_{T \in \mathsf{SSYT}(\lambda)} x^T,$$

where $x^T \coloneqq x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ where $\mu = (\mu_1, \ldots, \mu_k)$ is the content vector of T. Notice that $s_\emptyset = 1$.

[4]The Schur functions are central to the theory since they are the image of the irreducible representations of the symmetric group by the Frobenius characteristic map.

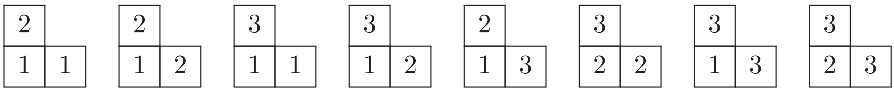


Figure 1.8: SSYT of shape $(2,1)$.

Remark 1.29. We may define in an analogous manner the notion of *skew Schur function*, $s_{\lambda/\mu}$ where the sum is taken over $\mathsf{SSYT}(\lambda/\mu)$, for λ/μ a skew partition. For $\mu \supset \lambda$, we define $s_{\lambda/\mu} = 0$.

Example. We have

$$s_{(2,1)}(X) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + \cdots$$

We draw in Figure 1.8 the semi-standard Young tableau corresponding to the monomials specified above.

It is easy to see that for all $n \in \mathbb{N}$

$$s_{(n)} = h_n \qquad\qquad s_{(1^n)} = e_n. \tag{1.30}$$

Contrary to the other families we introduced, it is not immediately clear from their definition that Schur functions are symmetric.

Proposition 1.31. *Schur functions are symmetric functions.*

Proof. We will describe a combinatorial involution

$$\alpha_i : \mathsf{SSYT}(\lambda) \to \mathsf{SSYT}(\lambda)$$

that maps a tableau T of content $\mu = (\mu_1, \ldots, \mu_k)$ to a tableau $\alpha_i(T)$ of content $\tilde{\mu} := (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \mu_i, \mu_{i+2}, \ldots, \mu_k)$. From the existence of this involution it will follow that

$$s_\lambda(X) = \sum_{T \in \mathsf{SSYT}(\lambda)} x^T = \sum_{\alpha_i(T) \in \mathsf{SSYT}(\lambda)} x^T = \tau_{i,i+1} \cdot s_\lambda.$$

Where $\tau_{i,i+1}$ is the consecutive transposition $(i, i+1)$. Since the consecutive transpositions generate $\mathfrak{S}_{(\infty)}$, this will imply that s_λ is symmetric (see Remark 1.2).

Let us now describe this involution. Take T a SSYT. Look for all the occurrences of i and $i+1$ in the filling of T. If i and $i+1$ occur as a pair in the same column, they are fixed by α_i. For the remaining occurrences, call them *free* occurrences, we can safely switch i and $i+1$ without breeching the condition on the columns of SSYT. We proceed as follows. For each row, if there are a free occurrences of i and b free occurrences of $i+1$, we replace them with from left to right with b occurrences of i and a occurrences of $i+1$. This operation will yield a SSYT and the multiplicities of i and $i+1$ are switched, indeed we made it so for the free occurrences and the fixed ones occur in pairs. It is clear that this operation is an involution. See Figure 1.9 for an example. $\square$

4	4							
2	2	3	3	3	4	4		
1	1	1	2	2	2	3	3	3

4	4							
2	3	3	3	3	4	4		
1	1	1	2	2	2	2	2	3

Figure 1.9: A SSYT (left) and its image by α_2 (right)

Remark 1.32. This argument also holds for skew shapes, thus skew Schur functions are likewise symmetric.

Corollary 1.33. *We have the following identity*

$$s_\lambda = \sum_{\mu \in \mathsf{Par}} K_{\lambda,\mu} m_\mu = \sum_{\mu \preceq \lambda} K_{\lambda,\mu} m_\mu$$

Proof. Take $\lambda, \mu = (\mu_1, \dots, \mu_k) \in \mathsf{Par}$. By definition of s_λ and $K_{\lambda,\mu}$ the coefficient of the monomial $x_1^{\mu_1} \cdots x_k^{\mu_k}$ in s_λ equals $K_{\lambda,\mu}$. Since by Proposition 1.31 s_λ is symmetric and all monomials in m_μ are obtained by acting on $x_1^{\mu_1} \cdots x_k^{\mu_k}$ with some element of $\mathfrak{S}_k$, we must have the first equality. The second equality follows from Proposition 1.23. □

Remark 1.34. Since the inverse of a lower-triangular matrix is lower-triangular, the fact that s_λ can be expanded in terms of $\{m_\mu\}_{\mu \preceq \lambda}$ implies that m_λ can be expanded in terms of $\{s_\mu\}_{\mu \preceq \lambda}$.

Example. We have

$$s_{3,2,1} = 16 m_{1,1,1,1,1,1} + 8 m_{2,1,1,1,1} + 4 m_{2,2,1,1} + 2 m_{2,2,2} + 2 m_{3,1,1,1} + m_{3,2,1}$$

As suggested by the title of this section, all the families we have encountered are of interest because they form basis of the space of symmetric functions.

Theorem 1.35. *For $n \in \mathbb{N}$*

(i) $\{e_\lambda\}_{\lambda \vdash n}$ *is a linear basis of* $\Lambda^{(n)}$*;*

(ii) $\{h_\lambda\}_{\lambda \vdash n}$ *is a linear basis of* $\Lambda^{(n)}$*;*

(iii) $\{p_\lambda\}_{\lambda \vdash n}$ *is a linear basis of* $\Lambda^{(n)}$*;*

(iv) $\{s_\lambda\}_{\lambda \vdash n}$ *is a linear basis of* $\Lambda^{(n)}$*.*

Proof. For the a proof of the statements (i), (ii) and (iii) we refer to [Sag01, Theorem 4.3.7].

Point (iv) can be deduced easily from Corollary 1.33 and Proposition 1.23. Indeed, the former suggests a matrix identity expressing the Schur functions in terms of the monomial basis (use the dominance order on partitions) and the latter implies that the matrix in question is unitriangular, and is thus invertible. □

Corollary 1.36. *The algebra* $\Lambda_{\mathbb{K}}$ *is minimally generated by* $\{e_i\}_{i \in \mathbb{N}}, \{h_i\}_{i \in \mathbb{N}}$ *and* $\{p_i\}_{i \in \mathbb{N}}$*.*

Generating functions

Upon some reflexion on the expansions of the right hand sides, we see that we have the following generating functions

$$E(\zeta) := \sum_{n\in\mathbb{N}} e_n\zeta^n = \prod_{i\in\mathbb{P}}(1+x_i\zeta) \tag{1.37}$$

$$H(\zeta) := \sum_{n\in\mathbb{N}} h_n\zeta^n = \prod_{i\in\mathbb{P}}\frac{1}{1-x_i\zeta} \tag{1.38}$$

$$P(\zeta) := \sum_{n\in\mathbb{N}} p_n\zeta^n = \sum_{i\in\mathbb{P}}\frac{1}{1-x_i\zeta}. \tag{1.39}$$

These formal power series can lead to elementary proofs of interesting identities.

Theorem 1.40. *We have the following expansions*

$$h_n = \sum_{\lambda\vdash n}\frac{p_\mu}{z_\mu} \tag{1.41}$$

$$e_n = \sum_{\lambda\vdash n}(-1)^{n-\ell(\mu)}\frac{p_\mu}{z_\mu}.; \tag{1.42}$$

Proof. We prove (1.41) here, (1.42) is obtained using similar techniques. Using the Taylor expansions of $\exp(x)$ and $\ln\left(\frac{1}{1-x}\right)$ and some formal power series magic (see [Wil94] for more details on why and how these manipulations work) we obtain the string of identities

$$\begin{aligned} H(\zeta) &= \exp\left(\ln\left(\prod_{i\in\mathbb{P}}\frac{1}{1-x_i\zeta}\right)\right) = \exp\left(\sum_{i\in\mathbb{P}}\ln\left(\frac{1}{1-x_i\zeta}\right)\right) \\ &= \exp\left(\sum_{i\in\mathbb{P}}\sum_{k\in\mathbb{P}}\frac{(x_i\zeta)^k}{k}\right) = \exp\left(\sum_{k\in\mathbb{P}}\frac{\zeta^k}{k}\left(\sum_{i\in\mathbb{P}}x_i^k\right)\right) \\ &= \exp\left(\sum_{k\in\mathbb{P}}\frac{p_k}{k}\zeta^k\right) = \prod_{k\in\mathbb{P}}\exp\left(\frac{p_k}{k}\zeta^k\right) = \prod_{k\in\mathbb{P}}\left(\sum_{l\in\mathbb{N}}\frac{p_k^l}{k^l}\frac{\zeta^{kl}}{l!}\right) \\ &= \left(1+p_1\zeta+\frac{p_1^2}{2!}\zeta^2+\cdots\right)\left(1+\frac{p_2}{2}\zeta^2+\frac{p_2^2}{2^2 2!}\zeta^{2\cdot 2}+\cdots\right)\cdots \end{aligned}$$

The coefficient of ζ^n on the left hand side is h_n by definition. On the right hand side a term of the series is obtained by multiplying a term from each parenthesis. Say we pick the n_i-th term from the i-th parenthesis. The resulting term equals

$$\prod_i \frac{p_i}{i^{n_i}n_i!}\zeta^{\sum_i n_i\cdot i}.$$

Let λ be the unique partition such that $m_i(\lambda) = n_i$ and we get that the coefficient of ζ^n on the right hand side is $\sum_{\lambda\vdash n}\frac{p_\lambda}{z_\lambda}$. □

Jacobi-Trudi identities

The expansion of the Schur functions in terms of complete homogeneous or elementary functions, is given by a pair of elegant determinant formulas.

Theorem 1.43 (Jacobi-Trudi identity)**.** *For any partition λ we have the two equivalent identities*

$$s_\lambda = \det\left((h_{\lambda_i+j-i})_{i,j=1}^{\ell(\lambda)} \right) \qquad\qquad s_{\lambda'} = \det\left((e_{\lambda_i+j-i})_{i,j=1}^{\ell(\lambda)} \right)$$

The fact that one is equivalent to the other is easily deduced by applying the ω-involution, see Section 1.4.

Pieri rules

Definition 1.44. A skew partition is said to be a *horizontal strip* if it does not contain a $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and a *vertical strip* if it does not contain a $\square\square$.

We state here the Pieri rules (see for example [Mac95, I.5.16 and 17]), which are are special cases of the classical Littlewood-Richardson rule (see for example [Sag01, Theorem 4.9.4]).

Proposition 1.45 (Pieri rules)**.** *Let $\lambda \in \mathsf{Par}$ and $n \in \mathbb{N}$*

$$h_n s_\lambda = \sum_\mu s_\mu \qquad\qquad e_n s_\lambda = \sum_\nu s_\nu$$

where the first sum is over all $\mu \supseteq \lambda$ such that μ/λ is a horizontal strip of size n and the second over all $\nu \supseteq \lambda$ such that μ/λ is a vertical strip of size n.

Murnaghan-Nakayama rule

Definitions 1.46. A skew partition λ/μ is said to be *connected* all of its squares share at least one edge with at least one other square. A *border strip* (some authors use *rim hook* or *ribbon*) is a connected skew partition that does not contain a $\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$. The *height* $ht(B)$ of a border strip B is its number of rows minus one.

The following is a special case of the more general result known as the Murnaghan-Nakayama rule. We refer to [Sta99, Theorem 7.17.1].

Theorem 1.47. *For $\lambda \in \mathsf{Par}$ and $n \in \mathbb{N}$*

$$s_\lambda p_n = \sum_\mu (-1)^{ht(\mu/\lambda)} s_\mu$$

where the sum is over all partitions $\mu \supseteq \lambda$ such that μ/λ is a border strip of size n.

1.3 Hall scalar product

We may further develop the structure of the symmetric function ring by defining a scalar product. We do this by specifying its effect on the power symmetric function basis.

Definition 1.48. Define the *Hall scalar product* on $\Lambda_{\mathbb{K}}$ by setting

$$\langle p_\lambda, p_\mu \rangle := z_\lambda \delta_{\lambda,\mu}$$

where $\delta_{a,b}$ is the widely used *Kronecker delta*, equal to 1 when $a = b$ and 0 when $a \neq b$.

Duality

Armed with a scalar product, certain basis of $\Lambda_{\mathbb{K}}$ are in special relation to each other.

Definition 1.49. A pair of basis $\{u_\lambda\}_{\lambda \vdash n}$, $\{v_\lambda\}_{\lambda \vdash n}$ of $\Lambda_{\mathbb{K}}^{(n)}$ is called *dual*, if for all $\lambda, \mu \vdash n$ we have $\langle u_\lambda, v_\lambda \rangle = \delta_{\lambda,\mu}$.

The defining feature of the Hall scalar product is that $\{p_\lambda\}_{\lambda \vdash n}$ and $\left\{\frac{p_\lambda}{z_\lambda}\right\}_{\lambda \vdash n}$ are dual basis of $\Lambda_{\mathbb{K}}^{(n)}$. We have some nice result characterising dual basis. For this, we will need two sets of variables $X := (x_1, x_2, \dots)$ and $Y := (y_1, y_2, \dots)$.

Proposition 1.50. *A pair of basis $\{u_\lambda\}_{\lambda \vdash n}$, $\{v_\lambda\}_{\lambda \vdash n}$ of $\Lambda_{\mathbb{K}}^{(n)}$ is dual if and only if*

$$\sum_{\lambda \vdash n} u_\lambda(X) v_\lambda(Y) = \sum_{\lambda \vdash n} \frac{p_\lambda(X) p_\lambda(Y)}{z_\lambda}.$$

Proof. Let A and B be the matrices of size $|\mathsf{Par}(n)| \times |\mathsf{Par}(n)|$ and coefficients in $\mathbb{K}$ defined by

$$u_\lambda = \sum_{\alpha \vdash n} A_{\alpha,\lambda} p_\alpha \qquad\qquad v_\mu = \sum_{\beta \vdash n} B_{\beta,\mu} \frac{p_\beta}{z_\beta}.$$

It follows that

$$\begin{aligned} \langle u_\lambda, v_\mu \rangle &= \sum_{\alpha \vdash n} \sum_{\beta \vdash n} A_{\alpha,\lambda} B_{\beta,\mu} \left\langle p_\alpha, \frac{p_\beta}{\beta} \right\rangle \\ &= \sum_{\alpha \vdash n} A_{\alpha,\lambda} B_{\alpha,\mu} \\ &= (B^T A)_{\mu,\lambda} \end{aligned}$$

It follows that $\{u_\lambda\}_{\lambda \vdash n}$ and $\{v_\lambda\}_{\lambda \vdash n}$ are dual if and only if $B^T A = Id$.

On the other hand we have

$$\begin{aligned}\sum_{\lambda\vdash n} u_\lambda(X)v_\lambda(Y) &= \sum_{\lambda\vdash n}\sum_{\alpha\vdash n}\sum_{\beta\vdash n} A_{\alpha,\lambda}B_{\beta,\lambda}\frac{p_\alpha(X)p_\beta(Y)}{z_\beta}\\ &= \sum_{\alpha\vdash n}\sum_{\beta\vdash n}\left(\sum_{\lambda\vdash n} A_{\alpha,\lambda}B_{\beta,\lambda}\right)\frac{p_\alpha(X)p_\beta(Y)}{z_\beta}\\ &= \sum_{\alpha\vdash n}\sum_{\beta\vdash n}(B^TA)_{\beta,\alpha}\frac{p_\alpha(X)p_\beta(Y)}{z_\beta}.\end{aligned}$$

Thus, the equality in the thesis holds if and only if $B^TA = Id$. This obviously implies that the two statements are equivalent. □

It turns out that there is a very nice factorisation of this common product of formula of dual basis.

Proposition 1.51. *We have the following formal power series equality*

$$\sum_{\lambda\in\mathsf{Par}}\frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda} = \prod_{i,j\in\mathbb{P}}\frac{1}{1-x_iy_j}.$$

Proof. Let's play with the formal power series:

$$\begin{aligned}\prod_{i,j\in\mathbb{P}}\frac{1}{1-x_iy_j} &= \prod_{i,j\in\mathbb{P}}\exp(-\ln(1-x_iy_i)) = \prod_{i,j\in\mathbb{P}}\exp\left(\sum_{k\in\mathbb{P}}\frac{x_iy_j}{k}\right)\\ &= \exp\left(\sum_{k,i,j\in\mathbb{P}}\frac{x_iy_j}{k}\right) = \exp\left(\sum_{k\in\mathbb{P}}\frac{p_k(X)p_k(Y)}{k}\right)\\ &= \sum_{n\in\mathbb{N}}\frac{1}{n!}\left(\sum_{k\in\mathbb{P}}\frac{p_k(X)p_k(Y)}{k}\right)^n\\ &= \sum_{n\in\mathbb{N}}\frac{1}{n!}\sum_{|\alpha|=n}\binom{n}{\alpha_1,\ldots,\alpha_\ell}\prod_{k=0}^{l}\left(\frac{p_k(X)p_k(Y)}{k}\right)^{\alpha_k}\\ &= \sum_{\lambda\in\mathsf{Par}}\frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda}.\end{aligned}$$

This last step was obtained by collecting all the compositions α whose rearrangement give the same partition λ. □

Let us summarise the duality results on the standard basis.[5]

Theorem 1.52. *The following pairs of basis are dual.*

[5]It might seem that we are *forgetting* a duality result of standard basis. Indeed we have not mentioned the dual basis of the elementary symmetric functions. Of course such a basis exists and is known as the *forgotten basis* of symmetric functions and are usually denoted by f_λ. We have not mentioned this basis since we have no need of it.

(i) $\{p_\lambda\}_{\lambda\vdash n}$ *and* $\left\{\frac{p_\lambda}{z_\lambda}\right\}_{\lambda\vdash n}$*;*

(ii) $\{h_\lambda\}_{\lambda\vdash n}$ *and* $\{m_\lambda\}_{\lambda\vdash n}$*;*

(iii) $\{s_\lambda\}_{\lambda\vdash n}$ *is dual to itself.*

Proof. (i) Trivial by definition.

(ii) Replacing ζ by y_j in the definition of the generating series for the homogeneous symmetric functions (1.38), we get

$$\sum_{n\in\mathbb{N}} h_n(X)y_j^n = \prod_{i\in\mathbb{P}} \frac{1}{1-x_iy_j}.$$

Taking the product over $j\in\mathbb{P}$, we obtain

$$\prod_{j\in\mathbb{P}}\sum_{n\in\mathbb{N}} h_n(X)y_j^n = \prod_{i,j\in\mathbb{P}} \frac{1}{1-x_iy_j}.$$

Notice that the right hand side of this equation is symmetric in the Y variables. We can thus expand the right hand side in terms of the monomial basis in the Y variables. It is not hard to see that the coefficient of $m_\lambda(Y)$ must be $h_\lambda(X)$. Thus, we obtain

$$\sum_{\lambda\in\mathsf{Par}} h_\lambda(X)m_\lambda(Y) = \prod_{i,j\in\mathbb{P}} \frac{1}{1-x_iy_j},$$

which by Proposition 1.50 and Proposition 1.51 implies that the monomial and homogeneous symmetric functions are dual to each other.

(iii) We do not prove it here. See for example [MR15, Theorem 5.6] for a nice combinatorial proof using the RSK algorithm.

□

A defining property of Schur basis

The orthogonality of the Schur basis turns out to be one of a pair of properties that uniquely defines them.

Proposition 1.53. *For any partition λ, the Schur function s_λ are the unique symmetric function satisfying*

(I) $\langle s_\lambda, s_\mu\rangle = 0$ *whenever* $\lambda\neq\mu$*;*

(II) $s_\lambda = m_\lambda + \sum_{\mu\prec\lambda}\square_{\lambda\mu}m_\mu$

where the $\square_{\lambda\mu}$ are some elements in $\mathbb{K}$.

Proof. We have already discussed the fact that Schur functions satisfy these two properties, see Theorem 1.52 and Corollary 1.33. What we must show is that these two properties in fact determine them. We will use an induction argument on $\preceq$ (see Remark 1.17).

Set $n = |\lambda|$. When we take the sum over $\mu \prec \lambda$, this implicitly supposes $\mu \vdash n$, since $\preceq$ is a partial order on $\mathsf{Par}(n)$. For the base case of the induction, we take $\lambda = (1^n)$ the minimal element of $(\mathsf{Par}(n), \preceq)$. Since there are no $\mu \vdash n$ such that $\mu \prec (1^n)$, the unique symmetric function satisfying (II) is $m_{(1^n)} = e_n = s_{(1^n)}$.

Now suppose that for all $\mu \prec \lambda$ we have shown that s_μ is the unique function satisfying the two properties. Using the fact that s_λ satisfies (II), we may write

$$s_\lambda = m_\lambda + \sum_{\mu \prec \lambda} \square_{\lambda\mu} m_\mu$$

for some $\square_{\lambda\mu} \in \mathbb{K}$. Using the induction hypothesis, we may expand

$$m_\mu = s_\mu - \sum_{\nu \prec \mu} \square_{\mu,\nu} m_\nu \qquad \text{for all } \mu \prec \lambda.$$

Iterating this process, we may express m_μ in terms of $\{s_\mu\}_{\mu \prec \lambda}$ (indeed after a finite number of steps the only term in the sum will be $m_{(1^n)} = s_{(1^n)}$). In other words, there exist coefficients $\blacksquare_{\lambda,\mu}$ such that

$$s_\lambda = m_\lambda + \sum_{\mu \prec \lambda} \blacksquare_{\lambda\mu} s_\mu.$$

Applying $\langle \cdot, s_\mu \rangle$ to this equation, with $\mu \prec \lambda$ and using (I) we get

$$\begin{aligned} 0 &= \langle m_\lambda, s_\mu \rangle + \blacksquare_{\lambda,\mu} \langle s_\mu, s_\mu \rangle \\ \Leftrightarrow \quad \blacksquare_{\lambda,\mu} &= -\frac{\langle m_\lambda, s_\mu \rangle}{\langle s_\mu, s_\mu \rangle}. \end{aligned}$$

(we know that $\langle s_\mu, s_\mu \rangle = 1$ but we do not actually need this fact to prove the result, we just need it to be $\neq 0$, which follows from the fact that the Hall scalar product is positive definite). Thus, the set $\{s_\mu\}_{\mu \prec \lambda}$ uniquely determines s_λ. □

Perp operator

Definition 1.54. Given $f \in \Lambda_\mathbb{K}$, we define $f^\perp$, pronounced f *perp*, as the adjoint operator of the multiplication. In other words, for all $g, h \in \Lambda_\mathbb{K}$

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle$$

This operator often has a nice combinatorial interpretation. The following result is an example of such.

Proposition 1.55. *For any* $\lambda \in \mathsf{Par}$ *and* $k \in \mathbb{N}$ *with* $k \leq \lambda_1$ *,*

$$s_{\lambda/(k)} = h_k^\perp s_\lambda.$$

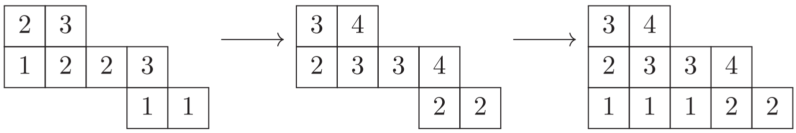
Proof. Take μ any partition and consider

$$\langle h_k^\perp s_\lambda, h_\mu \rangle = \langle s_\lambda, h_k h_\mu \rangle = \langle s_\lambda, h_\nu \rangle;$$

where ν is the partition obtained from μ by inserting k in the appropriate spot. Since the homogeneous and monomial symmetric functions are dual to each other (Proposition 1.50), $\langle s_\lambda, h_\nu \rangle$ is the coefficient of m_ν, which is the number SSYT of shape λ and content ν, which in turn –by symmetry– must be equal to the number of SSYT of shape λ and content $(k, \mu_1, \dots, \mu_{\ell(\mu)})$.

On the other hand, since $k \leq \lambda_1$, we have $(k) \subseteq \lambda$ and so we may consider $s_{\lambda/(k)}$. By similar reasoning $\langle s_{\lambda/(k)}, h_\mu \rangle$ is the number of SSYT of shape $\lambda/(k)$ and filling μ.

We can easily construct a bijection between the set of SSYT of shape λ and content (k, μ) and the set of SSYT of shape $\lambda/(k)$. Indeed, given a SSYT of the second kind, augment by 1 all the fillings. We obtain a SYT of shape $\lambda/(k)$ and content $(0, \mu)$. Complete this tableau by adding the k missing boxes and filling them with 1's and we obtain a SSYT of shape λ and content (k, μ). The inverse map is easily divined and so this transformation is bijective.



Thus we have shown that $\langle h_k^\perp s_\lambda, h_\mu \rangle = \langle s_{\lambda/(k)}, h_\mu \rangle$ for all partitions μ and so the thesis follows from the fact that the h_μ form a basis of Λ and the linearity of the scalar product. $\square$

1.4 The ω involution

In this section we discuss a very significant operator on the algebra of symmetric functions.

Definition 1.56. Set $\omega(p_n) \coloneqq (-1)^{n-1} p_n$ for all $n \in \mathbb{N}$ and extend $\omega : \Lambda_{\mathbb{K}} \to \Lambda_{\mathbb{K}}$ by requiring it to be an algebra morphism.

Clearly, ω is an involution. This map has some interesting properties.

Proposition 1.57. *Let λ be a partition. We have the following*

(i) $\omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$;

(ii) $\omega(e_\lambda) = h_\lambda$;

(iii) $\omega(s_\lambda) = s_{\lambda'}$;[6]

(iv) ω *is an isometry.*

Proof. (i) This is almost immediate from the definition:

$$\omega(p_\lambda) = \omega(p_{\lambda_1}) \cdots \omega(p_{\lambda_{\ell(\lambda)}}) = (-1)^{\lambda_1 - 1} p_{\lambda_1} \cdots (-1)^{\lambda_{\ell(\lambda)} - 1} p_{\lambda_{\ell(\lambda)}}$$
$$= (-1)^{|\lambda| - \ell(\lambda)} p_\lambda.$$

[6] One basis seems to be missing from this list, indeed we have not given $\omega(m_\lambda)$. In fact, the monomial symmetric functions get sent to the forgotten symmetric functions by ω, and of course vice versa.

(ii) We can easily show that $\omega(e_n) = \omega(h_n)$ for all $n \in \mathbb{N}$ by taking ω of both equations in Theorem 1.40 and using point (i). The full statement then follows by algebraic extension.

(iii) Now this is an easy consequence of Theorem 1.43 (the Jacobi-Trudi identities) and point (ii).

(iv) Using point (i), we get for any partitions λ, μ that

$$\begin{aligned}\langle \omega(p_\lambda), \omega(p_\mu)\rangle &= (-1)^{|\lambda|-\ell(\lambda)}(-1)^{|\mu|-\ell(\mu)}\langle p_\lambda, p_\mu\rangle \\ &= \left((-1)^{|\lambda|-\ell(\lambda)}\right)^2 z_\lambda \delta_{\lambda,\mu} = z_\lambda \delta_{\lambda,\mu} = \langle p_\lambda, p_\mu\rangle.\end{aligned}$$

And so the result follows from the linearity of the scalar product and the fact that the p_λ linearly generate $\Lambda_{\mathbb{K}}$.

□

1.5 Plethysm

Plethysm is a type of notation that will make many symmetric function identities easier to write down and play with. Its approach is to think of symmetric functions as processes on some expressions involving some set of variables. In particular, we interpret p_k as the process that elevates all the variables to the power k. We then use the fact that any symmetric function can be expressed in terms of the p_k, to extend this process to any element of $\Lambda_{\mathbb{K}}$. Let us now be a bit more precise.

Consider $f \in \Lambda_{\mathbb{K}}^{(n)}$. Since $\{p_\lambda\}_{\lambda \vdash n}$ linearly generates $\Lambda_{\mathbb{K}}^{(n)}$, we may write

$$f = \sum_{\lambda \vdash n} f_\lambda p_\lambda$$

for some $f_\lambda \in \mathbb{K}$. By $\mathbb{Q}((z_1, z_2, \dots))$ we denote the formal Laurent series with variables $z_1, z_2, \dots$ and coefficients in $\mathbb{Q}$.

Definition 1.58. Given $f \in \Lambda_{\mathbb{K}}^{(n)}$ and $A \in \mathbb{Q}((z_1, z_2, \dots))$ we define the *plethystic substitution* of A in f by

$$f[A] := \sum_{\lambda \vdash n} f_\lambda \prod_{i=1}^{\ell(\lambda)} A(z_1^{\lambda_i}, z_2^{\lambda_i}, \dots) \tag{1.59}$$

Clearly, for all k we have $p_k[A] = A(z_1^k, z_2^k, \dots)$ so we can rewrite (1.59) as

$$f[A] := \sum_{\lambda \vdash n} f_\lambda \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[A]$$

In this sense, we can think of plethystic evaluation in A as the unique process that is additive, multiplicative, and which is specified of the p_k generators of $\Lambda_{\mathbb{K}}$.

We insist on the fact that here we denote the variables of A by $z_1, z_2, \dots,$ but these are just arbitrary names of course. In what follows its variables might be named $x_1, x_2, \dots, q, t, u, y_1, y_2, \dots$ or anything else.

It is important to note that in general, plethystic substitution does not commute with other operations. For example, it does not commute with the evaluation of some variable at a number (see Remark 1.64). This strange behaviour often causes the mathematician first encountering plethysm some confusion. However, the tool is too useful and important to do without.

It is not hard to see that, given $f \in \Lambda_{\mathbb{K}}$, if $g \in \Lambda \subseteq \mathbb{Q}((x_1, x_2, \ldots))$ then $f[g] \in \Lambda$. Furthermore, this operation is associative.

Example 1.60. If $A = X := x_1 + x_2 + \cdots$ we have

$$f[X] = \sum_{\lambda \vdash n} f_\lambda \prod_{i=1}^{\ell(\lambda)} (\underbrace{x_1^{\lambda_i} + x_2^{\lambda_i} + \cdots}_{p_{\lambda_i}(x_1, x_2, \ldots)}) = \sum_{\lambda \vdash n} f_\lambda p_\lambda(x_1, x_2, \ldots) = f(x_1, x_2, \ldots)$$

Notation 1.61. Until now we have used the capital letters $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ to denote countably infinite alphabets of variables. In light of the example above, when using plethysm, we will from now on identify $X \coloneqq x_1 + x_2 + \cdots$ and $Y \coloneqq y_1 + y_2 + \cdots$ so that $f[X]$ indicates the symmetric function in this alphabet.

Definition 1.62. We introduce a special formal variable ϵ which has the property $\epsilon^d = (-1)^d$.

This is essentially an artifice to introduce a notion of -1 that is treated as a variable and not a number during plethystic substitution.

Let us prove some general facts about plethysm.

Proposition 1.63. *For $f \in \Lambda_{\mathbb{K}}^{(d)}$ and u a variable we have*

(i) $f[uX] = u^d f[X]$

(ii) $f[\epsilon X] = (-1)^d f[X]$

(iii) $f[-X] = (-1)^d \omega f[X]$

(iv) $f[-\epsilon X] = \omega f[X]$

Proof. (i) Since u is an variable, we understand uX as an element of $\mathbb{Q}((u, x_1, x_2, \ldots))$. It follows that

$$\begin{aligned} f[uX] &= \sum_{\lambda \vdash d} f_\lambda \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[uX] = \sum_{\lambda \vdash d} f_\lambda \prod_{i=1}^{\ell(\lambda)} u^{\lambda_i} p_{\lambda_i}[X] \\ &= \sum_{\lambda \vdash d} f_\lambda u^{\lambda_1 + \cdots + \lambda_{\ell(\lambda)}} p_\lambda[X] = u^d f[X] \end{aligned}$$

(ii) This now follows from the previous point and the definition of ϵ.

(iii) We have

$$\begin{aligned} f[-X] &= \sum_{\lambda \vdash d} f_\lambda \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[-X] = \sum_{\lambda \vdash d} f_\lambda \prod_{i=1}^{\ell(\lambda)} (-p_{\lambda_i}[X]) = \sum_{\lambda \vdash d} f_\lambda (-1)^{\ell(\lambda)} p_\lambda[X] \\ &= \sum_{\lambda \vdash d} f_\lambda (-1)^{2d} (-1)^{-\ell(\lambda)} p_\lambda[X] = (-1)^d \sum_{\lambda \vdash d} f_\lambda \underbrace{(-1)^{d-\ell(\lambda)} p_\lambda}_{\omega(p_\lambda)}[X] \\ &= (-1)^d \omega \left(\sum_{\lambda \vdash d} f_\lambda p_\lambda[X] \right) = (-1)^d \omega f[X] \end{aligned}$$

(iv) This follows easily from applying (iii) followed by (ii) for $X \mapsto \epsilon X$.

□

Remark 1.64. Here we see why, in general, plethystic substitution involving a variable, and substitution of that variable do not commute. For example take $f \in \Lambda^{(n)}$ then

$$f[u]\big|_{u\mapsto 2} = (u^n f[1])\big|_{u\mapsto 2} = 2^n f[1] \qquad\qquad f[2] = 2f[1].$$

However there are some variable substitutions that do commute with plethystic substitution. Consider $f[A(u)]$, for some $A(u) \in \mathbb{Q}((u, z_1, z_2, \dots))$. Then the substitution $u \mapsto v^k$ yields the same result when applied before or after the plethystic substitution, and this for all $k \in \mathbb{N}$. In particular, this holds for $u \mapsto 1$. Indeed, we see from the definition of plethysm that u and v^k get treated in the same way by plethystic evaluation. The same may be said for $u \mapsto 0$ or $u \mapsto v_1^k v_2^l$.

Plethystic Cauchy formula

A very simple application of plethysm coupled with the results concerning dual basis discussed above, yields the following nontrivial result.

Theorem 1.65 (Cauchy identity). *For every pair of dual bases $\{u_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}, \{v_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$ of $\Lambda_{\mathbb{K}}$ (with respect to the Hall scalar product), we have*

$$h_n[XY] = \sum_{\lambda\vdash n} u_\lambda[X]v_\lambda[Y].$$

Proof. It is clear from their definition (see Notation 1.61) that XY designates the expression $\sum_{i,j\in\mathbb{P}} x_i y_j$ and so $h_n[XY]$ is the symmetric function h_n evaluated in the alphabet $\{x_i y_j\}_{i,j\in\mathbb{P}}$. Evaluating the generating function of $h_n[XY]$ at $\zeta = 1$ then gives (see (1.38))

$$\sum_{n\in\mathbb{N}} h_n[XY] = \prod_{i,j\in\mathbb{P}} \frac{1}{1 - x_i y_j}.$$

Thus the conclusion follows from extracting the degree n part of Propositions 1.51 and applying 1.50. □

Corollary 1.66. *For all $n \in \mathbb{N}$,*

$$h_n[XY] = \sum_{\lambda\vdash n} s_\lambda[X]s_\lambda[Y] \qquad\qquad e_n[XY] = \sum_{\lambda\vdash n} s_{\lambda'}[X]s_\lambda[Y]$$

Proof. The first identity follows directly from Theorem 1.65 and Proposition 1.50. The second identity is obtained from the first by applying ω, *but only to the X variables.* In other words, consider the base field $\mathbb{K}$ to contain the Y variables. The result then follows from the fact that $\omega(s_\lambda) = s_{\lambda'}$ (Proposition 1.57). □

Corollary 1.67. *For any $f \in \Lambda_{\mathbb{K}}$ with $\mathbb{K}$ a field containing variables $y_1, y_2, \dots$ we have*

$$f[Y] = \left\langle f[X], \sum_{n\in\mathbb{N}} h_n[XY] \right\rangle.$$

Proof. Consider $\{u_\lambda\}_{\lambda\in \mathsf{Par}}$ and $\{v_\lambda\}_{\lambda\in \mathsf{Par}}$ a pair of dual basis of $\Lambda_{\mathbb{K}}$ then by Theorem 1.65

$$\left\langle f[X], \sum_{n\in\mathbb{N}} h_n[XY]\right\rangle = \sum_{n\in\mathbb{N}} v_\lambda[Y]\langle f[X], u_\lambda[X]\rangle = f[Y],$$

where the last equality comes from the duality of the u_λ and v_λ. □

1.6 Addition formula

It will often be useful to consider symmetric functions in the concatenation of two alphabets of variables, eg. $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$. Plethystically, we write this as $f[X+Y]$ (see Example 1.60). We denote the indices $1 < 2 < \cdots < \overline{1} < \overline{2} < \cdots$ of this concatenated alphabet, where the unbarred letters index X and the barred letters index Y. With this index convention, we give a sense to the definitions of m_λ, e_n, h_n, p_n above. For Schur functions, we extend the definition of semi-standard Young tableau to fillings in alphabet that is the union of barred and unbarred positive integers, ordered as above.

Proposition 1.68. *For $\lambda \in \mathsf{Par}$,*

$$s_\lambda[X+Y] = \sum_{\mu\subseteq\lambda} s_\mu[X] s_{\lambda/\mu}[Y]$$

Proof. Consider the semi-standard filling of the Young diagram of λ in the alphabet $1 < 2 < \cdots < \overline{1} < \overline{2} < \cdots$. The cells filled with an unbarred letter form a partition $\mu \subseteq \lambda$ and the cells filled with a barred letter form a skew tableau of shape λ/μ. The result now follows from the definition of Schur functions. □

See [LR11] for a more formal proof. Using the fact that $s_{(1^n)} = e_n$ and $s_{(n)} = h_n$, we immediately deduce the following special cases.

Corollary 1.69. *The following summation formulae hold*

$$e_n[X+Y] = \sum_{i=0}^{n} e_i[X] e_{n-i}[Y] \qquad\qquad h_n[X+Y] = \sum_{i=0}^{n} h_i[X] h_{n-i}[Y]$$

Corollary 1.70. *The following subtraction formula holds*

$$e_n[X-Y] = \sum_{i=0}^{n} e_i[X] h_{n-i}[Y].$$

Proof. This follows directly from Proposition 1.69 combined with point (ii) of Proposition 1.57 and point (iii) of Proposition 1.63. □

Proposition 1.68 may be used to establish the following non-trivial result.

Theorem 1.71. *For $\lambda, \mu, \nu \in \mathsf{Par}$ we have,*

$$\langle s_\mu s_\nu, s_\lambda\rangle = \langle s_\nu, s_{\lambda/\mu}\rangle.$$

Proof. Define the coefficients $c^\lambda_{\mu\nu}$ and $\tilde{c}^\lambda_{\mu\nu}$ via the expansions

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \qquad\qquad s_{\lambda/\mu} = \sum_\nu \tilde{c}^\lambda_{\mu\nu} s_\nu .$$

Consider three alphabets of variables X, Y and Z. Then we have

$$\begin{aligned}
\sum_{\lambda,\mu,\nu} \tilde{c}^\lambda_{\mu\nu} s_\mu[X] s_\nu[Y] s_\lambda[Z] &= \sum_{\lambda,\mu} s_\mu[X] s_{\lambda/\mu}[Y] s_\lambda[Z] \\
\text{(by 1.68)} &= \sum_\lambda s_\lambda[X+Y] s_\lambda[Z] \\
\text{(by 1.66)} &= \sum_{n\in\mathbb{N}} h_n[(X+Y)Z] = \sum_{n\in\mathbb{N}} h_n[XZ+YZ] \\
\text{(by 1.69)} &= \sum_{n\in\mathbb{N}} \sum_{i=1}^{n} h_i[XZ] h_{n-i}[YZ] \\
&= \left(\sum_{n\in\mathbb{N}} h_n[XZ]\right)\left(\sum_{m\in\mathbb{N}} h_n[YZ]\right) \\
\text{(by 1.66)} &= \left(\sum_\mu s_\mu[X] s_\mu[Z]\right)\left(\sum_\nu s_\nu[Y] s_\nu[Z]\right) \\
&= \sum_{\mu,\nu} s_\mu[X] s_\mu[Z] s_\nu[Z] s_\nu[Y] \\
&= \sum_{\mu,\nu,\lambda} c^\lambda_{\mu\nu} s_\mu[X] s_\lambda[Z] s_\nu[Y].
\end{aligned}$$

It follows that[7] $c^\lambda_{\mu\nu} = \tilde{c}^\lambda_{\mu\nu}$ and so the thesis is true. □

Remark 1.72. Theorem 1.71 gives another proof of Proposition 1.55.

Finally, we detail another special case of Proposition 1.68.

Corollary 1.73. *For all $\lambda \vdash n$ and variable z*

$$s_\lambda[1-z] = \begin{cases} (-z)^k(1-z) & \textit{if } \lambda = (n-k, 1^k) \textit{ for some } k \in \{0,\dots,n-1\} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Using Proposition 1.68, with $X = 1$ and $Y = -z$ we get

$$\begin{aligned}
s_\lambda[1-z] &= \sum_{\mu\subseteq\lambda} s_\mu[1] s_{\lambda/\mu}[-z] \\
\text{(by 1.63.(iii))} &= \sum_{\mu\subseteq\lambda} s_\mu[1](-1)^{|\lambda/\mu|} s_{(\lambda/\mu)'}[z]
\end{aligned}$$

By the definition of a Schur function (1.28), $s_\mu[1] = 0$ unless $\mu = (r)$ for some $r \in \mathbb{N}$. Similarly $s_{(\lambda/\mu)'}[z] = 0$ unless $(\lambda/\mu)' = (s)$ or equivalently, $\lambda/\mu = (1^s)$,

[7] These common coefficients are called the *Littlewood-Richardson coefficients.*

for some $s \in \mathbb{N}$. Given a $\lambda \vdash n$ that is not a hook partition, it is clear that for any $(r) \subseteq \lambda$, the skew partition $\lambda/(r)$ is not a column partition. This implies the second statement.

Now consider $\lambda = (n-k, 1^k)$, there are exactly two $r \in \mathbb{N}$ such that $\lambda/(r)$ is a column partition: $r = n-k$ in which case $\lambda/(r) = (1)^k$ and $r = n-k-1$ in which case $\lambda/(r) = (1^{k+1})$. So

$$\begin{aligned} s_{(n-k,1^k)} &= s_{n-k}[1](-1)^k s_{(1^k)'}[z] + s_{n-k-1}[1](-1)^{k+1} s_{(1^{k+1})'}[z] \\ &= (-1)^k s_{(k)}[z] + (-1)^{k+1} s_{(k+1)}[z] \\ &= (-1)^k z^k + (-1)^{k+1} z^{k+1} = (-z)^k (1-z). \end{aligned}$$

□

1.7 Translation and multiplication operator

This section follows [GHT99].

Definition 1.74. Define two linear operators $\partial_{s_\mu}, \underline{s}_\mu$ on Λ defined on the Schur basis by

$$\partial_{s_\mu} s_\lambda := s_{\lambda/\mu} \qquad\qquad \underline{s}_\mu s_\lambda := s_\mu s_\lambda$$

Definition 1.75. For Z an alphabet of variables, we define the *translation operator*, denoted τ_Z by $\tau_Z(f[X]) := f[X+Z]$[8].

Definition 1.76. For Z an alphabet of variables, we define the *multiplication operator*, denoted ρ_Z by $\rho_Z(f[X]) := \sum_{n\in\mathbb{N}} h_n[XZ]f[X]$.

We give an alternate description.

Proposition 1.77. *We have the following expansions*

(i) $\tau_Z = \sum_\mu s_\mu[Z]\partial_{s_\mu}$;

(ii) $\rho_Z = \sum_\mu s_\mu[Z]\underline{s}_\mu$.

Proof. We show (i) on the Schur basis and conclude by linearity of τ_Z. For all partitions λ, using Definition 1.75 and Proposition 1.68 we get

$$\tau_Z s_\lambda[X] = s_\lambda[X+Z] = \sum_\mu s_\mu[Z] s_{\lambda/\mu}[X] = \sum_\mu s_\mu[Z]\partial_{s_\mu} s_\lambda[X].$$

For (ii), take $f[X] \in \Lambda$. We have

$$\rho_Z f[X] = \sum_{n\in\mathbb{N}} h_n[XZ]f[X] = \sum_\lambda s_\lambda[X]s_\lambda[Z]f[X] = \sum_\lambda s_\lambda[Z]\underline{s}_\lambda(f[X]).$$

□

[8]The τ_z operator replaces X by $X+Z$ in *any* plethystic expression. For example we have $\tau_Z f[-X] = f[-(X+Z)] \neq f[-X+Z]$.

Combined with Theorem 1.71, these expansions imply that τ_Z et ρ_Z are adjoint operators for the Hall scalar product.

Corollary 1.78. *For any $f, g \in \Lambda$, we have $\langle \tau_Z f, g\rangle = \langle f, \rho_Z g\rangle$.*

Proof. We show that the equality holds on the Schur basis and extend by linearity. Consider $\lambda, \mu \in \mathsf{Par}$ and apply Proposition 1.77.(i)

$$\begin{aligned}
\langle \tau_Z s_\lambda, s_\mu \rangle &= \sum_\nu s_\nu[Z] \, \langle \partial_{s_\nu} s_\lambda, s_\mu \rangle \\
\text{(by 1.74)} &= \sum_\nu s_\nu[Z] \left\langle s_{\lambda/\nu}, s_\mu \right\rangle \\
\text{(by 1.71)} &= \sum_\nu s_\nu[Z] \, \langle s_\lambda, s_\nu s_\mu \rangle \\
\text{(by 1.74)} &= \sum_\nu s_\nu[Z] \, \langle s_\lambda, \underline{s}_\nu s_\mu \rangle \\
\text{(by 1.77.(ii))} &= \langle s_\lambda, \rho_Z s_\mu \rangle
\end{aligned}$$

□

Corollary 1.79. *For z a single variable we have $\tau_z = \sum_{k\in\mathbb{N}} z^k h_k^\perp$.*

Proof. For $Z = \{z\}$, Proposition1.77.(i) yields

$$\tau_z = \sum_\mu s_\mu[z] \partial_{s_\mu}$$

and $s_\mu[z] = 0$ except for $\mu = (k)$ for some $k \in \mathbb{N}$. The thesis thus follows from Proposition 1.55. □

Chapter 2

Macdonald Polynomials

In 1988, I. G. Macdonald introduced the family of symmetric functions that is most central to this thesis (see [Mac88] and Chapter VI of [Mac95]). These remarkable polynomials, depending on two parameters q and t, play a unifying role with respect to other important families of symmetric functions. Indeed, for suitable choices of q and t, they specialise to Schur functions, elementary symmetric functions, monomial symmetric functions, Hall-Littlewood symmetric functions, Jack symmetric functions and zonal symmetric functions. Furthermore, Macdonald's polynomials have deep relations to affine Hecke algebras and Hilbert schemes. Accommodating these parameters, from now on we will work over the field $\mathbb{Q}(q,t)$.

Definition 2.1. Let Λ denote the ring $\Lambda_{\mathbb{Q}(q,t)}$. The default variables of an element of Λ are $x_1, x_2, \ldots$. An element of this ring will thus be denoted by f, $f[X]$ or $f[X;q,t]$, depending on the context.

Careful! In this last notation, the order of q and t matters: in general $f[X;q,t] \neq f[X;t,q]$.

Example. If $f[X;q,t] = qs_{(1,1)} + ts_{(2)}$ then $f[X;t,q] = ts_{(1,1)} + qs_{(2)}$.

2.1 Definition

There are multiple closely related families of symmetric functions that are referred to as Macdonald polynomials. We present here Macdonald's original definition, followed by the variant we will be studying throughout the text, called modified Macdonald polynomials.

Original definition

Previously (Proposition 1.53), we proved that the Schur functions are the unique symmetric functions obtained from an orthogonalisation process using the Hall scalar product. Applying the same process but using a different scalar product engenders a number of well-known symmetric function families (e.g. Hall-Littlewood and Jack polynomials). Macdonald's original definition of his polynomials follows the same tune.

Let us start by extending the definition of the Hall scalar product (1.48) to the q,t-setting.

Definition 2.2. Given a partition λ, we set

$$Z_\lambda(q,t) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \in \mathbb{Q}(q,t).$$

Define the *q,t-scalar product* on Λ by setting

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} Z_\lambda(q,t)$$

and extending linearly.

We can rewrite this scalar product using plethysm. Indeed $Z_\lambda(q,t) = z_\lambda p_\lambda \left[\frac{1-q}{1-t}\right]$ and so by linearity, for all $f,g \in \Lambda$ we get

$$\langle f,g \rangle_{q,t} = \langle f[X], g[X] \rangle_{q,t} = \left\langle f[X], g\left[X\frac{1-q}{1-t}\right]\right\rangle.$$

One readily concludes that whenever $q = t$, the q,t-scalar product coincides with the Hall scalar product.

Theorem 2.3. *[Mac95, Chapter VI, (4.7)] There is a unique symmetric function basis $\{P_\lambda\}_{\lambda \in \mathsf{Par}}$ of $\Lambda_{\mathbb{Q}(q,t)}$ that satisfies the following properties*

(I) $\langle P_\lambda, P_\mu \rangle = 0$ *whenever* $\lambda \neq \mu$

(II) $P_\lambda = m_\lambda + \sum_{\mu \prec \lambda} \square_{\lambda\mu} m_\mu$

where the $\square_{\lambda\mu}$ are some elements in $\mathbb{Q}(q,t)$. These symmetric functions are known as Macdonald Polynomials*.*

The proof of this statement involves first showing the existence of such polynomials by constructing a self-adjoint linear operator on Λ of which the eigenvectors satisfy the wanted properties. Next, the argument for the unicity of these operators is entirely analogous to the one made for Schur functions; see Propostitiion 1.53.

This definition is quite useless for actually computing these objects, for which something called the *tableau formula* is much more suited. For a nice exposition of this formula see the “Macdonald P polynomials” entry in Alexandersson’s excellent symmetric function catalog [Ale].

We have the following specialisations

$$P_\lambda(X;q,q) = s_\lambda \qquad P_\lambda(X;q,1) = m_\lambda \qquad P_\lambda(X;1,t) = e_{\lambda'}$$

Example. We give here the (computer-generated) Schur expansions of the Macdonald polynomials of degree 3. Not because we will need them but just to get some idea of what they look like.

$$\begin{aligned} P_3 &= \left(\frac{-q^3+q^2t+qt-t^2}{-q^3t^2+q^2t+qt-1}\right) s_{1,1,1} + \left(\frac{q^2-qt+q-t}{-q^2t+1}\right) s_{2,1} + s_3 \\ P_{2,1} &= \left(\frac{qt-t^2+q-t}{-qt^2+1}\right) s_{1,1,1} + s_{2,1} \\ P_{1,1,1} &= s_{1,1,1} \end{aligned}$$

Modified version

It turns out that a slightly modified version of Macdonald's definition yields a family of even more interesting functions, that are more "combinatorial" than their counterparts.

Definition 2.4. Given a partition λ the *modified Macdonald polynomials*[1]are obtained from P_λ via the following normalisation and substitution:

$$H_\lambda[X;q,t] \coloneqq P_\lambda\left[\frac{X}{1-1/t};q,t^{-1}\right]\left(t^{n(\lambda)}\prod_{c\in\lambda}(1-q^{a(c)}t^{-l(c)-1})\right).$$

Remark 2.5. It is important to clarify some ambiguous notation. When $f\in\Lambda$ and $A(q,t,x_1,x_2,\dots)\in\mathbb{Q}((q,t,x_1,x_2,\dots))$, the expression

$$f[A(q,t,x_1,x_2,\dots);q,t^{-1}]$$

can reasonably refer to two *distinct* objects, depending if the substitution of t by t^{-1} occurs before or after the plethystic evaluation. We have encountered both rules in the literature. In this text, we follow [Hai99] and intend that the t substitution occurs *before* the plethystic evaluation, i.e.

$$\begin{aligned} f[A(q,t,x_1,x_2,\dots);q,t^{-1}] &\coloneqq f\big|_{t=t^{-1}}[[A(q,t,x_1,x_2,\dots)]\\ &\neq f[[A(q,t,x_1,x_2,\dots)]\big|_{t=t^{-1}}.\end{aligned}$$

We will use the same convention for other substitutions of q or t denoted in this manner. Here we have

$$P_\lambda\big|_{t=t^{-1}}\left[\frac{X}{1-1/t}\right]=P_\lambda\left[\frac{X}{1-t}\right]\Bigg|_{t=t^{-1}}.$$

Example. Let us look at the Schur expansions of the modified Macdonald polynomials of degree 3:

$$\begin{aligned} H_3 &= q^3s_{1,1,1}+\left(q^2+q\right)s_{2,1}+s_3\\ H_{2,1} &= qts_{1,1,1}+(q+t)\,s_{2,1}+s_3\\ H_{1,1,1} &= t^3s_{1,1,1}+\left(t^2+t\right)s_{2,1}+s_3\end{aligned}$$

We immediately notice that the coefficients look much nicer here, indeed they all live in $\mathbb{N}[q,t]$.

This observation holds in general. Set $K_{\lambda,\mu}(q,t)\in\mathbb{Q}(q,t)$ to be the coefficients such that

$$H_\lambda=\sum_\mu K_{\lambda,\mu}(q,t)s_\lambda, \tag{2.6}$$

[1]In the majority of the literature, these polynomials are denoted by $\tilde{H}_\lambda$. We depart from this notation in this thesis, since we will have no need for the other family of polynomials that are usually denoted by H_λ.

called the *modified q, t-Kostka coefficients*[2]. Then $K_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$, i.e. the modified Macdonald polynomials are *Schur positive*. This is a very deep fact that was eventually proved by Haiman, using tools from algebraic geometry [Hai01].

We record two important properties of these polynomials, of which we omit the proof.

- [Hai99, Proposition 2.5] For all $\lambda \in \mathsf{Par}$,
$$H_\lambda[X;q,t] = H_{\lambda'}[X;t;q]. \tag{2.7}$$

- [GH96, Theorem 2.7] For all $\lambda \in \mathsf{Par}$,
$$H_\lambda[X;q,t] = q^{n(\lambda')}t^{n(\lambda)}\omega H_\lambda[X;q^{-1},t^{-1}]. \tag{2.8}$$

The next result, see [Hai99, Proposostion 2.6], gives a characterisation of the modified Macdonald polynomials.

Proposition 2.9. *Let $\lambda \vdash n$. The modified Macdonald polynomials H_λ satisfy and are uniquely characterised by*

(I) $H_\lambda[X(1-q)] = \sum_{\mu\succeq\lambda} \square_{\lambda,\mu} s_\mu$;

(II) $H_\lambda[X(1-t)] = \sum_{\mu\succeq\lambda'} \blacksquare_{\lambda,\mu} s_\mu$;

(III) $\langle H_\lambda, s_{(n)}\rangle = 1$;

for some $\square_{\lambda,\mu}, \blacksquare_{\lambda,\mu} \in \mathbb{Q}(q,t)$.

Proof. We focus on the left hand side of (II).

$$\begin{aligned} H_\lambda[(1-t)X] &= H_\lambda[-t(1-1/t)X] \\ \text{(by 1.63.(i))} &= t^{|\lambda|}H_\lambda[(1-1/t)(-X)] \\ \text{(by 2.4)} &= t^n P_\lambda\left[\frac{(1-1/t)(-X)}{1-1/t};q,t^{-1}\right]\left(t^{n(\lambda)}\prod_{c\in\lambda}(1-q^{a(c)}t^{-l(c)-1})\right) \\ &= t^n\left(t^{n(\lambda)}\prod_{c\in\lambda}(1-q^{a(c)}t^{-l(c)-1})\right)P_\lambda[-X;q,t^{-1}]. \end{aligned}$$

In other words, $H_\lambda[(1-t)X]$ is merely a scalar multiple of $P_\lambda[-X;q,t^{-1}]$ and thus –using Proposition 1.63.(iii)– a multiple of $\omega P_\lambda[X;q,t^{-1}]$. By definition $P_\lambda[X,q,t]$ can be expanded in terms of $\{m_\mu\}_{\mu\preceq\lambda}$ and thus –using Remark 1.34– in terms of $\{s_\mu\}_{\mu\preceq\lambda}$. Since Schur functions do not depend on t the same holds for $P_\lambda[X;q,t^{-1}]$. Taking ω of the Schur expansion of $P_\lambda[X;q,t^{-1}]$ gives an expansion in terms of $\{s_{\mu'}\}_{\mu\preceq\lambda} = \{s_\mu\}_{\mu'\preceq\lambda} = \{s_\mu\}_{\mu\succeq\lambda'}$, where the second equality follows from Proposition 1.18; this proves (II). Now (I) can be deduced directly by switching q and t in (II) and using Equation (2.7). As for the final point, in [Mac95, page 362] one can find a fact about q,t-Kostka polynomials that when translated to their modified counterparts implies (III).

[2]Again, the more common notation for these coefficients is $\tilde{K}_{\lambda,\mu}(q,t)$

We conclude by proving that these three properties in fact determine H_λ uniquely. Suppose that there exist H'_λ that satisfy the same properties. Using (I) for H'_λ implies that for some coefficients $\square_{\lambda\mu} \in \mathbb{Q}(q,t)$

$$H'_\lambda[X(1-q)] = \sum_{\mu \succeq \lambda} \square_{\lambda\mu} s_\mu$$

From (I) applied to the H_λ and the fact that the inverse of an upper triangular matrix is upper triangular, we know that for all $\mu \vdash n$, s_μ may be expanded in the $\{H_\mu[X(1-q)]\}_{\nu \succeq \mu}$, so there exist $\tilde{\square}_{\mu\nu} \in \mathbb{Q}(q,t)$ such that

$$H'_\lambda[X(1-q)] = \sum_{\mu \succeq \lambda} \square_{\lambda\mu} \sum_{\nu \succeq \mu} \tilde{\square}_{\mu\nu} H_\nu[X(1-q)]$$

In other words $H'_\lambda[X(1-q)]$ can be expanded in $\{H_\nu[X(1-q)]\}_{\nu \succeq \lambda}$ and thus, setting $X \mapsto \frac{X}{1-q}$, we have that H'_λ can be expanded in $\{H_\nu\}_{\nu \succeq \lambda}$.

Similarly, using (II) for H'_λ implies that for some coefficients $\blacksquare_{\lambda\mu} \in \mathbb{Q}(q,t)$

$$H'_\lambda[X(1-t)] = \sum_{\mu \succeq \lambda'} \blacksquare_{\lambda\mu} s_\mu.$$

From (I) applied to the H_λ and the fact that the inverse of a lower anti-triangular[3] matrix is upper anti-triangular, we know that for all $\mu \vdash n$, s_μ may be expanded in the $\{H_\mu[X(1-t)]\}_{\nu \preceq \mu'}$, so there exist $\tilde{\blacksquare}_{\mu\nu} \in \mathbb{Q}(q,t)$ such that

$$H'_\lambda[X(1-t)] = \sum_{\mu \succeq \lambda'} \blacksquare_{\lambda\mu} \sum_{\nu \preceq \mu'} \tilde{\blacksquare}_{\mu\nu} H_\nu[X(1-t)].$$

By Proposition 1.18, this implies that H'_λ may be expanded in terms of $\{H_\mu\}_{\nu \preceq \lambda}$.

Therefore (III) implies $H'_\lambda = H_\lambda$. $\square$

Let us immediately put this characterisation to use and establish a nice little lemma.

Lemma 2.10. *For all $n \in \mathbb{N}$, we have*

$$h_n[X] = h_n\left[\frac{1}{1-q}\right] H_{(n)}[(1-q)X].$$

Proof. Since (n) is the maximal element of $(\mathsf{Par}(n), \preceq)$, Proposition 2.9 (I) implies that there exists some $c(q,t) \in \mathbb{Q}(q,t)$ such that $H_{(n)}[(1-q)X] = c(q,t)s_{(n)}[X]$. Replacing X by $\frac{X}{1-q}$ we obtain

$$H_{(n)}[X] = c(q,t)s_{(n)}\left[\frac{X}{1-q}\right].$$

[3] We define a lower anti-triangular matrix to be any $M \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ such that $M_{i,j} = 0$ for all $j \leq n-i$ and an upper anti-triangular matrix is any $N \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ such that $N_{i,j} = 0$ for all $j > n-i+1$.

Taking the scalar product with $s_{(n)}[X]$ and using Proposition 2.9 (III), we get

$$1 = c(q,t)\left\langle s_{(n)}[X], s_{(n)}\left[\frac{X}{1-q}\right]\right\rangle$$

and thus, since $s_{(n)} = h_n$

$$\frac{1}{c(q,t)} = \left\langle s_{(n)}[X], h_n\left[\frac{X}{1-q}\right]\right\rangle$$

Using 1.65 with $Y = \frac{1}{1-q}$

$$\begin{aligned}\frac{1}{c(q,t)} &= \left\langle s_{(n)}[X], \sum_{\lambda\vdash n} s_\lambda[X] s_\lambda\left[\frac{1}{1-q}\right]\right\rangle \\ \text{(by self-duality of the Schur)} &= s_{(n)}\left[\frac{1}{1-q}\right] = h_n\left[\frac{1}{1-q}\right]\end{aligned}$$

and the thesis follows. □

2.2 q-analogues

Since we are now working with symmetric functions over the two parameter field $\mathbb{Q}(q,t)$, we will often run into particular expressions involving one of these parameters that deserve some attention.

The q-analogue of a formula is a generalisation involving a parameter q that reduces to the original form when $q \to 1$. In this section, we introduce some standard q-analogues and some notable results about them.

Definition 2.11. Let $n, k \in \mathbb{N}$ with $0 \leq k \leq n$. We define

- $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{1-q^k}{1-q}$;
- $[n]_q! := \prod_{i=1}^n [i]_q$;
- $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$.

The following notation will facilitate the discussion.

Definition 2.12. For $n \in \mathbb{N}$ and any x we define the *q-Pochammer symbol*

$$(x;q)_n := \prod_{i=0}^{n-1}(1 - q^i x).$$

Cauchy gave us the following result (see [Hag08, Corollary 1.8.1] .

Theorem 2.13 (q-binomial Theorem)**.** *For any x and $n \in \mathbb{N}$*

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

Let us make some observations.

$$\begin{aligned}(q;q)_n &= (1-q)(1-q^2)\cdots(1-q^n)\\ [n]_q! &= \frac{(q;q)_n}{(1-q)^n}\\ \begin{bmatrix} n\\ k\end{bmatrix}_q &= \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{(q^{n-k+1};q)_k}{(q;q)_k}\end{aligned} \tag{2.14}$$

This motivates the following extension of the definition of $\begin{bmatrix} n\\ k\end{bmatrix}_q$.

Definition 2.15. For $n, k \in \mathbb{Z}$, we set

$$\begin{bmatrix} n\\ k\end{bmatrix}_q := \begin{cases} \frac{(q^{n-k+1};q)_k}{(q;q)_k} & \text{when } k \geq 0\\ 0 & \text{when } k < 0\end{cases}.$$

This is an extension of Definition 2.11, because both definitions coincide when $0 \leq k \leq n$.

We make some further observations.

- If $n \geq 0$ and $k > n$, then $k-n-1 \geq 0$ and $k-n-1 \leq k-1$ so the product $(q^{n-k+1};q)_k = \prod_{i=0}^{k-1}(1-q^{n-k+1+i})$ contains the factor $(1-q^0)$ and so $\begin{bmatrix} n\\ k\end{bmatrix}_q = 0$.

- For any x, $(x;q)_0$ is the empty product, equal to 1. So for all $n \in \mathbb{Z}$, $\begin{bmatrix} n\\ 0\end{bmatrix}_q = 1$.

- Clearly,

$$\text{for } 0 \leq k \leq n\text{:} \quad \begin{bmatrix} n\\ k\end{bmatrix}_q = \begin{bmatrix} n\\ n-k\end{bmatrix}_q. \tag{2.16}$$

  This does not always hold for more general k and n. For example if $n = -1$ and $k = 1$ then $\begin{bmatrix} n\\ k\end{bmatrix}_q = \begin{bmatrix} -1\\ 1\end{bmatrix}_q = \frac{(q^{-1};q)_1}{(q;q)_1} = \frac{1-q^{-1}}{1-q}$ and $\begin{bmatrix} n\\ n-k\end{bmatrix}_q = \begin{bmatrix} -1\\ -2\end{bmatrix} = 0$.

Proposition 2.17 (q-Pascal identities)**.** *For any $n, k \in \mathbb{Z}$*

$$\begin{bmatrix} n\\ k\end{bmatrix}_q = q^k\begin{bmatrix} n-1\\ k\end{bmatrix}_q + \begin{bmatrix} n-1\\ k-1\end{bmatrix}_q. \tag{2.18}$$

and,

$$\textit{for } 0 \leq k \leq n\text{:} \quad \begin{bmatrix} n\\ k\end{bmatrix}_q = \begin{bmatrix} n-1\\ k\end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1\\ k-1\end{bmatrix}_q. \tag{2.19}$$

Proof. Equation (2.19) is easily deduced from (2.18) and (2.16). Thus it suffices to prove (2.18).

- For $k < 0$, we get $0 + 0 = 0$.

- For $k = 0$ and any n, we get $1 = 1 + 0$.

- For $k > 0$ and $n = 0$, we already observed that $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$. So we have to show that
$$q^k \begin{bmatrix} -1 \\ k \end{bmatrix}_q + \begin{bmatrix} -1 \\ k-1 \end{bmatrix}_q = 0.$$
We do this by showing that for all $k \geq 0$
$$\begin{aligned}
\begin{bmatrix} -1 \\ k \end{bmatrix}_q &= \frac{(q^{-k};q)_k}{(q;q)_k} = \prod_{i=0}^{k-1} \frac{1-q^{-k+i}}{1-q^{i+1}} = q^{\sum_{i=0}^{k-1}(-k+i)} \prod_{i=0}^{k-1} \frac{q^{k-i}-1}{1-q^{i+1}} \\
&= q^{-\sum_{i=1}^{k} i}(-1)^k \prod_{i=0}^{k-1} \frac{1-q^{k-i}}{1-q^{i+1}} = q^{-\binom{k+1}{2}}(-1)^k \frac{\prod_{i=1}^{k} 1-q^i}{\prod_{i=1}^{k} 1-q^i} \\
&= (-1)^k q^{-\binom{k+1}{2}}.
\end{aligned}$$
Thus the result follows from the identity $k - \binom{k+1}{2} = -\binom{k}{2}$.

- For $k > 0$ and $n \neq 0$, we use the identity
$$1 = \frac{q^k(1-q^{n-k})}{1-q^n} + \frac{1-q^k}{1-q^n}$$
to write
$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{q^k(1-q^{n-k})}{1-q^n} \frac{(q^{n-k+1};q)_k}{(q;q)_k} + \frac{1-q^k}{1-q^n} \frac{(q^{n-k+1};q)_k}{(q;q)_k} \\
&= q^k \frac{(q^{n-k};q)_k}{(q;q)_k} + \frac{(q^{n-k+1};q)_{k-1}}{(q;q)_{k-1}} \\
&= q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .
\end{aligned}$$

□

We close the section with the following classical result (see [Sta99, Theorem 7.21.2, Corollary 7.21.3]).

Proposition 2.20. *For $n, k \in \mathbb{N}$*
$$e_k[[n]_q] = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \qquad h_k[[n]_q] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q .$$
Furthermore,
$$e_k\left[\frac{1}{1-q}\right] = q^{\binom{k}{2}} \frac{1}{(q;q)_k} \qquad h_k\left[\frac{1}{1-q}\right] = \frac{1}{(q;q)_k}.$$

2.3 Star scalar product

We define another scalar product on Λ for which the modified Macdonald polynomials are orthogonal.

Definition 2.21. We define the *star scalar product* on Λ by setting

$$\langle p_\lambda, p_\mu \rangle_* := (-1)^{|\lambda|-\ell(\lambda)} z_\lambda \left(\prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})(1-t^{\lambda_i}) \right) \delta_{\lambda,\mu}.$$

Definition 2.22. Set $M := (1-q)(1-t)$, we define the operator ϕ on Λ by the plethysm[4]

$$\phi f[X] := f[MX].$$

We will also use the following notation

$$f^* = f^*[X] := \phi^{-1} f[X] = f\left[\frac{X}{M}\right]$$

The following result [GHT99, Proposition 1.8] is an elegant way to express the star scalar product in terms of the Hall scalar product 1.48.

Proposition 2.23. *For all $f, g \in \Lambda$ we have*

$$\langle f, g \rangle_* = \langle \phi\omega f, g \rangle = \langle \omega\phi f, g \rangle$$

Proof. Let us first show that ϕ and ω commute, and thus that the second equality holds. Indeed, using Proposition 1.57.(iv)

$$\omega\phi f[X] = \omega f[MX] = f[-\epsilon MX] = f[M(-\epsilon X)] = \phi f[-\epsilon X] = \phi\omega f[X].$$

Now Consider λ, μ partitions and consider

$$\begin{aligned} \langle \omega\phi^{-1} p_\lambda, p_\mu \rangle_* &= \left\langle \omega\left(p_\lambda \left[\frac{X}{(1-q)(1-t)} \right] \right), p_\mu \right\rangle_* \\ &= \frac{1}{\prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})(1-t^{\lambda_i})} \langle \omega p_\lambda[X], p_\lambda \rangle_* \\ \text{by 1.57.(i)} &= \frac{(-1)^{|\lambda|-\ \ell(\lambda)}}{\prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})(1-t^{\lambda_i})} \langle p_\lambda[X], p_\mu \rangle_* \\ \text{by 2.21} &= \left((-1)^{|\lambda|-\ \ell(\lambda)} \right)^2 \frac{\prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})(1-t^{\lambda_i})}{\prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})(1-t^{\lambda_i})} z_\lambda \delta_{\lambda,\mu} \\ &= z_\lambda \delta_{\lambda\mu} = \langle p_\lambda, p_\mu \rangle. \end{aligned}$$

Since $\{p_\lambda\}_{\lambda \in \mathsf{Par}}$ linearly generates Λ and $\omega\phi^{-1}$ is a linear bijection, the result follows. □

Next, we translate Theorem 1.65 to the $\langle \cdot, \cdot \rangle_*$-setting.

Proposition 2.24. *For all $n \in \mathbb{N}$ and $\{u_\lambda\}_{\lambda \vdash n}$, $\{v_\lambda\}_{\lambda \vdash n}$ a pair of dual basis for the star scalar product we have*

$$e_n \left[\frac{XY}{M} \right] = \sum_{\lambda \vdash n} u_\lambda v_\lambda$$

[4] The ϕ operator replaces X by MX in *any* plethystic expression. For example we have $\phi f[X+1] = f[MX+1] \neq f[M(X+1)]$.

Proof. By duality and Proposition 2.23 we have

$$\delta_{\lambda,\mu} = \langle v_\lambda, u_\lambda \rangle_* = \langle \phi\omega v_\lambda, u_\mu \rangle$$

In other words $\{\phi\omega v_\lambda\}_{\lambda\vdash n}$ and $\{u_\lambda\}_{\lambda\vdash n}$ are dual for the Hall scalar product. Using the Cauchy identity 1.52 we may conclude that

$$h_n[XY] = \sum_{\lambda\vdash n} \phi\omega v_\lambda[X] \cdot u_\lambda[Y].$$

Finally, we will apply $\omega\phi^{-1}$ to this equation, but *only to the X variables.* More precisely, we consider $\mathbb{K} = \mathbb{Q}(q, t, y_1, y_2, \dots)$ and apply the morphism $\omega\phi^{-1} : \Lambda_{\mathbb{K}} \to \Lambda_{\mathbb{K}}$. For any $f \in \Lambda_{\mathbb{K}}$ we have

$$\omega\phi^{-1}(f[XY]) = f\left[-\epsilon\frac{X}{M}Y\right] = (\omega\phi^{-1}f)[XY]$$

so on the left hand side, we get $\omega h_n\left[\frac{XY}{M}\right] = e_n\left[\frac{XY}{M}\right]$. On the right hand side, by linearity, we get

$$\sum_{\lambda\vdash n} \omega\phi^{-1}\left(\phi\omega v_\lambda[X]\right) \cdot u_\lambda[Y] = \sum_{\lambda\vdash n} v_\lambda[X] u_\lambda[Y].$$

□

The following is the star scalar product equivalent of Corollary 1.67. Using Proposition 2.24, the proof is entirely analogous.

As mentioned above, our interest in the scalar product comes mainly from the following result (see [GHT99, Corollary 1.4]).

Proposition 2.25. *The modified Macdonald polynomials are orthogonal with respect to the star scalar product:*

$$\langle H_\lambda, H_\mu \rangle_* \coloneqq w_\lambda \delta_{\lambda,\mu}$$

where $w_\lambda \coloneqq \prod_{c\in\lambda}\left(q^{a_\lambda(c)} - t^{l_\lambda(c)+1}\right)\left(t^{l_\lambda(c)} - q^{a_\lambda(c)+1}\right)$

Therefore, using Proposition 2.24 we get the following formula.

Corollary 2.26. *For all $n \in \mathbb{N}$ we have*

$$e_n\left[\frac{XY}{M}\right] = \sum_{\lambda\vdash n} \frac{H_\lambda[X]H_\lambda[Y]}{w_\lambda}$$

Corollary 2.27. *For all $n \in \mathbb{N}$ we have*

$$e_n\left[\frac{XY}{M}\right] = \sum_{\lambda\vdash n} \frac{H_\lambda[X]H_\lambda[Y]}{w_\lambda}$$

We close this section by studying the effect of the $f^\perp$ operator in the star scalar product context.

Proposition 2.28. *For $f, g, h \in \Lambda$*

$$\langle (\phi\omega f)^{\perp} g, h \rangle_* = \langle g, fh \rangle_*.$$

In other words $(\phi\omega f)^{\perp}$ is the dual of the multiplication with respect to the star scalar product.

Proof. Indeed using Proposition 2.23 and Definition 1.54 we get

$$\langle (\phi\omega f)^{\perp} g, h \rangle_* = \langle (\phi\omega f)^{\perp} g, \phi\omega(h) \rangle = \langle g, \phi\omega(f)\phi\omega(h) \rangle = \langle g, \phi\omega(fh) \rangle = \langle g, fh \rangle_*.$$

□

2.4 Nabla, Delta and Theta

In this section, we introduce a number of interrelated operators, for whom the modified Macdonald polynomials are eigenvectors. These symmetric function operators, and the combinatorics related to them, form the central object of this text. We start by introducing shorthands for some partition constants (in $\mathbb{Q}(q,t)$) that will simplify their definition.

Definitions 2.29. For a nonempty partition λ we set

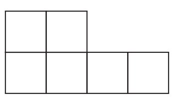
$$T_\lambda \coloneqq \prod_{c \in \lambda} q^{a'_\lambda(c)} t^{l'_\lambda(c)} = q^{n(\lambda')} t^{n(\lambda)}$$
$$B_\lambda \coloneqq \sum_{c \in \lambda} q^{a'_\lambda(c)} t^{l'_\lambda(c)}$$
$$D_\lambda \coloneqq (1-q)(1-t)B_\lambda - 1$$
$$\Pi_\lambda \coloneqq \prod_{c \in \lambda \setminus \{(0,0)\}} \left(1 - q^{a'_\lambda(c)} t^{l'_\lambda(c)}\right).$$

We also set $B_\emptyset \coloneqq 0$ and $T_\emptyset \coloneqq 1$. See Definitions 1.12 and 1.13 for a reminder on the notations used here.

Notice that

$$T_\lambda = e_{|\lambda|}[B_\lambda]. \tag{2.30}$$

Example. For $\lambda = (4,2)$ we have

$\lambda =$ 

$$T_{4,2} = q^7 t^2$$
$$B_{4,2} = q^3 + q^2 + qt + q + t + 1$$
$$\Pi_{4,2} = (1-q)(1-q^2)(1-q^3)(1-t)(1-qt).$$

In [BGHT99, BG99], the authors introduced the first incarnation of the aforementioned operators, opening up the field of study of q,t-combinatorics related to symmetric function coefficients.

Definition 2.31. We define the *nabla operator*, $\nabla : \Lambda \to \Lambda$ by setting

$$\nabla H_\lambda = T_\lambda H_\mu$$

and extending linearly.

A particularly interesting instance of this operator is ∇e_n, which is the bi-graded Frobenius characteristic of the module of diagonal harmonics [GH96, Hai02]. Furthermore, it has been conjectured in [HHL$^+$05] conjectured a combinatorial formula for this symmetric function, called the *shuffle conjecture*, now a theorem [CM18].

Also in [BGHT99], the authors introduced closely related, more general families of operators.

Definition 2.32. For $f \in \Lambda$ we define the *Delta operator* and *Delta prime operator* $\Delta_f, \Delta'_f : \Lambda \to \Lambda$ by setting

$$\Delta_f H_\lambda \coloneqq f[B_\lambda] H_\lambda \qquad \Delta'_f H_\lambda \coloneqq f[B_\lambda - 1] H_\lambda$$

and extending linearly.

Recently, these operators have been getting a lot of attention since [HRW18] conjectured combinatorial interpretations of $\Delta'_{e_{n-k-1}} e_n$. In [Zab19], Zabrocki proposed a module for which this symmetric function is conjectured to be its Frobenius characteristic.

Proposition 2.33. *For $f \in \Lambda^{(n)}$ and $1 \leq k \leq n$, we have*

(i) $\Delta_{e_n} f = \nabla f$*;*

(ii) $\Delta_{e_k} f = \Delta'_{e_k} f + \Delta'_{e_{k-1}} f$

(iii) $\Delta_{e_n} f = \Delta'_{e_{n-1}} f$

Proof. (i) Indeed for any $\lambda \vdash n$, B_λ has exactly n terms so

$$\Delta_{e_n} H_\lambda = e_n[B_\lambda] H_\lambda = T_\lambda H_\lambda = \nabla H_\lambda,$$

(see Example 1.60) which suffices since $\{H_\lambda\}_{\lambda \vdash n}$ is a basis of $\Lambda^{(n)}$.

(ii) This result relies on the elementary identity

$$\text{for } k \geq 1 \qquad e_k[B_\lambda] = e_k[B_\lambda - 1] + e_{k-1}[B_\lambda - 1], \tag{2.34}$$

which is easily obtained by noticing that $B_\lambda = 1 + \cdots$ and so we get $e_k[B_\lambda]$, by interpreting the two terms of $e_k[B_\lambda - 1] + e_{k-1}[B_\lambda - 1]$ as either "leaving or picking" this 1.

Thus checking on modified Macdonald polynomial basis

$$\begin{aligned} \Delta_{e_k} H_\lambda = e_k[B_\lambda] H_\lambda &= (e_k[B_\lambda - 1] + e_{k-1}[B_\lambda - 1]) H_\lambda \\ &= \Delta'_{e_k} H_\lambda + \Delta'_{e_{k-1}} H_\lambda \end{aligned}$$

(iii) This follows easily from the fact that

$$\lambda \vdash n \Rightarrow e_n[B_\lambda] = e_{n-1}[B_\lambda - 1] = T_\lambda.$$

□

The Delta and Nabla operators are self adjoint with respect to the star scalar product. This follows directly from the fact that they are diagonal on the Macdonald polynomials, who form an orthogonal basis with respect to the star scalar product (Proposition 2.25).

Lemma 2.35. *For α an operator in $\{\nabla, \Delta_f, \Delta'_f \mid f \in \Lambda\}$ and $g, h \in \Lambda$ we have* $\langle \alpha(g), h \rangle_* = \langle g, \alpha(h) \rangle$.

Finally, in [DIV20], we introduced a new family of related operators. First, a preliminary definition.

Definition 2.36. Define the linear *operator* $\mathbf{\Pi} : \oplus_{n>0}\Lambda^{(n)} \to \oplus_{n>0}\Lambda^{(n)}$ by setting, for any nonempty partition λ

$$\mathbf{\Pi} : H_\lambda \mapsto \Pi_\lambda H_\lambda$$

and extending linearly.

Definition 2.37. For $f \in \Lambda$, we define the *Theta operator* $\Theta_f : \Lambda \to \Lambda$ by setting,

$$\Theta_f g \coloneqq \begin{cases} \mathbf{\Pi} f^* \mathbf{\Pi}^{-1} g & \text{if } \deg(g) > 0 \\ \Theta_f g = 0 & \text{if } \deg(g) = 0 \text{ and } \deg(f) > 0 \\ \Theta_f g = fg & \text{if } \deg(g) = 0 \text{ and } \deg(f) = 0. \end{cases}$$

We will mostly use one particular instance of this operator, Θ_{e_k} which we will denote for short by Θ_k.

Here are some easy observations about this operator.

- Θ_f is linear for all $f \in \Lambda$.
- If $f \in \Lambda^{(k)}$ then $\Theta_f(\Lambda^{(n)}) \subseteq \Lambda^{(n+k)}$.
- $\Theta_0 = \mathrm{Id}$.

2.5 Refinements

This section is devoted to the introduction of some interesting refinements of the n-th elementary symmetric function. These particular refinements are so interesting because they inherit some key properties from e_n, e.g. some sort of positivity on application of ∇. More details will follow in later chapters.

The $E_{n,k}$ refinement

In [GH02], the authors introduced the following family of symmetric functions (see also [Hag08, 3.24]).

Definition 2.38. Take $n, k \in \mathbb{P}$ with $k \leq n$. We define the symmetric functions $E_{n,k}$ by way of the following expansion

$$e_n \left[X \frac{1-z}{1-q} \right] = \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}.$$

It is convenient to extend this definition to $n, k \in \mathbb{N}$ by setting $E_{n,0} = \delta_{n,0}$ and $E_{n,k} = 0$ for $k > n$. We thus have, for all $n, k \in \mathbb{N}$,

$$e_n \left[X \frac{1-z}{1-q} \right] = \sum_{k \in \mathbb{N}} \frac{(z;q)_k}{(q;q)_k} E_{n,k}$$

The following special cases are of particular interest.

- For $z = q^{s+1}$ (see Remark 1.64) we get, by (2.14)

$$e_n[X[s+1]_q] = \sum_{k=0}^{n} \frac{(q^{s+1};q)_k}{(q;q)_k} E_{n,k} = \sum_{k=0}^{n} \begin{bmatrix} k+s \\ k \end{bmatrix}_q E_{n,k}. \tag{2.39}$$

- For $z = q$ we get

$$e_n = \sum_{k=0}^{n} E_{n,k}. \tag{2.40}$$

Let us study the special case $k = n$.

Proposition 2.41. *For $n \in \mathbb{N}$*

$$E_{n,n} = \frac{(q;q)_n}{q^{\binom{n}{2}}} h_n \left[\frac{X}{1-q} \right].$$

Proof. Using Proposition 1.69, we may write

$$\begin{aligned} e_n \left[X \frac{1-z}{1-q} \right] &= \sum_{i=0}^{n} e_{n-i} \left[\frac{X}{1-q} \right] e_i \left[\frac{-zX}{1-q} \right] \\ \text{(by 1.63, 1.57)} \quad &= \sum_{i=0}^{n} e_{n-i} \left[\frac{X}{1-q} \right] (-z)^i h_i \left[\frac{X}{1-q} \right]. \end{aligned}$$

Thus, the coefficient of z^n in this expression is

$$(-1)^n h_n \left[\frac{X}{1-q} \right].$$

On the other hand, in the expression

$$\sum_{k=0}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}$$

the coefficient of z^n is equal to

$$\frac{\prod_{i=0}^{n-1}(-q^i)}{(q;q)_n} E_{n,n} = (-1)^n \frac{q^{\binom{n}{2}}}{(q;q)_n} E_{n,n}.$$

By Definition 2.38, these coefficients must coincide and the result follows by isolating $E_{n,n}$. $\square$

In [CL06], the authors prove an interesting link between the $E_{n,k}$ and $\omega(p_n)$, which we will be relevant later.

Proposition 2.42. *For $n \in \mathbb{N}$*

$$\omega(p_n) = \sum_{k=1}^{n} \frac{[n]_q}{[k]_q} E_{n,k}.$$

Proof. On the one hand, using Corollary 1.66 we have

$$\begin{aligned} e_n\left[X\frac{1-z}{1-q}\right] &= \sum_{\lambda\vdash n} s_\lambda\left[\frac{X}{1-q}\right] s_{\lambda'}[1-z] \\ \text{(by 1.73)} &= \sum_{\substack{\lambda\vdash n \\ \lambda'=(n-k,1^k):k\in\{0,\dots n-1\}}} s_\lambda\left[\frac{X}{1-q}\right](-z)^k(1-z) \\ &= \sum_{k=1}^{n} s_{(k,1^{n-k})}\left[\frac{X}{1-q}\right](-z)^{k-1}(1-z) \end{aligned}$$

On the other hand, using Definition 2.38 we have

$$\begin{aligned} e_n\left[X\frac{1-z}{1-q}\right] &= \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k} \\ \text{(by 2.12)} &= \sum_{k=1}^{n} \frac{(1-z)(zq;q)_{k-1}}{(q;q)_k} E_{n,k}. \end{aligned}$$

Combining these two expansions and simplifying $(1-z)$ before setting $z=1$, we get

$$\sum_{k=1}^{n}(-1)^{k-1}s_{(k,1^{n-k})}\left[\frac{X}{1-q}\right] = \sum_{k=1}^{n}\frac{(q;q)_{k-1}}{(q;q)_k}E_{n,k} = \sum_{k=1}^{n}\frac{1}{(1-q^k)}E_{n,k}. \tag{2.43}$$

Using Theorem 1.47 with $\lambda = \emptyset$ we know that

$$\omega(p_n) = (-1)^{n-1}p_n = (-1)^{n-1}\sum_{\mu}(-1)^{ht(\mu)}s_\mu$$

where the sum is over all border strips $\mu = \mu/\emptyset$ of size n. But a partition that is a border strip is simply a hook partition. So we get

$$\omega(p_n) = (-1)^{n-1}\sum_{k=1}^{n}(-1)^{n-k}s_{(k,1^{n-k})} = \sum_{k=1}^{n}(-1)^{k-1}s_{(k,1^{n-k})}$$

Injecting this into (2.43) and using the linearity of plethystic evaluation we get

$$\omega(p_n)\left[\frac{X}{1-q}\right] = \frac{\omega(p_n)[X]}{1-q^n}\sum_{k=1}^{n}\frac{1}{(1-q^k)}E_{n,k}$$

which clearly implies the result. □

The C_α refinement

In [HMZ12] the authors stated a further important refinement.

Definition 2.44. For $m \in \mathbb{N}$ and $f \in \Lambda$, set

$$\mathbb{C}_m f[X] \coloneqq -q^{-m+1} \sum_{r \in \mathbb{N}} q^{-r} h_{m+r}[X] h_r[X(1-q)]^{\perp} f[X].$$

Then for $\alpha \in \mathbb{P}^k$, define

$$C_\alpha \coloneqq \mathbb{C}_{\alpha_1} \mathbb{C}_{\alpha_2} \cdots \mathbb{C}_{\alpha_k}(1).$$

Example. We list C_α for all compositions α of 3:

$$\begin{aligned}
C_{(3)} &= \frac{1}{q^2} h_3 \\
C_{(2,1)} &= \left(\frac{1}{-q}\right) h_{(2,1)} + \left(\frac{-q+1}{-q^2}\right) h_3 \\
C_{(1,2)} &= -\frac{1}{q^2} h_{(2,1)} + \left(\frac{-q+1}{-q^3}\right) h_3 \\
C_{(1,1,1)} &= h_{(1,1,1)} + \left(\frac{-2q^2+q+1}{q^2}\right) h_{(2,1)} + \left(\frac{q^3-q^2-q+1}{q^3}\right) h_3.
\end{aligned}$$

In Section 5 of the same paper the authors prove that the C_α are a refinement of the $E_{n,k}$ and thus of e_n.

Theorem 2.45. *For $n, k \in \mathbb{N}$ and $k \leq n$*

$$E_{n,k} = \sum_{\substack{\alpha \vDash n \\ \ell(\alpha) = k}} C_\alpha$$

Using (2.40), the we deduce the following.

Corollary 2.46. *For $n \in \mathbb{N}$*

$$e_n = \sum_{\alpha \vDash n} C_\alpha.$$

Chapter 3

Symmetric function identities

The way one tends to get theorems about symmetric functions is by playing around with a whole lot of technical identities. This makes giving a self-contained presentation of contemporary research in this subject a Sisyphean task. All the same, omitting the ways in which the symmetric function identities we use are obtained would make the discussion sterile; since without them, there would be no results. We thus aim to walk a middle ground: we will state older and more involved identities without proof (but with a reference to one), whereas new or simpler ones will be accompanied by an explanation.

3.1 Classical identities

We start by stating [GHT99, Theorem I.2], which turns out to be quite powerful and underpins many a result in this thesis.

Theorem 3.1. *For any* $f \in \Lambda$ *and* $\lambda \in \mathsf{Par}$

$$\langle f, \tau_1 H_\lambda \rangle_* = (\nabla^{-1} \tau_{-\epsilon} f[X])\big|_{X \mapsto D_\lambda}.$$

Corollary 3.2. *For* $\lambda \in \mathsf{Par}$

$$H_\lambda[X+1] = \sum_{n,m \in \mathbb{N}} h_m \left[\frac{X}{M}\right] \nabla^{-1} e_n \left[\frac{XD_\lambda}{M}\right]$$

.

Proof. Using Corollary 1.67 with $Y = D_\lambda$ we get for any $f \in \Lambda$

$$\begin{aligned}
(\nabla^{-1} \tau_{-\epsilon} f[X])\big|_{X \mapsto D_\lambda} &= \sum_{n \in \mathbb{N}} \langle \nabla^{-1} \tau_{-\epsilon} f[X]), h_n[D_\lambda X] \rangle \\
\text{(by 2.23)} &= \sum_{n \in \mathbb{N}} \left\langle \nabla^{-1} \tau_{-\epsilon} f[X], e_n \left[\frac{D_\lambda X}{M}\right] \right\rangle_* \\
\text{(by 2.35)} &= \sum_{n \in \mathbb{N}} \left\langle \tau_{-\epsilon} f[X], \nabla^{-1} e_n \left[\frac{D_\lambda X}{M}\right] \right\rangle_*
\end{aligned}$$

$$
\begin{aligned}
(\text{by } 2.23) &= \sum_{n\in\mathbb{N}} \left\langle \tau_{-\epsilon} f[X], \phi\omega\nabla^{-1} e_n\left[\frac{D_\lambda X}{M}\right]\right\rangle \\
(\text{by } 1.78) &= \sum_{n\in\mathbb{N}} \left\langle f[X], \rho_{-\epsilon}\phi\omega\nabla^{-1} e_n\left[\frac{D_\lambda X}{M}\right]\right\rangle \\
(\text{by } 1.76) &= \sum_{n,m\in\mathbb{N}} \left\langle f[X], h_m[-\epsilon X]\phi\omega\nabla^{-1} e_n\left[\frac{D_\lambda X}{M}\right]\right\rangle \\
(\text{by } 1.63,\ 2.22) &= \sum_{n,m\in\mathbb{N}} \left\langle f[X], \phi\omega\left(h_m\left[\frac{X}{M}\right]\nabla^{-1} e_n\left[\frac{D_\lambda X}{M}\right]\right)\right\rangle \\
(\text{by } 2.23) &= \sum_{n,m\in\mathbb{N}} \left\langle f[X], h_m\left[\frac{X}{M}\right]\nabla^{-1} e_n\left[\frac{D_\lambda X}{M}\right]\right\rangle_* \\
&= \left\langle f[X], \sum_{n,m\in\mathbb{N}} h_m\left[\frac{X}{M}\right]\nabla^{-1} e_n\left[\frac{D_\lambda X}{M}\right]\right\rangle_* .
\end{aligned}
$$

Since this string of equalities holds for all $f \in \Lambda$, Theorem 3.1 implies the thesis. □

Macdonald-Koornwinder reciprocity

The following famous identity can be derived from [Mac95, VI (6.6)], but we give a simpler proof from [GHT99, Theorem 3.3].

Theorem 3.3. *(Macdonald-Koornwinder reciprocity) Let $\lambda, \mu \in \mathsf{Par}$ and z any variable, then*

$$
\frac{H_\lambda[1+zD_\mu]}{\prod_{c\in\lambda}(1-zq^{a'(c)}t^{l'(c)})} = \frac{H_\mu[1+zD_\lambda]}{\prod_{c\in\mu}(1-zq^{a'(c)}t^{l'(c)})}
$$

Proof. First of all, we remark that for all partitions λ

$$
\frac{1}{\prod_{c\in\lambda}(1-zq^{a'(c)}t^{l'(c)})} = \sum_{n\in\mathbb{N}} h_n[zB_\lambda]. \tag{3.4}
$$

To see this, consider (1.38) at $\zeta = 0$ and the definition of B_λ (Definition 2.29).

Next we calculate, using Corollary 1.69, that for any pair of alphabets of variables X and Y

$$
\sum_{n\in\mathbb{N}} h_n[X+Y] = \sum_{n\in\mathbb{N}}\sum_{i=0}^{n} h_i[X]h_{n-i}[Y] = \sum_{n,m\in\mathbb{N}} h_n[X]h_m[Y]. \tag{3.5}
$$

Using Corollary 1.67, with $Y = zD_\mu$ we have

$$
H_\lambda[1+zD_\mu] = \sum_{m\in\mathbb{N}} \langle H_\lambda[1+X], h_m[zD_\mu X]\rangle. \tag{3.6}
$$

So using (3.4) we get

$$\frac{H_\lambda[1+zD_\mu]}{\prod_{c\in\lambda}(1-zq^{a'(c)}t^{l'(c)})} = \sum_{n\in\mathbb{N}} h_n[zB_\lambda] H_\lambda[1+zD_\mu]$$

$$\text{(by 3.6)} = \sum_{n,m\in\mathbb{N}} h_n[zB_\lambda] \langle H_\lambda[X+1], h_m[zD_\mu X]\rangle$$

$$\text{(by 3.2)} = \sum_{n,m,k,l\in\mathbb{N}} h_n[zB_\lambda] \left\langle h_k\left[\frac{X}{M}\right]\nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], h_m[zD_\mu X]\right\rangle$$

$$\text{(by 1.76)} = \sum_{n,m,l\in\mathbb{N}} h_n[zB_\lambda] \left\langle \rho_{1/M}\nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], h_m[zD_\mu X]\right\rangle$$

$$\text{(by 1.78)} = \sum_{n,m,l\in\mathbb{N}} h_n[zB_\lambda] \left\langle \nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], \tau_{1/M}h_m[zD_\mu X]\right\rangle$$

$$= \sum_{n,m,l\in\mathbb{N}} h_n[zB_\lambda] \left\langle \nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], h_m\left[zD_\mu X+\frac{zD_\mu}{M}\right]\right\rangle$$

$$\text{(by 3.5)} = \sum_{n,m,l,k\in\mathbb{N}} h_n[zB_\lambda]h_k\left[\frac{zD_\mu}{M}\right] \left\langle \nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], h_m\left[zD_\mu X\right]\right\rangle$$

$$\text{(by 2.29)} = \sum_{n,m,l,k,r\in\mathbb{N}} h_n[zB_\lambda]h_k[zB_\mu]h_r\left[\frac{-z}{M}\right] \left\langle \nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], h_m\left[zD_\mu X\right]\right\rangle$$

(by 2.23)

$$\text{(by 1.63)} = \sum_{n,m,l,k,r\in\mathbb{N}} h_n[zB_\lambda]h_k[zB_\mu]h_r\left[\frac{-z}{M}\right] z^m\left\langle \nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], h_m\left[D_\mu X\right]\right\rangle$$

$$\text{(by 2.23)} = \sum_{n,m,l,k,r\in\mathbb{N}} h_n[zB_\lambda]h_k[zB_\mu]h_r\left[\frac{-z}{M}\right] z^m\left\langle \nabla^{-1}e_l\left[\frac{XD_\lambda}{M}\right], e_m\left[\frac{D_\mu X}{M}\right]\right\rangle_* .$$

By Lemma 2.35, this last expression is symmetric in λ and μ, which is what we wanted to show. □

Corollary 3.7. *For $\lambda, \mu \in \mathsf{Par}\setminus\{\emptyset\}$*

$$\frac{H_\lambda[MB_\mu]}{\Pi_\lambda} = \frac{H_\mu[MB_\lambda]}{\Pi_\mu}.$$

Proof. Use Theorem 3.3 and Definition 2.29, cancel the common factor $(1-z)$ (corresponding to $c=(0,0)$) from both denominators and evaluate at $z\mapsto 1$ (see Remark 1.64) . □

Macdonald expansions

The following is a consequence of Theorem 3.3.

Corollary 3.8. *For $\lambda \vdash n$ and $k\in\mathbb{N}$*

$$\langle H_\lambda, s_{(n-k,1^k)}\rangle = e_k[B_\lambda - 1].$$

Proof. On the one hand, for $\mu = \emptyset$, Theorem 3.3 becomes

$$H_\lambda[1-z] = \prod_{c\in\lambda}(1 - zq^{a'(c)}t^{l'(c)}) = (1-z)\prod_{c\in\lambda\setminus\{(0,0)\}}(1 - zq^{a'(c)}t^{l'(c)})$$

$$(\text{by } 2.29) = (1-z)\sum_{k=0}^{n-1}(-z)^k e_k[B_\lambda - 1].$$

On the other hand, setting $X = 1 - z$ (see Remark 1.64) in the defining equation of the q,t-Kostka coefficients (2.6)

$$H_\lambda[1-z] = \sum_{\mu\vdash n} K_{\lambda\mu}(q,t)s_\mu[1-z]$$

$$(\text{by Corollary 1.73}) = \sum_{k=0}^{n-1} K_{\lambda(n-k,1^k)}(q,t)(-z)^k(1-z).$$

Comparing the z-coefficients of these two equations we get that $K_{\lambda(n-k,1^k)} = \langle H_\lambda, s_{(n-k,1^k)}\rangle = e_k[B_\lambda - 1]$. □

Corollary 3.9. *For $\lambda \vdash n$ and $k \in \mathbb{N}$*

$$\langle H_\lambda, e_k h_{n-k}\rangle = e_k[B_\lambda].$$

Proof. We use an easy consequence of one of Pieri's rules (Proposition 1.45):

$$e_k h_{n-k} = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}. \tag{3.10}$$

So applying Corollary 3.8 twice, we get

$$\langle H_\lambda, e_k h_{n-k}\rangle = e_k[B_\lambda - 1] + e_{k-1}[B_\lambda - 1] = e_k[B_\lambda],$$

where the second equality is (2.34). □

Proposition 3.11. *For $n, k \in \mathbb{N}$*

$$h_k\left[\frac{X}{M}\right]e_{n-k}\left[\frac{X}{M}\right] = \sum_{\lambda\vdash n}\frac{e_k[B_\lambda]H_\lambda[X]}{w_\lambda}.$$

Proof. Since $\{H_\lambda\}_{\lambda\in\mathsf{Par}}$ and $\left\{\frac{H_\lambda}{w_\lambda}\right\}_{\lambda\in\mathsf{Par}}$ are dual basis of Λ with respect to the star scalar product, the expression is equivalent to

$$\left\langle h_k\left[\frac{X}{M}\right]e_{n-k}\left[\frac{X}{M}\right], H_\lambda\right\rangle_* = e_k[B_\lambda].$$

By Proposition 2.23

$$\left\langle h_k\left[\frac{X}{M}\right]e_{n-k}\left[\frac{X}{M}\right], H_\lambda\right\rangle_* = \langle\omega(h_k[X]e_{n-k}[X]), H_\lambda\rangle$$

$$(\text{By } 1.57\ ) = \langle e_k[X]h_{n-k}[X], H_\lambda\rangle$$

and so the result follows from Corollary 3.9. □

The following result was first stated in [GH02].

Lemma 3.12. *For $\lambda \in \mathsf{Par} \setminus \{\emptyset\}$ and $k \in \mathbb{N}$*

$$H_\lambda[(1-t)(1-q^k)] = (1-q^k)\Pi_\lambda h_k[(1-t)B_\lambda].$$

Proof. For $k = 0$ and $\lambda \in \mathsf{Par} \setminus \{\emptyset\}$ we get $0 = 0$, so we may assume $k > 0$.

By Lemma 2.10 and Proposition 2.20 we have, for all $k \in \mathbb{N}$

$$H_{(k)}[(1-q)X] = (q;q)_k h_k[X],$$

which implies

$$H_{(k)}[X] = (q;q)_k h_k \left[\frac{X}{1-q}\right].$$

Using Corollary 3.7 with $\mu = (k)$, we get

$$\begin{aligned} \Pi_{(k)} H_\lambda[MB_{(k)}] &= \Pi_\lambda H_{(k)}[MB_\lambda] \\ &= \Pi_\lambda (q;q)_k h_k \left[\frac{MB_\lambda}{1-q}\right] \\ \text{(by Definition 2.22)} &= \Pi_\lambda (q;q)_k h_k \left[(1-t)B_\lambda\right]. \end{aligned}$$

It is clear form their definitions (see 2.12 and 2.29) that

$$B_{(k)} = [k]_q = \frac{1-q^k}{1-q} \qquad\qquad \Pi_{(k)} = (q;q)_{k-1}.$$

The thesis thus follows from some easy cancellations. □

Proposition 3.13. *For $n, k \in \mathbb{N}$ with $n > 0$*

$$e_n\left[X[k]_q\right] = (1-q^k)\sum_{\lambda \vdash n} \frac{\Pi_\lambda h_k[(1-t)B_\lambda] H_\lambda[X]}{w_\lambda}$$

Proof. Using Corollary 2.27 with $Y = (1-t)(1-q^k)$ we get

$$\begin{aligned} \sum_{\lambda \vdash n} \frac{H_\lambda[X] H_\lambda[(1-t)(1-q^k)]}{w_\lambda} &= e_n \left[\frac{X(1-t)(1-q^k)}{M}\right] \\ &= e_n \left[X \frac{1-q^k}{1-q}\right] = e_n[X[k]_q] \end{aligned}$$

and so the result follows immediately from Lemma 3.12. □

We will often use the special case $k = 1$.

Corollary 3.14. *For $n \in \mathbb{P}$*

$$e_n = \sum_{\lambda \vdash n} \frac{M \Pi_\lambda B_\lambda H_\lambda}{w_\lambda}.$$

In the same vein, we have the following Macdonald expansion of $\omega(p_n)$, of which we omit the proof (which can be found here [GHS11, Proposition 2.3.d]).

Proposition 3.15. *For $n \in \mathbb{P}$*

$$\omega(p_n) = [n]_q [n]_t \sum_{\lambda \vdash n} \frac{M \Pi_\lambda H_\lambda}{w_\lambda}.$$

Nabla identities

Finally, we calculate the image by ∇ of some symmetric functions.

Proposition 3.16. *For $n \in \mathbb{N}$,*

(i) $\nabla e_n\left[\frac{X}{M}\right] = h_n\left[\frac{X}{M}\right]$;

(ii) $\nabla h_n\left[\frac{X}{1-q}\right] = q^{\binom{n}{2}} h_n\left[\frac{X}{1-q}\right]$.

Proof. (i) Write Proposition 3.11 for $k = 0$.

$$e_n\left[\frac{X}{M}\right] = \sum_{\lambda \vdash n} \frac{H_\lambda[X]}{w_\lambda}.$$

Taking ∇ of this expression and aplying Proposition 3.11 again, we conclude

$$\nabla e_n\left[\frac{X}{M}\right] = \sum_{\lambda \vdash n} \frac{T_\lambda H_\lambda[X]}{w_\lambda} = \sum_{\lambda \vdash n} \frac{e_n[B_\lambda] H_\lambda[X]}{w_\lambda} = h_n\left[\frac{X}{M}\right].$$

(ii) Using Lemma 2.10 with $X \mapsto \frac{X}{1-q}$ we get

$$h_n\left[\frac{X}{1-q}\right] = h_n\left[\frac{1}{1-q}\right] H_{(n)}[X].$$

Since $T_{(n)} = q^{\binom{n}{2}}$, applying ∇ to this equation yields

$$\nabla h_n\left[\frac{X}{1-q}\right] = h_n\left[\frac{1}{1-q}\right] q^{\binom{n}{2}} H_{(n)}[X] = q^{\binom{n}{2}} h_n\left[\frac{X}{1-q}\right].$$

□

3.2 Pieri coefficients

Another extremely useful tool for deriving identities are the following coefficients.

Definition 3.17. For $f \in \Lambda$ and $\lambda \in \mathsf{Par}$ the *Pieri coefficients*, denoted $c^{f^\perp}_{\lambda\mu}, d^f_{\mu\lambda} \in \mathbb{Q}(q,t)$ are defined by the expansions

$$f^\perp H_\lambda = \sum_\mu c^{f^\perp}_{\lambda\mu} H_\mu \qquad\qquad f H_\lambda = \sum_\mu d^f_{\mu\lambda} H_\mu.$$

We introduce special notation for two particular instances of these coefficients:

$$c^{(k)}_{\lambda\mu} := c^{h_k^\perp}_{\lambda\mu} \qquad\qquad d^{(k)}_{\mu\lambda} := d^{e_k[X/M]}_{\mu\lambda}.$$

These coefficients determine one another.

Proposition 3.18. *For all $f \in \Lambda$ and $\lambda,\ \mu \in \mathsf{Par}$,* $w_\mu c^{f[X]^\perp}_{\lambda\mu} = w_\lambda d^{\omega f[X/M]}_{\lambda\mu}$.

Proof. Since $\langle H_\lambda, H_\mu\rangle_* = w_\lambda \delta_{\lambda\mu}$ and by Proposition 2.28 we have

$$w_\mu c^{f[X]^\perp}_{\lambda\mu} = \langle f[X]^\perp H_\lambda, H_\mu\rangle_* = \left\langle H_\lambda, (\omega f)\left[\frac{X}{M}\right] H_\mu \right\rangle_* = w_\lambda d^{\omega f[X/M]}_{\lambda\mu}.$$

□

In particular we have

$$w_\mu c^{(k)}_{\lambda\mu} = w_\lambda d^{(k)}_{\lambda\mu}. \tag{3.19}$$

Theorem 3.20. *Take $f \in \Lambda^{(n)}$ and $\lambda, \mu \in \mathsf{Par}$. Let $\mu \subset_n \lambda$ denote $\mu \subseteq \lambda$ and $|\lambda| - |\mu| = n$. Then*

(i) $c^{f^\perp}_{\lambda\mu} \neq 0$ implies that $\mu \subset_n \lambda$. Thus $f^\perp H_\lambda = \sum_{\mu\subset_n\lambda} c^{f^\perp}_{\lambda\mu} H_\mu$.

(ii) $d^f_{\mu\lambda} \neq 0$ implies that $\mu \supset_n \lambda$. Thus $f H_\lambda = \sum_{\mu\supset_n\lambda} d^f_{\mu\lambda} H_\mu$.

The second point is a consequence of [Mac95, VI.6.7] and the first follows from the second combined with Proposition 3.18.

We record an important recursive formula that first appeared in [BH13, Proposition 5], but using the notation of [GHXZ16, Theorem 3.2].

Proposition 3.21. *For $k \in \mathbb{N}$ and $\lambda, \mu \in \mathsf{Par}$*

$$c^{(k+1)}_{\lambda\mu} = \frac{1}{B_{\lambda/\mu}} \sum_{\mu\subset_1\nu\subset_k\lambda} c^{(k)}_{\lambda\nu} c^{(1)}_{\nu\mu} \frac{T_\nu}{T_\mu}$$

where $B_{\lambda/\mu} = B_\lambda - B_\mu$.

For the remainder of this section we give the proofs of some identities involving Pieri coefficients that we will need later on.

Lemma 3.22. *For any $n \in \mathbb{N}$ and $\lambda \vdash n$*

$$B_\lambda = \sum_{\mu\ \subset_1\lambda} c^{(1)}_{\lambda\mu}$$

Proof. For $n \geq 1$, using Corollary 3.9 (twice) and Theorem 3.20 we readily obtain

$$\begin{aligned} B_\lambda = e_1[B_\lambda] &= \langle H_\lambda, e_1 h_{n-1}\rangle = \langle H_\lambda, h_1 h_{n-1}\rangle = \langle h_1^\perp H_\lambda, h_{n-1}\rangle \\ &= \sum_{\mu\subset_1\lambda} c^{(1)}_{\lambda\mu} \langle H_\mu, h_{n-1}\rangle = \sum_{\mu\subset_1\lambda} c^{(1)}_{\lambda\mu}, \end{aligned}$$

where the last equality uses Proposition 2.9.(III) and the identity $h_{n-1} = s_{(n-1)}$. For $n = 0$ the identity becomes $0 = 0$. □

The following is [GH02, Theorem 3.2]

Lemma 3.23. *Let $f \in \Lambda^{(d)}$ and $\lambda \vdash n$. Then*

$$\sum_{\mu\subset_d\lambda} c^{\omega f^\perp}_{\lambda\mu} = \left(\nabla^{-1} f\left[\frac{X-\epsilon}{M}\right]\right)\Bigg|_{X\mapsto D_\lambda}.$$

Proof. Replacing f by f^* in the formula of Theorem 3.1 we get.

$$\langle f^*, \tau_1 H_\lambda\rangle_* = (\nabla^{-1}\tau_{-\epsilon} f^*[X])\big|_{X=D_\lambda}.$$

Since $\tau_{-\epsilon} f^*[X] = \tau_{-\epsilon} f[X/M] = f[(X-\epsilon)/M]$, the right hand side coincides with the right hand side of the thesis. We consider the left hand side and apply Proposition 2.23

$$\begin{aligned}
\langle f^*, \tau_1 H_\lambda\rangle_* &= \langle \omega\phi f^*, \tau_1 H_\lambda\rangle \\
\text{(by def of } \phi \text{ and } f^* \text{ 2.22)} &= \langle \omega f[X], \tau_1 H_\lambda\rangle \\
\text{(by def of } \perp \text{ 1.54)} &= \langle 1, (\omega f)^\perp \tau_1 H_\lambda\rangle = \langle 1, \tau_1 (\omega f)^\perp H_\lambda\rangle \\
\text{(by 3.17, 3.20)} &= \left\langle 1, \tau_1\left(\sum_{\mu\subset_d\lambda} c_{\lambda\mu}^{\omega f^\perp} H_\mu\right)\right\rangle = \sum_{\mu\subset_d\lambda} c_{\lambda\mu}^{\omega f^\perp}\langle 1, \tau_1 H_\mu\rangle.
\end{aligned}$$

Now, using Proposition 1.79, we calculate

$$\langle 1, \tau_1 H_\mu\rangle = \left\langle 1, \sum_{k\in\mathbb{N}} h_k^\perp H_\mu\right\rangle = \sum_{k\in\mathbb{N}}\langle h_k, H_\mu\rangle = \sum_{k\in\mathbb{N}}\langle s_{(k)}, H_\mu\rangle = 1;$$

where the last equality comes from Proposition 2.9.(III). □

We can use this to deduce the following (see [Zab16, Lemma 12]).

Lemma 3.24. *For $k \in \mathbb{N}$ and $\lambda \in \mathsf{Par}$*

$$\sum_{\mu\subset_k\lambda} c_{\lambda\mu}^{e_k^\perp} = e_k[B_\lambda].$$

Proof. Applying Lemma 3.23 with $\omega f = e_k$, i.e. $f = h_k$, we obtain

$$\sum_{\mu\subset_k\lambda} c_{\lambda\mu}^{e_k^\perp} = \left(\nabla^{-1} h_k\left[\frac{X-\epsilon}{M}\right]\right)\Bigg|_{X\mapsto D_\lambda}$$

Using Proposition 1.69, we may write

$$\begin{aligned}
\nabla^{-1} h_k\left[\frac{X-\epsilon}{M}\right] &= \sum_{i=0}^{k}\nabla^{-1} h_i\left[\frac{X}{M}\right] h_{k-i}\left[\frac{-\epsilon}{M}\right] \\
\text{(by 3.16, 1.57)} &= \sum_{i=0}^{k} e_i\left[\frac{X}{M}\right] e_{k-i}\left[\frac{1}{M}\right] \\
\text{(by 1.69)} &= e_k\left[\frac{X+1}{M}\right].
\end{aligned}$$

Since by definition $D_\lambda = MB_\lambda - 1$, the thesis now follows easily. □

Lastly, we consider [GH02, Proposition 3.1]

Lemma 3.25. *For* $f = \sum_\lambda c_\lambda(q,t)s_\lambda \in \Lambda$ *set* $\tilde{f} := \sum_\lambda c(q^{-1},t^{-1})s_\lambda$*. Then*

$$c^{f^\perp}_{\lambda\mu}(q^{-1},t^{-1}) = \frac{T_\mu}{T_\lambda} c^{(\omega\tilde{f})^\perp}_{\lambda\mu}(q,t).$$

Proof. We start by showing the easy fact that for all $g,h \in \Lambda$, $\omega(g^\perp h) = (\omega g)^\perp(\omega h)$, indeed for any $b \in \Lambda$

$$\begin{aligned}\langle \omega(g^\perp h), b\rangle &= \langle g^\perp h, \omega(b)\rangle = \langle h, g\cdot\omega(b)\rangle = \langle h, \omega((\omega g)\cdot b)\rangle \\ &= \langle \omega(h), (\omega g)\cdot b\rangle = \langle (\omega g)^\perp \omega(h), b\rangle.\end{aligned}$$

By Definition 3.17, we have

$$f^\perp H_\lambda = \sum_\mu c^{f^\perp}_{\lambda\mu} H_\mu.$$

Applying ω to this equation, we get

$$(\omega f)^\perp \omega H_\lambda[X;q,t] = \sum_\mu c^{f^\perp}_{\lambda\mu}(q,t)\omega H_\mu[X;q,t]$$

We substitute $q \to q^{-1}$ and $t \to t^{-1}$. Notice that since s_λ does not depend on q,t, $\tilde{f}$ is just f after these substitutions.

$$(\omega\tilde{f})^\perp \omega H_\lambda[X;q^{-1},t^{-1}] = \sum_\mu c^{f^\perp}_{\lambda\mu}(q^{-1},t^{-1})\omega H_\mu[X;q^{-1},t^{-1}].$$

Multiply this equation by T_λ and use (2.8) to write

$$(\omega\tilde{f})^\perp H_\lambda[X;q,t] = T_\lambda \sum_\mu c^{f^\perp}_{\lambda\mu}(q^{-1},t^{-1})\frac{1}{T_\mu} H_\mu[X;q,t].$$

On the other hand, again by Definition 3.17, we have

$$(\omega\tilde{f})^\perp H_\lambda[X;q,t] = \sum_\mu c^{\omega\tilde{f}^\perp}_{\lambda\mu} H_\mu[X;q,t].$$

Comparing the coefficients of H_μ, the thesis follows. □

3.3 A summation formula

This section is dedicated to a summation formula involving Macdonald polynomials that underpins many of the strongest results concerning the Delta conjectures. It first appeared in [DIV18].

Theorem 3.26. *For* $n,m,s \in \mathbb{N}$

$$\begin{aligned}&\sum_{\lambda\vdash m+n} \frac{H_\lambda[X]}{w_\lambda} h_s[(1-t)B_\lambda]e_m[B_\lambda] \\ &= \sum_{l=0}^{m} t^{m-l} \sum_{k=0}^{s} q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-1 \\ s-k \end{bmatrix}_q h_{l+k}\left[\frac{X}{1-q}\right] h_{m-l}\left[\frac{X}{M}\right] e_{n-k}\left[\frac{X}{M}\right]\end{aligned}$$

The case $m = 0$ was know to Haglund [Hag04, Equation 2.38].

Lemma 3.27. *Let $n, m, s \in \mathbb{N}$. If $f \in \Lambda$ is such that*

$$\langle f, \tau_1 H_\lambda \rangle_* = h_s[(1-t)B_\lambda] e_m[B_\lambda] \tag{3.28}$$

then

$$\sum_{\lambda \vdash n} \frac{H_\lambda}{w_\lambda} h_s[(1-t)B_\lambda] e_m[B_\lambda] = \sum_{k=0}^{n} e^*_{n-k}(f)_k \tag{3.29}$$

where $(f)_k$ denotes the degree k homogeneous component of f.

Proof. Using (3.28) we get

$$\begin{aligned} \sum_{\lambda \vdash n} \frac{H_\lambda}{w_\lambda} h_s[(1-t)B_\lambda] e_m[B_\lambda] &= \sum_{\lambda \vdash n} \frac{H_\lambda}{w_\lambda} \langle f, \tau_1 H_\lambda \rangle_* \\ \text{(by 1.79)} &= \sum_{\lambda \vdash n} \frac{H_\lambda}{w_\lambda} \left\langle f, \sum_{k=0}^{n} h_k^\perp H_\lambda \right\rangle_* \\ \text{(by 2.28)} &= \sum_{\lambda \vdash n} \frac{H_\lambda}{w_\lambda} \left\langle \sum_{k=0}^{n} e^*_k f, H_\lambda \right\rangle_* = \cdots \end{aligned}$$

Since $\lambda \vdash n$, and $\langle H_\mu, H_\lambda \rangle_* = 0$ if $|\mu| \neq |\lambda|$, the scalar product of $e^*_k f$ with H_λ is only non-zero on one of the homogeneous components: $(e^*_k f)_n = e^*_k (f)_{n-k}$. It follows that

$$\cdots = \sum_{\lambda \vdash n} \frac{H_\lambda}{w_\lambda} \left\langle \sum_{k=0}^{n} e^*_k (f)_{n-k}, H_\lambda \right\rangle_* = \cdots$$

Using the fact that $\{H_\lambda\}_{\lambda \in \mathsf{Par}}$ and $\left\{ \frac{H_\lambda}{w_\lambda} \right\}_{\lambda \in \mathsf{Par}}$ are dual basis of Λ with respect to the star scalar product we may conclude

$$\cdots = \sum_{k=0}^{n} e^*_k (f)_{n-k} = \sum_{k=0}^{n} e^*_{n-k} (f)_k.$$

□

Proof of Theorem 3.26. Notice that the left hand side of (3.29) almost exactly coincides with the left hand side of the identity we want to prove. Our strategy therefore is to find a function f that satisfies (3.28); its thesis will then yield the desired identity. Thus recall the hypothesis of this lemma: $f \in \Lambda$ such that

$$\langle f, \tau_1 H_\lambda \rangle_* = h_s[(1-t)B_\lambda] e_m[B_\lambda].$$

The left hand side of the equation inspires us to apply Theorem 3.1:

$$\langle f, \tau_1 H_\lambda \rangle_* = (\nabla^{-1} \tau_{-\epsilon} f[X])\big|_{X \mapsto D_\lambda}.$$

So, let us look for a symmetric function f such that

$$\nabla^{-1} \tau_{-\epsilon} f[X] = h_s \left[\frac{X+1}{1-q} \right] e_m \left[\frac{X+1}{M} \right];$$

indeed, setting $X \mapsto D_\lambda$ in this equation gets us the desired hypothesis. In other words

$$\begin{aligned} f[X] =& \tau_\epsilon \nabla \left(h_s \left[\frac{X+1}{1-q} \right] e_m \left[\frac{X+1}{M} \right] \right) \\ \text{(by 1.69)} =& \tau_\epsilon \nabla \left(\sum_{i=0}^{s} \sum_{j=0}^{m} h_{s-i} \left[\frac{1}{1-q} \right] e_{m-j} \left[\frac{1}{M} \right] h_i \left[\frac{X}{1-q} \right] e_j \left[\frac{X}{M} \right] \right) = \cdots \end{aligned}$$

To continue, we need the following formula, which we will prove on page 55.

Lemma 3.30. *For $i, j \in \mathbb{N}$*

$$\nabla \left(h_i \left[\frac{X}{1-q} \right] e_j \left[\frac{X}{M} \right] \right) = \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{i+r} \left[\frac{X}{1-q} \right] h_{j-r} \left[\frac{X}{M} \right].$$

Using this result, we get

$$\begin{aligned} \cdots =& \tau_\epsilon \Bigg(\sum_{i=0}^{s} \sum_{j=0}^{m} h_{s-i} \left[\frac{1}{1-q} \right] e_{m-j} \left[\frac{1}{M} \right] \\ & \times \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{i+r} \left[\frac{X}{1-q} \right] h_{j-r} \left[\frac{X}{M} \right] \Bigg) \\ \text{(by def of } \tau_\epsilon) =& \sum_{i=0}^{s} \sum_{j=0}^{m} h_{s-i} \left[\frac{1}{1-q} \right] e_{m-j} \left[\frac{1}{M} \right] \\ & \times \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{i+r} \left[\frac{X+\epsilon}{1-q} \right] h_{j-r} \left[\frac{X+\epsilon}{M} \right] \\ \text{(by 1.69)} =& \sum_{i=0}^{s} \sum_{j=0}^{m} h_{s-i} \left[\frac{1}{1-q} \right] e_{m-j} \left[\frac{1}{M} \right] \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} \\ & \times \sum_{u=0}^{i+r} h_{i+r-u} \left[\frac{\epsilon}{1-q} \right] h_u \left[\frac{X}{1-q} \right] \sum_{v=0}^{j-r} h_{j-r-v} \left[\frac{\epsilon}{M} \right] h_v \left[\frac{X}{M} \right] \\ \text{(by 1.63, 1.57)} =& \sum_{i=0}^{s} \sum_{j=0}^{m} h_{s-i} \left[\frac{1}{1-q} \right] e_{m-j} \left[\frac{1}{M} \right] \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} \\ & \times \sum_{u=0}^{i+r} e_{i+r-u} \left[\frac{-1}{1-q} \right] h_u \left[\frac{X}{1-q} \right] \sum_{v=0}^{j-r} e_{j-r-v} \left[\frac{-1}{M} \right] h_v \left[\frac{X}{M} \right]. \end{aligned}$$

We extract the homogeneous part of degree d, i.e. $u + v = d$. Recall that $0 \leq d \leq i + j$. In the notation of Lemma 3.27 we write

$$\begin{aligned} (f[X])_d =& \sum_{i=0}^{s} \sum_{j=0}^{m} h_{s-i} \left[\frac{1}{1-q} \right] e_{m-j} \left[\frac{1}{M} \right] \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} \\ & \times \sum_{u=0}^{i+r} e_{i+r-u} \left[\frac{-1}{1-q} \right] h_u \left[\frac{X}{1-q} \right] e_{j-r-d+u} \left[\frac{-1}{M} \right] h_{d-u} \left[\frac{X}{M} \right] \end{aligned}$$

We make the substitutions $d \mapsto m+k$ and $u \mapsto l+k$, where k and l are new indices.

$$(f[X])_{m+k} = \sum_{i=0}^{s}\sum_{j=0}^{m} h_{s-i}\left[\frac{1}{1-q}\right] e_{m-j}\left[\frac{1}{M}\right] \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}$$
$$\times \sum_{l+k=0}^{i+r} e_{i+r-l-k}\left[\frac{-1}{1-q}\right] h_{l+k}\left[\frac{X}{1-q}\right] e_{j-r-m+l}\left[\frac{-1}{M}\right] h_{m-l}\left[\frac{X}{M}\right].$$

Since for $l > i+r-k$, $e_{i+r-l-k}\left[\frac{-1}{1-q}\right] = 0$ and for $l > m$, $h_{m-l}\left[\frac{X}{M}\right] = 0$, we may replace $\sum_{l+k=0}^{i+r}$ by $\sum_{l=-k}^{m}$.

$$(f[X])_{m+k} = \sum_{i=0}^{s}\sum_{j=0}^{m}\sum_{r=0}^{j}\sum_{l=-k}^{m} h_{s-i}\left[\frac{1}{1-q}\right] e_{m-j}\left[\frac{1}{M}\right] t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}$$
$$\times e_{i+r-l-k}\left[\frac{-1}{1-q}\right] h_{l+k}\left[\frac{X}{1-q}\right] e_{j-r-m+l}\left[\frac{-1}{M}\right] h_{m-l}\left[\frac{X}{M}\right]$$
$$(j \mapsto m-j) = \sum_{i=0}^{s}\sum_{j=0}^{m}\sum_{r=0}^{m-j}\sum_{l=-k}^{m} h_{s-i}\left[\frac{1}{1-q}\right] e_{j}\left[\frac{1}{M}\right] t^{m-j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}$$
$$\times e_{i+r-l-k}\left[\frac{-1}{1-q}\right] h_{l+k}\left[\frac{X}{1-q}\right] e_{l-r-j}\left[\frac{-1}{M}\right] h_{m-l}\left[\frac{X}{M}\right]$$

We have $l-j \le m-j$ so $r > m-j$ gives $e_{l-r-j}\left[\frac{-1}{M}\right] = 0$. Thus we may replace $\sum_{r=0}^{m-j}$ with $\sum_{r=0}^{m}$. For $j > l-r$ we again have $e_{l-r-j}\left[\frac{-1}{M}\right] = 0$. Furthermore $r \ge 0$, $l \le m$ implies $l-r \le m$. Thus, we may replace $\sum_{j=0}^{m}$ by $\sum_{j=0}^{l-r}$.

$$(f[X])_{m+k} = \sum_{i=0}^{s}\sum_{r=0}^{m}\sum_{l=-k}^{m} h_{s-i}\left[\frac{1}{1-q}\right] t^{m-l} \left(\sum_{j=0}^{l-r} e_j\left[\frac{1}{M}\right] t^{l-r-j} e_{l-r-j}\left[\frac{-1}{M}\right]\right)$$
$$\times q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}_q e_{i+r-l-k}\left[\frac{-1}{1-q}\right] h_{l+k}\left[\frac{X}{1-q}\right] h_{m-l}\left[\frac{X}{M}\right]$$
$$(\text{by } 1.63) = \sum_{i=0}^{s}\sum_{r=0}^{m}\sum_{l=-k}^{m} h_{s-i}\left[\frac{1}{1-q}\right] t^{m-l} \left(\sum_{j=0}^{l-r} e_j\left[\frac{1}{M}\right] e_{l-r-j}\left[\frac{-t}{M}\right]\right)$$
$$\times q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}_q e_{i+r-l-k}\left[\frac{-1}{1-q}\right] h_{l+k}\left[\frac{X}{1-q}\right] h_{m-l}\left[\frac{X}{M}\right]$$

$$(\text{by } 1.69) = \sum_{i=0}^{s}\sum_{r=0}^{m}\sum_{l=-k}^{m} h_{s-i}\left[\frac{1}{1-q}\right] t^{m-l} e_{l-r}\left[\frac{1-t}{M}\right]$$
$$\times q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}_q e_{i+r-l-k}\left[\frac{-1}{1-q}\right] h_{l+k}\left[\frac{X}{1-q}\right] h_{m-l}\left[\frac{X}{M}\right]$$

Since for $l < r$, $e_{l-r}\left[\frac{1-t}{M}\right] = 0$, and $r \ge 0$ we may replace $\sum_{l=-k}^{m}$ by $\sum_{l=0}^{m}$. Also,

$l \leq m$ so we may replace $\sum_{r=0}^{m}$ with $\sum_{r=0}^{l}$.

$$(f[X])_{m+k} = \sum_{l=0}^{m} h_{l+k}\left[\frac{X}{1-q}\right] h_{m-l}\left[\frac{X}{M}\right] t^{m-l}$$
$$\times \sum_{i=0}^{s}\sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] e_{i+r-l-k}\left[\frac{-1}{1-q}\right]$$

The second line of this formula can be simplified considerably. We state the needed identity here. Since its proof is quite technical and not particularly insightful, we postpone it to Appendix A.

Lemma 3.31. *For $l, s, k \in \mathbb{N}$ we have*

$$q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-1 \\ s-k \end{bmatrix}_q$$
$$= \sum_{i=0}^{s}\sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] e_{i+r-l-k}\left[\frac{-1}{1-q}\right].$$

Using this lemma, we get

$$(f[X])_{m+k} = \sum_{l=0}^{m} t^{m-l} q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-1 \\ s-k \end{bmatrix}_q h_{l+k}\left[\frac{X}{1-q}\right] h_{m-l}\left[\frac{X}{M}\right].$$

By construction, this f satisfies (3.28), so by Lemma 3.27 we have (3.29):

$$\sum_{\lambda \vdash m+n} \frac{H_\lambda[X]}{w_\lambda} h_s[(1-t)B_\lambda] e_m[B_\lambda] = \sum_{k=0}^{m+n} e_{m+n-k}\left[\frac{X}{M}\right] (f[X])_k$$
$$(k \mapsto m+k\ ) = \sum_{k=-m}^{n} e_{n-k}\left[\frac{X}{M}\right] (f[X])_{m+k}$$
$$= \sum_{k=0}^{s} e_{n-k}\left[\frac{X}{M}\right] (f[X])_{m+k};$$

where the last equality is justified by the fact that for $k > s$ or $k < 0$, we have $(f[X])_{m+k} = 0$, since one of the two q-binomials becomes 0, and when $k > n$, $e_{n-k}\left[\frac{X}{M}\right] = 0$. Combining the last two equations, we get exactly what we wanted to show. □

We still have to prove Lemma 3.30, for which we need two identities, the first one of which is the case $n = 0$ of the main theorem of this section.

Lemma 3.32. *For $m, s \in \mathbb{N}$*

$$\sum_{\lambda \vdash m} \frac{H_\lambda[X]}{w_\lambda} h_s[(1-t)B_\lambda] T_\lambda = \sum_{r=0}^{m} t^{m-r} \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q h_r\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right]$$

Proof. First notice that for $m = 0$ the equation becomes $\delta_{s,0} = \delta_{s,0}$ as on the left $B_\emptyset = 0$ and so $h_s[(1-t)B_\emptyset] = h_s[0] = \delta_{s,0}$, and on the right $\left[{s-1 \atop s}\right]_q = \delta_{s,0}$ (see the comments after Definition 2.15). Thus we may suppose $m > 1$.

We need the following equation [Hag08, Theorem 7.2]. For $m, r \in \mathbb{N}$ with $0 < r < m$

$$\nabla E_{m,r} = t^{m-r}(1-q^r) \sum_{\mu \vdash m-r} \frac{T_\mu}{w_\mu} \sum_{\lambda \supset_r \mu} \Pi_\lambda H_\lambda d_{\lambda\mu}^{h_r[X/(1-q)]};$$

which can be rewritten, using 3.20, 3.11 and 2.36

$$\nabla E_{m,r} = t^{m-r}(1-q^r)\mathbf{\Pi}\left(h_{m-r}\left[\frac{X}{M}\right] h_r\left[\frac{X}{1-q}\right]\right). \tag{3.33}$$

We show that this expression also holds for $r = 0$ and $r = m$. Indeed

- for $r = 0$ we get $0 = 0$ (by definition of $E_{m,0}$ 2.38);
- for $r = m$, combine Proposition 2.41 and Proposition 3.16 to deduce that $\nabla E_{m,m} = (q;q)_m h_m\left[\frac{X}{1-q}\right]$. Now thanks to Lemma 2.10 and Proposition 2.20 we know $(q;q)_m h_m\left[\frac{X}{1-q}\right] = H_{(m)}[X]$. Since $\Pi_{(m)} = (q;q)_{m-1}$, we may conclude that $\nabla E_{m,m} = (1-q^m)\mathbf{\Pi}\left(h_m\left[\frac{X}{1-q}\right]\right)$.

Applying ∇ to (2.39) gives:

$$\nabla e_m[X[s]_q] = \sum_{r=0}^{m} \frac{(q^s;q)_r}{(q;q)_r} \nabla E_{m,r};$$

which combined with (3.33) implies

$$\begin{aligned}
\nabla e_m[X[s]_q] &= \sum_{r=0}^{m} \frac{(q^s;q)_r}{(q;q)_r} t^{m-r}(1-q^r)\mathbf{\Pi}\left(h_{m-r}\left[\frac{X}{M}\right] h_r\left[\frac{X}{1-q}\right]\right) \\
\text{(by 2.12)} \quad &= \sum_{r=0}^{m} \frac{(q^{s+1};q)_{r-1}(1-q^s)}{(q;q)_{r-1}} t^{m-r}\mathbf{\Pi}\left(h_{m-r}\left[\frac{X}{M}\right] h_r\left[\frac{X}{1-q}\right]\right) \\
\text{(by 2.15)} \quad &= (1-q^s)\sum_{r=0}^{m} \begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix}_q t^{m-r}\mathbf{\Pi}\left(h_{m-r}\left[\frac{X}{M}\right] h_r\left[\frac{X}{1-q}\right]\right).
\end{aligned}$$

On the other hand applying ∇ to the formula of Proposition 3.13 gives

$$\nabla e_m\left[X[s]_q\right] = (1-q^s)\sum_{\lambda \vdash m} \frac{\Pi_\lambda H_\lambda[X]}{w_\lambda} h_s[(1-t)B_\lambda]T_\lambda.$$

Thus the thesis follows from equating the right hand sides of the last two equations, dividing by $(1-q^s)$ and applying $\mathbf{\Pi}^{-1}$. $\square$

The second result we need to prove Lemma 3.30 is [GHS11, Proposition 2.6]. Its proof is somewhat technical and not new, thus we postpone it to Appendix A.

Proposition 3.34. *For $i, j \in \mathbb{N}$*

$$h_i \left[\frac{X}{1-q}\right] e_j \left[\frac{X}{M}\right] = \sum_{\lambda \vdash i+j} \frac{H_\lambda[X]}{w_\lambda} q^{-\binom{i}{2}} \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q h_k[(1-t)B_\lambda]$$

Let us restate the lemma we want to prove.

Lemma 3.30. *For $i, j \in \mathbb{N}$*

$$\nabla \left(h_i \left[\frac{X}{1-q}\right] e_j \left[\frac{X}{M}\right] \right) = \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{i+r} \left[\frac{X}{1-q}\right] h_{j-r} \left[\frac{X}{M}\right].$$

Proof of Lemma 3.30. Clearly, a good place to start is applying ∇ to the formula in Proposition 3.34.

$$\begin{aligned} &\nabla \left(h_i \left[\frac{X}{1-q}\right] e_j \left[\frac{X}{M}\right] \right) \\ &= \sum_{\lambda \vdash i+j} \frac{T_\lambda H_\lambda[X]}{w_\lambda} q^{-\binom{i}{2}} \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q h_k[(1-t)B_\lambda] \\ \text{(by 3.32)} &= q^{-\binom{i}{2}} \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q \\ &\quad \times \sum_{r=0}^{i+j} t^{i+j-r} \begin{bmatrix} r+k-1 \\ k \end{bmatrix}_q h_r \left[\frac{X}{1-q}\right] h_{i+j-r} \left[\frac{X}{M}\right] = \cdots \end{aligned}$$

We need the following q-binomial identity, the proof of which can be found in Appendix A.

Lemma 3.35. *For $r, i \in \mathbb{N}$ we have*

$$q^{i(i-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q = \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q \begin{bmatrix} r+k-1 \\ k \end{bmatrix}_q.$$

Thanks to this lemma, we get

$$\begin{aligned} \cdots &= q^{-\binom{i}{2}} \sum_{r=0}^{i+j} q^{i(i-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q t^{i+j-r} h_r \left[\frac{X}{1-q}\right] h_{i+j-r} \left[\frac{X}{M}\right] \\ &= \sum_{r=0}^{i+j} q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}_q t^{i+j-r} h_r \left[\frac{X}{1-q}\right] h_{i+j-r} \left[\frac{X}{M}\right] \end{aligned}$$

and so the thesis follows by substituting $r \mapsto r + i$. □

3.4 Theta identities

The goal of this section is to prove the following significant results, which relate the Theta to the Delta operators. They first appeared in [DIV20].

Theorem 3.36. *For $n, k \in \mathbb{N}$ with $n > 0$ and $k < n$*

$$\Theta_k \nabla e_{n-k} = \Delta'_{e_{n-k-1}} e_n$$

Theorem 3.37. *For $n, k \in \mathbb{N}$ with $n > 0$ and $k < n$*

$$\frac{[n]_q}{[n-k]_q} \Theta_k \nabla \omega(p_{n-k}) = \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n).$$

They require a lemma each.

Lemma 3.38. *For $n, k \in \mathbb{N}$ with $0 < k < n$ and $\lambda \vdash n$*

$$\sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{(k)} B_\mu T_\mu = e_{n-k-1}[B_\lambda - 1] B_\lambda$$

Proof. We will prove this result by induction on $n-k$. Since by hypothesis $0 < k < n$, we have $n - k \geq 1$.

For $n - k = 1$, we have $k = n - 1$. So

$$\sum_{\mu \subset_{n-1} \lambda} c_{\lambda\mu}^{(n-1)} B_\mu T_\mu = c_{\lambda(1)}^{(n-1)} B_{(1)} T_{(1)} = c_{\lambda(1)}^{(n-1)}$$

By Theorem 3.20 we have $c_{\lambda(1)}^{n-1} H_{(1)} = \sum_{\mu \subset_{n-1} \lambda} c_{\lambda\mu}^{n-1} H_\mu = h_{n-1}^\perp H_\lambda$. Thus using Proposition 2.9.(III), we compute

$$\begin{aligned} c_{\lambda(1)}^{(n-1)} &= c_{\lambda(1)}^{(n-1)} \langle H_{(1)}, s_{(1)} \rangle = \langle h_{n-1}^\perp H_\lambda, e_1 \rangle \\ (\text{by def of } \perp) &= \langle H_\lambda, h_{n-1} e_1 \rangle \\ (\text{by 3.9}) &= e_1[B_\lambda] = B_\lambda = e_0[B_\lambda - 1] B_\lambda; \end{aligned}$$

which is what we wanted to prove.

For $n - k \geq 2$ start by using Proposition 3.22

$$\begin{aligned} \sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{(k)} B_\mu T_\mu &= \sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{(k)} \left(\sum_{\nu \subset_1 \mu} c_{\mu\nu}^{(1)} \right) T_\mu = \sum_{\nu \subset_{k+1} \lambda} \left(\sum_{\nu \subset_1 \mu \subset_k \lambda} c_{\lambda\mu}^{(k)} c_{\mu\nu}^{(1)} T_\mu \right) \\ &= \sum_{\nu \subset_{k+1} \lambda} T_\nu B_{\lambda/\nu} \left(\frac{1}{B_{\lambda/\nu}} \sum_{\nu \subset_1 \mu \subset_k \lambda} c_{\lambda\mu}^{(k)} c_{\mu\nu}^{(1)} \frac{T_\mu}{T_\nu} \right) \\ (\text{by 3.21}) &= \sum_{\nu \subset_{k+1} \lambda} T_\nu B_{\lambda/\nu} c_{\lambda\nu}^{(k+1)} \\ (\text{by def of } B_{\lambda/\nu}) &= B_\lambda \sum_{\nu \subset_{k+1} \lambda} T_\nu c_{\lambda\nu}^{(k+1)} - \sum_{\nu \subset_{k+1} \lambda} T_\nu B_\nu c_{\lambda\nu}^{(k+1)} \end{aligned}$$

This formula contains the following summation which we develop using (2.30)

$$\begin{aligned}\sum_{\nu\subset_{k+1}\lambda} T_\nu c^{(k+1)}_{\lambda\nu} &= \sum_{\nu\subset_{k+1}\lambda} c^{(k+1)}_{\lambda\nu} e_{n-k-1}[B_\nu] \\ \text{(by 3.9)} &= \sum_{\nu\subset_{k+1}\lambda} c^{(k+1)}_{\lambda\nu}\langle H_\nu, e_{n-k-1}\rangle \\ \text{(by 3.17, 3.20)} &= \langle h^\perp_{k+1} H_\lambda, e_{n-k-1}\rangle \\ \text{(by def of } \perp \text{ 1.54)} &= \langle H_\lambda, e_{n-k-1}h_{k+1}\rangle \\ \text{(by 3.9)} &= e_{n-k-1}[B_\lambda]\end{aligned}$$

So combining these two calculations we get

$$\sum_{\mu\subset_k\lambda} c^{(k)}_{\lambda\mu} B_\mu T_\mu = B_\lambda e_{n-k-1}[B_\lambda] - \sum_{\nu\subset_{k+1}\lambda} T_\nu B_\nu c^{(k+1)}_{\lambda\nu}. \tag{3.39}$$

By the induction hypothesis, we know that

$$\sum_{\nu\subset_{k+1}\lambda} c^{(k+1)}_{\lambda\nu} B_\nu T_\nu = e_{n-k-2}[B_\lambda - 1]B_\lambda. \tag{3.40}$$

Thus, using (3.39) and (3.40) give

$$\begin{aligned}\sum_{\mu\subset_k\lambda} c^{(k)}_{\lambda\mu} B_\mu T_\mu &= B_\lambda e_{n-k-1}[B_\lambda] - e_{n-k-2}[B_\lambda - 1]B_\lambda \\ &= (e_{n-k-1}[B_\lambda] - e_{n-k-2}[B_\lambda - 1])B_\lambda \\ \text{(by (2.34))} &= e_{n-k-1}[B_\lambda - 1]B_\lambda\end{aligned}$$

□

Proof of Theorem 3.36. For $k = 0$ we have, by Proposition 2.33

$$\Theta_0\nabla e_n = \nabla e_n = \Delta_{e_n} e_n = \Delta'_{e_{n-1}} e_n.$$

So we may suppose $0 < k < n$, (and thus apply Lemma 3.38). We start by using

Corollory 3.14 to write

$$\begin{aligned}
\Theta_k \nabla e_{n-k} &= \Theta_k \nabla \left(\sum_{\lambda \vdash n-k} \frac{M\Pi_\lambda B_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by definition of } \nabla) &= \Theta_k \left(\sum_{\lambda \vdash n-k} \frac{M\Pi_\lambda B_\lambda T_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by definition of } \Theta_k) &= \mathbf{\Pi} e_k^* \mathbf{\Pi^{-1}} \left(\sum_{\lambda \vdash n-k} \frac{M\Pi_\lambda B_\lambda T_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by definition of } \mathbf{\Pi}) &= \mathbf{\Pi} e_k^* \left(\sum_{\lambda \vdash n-k} \frac{M B_\lambda T_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by 3.17, 3.20)} \ &= \mathbf{\Pi} \left(\sum_{\lambda \vdash n-k} \frac{M B_\lambda T_\lambda}{w_\lambda} \sum_{\mu \supset_k \lambda} d_{\mu\lambda}^{(k)} H_\mu \right) \\
\text{(by (3.19))} &= \mathbf{\Pi} \left(\sum_{\lambda \vdash n-k} \frac{M B_\lambda T_\lambda}{w_\mu} \sum_{\mu \supset_k \lambda} c_{\mu\lambda}^{(k)} H_\mu \right) \\
&= \mathbf{\Pi} \left(\sum_{\mu \vdash n} \frac{M}{w_\mu} H_\mu \sum_{\lambda \subset_k \mu} B_\lambda T_\lambda c_{\mu\lambda}^{(k)} \right) \\
\text{(by 3.38)} &= \mathbf{\Pi} \left(\sum_{\mu \vdash n} \frac{M}{w_\mu} H_\mu e_{n-k-1}[B_\mu - 1] B_\mu \right) \\
\text{(by definition of } \mathbf{\Pi}) &= \sum_{\mu \vdash n} e_{n-k-1}[B_\mu - 1] \frac{M B_\mu \Pi_\mu H_\mu}{w_\mu}
\end{aligned}$$

$$\text{(by definition of } \Delta_f \text{ and 3.14)} = \Delta'_{e_{n-k-1}} \left(\sum_{\mu \vdash n} \frac{M B_\mu \Pi_\mu H_\mu}{w_\mu} \right) = \Delta'_{n-k-1} e_n.$$

□

Lemma 3.41. *For $n, k \in \mathbb{N}$, with $0 < k < n$ and $\lambda \vdash n$*

$$e_{n-k}[B_\lambda] = \sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{(k)} T_\mu.$$

Proof. Using the definitions of B_λ and T_λ (Definition 2.29), we may write

$$\begin{aligned}
e_{n-k}[B_\lambda] &= T_\lambda e_k[B_\lambda(q^{-1}, t^{-1})] \\
\text{(by 3.24)} \ &= T_\lambda \sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{e_k^\perp}(q^{-1}, t^{-1})
\end{aligned}$$

We will now use Lemma 3.25. Using the notation of that lemma. Since e_k does not depend on q,t, we have $\tilde{e}_k = e_k$. Thus

$$e_{n-k}[B_\lambda] = T_\lambda \sum_{\mu \subset_k \lambda} \frac{T_\mu}{T_\lambda} c_{\lambda\mu}^{h_k^\perp}(q,t) = \sum_{\mu \subset_k \lambda} T_\mu c_{\lambda\mu}^{(k)}.$$

□

Proof of Theorem 3.37. For $k = 0$, the equation reduces to the following identity, which holds thanks to Proposition 2.33:

$$\nabla \omega(p_n) = \Delta_{e_n} \omega(p_n).$$

Let us now suppose $0 < k < n$. Using Proposition 3.15

$$\begin{aligned}
\Theta_k \nabla \frac{[n]_q}{[n-k]_q} \omega(p_{n-k}) &= [n-k]_t [n]_q \Theta_k \nabla \left(\sum_{\lambda \vdash n-k} \frac{M \Pi_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by definition of } \nabla) &= [n-k]_t [n]_q \Theta_k \left(\sum_{\lambda \vdash n-k} \frac{M \Pi_\lambda T_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by definition of } \Theta_k) &= [n-k]_t [n]_q \mathbf{\Pi} e_k^* \mathbf{\Pi}^{-1} \left(\sum_{\lambda \vdash n-k} \frac{M \Pi_\lambda T_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by definition of } \mathbf{\Pi}) &= [n-k]_t [n]_q \mathbf{\Pi} e_k^* \left(\sum_{\lambda \vdash n-k} \frac{M T_\lambda H_\lambda}{w_\lambda} \right) \\
\text{(by 3.17, 3.20)} &= [n-k]_t [n]_q \mathbf{\Pi} \left(\sum_{\lambda \vdash n-k} \frac{M T_\lambda}{w_\lambda} \sum_{\mu \supset_k \lambda} d_{\mu\lambda}^{(k)} H_\mu \right) \\
\text{(by (3.19))} &= [n-k]_t [n]_q \mathbf{\Pi} \left(\sum_{\lambda \vdash n-k} \frac{M T_\lambda}{w_\mu} \sum_{\mu \supset_k \lambda} c_{\mu\lambda}^{(k)} H_\mu \right) \\
&= [n-k]_t [n]_q \mathbf{\Pi} \left(\sum_{\mu \vdash n} \frac{M}{w_\mu} H_\mu \sum_{\lambda \subset_k \mu} T_\lambda c_{\mu\lambda}^{(k)} \right) \\
\text{(by 3.41)} &= [n-k]_t [n]_q \mathbf{\Pi} \left(\sum_{\mu \vdash n} \frac{M}{w_\mu} e_{n-k}[B_\mu] H_\mu \right) \\
\text{(by definition of } \mathbf{\Pi} \text{ and } \Delta_f) &= [n-k]_t [n]_q \Delta_{e_{n-k}} \left(\sum_{\mu \vdash n} \frac{M \Pi_\mu}{w_\mu} H_\mu \right) \\
\text{(by 3.15)} &= [n-k]_t [n]_q \Delta_{e_{n-k}} \left(\frac{\omega(p_n)}{[n]_q [n]_t} \right) \\
&= \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n)
\end{aligned}$$

□

Chapter 4

Lattice paths

In this chapter, we introduce families of combinatorial objects, whose enumeration will give nice formulas for the symmetric functions that are the focus of this text, that is $\Delta'_{e_{n-k-1}} e_n$ and $\Delta_{e_{n-k}} \omega(p_n)$ and related symmetric functions.

4.1 The objects

Definition 4.1. Given $S \subseteq \mathbb{N}^2$, a *lattice path* in the plane with steps in S is some sequence $p_0 p_1 \cdots p_k$ such that $p_i \in \mathbb{N}^2$ and $p_i - p_{i-1} \in S$ for all $i \in \{1, \dots, k\}$.

Definition 4.2. Let $n \in \mathbb{N}$. A *square path* of size n, is a lattice path $p_0 \dots p_{2n}$ with steps in $\{E := (1,0), N := (0,1)\}$ such that $p_0 = (0,0)$, $p_{2n} = (n,n)$ and $p_{2n} - p_{2n-1} = E$. Given any such path, its *E, N-sequence* is a sequence of length $2n$ whose i-th element is $p_i - p_{i-1} \in \{E, N\}$. We refer to the elements of this sequence as *steps* of the path, call steps equal to N *north* or *vertical* steps and steps equal to E *east* or *horizontal* steps. We call p_{i-1} and p_i the *starting point* and *endpoint* of the step $p_i - p_{i-1}$, respectively. It is clear that a square path is entirely determined[1] by its E, N-sequence. The set of square paths of size n is denoted by $\mathsf{SQ}(n)$.

It is not hard to see that there are exactly $\binom{2n-1}{n}$ square paths of size n, indeed its E, N-sequence is a sequence of $2n$ steps, with n east steps and n north steps, the last of which must be an east step, so choosing n of the remaining $2n-1$ steps to be north steps uniquely determines the path.

Definition 4.3. Let $n \in \mathbb{N}$. A *Dyck path* of size n is a square path $p_0 \cdots p_{2n}$ such that $p_i \in \{(k,l) \in \mathbb{N}^2 \mid l \geq k\}$ for all $i \in \{0, \dots, 2n\}$. We denote the set of such paths by $\mathsf{D}(n)$.

We think of $\mathsf{D}(n)$ as a subset of $\mathsf{SQ}(n)$.

The number of Dyck paths of size n is one of the many incarnations of the famous *Catalan numbers*. That is $\#\mathsf{D}(n) = \frac{1}{n+1}\binom{2n}{n}$. See [Sta15] for a long list of objects counted by these numbers.

[1] We will often identify a square path with its E, N-sequence.

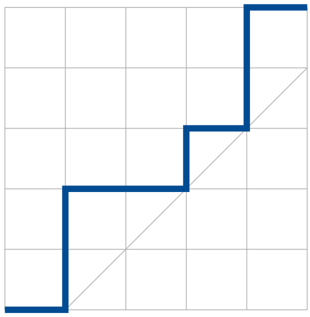


Figure 4.1: An element of $\mathsf{SQ}(5)$ (left) and of $\mathsf{D}(5)$ (right).

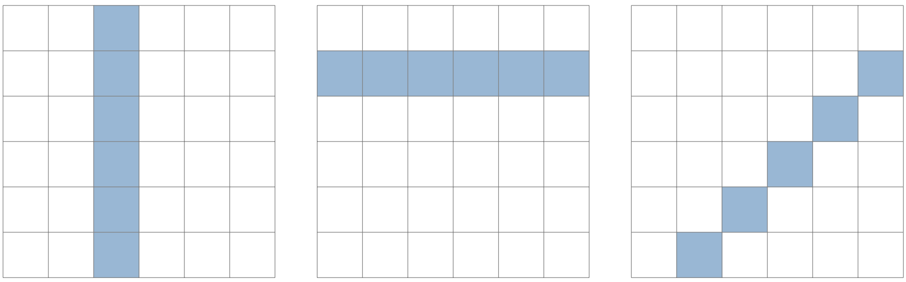


Figure 4.2: Column, row and diagonal of $(\mathbb{R}_{\geq 0})^2$.

We encourage the reader to think about these objects visually. We draw a square path $p_0 \dots p_{2n}$ by plotting the p_i in the first quadrant of the plane, $(\mathbb{R}_{\geq 0})^2$, and connecting p_{i-1} to p_i with a line segment. Thus, an E (respectively N) in the E, N-sequence of a path corresponds to a horizontal (respectively vertical) line segment of length 1. Visualised in this way, Dyck paths are square paths that stay weakly above the line $x = y$.

Example. Figure 4.1 contains and example of a square path that is not a Dyck path (left) and a Dyck path (right). The E, N-sequence of the path on the left is $ENNEENENNE$.

Definition 4.4.
- A *square* of $(\mathbb{R}_{\geq 0})^2$ is any square of area 1 contained in $(\mathbb{R}_{\geq 0})^2$ whose vertices have integer coordinates.
- For $i \in \mathbb{P}$, the i-th *column* of $(\mathbb{R}_{\geq 0})^2$ is $\{(x, y) \in (\mathbb{R}_{\geq 0})^2 \mid x \in [i-1, i]\}$.
- For $i \in \mathbb{P}$, the i-th *row* of $(\mathbb{R}_{\geq 0})^2$ is $\{(x, y) \in (\mathbb{R}_{\geq 0})^2 \mid y \in [i-1, i]\}$.
- For $i \in \mathbb{Z}$, the i-th *diagonal* of $(\mathbb{R}_{\geq 0})^2$ is the union of its squares that have two vertices on the line $y = x + i$. The 0-th diagonal is called the *main diagonal*.

Example. In Figure 4.2, we shaded the 3-rd column (left), the 5-th row (middle) and -1-th diagonal (right) of $(\mathbb{R}_{\geq 0})^2$.

Definition 4.5. A vertical step of a square path π is said to be at *height* i if its starting and endpoint are vertices of the left edge of a square in the i-th diagonal of

$(\mathbb{R}_{\geq 0})^2$. Take m to be the minimum of the heights of all vertical steps of a square path π then the m-th diagonal is called the *base diagonal* of π, and its *shift* is $\mathsf{shift}(\pi) := |m|$.

Dyck paths are exactly the square paths of shift 0, thus their main and base diagonal always coincide.

Example. The square path on the left in Figure 4.1 has base diagonal $y = x - 1$.

Definition 4.6. Given $\pi \in \mathsf{SQ}(n)$, let $a_i(\pi) \in \mathbb{Z}$ be such that the i-th north step of π lies at height $a_i(\pi)$. The tuple $a(\pi) := (a_1(\pi), \dots, a_n(\pi)) \in \mathbb{Z}^n$ is called the *area word* of the path and its components $a_i(\pi)$ are called its letters.

Example. The area word of the path on the right in Figure 4.1 is 01011.

We make some observations about area words.

- A square path is completely determined by its area word.
- If $(a_1, \dots, a_n)$ is the area word of a square path then $a_{i+1} \leq a_i + 1$ for all $1 \leq i \leq n-1$; and $a_1 \leq 0$.
- The last letter of the area word of a square path must be ≥ 0.
- Any sequence of n integers that satisfies the previous two properties is the area word of some square path.
- The area word of a square path is the area word of a Dyck path if and only if it contains only non-negative letters.

Definition 4.7. Take $\pi \in \mathsf{SQ}(n)$ of shift s. A *partial labelling* of π is a tuple $w \in \mathbb{N}^n$ such that

- $a_i(\pi) < a_{i+1}(\pi)$ implies $w_i < w_{i+1}$;
- $a_1(\pi) = 0$ implies $w_1 \neq 0$;
- there exists i such that $a_i(\pi) = -s$ and $w_i \neq 0$.

A *labelling* of a square path is a partial labelling whose entries are non-zero (thus the two last conditions are trivially satisfied). Denote the set of partial labellings of π with exactly m zeros by $\mathsf{La}(\pi, m)$ and the set of labellings of π by $\mathsf{La}(\pi)$.

We distinguish between partial labellings (using 0) and labellings (not using 0), as the zero and non-zero labels are really intended to be two distinct attributes. Indeed, the word "partial" comes from the fact that one might think of steps labelled 0 as unlabelled steps (see Definition 4.17). However, the zero label is a useful artifice, as it is smaller than all the other labels, a property which we will often use (see Defintion 4.19).

Visually, we will draw w_i in the square directly to the right of the i-th north step of the path. Thus, the first condition of the definition of says that labels lying in the same column are strictly increasing from bottom to top. The second condition states that if a path starts with a vertical step, the label of this step may not be 0. The third condition ensures that at least one of the vertical steps starting from the base diagonal has a non-zero label. See Figure 4.3 for an example of a partially labelled square path (left) and a labelled Dyck path (right).

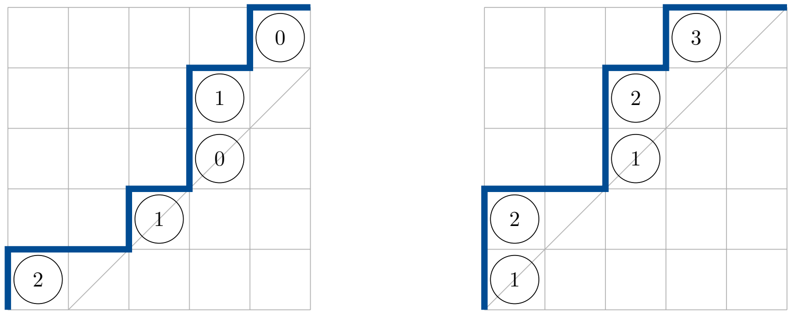


Figure 4.3: An element of $\mathsf{LSQ}(2,3)$ (left) and of $\mathsf{LD}(5)$ (right).

Definition 4.8. A *rise* of $\pi \in \mathsf{SQ}(n)$ is an index $i \in \{2, \dots, n\}$ such that $a_i(\pi) > a_{i-1}(\pi)$. We identify this index i with the i-th vertical step of the path, so that a rise is a vertical step preceded by another vertical step. The set of these indices is denoted by $\mathsf{Rise}(\pi)$.

Definition 4.9. A *valley* of a $\pi \in \mathsf{SQ}(n)$ is an index $i \in \{2, \dots, n\}$ such that $a_i(\pi) \leq a_{i-1}(\pi)$. We identify this index i with the i-th vertical step of the path, so that a valley is a vertical step preceded by horizontal step. The set of these indices is denoted by $\mathsf{Val}(\pi)$.

Definition 4.10. A *(contractible) valley* of a pair (π, w) where w is a partial labelling of a square path π is an index $i \in \{1, \dots, n\}$ such that one of the following holds:

- $i = 1$ and either $a_1 < -1$, or $a_1(\pi) = -1$ and $w_1 > 0$;
- $i > 1$ and $a_i < a_{i-1}$;
- $i > 1$ and $a_i = a_{i-1}$ and $w_i > w_{i-1}$.

We identify this index i with the i-th vertical step of the path. The set of these indices is denoted by $\mathsf{Val}(\pi, w)$.

Thus (disregarding the first vertical step which is special) a vertical step is a valley if it is either preceded by two horizontal steps or preceded by a single horizontal step which is preceded by a vertical step whose label is strictly smaller than its own label. Thus if we were to remove the horizontal step preceding valley and add it to the end of the path, we would obtain a valid (partially) labelled square path, hence the word "contractible".

Definition 4.11. A *rise-valley decorated square path* is a triple (π, dr, dv) where π is a square path, dr is a subset of $\mathsf{Rise}(\pi)$ and dv a subset of $\mathsf{Val}(\pi)$. The set of such triples where π is of size n, $\#dr = k$ and $\#dv = l$ is denoted by $\mathsf{SQ}(n)^{*k \bullet l}$. Define the subset $\mathsf{D}(n)^{*k \bullet l} \subseteq \mathsf{SQ}(n)^{*k \bullet l}$ of elements (π, dr, dv) such that π is a Dyck path.

Definition 4.12. A *rise-valley decorated partially labelled square path* is a quadruple (π, w, dr, dv) where π is a square path, w is a partial labelling of π, dr is a subset of $\mathsf{Rise}(\pi)$ and dv a subset of $\mathsf{Val}(\pi, w)$. The set of such quadruples

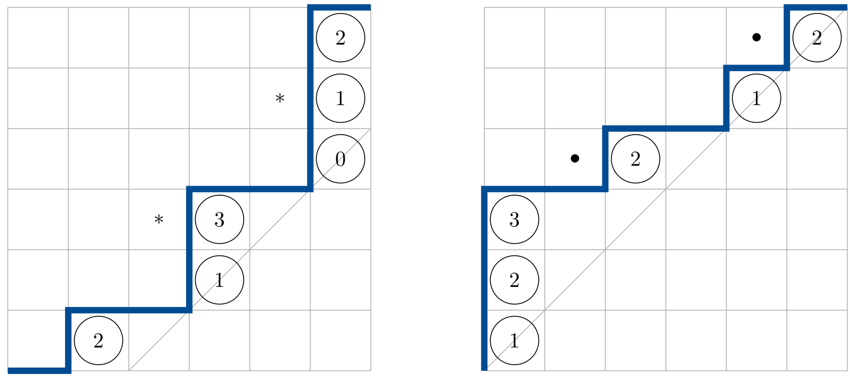


Figure 4.4: An element of $\mathsf{LSQ}(1,5)^{*2}$ (left) and of $\mathsf{LD}(6)^{\bullet k}$ (right).

where π is of size $m+n$, w has exactly m zero entries (and thus n positive entries), $\#dr = k$ and $\#dv = l$ is denoted by $\mathsf{LSQ}(m,n)^{*k\bullet l}$. Define the subset $\mathsf{LD}(m,n)^{*k\bullet l} \subseteq \mathsf{LSQ}(m,n)^{*k\bullet l}$ of elements (π, w, dr, dv) such that π is a Dyck path.

Visually, for all $i \in dr$ we draw a $*$ in the square directly to the left of the i-th vertical step of π. And for all $i \in dv$ we draw a $\bullet$ in the same way.

The sets of objects whose enumeration will provide combinatorial interpretations of our symmetric functions are all special cases or subsets of $\mathsf{SQ}(n)^{*k\bullet l}$ or $\mathsf{LSQ}(m,n)^{*k\bullet l}$. In particular, in this text we never actually decorate both rises and valleys simultaneously. We list here all such sets, whose notation we aimed to make intuitive.

$$
\begin{aligned}
\mathsf{SQ}(n)^{*k} &:= \mathsf{SQ}(n)^{*k\bullet 0} & \mathsf{D}(n)^{*k} &:= \mathsf{D}(n)^{*k\bullet 0} \\
\mathsf{SQ}(n)^{\bullet k} &:= \mathsf{SQ}(n)^{*0\bullet k} & \mathsf{D}(n)^{\bullet k} &:= \mathsf{D}(n)^{*0\bullet k} \\
\mathsf{LSQ}(n) &:= \mathsf{LSQ}(0,n)^{*0\bullet 0} & \mathsf{LD}(n) &:= \mathsf{LD}(0,n)^{*0\bullet 0} \\
\mathsf{LSQ}(m,n) &:= \mathsf{LSQ}(m,n)^{*0\bullet 0} & \mathsf{LD}(m,n) &:= \mathsf{LD}(m,n)^{*0\bullet 0} \\
\mathsf{LSQ}(n)^{*k} &:= \mathsf{LSQ}(0,n)^{*k\bullet 0} & \mathsf{LD}(n)^{*k} &:= \mathsf{LD}(0,n)^{*k\bullet 0} \\
\mathsf{LSQ}(n)^{\bullet k} &:= \mathsf{LSQ}(0,n)^{*0\bullet k} & \mathsf{LD}(n)^{\bullet k} &:= \mathsf{LD}(0,n)^{*0\bullet k} \\
\mathsf{LSQ}(m,n)^{*k} &:= \mathsf{LSQ}(m,n)^{*k\bullet 0} & \mathsf{LD}(m,n)^{*k} &:= \mathsf{LD}(m,n)^{*k\bullet 0} \\
\mathsf{LSQ}(m,n)^{\bullet k} &:= \mathsf{LSQ}(m,n)^{*0\bullet k} & \mathsf{LD}(m,n)^{\bullet k} &:= \mathsf{LD}(m,n)^{*0\bullet k}.
\end{aligned}
$$

Remark 4.13. The elements of these sets are technically either triples (π, dr, dv) or quadruples (π, w, dr, dv), but in a context where dv or dr is always empty, we will often consider them as pairs or triples. This should not lead to any confusion when the set is clearly specified.

We will also need the following slightly modified version of $\mathsf{LSQ}(m,n)^{\bullet k}$.

Definition 4.14. Set $\mathsf{LSQ}'(m,n)^{\bullet k} \subseteq \mathsf{LSQ}(m,n)^{\bullet k}$, to be the subset of $(\pi, w, \emptyset, dv)$ such that there exists $1 \leq i \leq m+n$ such that $a_i(\pi) = -\mathsf{shift}(\pi)$, $w_i \neq 0$ and $i \notin dv$. In other words, there exists a vertical step starting from the base diagonal that is neither a decorated valley nor labelled with a 0.

Next, we introduce definitions that will be used to partition these combinatorial sets.

Definition 4.15. Take $P \in \mathsf{D}(n)^{*k\bullet l} \sqcup \mathsf{LD}(m,n)^{*k\bullet l}$. A *touching point* is the starting point of a vertical step at height 0 that is neither a decorated valley nor labelled with a 0 (if P is labelled). The number of touching points of a path is denoted as $\mathsf{touch}(P)$.

Definition 4.16. Given $P \in \mathsf{D}(n)^{*k\bullet l} \sqcup \mathsf{LD}(m,n)^{*k\bullet l}$, consider $\{t_0, t_1, \dots\}$ to be the increasing set of abscisse coordinates of the touching points of P. For $P \in \mathsf{LD}(m,n)^{*k,\bullet l}$ with[2] $m \cdot l = 0$ and labelling w, set

$$\alpha_i := \#\{j \in \{t_{i-1}, \dots, t_i\} \mid j \notin \mathsf{Rise}(P) \cup \mathsf{Val}(P), w_j \neq 0\}.$$

For $P \in \mathsf{D}(n)^{*k,\bullet l}$, set

$$\alpha_i := \#\{j \in \{t_{i-1}, \dots, t_i\} \mid j \notin \mathsf{Rise}(P) \cup \mathsf{Val}(P)\}.$$

The *diagonal composition* of P is defined as $\mathsf{dcomp}(P) := (\alpha_1, \alpha_2, \dots)$, a composition of $n-k-l$. Clearly, we have $\ell(\mathsf{dcomp}(P)) = \mathsf{touch}(P)$.

Given $n, m, k, l \in \mathbb{N}$ with $m \cdot l = 0$ and a composition $\alpha \vDash n-k-l$, we define the following sets

$$\begin{aligned} \mathsf{D}(\alpha)^{*k\bullet l} &:= \{P \in \mathsf{D}(n)^{*k\bullet l} \mid \mathsf{dcomp}(P) = \alpha\} \\ \mathsf{LD}(m,\alpha)^{*k\bullet l} &:= \{P \in \mathsf{LD}(m,n)^{*k\bullet l} \mid \mathsf{dcomp}(P) = \alpha\} \end{aligned}$$

of which we will use only the following special cases.

$$\begin{array}{lll} \mathsf{D}(\alpha)^{*k} := \mathsf{D}(\alpha)^{*k\bullet 0} & \mathsf{D}(\alpha)^{\bullet k} := \mathsf{D}(\alpha)^{*0\bullet k} & \\ \mathsf{LD}(m,\alpha)^{*k} := \mathsf{LD}(m,\alpha)^{*k\bullet 0} & \mathsf{LD}(\alpha)^{*k} := \mathsf{LD}(0,\alpha)^{*k} & \mathsf{LD}(\alpha)^{\bullet k} := \mathsf{LD}(0,\alpha)^{*0\bullet k}. \end{array}$$

We will use these objects to build symmetric functions via the following construction.

Definition 4.17. For $w \in \mathbb{N}^n$ the *monomial associated* to w is

$$x^w := \prod_{i=1}^{m+n} x_{w_i} \Big|_{x_0=1}.$$

For a path $P := (\pi, w, dr, dv) \in \mathsf{LSQ}(m,n)^{*k,\bullet l}$ the monomial associated to it is the monomial associated to its labelling: $x^P := x^w$. Since $w \in \mathbb{N}^{m+n}$ and there are m labels equal to 0, x^P is a monomial of degree n.

Example. The monomial of the path on the left in Figure 4.4 is $x_1^2 x_2^2 x_3$.

4.2 Statistics

Now that we have defined the sets of combinatorial objects, we introduce some *statistics* on these sets. Given a combinatorial set S, a statistic on this set is just a function $S \to \mathbb{N}$, essentially a counting function of some attribute of the object.

[2] We do not venture a definition for the diagonal decomposition of paths that have both zero labels and decorations on valleys, as we have not yet found a valley version of Conjecture 5.12.

Definition 4.18. Given π a square path and $dr \subseteq \mathsf{Rise}(\pi)$, we define

$$\mathsf{area}(\pi, dr) \coloneqq \sum_{i \notin dr} (a_i(\pi) + \mathsf{shift}(\pi)) .$$

In other words, the area is the number of whole squares between the path and the base diagonal, that are not in a row containing a rise. For $P \coloneqq (\pi, dr, dv) \in \mathsf{SQ}(n)^{*k \bullet l}$ or $Q \coloneqq (\pi, w, dr, dv) \in \mathsf{LSQ}(m,n)^{*k \bullet l}$, we define $\mathsf{area}(P) = \mathsf{area}(Q) = \mathsf{area}(\pi, dr)$ so that the area of a path is independent of its valley decorations and partial labelling.

Example. The path on the left in Figure 4.4 has area equal to 3.

Definition 4.19. Given $\pi \in \mathsf{SQ}(m+n)$, $w \in \mathsf{La}(\pi, m)$ and $dv \subseteq \mathsf{Val}(\pi, w)$ define

- a *primary inversion* as a pair (i, j) with $i \notin dv$ such that $1 \leq i < j \leq m+n$, $a_i(\pi) = a_j(\pi)$ and $w_i < w_j$;
- a *secondary inversion* as a pair (i, j) with $i \notin dv$ such that $1 \leq i < j \leq m+n$, $a_i(\pi) = a_j(\pi) + 1$ and $w_i > w_j$;
- *bonus dinv* as an index i such that $a_i(\pi) < 0$ and $w_i > 0$.

We then set

$$\begin{aligned} \mathsf{dinv}(\pi, w, dv) \coloneqq & \#\text{primary inversions} + \#\text{secondary inversions} \\ & + \text{bonus dinv} - \#dv \end{aligned}$$

For $P \coloneqq (\pi, w, dr, dv) \in \mathsf{LSQ}(m,n)^{*k \bullet l}$ we set $\mathsf{dinv}(P) \coloneqq \mathsf{area}(\pi, w, dv)$ so that the dinv of a path does not depend on its rise decorations.

Example. The path on the right in Figure 4.4 has 2 primary inversions, $(1, 6)$ and $(5, 6)$, 2 secondary inversions, $(2, 5)$ and $(3, 4)$, no bonus dinv (no Dyck path does) and two decorated valleys. Thus its dinv is 2.

From the definition, it is not immediately clear that the dinv of a path is always a non-negative quantity.

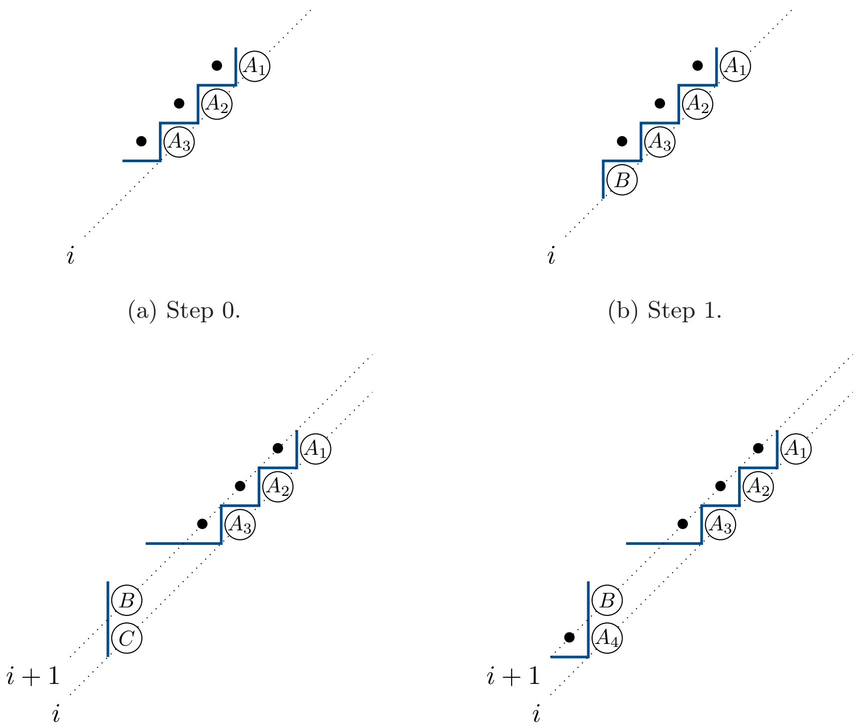
Proposition 4.20. *For all $P \in \mathsf{LSQ}(m,n)^{*k \bullet l}$, $\mathsf{dinv}(P) \geq 0$.*

Proof. We will show that each decorated valley of P implies at least one unit of primary, secondary or bonus dinv.

Consider a decorated valley at height i. By definition, it is preceded by a horizontal step. For the remainder of the proof, "decorated valley" will refer to the decorated vertical step *and* the horizontal step that precedes it.

Step 0. Suppose the valley is part of a string of decorated valleys, labelled $A_s, \dots, A_1$ from left to right, see Figure 4.5a. Since the valleys are contractible we must have $A_s < \cdots < A_1$. This string is then directly preceded either by a vertical step that is not a decorated valley (as otherwise this would be part of the string), or by a horizontal step.

Step 1. If the string is preceded by a vertical step, then this step's label, say B, must be such that $B < A_s < \cdots < A_1$ since the step labelled by A_s is a contractible



(a) Step 0. (b) Step 1.

(c) Step 2.1. (d) Step 2.1.2.

Figure 4.5: Dinv is non-negative.

valley, see Figure 4.5b. Thus, the step labelled B creates primary dinv with each of the decorated valleys in the string following it

Step 2. If the string is preceded by a horizontal step, consider two subcases.

Step 2.1. First suppose that the valley labelled A_s is preceded by a leftmost vertical step at height i that is not a decorated valley (which is always true for $i \geq 0$). This implies that the step labelled A_s must be preceded at some point by two consecutive vertical steps, at height i and $i+1$, labelled C and B respectively, see Figure 4.5c. For all j, if $B > A_j$, then the step labelled B creates secondary dinv with the step labelled A_j. If $B \leq A_j$ then $C < A_j$.

Step 2.1.1. If the step labelled C is not a decorated valley then it creates primary dinv with the step labelled A_j.

Step 2.1.2. If, however the step labelled C is a decorated valley, rename its label A_{s+1} and consider it as part of the "string" of decorated valleys, see Figure 4.5d. Restart the argument from Step 1 (since the path is finite, this loop must terminate).

Step 2.2. The step labelled A_s is not preceded by a vertical step at height i that is not a decorated valley. This implies that $i < 0$. Thus, decorated valleys at height i that are not labelled 0 contribute to the bonus dinv. So we are exclusively concerned with the decorated valleys labelled 0. Decorated valleys labelled 0 that are not the first step at height i must create secondary dinv with a step to its left: indeed, they must be preceded by two horizontal steps, otherwise they would not be contractible. Since they are not the first step at height i, they must be preceded by two consecutive vertical steps, at height i and $i+1$, labelled B and C respectively, as in Figure 4.5c. Since B labels a rise, it must be positive and therefore must create secondary dinv with steps labelled 0 to its right.

Thus, we are left with a decorated valley labelled 0 that is the first step at height i. By the definition of a contractible valley (4.10), this implies that $i \neq 1$. Since the square path must end east, there must be a rise at height $i+1 < 0$, which creates one unit of bonus dinv. □

4.3 Combinatorics of q-analogues

In this section, we develop a general combinatorial framework for some q-binomial expressions that will come up often in our discussion. The formulas are q-analogues of classical combinatorial identities.

Definition 4.21. Given a set S and a statistic $\mathsf{stat} : S \to \mathbb{N}$ on that set, we say that stat is *q-counted* on S by $\sum_{s \in S} q^{\mathsf{stat}(s)}$.

Definition 4.22. Suppose $a, b \in \mathbb{N}$. An *interlacing* of a and b is a function $f : \{1, \dots, a+b\} \to \{0,1\}$ such that $\# f^{-1}(0) = a$ and $\# f^{-1}(1) = b$. In other words, it is a word in of length $a+b$ with a letters 0 and b letters 1. When $a = b = 0$ there is one interlacing which is the empty function (or word). An *inversion* of f is a pair (i,j) such that $1 \leq i < j \leq a+b$ and $f(i) > f(j)$. We denote by $\mathsf{inv}(f)$ the number of inversions of f. In other words, $\mathsf{inv}(f)$ is the number of times a 1 precedes a 0 in the interlacing.

Proposition 4.23. *For $a, b \in \mathbb{N}$*

$$\sum_f q^{\mathsf{inv}(f)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q = \begin{bmatrix} a+b \\ b \end{bmatrix}_q,$$

where the sum is over all interlacings f of a and b.

Proof. The second equality follows directly from the q-binomial definition (2.11). We show the fist equality by double induction on a and b. The case where $a = 0$ or $b = 0$ trivially gives $1 = 1$. Now suppose $a > 1$ and $b > 1$. The set of interlacings f of a and b may be divided into two parts: the first containing the interlacings such that $f(1) = 1$ and the second the ones with $f(1) = 0$. Interlacings of the former kind may be uniquely obtained from an interlacing of a and $b-1$ by placing a 1 in front of the word. Adding this 1 creates exactly a inversions. Inversions of the second kind may be obtained uniquely from an interlacing of $a-1$ and b by placing a 0 in front, which creates no inversions. The q-binomial identity of Proposition 2.17,

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_q = q^a \begin{bmatrix} a+b-1 \\ a \end{bmatrix}_q + \begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix}_q,$$

combined with the induction hypothesis thus implies the thesis. □

In other words, the inv statistic on the set of interlacings of a and b is q-counted by $\begin{bmatrix} a+b \\ a \end{bmatrix}_q$.

Definition 4.24. For $a, b \in \mathbb{N}$, a *strict interlacing* of a and b is an interlacing that contains no 00 substring.

Proposition 4.25. *For $a, b \in \mathbb{N}$*

$$\sum_f q^{\mathsf{inv}(f)} = q^{\binom{b}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_q$$

where the sum is over all strict interlacings f of b and a such that $f(a+b) = 1$.

Proof. If $a < b$ then there are no strict interlacings of a and b and so the identity is $0 = 0$. So we may suppose $a - b \geq 0$. By Proposition 4.23,

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} b + (a-b) \\ b \end{bmatrix}_q = \sum_f q^{\mathsf{inv}(f)},$$

where the sum is over all interlacings f of b and $a - b$. Given such an interlacing, insert a 1 immediately after every occurrence of a 0. It is clear that we obtain a strict interlacing between b and a ending with a 1; and that any such interlacing may be obtained in this way. Adding the 1 after the first occurrence of a 0 creates an inversion with the $b-1$ remaining 0's to its right. Similarly, the insertion of a 1 after the second 0 creates $b-2$ inversions. We may induce that the total number of inversions that were created by the insertion is $(b-1)+(b-2)+\cdots+2+1 = \binom{b}{2}$; and the thesis is proved. $\square$

Chapter 5

Delta conjectures and Theta refinements

We give an overview of (conjectured) combinatorial formulas for symmetric functions obtained via Delta or Theta operators.

5.1 Delta conjectures

In [HHL$^+$05], the authors conjectured a combinatorial formula in terms of labelled Dyck paths for the symmetric function ∇e_n. Their formula was know as the *shuffle conjecture* and became the shuffle theorem after the proof was found by Carlsson and Mellit [CM18]. See Section 5.3 for an explanation of the term "shuffle".

Theorem 5.1 (shuffle theorem)**.** *For $n \in \mathbb{N}$*

$$\nabla e_n = \sum_{P \in \mathsf{LD}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

This formula was conjecturally generalised in [HRW18], where the authors propose two different combinatorial formulas for the symmetric function $\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n$, which reduces to ∇e_n for $m = k = 0$. The combinatorial objects are rise decorated and valley decorated labelled Dyck paths. These twin conjectures are known as the *generalised Delta conjecture*, the case $m = 0$ being the *Delta conjecture*.

Conjecture 5.2 (generalised Delta conjecture, rise version)**.** *For $m, n, k \in \mathbb{N}$ with $k < n$*

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{P \in \mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Conjecture 5.3 (generalised Delta conjecture, valley version)**.** *For $m, n, k \in \mathbb{N}$ with $k < n$*

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{P \in \mathsf{LD}(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

In [LW07], the authors posited that the symmetric function $\nabla\omega(p_n)$ can be described combinatorially in terms of labelled square paths: the *square conjecture*. In [Ser17], Sergel proved that the Shuffle theorem implies the square conjecture, which thus became a theorem.

Theorem 5.4 (square theorem). *For $n \in \mathbb{N}$*

$$\nabla\omega(p_n) = \sum_{P \in \mathsf{LSQ}(n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

It was natural to look for a generalisation of the square theorem, analogous to the (generalised) Delta conjecture. This took some tinkering, because the simply decorated version of the combinatorial objects does not give the expected symmetric function. In [DIV19], we state the anticipated generalisation of the rise version as follows.

Conjecture 5.5 (generalised Delta square conjecture, rise version).
For $m, n, k \in \mathbb{N}$ with $k < n$

$$\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{P \in \mathsf{LSQ}(m,n)^{*k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

In [IV20], we conjecture two formulas that may reasonable be called valley versions of the generalised Delta square conjecture.

Conjecture 5.6 (generalised Delta square conjecture, valley version).
For $m, n, k \in \mathbb{N}$ with $k < n$

$$\frac{[n-k]_q}{[n]_q} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{P \in \mathsf{LSQ}(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

By Theorem 3.37, we know that $\Delta_{e_{n-k}}\omega(p_n)$ and $\Theta_k \nabla\omega(p_{n-k})$ are the same, up to a scalar. Slightly adapting the set of combinatorial objects (see Definition 4.14) seems to give the following formula.

Conjecture 5.7 (modified generalised Delta square conjecture, valley version).
For $m, n, k \in \mathbb{N}$ with $k < n$

$$\Delta_{h_m} \Theta_k \nabla\omega(p_{n-k}) = \sum_{P \in \mathsf{LSQ}'(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

5.2 Refinements

What Carlsson and Mellit actually proved in [CM18] is the compositional refinement of the shuffle formula, conjectured in [HMZ12].

Theorem 5.8 (compositional shuffle theorem). *For $n \in \mathbb{N}$ and $\alpha \vDash n$*

$$\nabla C_\alpha = \sum_{P \in \mathsf{LD}(\alpha)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

This result clearly implies the Shuffle theorem by Corollary 2.46. Using Theorem 2.45 and Definitions 4.15 and 4.16, we also get the following.

Corollary 5.9 (touching shuffle theorem)**.** *For $n, r \in \mathbb{N}$*

$$\nabla E_{n,r} = \sum_{\substack{P \in \mathsf{LD}(n) \\ \mathsf{touch}(P) = r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$$

How do these refinements generalise to the Delta context? Simply applying $\Delta'_{e_{n-k-1}}$ to C_α or $E_{n,k}$ does not coincide with an obvious combinatorial interpretation analogous to the undecorated case. This is where the Theta operators of [DIV20] come into play. This paper contains all the remaining conjectures of the current section. The symmetric function of the Delta conjecture is $\Delta'_{e_{n-k-1}} e_n = \Theta_k \nabla e_{n-k}$, by Theorem 3.36. Therefore the following formulas, refine the Delta conjecture.

Conjecture 5.10 (compositional Delta conjecture, rise version)**.** *For $n, k \in \mathbb{N}$ with $k < n$ and $\alpha \vDash n - k$*

$$\Theta_k \nabla C_\alpha = \sum_{P \in \mathsf{LD}(\alpha)^{*k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Conjecture 5.11 (compositional Delta conjecture, valley version)**.** *For $n, k \in \mathbb{N}$ with $k < n$ and $\alpha \vDash n - k$*

$$\Theta_k \nabla C_\alpha = \sum_{P \in \mathsf{LD}(\alpha)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Now we would like to apply Δ_{h_m} to get a generalised version, but unfortunately, this does not quite work: the dinv does not match. We can state only a partial conjecture.

Conjecture 5.12. *For $m, n, k \in \mathbb{N}$ with $k < n$ and $\alpha \vDash n - k$*

$$\Delta_{h_m} \Theta_k \nabla C_\alpha \big|_{q=1} = \sum_{P \in \mathsf{LD}(m,\alpha)^{*k}} t^{\mathsf{area}(P)} x^P.$$

We do not have a valley version of this partial conjecture. However, we do have a touching version of the generalised Delta conjecture.

Conjecture 5.13 (touching generalised Delta conjecture, rise version)**.** *For $m, n, k, r \in \mathbb{N}$ with $k < n$*

$$\Delta_{h_m} \Theta_k \nabla E_{n-k,r} = \sum_{\substack{P \in \mathsf{LD}(m,n)^{*k} \\ \mathsf{touch}(P) = r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$$

Conjecture 5.14 (touching generalised Delta conjecture, valley version)**.** *For $m, n, k, r \in \mathbb{N}$ with $k < n$*

$$\Delta_{h_m} \Theta_k \nabla E_{n-k,r} = \sum_{\substack{P \in \mathsf{LD}(m,n)^{\bullet k} \\ \mathsf{touch}(P) = r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Next, we shift our attention to refinements of the (generalised) Delta square conjecture. We do not, as of yet, have a "compositional" version. However, a touching version seems to hold, but only for the rise version. Proposition 2.42 states that the $\frac{[n]_q}{[r]_q} E_{n,r}$ are building blocks of $\omega(p_n)$. Therefore the following is a refinement of the generalised Delta square conjecture.

Conjecture 5.15 (touching generalised Delta square conjecture, valley version)**.** *For $m, n, k, r \in \mathbb{N}$ with $k < n$*

$$\frac{[n]_q}{[r]_q} \Delta_{h_m} \Theta_k \nabla E_{n-k,r} = \sum_{\substack{P \in \mathsf{LSQ}(m,n)^{*k} \\ \mathsf{touch}(P) = r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

[DIV18] [DIV20] [IV20]

5.3 Shuffle theory

The conjectures of the previous sections can be rephrased using shuffle theory. To this end, we need some definitions.

Definition 5.16. Given two pairwise distinct finite sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$, their *shuffle* $(a_1, \dots, a_n) \sqcup\!\sqcup (b_1, \dots, b_m)$ is the set of sequences $(c_1, \dots, c_{m+n})$ such that

- $\{c_i \mid 1 \leq i \leq m+n\} = \{a_i \mid 1 \leq i \leq n\} \cup \{b_i \mid 1 \leq i \leq m\}$;
- if $j < k$ and $c_j = a_r, c_k = a_s$ then $r < s$;
- if $j < k$ and $c_j = b_r, c_k = b_s$ then $r < s$.

Example. The shuffle of $(1, 2)$ and $(3, 4)$ is

$$\{(12, 3, 4), (1, 3, 2, 4), (1, 3, 4, 2), (3, 1, 2, 4), (3, 1, 4, 2), (3, 4, 1, 2)\}.$$

Definition 5.17. Given a composition α a *α-shuffle* is an element of $(1, \dots, \alpha_1) \sqcup\!\sqcup (\alpha_1 + 1, \dots, \alpha_1 + \alpha_2) \sqcup\!\sqcup \cdots$.

Definition 5.18. Consider a path $P = (\pi, w, dr, dv) \in \mathsf{LSQ}(m,n)^{*k \bullet l}$ with shift s. The *reading word* of P is the permutation of w obtained as follows: read the labels in the i-th diagonal, from left to right, for $i = -s, -s+1, \dots$; then reverse this word (i.e. read it from right to left).

Example. The path on the left in Figure 5.1 has reading word 213201.

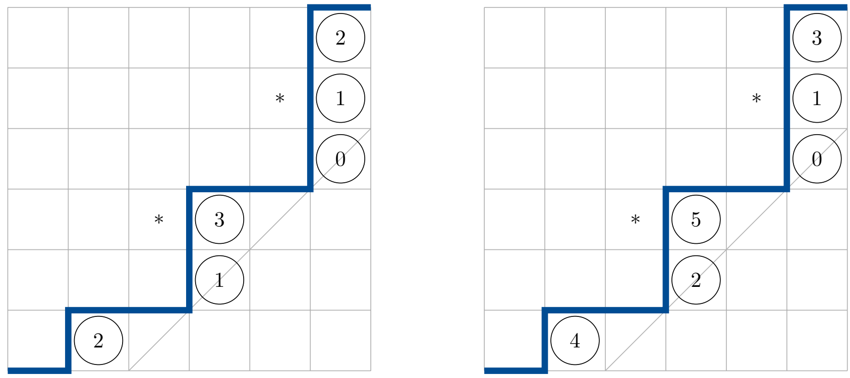


Figure 5.1: An element of $\mathsf{LSQ}(1,5)^{*2}$ (left) and its standardisation (right) in $\mathsf{Pref}(1,5)^{*2}$.

Definition 5.19. Given a composition α, an element of $\mathsf{LSQ}(m,n)^{*k\bullet l}$ is said to be an *α-shuffle* if its reading word, without the zeros, is an α-shuffle.

Definition 5.20. A (generalised decorated) *preference function* is an element of $\mathsf{LSQ}(m,n)^{*k\bullet l}$ such that its non-zero labels are exactly $1, 2, \dots, n$. The set of such paths is denoted by $\mathsf{Pref}(m,n)^{*k\bullet l}$. A preference function whose underlying lattice path is a Dyck path is called a *parking function* and the subset of such paths is denoted by $\mathsf{Park}(m,n)^{*k\bullet l}$.

Definition 5.21. Consider $P = (\pi, w, dr, dv) \in \mathsf{LSQ}(m,n)^{*k\bullet l}$ with reading word u. Suppose α is the composition of n such that w contains α_1 letters equal to 1, α_2 letters equal to 2 and so forth. Let $\tilde{u}$ by the word obtained from u by replacing, from left to right, its 1's by $1, 2, \dots \alpha_1$; its 2's by $\alpha_1 + 1, \dots, \alpha_1 + \alpha_2$ and so forth. The *standardisation* of P the unique element $(\pi, \tilde{w}, dr, dv) \in \mathsf{Pref}(m,n)^{*k\bullet l}$ whose reading word is $\tilde{u}$.

Proposition 5.22. *Consider $P \in \mathsf{LSQ}(m,n)^{*k\bullet l}$ and Q its standardisation. Then $\mathsf{area}(P) = \mathsf{area}(Q)$ and $\mathsf{dinv}(P) = \mathsf{dinv}(Q)$.*

The proof is simple: since Q is simply a relabelling of P and the area of a path does not depend on its labels, the first statement is obvious. The second statement follows easily from the definitions of dinv (4.19), reading word (5.18) and standardisation (5.21).

Proposition 5.23. *Consider $f \in \Lambda^{(n)}$ and let S be $\mathsf{LSQ}(m,n)^{*k\bullet l}$, $\mathsf{LSQ}'(m,n)^{*k\bullet l}$ or $\mathsf{LD}(m,n)^{*k\bullet l}$. Let $\tilde{S}$ be the set obtained from S by standardising its elements. The following statements are equivalent.*

(i) $f = \sum_{P \in S} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$;

(ii) *For all $\lambda \vdash n$,*

$$\langle f, h_\lambda \rangle = \sum_{\substack{P \in \tilde{S} \\ P \text{ is a } \lambda\text{-shuffle}}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}.$$

Proof. Since $\{h_\lambda\}_{\lambda\vdash n}$ and $\{m_\lambda\}_{\lambda\vdash n}$ are dual basis of $\Lambda^{(n)}$ with respect to the Hall scalar product (see Proposition 1.50), $\langle f, h_\lambda\rangle$ corresponds to the coefficient of m_λ in the monomial basis expansion of f. By definition of m_λ, this must be the same coefficient found in front of the monomial $x_1^{\lambda_1}\cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$ of f.

Suppose that (i) holds. Then for any $\lambda\vdash n$ the coefficient in front of $x_1^{\lambda_1}\cdots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}}$ of f is

$$\sum_{P\in S(\lambda)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)},$$

where $S(\lambda)$ is the subset of S whose paths have λ_1 labels equal to 1, λ_2 labels equal to 2 and so forth. Standardising the elements of $S(\lambda)$, we get exactly the elements of $\tilde{S}$ that are λ-shuffles. Thus using Proposition 5.22, we get (ii).

Now suppose that (ii) holds. By the same arguments as before we have

$$\left\langle \sum_{P\in S} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P, h_\lambda \right\rangle = \sum_{\substack{P\in\tilde{S} \\ P \text{ is a } \lambda\text{-shuffle}}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}.$$

Since $\{h_\lambda\}_{\lambda\vdash n}$ is a basis of $\Lambda^{(n)}$, this implies that (ii) ensures (i). $\square$

Chapter 6

The touching generalised shuffle theorem

In this chapter we will prove the case $k = 0$ of the touching generalised Delta conjecture (5.13 and 5.15).

Theorem 6.1 (touching generalised shuffle theorem). *For $m, n, r \in \mathbb{N}$*

$$\Delta_{h_m} \nabla E_{n,r} = \sum_{\substack{P \in \mathsf{LD}(m,n) \\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

As the $E_{n,r}$ are the building blocks of e_n (Equation (2.40)), taking the sum over r will yield the following immediate consequence of Theorem 6.1.

Corollary 6.2 (generalised shuffle theorem). *For $m, n \in \mathbb{N}$, with $n > 0$*

$$\Delta_{h_m} \nabla e_n = \sum_{P \in \mathsf{LD}(m,n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

In order to prove this result, we will provide a combinatorial interpretation of the symmetric function $\Delta_{h_m} \nabla e_n[X[s+1]_q]$ in terms of *augmented Dyck paths* by using a key symmetric function identity for $h_j^\perp \nabla e_n[X[s+1]_q]$.

6.1 Symmetric function identity

This section is dedicated to the proof of the following symmetric function identity.

Theorem 6.3. *For $n, m, s \in \mathbb{N}$ with $n > 0$ and $m \leq n$*

$$h_m^\perp \nabla e_n[X[s+1]_q] = \sum_{l=0}^{m} t^{m-l} \begin{bmatrix} s+l \\ l \end{bmatrix}_q \Delta_{h_{m-l}} \nabla e_{n-m}[X[s+l+1]_q].$$

Proof. For $m = n$, the right hand side is a constant

$$\begin{aligned} h_n^\perp \nabla e_n[X[s+1]_q] &= \left\langle \nabla e_n \left[\frac{XM[s+1]_q}{M} \right], h_n \right\rangle \\ \text{(by 2.27)} &= \sum_{\lambda \vdash n} \frac{H_\lambda[M[s+1]_q]}{w_\lambda} \langle \nabla H_\lambda[X], h_n \rangle \\ \text{(by 2.31)} &= \sum_{\lambda \vdash n} \frac{H_\lambda[M[s+1]_q]}{w_\lambda} \langle T_\lambda H_\lambda[X], s_{(n)} \rangle \\ \text{(by 2.9.(III))} &= \sum_{\lambda \vdash n} \frac{T_\lambda H_\lambda[M[s+1]_q]}{w_\lambda} \\ \text{(by 3.11)} &= h_n \left[\frac{M[s+1]_q}{M} \right] = h_n[[s+1]_q] \\ \text{(by 2.20)} &= \begin{bmatrix} n+s \\ n \end{bmatrix}_q \end{aligned}$$

Since $\Delta_{h_{m-l}} \nabla e_0 = \delta_{l,m}$, this is what we wanted to show.

Let us now assume that $m < n$. To begin we use the Macdonald expansion of $e_n[X[k]_q]$ (Proposition 3.13) and the definition of ∇ (2.31) .

$$\begin{aligned} h_m^\perp \nabla e_n \left[X[s+1]_q \right] &= (1-q^{s+1}) \sum_{\lambda \vdash n} \Pi_\lambda h_{s+1}[(1-t)B_\lambda] T_\lambda \frac{h_m^\perp H_\lambda[X]}{w_\lambda} \\ \text{(by 3.17, 3.20)} &= (1-q^{s+1}) \sum_{\lambda \vdash n} \sum_{\mu \subset_m \lambda} \Pi_\lambda h_{s+1}[(1-t)B_\lambda] T_\lambda \frac{c_{\lambda\mu}^{(m)} H_\mu[X]}{w_\lambda} \\ \text{(by (3.19))} &= (1-q^{s+1}) \sum_{\mu \vdash n-m} \frac{H_\mu[X]}{w_\mu} \sum_{\lambda \supset_m \mu} d_{\lambda\mu}^{(m)} \Pi_\lambda h_{s+1}[(1-t)B_\lambda] T_\lambda \\ &= \cdots \end{aligned}$$

We need an extra identity, [Hag04, Equation 79], the proof of which we included in Appendix A.

Lemma 6.4. *For f, g homogeneous elements of Λ with $\deg(f) = m$ and $\mu \in \mathsf{Par} \setminus \{\emptyset\}$, we have*

$$\sum_{\lambda \supset_m \mu} d_{\lambda\mu}^f \Pi_\lambda g[MB_\lambda] = \Pi_\mu (\Delta_{f[MX]} g)[MB_\mu].$$

We apply this formula with $f[X] = e_m \left[\frac{X}{M} \right]$ and $g[X] = h_{s+1} \left[\frac{X}{1-q} \right] e_n \left[\frac{X}{M} \right]$ so that $d_{\lambda\mu}^f = d_{\lambda\mu}^{(m)}$ (see Definition 3.17) and $g[MB_\lambda] = h_{s+1}[(1-t)B_\lambda] T_\lambda$. Thus we get

$$\cdots = (1-q^{s+1}) \sum_{\mu \vdash n-m} \frac{H_\mu[X]}{w_\mu} \Pi_\mu \Delta_{e_m} \left(h_{s+1} \left[\frac{Z}{1-q} \right] e_n \left[\frac{Z}{M} \right] \right) \Big|_{Z \mapsto MB_\mu}$$

Proposition 3.34 gives the Macdonald expansion of the parenthetical, which allows for an explicit formulation of its image by Δ_{e_m}. We recall the formula here:

$$h_i\left[\frac{X}{1-q}\right]e_j\left[\frac{X}{M}\right]=\sum_{\lambda\vdash i+j}\frac{H_\lambda[X]}{w_\lambda}q^{-\binom{i}{2}}\sum_{k=0}^{i}(-1)^{i-k}q^{\binom{i-k}{2}}\begin{bmatrix}i-1\\ i-k\end{bmatrix}_q h_k[(1-t)B_\lambda].$$

Using this, we get

$$\begin{aligned}\cdots=&(1-q^{s+1})\sum_{\mu\vdash n-m}\frac{H_\mu[X]}{w_\mu}\Pi_\mu\sum_{\nu\vdash s+n+1}e_m[B_\nu]\frac{H_\nu[MB_\mu]}{w_\lambda}q^{-\binom{s+1}{2}}\\ &\times\sum_{k=0}^{s+1}(-1)^{s+1-k}q^{\binom{s+1-k}{2}}\begin{bmatrix}s\\ s+1-k\end{bmatrix}_q h_k[(1-t)B_\nu].\end{aligned}$$

Since $\begin{bmatrix}s\\ s+1\end{bmatrix}_q=0$ and $\begin{bmatrix}s\\ s+1-k\end{bmatrix}_q=\begin{bmatrix}s\\ k-1\end{bmatrix}_q$ for all k such that $0\leq s+1-k\leq s$, i.e. $1\leq k\leq s+1$ we may rewrite

$$\begin{aligned}\cdots=&(1-q^{s+1})\sum_{\mu\vdash n-m}\frac{H_\mu[X]}{w_\mu}\Pi_\mu\sum_{\nu\vdash s+n+1}e_m[B_\nu]\frac{H_\nu[MB_\mu]}{w_\lambda}q^{-\binom{s+1}{2}}\\ &\times\sum_{k=1}^{s+1}(-1)^{s+1-k}q^{\binom{s+1-k}{2}}\begin{bmatrix}s\\ k-1\end{bmatrix}_q h_k[(1-t)B_\nu]\\ =&(1-q^{s+1})\sum_{\mu\vdash n-m}\frac{H_\mu[X]}{w_\mu}\Pi_\mu\sum_{k=1}^{s+1}q^{-\binom{s+1}{2}}(-1)^{s+1-k}q^{\binom{s+1-k}{2}}\begin{bmatrix}s\\ k-1\end{bmatrix}_q\\ &\times\sum_{\nu\vdash s+n+1}\frac{H_\nu[MB_\mu]}{w_\lambda}h_k[(1-t)B_\nu]e_m[B_\nu]=\cdots\end{aligned}$$

Next, we apply the summation formula of Theorem 3.26:

$$\begin{aligned}&\sum_{\lambda\vdash m+n}\frac{H_\lambda[X]}{w_\lambda}h_s[(1-t)B_\lambda]e_m[B_\lambda]\\ &=\sum_{l=0}^{m}t^{m-l}\sum_{k=0}^{s}q^{\binom{k}{2}}\begin{bmatrix}l+k\\ k\end{bmatrix}_q\begin{bmatrix}l+s-1\\ s-k\end{bmatrix}_q h_{l+k}\left[\frac{X}{1-q}\right]h_{m-l}\left[\frac{X}{M}\right]e_{n-k}\left[\frac{X}{M}\right]\end{aligned}$$

which gives

$$\begin{aligned}\cdots=&(1-q^{s+1})\sum_{\mu\vdash n-m}\frac{H_\mu[X]}{w_\mu}\Pi_\mu\sum_{k=1}^{s+1}q^{-\binom{s+1}{2}}(-1)^{s+1-k}q^{\binom{s+1-k}{2}}\begin{bmatrix}s\\ k-1\end{bmatrix}_q\\ &\times\sum_{l=0}^{m}t^{m-l}\sum_{r=0}^{k}q^{\binom{r}{2}}\begin{bmatrix}l+r\\ r\end{bmatrix}_q\begin{bmatrix}l+k-1\\ k-r\end{bmatrix}_q h_{l+r}[(1-t)B_\mu]h_{m-l}[B_\mu]e_{s+n+1-m-r}[B_\mu].\end{aligned}$$

Since $e_{s+n+1-m-r}[B_\mu]=0$ if $s+n+1-m-r>|\mu|=n-m$, i.e. if $r<s+1$, all the terms are 0 except for $k=r=s+1$. Thus

$$\begin{aligned}\cdots=&(1-q^{s+1})\sum_{\mu\vdash n-m}\frac{H_\mu[X]}{w_\mu}\Pi_\mu\sum_{l=0}^{m}t^{m-l}\begin{bmatrix}l+s+1\\ s+1\end{bmatrix}_q\\ &\times h_{l+s+1}[(1-t)B_\mu]h_{m-l}[B_\mu]e_{n-m}[B_\mu]=\cdots\end{aligned}$$

We now use the following elementary q-binomial identity, whose proof can be found in Appendix A.

Lemma 6.5. *For $l, s \in \mathbb{N}$*

$$(1-q^{s+1})\begin{bmatrix} l+s+1 \\ l \end{bmatrix}_q = (1-q^{s+l+1})\begin{bmatrix} s+l \\ l \end{bmatrix}_q.$$

Since $\begin{bmatrix} l+s+1 \\ s+1 \end{bmatrix}_q = \begin{bmatrix} l+s+1 \\ l \end{bmatrix}_q$ for all $l, s \in \mathbb{N}$ we may use this lemma, and the fact that $e_{n-m}[B_\mu] = T_\mu$ for all $\mu \vdash n-m$ to write

$$\begin{aligned} \cdots &= \sum_{\mu \vdash n-m} \frac{H_\mu[X]}{w_\mu} \Pi_\mu \sum_{l=0}^{m} t^{m-l}(1-q^{s+l+1})\begin{bmatrix} l+s \\ l \end{bmatrix}_q h_{l+s+1}[(1-t)B_\mu] h_{m-l}[B_\mu] T_\mu \\ &= \sum_{l=0}^{m} t^{m-l}\begin{bmatrix} l+s \\ l \end{bmatrix}_q (1-q^{s+l+1}) \sum_{\mu \vdash n-m} \frac{h_{m-l}[B_\mu] T_\mu H_\mu[X]}{w_\mu} \Pi_\mu h_{l+s+1}[(1-t)B_\mu] \\ &= \cdots \end{aligned}$$

Using the definitions of Δ_f and ∇ (2.31, 2.32), combined with the Macdonald expansion of $e_n[X[k]_q]$ (Proposition 3.13), we may conclude

$$\cdots = \sum_{l=0}^{m} t^{m-l}\begin{bmatrix} l+s \\ l \end{bmatrix}_q \Delta_{h_{m-l}} \nabla e_{n-m}[X[s+l+1]_q].$$

□

6.2 Augmented Dyck paths

Definition 6.6. An *augmented Dyck path* is a pair (π, w) with π a Dyck path and w a labelling of π with entries in $\mathbb{N} \cup \{\infty\}$, such that the steps labelled ∞ are all at height zero and so they are the only labels in their columns. Furthermore, we require that ∞ labels are not directly followed by a 0 label and that the leftmost finite label on the main diagonal is non-zero. The set of such paths of size $m+n+s$ with m labels equal to 0, n positive finite labels and s infinite labels is denoted by $\mathsf{LD}(m, n, s)$.

We draw an example in Figure 6.1.

Definition 6.7. Let $P \in \mathsf{LD}(m, n, s)$ we define $\mathsf{dinv}(P)$ in exactly the same way as Definition 4.19, where we consider that

- $i < \infty$ for $i \in \mathbb{P}$;
- 0 and ∞ are incomparable.

As usual, the area is not influenced by the labels so for $(\pi, w) \in \mathsf{LD}(m, n, s)$, we define $\mathsf{area}(\pi, w)$ to be the same expression as in Definition 4.18, with $\mathsf{shift}(\pi) = 0$ and $dv = \emptyset$, i.e. the number of whole squares between the path and the main diagonal.

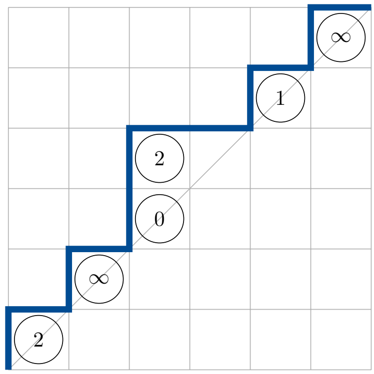


Figure 6.1: An element of $\mathsf{LD}(1,3,2)$.

Example. The path in Figure 6.1 has 3 primary dinv, $(1,2),(3,5)$ and $(5,6)$, and 1 secondary dinv, $(4,5)$, so total dinv 4. Its area is 1.

Finally, we define the monomial of an augmented Dyck path.

Definition 6.8. If $P \in \mathsf{LD}(m,n,s)$ then the *monomial associated to* P is

$$x^P := \prod_{i=1}^{m+n} x_{w_i} \Big|_{\substack{x_0=1 \\ x_\infty=1}}.$$

Since there are m labels equal to 0 and s labels equal to ∞, this is a degree n monomial.

Example. The monomial of the path in Figure 6.1 is $x_1 x_2^2$.

6.3 The proof

In this section, we prove Theorem 6.2, passing by a combinatorial interpretation in terms of augmented Dyck paths of $\Delta_{h_m} \nabla e_{m+n+n}[X[s+1]_q]$. To begin, we apply ∇ to Equation (2.39), which gives

$$\nabla e_n[X[s+1]_q] = \sum_{k=0}^{n} \begin{bmatrix} k+s \\ k \end{bmatrix}_q \nabla E_{n,k}. \tag{6.9}$$

The touching shuffle theorem (Corollary 5.9) states that

$$\nabla E_{n,k} = \sum_{\substack{P \in \mathsf{LD}(n) \\ \mathsf{touch}(P)=k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P. \tag{6.10}$$

Combinatorially, we have the following.

Proposition 6.11. *For $n, s \in \mathbb{N}$ with $n > 0$*

$$\sum_{P \in \mathsf{LD}(m,n,s)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P = \sum_{k=0}^{n} \begin{bmatrix} k+s \\ k \end{bmatrix}_q \sum_{\substack{P \in \mathsf{LD}(m,n) \\ \mathsf{touch}(P)=k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Proof. Start from a path $P \in \mathsf{LD}(m,n)$ with $\mathsf{touch}(P) = k$. We will insert s vertical steps labelled ∞, directly followed by a horizontal step. In this way we obtain all the elements of $\mathsf{LD}(m,n,s)$. Since all steps labelled ∞ must by definition lie on the main diagonal, the only places we may insert (a consecutive string of) ∞'s is right before a vertical step at height 0 with a non-zero label, i.e. at the touching points of P, of which there are k. Encode the choice of where to insert the ∞ steps with an interlacing of s and k where the 0's correspond to the ∞'s and the 1's to the touching points. Since a touching point is followed by a vertical step that is labelled by a positive integer, each time a 1 precedes a 0 a unit of primary dinv is created. Since the ∞'s do not create secondary dinv, nor primary dinv with 0's, this is the only contribution to the dinv. Thus the inv of the interlacing corresponds exactly to the dinv added to the path by the insertion. Clearly, this insertion does not change the area of the path and thus Proposition 4.23 concludes the proof. □

Combining equations (6.9), (6.10) and Proposition 6.11 with $m = 0$, we may conclude the following

Proposition 6.12. *For $n, s \in \mathbb{N}$ with $n > 0$*

$$\nabla e_n[X[s+1]_q] = \sum_{P \in \mathsf{LD}(0,n,s)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Next, we use the theory of shuffles to deduce a combinatorial formula for $h_m^\perp \nabla e_{m+n}[X[s+1]_q]$. We need to extend the vocabulary of Section 5.3 to augmented paths: given a path P in $\mathsf{LD}(m,n,s)$, we define its *reading word* in exactly the same way as regular Dyck paths (5.18). Define also its *standardisation* and what is means for P to be a α-shuffle analogously to 5.21 and 5.19, where the ∞'s are disregarded, like the 0's. Lastly an element of $\mathsf{LD}(m,n,s)$ is an *augmented parking function* if its non-zero non-infinity labels are exactly $1, \dots, n$ and the set of such paths is denoted by $\mathsf{Park}(m,n,s)$.

The same[1] argument as in the proof of Propostition 5.23, applied to Proposition 6.11 implies

$$\begin{aligned} \langle h_m^\perp \nabla e_{m+n}[X[s+1]_q], h_\lambda \rangle &= \langle \nabla e_{m+n}[X[s+1]_q], h_m h_\lambda \rangle \\ &= \sum_{\substack{P \in \mathsf{Park}(0,n,s) \\ P \text{ is a } \lambda(m)\text{-shuffle}}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}; \end{aligned}$$

for all $\lambda \vdash n$, where $\lambda(m)$ is the concatenation $(\lambda_1, \dots, \lambda_{\ell(\lambda)}, m)$. Since

$$\langle h_m^\perp \nabla e_{m+n}[X[s+1]_q], h_\lambda \rangle$$

is the coefficient of m_λ in the monomial basis expansion of $h_m^\perp \nabla e_{m+n}[X[s+1]_q]$ we may immediately deduce the following.

[1] There is one subtlety: (λ, m) might not be a partition but a composition; but since $\nabla e_n[X[s+1]_q]$ is symmetric, permuting the variables does not influence the coefficient.

Proposition 6.13. *Consider $m, n, s \in \mathbb{N}$ with $n > 0$. Let $S \subseteq \mathsf{LD}(0, m+n, s)$ be such that $P \in S$ if P has exactly m maximal finite labels and set $\max_P$ to be the value of this maximal finite label, then*

$$h_m^\perp \nabla e_{m+n}[X[s+1]_q] = \sum_{P \in S} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \big|_{x_{\max_P} \mapsto 1}.$$

Let us reformulate the right hand side of this equation.

Lemma 6.14. *For $m, n, s \in \mathbb{N}$ with $n > 0$, using the notation of Proposition 6.13, we have*

$$\sum_{P \in S} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \big|_{x_{\max_P} \mapsto 1} = \sum_{l=0}^{m} t^{m-l} \begin{bmatrix} s+l \\ l \end{bmatrix}_q \times \sum_{P \in \mathsf{LD}(m-l, n, s+l)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Proof. We start from $P \in S$ and let l be the number of maximal finite labels of P at height 0. We will transform P via a "pushing algorithm" and obtain a path in $\mathsf{LD}(m-l, n, s+l)$, keeping track of the modifications to the statistics. See Figure 6.2 for a visual aid. By their nature, the vertical steps labelled with a maximal finite label must be followed by a horizontal step. The pushing algorithm consists of two operations.

- Replace the l maximal finite labels at height 0 with ∞ labels.
- By definition of S, $0 \leq l \leq m$ and P has exactly $m - l$ maximal finite labels that are not at height 0. Replace the vertical step labelled with these maximal finite labels, and the horizontal step that follows it, by a horizontal step followed by a vertical step, labelled with 0.

Thus we obtain a path in $\mathsf{LD}(m-l, n-m, s+l)$, indeed: any 0 on the main diagonal came from a rise at height 1, so the label right before it must be finite. It is not hard to see that any path of this set can obtained via the pushing algorithm.

The first operation is simply a relabelling so does not affect the area. Nor is the secondary dinv affected since maximal finite labels on the main diagonal do not create any. The primary dinv with finite labels in not affected either as it is conserved when changing the label to infinity. However, the primary dinv between maximal finite labels and infinity labels disappears since infinity labels do not create dinv among each other. Given an interlacing of s (the number of ∞'s that were already present) and l, we may interpret it as the relative positioning of these two kind of steps. Each time a maximal finite label precedes an infinity label, a unit of dinv will be lost after the pushing algorithm. Thus the inv of the interlacing corresponds exactly to the lost dinv, which accounts for the factor $\begin{bmatrix} s+l \\ l \end{bmatrix}_q$ (see Proposition 4.23).

The second operation does not affect the dinv since primary becomes secondary and vice versa. It affects the area though: it is easy to see that the area goes down by exactly $m - l$ units.

□

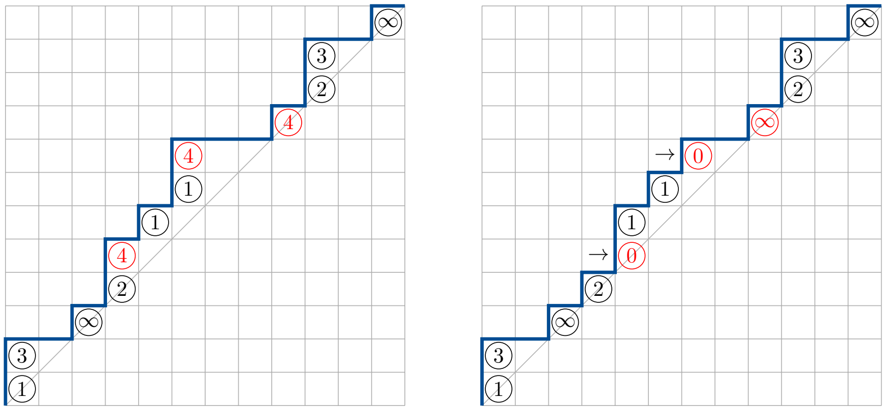


Figure 6.2: Pushing algorithm.

We are now ready to prove the generalised version of Proposition 6.12.

Theorem 6.15. *For all $m, n, s \in \mathbb{N}$ with $n > 0$*

$$\Delta_{h_m} \nabla e_n[X[s+1]_q] = \sum_{P \in \mathsf{LD}(m,n,s)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Proof. We proceed by induction on m. The base case, $m = 0$ is exactly Proposition 6.12. Now suppose $m > 0$. We will provide a combinatorial interpretation for the symmetric function identity of Theorem 6.3 with $n \mapsto m + n$:

$$h_m^\perp \nabla e_{m+n}[X[s+1]_q] = \sum_{l=0}^{m} t^{m-l} \begin{bmatrix} s+l \\ l \end{bmatrix}_q \Delta_{h_{m-l}} \nabla e_n[X[s+l+1]_q]. \tag{6.16}$$

Combining Proposition 6.13 and Lemma 6.14, we know that the left hand side equals

$$h_m^\perp \nabla e_{m+n}[X[s+1]_q] = \sum_{l=0}^{m} t^{m-l} \begin{bmatrix} s+l \\ l \end{bmatrix}_q \sum_{P \in \mathsf{LD}(m-l,n,s+l)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P. \tag{6.17}$$

By the induction hypothesis, we know that

$$\Delta_{h_{m-l}} \nabla e_n[X[s+l+1]_q] = \sum_{P \in \mathsf{LD}(m-l,n,s+l)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \quad \text{for } l > 0. \tag{6.18}$$

Combining (6.16), (6.17) and (6.18) we get

$$\begin{aligned} \sum_{l=0}^{m} t^{m-l} \begin{bmatrix} s+l \\ l \end{bmatrix}_q \sum_{P \in \mathsf{LD}(m-l,n,s+l)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P &= t^m \Delta_{h_m} \nabla e_n[X[s+1]_q] \\ &+ \sum_{l=1}^{m} \sum_{P \in \mathsf{LD}(m-l,n,s+l)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P, \end{aligned}$$

which readily implies our thesis. $\square$

Now, applying Δ_{h_m} to Equation (6.9), we get

$$\Delta_{h_m}\nabla e_n[X[s+1]_q] = \sum_{k=0}^{n} \begin{bmatrix} k+s \\ k \end{bmatrix}_q \Delta_{h_m}\nabla E_{n,k}. \tag{6.19}$$

This gives a system of linear equations expressing $\{\Delta_{h_m}\nabla e_n[X[s+1]_q]\}_{0\leq s\leq n}$ in terms of $\{\Delta_{h_m}\nabla E_{n,k}\}_{0\leq k\leq n}$ with transition matrix $\left[\begin{bmatrix} k+s \\ s \end{bmatrix}_q\right]_{k,s=0,\dots,n}$. Call this system L.

The following is an elementary fact about Pascal matrices (see [Wik20]), a proof of which can be found in Appendix A.

Lemma 6.20. *If* $M := \left[\binom{i+j}{i}\right]_{i,j=0,\dots,n} \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{N})$ *then* $\det(M) = 1$.

By definition of the q-binomials we obtain the determinant $\left[\binom{k+s}{k}\right]_{k,s=0,\dots,n}$ by setting $q = 1$ in the determinant of $\left[\begin{bmatrix} k+s \\ s \end{bmatrix}_q\right]_{k,s=0,\dots,n}$. So this lemma implies that both these matrices are invertible. So the linear system L has a unique solution. Combining Theorem 6.15 and Proposition 6.11 (the combinatorial counterpart of (6.19)) determines this unique solution. Therefore, we may deduce the result stated at the top of this chapter.

Theorem 6.1 (touching generalised shuffle theorem)**.** *For* $m, n, r \in \mathbb{N}$

$$\Delta_{h_m}\nabla E_{n,r} = \sum_{\substack{P\in\mathsf{LD}(m,n)\\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Chapter 7

The valley Delta square

In this chapter we will show an implication between two of the conjectures stated in Chapter 5: Conjecture 5.14 implies Conjecture 5.7. We restate them here.

Conjecture 5.14 (touching generalised Delta conjecture, valley version)**.** *For $m, n, k, r \in \mathbb{N}$ with $k < n$*

$$\Delta_{h_m} \Theta_k \nabla E_{n-k,r} = \sum_{\substack{P \in \mathsf{LD}(m,n)^{\bullet k} \\ \mathsf{touch}(P) = r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Conjecture 5.7 (modified generalised Delta square conjecture, valley version)**.** *For $m, n, k \in \mathbb{N}$ with $k < n$*

$$\Delta_{h_m} \Theta_k \nabla \omega(p_{n-k}) = \sum_{P \in \mathsf{LSQ}'(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Chapter 6 is dedicated to the proof of the case $k = 0$ the former statement. Thus this implication will establish the case $k = 0$ of the latter one: the *generalised square theorem.*

Theorem 7.1 (generalised square theorem)**.** *For $m, n \in \mathbb{N}$*

$$\Delta_{h_m} \nabla \omega(p_n) = \sum_{P \in \mathsf{LSQ}'(m,n)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

7.1 Schedule numbers

In this section we provide a combinatorial formula for the right hand side of Conjecture 5.7. The formula uses *schedule numbers*, a notion that was developed in [HL05], [Hic13], [Ser17], [HS19]. Contrary to these publications, our formula enumerates labelled square paths and not preference functions, i.e. it allows for repeated labels. It thus provides a new factorisation of previous schedule formulas.

Definition 7.2. Let $P \in \mathsf{LSQ}(m,n)^{\bullet k}$ and $s := \mathsf{shift}(P)$. For $i \geq 0$, consider the multiset of labels contained in the $(s+i)$-th diagonal of P, where the labels of

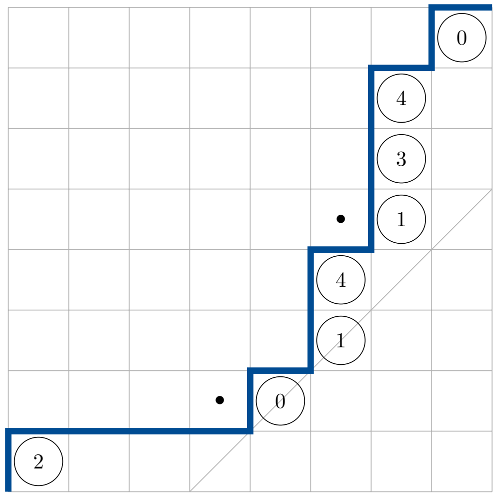


Figure 7.1: An element of $\mathsf{LD}(2,6)^{\bullet 2}$

decorated valleys are decorated with a $\bullet$. Let ρ_i be the *decorated word* obtained from arranging this multiset in increasing order, considering $c < \overset{\bullet}{c} < c+1$. The *diagonal word* of P is $\mathsf{dw}(P) \coloneqq \rho_\ell \dots \rho_1 \rho_0$, where $\ell = \max\{i \mid \rho_i \neq \emptyset\}$.

Example. The path in Figure 7.1 has $\rho_0 = \overset{\bullet}{0}1$, $\rho_1 = \overset{\bullet}{1}4$, $\rho_2 = 3$ and $\rho_3 = 024$ and so its diagonal word is $0243\overset{\bullet}{1}4\overset{\bullet}{0}1$.

A *run* of a word is a maximal increasing substring of that word. Since square paths end east, all its diagonals (except the base diagonal) contain a label that lies on top of a (strictly smaller) label contained in the diagonal right below it. It follows that the ρ_i are the runs of the diagonal word of P.

Definition 7.3. Consider $P \in \mathsf{LD}(m,n)^{\bullet k}$ and set $\mathsf{dw}(P) = \rho_\ell \cdots \rho_0$, where the ρ_i's are its runs. We define its *i-th run multiplicity functions* $z_i, z_i^\bullet : \mathbb{N} \to \mathbb{N}$, where for any $c \in \mathbb{N}$

$$\begin{aligned} z_i(c) &= \quad \#\text{ of undecorated } c\text{'s in } \rho_i \\ z_i^\bullet(c) &= \quad \#\text{ of decorated } c\text{'s in } \rho_i. \end{aligned}$$

Clearly, each function z_i has finite support.

Definition 7.4. Consider $P \in \mathsf{LSQ}(m,n)^{\bullet k}$ and set $\mathsf{dw}(P) = \rho_\ell \cdots \rho_0$, where the ρ_i's are its runs. For $c \in \mathbb{N}$, we define its *schedule numbers* $w_{i,s}(c)$ as follows:

$$w_{i,s}(c) \coloneqq \begin{cases} \sum_{d>c} z_i(d) + \sum_{d<c} z_{i-1}(d) & \text{if } i \in \{s+1, \dots, \ell\} \\ \sum_{d>c} z_i(d) + 1 - \delta_{c,0} & \text{if } i = s \\ \sum_{d<c} z_i(d) + \sum_{d>c} z_{i+1}(d) & \text{if } i \in \{0, \dots, s-1\} \end{cases}$$

$$w_{i,s}^\bullet(c) \coloneqq \sum_{d<c} z_i(d) + \sum_{d>c} z_{i+1}(d) - \delta_{c,0}\delta_{i,s-1}$$

At this point, this definition seems technical and mysterious. The proof of the main result of this section will clarify this choice.

Definition 7.5. Let $p_1, \dots, p_k$ be a sequence of integers. We define its *descent set*

$$\mathsf{Des}(p_1, \dots, p_k) \coloneqq \{1 \le i \le k-1 \mid p_i > p_{i+1}\}$$

and its *major index* $\mathsf{maj}(p_1, \dots, p_k)$ to be the sum of the elements of the descent set.

Convention 7.6. If w is the diagonal word of some path in $\mathsf{LSQ}(m,n)^{\bullet k}$ it is a decorated word with letters in the alphabet $\mathbb{N}$. For a decorated word w, we define $\mathsf{maj}(w)$ and x^w to be computed as usual, simply ignoring the decorations.

Theorem 7.7. *Let $z = \rho_\ell \cdots \rho_0$ be a decorated word in the alphabet $\mathbb{N}$ so that the ρ_i are its runs. Let $b(z,s) \coloneqq \sum_{c>0} \sum_{i<s} z_i(c) + \sum_{i<s-1}(-z_i^\bullet(0))$. Then*

$$\sum_{\substack{P \in \mathsf{LSQ}(m,n)^{\bullet k} \\ \mathsf{shift}(P)=s \\ \mathsf{dw}(P)=z}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$$

$$= t^{\mathsf{maj}(z)} q^{b(z,s)} \prod_{i=0}^{\ell} \left(\prod_{c \in \mathbb{N}} \begin{bmatrix} w_{i,s}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q q^{\binom{z_i^\bullet(c)}{2}} \begin{bmatrix} w_{i,s}^\bullet(c) \\ z_i^\bullet(c) \end{bmatrix}_q \right) x^z.$$

Proof. The right hand side of this equation consists of a finite number of terms different from 1. Indeed $z_i(c) = z_i^\bullet(c) = 0$ for all but a finite number of elements of $\mathbb{N}$ and thus all but a finite number of q-binomials are equal to 1, which means that the product is actually finite.

For any $P \in \mathsf{LSQ}(m,n)^{\bullet k}$ with $\mathsf{dw}(P) = z$ we trivially have $x^P = x^z$. It is also not difficult to see that for any such path $\mathsf{maj}(z) = \mathsf{area}(P)$, indeed

$$\begin{aligned} \mathsf{area}(P) &= \ell \cdot \#\rho_\ell + (\ell-1) \cdot \#\rho_{\ell-1} + \cdots + 1 \cdot \#\rho_1 \\ &= \rho_\ell + (\rho_\ell + \rho_{\ell-1}) + \cdots + (\rho_\ell + \rho_{\ell-1} + \cdots + \rho_1) = \mathsf{maj}(z). \end{aligned}$$

This takes care of the factor $t^{\mathsf{maj}(z)}$.

For the dinv, we will construct all the paths of a given diagonal word and shift, starting from the empty path, all the while keeping track of the dinv. We outline the different steps of the construction. We only describe the placement of the (decorated) labels in the lattice, as each such placement is the labelling of a unique square path.

1. For $i = s, s+1, \dots \ell$ insert the $z_i(c)$ labels equal to c into the $(i-s)$-th diagonal, for all $c \in \mathbb{N}$, in decreasing order.

2. For $i = s-1, s-2, \dots 0$ insert the $z_i(c)$ labels equal to c into the $(i-s)$-th diagonal, for all $c \in \mathbb{N}$, in increasing order.

3. For all i insert the $z_i^\bullet(c)$ decorated labels equal to c into the $(i-s)$-th diagonal, for all $c \in \mathbb{N}$, in decreasing order (the order of i is unimportant).

In other words in the first step we construct undecorated Dyck paths, in the second we turn them into undecorated square paths and in the third we add decorated labelled steps.

Call a (i,c)-*insertion* (respectively $(i,c)^{\bullet}$-*insertion*) the insertion of $z_i(c)$ (respectively $z_i^{\bullet}(c)$) labels equal to c into the $(i-s)$-th diagonal. We will now study, for each insertion the numbers of ways it may be executed, and the contribution to the dinv each of these ways engenders.

We made figures illustrating the construction of some of the paths with diagonal word $44223\overset{\bullet}{3}\overset{\bullet}{3}\overset{\bullet}{0}11\overset{\bullet}{2}$ and shift 1. We included them in Appendix B.

Dyck paths. First consider $i = s$. Right before the (s,c)-insertion, there are $\sum_{d>c} z_s(d)$ labels in the 0-th diagonal. If $c \neq 0$ the $z_i(c)$ labels may be inserted anywhere between them, i.e. the number of insertions is equal to the number of interlacings of $\sum_{d>c} z_s(d) = w_{s,s}(c) - 1$ and $z_i(c)$. If $c = 0$, since the leftmost label in the 0-th diagonal may never be 0, the number of insertions equals the number or interlacings between $\sum_{d>0} z_s(d) - 1 = w_{s,s}(0) - 1$ and $z_i(0)$. In both cases, any time one of the inserted c precedes one of the d with $d > c$, a unit of dinv is created. Thus the dinv created by an insertion is the inv of the corresponding interlacing. By Proposition 4.23, the dinv of all possible insertions is q-counted by $\begin{bmatrix} w_{s,s}(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q$. See Figure B.1.

For $i > s$, consider the path right before the (i,c)-insertion. We identify two kinds of *insertion spots*: a smaller label in the $(i-s-1)$-th diagonal *or* a label in the $(i-s)$-th one (which must be bigger than c because of the insertion order). There are $\sum_{d<c} z_{i-1}(d)$ labels of the first and $\sum_{d>c} z_i(d)$ of the second kind, and so the total number of insertion spots comes to $w_{i,s}(c)$. Any (i,c)-insertion corresponds uniquely to an interlacing of $w_{i,s}(c) - 1$ and $z_i(c)$: indeed

- the first occurrence of c in the $(i-s)$-th diagonal must be preceded by an insertion spot;
- there is a unique way of inserting a string of consecutive c's right after any insertion spot. Say we want to insert k consecutive c's. Shift the labels following the insertion spot k squares to the north-east. Then insert the first of the string of c's into the square on top of an insertion spot of the first kind or in the square north-east of an insertion spot of the second kind;
- between two strings of consecutive c's there must be an insertion spot.

Any time an occurrence of c precedes an insertion spot of the first (respectively second) kind, a unit of secondary (respectively primary) dinv is created. So as before, by Proposition 4.23, the dinv of all possible insertions is q-counted by $\begin{bmatrix} w_s(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q$. See Figure B.2.

Square paths. For the (i,c)-insertion with $i < s$ the insertions spots are either bigger labels in the diagonal directly above the $(i-s)$-th one, of which there are $\sum_{d>c} z_{i+1}(d)$, or labels in the $(i-s)$-th diagonal (which are smaller than c due to the insertion order), of which there are $\sum_{d<c} z_i(d)$. Thus there are $w_{i,s}(c)$ insertion spots. We have that

- the last occurrence of c must be followed by an insertion spot;

- there is an unique way of inserting a string of consecutive c's right *before* any insertion spot. Say we want to insert k consecutive c's. Shift the insertion spot and the labels following it k squares to the north-east. Insert the last of the string of c's in the square below an insertion spot of the first kind or in the square south-west of the insertion spot of the second kind;
- between two strings of consecutive c's there must be an insertion spot.

So the (i,c)-insertion corresponds uniquely to an interlacing of $z_i(c)$ and $w_{i,s}(c)-1$. An insertion spot of the first (respectively second) creates secondary (respectively primary) dinv with all following c's. Furthermore, any non-zero label that gets inserted under the main diagonal creates a unit of bonus dinv. Thus the dinv of all possible insertions is q-counted by $q^{(1-\delta_{c,0})z_i(c)} \begin{bmatrix} w_s(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q$. See Figure B.3.

Decorations. Now we treat $(i,c)^\bullet$-insertions. For all i define the *dinv markers* to be the $\sum_{d<c} z_i(d)$ undecorated labels smaller that c in the $(i-s)$-th diagonal and the $\sum_{d>c} z_{i+1}(d)$ undecorated labels bigger than c in the $(i-s+1)$-th diagonal. These dinv markers are exactly the labels with which a decorated c inserted to its right would create primary or secondary dinv.

First consider $i \geq s$. The number of dinv markers equals $w^\bullet_{i,s}(c)$. We claim that $(i,c)^\bullet$-insertions correspond bijectively to strict interlacings of $z^\bullet_i(c)$ and $w^\bullet_{i,s}(c)$, *starting* with a 1. The map is naturally defined: the relative order of the $w^\bullet_{i,s}(c)$ dinv markers and $z^\bullet_i(c)$ inserted c's defines the interlacing. We show that this map is a well defined bijection.

Well defined. We have to show that for any $(i,c)^\bullet$-insertion, the corresponding interlacing is always a strict interlacing beginning with a 1. In the proof of Proposition 4.20, it is argued that a decorated valley at height ≥ 0 is alway preceded by a label with which it creates primary or secondary dinv, i.e. a dinv marker; and so the interlacing must start with 1. Next, we need to argue that the interlacing is always strict, i.e. that there is always a dinv marker between two inserted c's. If the step labelled by an inserted c is followed by a vertical step, its label must be bigger that c and so it is a dinv marker. If it is followed by a horizontal step, it might be followed by a string of decorated labels at the same height: $B_1, \dots, B_l$. We must have $c < B_1 < \cdots < B_l$ since the valleys are contractible. If the step labelled B_l is followed by a vertical step, its label must be bigger than B_l and so an dinv marker. If the step labelled B_l is followed by a horizontal step the step after this horizontal step cannot be a decorated valley labelled c (not contractible) so it must either be a vertical, undecorated step, or another horizontal step. In the latter case, the next label at height $i-s$ is a rise and so undecorated. Thus, there is an undecorated label at height $i-s$ between our inserted c and the next one. Again, we may use the arguments in the proof of Proposition 4.20 to conclude that there must be an dinv marker before the next occurrence of an inserted decorated c.

Injectivity. Suppose that there are two different insertions with the same interlacing. This implies that between two (or after all) dinv markers there are two different ways to insert a decorated c. Combining these two ways, one would obtain a path with two inserted c's that are not separated by a dinv marker, in contradiction to what is shown in the previous paragraph.

Surjectivity. We must show that it is always possible to insert a decorated c

between two (or after all) dinv markers. We describe an insertion procedure for all possibilities.

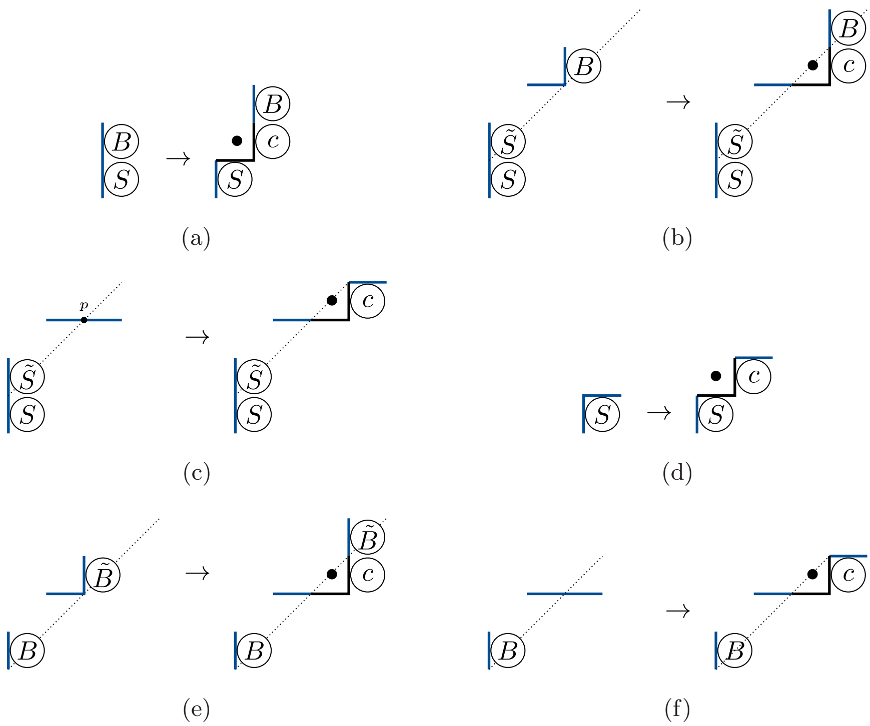


Figure 7.2: Surjectivity for $i \geq s$.

First consider a dinv marker of the first kind, i.e. a label S at height $(i - s)$ smaller than c.

- If the dinv marker is followed by a vertical step, whose label B is bigger than c, then insert the decorated label c directly north-east of S, right under B. See Figure 7.2a.
- Suppose that the dinv marker is followed by a vertical step, whose label $\tilde{S}$ is smaller than c and before the path crosses the $(i - s + 1)$-th diagonal horizontally, there is dinv marker of the second kind, i.e. a label B bigger than c at height $i - s + 1$. If the step labelled S is followed by another vertical step, the path crosses the $(i - s + 1)$-th diagonal vertically. Thus, since $i \geq s$, the path will cross the same diagonal horizontally, after the vertical crossing. Insert the decorated c such that it lies right below this B. See Figure 7.2b.
- Suppose that the dinv marker is followed by a vertical step, whose label $\tilde{S}$ is smaller than c and there is no dinv marker of the second kind between $\tilde{S}$ and the point p where the path crosses the $(i - s + 1)$-th diagonal horizontally. At p, insert a horizontal step followed by a decorated vertical step labelled c. See Figure 7.2c.

- Suppose that the dinv marker is followed by a horizontal step. Then insert the decorated label c in the square north-east of S. See Figure 7.2d.

Next, consider a dinv marker of the second kind, i.e. a label B at height $(i-s+1)$, bigger than c. Since $i \geq s$, we know the path will cross the $(i-s+1)$-th diagonal horizontally, after the dinv marker.

- Suppose that before the path crosses the $(i-s+1)$-th diagonal horizontally, there is a second dinv marker labelled $\tilde{B}$ of the second kind. Insert the decorated c such that it lies right below $\tilde{B}$. See Figure 7.2e.

- Suppose that there is no dinv marker of the second kind between B and the point p where the path crosses the $(i-s+1)$-th diagonal horizontally. At p, insert a horizontal step followed by a decorated vertical step labelled c. See Figure 7.2f.

This completes the list of possibilities and thus the the argument for bijectivity. By the definition of dinv markers, each time a dinv marker precedes an inserted c a unit of secondary or primary dinv is created, which corresponds to the inv of the interlacing. Furthermore, for each $z_i^\bullet(c)$ decorated c's that are inserted there is a -1 contribution to the dinv. So the total contribution to the dinv is q-counted by

$$q^{-z_i^\bullet(c)} \sum_f q^{\mathsf{inv}(f)}$$

where the sum is over strict interlacings of $z_i^\bullet(c)$ and $w_{i,s}^\bullet(c)$, starting with a 1. This first 1 contributes $z_i^\bullet(c)$ to the inv and so if we change the sum to be over the strict interlacings *ending* with a 1 we get

$$q^{-z_i^\bullet(c)} q^{z_i^\bullet(c)} \sum_f q^{\mathsf{inv}(f)} = q^{\binom{z_i^\bullet(c)}{2}} \begin{bmatrix} w_{i,s}^\bullet \\ z_i^\bullet(c) \end{bmatrix}_q,$$

where the equality comes from Proposition 4.25.

Now for $i < s$. Using the same map as for the previous case, we will show that $(i,c)^\bullet$-insertions correspond bijectively to strict interlacings of $z_i^\bullet(c)$ and $w_{i,s}^\bullet(c)$, ending with a 1.

Well defined. There are three things to show. First, that the interlacing corresponding to an insertion is always strict. Exactly the same argument as for $i \geq s$ applies. Second, we show that the interlacing corresponding to any insertion ends with a 1. Consider c an inserted label at height $i-s$. If the step c labels is followed by a vertical step, this must be labelled with a label bigger that c and so this is a dinv marker. Suppose that the inserted c is followed by a horizontal step.

Since the path must end east, there must be two consecutive vertical steps, at height $i-s$ and $i-s+1$, after c. If the label of the second of these steps is bigger than c it is a dinv marker. If not, the label S_1 of the first vertical step must be smaller than c, so if it is not decorated, it is a dinv marker. If it is decorated it may be preceded by a string of decorated valleys at height $i-s$, labelled $S_2,\dots,S_l$ with $S_1 > \cdots > S_l$ (by contractibility). The step labelled S_l is preceded by a horizontal step; if this step is preceded by an undecorated vertical step its label must be smaller than c and is thus a dinv marker. If it is preceded by a second horizontal

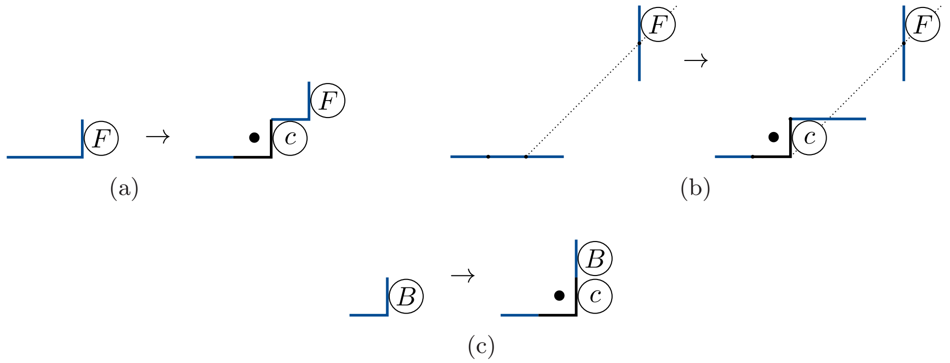


Figure 7.3: Surjectivity for $i < s$.

step we may deduce the existence of two consecutive vertical steps (at height $i - s$ and $i - s + 1$) between c and S_l. We have arrived at the same situation as at the beginning of the paragraph. Since the path is finite this loop must terminate and a dinv marker exists after c.

Finally, for $i = s - 1$ and $c = 0$, $w^{\bullet}_{s-1,s}(0)$ is equal to the number of dinv markers minus 1. Indeed, the interlacing corresponding to the $(s-1, 0)^{\bullet}$-insertion must start with a 1: by definition the path may not start with a decorated 0 at height -1 so the first decorated 0 at height -1 must be preceded by two horizontal step and thus a positive label at height 0. Therefore, disregarding this first 1 of the interlacing, an $(s-1, 0)^{\bullet}$-insertion corresponds to an interlacing of $z^{\bullet}_{s-1}(0)$ and $w^{\bullet}_{s-1,s}(0)$.

Remark 7.8. Keep in mind that this disregarded 1 creates $z^{\bullet}_{s-1}(0)$ units of dinv with all the 0's that followed it in the interlacing.

Injectivity. The argument is the same as for $i \geq s$.

Surjectivity. The fact that there must be a dinv marker to the right of all inserted c's ensures that the insertion algorithms for $i \geq s$ also apply here. So the only thing left to show is that, if $i \neq s - 1$ or $c \neq 0$, we may always insert a decorated c to the left of all dinv markers. We consider the first label at height $i - s$, denote it F and consider the following cases.

- Suppose that F is a dinv marker or appears before all dinv markers, is preceded by a horizontal step. Then this step must be preceded by another horizontal step, else the step labelled F would not be the first at its height. Insert a horizontal step followed by a decorated vertical step labelled c between these two horizontal steps. If F is decorated, the insertion order ensures that $c < F$ and so F labels a contractible valley. See Figure 7.3a.

- Suppose F is a dinv marker or appears before all dinv markers and is preceded by a vertical step. Since the path starts at $(0, 0)$ this implies that before F there must be point were the path crosses the $(i-s)$-th diagonal horizontally. The two consecutive horizontal steps of this crossing must be preceded by a third horizontal step, since if there was a vertical step preceding them, F would not be the first label at its height. Insert a horizontal step followed by a decorated vertical step labelled c after the first (from the left) of these three horizontal steps. See Figure 7.3b.

- Suppose F is preceded by a dinv marker, a label $B > c$ at height $i - s + 1$. Then the step labelled B must be preceded by a horizontal step, for if it were preceded by a vertical one, F would not be the first label at its height. Insert a horizontal step followed by a decorated vertical step labelled c after this horizontal step, underneath B. See Figure 7.3c.

So the bijective correspondence between $(i,c)^{\bullet}$-insertions and strict interlacings of $z_i^{\bullet}(c)$ and $w_{i,s}^{\bullet}(c)$, ending with a 1 is established. Clearly, the inv of the interlacing equals the primary and secondary dinv created by the insertion, with the exception of the $z_{s-1}^{\bullet}(0)$ units of primary dinv created with the first dinv marker and the 0's at height -1 (see Remark 7.8), which is not accounted for in the interlacing. Next, for $c \neq 0$ and $i < s$ any $(i,c)^{\bullet}$-insertion creates $z_i^{\bullet}(c)$ units of bonus dinv. Furthermore, each inserted decorated valley contributes -1 to the dinv. It follows that the contribution to the dinv for all possible $(i,c)^{\bullet}$-insertions is q-counted by

$$q^{z_{s-1}^{\bullet}(0)\delta_{i,s-1}\delta_{c,0}} q^{(1-\delta_{c,0})z_i^{\bullet}(c)} q^{-z_i^{\bullet}(c)} q^{\binom{z_i^{\bullet}(c)}{2}} \begin{bmatrix} w_{i,s}^{\bullet} \\ z_i^{\bullet}(c) \end{bmatrix}_q$$

See Figures B.4 and B.5.

Taking the product over all possible i and c and using

$$\begin{aligned}
&\sum_{i<s}\sum_{c\in\mathbb{N}} \left((1-\delta_{c,0})z_i(c) + z_{s-1}^{\bullet}(0)\delta_{i,s-1}\delta_{c,0} + (1-\delta_{c,0})z_i^{\bullet}(c) - z_i^{\bullet}(c)\right) \\
&= \sum_{i<s}\sum_{c>0} (z_i(c) + z_i^{\bullet}(c) - z_i^{\bullet}(c)) + \sum_{i<s-1} (-z_i^{\bullet}(0)) + z_{s-1}^{\bullet}(0) - z_{s-1}^{\bullet}(0) \\
&= \sum_{c>0}\sum_{i<s} z_i(c) + \sum_{i<s-1} (-z_i^{\bullet}(0)) = b(z,s).
\end{aligned}$$

we finally obtain the announced formula.

□

7.2 The implication

We relate the combinatorics of square paths to the combinatorics of Dyck paths, using the schedule formula of the previous section.

Notation 7.9. For the remainder of the chapter, let z be the diagonal word of an element of $\mathsf{LSQ}'(m,n)^{\bullet k}$. We suppose $z = \rho_\ell \cdots \rho_0$ where the ρ_i are its runs. We also set $r_i \coloneqq \sum_{c>0} z_i(c)$, i.e. the number of undecorated, non-zero numbers in ρ_i. Since z is the diagonal word of a square path $r_i > 0$ for all $0 < i \leq \ell$. Indeed any but the base diagonal must contain a rise, which may not be decorated or labelled 0. The main diagonal must contain an undecorated step that is not labelled 0 by definition. If z is the area word of an element in $\mathsf{LSQ}'(m,n)^{\bullet k}$ then we also have $r_0 > 0$.

Furthermore $0 \leq s \leq \ell$ define

$$\mathsf{LSQ}_{q,t;x}(z,s) := \sum_{\substack{P \in \mathsf{LSQ}(m,n)^{\bullet k} \\ \mathsf{shift}(P)=s \\ \mathsf{dw}(P)=z}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Using the schedule formula of Theorem 7.7, we will establish the following result, relating $\mathsf{LSQ}_{q,t;x}(z,s)$ and $\mathsf{LSQ}_{q,t;x}(z,s')$. For this we need some lemmas.

Lemma 7.10. *Let $0 < s \leq \ell$ and suppose that $r_{s-1} > 0$. Then we have*

$$\prod_{c \in \mathbb{N}} \frac{\begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q} = \frac{[r_s]_q}{[r_{s-1}]_q} \frac{[r_s + z_s(0) - 1]_q!}{[r_{s-1} + z_{s-1}(0) - 1]_q!} \prod_{c=0}^{m} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}.$$

Proof. Recall that

$$w_{s,s}(c) = \sum_{d>c} z_s(d) + 1 - \delta_{c,0} \qquad w_{s-1,s-1}(c) = \sum_{d>c} z_{s-1}(d) + 1 - \delta_{c,0}.$$

And so we have

$$\begin{aligned}
&\prod_{c \in \mathbb{N}} \frac{\begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q} \\
&= \prod_{c \in \mathbb{N}} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!} \frac{[w_{s,s}(c) + z_s(c) - 1]_q!}{[w_{s,s}(c) - 1]_q!} \frac{[w_{s-1,s-1}(c) - 1]_q!}{[w_{s-1,s-1}(c) + z_{s-1}(c) - 1]_q!} \\
&= \prod_{c \in \mathbb{N}} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!} \frac{[\sum_{d \geq c} z_s(d) - \delta_{c,0}]_q!}{[\sum_{d>c} z_s(d) - \delta_{c,0}]_q!} \frac{[\sum_{d>c} z_{s-1}(d) - \delta_{c,0}]_q!}{[\sum_{d \geq c} z_{s-1}(d) - \delta_{c,0}]_q!} \\
&= \frac{[\sum_{d \geq 0} z_s(d) - 1]_q!}{[\sum_{d>0} z_s(d) - 1]_q!} \frac{[\sum_{d>0} z_{s-1}(d) - 1]_q!}{[\sum_{d \geq 0} z_{s-1}(d) - 1]_q!} \prod_{c \in \mathbb{N}} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!} \\
&\quad \times \prod_{c>0} \frac{[\sum_{d \geq c} z_s(d)]_q!}{[\sum_{d>c} z_s(d)]_q!} \frac{[\sum_{d>c} z_{s-1}(d)]_q!}{[\sum_{d \geq c} z_{s-1}(d)]_q!} = \cdots
\end{aligned}$$

Since $[\sum_{d \geq c} z_s(d)]_q! = (\prod_{j=1}^{z_s(c)} [\sum_{d>c} z_s(d) + j]_q)[\sum_{d>c} z_s(d)]_q!$ and

$$\prod_{c>1} \prod_{j=1}^{z_s(c)} \left[\sum_{d>c} z_s(d) + j \right]_q = \left[\sum_{c>0} z_s(c) \right]_q ! = [r_s]_q!$$

we may simplify the last product to obtain

$$\begin{aligned}
\cdots &= \frac{[r_s + z_s(0) - 1]_q!}{[r_s - 1]_q!} \frac{[r_{s-1} - 1]_q!}{[r_{s-1} + z_{s-1}(0) - 1]_q!} \prod_{c=0}^{m} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!} \frac{[r_s]_q!}{[r_{s-1}]_q!} \\
&= \frac{[r_s]_q}{[r_{s-1}]_q} \frac{[r_s + z_s(0) - 1]_q!}{[r_{s-1} + z_{s-1}(0) - 1]_q!} \prod_{c=0}^{m} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!}.
\end{aligned}$$

The denominators are clearly non-zero since they are either the q-analogue of positive integers or the q-factorials of non-negative integers. $\square$

Lemma 7.11. *Let $0 < s \leq \ell$. Then we have*

$$\prod_{c \in \mathbb{N}} \frac{\begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q} = \frac{[r_{s-1} + z_{s-1}(0) - 1]_q!}{[r_s + z_s(0) - 1]_q!} \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!}$$

Proof. Recall that

$$w_{s-1,s}(c) = w_{s,s-1}(c) = \sum_{d<c} z_{s-1}(d) + \sum_{d>c} z_s(d)$$

Set $m \coloneqq \max\{c \mid z_s(c) \neq 0 \text{ or } z_{s-1}(c)\}$. We have the following

$$\begin{aligned}
\prod_{c \in \mathbb{N}} \frac{\begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q}{\begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q} &= \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \cdot \frac{[w_{s-1,s}(c) + z_{s-1}(c) - 1]_q!}{[w_{s,s-1}(c) + z_s(c) - 1]_q!} \\
&= \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \cdot \frac{[\sum_{d \leq c} z_{s-1}(d) + \sum_{d>c} z_s(d) - 1]_q!}{[\sum_{d<c} z_{s-1}(d) + \sum_{d \geq c} z_s(d) - 1]_q!} \\
&= \frac{\prod_{c=0}^{m} [\sum_{d>c} z_s(d) + \sum_{d \leq c} z_{s-1}(d) - 1]_q!}{\prod_{c=0}^{m} [\sum_{d>c-1} z_s(d) + \sum_{d \leq c-1} z_{s-1}(d) - 1]_q!} \cdot \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{\prod_{c=0}^{m} [\sum_{d>c} z_s(d) + \sum_{d \leq c} z_{s-1}(d) - 1]_q!}{\prod_{c=-1}^{m-1} [\sum_{d>c} z_s(d) + \sum_{d \leq c} z_{s-1}(d) - 1]_q!} \cdot \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{[\sum_{d>m} z_s(d) + \sum_{d \leq m} z_{s-1}(d) - 1]_q!}{[\sum_{d>-1} z_s(d) + \sum_{d \leq -1} z_{s-1}(d) - 1]_q!} \cdot \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{[\sum_{d \leq m} z_{s-1}(d) - 1]_q!}{[\sum_{d \geq 0} z_s(d) - 1]_q!} \cdot \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \\
&= \frac{[r_{s-1} + z_{s-1}(0) - 1]_q!}{[r_s + z_s(0) - 1]_q!} \prod_{c=0}^{m} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!}.
\end{aligned}$$

As the denominators are q-factorials of non-negative integers, they are non-zero. □

We now combine these two lemmas to get the following.

Lemma 7.12. *Let $0 \leq s' < s \leq \ell$ and suppose that $r_{s'} > 0$. Then we have*

$$\prod_{i=0}^{\ell} \prod_{c \in \mathbb{N}} \begin{bmatrix} w_{i,s}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q = \frac{[r_s]_q}{[r_{s'}]_q} \cdot \prod_{i=0}^{\ell} \prod_{c \in \mathbb{N}} \begin{bmatrix} w_{i,s'}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q.$$

Proof. First suppose $s' = s - 1$. By definition, for all $c \in \mathbb{N}$ and $i \notin \{s-1, s\}$ we have $w_{i,s}(c) = w_{i,s-1}(c)$, thus it suffices to show that

$$\begin{aligned}
&\prod_{c \in \mathbb{N}} \begin{bmatrix} w_{s-1,s}(c) + z_{s-1}(c) - 1 \\ z_{s-1}(c) \end{bmatrix}_q \begin{bmatrix} w_{s,s}(c) + z_s(c) - 1 \\ z_s(c) \end{bmatrix}_q \\
&\quad = \frac{[r_s]_q}{[r_{s-1}]_q} \cdot \prod_{c \in \mathbb{N}} \begin{bmatrix} w_{s-1,s-1}(c) + z_{s-1}(c) - 1 \\ z_{s-1}(c) \end{bmatrix}_q \begin{bmatrix} w_{s,s-1}(c) + z_s(c) - 1 \\ z_s(c) \end{bmatrix}_q.
\end{aligned} \tag{7.13}$$

By Lemma 7.10 we have

$$\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{s,s}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q = \frac{[r_s]_q}{[r_{s-1}]_q} \cdot \frac{[r_s+z_s(0)-1]_q!}{[r_{s-1}+z_{s-1}(0)-1]_q!} \cdot \prod_{c\in\mathbb{N}} \frac{[z_{s-1}(c)]_q!}{[z_s(c)]_q!} \times \begin{bmatrix} w_{s-1,s-1}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q$$

and by Lemma 7.11 we have

$$\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{s-1,s}(c)+z_{s-1}(c)-1 \\ z_{s-1}(c) \end{bmatrix}_q = \frac{[r_{s-1}+z_{s-1}(0)-1]_q!}{[r_s+z_s(0)-1]_q!} \cdot \prod_{c\in\mathbb{N}} \frac{[z_s(c)]_q!}{[z_{s-1}(c)]_q!} \cdot \begin{bmatrix} w_{s,s-1}(c)+z_s(c)-1 \\ z_s(c) \end{bmatrix}_q$$

so after the obvious simplifications the statement for $s'=s-1$ follows. Now, applying Equation 7.13 repeatedly, we get

$$\begin{aligned}
\prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s}(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q &= \frac{[r_s]_q}{[r_{s-1}]_q} \cdot \prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s-1}(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q \\
&= \frac{[r_s]_q}{[r_{s-1}]_q}\frac{[r_{s-1}]_q}{[r_{s-2}]_q} \cdot \prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s-2}(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q \\
&= \dots \\
&= \frac{[r_s]_q}{[r_{s-1}]_q} \cdots \frac{[r_{s'+1}]_q}{[r_{s'}]_q} \cdot \prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s'}(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q \\
&= \frac{[r_s]_q}{[r_{s'}]_q} \cdot \prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s'}(c)+z_i(c)-1 \\ z_i(c) \end{bmatrix}_q
\end{aligned}$$

as desired. ☐

We need one final lemma before proving the theorem.

Lemma 7.14. *Let $0 \leq s' < s \leq \ell$ and suppose that $r_s - z^{\bullet}_{s-1}(0) > 0$ and $r_{s'} - z^{\bullet}_{s'-1}(0) > 0$. Then we have*

$$\frac{[r_s]_q}{[r_s - z^{\bullet}_{s-1}(0)]_q} \prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w^{\bullet}_{i,s}(c) \\ z^{\bullet}_i(c) \end{bmatrix}_q = \frac{[r_{s'}]_q}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q} \prod_{i=0}^{\ell}\prod_{c\in\mathbb{N}} \begin{bmatrix} w^{\bullet}_{i,s'}(c) \\ z^{\bullet}_i(c) \end{bmatrix}_q.$$

Proof. Recall that

$$w^{\bullet}_{i,s}(c) = \sum_{d<c} z_i(d) + \sum_{d>c} z_{i+1}(d) - \delta_{c,0}\delta_{i,s-1}.$$

For $c \neq 0$, $w^{\bullet}_{i,s}(c)$ does not depend on s, and $w^{\bullet}_{i,s}(0) = w^{\bullet}_{i,s'}(0) \geq 0$ for $i \notin \{s-1, s'-1\}$. Thus it suffices to show that

$$\frac{[r_s]_q}{[r_s - z^{\bullet}_{s-1}(0)]_q} \begin{bmatrix} w^{\bullet}_{s-1,s}(0) \\ z^{\bullet}_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^{\bullet}_{s'-1,s}(0) \\ z^{\bullet}_{s'-1}(0) \end{bmatrix}_q = \frac{[r_{s'}]_q}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q} \begin{bmatrix} w^{\bullet}_{s-1,s'}(0) \\ z^{\bullet}_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^{\bullet}_{s'-1,s'}(0) \\ z^{\bullet}_{s'-1}(0) \end{bmatrix}_q.$$

We have

$$w^{\bullet}_{s-1,s}(0) = r_s - 1 \quad w^{\bullet}_{s'-1,s}(0) = r_{s'} \quad w^{\bullet}_{s-1,s'}(0) = r_s \quad w^{\bullet}_{s'-1,s'}(0) = r_{s'} - 1$$

so

$$\begin{aligned}
&\frac{[r_s]_q}{[r_s - z^{\bullet}_{s-1}(0)]_q} \begin{bmatrix} w^{\bullet}_{s-1,s}(0) \\ z^{\bullet}_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^{\bullet}_{s'-1,s}(0) \\ z^{\bullet}_{s'-1}(0) \end{bmatrix}_q \\
&= \frac{[r_s]_q}{[r_s - z^{\bullet}_{s-1}(0)]_q} \begin{bmatrix} r_s - 1 \\ z^{\bullet}_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} r_{s'} \\ z^{\bullet}_{s'-1}(0) \end{bmatrix}_q \\
&= \frac{[r_s]_q}{[r_s - z^{\bullet}_{s-1}(0)]_q} \frac{[r_s - 1]_q!}{[r_s - z^{\bullet}_{s-1}(0) - 1]_q! [z^{\bullet}_{s-1}(0)]_q!} \frac{[r_{s'}]_q!}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q! [z^{\bullet}_{s'-1}(0)]_q!} \\
&= \frac{[r_s]_q!}{[r_s - z^{\bullet}_{s-1}(0)]_q! [z^{\bullet}_{s-1}(0)]_q!} \frac{[r_{s'}]_q!}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q! [z^{\bullet}_{s'-1}(0)]_q!} \\
&= \frac{[r_{s'}]_q}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q} \frac{[r_s]_q!}{[r_s - z^{\bullet}_{s-1}(0)]_q! [z^{\bullet}_{s-1}(0)]_q!} \frac{[r_{s'} - 1]_q!}{[r_{s'} - z^{\bullet}_{s'-1}(0) - 1]_q! [z^{\bullet}_{s'-1}(0)]_q!} \\
&= \frac{[r_{s'}]_q}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q} \begin{bmatrix} r_s \\ z^{\bullet}_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} r_{s'} - 1 \\ z^{\bullet}_{s'-1}(0) \end{bmatrix}_q \\
&= \frac{[r_{s'}]_q}{[r_{s'} - z^{\bullet}_{s'-1}(0)]_q} \begin{bmatrix} w^{\bullet}_{s-1,s'}(0) \\ z^{\bullet}_{s-1}(0) \end{bmatrix}_q \begin{bmatrix} w^{\bullet}_{s'-1,s'}(0) \\ z^{\bullet}_{s'-1}(0) \end{bmatrix}_q
\end{aligned}$$

as desired.

□

We now have all the preliminary result necessary to prove our theorem.

Theorem 7.15. *Let* $0 \leq s' < s \leq \ell$ *and suppose that* $r_{s'} > 0$

$$[r_{s'} - z^{\bullet}_{s'-1}(0)]_q \cdot \mathsf{LSQ}_{q,t;x}(z, s) = q^{\sum_{i=s'}^{s-1}(r_i - z^{\bullet}_{i-1}(0))}[r_s - z^{\bullet}_{s-1}(0)]_q \cdot \mathsf{LSQ}_{q,t;x}(z, s').$$

Proof. For any square path there must be an undecorated, positively labelled step at height 0 preceding all, between any pair, and following all decorated valleys labelled 0 at height -1. Indeed every decorated valley labelled 0 is preceded by two horizontal steps so there must be a rise in between any two such steps. Furthermore by definition a square path may not start with two horizontal steps followed by a decorated valley labelled 0. Thus there must be a label at height 0 before the first occurence of such a step at height -1. The first label at height 0 of a path may never be zero. Finally, since the path must end with an east step, a rise must follow all steps at height -1.

So if z is the diagonal word of some path of shift s, $r_s - z^\bullet_{s-1}(0) > 0$. Thus, if $r_s - z^\bullet_{s-1}(0) = 0$ we have $\mathsf{LSQ}_{q,t;x}(z,s) = \emptyset$ and the equation in the thesis is $0 = 0$. Since the the same argument applies to s', we may suppose $r_s - z^\bullet_{s-1}(0) > 0$ and $r_{s'} - z^\bullet_{s'-1}(0) > 0$ from now on.

By Theorem 7.7, we have

$$\begin{aligned}&\mathsf{LSQ}_{q,t;x}(z,s)\\ &\quad = t^{\mathsf{maj}(z)} q^{b(z,s)} \prod_{i=0}^{\ell} \left(\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q q^{\binom{z^\bullet_i(c)}{2}} \begin{bmatrix} w^\bullet_{i,s}(c) \\ z^\bullet_i(c) \end{bmatrix}_q \right) x^z.\end{aligned}$$

and

$$\begin{aligned}&\mathsf{LSQ}_{q,t;x}(z,s')\\ &\quad = t^{\mathsf{maj}(z)} q^{b(z,s')} \prod_{i=0}^{\ell} \left(\prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s'}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q q^{\binom{z^\bullet_i(c)}{2}} \begin{bmatrix} w^\bullet_{i,s'}(c) \\ z^\bullet_i(c) \end{bmatrix}_q \right) x^z.\end{aligned}$$

By definition we have

$$\begin{aligned} b(z,s) &= b(z,s') + \sum_{c>0} \sum_{i=s'}^{s-1} z_i(c) + \sum_{i=s'-1}^{s-2} (-z^\bullet_i(0)) \\ &= b(z,s') + \sum_{i=s'}^{s-1} r_i + \sum_{i=s'}^{s-1} (-z^\bullet_{i-1}(0)) = b(z,s') + \sum_{i=s'}^{s-1} (r_i - z^\bullet_{i-1}(0)). \end{aligned}$$

Lemma 7.12 states that

$$\prod_{i=0}^{\ell} \prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q = \frac{[r_s]_q}{[r_{s'}]_q} \cdot \prod_{i=0}^{\ell} \prod_{c\in\mathbb{N}} \begin{bmatrix} w_{i,s'}(c) + z_i(c) - 1 \\ z_i(c) \end{bmatrix}_q.$$

Lemma 7.14 states that

$$\frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \prod_{i=0}^{\ell} \prod_{c\in\mathbb{N}} \begin{bmatrix} w^\bullet_{i,s}(c) \\ z^\bullet_i(c) \end{bmatrix}_q = \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \prod_{i=0}^{\ell} \prod_{c\in\mathbb{N}} \begin{bmatrix} w^\bullet_{i,s'}(c) \\ z^\bullet_i(c) \end{bmatrix}_q.$$

Combining these last five equations we get

$$\begin{aligned}&\frac{[r_s]_q}{[r_s - z^\bullet_{s-1}(0)]_q} \cdot \mathsf{LSQ}_{q,t;x}(z,s)\\ &\quad = q^{\sum_{i=s'}^{s-1}(r_i - z^\bullet_{i-1}(0))} \frac{[r_s]_q}{[r_{s'}]_q} \cdot \frac{[r_s - z^\bullet_{s-1}(0)]_q}{[r_s]_q} \cdot \frac{[r_{s'}]_q}{[r_{s'} - z^\bullet_{s'-1}(0)]_q} \cdot \mathsf{LSQ}_{q,t;x}(z,s'),\end{aligned}$$

which gives the thesis after obvious cancellations. □

Corollary 7.16. *If $r_0 \neq 0$, then*

$$\mathsf{LSQ}_{q,t;x}(z,s) = q^{b(z,s)} \frac{[r_s - z^\bullet_{s-1}(0)]_q}{[r_0]_q} \mathsf{LSQ}_{q,t;x}(z,0).$$

Proof. It follows immediately by applying Theorem 7.15 with $s' = 0$ (using $z^{\bullet}_{-1}(0) = 0$). $\square$

Corollary 7.17. *For $n, k, r \in \mathbb{N}$ with $r > 0$,*

$$\sum_{\substack{P \in \mathsf{LSQ}'(m,n)^{\bullet k} \\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P = \frac{[n-k]_q}{[r]_q} \sum_{\substack{P \in \mathsf{LD}(m,n)^{\bullet k} \\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Proof. Summing the equation from Corollary 7.16 over all possible values of the shift we get

$$\begin{aligned} \sum_{s=0}^{\ell} \sum_{\substack{P \in \mathsf{LSQ}(m,n)^{\bullet k} \\ \mathsf{shift}(P)=s \\ \mathsf{dw}(P)=z}} & q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \\ &= \sum_{s=0}^{\ell} q^{b(z,s)} \frac{[r_s - z^{\bullet}_{s-1}(0)]_q}{[r_0]_q} \sum_{\substack{P \in \mathsf{LD}(m,n)^{\bullet k} \\ \mathsf{dw}(P)=z}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \\ &= \frac{[\sum_{s=0}^{\ell}(r_s - z^{\bullet}_{s-1}(0))]_q}{[r_0]_q} \sum_{\substack{P \in \mathsf{LD}(m,n)^{\bullet k} \\ \mathsf{dw}(P)=z}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P. \end{aligned}$$

By definition, for any path P, $\mathsf{touch}(P) = r_0$. So now take the sum over all the decorated words z of length n with k decorations and $r_0 = r > 0$. In this way we obtain all the elements of $\mathsf{LSQ}'(m,n)^{*k}$ on the left hand side.

We have, for any such z, that $\sum_{s=0}^{\ell} r_s = n - k + \sum_{s=0}^{\ell} z^{\bullet}_{s-1}(0)$; indeed it is the total number of nondecorated positive labels ($z^{\bullet}_{\ell}(0) = 0$ since a decorated step labelled 0 must always be preceded by two horizontal steps, which is not possible in the top diagonal). $\square$

Theorem 7.18. *If Conjecture 5.14 holds, then so does Conjecture 5.7*

Proof. Conjecture 5.14 is

$$\Delta_{h_m} \Theta_k \nabla E_{n-k,r} = \sum_{\substack{P \in \mathsf{LD}(m,n)^{\bullet k} \\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Using Corollary 7.17, we get

$$\frac{[n-k]_q}{[r]_q} \Delta_{h_m} \Theta_k \nabla E_{n-k,r} = \sum_{\substack{P \in \mathsf{LSQ}(m,n)^{\bullet k} \\ \mathsf{touch}(P)=r}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

Since $\mathsf{LSQ}'(m,n)$ contains exactly the paths with at least one touching point, taking the sum over $r > 0$ and applying Proposition 2.42 gives exactly Conjecture 5.7:

$$\Delta_{h_m} \Theta_k \nabla \omega(p_{n-k}) = \sum_{P \in \mathsf{LSQ}'(m,n)^{\bullet k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P.$$

$\square$

Chapter 8

The compositional Delta conjecture

We extend the combinatorial framework of the proof of the compositional shuffle conjecture [CM18] to rise decorated Dyck paths. In particular, we prove an extension of the "main recursion" of their paper that relates decorated Dyck paths to their "raising and lowering" operators. In this way, we reduce the compositional Delta conjecture 5.10 (rise version) to a conjectural identity of operators.

This chapter is the least self-contained of this thesis and should really be read in tandem with [CM18]. See also [HX17]: a more detailed exposition of the same paper.

8.1 Diagonally labelled decorated Dyck paths

In this section, we introduce a new type of lattice path: the diagonally labelled decorated Dyck paths. As the name suggests, this is a kind of labelled Dyck path, but instead of labelling vertical steps, the labels are contained in the main diagonal. The interest in these paths comes from a bijection between rise decorated labelled Dyck paths and valley decorated diagonally labelled Dyck paths. We will introduce a new statistic, ninv, on this new set and recall the definition of the classic bounce statistic on Dyck paths. The bijection will send dinv to ninv and area to bounce.

Definition 8.1. A *diagonal labelling* of $\pi \in \mathsf{D}(n)$ is an element $w \in \mathbb{P}^n$ such that for all $i \in \mathsf{Val}(\pi)$, $w_i > w_{j(i)}$ where $j(i)$ is the index of the column containing the horizontal step preceding the i-th vertical step of π. The set of diagonal labellings of π is denoted by $\mathsf{DLa}(\pi)$.

Definition 8.2. A (valley) *decorated, diagonally labelled Dyck path* is a triple $P \coloneqq (\pi, w, dv)$ where π is a Dyck path, w a diagonal labelling of π, and dv a subset of $\mathsf{Val}(\pi)$. The set of such triples where π is of size n, and $\#dv = k$ is denoted by $\mathsf{DLD}(n)^{\bullet k}$. As usual, the monomial of P is defined by $x^P \coloneqq x^w$.

See Figure 8.1 for an example of such a path.

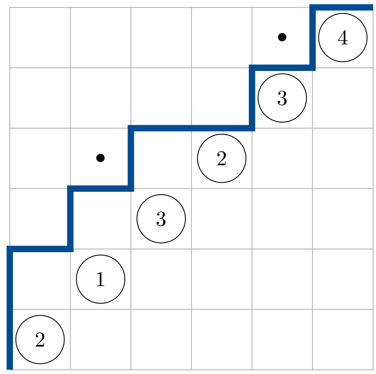


Figure 8.1: An element of $\mathsf{DLD}(6)^{\bullet 2}$.

Definition 8.3. Let (π, w) be a pair in $\mathsf{D}(n) \times \mathbb{P}^n$ and represent w in the main diagonal of π. A pair of indices (i, j) with $1 \leq i < j \leq n$ is called a *non-inversion* of (π, w) if $w_i < w_j$ and the cell in the i-th column and j-th row lies underneath π. The number of non-inverions of (π, w) is denoted by $\mathsf{ninv}(\pi, w)$.

For $P := (\pi, w, dr, dv) \in \mathsf{DLD}(n)^{*k\bullet l}$ we set $\mathsf{ninv}(P) := \mathsf{ninv}(\pi, w)$ so that the ninv of a path is independent of its decorations.

Example. For example, the path in Figure 8.1 has only one non-inversion: $(2, 3)$. Thus its ninv is 1.

Next, we define a decorated version of Haglund's bounce statistic (see [Hag03]).

Definition 8.4. Take $\pi \in \mathsf{D}(n)$. Its *bounce path* is a lattice path from $(0, 0)$ to (n, n) defined as follows: it starts at $(0, 0)$ and travels north until it hits an east step of π, whereupon it changes direction and travels eastward until it hits the main diagonal. Then it travels north again and repeats this process until it arrives at (n, n). The *sections* of a bounce path are the portions between its touching points.

The *bounce word* is an increasing word $(b_1(\pi), \dots, b_n(\pi)) \in \mathbb{N}^n$ that has as many 0's as the height of the first section of the bounce path, as many 1's as the height of its second section and so forth. Take $dv \subseteq \mathsf{Val}(\pi)$ and define

$$\mathsf{bounce}(\pi, dv) := \sum_{i \notin dv} b_i(\pi).$$

For $P := (\pi, dr, dv) \in \mathsf{D}(n)^{*k\bullet l}$ or $Q := (\pi, w, dr, dv) \in \mathsf{DLD}(n)^{*k\bullet l}$ we set $\mathsf{bounce}(P) = \mathsf{bounce}(Q) := \mathsf{bounce}(\pi, dv)$ so that the bounce of a path is independent of its decorated rises or diagonal labelling.

The ζ map

The following map first appeared in [HL05].

Theorem 8.5. *There exists a bijection*

$$\zeta : \mathsf{LD}(n)^{*k} \to \mathsf{DLD}(n)^{\bullet k}$$

such that for all $P \in \mathsf{LD}(n)^{*k}$

$$\mathsf{dinv}(P) = \mathsf{inv}(\zeta(P)) \qquad \mathsf{area}(P) = \mathsf{bounce}(\zeta(P)).$$

Proof. Figure 8.2 contains a path (left) and its image by ζ (right) as a visual aid. Let us define an auxiliary map

$$\zeta_0 : \mathsf{D}(n) \to \mathsf{D}(n).$$

Take $\pi \in \mathsf{D}(n)$ and rearrange its area word in ascending order. This new word, call it u, will be the bounce word of $\zeta_0(\pi)$. We construct $\zeta_0(\pi)$ as follows. First draw the bounce path corresponding to u. The first vertical stretch and last horizontal stretch of $\zeta_0(\pi)$ are fixed by this bounce path. For the section of the path between consecutive peaks[1] of the bounce path we apply the following procedure: place a pen on the top of the i-th peak of the bounce path and scan the area word of D from left to right. Every time we encounter a letter equal to $i-1$ we draw an east step and when we encounter a letter equal to i we draw a north step. By construction of the bounce path, we end up with our pen on top of the $(i+1)$-th peak of the bounce path. Note that in an area word a letter equal to $i \neq 0$ cannot appear unless it is preceded somewhere by a letter equal to $i-1$. This means that starting from the i-th peak, we always start with a horizontal step which explains why u is the bounce word of $\zeta_0(\pi)$. It is not hard to describe the inverse of this map and thereby conclude that it is bijective.

Next, we show that ζ_0 induces a bijection $\zeta^*_\pi : \mathsf{Rise}(\pi) \to \mathsf{Val}(\zeta_0(\pi))$ for all $\pi \in \mathsf{D}(n)$. Let $j \in \mathsf{Rise}(\pi)$. It follows that $a_j(\pi) = a_{j-1}(\pi) + 1$. Take i such that $a_{j-1}(\pi) = i - 1$. While scanning the area word to construct the path between the i-th and $(i+1)$-th peak of the bounce path, we will encounter $a_{j-1}(\pi) = i-1$, directly followed by $a_j(\pi) = i$. This will correspond to a horizontal step followed by a vertical step in $\zeta_0(\pi)$ and thus to an element of $\zeta^*_\pi \in \mathsf{Val}(\zeta_0(\pi))$. Again, the inverse map is easily divined.

Fix $P \in \mathsf{LD}(n)^{*k}$ and P_1 its first component, i.e. its underlying Dyck path. We now define each component of $\zeta(P)$.

- We set $\zeta_1(P) \coloneqq \zeta_0(P_1)$.
- Set $\zeta_2(P)$ to be the reverse reading word of P, i.e. the labels of P in the i-th diagonal read from left to right for $i = 0, 1 \dots$. We have to show that $\zeta_2(P) \in \mathsf{DLa}(\zeta_0(\pi))$. The labels of $\zeta(P)$ underneath $(i+1)$-th section of the bounce path of $\zeta_0(\pi)$ are exactly the labels of P at height i, in the same relative order left to right. In fact when representing $\zeta_2(P)$ in the main diagonal, we ensure that the label contained in the column (respectively row) of a horizontal step (respectively vertical step) s is the label of the step of P_1 encountered in the construction of $\zeta_0(P_1)$ at the moment of drawing s. Thus, under $\zeta^*_{P_1}$, the condition on the labels of rises of P_1 corresponds exactly to the condition on the labels of valleys of $\zeta_0(P_1)$.
- Naturally, we set $\zeta_3(P) = \zeta^*_{P_1}(dr)$.

[1] A *peak* of a path is the endpoint of a vertical step and the starting point of a horizontal step.

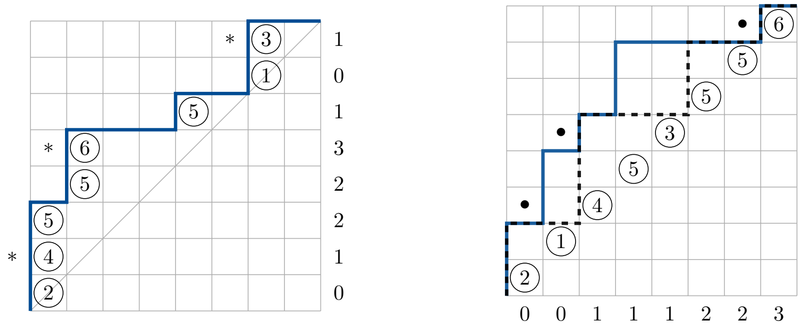


Figure 8.2: An element of $\mathsf{LD}(8)^{*3}$ with its area word (left) and its image by ζ in $\mathsf{DLD}(8)^{\bullet 3}$ with its bounce word.

So we have defined ζ, whose inverse is easily described following the procedures backwards. Let us now study the effect of ζ on dinv and area.

We start with the dinv statistic. Take $1 \leq i < j \leq n$ and conside the square s in the i-th column and j-th row of $(\mathbb{R}_{\geq 0})^2$. The pair (i, j) is a non-inversion of $\zeta(P)$ if and only if $w_i < w_j$ and s lies underneath $\zeta_1(P)$. Suppose s lies under the bounce path of $\zeta_1(P)$. Then w_i and w_j label steps at the same height in P, say the i'-th and j'-th vertical steps respectively, with $i' < j'$ (by the ζ correspondence). Thus (i', j') is a primary inversion of P if and only if $w_i < w_j$. Now suppose that s lies under $\zeta_1(P)$ and above its bounce path. Then in P, the step labelled w_j (say the j'-th vertical step) must lie one unit higher that the step labelled w_i (say the i'-th vertical step). The condition that s lies underneath $\zeta_1(P)$ is equivalent to $j' < i'$ and so (i', j') is a secondary inversion of P if and only if $w_i < w_j$.

Now for the area. Since the bounce word of $\zeta_1(P)$ is a rearrangement of the area word of P_1, $\sum_i a_i(P_1) = \sum_i b_i(\zeta_1(P))$. Furthermore, if $i \in \mathsf{Rise}(P_1)$ is at height j then $\zeta^*_{P_1}(i)$ decorates a step in the same row as a vertical step in the $(j+1)$-th section of the bounce path. Thus $b_{\zeta^*_{P_1}(i)}(\zeta_1(P)) = j$ and we must have

$$\sum_{i \not\in \mathsf{Rise}(P_1)} a_i(P_1) = \sum_{i \not\in \mathsf{Val}(\zeta_1(P))} b_i(\zeta_1(P)),$$

which is what we wanted to show.

□

We can also easily formulate an unlabelled version of the zeta map: it suffices to send $(\pi, dr) \in \mathsf{D}(n)^{*k}$ to $(\zeta_0(\pi), \zeta^*_\pi(dr))$. By slight abuse of notation, we use the same name for both the labelled and unlabelled version of the map.

Corollary 8.6. *There exists a bijection*

$$\zeta : \mathsf{D}(n)^{*k} \to \mathsf{D}(n)^{\bullet k}$$

such that for all $P \in \mathsf{D}(n)^{*k}$

$$\mathsf{dinv}(P) = \mathsf{area}(\zeta(P)) \qquad \mathsf{area}(P) = \mathsf{bounce}(\zeta(P)).$$

A second diagonal decomposition

We think of $\mathsf{DLD}(n)^{\bullet k}$ mainly as the image of $\mathsf{LD}(n)^{*k}$ by ζ. Hence we use the map to transfer a definition from the latter set to the former.

Definition 8.7. For a decorated Dyck path $(\pi, dv) \in \mathsf{D}(n)^{\bullet k}$, let $\mathsf{dcomp}'(\pi, dv) := \mathsf{dcomp}(\zeta^{-1}(P))$ (see Definition 4.16). For a diagonally labelled path $(\pi, w, dv) \in \mathsf{DLD}(n)^{\bullet k}$ we set $\mathsf{dcomp}'(\pi, dv, w) := \mathsf{dcomp}'(\pi, dv)$, i.e. the labelling does not affect the diagonal composition.

In Lemma 8.32, we will describe a way to compute $\mathsf{dcomp}'(P)$ directly from P.

Given a composition $\alpha \vDash n-k$, we set

$$\begin{aligned} \mathsf{D}'(\alpha)^{\bullet k} &:= \{P \in \mathsf{D}(n)^{\bullet k} \mid \mathsf{dcomp}'(P) = \alpha\} \\ \mathsf{DLD}'(\alpha)^{\bullet k} &:= \{P \in \mathsf{DLD}(n)^{\bullet k} \mid \mathsf{dcomp}'(P) = \alpha\}, \end{aligned}$$

where the prime indicates the use of dcomp' instead of dcomp.

We may now use Theorem 8.5 to reformulate the combinatorial side of the compositional Delta conjecture (Conjecture 5.10) as follows.

Corollary 8.8. *For $n, k \in \mathbb{N}$*

$$\sum_{P \in \mathsf{LD}(\alpha)^{*k}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P = \sum_{P \in \mathsf{DLD}'(\alpha)^{\bullet k}} q^{\mathsf{ninv}(P)} t^{\mathsf{bounce}(P)} x^P.$$

8.2 Weighted characteristic functions

To a Dyck path and a *weight function* on its valleys we associate a symmetric function called its weighted characteristic function. These can be used as the building blocks of the combinatorial side of the compositional Delta conjecture.

Definition 8.9. Take $\pi \in \mathsf{D}(n)$ and a function $wt : \mathsf{Val}(\pi) \to \mathbb{Q}(q,t)$. The *weighted characteristic function* of π is

$$\chi(\pi, wt) = \sum_{w \in \mathbb{P}^n} q^{\mathsf{ninv}(\pi, w)} \left(\prod_{\substack{i \in \mathsf{Val}(\pi) \\ w_i \leq w_{j(i)}}} wt(i) \right) x^w$$

where $j(i)$ is the column containing the horizontal step preceding the valley i (as in Definition 8.1).

Two special cases are of particular interest.

- For $wt = 0$, the constant zero function, by Definition 8.1, we get

$$\chi(\pi, 0) = \sum_{w \in \mathsf{DLa}(\pi)} q^{\mathsf{ninv}(\pi, w)} x^w.$$

It follows by Definitions 8.2 and 8.3 that

$$\sum_{P\in \mathsf{DLD}'(\alpha)^{\bullet k}} q^{\mathsf{ninv}(P)} t^{\mathsf{bounce}(P)} x^P = \sum_{(\pi,dv)\in \mathsf{D}'(\alpha)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)} \chi(\pi,0). \tag{8.10}$$

This is relevant because it is the combinatorial side of the compositional Delta conjecture (see Corollary 8.8).

- For the constant function $wt = 1$, we define the *unweighted* characteristic function $\chi(\pi) \coloneqq \chi(\pi,1)$. It happens that any weighted characteristic may be expressed in terms of unweighted characteristic functions of different paths.

The following is [CM18, Proposition 3.7].

Proposition 8.11. *For all $\pi \in \mathsf{D}(n)$ and $wt : \mathsf{Val}(\pi) \to \mathbb{Q}(q,t)$, the weighted characteristic function $\chi(\pi, wt)$ is a symmetric function.*

Next we establish [CM18, Example 3.8].

Definition 8.12. For $\pi \in \mathsf{D}(n)$ and $S \subseteq \mathsf{Val}(\pi)$ define π_S to be the path whose E,N-sequence is obtained from the E,N-sequence of π by replacing the EN subsequences corresponding to the valleys in S by NE. We refer to this process as "flipping the valleys in S".

Proposition 8.13. *For $\pi \in \mathsf{D}(n)$, we have*

$$\chi(\pi,0) = (1-q)^{-\#\mathsf{Val}(\pi)} \sum_{S\subseteq \mathsf{Val}(\pi)} (-1)^{\#S} \chi(\pi_S, 1).$$

Proof. Start from any weight function $wt : \mathsf{Val}(\pi) \to \mathbb{Q}(q,t)$ and let $k \in \mathsf{Val}(\pi)$. We define wt_1 to coincide with wt except that $wt(k) = 1$. Then define $wt_2 : \mathsf{Val}(\pi_{\{k\}}) \to \mathbb{Q}(q,t)$ to coincide with wt on all valleys of $\pi_{\{k\}}$ that are also valleys of π and to be 1 on the remaining valleys of $\pi_{\{k\}}$. We will show the following

$$\chi(\pi, wt) = \frac{q\cdot wt(k) - 1}{q-1}\chi(\pi, wt_1) + \frac{1-wt(k)}{q-1}\chi(\pi_{\{k\}}, wt_2). \tag{8.14}$$

Take $w \in \mathbb{P}^n$. On the left hand side, the coefficient of x^w is

$$q^{\mathsf{ninv}(\pi,w)} \prod_{\substack{i\in\mathsf{Val}(\pi)\\ w_i\leq w_{j(i)}}} wt(i).$$

On the right hand side we get, by the definitions of $wt_1, \pi_{\{k\}}$ and wt_2

$$\prod_{\substack{i\in\mathsf{Val}(\pi)\setminus\{k\}\\ w_i\leq w_{j(i)}}} wt(i) \left(q^{\mathsf{ninv}(\pi,w)} \frac{q\cdot wt(k)-1}{q-1} + q^{\mathsf{ninv}(\pi_{\{k\}},w)} \frac{1-wt(k)}{q-1}\right).$$

If $w_k \leq w_{j(k)}$ then $\mathsf{ninv}(\pi,w) = \mathsf{ninv}(\pi_{\{k\}},w)$ and so the parenthetical becomes $q^{\mathsf{ninv}(\pi,w)} wt(k)$. If $w_k > w_{j(k)}$ then $\mathsf{ninv}(\pi,w) + 1 = \mathsf{ninv}(\pi_{\{k\}},w)$ in which case the parenthetical is simply $q^{\mathsf{ninv}(\pi,w)}$. In both cases, the expression coincides with the left hand side of (8.14).

Now we apply Equation (8.14) with $wt = 0$, iteratively. We apply it $\#\mathsf{Val}(\pi)$ times, each time picking a different valley of π. Thus we end up with $2^{\#\mathsf{Val}(\pi)}$ terms, i.e. the number of subsets S of $\mathsf{Val}(\pi)$. For each iteration, think of the first term as "not putting k in S" and of the second as "putting k in S". Thus, the term corresponding to an $S \subseteq \mathsf{Val}(\pi)$ is $\frac{(-1)^{\#\mathsf{Val}(\pi)-S}}{(q-1)^{\#\mathsf{Val}(\pi)}}\chi(\pi_S, 1)$, which is what we wanted to show. □

Raising and lowering operators

We introduce some key operators from [CM18]. In this text, the only fact we will use about them is stated in Theorem 8.17, which expresses the combinatorics in terms of these algebraic operators.

Definition 8.15. Given a polynomial P depending on variables u, w, define the operator Υ_{uw} as[2]

$$(\Upsilon_{uw}P)(u,w) \coloneqq \frac{(q-1)vP(u,w) + (w-qu)P(w,u)}{w-u}$$

Definition 8.16. For $k \in \mathbb{N}$, define $V_k \coloneqq \Lambda[y_1,\ldots,y_k] = \Lambda \otimes \mathbb{Q}[y_1,\ldots,y_k]$. Let

$$T_i \coloneqq \Upsilon_{y_i y_{i+1}} \colon V_k \to V_k \text{ for } 1 \le i \le k-1.$$

We define the operators $d_+ \colon V_k \to V_{k+1}$ and $d_- \colon V_k \to V_{k-1}$: for $F[X] \in V_k$

$$\begin{aligned}(d_+F)[X] &\coloneqq T_1T_2\cdots T_k(F[X+(q-1)y_{k+1}])\\ (d_-F)[X] &\coloneqq -F[X-(q-1)y_k]\sum_{i\ge 0}(-1/y_k)^i e_i[X]\Big|_{y_k^{-1}}.\end{aligned}$$

The following is Theorem 4.4 of [CM18].

Theorem 8.17. *Let $\pi \in \mathsf{D}(n)$ and $\epsilon_1 \ldots \epsilon_{2n}$ the word in the alphabet $\{+,-\}$ obtained from the E,N-sequnece of π by the substitutions $E \mapsto +$ and $N \mapsto -$. Then*

$$\chi(\pi) = d_{\epsilon_1}\cdots d_{\epsilon_{2n}}(1)$$

Corollary 8.18. *Let $\pi \in \mathsf{D}(n)$ and $\tilde{\epsilon}_1 \ldots \tilde{\epsilon}_m$ the word in the alphabet $\{+,-,v\}$ obtained from the E,N-sequence of π by the substitutions $EN \mapsto v$ followed by $E \mapsto +$ and $N \mapsto -$. It follows that $m = 2n - \#\mathsf{Val}(\pi)$. Define*

$$d_v \coloneqq \frac{[d_-,d_+]}{q-1} = \frac{d_-d_+ - d_+d_-}{q-1}.$$

Then

$$\chi(\pi,0) = d_{\tilde{\epsilon}_1}\cdots d_{\tilde{\epsilon}_m}(1)$$

[2]In [CM18] this operator is called Δ_{uv}, but we changed the notation in order to avoid confusion with the Δ_f operator defined on Λ.

Proof. By Proposition 8.13, we have

$$\chi(\pi,0) = (1-q)^{-\#\mathsf{Val}(\pi)} \sum_{S\subseteq \mathsf{Val}(\pi)} (-1)^{\#S}\chi(\pi_S,1).$$

Take $k = \min(\mathsf{Val}(\pi))$, then

$$\chi(\pi,0) = (1-q)^{-\#\mathsf{Val}(\pi)} \sum_{S\subseteq \mathsf{Val}(\pi)\setminus\{k\}} (-1)^{\#S}\chi(\pi_S,1) + (-1)^{\#S+1}\chi(\pi_{S\cup\{k\}},1).$$

Let $\epsilon_1\dots\epsilon_{2n}$ be as in Theorem 8.17 and $\epsilon_l\epsilon_{l+1}$ the $+-$ corresponding to the EN sequence corresponding to the valley k and its preceding east step. Applying Theorem 8.17, we get

$$\begin{aligned}\chi(\pi,0) &= (1-q)^{-\#\mathsf{Val}(\pi)} \sum_{S\subseteq \mathsf{Val}(\pi)\setminus\{k\}} (-1)^{\#S}(d_{\epsilon_1}\cdots d_{\epsilon_l}d_{\epsilon_{l+1}}\cdots d_{\epsilon_{2n}}(1) \\ &\qquad\qquad - d_{\epsilon_1}\cdots d_{\epsilon_{l+1}}d_{\epsilon_l}\cdots d_{\epsilon_{2n}}(1)) \\ &= (1-q)^{-\#\mathsf{Val}(\pi)} \sum_{S\subseteq \mathsf{Val}(\pi)\setminus\{k\}} (-1)^{\#S}(-d_{\epsilon_1}\cdots [d_{\epsilon_{l+1}},d_{\epsilon_l}]\cdots d_{\epsilon_{2n}}(1)) \\ &= (1-q)^{-\#\mathsf{Val}(\pi)+1} \sum_{S\subseteq \mathsf{Val}(\pi)\setminus\{k\}} (-1)^{\#S} d_{\epsilon_1}\cdots d_v\cdots d_{\epsilon_{2n}}(1).\end{aligned}$$

Iterating this argument for all elements of $\mathsf{Val}(\pi)$ (from bottom to top), we get the desired result. □

Partial Dyck paths

Definition 8.19. For $n,\ell \in \mathbb{N}$, we define $\mathsf{PD}^\ell(n)$ to be the set of lattice paths $p_0\dots p_{2n-\ell}$ with steps in $\{E := (1,0), N := (0,1)\}$ such that $p_0 = (0,\ell)$, $p_{2n-\ell} = (n,n)$, $p_1 - p_0 = E$ and $p_i \in \{(k,l)\in\mathbb{N}^2 \mid l \geq k\}$. The elements of this set are called *partial Dyck paths*.

Convention 8.20. In this chapter, we will identify (partial) Dyck paths with their E,N-sequences.

Notation 8.21. The terminology of the previous definition is explained by the fact that for all $\pi \in \mathsf{D}(n)$ there exists a unique $\ell \in \mathbb{N}$ and $\mathsf{pd}(\pi) \in \mathsf{ED}^\ell$ such that $\pi = N^\ell \mathsf{pd}(\pi)$.

Definition 8.22. Take $\pi \in \mathsf{ED}^\ell(n)$. As in Corollary 8.18, let $\tilde\epsilon_1\dots\tilde\epsilon_m$ the word in the alphabet $\{+,-,v\}$ obtained from the E,N-sequence of π by the substitutions $EN \mapsto v$ followed by $E \mapsto +$ and $N \mapsto -$. We define

$$d(\pi) = d_{\tilde\epsilon_1}\cdots d_{\tilde\epsilon_m}(1) \in V_\ell.$$

Using this notation, Corrolary 8.18 states that for all $\pi \in \mathsf{D}(n) = \mathsf{PD}^0(n)$, $\chi(\pi,0) = d(\pi)$.

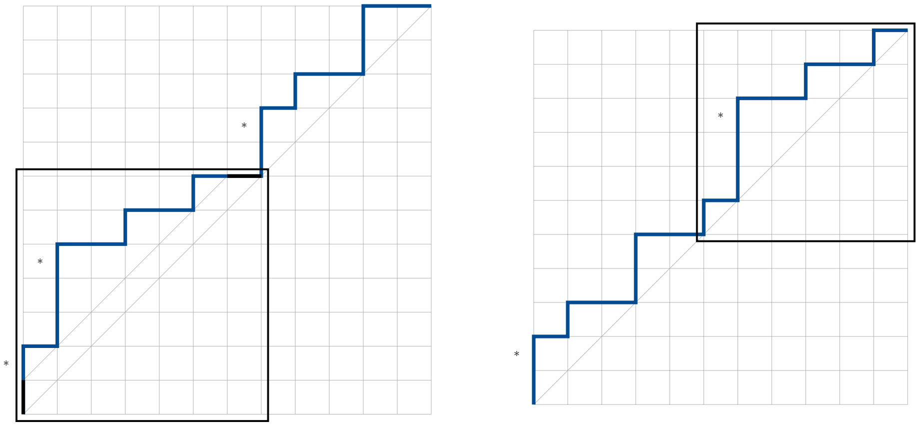


Figure 8.3: An element of $\mathsf{D}(12)^{*3}$ (left) and its image by ψ in $\mathsf{D}(11)^{*2}$.

8.3 ψ and γ maps

We will define two related families of maps, that will be essential in formulating the recursion that is the main result of this chapter.

The ψ maps

Definition 8.23. We define $\psi\colon \mathsf{D}(n)^{*k} \to \mathsf{D}(n-1)^{*k} \sqcup \mathsf{D}(n-1)^{*k-1}$ as follows: given $(\pi, dr) \in \mathsf{D}(n)^{*k}$ take the portion of π between the first two touching points (or the whole path if there is only one touching point), remove its first (north) step and its last (east) step, and attach it to the end of the path. If the first rise was decorated we remove the decoration since it is no longer a rise.

See Figure 8.3 for an example.

We are interested in the restriction of ψ to the compositional pieces of $\mathsf{D}(n)^{*k}$. Take $a \in \mathbb{P}$ and α a composition such that the concatenation $(a)\alpha \vDash n-k$. Then it is obvious from the definitions of ψ and dcomp (4.16) that

$$\psi : \mathsf{D}((a)\alpha)^{*k} \to \bigcup_{\beta \vDash a-1} \mathsf{D}(\alpha\beta)^{*k} \sqcup \bigcup_{\beta \vDash a} \mathsf{D}(\alpha\beta)^{*k-1}.$$

This map is not invertible. Indeed it is not injective: a path P with $\mathsf{touch}(P) > 1$ is the image by ψ of more than one path. However, there exists a family of maps that are essentially right inverses of ψ. To formulate them, we need to define some notation for what will be the diagonal composition of the image.

Notation 8.24. Take α a composition and $r \in \mathbb{N}$. Then we denote

$$\alpha^r := \left(\left(1 + \sum_{i>r} \alpha_i\right), \alpha_1, \alpha_2, \dots, \alpha_r\right) \qquad \text{for } 0 \le r \le \ell(\alpha)$$

$$\alpha^{r,*} = \alpha^{r,\bullet} := \left(\left(\sum_{i>r} \alpha_i\right), \alpha_1, \alpha_2, \dots, \alpha_r\right) \qquad \text{for } 0 \le r < \ell(\alpha)\ .$$

Definition 8.25. Let $k \in \mathbb{N}$ and α a composition. We define two similar maps:

$$\psi_r \colon \mathsf{D}(\alpha)^{*k} \to \mathsf{D}(\alpha^r)^{*k} \qquad \text{for } 0 \leq r \leq \ell(\alpha)$$
$$\psi_r^* \colon \mathsf{D}(\alpha)^{*k} \to \mathsf{D}(\alpha^{r,*})^{*k+1} \qquad \text{for } 0 \leq r < \ell(\alpha).$$

Given $(\pi, dr) \in \mathsf{D}(\alpha)^{*k}$ and $0 \leq r < \ell(\alpha)$, call π_1 and π_2 the portions of π below and above its $(r+1)$-th touching point, respectively. For $r = \ell(\alpha)$, set $\pi_1 = \pi$ and $\pi_2 = \emptyset$. Notice that if $\pi_2 \neq \emptyset$ then it necessarily starts with a north step. To define $\psi_r(\pi, dr) = (\pi', dr')$ we set

$$\pi' := N\pi_2 E \pi_1.$$

We use the same definition for $\psi_r^*(\pi, dr)$. For the decorations, we keep the decorations on the rises in the same place, relative to π_1 and π_2. When $\pi_2 \neq \emptyset$, i.e. $r < \ell(\alpha)$, π' must start with two north steps, so the second step of π' is a newly created rise, which we can choose to decorate or not. This choice is the difference between ψ_r and ψ_r^*: for the former we do not decorate the new rise while for the latter we do. It is clear from the definitions that $\mathsf{dcomp}(\psi_r(\pi, dr)) = \alpha^r$ and $\mathsf{dcomp}(\psi_r^*(\pi, dr)) = \alpha^{r,*}$.

By construction, we have the following.

Proposition 8.26. *For $k \in \mathbb{N}$ and α a composition*

$$\psi \circ \psi_r = \mathrm{Id}\big|_{\mathsf{D}(\alpha)^{*k}} \qquad \textit{for } 0 \leq r \leq \ell(\alpha)$$
$$\psi \circ \psi_r^* = \mathrm{Id}\big|_{\mathsf{D}(\alpha)^{*k}} \qquad \textit{for } 0 \leq r < \ell(\alpha).$$

The γ maps

We translate the ψ maps to the context of valley decorated Dyck paths using the ζ map.

Definition 8.27. We define $\gamma : \mathsf{D}(n)^{\bullet k} \to \mathsf{D}(n-1)^{\bullet k} \sqcup \mathsf{D}(n-1)^{\bullet k-1}$ which, for a path in $\mathsf{D}(n)^{\bullet k}$, deletes its first NE subsequence. If the E of this subsequence was the horizontal step preceding a decorated valley, then γ removes this decoration as it no longer decorates a valley. See Figure 8.4.

Lemma 8.28. *For $n, k \in \mathbb{N}$ the following diagram commutes*

$$\begin{array}{ccc} \mathsf{D}(n)^{*k} & \xrightarrow{\quad\zeta\quad} & \mathsf{D}(n)^{\bullet k} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\gamma} \\ \mathsf{D}(n-1)^{*k} \sqcup \mathsf{D}(n-1)^{*k-1} & \xrightarrow{\zeta} & \mathsf{D}(n-1)^{\bullet k} \sqcup \mathsf{D}(n-1)^{\bullet k-1} \end{array}.$$

Proof. Let P be an element of $\mathsf{D}(n)^{*k}$. It is easy to see that the first rise of P is decorated if and only if the first valley of $\zeta(P)$ is decorated. Thus the stipulations on decorations carry through nicely in the diagram. For the sake of completeness, we sketch the proof for the undecorated case, see [HX17, Corollary 2.7].

Consider $\pi \in \mathsf{D}(n)$. From its area word $a(\pi)$ construct the permutation $\sigma_\pi \in \mathfrak{S}_n$ by reading, from left to right, the indices of the 0's of $a(\pi)$, followed by the indices

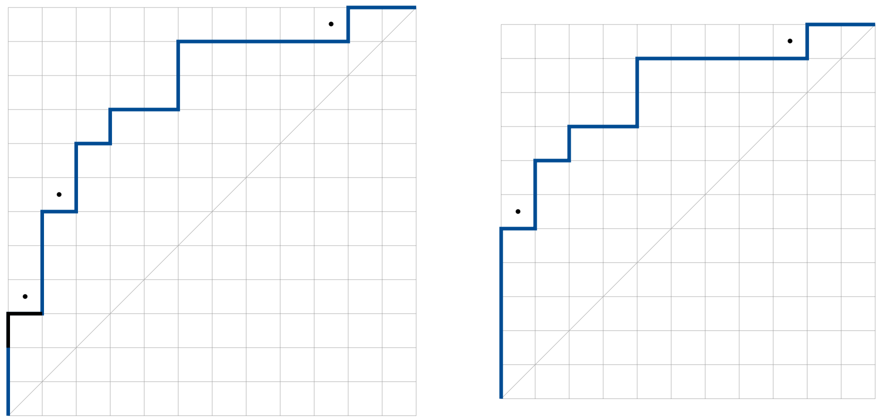


Figure 8.4: An element of $\mathsf{D}(12)^{\bullet 3}$ (left) and its image by γ in $\mathsf{D}(11)^{\bullet 2}$.

of the 1's, and so forth. Now construct the word $u(\pi) \in \mathbb{N}^n$ such that its i-th letter $u_i(\pi)$ is the number of $j > \sigma_\pi(i)$ such that $(\sigma_\pi(i), j)$ is a primary inversion plus the number $j < \sigma_\pi(i)$ such that $(j, \sigma_\pi(i))$ is a secondary inversion. Recalling the (unlabelled equivalent) of the proof of $\mathsf{dinv}(\pi) = \mathsf{area}(\zeta(\pi))$ (see Theorem 8.5), we know that $u_i(\pi)$ is exactly the number of area squares of π in the i-th column.

Consider the effect of ψ on the area word of a path. The area word of a Dyck path always starts with a 0. The area word of $\psi(\pi)$ is obtained from $a(\pi)$ by taking the all its letters strictly before the second occurrence of 0 (and all the letters if there is no such second occurrence), deleting the 0 at the start, subtracting 1 from the remaining letters and moving this modified subword to the end of the word. We have that $u_{i+1}(\pi) = u_i(\psi(\pi))$ for all $i = 1, \ldots, n-1$. We illustrate this fact with an example that shows how the inversion pairs get transferred from π to $\phi(\pi)$.

Example. Let $P := (\pi, dr)$ be the path on the left in Figure 8.3. We compute $a(\pi) = 0112321\,01101$. For $\psi(\pi)$, the path on the right in the same figure, we get $a(\psi(P)) = 01101\,001210$. We have $\sigma_\pi(6) = 7$ because there are three 0's in $a(\pi)$ and the third occurence of 1 in $a(\pi)$ has index 7. The primary inversions of π to the right of 7 are $(7,9), (7,10), (7,12)$ and the secondary inversions to the left of 7 are $(4,7)$, $(6,7)$. It follows that $u_6(\pi) = 5$. On the other hand $\sigma_{\psi(\pi)}(5) = 11$, the fifth occurence of 0. There are only secondary inversions of $\psi(\pi)$ involving 11: $(2,11)$, $(3,11)$, $(5,11)$, $(8,11)$, $(10,11)$. The first three come from the primary inversions of π to the right of 7 and the last two from the secondary inversions to the left of 7. So $u_5(\psi(P)) = 5$.

The equality $u_{i+1}(\pi) = u_i(\psi(\pi))$ ensures that the number of area squares in the $i+1$-th column of $\zeta(\pi)$ equals the number of area square is the i-th column of $\zeta(\psi(\pi))$. Since applying γ to $\zeta(\pi)$ essentially means deleting its first column, this implies that $\zeta(\psi(\pi))$ equals $\gamma(\zeta(\pi))$. □

Again, we have a family of right inverses. We start by describing them implicitly, using ψ_r and ζ. We will give an explicit description later.

Definition 8.29. For $k \in \mathbb{N}$ and α a composition.

$$\gamma_r := \zeta \circ \psi_r \circ \zeta^{-1} : \mathsf{D}'(\alpha)^{\bullet k} \to \mathsf{D}'(\alpha^r)^{\bullet k} \qquad \text{for } 0 \leq r \leq \ell(\alpha)$$
$$\gamma_r^\bullet := \zeta \circ \psi_r^* \circ \zeta^{-1} : \mathsf{D}'(\alpha)^{\bullet k} \to \mathsf{D}'(\alpha^{r,\bullet})^{\bullet k+1} \qquad \text{for } 0 \leq r < \ell(\alpha).$$

In other words, we define these maps so that the following diagrams commute

$$\begin{array}{ccc} \mathsf{D}(\alpha)^{*k} & \xrightarrow{\zeta} & \mathsf{D}'(\alpha)^{\bullet k} \\ \downarrow{\scriptstyle\psi_r} & & \downarrow{\scriptstyle\gamma_r} \\ \mathsf{D}(\alpha^r)^{*k} & \xrightarrow{\zeta} & \mathsf{D}'(\alpha^r)^{\bullet k} \end{array} \qquad \begin{array}{ccc} \mathsf{D}(\alpha)^{*k} & \xrightarrow{\zeta} & \mathsf{D}'(\alpha)^{\bullet k} \\ \downarrow{\scriptstyle\psi_r^*} & & \downarrow{\scriptstyle\gamma_r^\bullet} \\ \mathsf{D}(\alpha^{r,*})^{*k+1} & \xrightarrow{\zeta} & \mathsf{D}'(\alpha^{r,\bullet})^{\bullet k+1}. \end{array}$$

Proposition 8.30. *For $k \in \mathbb{N}$ and α a composition we have*

$$\gamma \circ \gamma_r = \mathrm{Id}\big|_{\mathsf{D}'(\alpha)^{\bullet k}} \qquad \textit{for } 0 \leq r \leq \ell(\alpha)$$
$$\gamma \circ \gamma_r^\bullet = \mathrm{Id}\big|_{\mathsf{D}'(\alpha)^{\bullet k}} \qquad \textit{for } 0 \leq r < \ell(\alpha).$$

Proof. By Proposition 8.26, Definition 8.29 and Lemma 8.28

$$\mathrm{Id}\big|_{\mathsf{D}(\alpha)^{*k}} = \psi \circ \psi_r = \psi \circ \zeta^{-1} \circ \gamma_r \circ \zeta = \zeta^{-1} \circ \gamma \circ \gamma_r \circ \zeta$$
$$\Leftrightarrow \gamma \circ \gamma_r = \zeta \circ \mathrm{Id}\big|_{\mathsf{D}(\alpha)^{*k}} \circ \zeta^{-1} = \mathrm{Id}\big|_{\mathsf{D}'(\alpha)^{\bullet k}}.$$

The proof for $\gamma_r^\bullet$ is exactly analogous. □

Lemma 8.31. *For any $P \in \mathsf{D}'(\alpha)^{\bullet k}$, $\gamma_r(P)$ and $\gamma_r^\bullet(P)$ start with $r+1$ vertical steps followed by a horizontal step.*

Proof. By definition $\gamma_r(P) = \zeta \circ \psi_r \circ \zeta^{-1}(P)$. We have $\psi_r(\zeta^{-1}(P)) \in \mathsf{D}(\alpha^r)^{*k}$ and so by definition of α^r, the area word of $\psi_r(\zeta^{-1}(P))$ contains $r+1$ letters 0. Thus, upon applying ζ we obtain a path starting with $r+1$ vertical steps followed by a horizontal step. The proof for $\gamma_r^\bullet(P)$ is the same. □

Proposition 8.30 and Lemma 8.31 uniquely determine γ_r and $\gamma_r^\bullet$. We give an explicit description of their effect.

Take $(\pi, dv) \in \mathsf{D}'(\alpha)^{\bullet k}$ and set $\ell := \ell(\alpha)$. By definition of ζ and dcomp', π starts with ℓ north steps followed by an east step. Define $\tilde{\pi}$ to be the portion of π following its ℓ first vertical steps. Set $dv^{+j} := \{i + j \mid i \in dv\}$. We define

$$\gamma_r(\pi, dc) := \left(N^r NEN^{\ell-r}\tilde{\pi}, dc^{+1}\right),$$

i.e. we add one NE sequence after the first r north steps and we keep the decorated valleys as they are, relative to π. Similarly

$$\gamma_r^\bullet(\pi, dc) := \left(N^r NEN^{\ell-r}\tilde{\pi}, \{r+2\} \cup dc^{+1}\right),$$

i.e. the path is defined in the same way as before, and we decorate the only new valley.

Next, we would like to explicitly describe the compositional pieces of the set $\gamma(D'(\alpha)^{\bullet k})$. To this end we describe how to compute $\mathsf{dcomp}'(P)$ directly from $P \in \mathsf{D}(n)^{\bullet k}$ (without passing by ζ).

Lemma 8.32. *Given $k \in \mathbb{N}$, a composition α , $P \in \mathsf{D}'(\alpha)^{\bullet k}$ and $1 \le r \le \ell(\alpha)$*

$$\alpha_r = \mathsf{bounce}(\gamma_{r-1}(P)) - \mathsf{bounce}(\gamma_r(P)).$$

Furthermore, if $r \neq \ell(\alpha)$ we have

$$\alpha_r = \mathsf{bounce}(\gamma^\bullet_{r-1}(P)) - \mathsf{bounce}(\gamma^\bullet_r(P)).$$

Proof. It follows easily from ψ_r's definition that for any $Q \in \mathsf{D}(\alpha)^{*k}$

$$\mathsf{area}(\psi_r(Q)) = \mathsf{area}(Q) + \sum_{i>r} \alpha_i.$$

And so it follows that for $1 \le r \le \ell(\alpha)$

$$\alpha_r = \mathsf{area}(\psi_{r-1}(Q)) - \mathsf{area}(\psi_r(Q)).$$

Now given $P \in \mathsf{D}'(\alpha)^{\bullet k}$ we have $\alpha = \mathsf{dcomp}'(P) = \zeta^{-1}(P)$ by definition. Taking $Q = \zeta^{-1}(P)$ in the last equation we get

$$\begin{aligned} \alpha_r &= \mathsf{area}(\psi_{r-1} \circ \zeta^{-1}(P)) - \mathsf{area}(\psi_r \circ \zeta^{-1}(P)) \\ \text{(by 8.29)} &= \mathsf{area}(\zeta^{-1} \circ \gamma_{r-1}(P)) - \mathsf{area}(\zeta^{-1} \circ \gamma_r(P)) \\ \text{(by 8.6)} &= \mathsf{bounce}(\gamma_{r-1}(P)) - \mathsf{bounce}(\gamma_r(P)). \end{aligned}$$

The second affirmation is implied by the first one and the fact that

$$\mathsf{bounce}(\gamma_r(P)) = \mathsf{bounce}(\gamma^\bullet_r(P)) + 1.$$

□

Lemma 8.33. *For $a, k \in \mathbb{N}$ with $a > 0$ and α a composition*

$$\gamma : \mathsf{D}'((a)\alpha)^{\bullet k} \to \bigcup_{\beta \vDash a-1} \mathsf{D}'(\alpha\beta)^{\bullet k} \sqcup \bigcup_{\beta \vDash a} \mathsf{D}'(\alpha\beta)^{\bullet k-1}.$$

Proof. Take $P \in \mathsf{D}'((a)\alpha)$. Set $\mathsf{dcomp}'(\gamma(P)) = \tilde{\alpha}$. We have to prove that $\tilde{\alpha}_r = \alpha_r$ for $1 \le r \le \ell(\alpha)$; in fact, if this is true, then we necessarily get $\tilde{\alpha} = \alpha\beta$ for some $\beta \vDash a - 1$ if the first valley of π is decorated, or $\beta \vDash a$ if it is not. Indeed, the size of a path is the size of its composition plus the number of decorations, we have $\alpha \vDash n - k - a$, and applying γ decreases the size of the path by exactly one unit.

Since α_r is the $(r+1)$-th part of the composition $(a)\alpha$, by Lemma 8.32 we have for $1 \le r \le \ell(\alpha)$

$$\alpha_r = \mathsf{bounce}(\gamma_r(P)) - \mathsf{bounce}(\gamma_{r+1}(P)).$$

Using the same lemma we have

$$\tilde{\alpha}_r = \mathsf{bounce}(\gamma_{r-1}(\gamma(P))) - \mathsf{bounce}(\gamma_r(\gamma(P))).$$

So it will be sufficient to show that

$$\mathsf{bounce}(\gamma_r(P)) = \mathsf{bounce}(\gamma_{r-1}(\gamma(P)))$$

for $1 \le r \le \ell(\alpha)+1$, as it implies our thesis by simply taking the relevant differences. The two bounce paths are identical from the second section onwards by construction (as the extra column lies above the first section, see the black steps of Figure 8.5). Since the first section does not contribute to the bounce the result follows. □

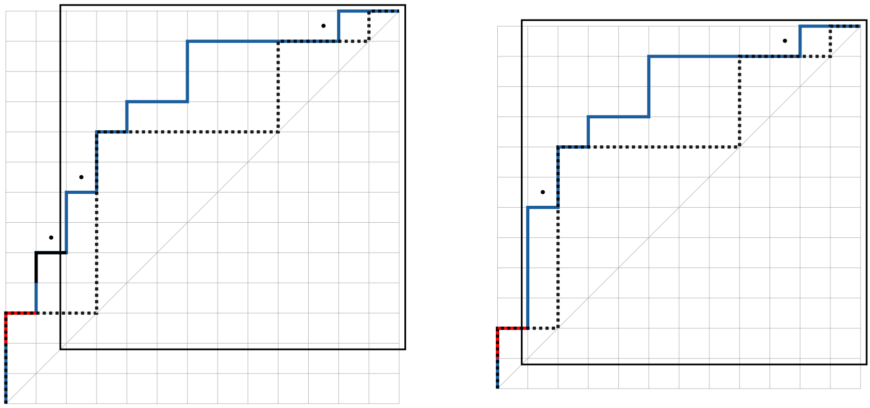


Figure 8.5: $\gamma_2(P)$ (left) and $\gamma_1(\gamma(P))$ (right). The red steps are inserted by γ_r, the black steps are deleted by γ.

From this lemma we may deduce one last fact about these combinatorial sets and maps.

Corollary 8.34. *For $k, a \in \mathbb{N}$ with $a > 0$ and $\ell \coloneqq \ell(\alpha)$*

$$\mathsf{D}'((a)\alpha)^{\bullet k} = \bigsqcup_{\beta \vDash a-1} \gamma_\ell(\mathsf{D}'(\alpha\beta)^{\bullet k}) \sqcup \bigsqcup_{\beta \vDash a} \gamma_\ell^\bullet(\mathsf{D}'(\alpha\beta)^{\bullet k-1})$$

Proof. Take $P \in \mathsf{D}'((a)\alpha)^{\bullet k}$. Then P starts with $\ell + 1$ north steps followed by an east step. By Lemma 8.33, there are two possibilities.

- Either $\gamma(P) \in \bigsqcup_{\beta \vDash a-1} \mathsf{D}'(\alpha\beta)^{\bullet k}$ in which case $P = \gamma_\ell(\gamma(P))$.
- Or, $\gamma(P) \in \bigsqcup_{\beta \vDash a} \mathsf{D}'(\alpha\beta)^{\bullet k-1}$ and so $P = \gamma_\ell^\bullet(\gamma(P))$.

This implies that "$\subseteq$" holds. For the other inclusion, if $P \in \mathsf{D}'(\alpha\beta)^{\bullet k}$, for some $\beta \vDash a-1$, then $\gamma_r(P) \in \mathsf{D}'((a)\alpha)^{\bullet k}$ since $(\alpha\beta)^r = (1+\sum_i \beta_i, \alpha_1, \dots, \alpha_r)$. Similarly, if $P \in \mathsf{D}'(\alpha\beta)^{\bullet k-1}$, for some $\beta \vDash a$, then $\gamma_r^\bullet(P) \in \mathsf{D}'((a)\alpha)^{\bullet k}$ since $(\alpha\beta)^{r,\bullet} = (\sum_i \beta_i, \alpha_1, \dots, \alpha_r)$. □

8.4 The recursion

With the combinatorial tools from the previous sections in hand, we now get to the goal of this chapter: proving the following recursion.

Theorem 8.35. *Take $k \in \mathbb{N}$, α a composition and $\ell \coloneqq \ell(\alpha)$, then we have*

$$\sum_{P \in \mathsf{DLD}'(\alpha)^{\bullet k}} q^{\mathsf{ninv}(P)} t^{\mathsf{bounce}(P)} x^P = d_-^\ell M_\alpha^{*k} \tag{8.36}$$

where $M_\alpha^{*k} \in V_\ell$ is defined by the recursive relations

$$M_{(1)\alpha}^{*k} = d_+ M_\alpha^{*k} + d_v M_{\alpha(1)}^{*k-1}, \tag{8.37}$$

and for $a > 1$

$$M^{*k}_{(a)\alpha} = t^{a-1} d_v \left(\sum_{\beta \vDash a-1} d_-^{\ell(\beta)-1} M^{*k}_{\alpha\beta} + \sum_{\beta \vDash a} d_-^{\ell(\beta)-1} M^{*k-1}_{\alpha\beta} \right), \tag{8.38}$$

with initial conditions $M^{*k}_{\emptyset} = \delta_{k,0}$.

Proof. By Equation (8.10) Corollary 8.18 and Definition 8.22 and the fact that all elements of $\mathsf{DLD}'(\alpha)^{*k}$ start with $\ell(\alpha) = \ell$ vertical steps followed by a horizontal step, we have

$$\begin{aligned} \sum_{P \in \mathsf{DLD}'(\alpha)^{\bullet k}} q^{\mathsf{ninv}(P)} t^{\mathsf{bounce}(\pi,dv)} x^P &= \sum_{(\pi,dv) \in \mathsf{DLD}'(\alpha)^{\bullet k}} t^{\mathsf{bounce}(P)} d(\pi) \\ (\text{by } 8.21) &= \sum_{(\pi,dv) \in \mathsf{DLD}'(\alpha)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)} d(N^\ell \mathsf{pd}(\pi)) \\ (\text{by } 8.22) &= d_-^\ell \sum_{(\pi,dv) \in \mathsf{DLD}'(\alpha)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)} d(\mathsf{pd}(\pi)). \end{aligned}$$

So, in view of (8.36), it suffices to prove

$$M^{*k}_{\alpha} = \sum_{(\pi,dv) \in \mathsf{DLD}'(\alpha)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)} d(\mathsf{pd}(\pi)). \tag{8.39}$$

We prove this by induction on $|\alpha| + k$, i.e. the size of the paths.

We start with the initial conditions. If $|\alpha| + k = 0$ then $\alpha = \emptyset$. We have

$$\mathsf{LD}(\emptyset)^{*k} = \begin{cases} \{\text{empty path}\} & \text{if } k = 0 \\ \emptyset & \text{if } k \neq 0 \end{cases}$$

because the empty path has no decorations and any nonempty path starts with a vertical step that may not be decorated and so its diagonal decomposition is nonempty. Taking the image by ζ we get the same for $\mathsf{DLD}'(\alpha)^{\bullet k}$. Since $d(\text{empty path}) = 1$ the right hand side of (8.39) is $\delta_{k,0}$ which matches $M^{*k}_{\emptyset}$.

Suppose $|\alpha| + k > 0$. We may suppose $|\alpha| > 1$ because if $\alpha = \emptyset$ we may apply the previous argument. So take any nonempty composition $(a)\alpha$ with $a \in \mathbb{P}$, where α is some composition which may be empty. Set $r = \ell(\alpha)$. The right hand side of (8.39) is

$$\sum_{(\pi,dv) \in \mathsf{DLD}'((a)\alpha)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)} d(\mathsf{pd}(\pi)) = \cdots$$

Given $(\pi, dv) \in \mathsf{DLD}'((a)\alpha)^{\bullet k}$, by Corollary 8.34 we have

- either $(\pi, dv) = \gamma_r(\pi', dv')$ for some $(\pi', dv') \in \mathsf{D}'(\alpha\beta)^{\bullet k}$, $\beta \vDash a - 1$,
- or $(\pi, dv) = \gamma_r^\bullet(\pi', dv')$ for some $(\pi', dv') \in \mathsf{D}'(\alpha\beta)^{\bullet k-1}$, $\beta \vDash a$.

In both cases the underlying Dyck path is

$$\gamma_r(\pi', dv')_1 = \gamma_r^\bullet(\pi', dv')_1 = N^{\ell(\alpha)}NEN^{\ell(\beta)}\mathsf{pd}(\pi).$$

Since $\mathsf{pd}(N^{\ell(\alpha)}NEN^{\ell(\beta)}\mathsf{pd}(\pi)) = EN^{\ell(\beta)}\mathsf{pd}(\pi)$, it follows that

$$\begin{aligned}\cdots = &\sum_{\beta\vDash a-1}\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k}} t^{\mathsf{bounce}(\gamma_r(\pi,dv))}d(EN^{\ell(\beta)}\mathsf{pd}(\pi))\\ &+\sum_{\beta\vDash a}\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k-1}} t^{\mathsf{bounce}(\gamma_r^\bullet(\pi,dv))}d(EN^{\ell(\beta)}\mathsf{pd}(\pi))\\ =&\cdots\end{aligned}$$

Let us compute the bounce of $\gamma_r(\pi, dv)$ and $\gamma_r^\bullet(\pi, v)$ in terms of the bounce of $(\pi, dv) \in \mathsf{D}'(\alpha\beta)^{\bullet k} \sqcup \mathsf{D}'(\alpha\beta)^{\bullet k-1}$. By Lemma 8.32, we have

$$\begin{aligned}\mathsf{bounce}(\gamma_r(\pi,dv)) &= \mathsf{bounce}(\gamma_{r-1}(\pi,dv)) - \alpha_r = \mathsf{bounce}(\gamma_{r-2}(\pi,dv)) - \alpha_{r-1} - \alpha_r\\ &= \cdots = \mathsf{bounce}(\gamma_0(\pi,dv)) - \sum_{i=1}^r \alpha_i = \mathsf{bounce}(\gamma_0(\pi,dv)) - |\alpha|\end{aligned}$$

The same identity holds when replacing γ_r with $\gamma_r^\bullet$ and γ_0 with $\gamma_0^\bullet$.

- Clearly, for $\beta \vDash a-1$ and $(\pi, dv) \in \mathsf{D}'(\alpha\beta)^{\bullet k}$, we have $\mathsf{bounce}(\gamma_0(\pi, dv)) = \mathsf{bounce}(\pi, dv) + |\alpha\beta|$ and so

$$\begin{aligned}\mathsf{bounce}(\gamma_r(\pi,dv)) &= \mathsf{bounce}(\pi,dv) + |\beta|\\ &= \mathsf{bounce}(\pi,dv) + a - 1.\end{aligned}$$

- Similarly, for $\beta \vDash a$ and $(\pi, dv) \in \mathsf{D}'(\alpha\beta)^{\bullet k-1}$ we have $\mathsf{bounce}(\gamma_0^\bullet(\pi, dv)) = \mathsf{bounce}(\pi, dv) + |\alpha\beta| - 1$ and so we get

$$\begin{aligned}\mathsf{bounce}(\gamma_r^\bullet(\pi,dv)) &= \mathsf{bounce}(\pi,dv) - 1 + |\beta|\\ &= \mathsf{bounce}(\pi,dv) + a - 1.\end{aligned}$$

Therefore we may continue

$$\begin{aligned}\cdots =&t^{a-1}\Bigg(\sum_{\beta\vDash a-1}\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv))}d(EN^{\ell(\beta)}\mathsf{pd}(\pi))\\ &+\sum_{\beta\vDash a}\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k-1}} t^{\mathsf{bounce}(\pi,dv)}d(EN^{\ell(\beta)}\mathsf{pd}(\pi))\Bigg)\\ =&\cdots\end{aligned} \tag{8.40}$$

If $a > 1$ then for all $\beta \vDash a-1$ or $\beta \vDash a$, $\ell(\beta) \geq 1$ and so

$$EN^{\ell(\beta)}\mathsf{pd}(\pi) = ENN^{\ell(\beta)-1}\mathsf{pd}(\pi).$$

The definition of d (Definition 8.22) thus gives

$$\begin{aligned}\cdots =&t^{a-1}d_v\left(\sum_{\beta\vDash a-1} d_-^{\ell(\beta)-1}\left(\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv))}d(\mathsf{pd}(\pi))\right)\right.\\ &\left.+\sum_{\beta\vDash a} d_-^{\ell(\beta)-1}\left(\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k-1}} t^{\mathsf{bounce}(\pi,dv)}d(\mathsf{pd}(\pi))\right)\right)\\ =&\cdots\end{aligned}$$

Using the induction hypothesis we have

$$M^{*k}_{\alpha\beta} = \sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)}d(\mathsf{pd}(\pi)) \qquad \text{for } \beta\vDash a-1 \tag{8.41}$$

$$M^{*k-1}_{\alpha\beta} = \sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k-1}} t^{\mathsf{bounce}(\pi,dv)}d(\mathsf{pd}(\pi)) \qquad \text{for } \beta\vDash a. \tag{8.42}$$

and so

$$\begin{aligned}\cdots =&t^{a-1}d_v\left(\sum_{\beta\vDash a-1} d_-^{\ell(\beta)-1}M^{*k}_{\alpha\beta}+\sum_{\beta\vDash a} d_-^{\ell(\beta)-1}M^{*k-1}_{\alpha\beta}\right)\\ \text{(by (8.38))} =& M^{*k}_{(a)\alpha},\end{aligned}$$

which is exactly the left hand side of (8.39) for our case.

Finally we must prove the case $a=1$. If $\beta\vDash a-1$ then $\ell(\beta)=0$ and if $\beta\vDash a$, $\ell(\beta)=1$ and so continuing from (8.40) we get

$$\begin{aligned}\cdots =&t^{a-1}\left(\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)}d(E\mathsf{pd}(\pi))\right.\\ &\left.+\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k-1}} t^{\mathsf{bounce}(\pi,dv)}d(EN\mathsf{pd}(\pi))\right)\\ \text{(by 8.22)} =&t^{a-1}\left(d_+\left(\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k}} t^{\mathsf{bounce}(\pi,dv)}d(\mathsf{pd}(\pi))\right)\right.\\ &\left.+d_v\left(\sum_{(\pi,dv)\in\mathsf{DLD}'(\alpha\beta)^{\bullet k-1}} t^{\mathsf{bounce}(\pi,dv)}d(\mathsf{pd}(\pi))\right)\right)\\ \text{(by (8.41),(8.42))} =&d_+M^{*k}_{\alpha}+d_vM^{*k-1}_{\alpha(1)}\\ \text{(by (8.37))} =&M^{*k}_{(1)\alpha}.\end{aligned}$$

We have now shown that (8.39) holds for any composition α. □

8.5 Operator Delta conjecture

The result of the previous section generalises the combinatorial framework in [CM18] to rise decorated Dyck paths. It therefore reduces the problem of the (compositional) Delta conjecture to an identity of operators.

Conjecture 8.43 (Operator compositional Delta conjecture, rise version)**.** *If α is a composition of length ℓ, then*

$$\Theta_k \nabla C_\alpha = d_-^\ell M_\alpha^{*k}, \tag{8.44}$$

with M_α^{*k} defined as in Theorem 8.35.

The following proposition is an immediate consequence of Theorem 8.35 and Corollary 8.8.

Proposition 8.45. *The rise version of the compositional Delta conjecture, i.e. Conjecture 5.10, is equivalent to Conjecture 8.43.*

Future directions

It suffices to look at the list of conjectures in Chapter 5 to realise that a lot of problems remain open. One obvious next step towards solving some of these would be to generalise Carlsson and Mellit's algebraic argument in [CM18] to prove Conjecture 8.43. This would establish the rise version of the Delta conjecture and its refinements.

In Chapter 6, we showed that the touching shuffle theorem implies the generalised shuffle theorem. It might be possible to use similar techniques to show that the Delta conjecture implies the generalised Delta conjecture. For the valley version, we have some (conjectural) symmetric function identities suggesting that this might be a fruitful avenue. Some of these identities are strongly suggested by the combinatorics. The truth of this implication combined with Chapter 7, would make the valley version of the generalised Delta square conjecture conditional only upon the valley version of the Delta conjecture.

The rise version of the (generalised) Delta (square) conjecture, lacks a schedule formula. Finding such a formula might make the rise equivalent of Theorem 7.18 accessible, thus reducing the proof of the rise version of the (generalised) Delta square conjecture to the rise version of the (generalised) Delta conjecture.

Furthermore, using the Theta operators, a unified formula for both the rise and valley version of the Delta conjecture might be achievable. We have computational evidence suggesting that for $n, k, l \in \mathbb{N}$ with $n > 0$ and $k + l < n$ the following equality holds

$$\sum_{P \in \mathsf{LD}(n)^{*k, \bullet l}} t^{\mathsf{area}(P)} x^P = \Theta_l \Theta_k \nabla e_{n-k-l} \big|_{q=1}.$$

Finding a q-statistic to complete this formula would give what we call a *Theta conjecture*, lifting the Delta conjecture to a more general framework.

In general, we believe that the study of the Theta operators will yield more interesting mathematics. Some computer experiments seem to indicate that applying Θ_{s_λ} to some Schur positive images of nabla yields more Schur positive symmetric functions. For example $\Theta_{s_\lambda} \nabla C_\alpha$ and $(-1)^{|\mu| - \ell(\mu)} \Theta_{s_\lambda} \nabla m_\mu$ seem to be Schur positive.

On the representation theoretic side, there is our conjecture of the Frobenius characteristic of the module $\mathcal{M}_{n,2}$ in terms of Theta operators (see page xi). It might be feasible to get at least a partial proof of this result (e.g. for t or $q = 0$), in the vein of [HRS19] and [HRS18]. Also, it is a long standing open problem is to find a module whose Frobenius characteristic gives $\omega(p_n)$.

Appendix A

Missing proofs

Lemma 3.31. *For $l, s, k \in \mathbb{N}$ we have*

$$q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-1 \\ s-k \end{bmatrix}_q$$

$$= \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right].$$

Proof. We will actually prove that the statement holds for all $k \in \mathbb{Z}$ such that $l + k \geq 0$, which is clearly sufficient. The argument will be a double induction on s and $l + k$. We start with the base cases.

For $l = 0$, the statement becomes

$$q^{\binom{k}{2}} \begin{bmatrix} s-1 \\ s-k \end{bmatrix}_q = \sum_{i=0}^{s} q^{\binom{i}{2}} h_{s-i} \left[\frac{1}{1-q}\right] e_{i-k} \left[\frac{-1}{1-q}\right].$$

Using Definition 2.15,

$$q^{\binom{k}{2}} \begin{bmatrix} s-1 \\ s-k \end{bmatrix}_q = q^{\binom{k}{2}} \frac{(q^k;q)_{s-k}}{(q;q)_{s-k}}$$

$$(\text{by } 2.13) = q^{\binom{k}{2}} \frac{1}{(q;q)_{s-k}} \sum_{i=0}^{s-k} (-1)^i (q^k)^i q^{\binom{i}{2}} \begin{bmatrix} s-k \\ i \end{bmatrix}_q;$$

using the easy identity $\binom{k}{2} + \binom{i}{2} + ik = \binom{i+k}{2}$, gives

$$\cdots = \frac{1}{(q;q)_{s-k}} \sum_{i=0}^{s-k} (-1)^i q^{\binom{i+k}{2}} \begin{bmatrix} s-k \\ i \end{bmatrix}_q$$

Now, substituting $i \mapsto i - k$, we get

$$\cdots = \frac{1}{(q;q)_{s-k}} \sum_{i=k}^{s} (-1)^{i-k} q^{\binom{i}{2}} \begin{bmatrix} s-k \\ i-k \end{bmatrix}_q = \frac{1}{(q;q)_{s-k}} \sum_{i=0}^{s} (-1)^{i-k} q^{\binom{i}{2}} \begin{bmatrix} s-k \\ i-k \end{bmatrix}_q;$$

where this last equality holds because $\begin{bmatrix} s-k \\ i-k \end{bmatrix}_q = 0$ for $i < k$. Using (2.14), we rewrite

$$\begin{aligned}
\cdots &= \frac{1}{(q;q)_{s-k}} \sum_{i=0}^{s} (-1)^{i-k} q^{\binom{i}{2}} \frac{(q;q)_{s-k}}{(q;q)_{i-k}(q;q)_{s-i}} \\
\text{(by 2.20)} &= \sum_{i=0}^{s} (-1)^{i-k} q^{\binom{i}{2}} h_{i-k}\left[\frac{1}{1-q}\right] h_{s-i}\left[\frac{1}{1-q}\right] \\
\text{(by 1.63,1.57)} &= \sum_{i=0}^{s} q^{\binom{i}{2}} e_{i-k}\left[\frac{-1}{1-q}\right] h_{s-i}\left[\frac{1}{1-q}\right],
\end{aligned}$$

as desired.

For $s = 0$ the statement becomes

$$\delta_{k,0} = \sum_{r=0}^{l} e_{l-r}\left[\frac{1}{1-q}\right] e_{r-l-k}\left[\frac{-1}{1-q}\right]$$

since for $k \neq 0$ the product of binomials $\begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l-1 \\ -k \end{bmatrix}_q$ must be 0. The right hand side is almost the addition formula 1.69. Indeed substituting $r \mapsto l-r$ we get

$$\begin{aligned}
\sum_{r=0}^{l} e_{l-r}\left[\frac{1}{1-q}\right] e_{r-l-k}\left[\frac{-1}{1-q}\right] &= \sum_{r=0}^{l} e_{r}\left[\frac{1}{1-q}\right] e_{-k-r}\left[\frac{-1}{1-q}\right] \\
\text{(by 1.69)} &= e_{-k}\left[\frac{1}{1-q} + \frac{-1}{1-q}\right] = e_{-k}[0] = \delta_{k,0};
\end{aligned}$$

which is what we wanted to show.

For $l + k = 0$ the statement becomes

$$\delta_{k,0}\delta_{l,0} = \sum_{i=0}^{s}\sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] e_{i+r}\left[\frac{-1}{1-q}\right].$$

We will use the following fact: for all $a, b \in \mathbb{Z}$

$$\begin{aligned}
\begin{bmatrix} a \\ b \end{bmatrix} h_a\left[\frac{1}{1-q}\right] &= \frac{(q^{a-b+1};q)_b}{(q;q)_b(q;q)_a} = \frac{(q^{a-b+1};q)_b(q;q)_{a-b}}{(q;q)_b(q;q)_a(q;q)_{a-b}} \\
&= \frac{(q;q)_a}{(q;q)_b(q;q)_a(q;q)_{a-b}} = h_b\left[\frac{1}{1-q}\right] h_{a-b}\left[\frac{1}{1-q}\right]. \qquad \text{(A.1)}
\end{aligned}$$

First, by 1.63 and 1.57 we have

$$\begin{aligned}
&\sum_{i=0}^{s}\sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] e_{i+r}\left[\frac{-1}{1-q}\right] \\
&= \sum_{i=0}^{s}\sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ r \end{bmatrix} h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] (-1)^{i+r} h_{i+r}\left[\frac{1}{1-q}\right]
\end{aligned}$$

(by (A.1))

$$= \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i}{2}} h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] (-1)^{i+r} h_i\left[\frac{1}{1-q}\right] h_r\left[\frac{1}{1-q}\right]$$

(by 1.63, 1.57)

$$= \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i}{2}} h_{s-i}\left[\frac{1}{1-q}\right] e_{l-r}\left[\frac{1}{1-q}\right] (-1)^{i} h_i\left[\frac{1}{1-q}\right] e_r\left[\frac{-1}{1-q}\right]$$

$$= \left(\sum_{i=0}^{s} (-1)^i q^{\binom{i}{2}} h_{s-i}\left[\frac{1}{1-q}\right] h_i\left[\frac{1}{1-q}\right]\right)\left(\sum_{r=0}^{l} e_{l-r}\left[\frac{1}{1-q}\right] e_r\left[\frac{-1}{1-q}\right]\right)$$

(by (A.1))

$$= \left(h_s\left[\frac{1}{1-q}\right] \sum_{i=0}^{s} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} s \\ i \end{bmatrix}_q\right)\left(\sum_{r=0}^{l} e_{l-r}\left[\frac{1}{1-q}\right] e_r\left[\frac{-1}{1-q}\right]\right)$$

(by 2.13, 1.69)

$$= h_s\left[\frac{1}{1-q}\right] (1;q)_s e_l\left[\frac{1}{1-q} + \frac{-1}{1-q}\right] = h_s\left[\frac{1}{1-q}\right] (1;q)_s e_l\left[0\right]$$

(by definition of $(x;q)_n$)

$$= \delta_{s,0}\delta_{l,0}$$

Now for the inductive step, consider $s, l+k > 0$. We will use the easy identity $\binom{k}{2} - k + 1 = \binom{k-1}{2}$ a few times.

$$q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-1 \\ s-k \end{bmatrix}_q$$

$$\text{(by (2.18))} = q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \left(q^{s-k} \begin{bmatrix} l+s-2 \\ s-k \end{bmatrix}_q + \begin{bmatrix} l+s-2 \\ s-k-1 \end{bmatrix}_q\right)$$

$$\text{(by (2.18))} = q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-2 \\ s-k-1 \end{bmatrix}_q$$

$$+ q^{\binom{k}{2}} q^{s-k} \begin{bmatrix} l+s-2 \\ s-k \end{bmatrix}_q \left(q^k \begin{bmatrix} l+k-1 \\ k \end{bmatrix}_q + \begin{bmatrix} l+k-1 \\ k-1 \end{bmatrix}_q\right)$$

$$= q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-2 \\ s-k-1 \end{bmatrix}_q$$

$$+ q^{\binom{k-1}{2}} q^{s-1} \begin{bmatrix} l+s-2 \\ s-k \end{bmatrix}_q \left(q^k \begin{bmatrix} l+k-1 \\ k \end{bmatrix}_q + \begin{bmatrix} l+k-1 \\ k-1 \end{bmatrix}_q\right)$$

$$= q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+s-2 \\ s-k-1 \end{bmatrix}_q + q^{\binom{k}{2}} q^{s} \begin{bmatrix} l+s-2 \\ s-k \end{bmatrix}_q \begin{bmatrix} l+k-1 \\ k \end{bmatrix}_q$$

$$+ q^{\binom{k-1}{2}} q^{s-1} \begin{bmatrix} l+s-2 \\ s-k \end{bmatrix}_q \begin{bmatrix} l+k-1 \\ k-1 \end{bmatrix}_q$$

$$= q^{\binom{k}{2}} \begin{bmatrix} l+k \\ k \end{bmatrix}_q \begin{bmatrix} l+(s-1)-1 \\ (s-1)-k \end{bmatrix}_q + q^s q^{\binom{k}{2}} \begin{bmatrix} (l-1)+k \\ k \end{bmatrix}_q \begin{bmatrix} (l-1)+s-1 \\ s-k \end{bmatrix}_q$$
$$+ q^{s-1} q^{\binom{k-1}{2}} \begin{bmatrix} l+(k-1) \\ (k-1) \end{bmatrix}_q \begin{bmatrix} l+(s-1)-1 \\ (s-1)-(k-1) \end{bmatrix}_q .$$

We can now invoke the induction hypothesis–and use $\left[\begin{smallmatrix} i+r \\ r \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} i+r \\ i \end{smallmatrix}\right]_q$– to write

$$\cdots = \sum_{i=0}^{s-1} \sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q h_{s-1-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$+ q^s \sum_{i=0}^{s} \sum_{r=0}^{l-1} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{l-1-r} \left[\frac{1}{1-q}\right] e_{i+r-(l-1)-k} \left[\frac{-1}{1-q}\right]$$
$$+ q^{s-1} \sum_{i=0}^{s-1} \sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q h_{s-1-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-(k-1)} \left[\frac{-1}{1-q}\right].$$

In the first line, for $i = s$, $h_{s-1+i}\left[\frac{1}{1-q}\right] = 0$, so we may replace $\sum_{i=0}^{s-1}$ by $\sum_{i=0}^{s}$. In the second line, shift the index $r \mapsto r-1$. In the third line, shift the index $i \mapsto i-1$.

$$\cdots = \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q h_{s-1-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$+ q^s \sum_{i=0}^{s} \sum_{r=1}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r-1 \\ i \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$+ q^{s-1} \sum_{i=1}^{s} \sum_{r=0}^{l} q^{\binom{i-1}{2}} \begin{bmatrix} i-1+r \\ i-1 \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right].$$

Next, in the first line, we use $h_{s-i-1}\left[\frac{1}{1-q}\right] = (1-q^{s-i}) h_{s-i}\left[\frac{1}{1-q}\right]$. Using the fact that $\left[\begin{smallmatrix} i-1 \\ i \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} r-1 \\ -1 \end{smallmatrix}\right]_q = 0$, we may replace $\sum_{r=1}^{l}$ by $\sum_{r=0}^{l}$ in the second line and $\sum_{i=1}^{s}$ by $\sum_{i=0}^{s}$ in the third line.

$$\cdots = \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$+ q^s \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i}{2}} \begin{bmatrix} i+r-1 \\ i \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$+ q^{s-1} \sum_{i=0}^{s} \sum_{r=0}^{l} q^{\binom{i-1}{2}} \begin{bmatrix} i-1+r \\ i-1 \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$= \sum_{i=0}^{s} \sum_{r=0}^{l} h_{s-i} \left[\frac{1}{1-q}\right] e_{l-r} \left[\frac{1}{1-q}\right] e_{i+r-l-k} \left[\frac{-1}{1-q}\right]$$
$$\times \left(q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) + q^s q^{\binom{i}{2}} \begin{bmatrix} i+r-1 \\ i \end{bmatrix}_q + q^{s-1} q^{\binom{i-1}{2}} \begin{bmatrix} i+r-1 \\ i-1 \end{bmatrix}_q \right).$$

Since

$$
\begin{aligned}
&q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) + q^s q^{\binom{i}{2}}\begin{bmatrix} i+r-1 \\ i \end{bmatrix}_q + q^{s-1}q^{\binom{i-1}{2}}\begin{bmatrix} i+r-1 \\ i-1 \end{bmatrix}_q \\
&= q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) + q^{s-1}q^i q^{\binom{i-1}{2}}\begin{bmatrix} i+r-1 \\ i \end{bmatrix}_q + q^{s-1}q^{\binom{i-1}{2}}\begin{bmatrix} i+r-1 \\ i-1 \end{bmatrix}_q \\
&= q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) + q^{s-1}q^{\binom{i-1}{2}}\left(q^i\begin{bmatrix} i+r-1 \\ i \end{bmatrix}_q + \begin{bmatrix} i+r-1 \\ i-1 \end{bmatrix}_q\right) \\
(\text{by } (2.18)) &= q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) + q^{s-1}q^{\binom{i-1}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q \\
&= q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q (1-q^{s-i}) + q^{s-i}q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q = q^{\binom{i}{2}}\begin{bmatrix} i+r \\ i \end{bmatrix}_q ;
\end{aligned}
$$

this concludes the proof. □

Proposition 3.34. *For $i, j \in \mathbb{N}$*

$$
h_i\left[\frac{X}{1-q}\right] e_j\left[\frac{X}{M}\right] = \sum_{\lambda \vdash i+j} \frac{H_\lambda[X]}{w_\lambda} q^{-\binom{i}{2}} \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q h_k[(1-t)B_\lambda]
$$

Proof. Consider f a homogeneous symmetric function and λ a partition. We show that

$$
\langle f h_j, H_\lambda \rangle = \left(\nabla^{-1}\tau_{-\epsilon}\omega f^*[X]\right)\big|_{X \mapsto D_\lambda} \qquad \text{where } j + \deg(f) = |\lambda| \tag{A.2}
$$

Indeed, using Theorem 3.1 with $f \mapsto \omega f^*$

$$
\begin{aligned}
(\nabla^{-1}\tau_{-\epsilon}\omega f^*[X])\big|_{X=D_\lambda} &= \langle \omega f^*, \tau_1 H_\lambda \rangle_* \\
(\text{by } 2.23) &= \langle f, \tau_1 H_\lambda \rangle \\
(\text{by } 1.79,\ 1.54\ ) &= \sum_{k \in \mathbb{N}} \langle h_k f, \tau_1 H_\lambda \rangle .
\end{aligned}
$$

By homogeneity, only one of these summands is different from 0, i.e. $k = j$ where $j + \deg(f) = |\lambda|$. Now using (A.2) with $f = e_i[(1-t)X]$ gives

$$
\langle e_i[(1-t)X] h_j[X], H_\lambda \rangle = \left(\nabla^{-1} h_i\left[\frac{X-\epsilon}{1-q}\right]\right)\Bigg|_{X \mapsto D_\lambda} \qquad i + j = |\lambda|.
$$

This identity will be helpful since, by the duality of the bases $\{H_\lambda\}_{\lambda \in \mathsf{Par}}$ and $\left\{\frac{H_\lambda}{w_\lambda}\right\}_{\lambda \in \mathsf{Par}}$ for the star scalar product,

$$
\begin{aligned}
h_i\left[\frac{X}{1-q}\right] e_j\left[\frac{X}{M}\right] &= \sum_{\lambda \vdash i+j} \frac{H_\lambda}{w_\lambda} \left\langle h_i\left[\frac{X}{1-q}\right] e_j\left[\frac{X}{M}\right], H_\lambda \right\rangle_* \\
(\text{by } 2.23) &= \sum_{\lambda \vdash i+j} \frac{H_\lambda}{w_\lambda} \langle e_i\left[X(1-t)\right] h_j\left[X\right], H_\lambda \rangle
\end{aligned}
$$

Thus, using the previous equation

$$h_i\left[\frac{X}{1-q}\right]e_j\left[\frac{X}{M}\right]=\sum_{\lambda\vdash i+j}\frac{H_\lambda}{w_\lambda}\left(\nabla^{-1}h_i\left[\frac{X-\epsilon}{1-q}\right]\right)\Bigg|_{X\mapsto D_\lambda} \tag{A.3}$$

Using Corollary 1.69, we may write

$$\begin{aligned}
\nabla^{-1}h_i\left[\frac{X-\epsilon}{1-q}\right] &= \nabla^{-1}\left(\sum_{k=0}^{i}h_{i-k}\left[\frac{X}{1-q}\right]h_k\left[\frac{-\epsilon}{1-q}\right]\right)\\
\text{(by 1.63, 1.57)} &= \nabla^{-1}\left(\sum_{k=0}^{i}h_{i-k}\left[\frac{X}{1-q}\right]e_k\left[\frac{1}{1-q}\right]\right)\\
\text{(by 3.16, 2.20)} &= \sum_{k=0}^{i}q^{-\binom{i-k}{2}}h_{i-k}\left[\frac{X}{1-q}\right]\frac{q^{\binom{k}{2}}}{(q;q)_k}.
\end{aligned}$$

Since $D_\lambda \coloneqq MB_\lambda-1$ it follows that

$$\begin{aligned}
\left(\nabla^{-1}h_i\left[\frac{X-\epsilon}{1-q}\right]\right)\Bigg|_{X\mapsto D_\lambda} &= \sum_{k=0}^{i}\frac{q^{-\binom{i-k}{2}+\binom{k}{2}}}{(q;q)_k}h_{i-k}\left[\frac{MB_\lambda-1}{1-q}\right]\\
\text{(by 1.69)} &= \sum_{k=0}^{i}\frac{q^{-\binom{i-k}{2}+\binom{k}{2}}}{(q;q)_k}\sum_{l=0}^{i-k}h_l\left[B_\lambda(1-t)\right]h_{i-k-l}\left[\frac{-1}{1-q}\right]\\
\text{(by 1.63, 1.57, 2.20)} &= \sum_{k=0}^{i}\frac{q^{-\binom{i-k}{2}+\binom{k}{2}}}{(q;q)_k}\sum_{l=0}^{i-k}h_l\left[B_\lambda(1-t)\right](-1)^{i-k-l}\frac{q^{\binom{i-k-l}{2}}}{(q;q)_{i-k-l}}\\
&= \sum_{l=0}^{i}(-1)^{i-l}h_l\left[B_\lambda(1-t)\right]\sum_{k=0}^{i-l}(-1)^k q^{\binom{k}{2}}\frac{q^{\binom{i-k-l}{2}-\binom{i-k}{2}}}{(q;q)_{i-k-l}(q;q)_k}=\cdots
\end{aligned}$$

We use the identity $\binom{i-k-l}{2}-\binom{i-k}{2}=\binom{l+1}{2}+kl-li$ to write

$$\begin{aligned}
\cdots &= \sum_{l=0}^{i}(-1)^{i-l}q^{\binom{l+1}{2}-li}h_l\left[B_\lambda(1-t)\right]\sum_{k=0}^{i-l}(-1)^k q^{\binom{k}{2}}(q^l)^k\frac{1}{(q;q)_{i-k-l}(q;q)_k}\\
&= \sum_{l=0}^{i}(-1)^{i-l}\frac{q^{\binom{l+1}{2}-li}}{(q;q)_{i-l}}h_l\left[B_\lambda(1-t)\right]\sum_{k=0}^{i-l}(-1)^k q^{\binom{k}{2}}(q^l)^k\begin{bmatrix}i-l\\k\end{bmatrix}_q\\
\text{(by 2.13)} &= \sum_{l=0}^{i}(-1)^{i-l}\frac{q^{\binom{l+1}{2}-li}}{(q;q)_{i-l}}h_l\left[B_\lambda(1-t)\right](q^l;q)_{i-l}\\
\text{(by 2.15)} &= \sum_{l=0}^{i}(-1)^{i-l}q^{\binom{l+1}{2}-li}\begin{bmatrix}i-1\\i-l\end{bmatrix}h_l\left[B_\lambda(1-t)\right]\\
&= q^{-\binom{i}{2}}\sum_{l=0}^{i}(-1)^{i-l}q^{\binom{i-l}{2}}\begin{bmatrix}i-1\\i-l\end{bmatrix}h_l\left[B_\lambda(1-t)\right];
\end{aligned}$$

where the last equality follows from $\binom{l+1}{2} - li = \binom{i-l}{2} - \binom{i}{2}$. Thus, setting $l \mapsto k$ we have

$$\left(\nabla^{-1} h_i \left[\frac{X-\epsilon}{1-q}\right]\right)\Bigg|_{X \mapsto D_\lambda} = q^{-\binom{i}{2}} \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix} h_k \left[B_\lambda(1-t)\right];$$

which–combined with (A.3)– gives the thesis. □

Lemma 3.35. *For $r, i \in \mathbb{N}$ we have*

$$q^{i(i-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q = \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q \begin{bmatrix} r+k-1 \\ k \end{bmatrix}_q .$$

Proof. We start by applying Proposition 2.20.

$$\begin{aligned}
q^{i(i-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q &= q^{i(i-1)} h_i[[r-i+1]_q] \\
\text{(by def of } h_k \text{ and } [\cdot]_q) &= h_i[q^{i-1}[r-i+1]_q] \\
\text{(by def of } [\cdot]_q) &= h_i \left[q^{i-1} \cdot \frac{1-q^{r-i+1}}{1-q}\right] = h_i \left[\frac{q^{i-1}-q^r}{1-q}\right] \\
\text{(by 1.69)} &= \sum_{k=0}^{i} h_k \left[\frac{1-q^r}{1-q}\right] h_{i-k} \left[-\left(\frac{1-q^{i-1}}{1-q}\right)\right] \\
\text{(by 1.63,1.57)} &= \sum_{k=0}^{k} h_k \left[\frac{1-q^r}{1-q}\right] (-1)^{i-k} e_{i-k} \left[\frac{1-q^{i-1}}{1-q}\right] \\
\text{(by def of } [\cdot]_q) &= \sum_{k=0}^{i} h_k[[r]_q](-1)^{i-k} e_{i-k}[[i-1]_q]
\end{aligned}$$

Thus the conclusion follows from Proposition 2.20. □

Lemma 6.4. *For f, g homogeneous elements of Λ with $\deg(f) = m$ and $\mu \in \mathsf{Par} \setminus \{\emptyset\}$, we have*

$$\sum_{\lambda \supset_m \mu} d^f_{\lambda\mu} \Pi_\lambda g[MB_\lambda] = \Pi_\mu (\Delta_{f[MX]} g)[MB_\mu].$$

Proof. By Definition 3.17 and Theorem 3.20 we have

$$\sum_{\lambda \supset_m \mu} d^f_{\lambda\mu} H_\lambda = f H_\mu.$$

Let ν be any partition. Evaluate the above equation at $X = 1 + z(MB_\nu - 1)$ and apply Macdonald-Koornwinder reciprocity (3.3) on both sides:

$$\begin{aligned}
&\sum_{\lambda \supset_m \mu} d^f_{\lambda\mu} H_\nu[1 + z(MB_\lambda - 1)] \frac{\prod_{c \in \lambda}(1 - zq^{a'(c)} t^{l'(c)})}{\prod_{c \in \nu}(1 - zq^{a'(c)} t^{l'(c)})} \\
&= f[1 + z(MB_\nu - 1)] H_\nu[1 + z(MB_\mu - 1)] \frac{\prod_{c \in \mu}(1 - zq^{a'(c)} t^{l'(c)})}{\prod_{c \in \nu}(1 - zq^{a'(c)} t^{l'(c)})}
\end{aligned}$$

The denominators cancel out. Next, since $\mu \neq \emptyset$ and thus $\lambda \neq \emptyset$, there is a common factor $1-z$ on both sides. Cancelling this factor and then setting $z \mapsto 1$ (see Remark 1.64) we get

$$\begin{aligned}\sum_{\lambda \supset_m \mu} d^f_{\lambda\mu} H_\nu[MB_\lambda]\Pi_\lambda &= f[MB_\nu]H_\nu[MB_\mu]\Pi_\mu \\ \text{(by definition of } \Delta_f \text{ 2.32)} &= \left(\Delta_{f[MX]} H_\nu\right)[MB_\mu]\Pi_\mu.\end{aligned}$$

Since $\{H_\nu\}_{\nu \vdash i}$ forms a basis of $\Lambda^{(i)}$ this identity can be extended linearly to hold for any $g \in \Lambda^{(i)}$. □

Lemma 6.5. *For $l, s \in \mathbb{N}$*

$$(1-q^{s+1})\begin{bmatrix} l+s+1 \\ l \end{bmatrix}_q = (1-q^{s+l+1})\begin{bmatrix} s+l \\ l \end{bmatrix}_q.$$

Proof. For $l = 0$, the statement is clearly true so we may suppose $l > 0$. We will use Equations (2.18) and (2.16).

$$\begin{aligned}(1-q^{s+1})\begin{bmatrix} l+s+1 \\ l \end{bmatrix}_q &= \begin{bmatrix} l+s+1 \\ l \end{bmatrix}_q - q^{s+1+l}\begin{bmatrix} l+s \\ l \end{bmatrix}_q - q^{s+1}\begin{bmatrix} l+s \\ l-1 \end{bmatrix}_q \\ &= \begin{bmatrix} l+s+1 \\ s+1 \end{bmatrix}_q - q^{s+1+l}\begin{bmatrix} l+s \\ l \end{bmatrix}_q - q^{s+1}\begin{bmatrix} l+s \\ s+1 \end{bmatrix}_q \\ &= q^{s+1}\begin{bmatrix} l+s \\ s+1 \end{bmatrix}_q + \begin{bmatrix} l+s \\ s \end{bmatrix}_q - q^{s+1+l}\begin{bmatrix} l+s \\ l \end{bmatrix}_q - q^{s+1}\begin{bmatrix} l+s \\ s+1 \end{bmatrix}_q \\ &= (1-q^{s+l+1})\begin{bmatrix} s+l \\ l \end{bmatrix}_q.\end{aligned}$$

□

Lemma 6.20. *If $M := \left[\binom{i+j}{i}\right]_{i,j=0,\dots,n} \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{N})$ then $\det(M) = 1$.*

Proof. We define two more matrices

$$L = \left[\binom{i}{j}\right]_{i,j=0,\dots,n} \qquad U = \left[\binom{j}{i}\right]_{i,j=0,\dots,n}.$$

Clearly L and U are lower and upper triangular, respectively, and all their diagonal entries are 1. Thus, $\det(L) = \det(U) = 1$. We will show that $M = LU$, which readily implies the thesis. We have to show that for all $i, j = 0, \dots, n$

$$\begin{aligned}(LU)_{i,j} = \sum_{k=0}^{n}\binom{i}{k}\binom{j}{k} &= \binom{i+j}{i} \\ \Leftrightarrow \sum_{k=0}^{n}\binom{i}{i-k}\binom{j}{k} &= \binom{i+j}{i};\end{aligned}$$

which clearly holds since a choice i among $i+j$ can be decomposed into $i-k$ choices among i and k choices among j for some $k = 0, \dots, i$ and the rest of the terms are zero. □

Appendix B

Figures for schedule numbers

This appendix contains figures illustrating the construction of some of the square paths of $\mathsf{LSQ}(1,8)^{\bullet 2}$ with diagonal word $44\ 223\ \dot{0}11\dot{2}$ and shift 1. They serve as visuals for the proof of Theorem 7.7.

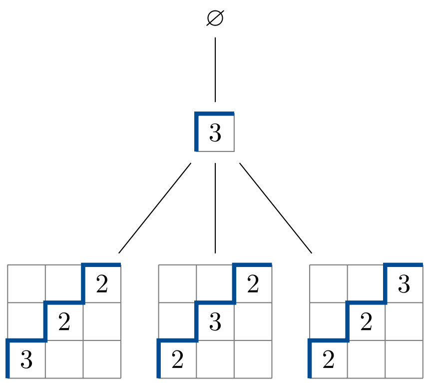


Figure B.1: $(1,3)$ and $(1,2)$-insertion

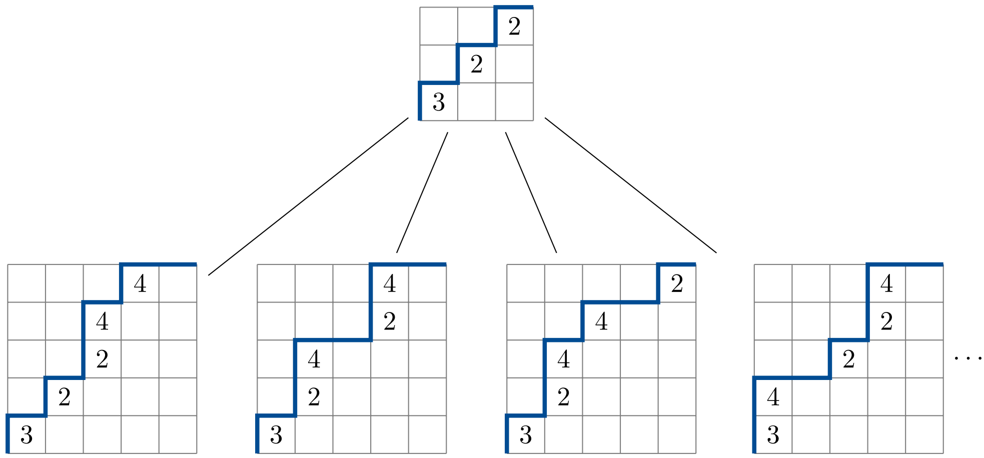


Figure B.2: (2, 4)-insertion

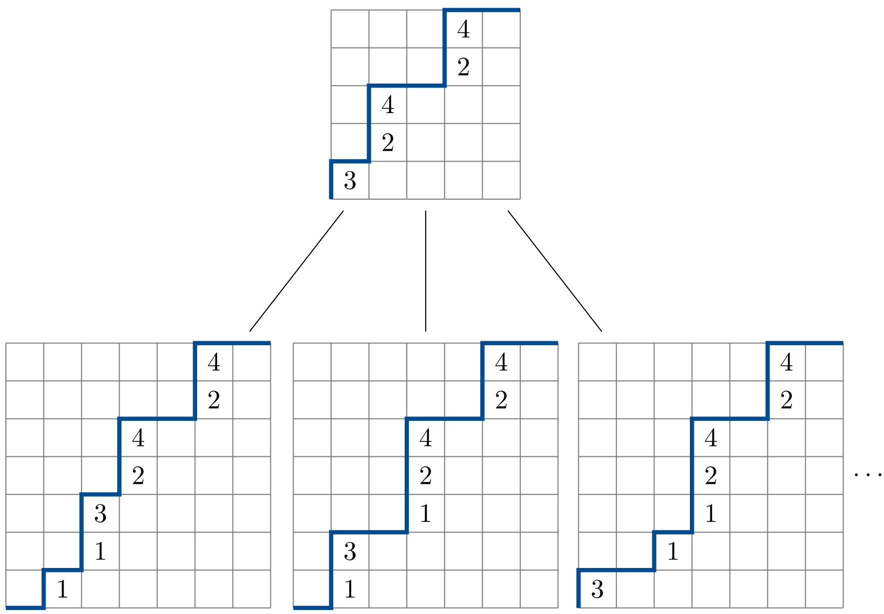


Figure B.3: (0, 1)-insertion

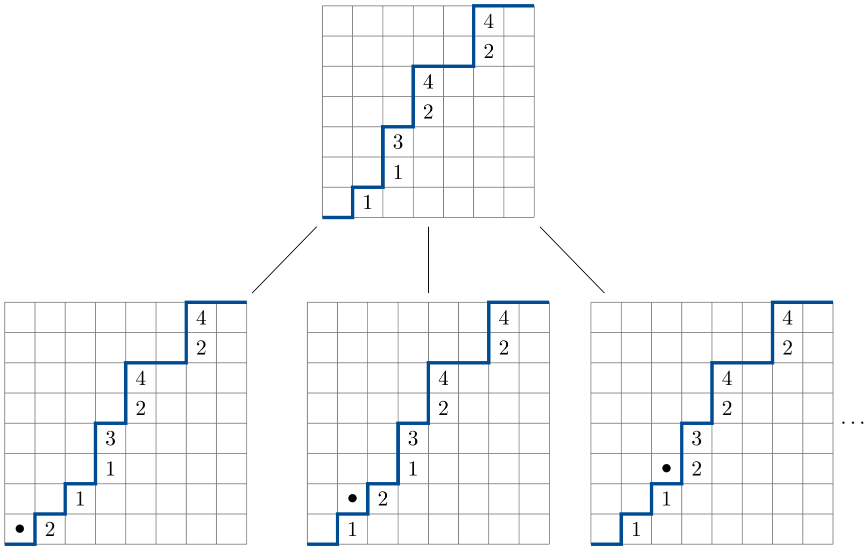


Figure B.4: $(0,2)^{\bullet}$-insertion

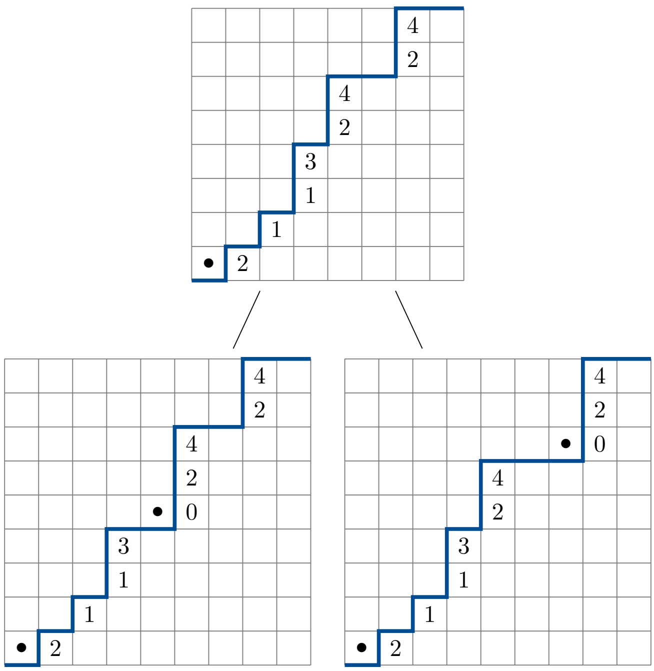


Figure B.5: $(0,0)^{\bullet}$-insertion

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List of symbols and notations

(1^n)	$\underbrace{(1,\dots,1)}_{n\text{times}}$	3
$a'_\lambda(c)$	co-arm of a cell c of a partition λ	4
$a_\lambda(c)$	arm of a cell c of a partition λ	4
$\alpha \vDash n$	composition of n	2
area	area statistic	67
B_λ	$\sum_{c\in\lambda} q^{a'_\lambda(c)} t^{l'_\lambda(c)}$	35
$B_{\lambda/\mu}$	$B_\lambda - B_\mu$	47
C_α	compositional refinement of e_n	40
$\chi(\pi, wt)$	weighted characteristic function	107
$c^{f^\perp}_{\lambda\mu}$	Pieri coefficient $f^\perp H_\lambda = \sum_{\mu\subset_n\lambda} c^{f^\perp}_{\lambda\mu} H_\mu$	46
$c^{(k)}_{\lambda\mu}$	Pieri coefficient $h_k^\perp H_\lambda = \sum_{\mu\subset_n\lambda} c^{(k)}_{\lambda\mu} H_\mu$	46
$\mathsf{D}(\alpha)^{*k}$	$P \in \mathsf{D}(n)^{*k}$ such that $\mathsf{dcomp}(P) = \alpha$	66
$\mathsf{D}(\alpha)^{\bullet k}$	$P \in \mathsf{D}(n)^{\bullet k}$ such that $\mathsf{dcomp}(P) = \alpha$	66
$\mathsf{D}(n)$	Dyck paths of size n	61
$\mathsf{D}(n)^{*k}$	Dyck paths of size n with k decorated rises	65
$\mathsf{D}(n)^{\bullet k}$	Dyck paths of size n with k decorated valleys	65
dcomp	diagonal composition of a path	66
dcomp'	second diagonal composition of a path	107
$\delta_{a,b}$	Kronecker delta, equal to 1 when $a = b$ and 0 when $a \neq b$	13
Δ_f	symmetric function operator defined by $\Delta_f H_\lambda = f[B_\lambda] H_\lambda$	36
Δ'_f	symmetric function operator defined by $\Delta'_f H_\lambda = f[B_\lambda - 1] H_\lambda$	36
dinv	dinv statistic	67
$\mathsf{DLa}(\pi)$	Diagonal labellings of π	103
D_λ	$(1-q)(1-t)B_\lambda - 1$	35
$\mathsf{DLD}(n)^{\bullet k}$	diagonally labelled square paths of size n and k decorated valleys	103
d_-	lowering operator	109
$d^f_{\mu\lambda}$	Pieri coefficient $f H_\lambda = \sum_{\mu\supset_n\lambda} d^f_{\mu\lambda} H_\mu$	46
$d^{(k)}_{\mu\lambda}$	Pieri coefficient $e\left[\frac{X}{M}\right] H_\lambda = \sum_{\mu\supset_n\lambda} d^{(k)}_{\mu\lambda} H_\mu$	46
d_+	raising operator	109
d_v	$\frac{[d_-, d_+]}{1-q}$	109

$E(\zeta)$	generating function of elementary symmetric functions	11
e_λ	elementary symmetric function	8
$\ell(\lambda)$	length, i.e. number of parts of a partition	2
e_n	n-th elementary symmetric function	7
$E_{n,k}$	defined by $e_n\left[X\frac{1-z}{1-q}\right]=\sum_{k=0}^{n}\frac{(z;q)_k}{(q;q)_k}E_{n,k}$	38
ϵ	formal variable such that $\epsilon^d=(-1)^d$	19
f^*	$f\left[\frac{X}{M}\right]$	33
$H(\zeta)$	generating function of homogeneous symmetric functions	11
H_λ	modified Macdonald polynomial	27
h_λ	homogeneous symmetric function	8
h_n	n-th homogeneous symmetric function	8
$\mathbb{K}$	a field	1
$K_{\lambda,\mu}$	Kostka number, i.e. number of semi-standard Young tableau of shape λ and weight μ	6
$K_{\lambda,\mu}(q,t)$	modified q,t-Kostka constant	27
$a'_\lambda(c)$	co-leg of a cell c of a partition λ	4
$\mathsf{La}(\pi)$	Labellings of π	63
$\mathsf{La}(\pi,m)$	Partial labellings of π with m zeros	63
Λ	$\Lambda_{\mathbb{Q}(q,t)}$	25
λ	always denotes a partition	2
$\Lambda_{\mathbb{K}}$	ring of symmetric functions with coefficients in $\mathbb{K}$	1
$\Lambda_{\mathbb{K}}^{(n)}$	space of symmetric functions, homogeneous of degree n, with coefficients in $\mathbb{K}$	2
$\lambda\vdash n$	partition of n	2
$\mathsf{LD}(\alpha)^{*k}$	$P\in\mathsf{LD}(n)^{*k}$ such that $\mathsf{dcomp}(P)=\alpha$	66
$\mathsf{LD}(\alpha)^{\bullet k}$	$P\in\mathsf{LD}(n)^{\bullet k}$ such that $\mathsf{dcomp}(P)=\alpha$	66
$\mathsf{LD}(m,\alpha)^{*k}$	$P\in\mathsf{LD}(m,n)^{*k}$ such that $\mathsf{dcomp}(P)=\alpha$	66
$\mathsf{LD}(m,n)$	Labelled Dyck paths of size $m+n$, with m labels equal to 0	65
$\mathsf{LD}(m,n)^{*k}$	labelled Dyck paths of size $m+n$, with m labels equal to 0 and k decorated rises	65
$\mathsf{LD}(m,n)^{\bullet k}$	labelled Dyck paths of size $m+n$, with m labels equal to 0 and k decorated valleys	65
$\mathsf{LD}(m,n,s)$	augmented Dyck paths with m zero labels, n finite positive labels and s infinity labels	80
$\mathsf{LD}(n)$	Labelled Dyck paths of size n	65
$\mathsf{LD}(n)^{*k}$	labelled Dyck paths of size n with k decorated rises	65
$\mathsf{LD}(n)^{\bullet k}$	labelled Dyck paths of size n with k decorated valleys	65
$l_\lambda(c)$	leg of a cell c of a partition λ	4
$\mathsf{LSQ}'(m,n)^{\bullet k}$	$(\pi,w,\emptyset,dv)\in\mathsf{LSQ}(m,n)^{\bullet k}$ such that there exists $1\le i\le m+n$ such that $a_i(\pi)=-\mathsf{shift}(\pi)$, $w_i\neq 0$ and $i\notin dv$.	65

$\mathsf{LSQ}(m,n)$	Labelled square paths of size $m+n$ with m labels equal to 0	65
$\mathsf{LSQ}(m,n)^{*k}$	labelled square paths of size $m+n$ with m labels equal to 0 and k decorated rises	65
$\mathsf{LSQ}(m,n)^{\bullet k}$	labelled square paths of size $m+n$ with m labels equal to 0 and k decorated valleys	65
$\mathsf{LSQ}(n)$	Labelled square paths of size n	65
$\mathsf{LSQ}(n)^{*k}$	labelled square paths of size n with k decorated rises	65
$\mathsf{LSQ}(n)^{\bullet k}$	labelled square paths of size n with k decorated valleys	65
M	$(1-q)(1-t)$	33
maj	maj statistic on words	89
m_λ	monomial symmetric function	2
$n(\lambda)$	$\sum_{i=1}^{k}(i-1)\lambda_i$	4
$(n-k,1^k)$	hook shape partition	4
∇	symmetric function operator defined by $\nabla H_\lambda = T_\lambda H_\lambda$	35
$m_i(\lambda)$	$\lvert\{j \mid \lambda_j = i\}\rvert$	4
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	$\frac{[n]_q!}{[k]_q![n-k]_q!}$	30
$[n]_q$	$[n]_q$	30
$[n]_q!$	$\prod_{i=1}^{n}[i]_q!$	30
ω	algebra morphism of Λ defined by $\omega(p_n) = (-1)^{n-1}p_n$	17
$\mathbb{P}$	stricly positive, natural numbers, i.e. $\mathbb{N}\setminus\{0\}$	1
$p(n)$	size of the set of partitions of n	2
$P(\zeta)$	generating function of power symmetric functions	11
Par	set of partitions	2
$\mathsf{Par}(n)$	set of partitions of n	2
$\mathsf{pd}(\pi)$	partial Dyck path associated to π	110
ϕ	symmetric function operator defined by $\phi f[X] = f[MX]$	33
$\mathbf{\Pi}$	symmetric function operator defined by $\mathbf{\Pi} H_\lambda = \Pi_\lambda H_\lambda$	37
Π_λ	$\prod_{c\in\lambda\setminus(0,0)}\left(1-q^{a'_\lambda(c)}t^{l'_\lambda(c)}\right)$	35
P_λ	Macdonald polynomial	26
p_λ	power symmetric function	8
p_n	n-th power symmetric function	8
$\preceq$	dominance orded on $\mathsf{Par}(n)$	5
ρ_Z	$\rho_Z(f[X]) = \sum_{n\in\mathbb{N}} h_n[XZ]f[X]$	23
$\mathsf{Rise}(\pi)$	the set of rises of the path π	64
s_λ	Schur symmetric function	8
$\mathfrak{S}_n$	n-th symmetric group	2
$\mathsf{SQ}(n)$	square paths of size n	61
$\mathsf{SQ}(n)^{*k}$	square paths of size n with k decorated rises	65
$\mathsf{SQ}(n)^{\bullet k}$	square paths of size n	65
$\mathsf{SSYT}(\lambda/\mu)$	set of semi-standard Young tableau of shape λ/μ	6

$\subset_n$	$\mu \subseteq \lambda$ and $\lvert\lambda\rvert - \lvert\mu\rvert = n$	47
$\mathsf{SYT}(\lambda/\mu)$	set of standard Young tableau of shape λ/μ	6
τ_Z	$\tau_Z(f[X]) = f[X+Z]$	23
Θ_f	symmetric function operator	37
T_λ	$\sum_{c\in\lambda} q^{a'_\lambda(c)} t^{l'_\lambda(c)}$	35
touch	number of touching points of a path	66
$\mathsf{Val}(\pi)$	the set of valleys of the path π	64
$\mathsf{Val}(\pi, w)$	the set of contractible valleys of the path π with (partial) labelling w	64
$w_{i,s}(c)$	schedule number for an undecorated label c in the $(i-s)$-th diagonal of a square path of shift s	88
$w^\bullet_{i,s}(c)$	schedule number for an decorated label c in the $(i-s)$-th diagonal of a square path of shift s	88
w_λ	$\prod_{c\in\lambda} \left(q^{a_\lambda(c)} - t^{l_\lambda(c)+1}\right)\left(t^{l_\lambda(c)} - q^{a_\lambda(c)+1}\right)$	34
$(x;q)_n$	$\prod_{i=0}^{n-1}(1-xq^i)$	30
ζ	the zeta map	104, 106
$z_i(c)$	i-th undecorated multiplicity function	88
$z^\bullet_i(c)$	i-th decorated multiplicity function	88
z_λ	$\prod_{i=1}^k i^{m_i(\lambda)} m_i(\lambda)!$	4

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