

THETA OPERATORS, REFINED DELTA CONJECTURES, AND COINVARIANTS

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ABSTRACT. We introduce the family of Theta operators Θ_f indexed by symmetric functions f that allow us to conjecture a compositional refinement of the Delta conjecture of Haglund, Remmel and Wilson [?Haglund-Remmel-Wilson-2015] for $\Delta'_{e_{n-k-1}} e_n$. We show that the 4-variable Catalan theorem of Zabrocki [?Zabrocki-4Catalan-2016] is precisely the Schröder case of our compositional Delta conjecture, and we show how to relate this conjecture to the Dyck path algebra introduced by Carlsson and Mellit in [?Carlsson-Mellit-ShuffleConj-2015], extending one of their results.

Again using the Theta operators, we conjecture a touching refinement of the generalized Delta conjecture for $\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n$, and prove the case $k = 0$, which was also conjectured in [?Haglund-Remmel-Wilson-2015], extending the shuffle theorem of Carlsson and Mellit to a *generalized shuffle theorem* for $\Delta_{h_m} \nabla e_n$. Moreover we show how this implies the case $k = 0$ of our generalized Delta square conjecture for $\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n)$, extending the square theorem of Sergel [?Leven-2016] to a *generalized square theorem* for $\Delta_{h_m} \nabla \omega(p_n)$.

Still the Theta operators will provide a conjectural formula for the Frobenius characteristic of super-diagonal coinvariants with two sets of Grassmannian variables, extending the one of Zabrocki in [?Zabrocki_Delta_Module] for the case with one set of such variables. We propose a combinatorial interpretation of this last formula at $q = 1$, leaving open the problem of finding a *dinv* statistic that gives the whole symmetric function.

Contents

1. INTRODUCTION

In the 90's Garsia and Haiman introduced the \mathfrak{S}_n -module of *diagonal harmonics*, i.e. the coinvariants of the diagonal action of \mathfrak{S}_n on polynomials in two sets of n variables, and they conjectured that its Frobenius characteristic was given by ∇e_n , where ∇ is the *nabla* operator on symmetric functions introduced in [?Bergeron-Garsia-Haiman-Tesler-Positivity-1999]. In 2002 Haiman proved this conjecture (see [?Haiman-Vanishing-2002]). Later the authors of [?HHLRU-2005] formulated the so called *shuffle conjecture*, i.e. they predicted a combinatorial formula for ∇e_n in terms of labelled Dyck paths. Several years later in [?Haglund-Morse-Zabrocki-2012] Haglund, Morse and Zabrocki conjectured a *compositional* refinement of the shuffle conjecture, which specified also the points where the Dyck paths touch the main diagonal. Recently Carlsson and Mellit in [?Carlsson-Mellit-ShuffleConj-2015] proved precisely this refinement, thanks to the introduction of what they called the *Dyck path algebra*. See [?Willigenburg_History_Shuffle] for more on this story.

In [?Haglund-Remmel-Wilson-2015] Haglund, Remmel and Wilson formulated the *Delta conjecture*, which can be stated as

$$\Delta'_{e_{n-k-1}} e_n = \sum_{P \in \text{LD}(0,n)^{*k}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P,$$

where the sum is over labelled Dyck paths of size n with positive labels and k decorated rises. It turns out that for $k = 0$ this formula reduces to the shuffle conjecture. Recently this conjecture attracted quite a bit of interest: see [?TheBible, Section 2] for the state of the art on this problem.

Even more recently Zabrocki in [?Zabrocki_Delta_Module] conjectured that this formula gives the Frobenius characteristic for *super-diagonal coinvariants* of degree k in the Grassmannian variables (see Section ?? for the missing definitions). It turns out that the submodule of degree 0 in the Grassmannian variables is precisely the module of diagonal harmonics. So the whole framework of “diagonal harmonics + shuffle conjecture” got generalized to this new setting.

In this work we add a new piece to the puzzle, by extending the compositional shuffle conjecture to a *compositional Delta conjecture*. In order to do so, we introduce a new family of *Theta operators* Θ_f on symmetric functions, indexed by symmetric functions f . In fact with this article we want to make the case for the Theta operators.

Here are the highlights of the present paper:

- We state a *compositional Delta conjecture*, which will read as follows: for a composition $\alpha \vDash n-k$

$$\Theta_{e_k} \nabla C_\alpha = \sum_{\substack{P \in \text{LD}(0,n)^{*k} \\ \text{dcomp}(P)=\alpha}} q^{\text{div}(P)} t^{\text{area}(P)} x^P,$$

where ∇C_α are the symmetric functions appearing in the compositional shuffle conjecture, and $\text{dcomp}(P)$ is the composition given by the distances between the consecutive points where the Dyck path touches the diagonal, ignoring the rows containing a decorated rise.

- We show that Zabrocki's 4-variable Catalan theorem [?Zabrocki-4Catalan-2016] is actually the Schröder case of our compositional Delta conjecture.
- We show that the combinatorial side satisfies a recursion that can be described by the Dyck path algebra in the same way as Carlsson and Mellit proved it for the compositional shuffle conjecture. In this case our theorem will read as

$$d_-^{\ell(\alpha)} M_\alpha^{*k} = \sum_{\substack{P \in \text{LD}(0,n)^{*k} \\ \text{dcomp}(P)=\alpha}} q^{\text{div}(P)} t^{\text{area}(P)} x^P,$$

where M_α^{*k} is defined recursively, and it coincides with N_α in [?Carlsson-Mellit-ShuffleConj-2015] for $k=0$. This reduces our refinement of the Delta conjecture to an identity of operators.

- We state a *touching generalized Delta conjecture*, which refines the generalized Delta conjecture in [?Haglund-Remmel-Wilson-2015] for $\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n$ to

$$\Delta_{h_m} \Theta_{e_k} \nabla E_{n-k,r} = \sum_{\substack{P \in \text{LD}(m,n)^{*k} \\ \text{touch}(P)=r}} q^{\text{div}(P)} t^{\text{area}(P)} x^P,$$

where $\nabla E_{n-k,r}$ are the symmetric functions already appearing in the shuffle conjecture and the sum of the righthand side is over labelled Dyck paths with nonnegative labels of size $m+n$, with m zero labels, k decorated rises and touching the diagonal r times. Furthermore we state a *touching generalized Delta square conjecture*, which refines our generalized Delta square conjecture [?DAdderio-Iraci-VandenWyngaerd-Delta-Square] for $[n-k]_t/[n]_t \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n)$ to

$$\frac{[n]_q}{[r]_q} \Delta_{h_m} \Theta_{e_k} \nabla E_{n-k,r} = \sum_{\substack{P \in \text{LSQ}(m,n)^{*k} \\ \text{touch}(P)=r}} q^{\text{div}(P)} t^{\text{area}(P)} x^P,$$

where the sum is over labelled square paths ending east with nonnegative labels, k decorated rises and touching the diagonal r times.

- We prove the case $k=0$ of our touching generalized Delta conjecture, which was already conjectured in [?Haglund-Remmel-Wilson-2015, Conjecture 7.5]. This extends the shuffle theorem of Carlsson and Mellit [?Carlsson-Mellit-ShuffleConj-2015] to a *generalized shuffle theorem* for $\Delta_{h_m} \nabla e_n$.
- We prove the case $k=0$ of our touching generalized Delta square conjecture. This extends the square theorem of Sergel [?Leven-2016] to a *generalized square theorem* for $\Delta_{h_m} \nabla \omega(p_n)$.
- We extend Zabrocki's conjecture [?Zabrocki_Delta_Module] to the module $M_n^{(2)}$ of super-diagonal coinvariants of the diagonal action of \mathfrak{S}_n on polynomials in two sets of n commutative variables and two sets of n Grassmanian variables:

$$\mathcal{F}_{q,t,\tilde{z}}(M_n^{(2)}) = \sum_{\substack{i,j \geq 0 \\ 1 \leq i+j < n}} z_1^i z_2^j \Theta_{e_i} \Theta_{e_j} \nabla e_{n-(i+j)}.$$

- We conjecture a combinatorial interpretation for the formula in the previous item at $q=1$:

$$\Theta_{e_r} \Theta_{e_k} \nabla e_{n-r-k} \Big|_{q=1} = \sum_{P \in \text{LD}(0,n)^{*k,or}} t^{\text{area}(P)} x^P$$

where $\text{LD}(0,n)^{*k,or}$ is the set of labelled Dyck paths with positive labels of size n with k decorated rises and r decorated contractible valleys. We leave open the outstanding problem of finding a div statistic that gives the whole symmetric function $\Theta_{e_r} \Theta_{e_k} \nabla e_{n-r-k}$.

The rest of this paper is organized in the following way. In Section 2 we introduce the basic ingredients of symmetric functions, that will be needed in Section 3 to introduce the Theta operators and to state the basic theorems that led us to their definition. In Section 4 we recall the combinatorial definitions needed in Section 5 to state our refined Delta conjectures. In Section 6 we prove our results about the compositional Delta conjecture, while in Section 7 we prove our results about the touching generalized Delta conjectures. In Section 8 we state our conjectures about the Frobenius characteristics of super-diagonal coinvariants, while in Section 9 we state our combinatorial interpretation of those at $q = 1$. In Section 10 we state more conjectures about the Theta operators. Finally in Section 11 we give the details of the technical proofs of symmetric function theory that we left out in the previous sections.

ACKNOWLEDGEMENTS

We are happy to thank Mike Zabrocki for providing us useful references, programs and information to check our conjecture on the Frobenius characteristic of super-diagonal coinvariants. We also thank the anonymous referee for carefully reading a previous draft of the present work and for pointing out several imprecisions.

2. SYMMETRIC FUNCTIONS: BASICS

In this section we limit ourselves to introduce the necessary notation to state our main theorems and conjectures. We refer to Section ?? for more on symmetric functions.

The main references that we will use for symmetric functions are [?Macdonald-Book-1995], [?Stanley-Book-1999] and [?Haglund-Book-2008].

The standard bases of the symmetric functions that will appear in our calculations are the complete $\{h_\lambda\}_\lambda$, elementary $\{e_\lambda\}_\lambda$, power $\{p_\lambda\}_\lambda$ and Schur $\{s_\lambda\}_\lambda$ bases.

We will use the usual convention that $e_0 = h_0 = 1$ and $e_k = h_k = 0$ for $k < 0$.

The ring Λ of symmetric functions can be thought of as the polynomial ring in the power sum generators p_1, p_2, p_3, \dots . This ring has a grading $\Lambda = \bigoplus_{n \geq 0} \Lambda^{(n)}$ given by assigning degree i to p_i for all $i \geq 1$. As we are working with Macdonald symmetric functions involving two parameters q and t , we will consider this polynomial ring over the field $\mathbb{Q}(q, t)$. We will make extensive use of the *plethystic notation*.

With this notation we will be able to add and subtract alphabets, which will be represented as sums of monomials $X = x_1 + x_2 + x_3 + \dots$. Then, given a symmetric function f , and thinking of it as an element of Λ , we denote by $f[X]$ the expression f with p_k replaced by $x_1^k + x_2^k + x_3^k + \dots$, for all k . More generally, given any expression $Q(z_1, z_2, \dots)$, we define the plethystic substitution $f[Q(z_1, z_2, \dots)]$ to be f with p_k replaced by $Q(z_1^k, z_2^k, \dots)$.

We denote by $\langle \cdot, \cdot \rangle$ the *Hall scalar product* on symmetric functions, which can be defined by saying that the Schur functions form an orthonormal basis. We denote by ω the fundamental algebraic involution which sends e_k to h_k , s_λ to $s_{\lambda'}$ and p_k to $(-1)^{k-1} p_k$.

With the symbol “ \perp ” we denote the operation of taking the adjoint of an operator with respect to the Hall scalar product, i.e.

$$(1) \quad \langle f^\perp g, h \rangle = \langle g, fh \rangle \quad \text{for all } f, g, h \in \Lambda.$$

For a partition $\mu \vdash n$, we denote by

$$(2) \quad \tilde{H}_\mu := \tilde{H}_\mu[X] = \tilde{H}_\mu[X; q, t] = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q, t) s_\lambda$$

the (*modified*) *Macdonald polynomials*, where

$$(3) \quad \tilde{K}_{\lambda\mu} := \tilde{K}_{\lambda\mu}(q, t) = K_{\lambda\mu}(q, 1/t) t^{n(\mu)} \quad \text{with} \quad n(\mu) = \sum_{i \geq 1} \mu_i(i-1)$$

are the (*modified*) *Kostka coefficients* (see [?Haglund-Book-2008, Chapter 2] for more details).

The set $\{\tilde{H}_\mu[X; q, t]\}_\mu$ is a basis of the ring of symmetric functions Λ with coefficients in $\mathbb{Q}(q, t)$. This is a modification of the basis introduced by Macdonald [?Macdonald-Book-1995], and they are the Frobenius characteristic of the so called Garsia-Haiman modules (see [?Garsia-Haiman-PNAS-1993]).

If we identify the partition μ with its Ferrers diagram, i.e. with the collection of cells $\{(i, j) \mid 1 \leq i \leq \mu_j, 1 \leq j \leq \ell(\mu)\}$, then for each cell $c \in \mu$ we refer to the *arm*, *leg*, *co-arm* and *co-leg* (denoted

respectively as $a_\mu(c)$, $l_\mu(c)$, $a_\mu(c)'$, $l_\mu(c)'$ as the number of cells in μ that are strictly to the right, above, to the left and below c in μ , respectively (see Figure ??).

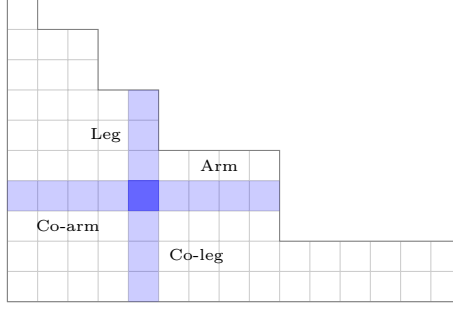


FIGURE 1. Arm, co-arm, leg and co-leg.

We set

$$(4) \quad M := (1 - q)(1 - t),$$

and we define for every partition μ

$$(5) \quad B_\mu := B_\mu(q, t) = \sum_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)}$$

$$(6) \quad D_\mu := MB_\mu(q, t) - 1$$

$$(7) \quad T_\mu := T_\mu(q, t) = \prod_{c \in \mu} q^{a'_\mu(c)} t^{l'_\mu(c)}$$

$$(8) \quad \Pi_\mu := \Pi_\mu(q, t) = \prod_{c \in \mu / (1)} (1 - q^{a'_\mu(c)} t^{l'_\mu(c)})$$

$$(9) \quad w_\mu := w_\mu(q, t) = \prod_{c \in \mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}).$$

Notice that

$$(10) \quad B_\mu = e_1[B_\mu] \quad \text{and} \quad T_\mu = e_{|\mu|}[B_\mu].$$

For every symmetric function $f[X]$ we set

$$(11) \quad f^* = f^*[X] := f \left[\frac{X}{M} \right].$$

The following linear operators were introduced in [?Bergeron-Garsia-ScienceFiction-1999, ?Bergeron-Garsia-Hairer-2000] and they are at the basis of the conjectures relating symmetric function coefficients and q, t -combinatorics in this area.

We define the *nabla* operator on Λ by

$$(12) \quad \nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu \quad \text{for all } \mu,$$

and we define the *Delta* operators Δ_f and Δ'_f on Λ by

$$(13) \quad \Delta_f \tilde{H}_\mu = f[B_\mu(q, t)] \tilde{H}_\mu \quad \text{and} \quad \Delta'_f \tilde{H}_\mu = f[B_\mu(q, t) - 1] \tilde{H}_\mu, \quad \text{for all } \mu.$$

Observe that on the vector space of symmetric functions homogeneous of degree n , denoted by $\Lambda^{(n)}$, the operator ∇ equals Δ_{e_n} . Moreover, for every $1 \leq k \leq n$,

$$(14) \quad \Delta_{e_k} = \Delta'_{e_k} + \Delta'_{e_{k-1}} \quad \text{on } \Lambda^{(n)},$$

and for any $k > n$, $\Delta_{e_k} = \Delta'_{e_{k-1}} = 0$ on $\Lambda^{(n)}$, so that $\Delta_{e_n} = \Delta'_{e_{n-1}}$ on $\Lambda^{(n)}$.

Recall the standard notation for q -analogues: for $n \in \mathbb{N}$,

$$(15) \quad [0]_q := 0, \quad \text{and} \quad [n]_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1} \quad \text{for } n \geq 1,$$

$$(16) \quad [0]_q! := 1 \quad \text{and} \quad [n]_q! := [n]_q [n-1]_q \cdots [2]_q [1]_q \quad \text{for } n \geq 1,$$

and

$$(17) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad \text{for } n \geq k \geq 0, \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := 0 \quad \text{for } n < k,$$

and also the standard notation for the *q-rising factorial*

$$(18) \quad (a; q)_n := (1-a)(1-qa)(1-q^2a) \cdots (1-q^{n-1}a).$$

In [?Haglund-Morse-Zabrocki-2012] the following operators were introduced: for any $m \geq 0$ and any $F[X] \in \Lambda$

$$(19) \quad \mathbb{C}_m F[X] := (-1/q)^{m-1} \sum_{r \geq 0} q^{-r} h_{m+r}[X] h_r[X(1-q)]^\perp F[X],$$

and for any composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ of n , denoted $\alpha \vDash n$, we set

$$(20) \quad C_\alpha = C_\alpha[X; q] := \mathbb{C}_{\alpha_1} \mathbb{C}_{\alpha_2} \cdots \mathbb{C}_{\alpha_l}(1).$$

The symmetric functions $E_{n,k}$ were introduced in [?Garsia-Haglund-qtCatalan-2002] by means of the following expansion:

$$(21) \quad e_n \left[X \frac{1-z}{1-q} \right] = \sum_{k=1}^n \frac{(z; q)_k}{(q; q)_k} E_{n,k}.$$

Notice that setting $z = q^j$ in (??) we get

$$(22) \quad e_n [X[j]_q] = e_n \left[X \frac{1-q^j}{1-q} \right] = \sum_{k=1}^n \frac{(q^j; q)_k}{(q; q)_k} E_{n,k} = \sum_{k=1}^n \begin{bmatrix} k+j-1 \\ k \end{bmatrix}_q E_{n,k}.$$

In particular, for $z = q$ we get

$$(23) \quad e_n = E_{n,1} + E_{n,2} + \cdots + E_{n,n}.$$

The following identity is proved in [?Haglund-Morse-Zabrocki-2012, Section 5]:

$$(24) \quad E_{n,r} = \sum_{\substack{\alpha \vDash n \\ \ell(\alpha)=r}} C_\alpha \quad \text{for all } r = 1, 2, \dots, n,$$

where $\ell(\alpha)$ denotes the length of the composition α .

Together with (??) it gives immediately

$$(25) \quad e_n = \sum_{\alpha \vDash n} C_\alpha.$$

The following identity is proved in [?Can-Loehr-2006, Theorem 4]:

$$(26) \quad \omega(p_n) = \sum_{k=1}^n \frac{[n]_q}{[k]_q} E_{n,k}.$$

3. THE THETA OPERATORS

In this section we introduce the family of operators Θ_f and we state a few results about them: *we will give the missing proofs in Section ??*. See also Section ?? for some conjectures about them.

Recall the definition of the invertible linear operator $\mathbf{\Pi}$ on $\oplus_{n \geq 1} \Lambda^{(n)}$ defined by

$$(27) \quad \mathbf{\Pi} \tilde{H}_\mu := \Pi_\mu \tilde{H}_\mu$$

for any non-empty partition μ .

For any symmetric function $f \in \Lambda^{(n)}$ we introduce the following *Theta operators* on Λ : for every $F \in \Lambda^{(m)}$ we set

$$(28) \quad \Theta_f F := \begin{cases} 0 & \text{if } n \geq 1 \text{ and } m = 0 \\ f \cdot F & \text{if } n = 0 \text{ and } m = 0 \\ \mathbf{\Pi} f^* \mathbf{\Pi}^{-1} F & \text{otherwise} \end{cases},$$

and we extend by linearity the definition to any $f, F \in \Lambda$.

It is clear that Θ_f is linear, and moreover, if f is homogenous of degree k , then so is Θ_f , i.e.

$$\Theta_f \Lambda^{(n)} \subseteq \Lambda^{(n+k)} \quad \text{for } f \in \Lambda^{(k)}.$$

Since in the present article we will use mostly a special case of these operators, we introduced the following shorter notation: for $k \geq 0$, we set

$$(29) \quad \Theta_k := \Theta_{e_k}.$$

Notice that Θ_0 is the identity operator on Λ .

The following theorems, which we prove in Section ??, led to the definition of the Theta operators.

Theorem 3.1. For $n \geq 1$ and $k \geq 0$,

$$(30) \quad \Theta_k \nabla e_{n-k} = \Delta'_{e_{n-k-1}} e_n.$$

Corollary 3.2. For $n \geq 1$ and $k \geq 0$,

$$(31) \quad \sum_{r=1}^{n-k} \Theta_k \nabla E_{n-k,r} = \Delta'_{e_{n-k-1}} e_n.$$

Proof. Using (??), we have

$$\begin{aligned} \sum_{r=1}^{n-k} \Theta_k \nabla E_{n-k,r} &= \Theta_k \nabla e_{n-k} \\ &\text{(using (??))} = \Delta'_{e_{n-k-1}} e_n. \end{aligned}$$

□

Theorem 3.3. For $n \geq 1$ and $k \geq 0$,

$$(32) \quad \Theta_k \nabla \frac{[n]_q}{[n-k]_q} \omega(p_{n-k}) = \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n).$$

Corollary 3.4. For $n \geq 1$ and $k \geq 0$,

$$(33) \quad \sum_{r=1}^{n-k} \frac{[n]_q}{[r]_q} \Theta_k \nabla E_{n-k,r} = \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n).$$

Proof. We have

$$\begin{aligned} \sum_{r=1}^{n-k} \frac{[n]_q}{[r]_q} \Theta_k \nabla E_{n-k,r} &= \frac{[n]_q}{[n-k]_q} \sum_{r=1}^{n-k} \Theta_k \nabla \frac{[n-k]_q}{[r]_q} E_{n-k,r} \\ &\text{(using (??))} = \Theta_k \nabla \frac{[n]_q}{[n-k]_q} \omega(p_{n-k}) \\ &\text{(using (??))} = \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n). \end{aligned}$$

□

4. COMBINATORIAL DEFINITIONS

Definition 4.1. A *square path ending east* of size n is a lattice path going from $(0, 0)$ to (n, n) consisting of east or north unit steps, always ending with an east step. The set of such paths is denoted by $\text{SQ}(n)$. We call *base diagonal* of a square path the diagonal $y = x + l$, where l is the smallest integer such that the path touches that diagonal (so that $l \leq 0$). The *shift* of the square path is the non-negative value $-l$. We refer to the line $x = y$ as the *main diagonal*. A *Dyck path* is a square path whose shift is 0, i.e. its base diagonal is the main diagonal. The set of Dyck paths is denoted by $\text{D}(n)$. Of course $\text{D}(n) \subseteq \text{SQ}(n)$.

For example, the path in Figure ?? has shift 3.

Definition 4.2. A *labelling* or *word* of a square path π of size $m+n$ ending east is an element $w \in \mathbb{N}^{m+n}$ such that when we label the i -th vertical step of π with w_i

- the labels appearing in each column of π are strictly increasing from bottom to top;
- there is at least one nonzero label along the base diagonal of π ;
- if π starts with a vertical step, then this first step has a nonzero label.

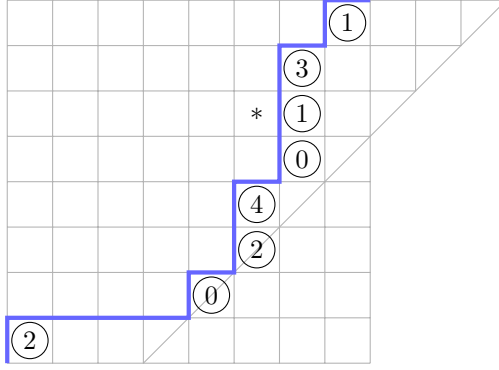


FIGURE 2. Example of an element in $\text{LSQ}(2, 6)^{*1}$ with reading word 241231.

The set of such labellings with m labels equal to 0 is denoted by $W(\pi, m)$, and we set $W(\pi) := W(\pi, 0)$.

A *partially labelled square path ending east* is an element $P = (\pi, w)$ of

$$\text{LSQ}(m, n) := \{(\pi, w) \mid \pi \in \text{SQ}(m+n), w \in W(\pi, m)\}.$$

We also define the subset of *labelled Dyck paths* as

$$\text{LD}(m, n) := \{(\pi, w) \in \text{LSQ}(m, n) \mid \pi \in \text{D}(m+n)\} \subseteq \text{LSQ}(m, n).$$

Definition 4.3. Let π be a square path ending east of size $m+n$. We define its *area word* to be the sequence of integers $a(\pi) = (a_1(\pi), a_2(\pi), \dots, a_{m+n}(\pi))$ such that the i -th vertical step of the path starts from the diagonal $y = x + a_i(\pi)$. For example the path in Figure ?? has area word $(0, -3, -3, -2, -2, -1, 0, 0)$.

Definition 4.4. We define for each $P \in \text{LSQ}(m, n)$ a monomial in the variables x_1, x_2, \dots : we set

$$x^P := \prod_{i=1}^{m+n} x_{l_i(P)}$$

where $l_i(P)$ is the label of the i -th vertical step of P (the first being at the bottom), and where we conventionally set $x_0 = 1$. The fact that x_0 does not appear in the monomial explains the word *partially*.

Definition 4.5. The *risers* of a square path ending east π are the indices

$$r(\pi) := \{2 \leq i \leq m+n \mid a_i(\pi) > a_{i-1}(\pi)\},$$

or the vertical steps that are directly preceded by another vertical step.

A *decorated square path* (respectively Dyck path) is a pair $P = (\pi, dr)$ where π is a square path (respectively Dyck path) and $dr \subseteq r(\pi)$. We set

$$\text{SQ}(n)^{*k} := \{(\pi, dr) \mid \pi \in \text{SQ}(n), dr \subseteq r(\pi), |dr| = k\}$$

$$\text{D}(n)^{*k} := \{(\pi, dr) \in \text{SQ}(n)^{*k} \mid \pi \in \text{D}(n)\}$$

A *partially labelled decorated square path* (respectively Dyck path) is a triple (π, dr, w) where π is a square path (respectively Dyck path), $dr \subseteq r(\pi)$ and w is a labelling of π . We set

$$\text{LSQ}(m, n)^{*k} := \{(\pi, dr, w) \mid (\pi, w) \in \text{LSQ}(m, n), dr \subseteq r(\pi), |dr| = k\}$$

$$\text{LD}(m, n)^{*k} := \{(\pi, dr, w) \in \text{LSQ}(m, n)^{*k} \mid \pi \in \text{D}(m+n)\} \subseteq \text{LSQ}(m, n)^{*k}.$$

We will also use the following natural identifications

$$\text{LSQ}(n, m)^{*0} = \text{LSQ}(n, m)$$

$$\text{LD}(m, n)^{*0} = \text{LD}(m, n)$$

Definition 4.6. Given a partially labelled square path, the vertical steps with label 0 will be called *zero valleys* (they are necessarily preceded by a horizontal step, hence the name valleys).

Definition 4.7. Given a partially labelled square path P of shift s , we define its *reading word* $\sigma(P)$ as the sequence of *nonzero* labels, read starting from the base diagonal $y = x - s$ going bottom left to top right, then moving to the next diagonal, $y = x - s + 1$ again going bottom left to top right, and so on.

If the reading word of P is $r_1 \dots r_n$ then the *reverse reading word* of P is $r_n \dots r_1$.

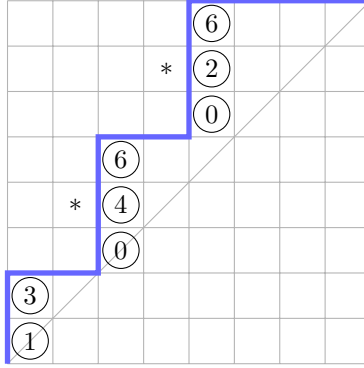


FIGURE 3. Example of an element in $\text{LD}(2, 6)^{*2}$ with reading word 134626.

See Figure ?? and Figure ?? for an example.

We define two statistics on this set that reduce to the same statistics as defined in [?Loehr-Warrington-square-2007] when $m = k = 0$.

Definition 4.8. Let $P = (\pi, dr) \in \text{SQ}(m, n)^{*k}$ and s be its shift. Define

$$\text{area}(P) := \sum_{i \notin dr} (a_i(\pi) + s).$$

More visually, the area is the number of whole squares between the path and the base diagonal that are not contained in rows with a decorated rise.

If $P = (\pi, dr, w) \in \text{LSQ}(m, n)^{*k}$ then we set $\text{area}(P) = \text{area}((\pi, dr))$. In other words, the area of a path does not depend on its labelling.

For example, the path in Figure ?? has area 11.

Definition 4.9. Let $P = (\pi, dr, w) \in \text{LSQ}(m, n)^{*k}$. For $1 \leq i < j \leq m + n$, we say that the pair (i, j) is an *inversion* if

- either $a_i(\pi) = a_j(\pi)$ and $w_i < w_j$ (*primary inversion*),
- or $a_i(\pi) = a_j(\pi) + 1$ and $w_i > w_j$ (*secondary inversion*),

where w_i denotes the i -th letter of w , i.e. the label of the vertical step in the i -th row.

Then we define

$$\begin{aligned} \text{dinv}(P) := & \#\{0 \leq i < j \leq m + n \mid (i, j) \text{ is an inversion}\} \\ & + \#\{0 \leq i \leq m + n \mid a_i(\pi) < 0 \text{ and } w_i \neq 0\}. \end{aligned}$$

This second term is referred to as *bonus dinv*.

For example, the path in Figure ?? has dinv 6: 2 primary inversions, i.e. $(1, 7)$ and $(2, 3)$, 1 secondary inversion, i.e. $(1, 6)$, and 3 bonus dinv , coming from the rows 3, 4 and 6. Notice that Dyck paths coincide with the square paths with no bonus dinv .

Definition 4.10. Given $P \in \text{LD}(m, n)^{*k}$ we define its *diagonal composition* $\text{dcomp}(P)$ to be the composition of $n - k$ whose i -th part is the number of rows of P without a 0 label or a decoration $*$ that lie between the i -th and the $(i + 1)$ -th vertical step of P on the base diagonal not labelled by a 0 (or from the i -th step onwards if it is the last such step). See Figure ?? for an example. If $P \in \text{D}(n)^{*k}$, its diagonal composition is defined identically, without the conditions concerning the labels (since there are none).

Definition 4.11. Given $P \in \text{LSQ}(m, n)^{*k}$ we define a *touching point* of P to be a starting point of a vertical step of P that lies on the base diagonal, and whose label is not zero (this includes $(0, 0)$ but not (n, n)). The *touching number* $\text{touch}(P)$ of P is defined as the number of touching points of P or equivalently as the length of its diagonal composition. See Figure ?? for an example. For $P \in \text{D}(n)^{*k}$ these definitions are the same, without the condition concerning the labels (since there are none).

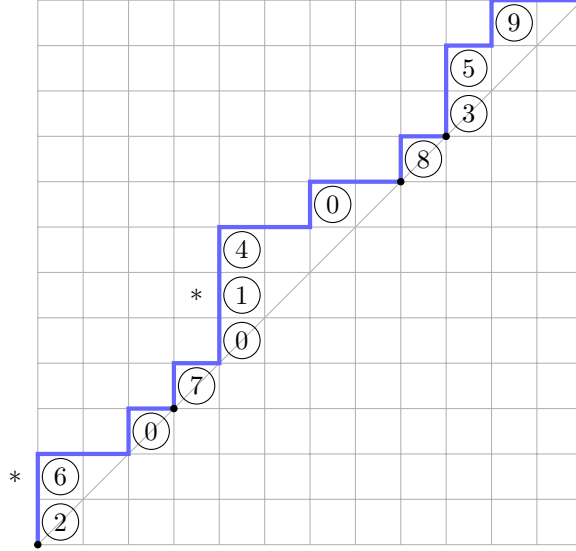


FIGURE 4. A partially labelled Dyck path with diagonal composition $\alpha = (1, 2, 1, 3)$ and touching number $\ell(\alpha) = 4$. The touching points are highlighted with a black dot.

5. STATEMENTS OF REFINED DELTA CONJECTURES

In this section we state our refined conjectures.

The following conjecture is due to Haglund, Remmel and Wilson [?Haglund-Remmel-Wilson-2015].

Conjecture 5.1 (Generalized Delta). *Given $n, k, m \in \mathbb{N}$ with $n > k \geq 0$,*

$$(34) \quad \Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{P \in \text{LSQ}(m, n)^{*k}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

We state our “touching” refinement of this conjecture.

Conjecture 5.2 (Touching generalized Delta). *Given $n, k, m, r \in \mathbb{N}$, $n > k \geq 0$ and $n - k \geq r \geq 1$,*

$$(35) \quad \Delta_{h_m} \Theta_k \nabla E_{n-k, r} = \sum_{\substack{P \in \text{LD}(m, n)^{*k} \\ \text{touch}(P) = r}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

Remark 5.3. It follows immediately from Corollary ?? that our touching generalized Delta conjecture implies the generalized Delta conjecture.

We now state our *compositional* refinement of the *Delta conjecture*, i.e. of the case $k = 0$ of Conjecture ??.

Conjecture 5.4 (Compositional Delta). *Given $n, k \in \mathbb{N}$, $n > k \geq 0$ and $\alpha \vDash n - k$,*

$$(36) \quad \Theta_k \nabla C_\alpha = \sum_{\substack{P \in \text{LD}(0, n)^{*k} \\ \text{dcomp}(P) = \alpha}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

Remark 5.5. We observe immediately that the compositional Delta conjecture implies the touching Delta conjecture: combinatorially we clearly have $\text{touch}(P) = \ell(\text{dcomp}(P))$, while on the symmetric function side we use (??).

Remark 5.6. Notice that we have a compositional version only of the Delta conjecture, and not of the generalized Delta conjecture, i.e. only for the number m of zero labels equal to 0. For $m > 0$ the dinv seems to be off at the compositional level. This situation should be compared with the final comments of [?DAdderio-Iraci-VandenWyngaerd-GenDeltaSchroeder].

Still, we can state the following conjecture.

Conjecture 5.7. Given $n, k, m \in \mathbb{N}$, $m > 0$, $n > k \geq 0$ and $\alpha \vDash n - k$,

$$(37) \quad \Delta_{h_m} \Theta_k \nabla C_\alpha |_{q=1} = \sum_{\substack{P \in \text{LD}(m, n)^{*k} \\ \text{dcomp}(P) = \alpha}} t^{\text{area}(P)} x^P.$$

The following conjecture appeared in our work [?DAdderio-Iraci-VandenWyngaerd-Delta-Square].

Conjecture 5.8 (Generalized Delta square). Given $n, k, m \in \mathbb{N}$ with $n > k \geq 0$,

$$(38) \quad \frac{[n-k]_q}{[n]_q} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{P \in \text{LSQ}(m, n)^{*k}} q^{\text{div}(P)} t^{\text{area}(P)} x^P.$$

We state our ‘‘touching’’ refinement of this conjecture.

Conjecture 5.9 (Touching generalized Delta square). Given $n, k, m, r \in \mathbb{N}$, $n > k \geq 0$ and $n - k \geq r \geq 1$,

$$(39) \quad \frac{[n]_q}{[r]_q} \Delta_{h_m} \Theta_k \nabla E_{n-k, r} = \sum_{\substack{P \in \text{LSQ}(m, n)^{*k} \\ \text{touch}(P) = r}} q^{\text{div}(P)} t^{\text{area}(P)} x^P.$$

Remark 5.10. It follows immediately from Corollary ?? that our touching generalized Delta square conjecture implies the generalized Delta square conjecture.

6. ABOUT THE COMPOSITIONAL VERSION

In this section we prove some results about our compositional Delta conjecture.

6.1. Relation to the 4-variable Catalan theorem. In [?Zabrocki-4Catalan-2016] Zabrocki showed that for any composition $\alpha \vDash n - \ell$

$$\langle \Delta_{h_\ell} \nabla C_\alpha, h_k e_{n-\ell-k} \rangle$$

is the $(\text{div}, \text{area})$ q, t -enumerator of Dyck paths P of size n with ℓ decorated rises and k decorated peaks with $\text{dcomp}(P) = \alpha$. So this should match the Schröder case of our compositional Delta conjecture, i.e. it should be equal to

$$\langle \Theta_\ell \nabla C_\alpha, h_k e_{n-k} \rangle.$$

Indeed, this follows immediately from the following lemma, which we prove in Section ??.

Lemma 6.1. For any $f \in \Lambda^{(n-\ell)}$,

$$(40) \quad \langle \Delta_{h_\ell} f, h_k e_{n-\ell-k} \rangle = \langle \Theta_\ell f, h_k e_{n-k} \rangle.$$

This shows that the 4-variable Catalan result of Zarbrocki is really the Schröder case of our compositional Delta conjecture.

6.2. Relation to the Dyck path algebra. In [?Carlsson-Mellit-ShuffleConj-2015] the authors construct the Dyck path algebra by defining combinatorial operators that can be used to compute the q, t, \underline{x} -enumerators of specific subsets of labelled Dyck paths. The purpose of this subsection is to use these operators to extend their results to decorated labelled Dyck paths.

6.2.1. Combinatorial translation. The goal is to relate our compositional Delta conjecture to the operators of the Dyck path algebra from [?Carlsson-Mellit-ShuffleConj-2015]. Following [?Carlsson-Mellit-ShuffleConj-2015] first we need to translate our q, t, \underline{x} -enumerator of $(\text{div}, \text{area})$ into one of another bivariate $(\text{inv}, \text{bounce})$.

In order to do this we need to extend some definitions and bijections in [?Haglund_Loehr_conj_Hilbert] to the decorated setting (cf. also [?Haglund-Xin_Lecture-Notes, Chapters 2 and 3]).

Definition 6.2. Let $\pi \in \text{D}(n)$. We define its *bounce path* as a lattice path from $(0, 0)$ to (n, n) computed in the following way: it starts in $(0, 0)$ and travels north until it encounters the beginning of an east step of π , then it turns east until it hits the main diagonal, then it turns north again, and so on; thus it continues until it reaches (n, n) .

We label the vertical steps of the bounce path starting from 0 and increasing the labels by 1 every time the path hits the main diagonal (so the steps in the first vertical segment of the path are labelled with 0, the ones in the next vertical segment are labelled with 1, and so on). We define the *bounce word* of π to be the string $b(\pi) = b_1(\pi) \cdots b_n(\pi)$ where $b_i(\pi)$ is the label attached to the i -th vertical step of the bounce path.

We define the statistic *bounce* on $D(n)$ and $LD(n)$ as

$$\text{bounce}(\pi) := \sum_{i=1}^n b_i(\pi).$$

Let us start by describing Haglund's classical bijection (Theorem 3.15 in [?Haglund-Book-2008])

$$\zeta_0 : D(n) \rightarrow D(n)$$

that transforms $(\text{dinv}, \text{area})$ into $(\text{area}, \text{bounce})$.

Take $\pi \in D(n)$ and rearrange its area word in ascending order. This new word, call it u , will be the bounce word of $\zeta_0(\pi)$. We construct $\zeta_0(\pi)$ as follows. First draw the bounce path corresponding to u . The first vertical stretch and last horizontal stretch of $\zeta_0(\pi)$ are fixed by this path. For the section of the path between consecutive peaks of the bounce path we apply the following procedure: place a pen on the top of the i -th peak of the bounce path and scan the area word of D from left to right. Every time we encounter a letter equal to $i - 1$ we draw an east step and when we encounter a letter equal to i we draw a north step. By construction of the bounce path, we end up with our pen on top of the $(i + 1)$ -th peak of the bounce path. Note that in an area word a letter equal to $i \neq 0$ cannot appear unless it is preceded somewhere by a letter equal to $i - 1$. This means that starting from the i -th peak, we always start with a horizontal step which explains why u is indeed the bounce word of $\zeta_0(\pi)$.

We want to extend ζ_0 to $LD(n)^{*k}$. For an element $(\pi, dr, w) \in LD(n)^{*k}$ we apply ζ_0 to π . We must now specify what happens to the decorated rises and the labelling of π .

Definition 6.3. A *corner* (or *valley*) of a Dyck path π is one of the indices

$$c(\pi) := \{2 \leq i \leq n \mid a_i(\pi) \leq a_{i-1}(\pi)\}.$$

Corners will often be identified with the vertical steps of π that are directly preceded by a horizontal step.

Proposition 6.4. *If $\pi \in D(n)$ then there exists a bijection between $r(\pi)$ and $c(\zeta_0(\pi))$.*

Proof. Let $j \in r(\pi)$. It follows that $a_j(\pi) = a_{j-1}(\pi) + 1$. Take i such that $a_{j-1}(\pi) = i - 1$. While scanning the area word to construct the path between the i -th and $(i + 1)$ -th peak of the bounce path, we will encounter $a_{j-1}(\pi) = i - 1$, directly followed by $a_j(\pi) = i$. This will correspond to a horizontal step followed by a vertical step in $\zeta_0(\pi)$ and thus to an element of $c(\zeta_0(\pi))$. Arguing backwards, it is easy to see that this gives the desired bijection. \square

Definition 6.5. Given $\pi \in D(n)$, we define $W'(\pi)$ to be the elements $(w_1, \dots, w_n) \in \mathbb{N}_0^n$ such that for every $i \in c(\pi)$, $w_i > w_{j(i)}$, where $j(i)$ is the index of the column containing the horizontal step preceding the i -th vertical step of π .

Definition 6.6. The set of pairs (π, dc) with $\pi \in D(n)$, $dc \subseteq c(\pi)$ and $|dc| = k$ will be denoted $D'(n)^{\bullet k}$. The indices in dc will be referred to as *decorated corners*: we decorate the i -th vertical step of π with a \bullet for all $i \in dc$.

The set of triples (π, dc, w') with $(\pi, dc) \in D'(n)^{\bullet k}$ and $w' \in W'(\pi)$ will be denoted by $LD'(n)^{\bullet k}$. In this set, we represent w' inside the squares containing the main diagonal $x = y$, starting from the bottom. See Figure ?? on the right for an example.

The following result (without decorations) first appeared in [?Haglund_Loehr_conj_Hilbert] (see also [?Haglund-Book-2008, Chapter 5]). We sketch its proof for completeness.

Proposition 6.7. *There exists a bijection*

$$\zeta : LD(n)^{*k} \rightarrow LD'(n)^{\bullet k}.$$

Sketch of the proof. Take $(\pi, dr, w) \in LD(n)^{*k}$. We want to define $\zeta(\pi, dr, w) := (\pi', dc, w') \in LD'(n)^{\bullet k}$. Naturally, we set $\pi' = \zeta_0(\pi)$. The decorated corners dc are given by the bijection described in Proposition ?. Lastly, we set w' to be the *dinv* reading word of w . We claim that $w' \in W'(\pi)$. Indeed, $w \in \mathbb{N}^n$ is an element of $W(\pi)$ if and only if for every $i \in r(\pi)$, $w_{i-1} < w_i$. These rises correspond to the corners of $\zeta_0(\pi)$ by the bijection of Proposition ?, and the condition on the labels translates into the condition that the *dinv* reading word of π is in $W'(\pi)$: indeed, when representing w' in the squares of the main diagonal, we ensure that the label contained in the column (respectively row) of a horizontal step (respectively vertical step) s is the label of the step of π encountered in the construction of $\zeta_0(\pi)$ at the moment of drawing s . It follows that a corner c of $\zeta(\pi, dr, w) = (\pi', dc, w')$ is constructed by reading

the two consecutive vertical steps of π whose labels are the ones contained in the column and the row of the horizontal and vertical step of c , respectively.

The previous argument works also backwards, ensuring the bijectivity of ζ .

See Example ?? for an illustration of this map. \square

We now extend the statistics on (labelled) Dyck paths to the decorated setting.

Definition 6.8. Take $(\pi, dc) \in D'(n)^{\bullet k}$ or $(\pi, dc, w') \in LD'(n)^{\bullet k}$ and let $b_1(\pi) \cdots b_n(\pi)$ be the bounce word of π . Then we set

$$\text{bounce}(\pi, dc) = \text{bounce}(\pi, dc, w') := \sum_{i \notin dc} b_i(\pi).$$

Definition 6.9. Consider $(\pi, dc, w') \in LD'(n)^{\bullet k}$. An *inversion* of (π, w') is a pair (i, j) with $i < j$ and $w'_i < w'_j$ such that the square whose upper right corner is the intersection of the lines $x = i$ and $y = j$ (i.e. the square lying in the same column as the label w'_i and the same row as w'_j) lies under the path π . We set $\text{inv}(\pi, w') = \text{inv}(\pi, dc, w')$ to be the total number of inversions of (π, w') (the inv statistic is also called area' in the literature).

Proposition 6.10. The ζ map of Proposition ?? is such that for $(\pi, dr, w) \in LD(n)^{*k}$

$$\text{dinv}((\pi, dr, w)) = \text{inv}(\zeta(\pi, dr, w))$$

$$\text{area}((\pi, dr, w)) = \text{bounce}(\zeta(\pi, dr, w)).$$

Proof. For $k = 0$, this result is exactly Remark 2.3 in [?Carlsson-Mellit-ShuffleConj-2015]. For $k > 0$, it is enough to show that the number of squares in a row containing a decorated rise is equal to the label attached to the step of the bounce path that is in the same row of the vertical step that is part of the corresponding valley. But this is clear as it holds by construction for any vertical step of the preimage (not only for decorated rises). \square

Example 6.11. The left path of Figure ??, is an element of $LD(8)^{*3}$ with area word $(0, 1, 2, 2, 3, 1, 0, 1)$, $dr = \{2, 5, 8\}$ and $w = (2, 4, 5, 5, 6, 5, 1, 3)$. It follows that its area is 5. Its primary inversions $\{(2, 6)\}$, its secondary inversions $\{(2, 7), (6, 7), (3, 8), (4, 8)\}$, so its dinv equals 5.

The path on the right is an element of $LD'(8)^{\bullet 3}$ and the image by ζ of the path on the left. It has $w' = (2, 1, 4, 5, 3, 5, 5, 6)$. Its bounce word is 00111223 and $dc = \{3, 5, 8\}$, so its bounce is 5. Its inversions are $\{(2, 3), (2, 4), (3, 4), (5, 6), (5, 7)\}$ so its inv equals 5.

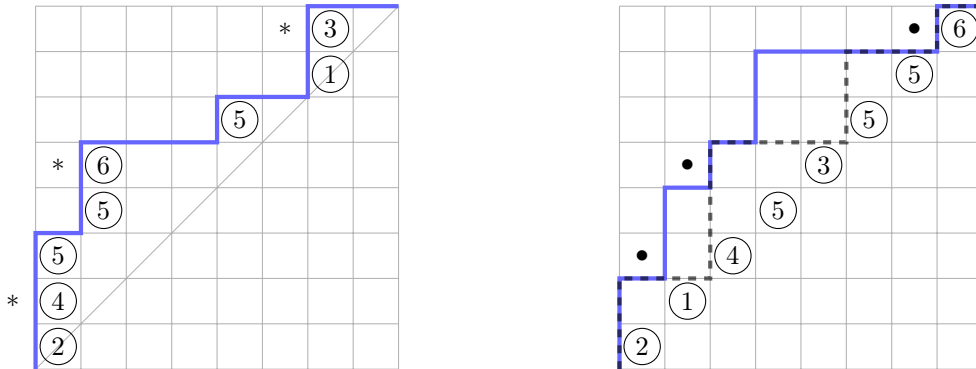


FIGURE 5. An element in $LD(n)^{*k}$ (left) and its image by the zeta map (right).

In order to complete our translation, we need to refine our sets according to the compositions.

Definition 6.12. For $P \in D'(n)^{\bullet k} \cup LD'(n)^{\bullet k}$, let $\text{dcomp}'(P) := \text{dcomp}(\zeta^{-1}(P))$. We will see in Lemma ?? a way to compute $\text{dcomp}'(P)$ directly on P .

Given a composition $\alpha \vDash n - k$, we set

$$D(\alpha)^{*k} := \{P \in D(n)^{*k} \mid \text{dcomp}(P) = \alpha\}$$

$$D'(\alpha)^{\bullet k} := \{P \in D'(n)^{\bullet k} \mid \text{dcomp}'(P) = \alpha\}$$

$$LD(\alpha)^{*k} := \{P \in LD(n)^{*k} \mid \text{dcomp}(P) = \alpha\}$$

$$LD'(\alpha)^{\bullet k} := \{P \in LD'(n)^{\bullet k} \mid \text{dcomp}'(P) = \alpha\}.$$

Remark 6.13. Observe that by definition of dcomp' , if $(\pi, dc) \in \mathcal{D}'(\alpha)^{\bullet k}$, then π starts with $\ell := \ell(\alpha)$ many north steps followed by an east step, i.e. π is of the form $\pi = N^\ell \tilde{\pi}$ for a unique path $\tilde{\pi}$ from $(0, \ell)$ to $(|\alpha| + k, |\alpha| + k)$ that starts with an east step and stays weakly above the diagonal $x = y$.

We define the q, t, \underline{x} -enumerator of this set as

$$\text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k} := \sum_{(\pi, dc, w) \in \text{LD}'(\alpha)^{\bullet k}} t^{\text{bounce}(\pi, dc)} q^{\text{inv}(\pi, w)} x^w,$$

where $x^w = \prod_i x_{w_i}$.

From the preceding discussion, the following identity is now clear:

$$(41) \quad \text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k} = \sum_{\substack{P \in \text{LD}(0, n)^{\bullet k} \\ \text{dcomp}(P) = \alpha}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P,$$

where the right hand side is precisely what appears in our compositional Delta conjecture. This completes our translation. Now we indicate a decomposition of these enumerators.

The following definitions and statements are from [?Carlsson-Mellit-ShuffleConj-2015].

Definition 6.14. For $\pi \in \mathcal{D}(n)$, we define the *zero-weight characteristic function* of π as

$$\bar{\chi}(\pi) := \sum_{w \in \mathcal{W}'(\pi)} q^{\text{inv}(\pi, w)} x^w.$$

The zero-weight characteristic function is a special case of a more general weighted characteristic function $\chi(\pi, wt)$ (see [?Carlsson-Mellit-ShuffleConj-2015, Subsection 3.2]) where $wt: c(\pi) \rightarrow \mathbb{Q}(q, t)$ is any function: it corresponds to the case $wt = 0$. It is proved in [?Carlsson-Mellit-ShuffleConj-2015, Proposition 3.7] that the $\chi(\pi, wt)$'s are symmetric functions.

The following proposition follows immediately from the definitions.

Proposition 6.15. *We have*

$$(42) \quad \text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k} = \sum_{(\pi, dc, w) \in \text{LD}'(\alpha)^{\bullet k}} t^{\text{bounce}(\pi, dc)} q^{\text{inv}(\pi, w)} x^w = \sum_{(\pi, dc) \in \mathcal{D}'(\alpha)^{\bullet k}} t^{\text{bounce}(\pi, dc)} \bar{\chi}(\pi).$$

6.2.2. Dyck path algebra operators. Again following [?Carlsson-Mellit-ShuffleConj-2015], we now introduce the operators of the *Dyck path algebra* in order to give an expression of $\text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k}$ in terms of them.

Given a polynomial P depending on variables u, v , define the operator Υ_{uv} as

$$(\Upsilon_{uv}P)(u, v) := \frac{(q-1)vP(u, v) + (v-qu)P(v, u)}{v-u}$$

In [?Carlsson-Mellit-ShuffleConj-2015] this operator is called Δ_{uv} , but we changed the notation in order to avoid confusion with the Δ_f operator defined on Λ .

Definition 6.16 ([?Carlsson-Mellit-ShuffleConj-2015, Definition 4.2]). For $k \in \mathbb{N}$, define $V_k := \Lambda[y_1, \dots, y_k] = \Lambda \otimes \mathbb{Q}[y_1, \dots, y_k]$. Let

$$T_i := \Upsilon_{y_i y_{i+1}}: V_k \rightarrow V_k \text{ for } 1 \leq i \leq k-1.$$

We define the operators $d_+: V_k \rightarrow V_{k+1}$ and $d_-: V_k \rightarrow V_{k-1}$: for $F[X] \in V_k$

$$\begin{aligned} (d_+F)[X] &:= T_1 T_2 \cdots T_k (F[X + (q-1)y_{k+1}]) \\ (d_-F)[X] &:= -F[X - (q-1)y_k] \sum_{i \geq 0} (-1/y_k)^i e_i[X] \Big|_{y_k-1}. \end{aligned}$$

The idea is to use these operators to get the zero-weight characteristic functions $\bar{\chi}(\pi)$. In fact for our purposes it will be convenient to define the corresponding operators for *partial Dyck paths*.

Definition 6.17. Let $\text{ED}^\ell(n)$ be the set of paths from $(0, \ell)$ to (n, n) consisting of east or north unit steps, starting with an east step, and staying weakly above the diagonal $x = y$, together with their translates by vectors of the form (v, v) with $v \in \mathbb{N}$.

Set $\text{ED}^\ell = \bigsqcup_n \text{ED}^\ell(n)$ and $\text{ED} = \bigsqcup_\ell \text{ED}^\ell$, and let $d: \text{ED} \rightarrow V$ be defined as

$$d(\emptyset) = 1, \quad d(E\tilde{\pi}) = d_+d(\tilde{\pi}), \quad d(EN^i\tilde{\pi}) = \frac{1}{q-1} [d_-, d_+] d_-^{-1} d(\tilde{\pi})$$

for $\tilde{\pi} \in \text{ED}$, where $[d_-, d_+] = d_-d_+ - d_+d_-$ is the usual Lie bracket.

Definition 6.18. Given a Dyck path $\pi \in D(n)$, let $\tilde{\pi} \in \text{ED}$ be the unique element such that $\pi = N^\ell \tilde{\pi}$ for some ℓ . Then we define $d(\pi) := d_-^\ell d(\tilde{\pi})$.

We have the following fundamental theorem.

Theorem 6.19 ([?Carlsson-Mellit-ShuffleConj-2015, Corollary 4.6]). *For any Dyck path π ,*

$$(43) \quad \overline{\chi}(\pi) = d(\pi).$$

The following corollary is now immediate.

Corollary 6.20. *We have*

$$(44) \quad \text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k} = \sum_{(\pi, dc) \in \text{D}'(\alpha)^{\bullet k}} t^{\text{bounce}(\pi, dc)} d(\pi).$$

Proof. Just combine (??) with (??). □

Remark 6.21. Observe that, from (??), the Remark ??, and the definition of the function d , for $\ell := \ell(\alpha)$ we have

$$(45) \quad \begin{aligned} \text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k} &= \sum_{(\pi, dc) \in \text{D}'(\alpha)^{\bullet k}} t^{\text{bounce}(\pi, dc)} d(\pi) \\ &= \sum_{\substack{(\pi, dc) \in \text{D}'(\alpha)^{\bullet k} \\ \pi = N^\ell \tilde{\pi}: \tilde{\pi} \in \text{ED}^\ell}} t^{\text{bounce}(\pi, dc)} d(N^\ell \tilde{\pi}) \\ &= d_-^\ell \sum_{\substack{(\pi, dc) \in \text{D}'(\alpha)^{\bullet k} \\ \pi = N^\ell \tilde{\pi}: \tilde{\pi} \in \text{ED}^\ell}} t^{\text{bounce}(\pi, dc)} d(\tilde{\pi}) \end{aligned}$$

We need one more definition: given two compositions $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_s)$, we define their *concatenation* as $\alpha\beta := (\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s)$, which is obviously also a composition.

We are now ready to express $\text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k}$ in terms of the operators of the Dyck path algebra. This is the main theorem of this section, and it extends [?Carlsson-Mellit-ShuffleConj-2015, Theorem 4.1] to the decorated setting.

Theorem 6.22. *If α is a composition of length ℓ , then we have*

$$(46) \quad \text{LD}'_{q,t,\underline{x}}(\alpha)^{\bullet k} = d_-^\ell M_\alpha^{*k}$$

where $M_\alpha^{*k} \in V_\ell$ is defined by the recursive relations

$$(47) \quad M_{(1)\alpha}^{*k} = d_+ M_\alpha^{*k} + \frac{1}{q-1} [d_-, d_+] M_{\alpha(1)}^{*k-1},$$

and for $a > 1$

$$(48) \quad M_{(a)\alpha}^{*k} = \frac{t^{a-1}}{q-1} [d_-, d_+] \left(\sum_{\beta=a-1} d_-^{\ell(\beta)-1} M_{\alpha\beta}^{*k} + \sum_{\beta=a} d_-^{\ell(\beta)-1} M_{\alpha\beta}^{*k-1} \right),$$

with initial conditions $M_\emptyset^{*k} = \delta_{k,0}$.

In order to prove this theorem, we need to define some combinatorial maps, and prove some lemmas concerning them.

6.2.3. Combinatorial recursion.

Definition 6.23. We define $\psi: D(n)^{*k} \rightarrow D(n-1)^{*k} \sqcup D(n-1)^{*k-1}$ as follows: given $(\pi, dr) \in D(n)^{*k}$ take the portion of π between the first two touching points (or the whole path if there is only one touching point), remove its first (north) step and its last (east) step, and attach it to the end of the path. If the first rise was decorated we remove the decoration since it is no longer a rise. See Figure ??.

This map is linked to a family of maps, all of which are essentially its right inverses.

For a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ and an integer $0 \leq r \leq \ell$ we set

$$\alpha^r := \left(\left(1 + \sum_{i>r} \alpha_i \right), \alpha_1, \alpha_2, \dots, \alpha_r \right), \quad \alpha^{r,*} = \alpha^{r,\bullet} := \left(\left(\sum_{i>r} \alpha_i \right), \alpha_1, \alpha_2, \dots, \alpha_r \right).$$

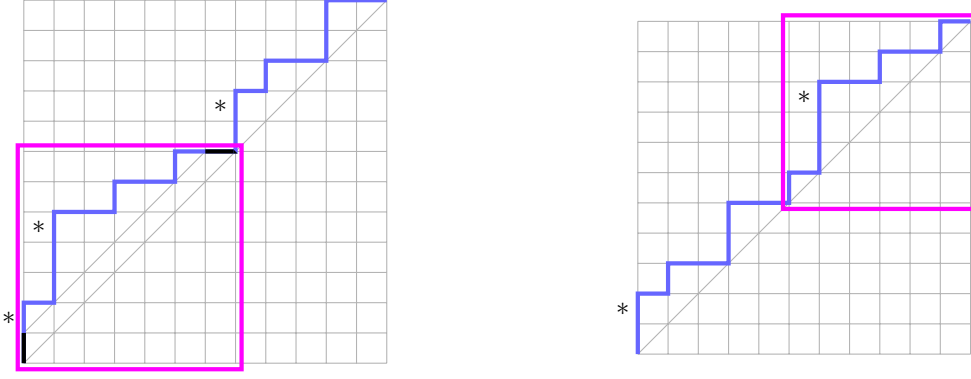


FIGURE 6. A path $(\pi, dr) \in D((5, 2, 2))^{*3}$ and its image $\psi(\pi, dr) \in D((2, 2, 1, 3, 1))^{*2}$. The first (north) and the last (east) steps (in black) in the highlighted section of the path are removed, and then the whole section is moved to the end. Since the first step of the section does not form a rise in the image, the corresponding decoration is removed.

We define two similar maps:

$$\begin{aligned}\psi_r &: D(\alpha)^{*k} \rightarrow D(\alpha^r)^{*k} \\ \psi_r^* &: D(\alpha)^{*k} \rightarrow D(\alpha^{r,*})^{*k+1}.\end{aligned}$$

Given $(\pi, dr) \in D(\alpha)^{*k}$ and $0 \leq r \leq \ell(\alpha)$, call π_1 and π_2 the portions of π below and above its $(r+1)$ -th touching point, respectively, if $r \neq \ell(\alpha)$, while $\pi_1 = \pi$ and $\pi_2 = \emptyset$ if $r = \ell(\alpha)$. Notice that if $\pi_2 \neq \emptyset$ then it necessarily starts with a north step. To define $\psi_r(\pi, dr) = (\pi', dr')$ we set

$$\pi' := N\pi_2 E\pi_1$$

i.e. the path that starts at $(0,0)$ with a north step, followed by π_2 , an east step and finally π_1 . We use the same definition for $\psi_r^*(\pi, dr)$. For the decorations, we keep the decorations on the rises in the same place, relative to π_1 and π_2 . When $\pi_2 \neq \emptyset$, i.e. $r \neq \ell(\alpha)$, π_2 starts with a north step, and so π' must start with two north steps, so the second step of π' is a newly created rise, which we can choose to decorate or not. This choice is the difference between ψ_r and ψ_r^* : for the former we do not decorate the new rise while for the latter we do. It is clear from the definitions that $\text{dcomp}(\psi_r(\pi, dr)) = \alpha^r$ and $\text{dcomp}(\psi_r^*(\pi, dr)) = \alpha^{r,*}$.

Definition 6.24. We define $\gamma : D'(n)^{\bullet k} \rightarrow D'(n-1)^{\bullet k} \sqcup D'(n-1)^{\bullet k-1}$ which takes $(\pi, dc) \in D'(n)^{\bullet k}$ and deletes the first NE sequence of the path and if this E step was part of a decorated corner, it removes its decoration since it is no longer a corner. See Figure ??.

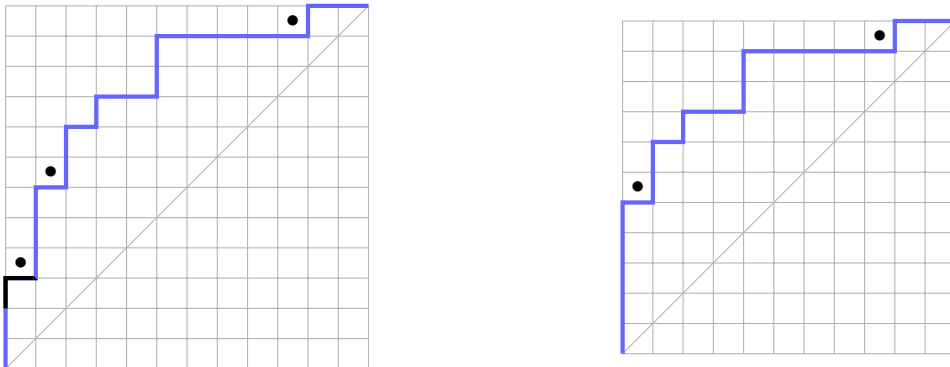


FIGURE 7. A path $(\pi, dc) \in D'((5, 2, 2))^{\bullet 3}$ and its image $\gamma(\pi, dc) \in D'((2, 2, 1, 4))^{\bullet 2}$. The first NE pair (in black) is removed. Since the removed east step was part of a decorated valley, the corresponding decoration is removed. These two paths are actually the images of the paths in Figure ?? via ζ .

Again, we have a family of right inverses.

$$\begin{aligned}\gamma_r &: D'(\alpha)^{\bullet k} \rightarrow D'(\alpha^r)^{\bullet k} \\ \gamma_r^\bullet &: D'(\alpha)^{\bullet k} \rightarrow D'(\alpha^{r,\bullet})^{\bullet k+1}\end{aligned}$$

Take $(\pi, dc) \in D'(\alpha)^{\bullet k}$. Set ℓ to be the number such that π starts from the bottom with ℓ north steps followed by an east step. Define $\tilde{\pi}$ to be the portion of π following its ℓ first vertical steps. Notice that by definition of the map ζ and $d\text{comp}'$, we have $\ell = \ell(\alpha)$. Set $dc^{+j} := \{i + j \mid i \in dc\}$. For $0 \leq r \leq \ell$, we define

$$\gamma_r(\pi, dc) := (N^{r+1}EN^{\ell-r}\tilde{\pi}, dc^{+1}),$$

i.e. we add one NE sequence after the first r North steps and we keep the decorated corners as they are, relative to π .

If $r \neq \ell$ we also have a map $\gamma_r^\bullet: D'(\alpha)^{\bullet k} \rightarrow D'(\alpha^{r,\bullet})^{\bullet k+1}$ defined as

$$\gamma_r^\bullet(\pi, dc) := (N^{r+1}EN^{\ell-r}\tilde{\pi}, \{r+2\} \cup dc^{+1}),$$

i.e. the path is defined in the same way as before, and we decorate the only new corner.

It is not immediately clear that for $(\pi, dc) \in D'(\alpha)^{\bullet k}$ we have $\gamma_r(\pi, dc) \in D'(\alpha^r)^{\bullet k}$ and $\gamma_r^\bullet(\pi, dc) \in D'(\alpha^{r,\bullet})^{\bullet k}$, but it follows from the following lemma.

Lemma 6.25.

(i) For all $\alpha \vDash n - k$ and $0 \leq r \leq \ell(\alpha)$ we have

$$\begin{aligned}\psi \circ \psi_r &= \text{Id}|_{D(\alpha)^{*k}} & \text{and if } r \neq \ell(\alpha) \text{ then} & & \psi \circ \psi_r^* &= \text{Id}|_{D(\alpha)^{*k}} \\ \gamma \circ \gamma_r &= \text{Id}|_{D'(\alpha)^{\bullet k}} & & & \gamma \circ \gamma_r^\bullet &= \text{Id}|_{D'(\alpha)^{\bullet k}}\end{aligned}$$

(ii) We have $\zeta \circ \psi = \gamma \circ \zeta : D(n)^{*k} \rightarrow D'(n-1)^{\bullet k} \sqcup D'(n-1)^{\bullet k-1}$.

(iii) The following diagrams commute

$$\begin{array}{ccc} D(\alpha)^{*k} & \xrightarrow{\zeta} & D'(\alpha)^{\bullet k} \\ \psi_r \downarrow & & \downarrow \gamma_r \\ D(\alpha^r)^{*k} & \xrightarrow{\zeta} & D'(\alpha^r)^{\bullet k} \end{array} \qquad \begin{array}{ccc} D(\alpha)^{*k} & \xrightarrow{\zeta} & D'(\alpha)^{\bullet k} \\ \psi_r^* \downarrow & & \downarrow \gamma_r^\bullet \\ D(\alpha^{r,*})^{*k+1} & \xrightarrow{\zeta} & D'(\alpha^{r,\bullet})^{\bullet k+1}. \end{array}$$

Proof. For no decorations, this is exactly [Haglund-Xin_Lecture-Notes, Corollary 2.7]. The exact same argument generalizes to our case, keeping track of the decorations, so we omit it. \square

Lemma 6.26. Given $(\pi', dc) \in D'(\alpha)^{\bullet k}$ we have

$$\begin{aligned}\alpha_i &= \text{bounce}(\gamma_{i-1}(\pi, dc)) - \text{bounce}(\gamma_i(\pi, dc)), \\ &= \text{bounce}(\gamma_{i-1}^\bullet(\pi, dc)) - \text{bounce}(\gamma_i^\bullet(\pi, dc)).\end{aligned}$$

Proof. Consider $(\pi, dr) = \zeta^{-1}(\pi', dc) \in D(\alpha)^{*k}$. It is easy to see from the definition of ψ_r that

$$\text{area}(\psi_r(\pi, dr)) = \text{area}(\pi, dr) + \sum_{j>r} \alpha_j.$$

It follows that

$$\alpha_i = \text{area}(\psi_{i-1}(\pi, dr)) - \text{area}(\psi_i(\pi, dr)).$$

The result now follows from Lemma ?? (iii) and Proposition ?. This proves the first equality. The second one follows from the first one together with the obvious

$$\text{bounce}(\gamma_r(\pi, dc)) = \text{bounce}(\gamma_r^\bullet(\pi, dc)) + 1.$$

\square

Proposition 6.27. Let $a \in \mathbb{N} \setminus \{0\}$, and consider $(\pi, dc) \in D'((a)\alpha)^{\bullet k}$. Then either $\gamma(\pi, dc) \in D'(\alpha\beta)^{\bullet k}$ for some $\beta \vDash a - 1$ or $\gamma(\pi, dc) \in D'(\alpha\beta)^{\bullet k-1}$ for some $\beta \vDash a$.

Proof. Let $d\text{comp}'(\gamma(\pi, dc)) = \alpha'$. We have to prove that $\alpha'_i = \alpha_i$ for $1 \leq i < \ell(\alpha)$; in fact, if this is true, then necessarily $\alpha' = \alpha\beta$ for some $\beta \vDash a - 1$ if the first corner in π is decorated, or $\beta \vDash a$ if it is not (as the total size is the size of the composition plus the number of decorations, and applying γ decreases the size by exactly one unit).

Recall that

$$\alpha_i = \text{bounce}(\gamma_i(\pi, dc)) - \text{bounce}(\gamma_{i+1}(\pi, dc))$$

$$(50) \quad D'((a)\alpha)^{\bullet k} = \bigsqcup_{\beta \vDash a-1} \gamma_r(D'(\alpha\beta)^{\bullet k}) \sqcup \bigsqcup_{\beta \vDash a} \gamma_r^\bullet(D'(\alpha\beta)^{\bullet k-1}).$$

Indeed if $(\pi, dc) \in D'((a)\alpha)^{\bullet k}$ then, by Proposition ??, either $\gamma(\pi, dc) \in \bigsqcup_{\beta \vDash a-1} D'(\alpha\beta)^{\bullet k}$ or $\gamma(\pi, dc) \in \bigsqcup_{\beta \vDash a} D'(\alpha\beta)^{\bullet k-1}$. Since $r = \ell(\alpha)$, we have $(\pi, dc) = \gamma_r(\gamma(\pi, dc))$ in the former case and $(\pi, dc) = \gamma_r^\bullet(\gamma(\pi, dc))$ in the latter and so the inclusion “ \subseteq ” holds. For the other inclusion, if $(\pi, dc) \in D'(\alpha\beta)^{\bullet k}$, for some $\beta \vDash a-1$, then $\gamma_r(\pi, dc) \in D'((a)\alpha)^{\bullet k}$ since $(\alpha\beta)^r = (1 + \sum_i \beta_i, \alpha_1, \dots, \alpha_r)$. Similarly, if $(\pi, dc) \in D'(\alpha\beta)^{\bullet k-1}$, for some $\beta \vDash a$, then $\gamma_r^\bullet(\pi, dc) \in D'((a)\alpha)^{\bullet k}$ since $(\alpha\beta)^{r, \bullet} = (\sum_i \beta_i, \alpha_1, \dots, \alpha_r)$.

So by the induction hypothesis we know that

$$(51) \quad M_{\alpha\beta}^{*k} = \sum_{\substack{(\pi, dc) \in D'(\alpha\beta)^{\bullet k} \\ \pi = N^{\ell(\alpha\beta)} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}}} t^{\text{bounce}(\pi, dc)} d(\tilde{\pi}) \quad \text{for } \beta \vDash a-1, \text{ and}$$

$$(52) \quad M_{\alpha\beta}^{*k-1} = \sum_{\substack{(\pi, dc) \in D'(\alpha\beta)^{\bullet k-1} \\ \pi = N^{\ell(\alpha\beta)} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}}} t^{\text{bounce}(\pi, dc)} d(\tilde{\pi}) \quad \text{for } \beta \vDash a.$$

Consider $(\pi, dc) \in D'(\alpha\beta)^{\bullet k}$. Provided that $a > 1$, we get

$$(53) \quad \gamma_r(\pi, dc) = (N^{\ell(\alpha)+1} E N N^{\ell(\beta)-1} \tilde{\pi}, dc^{+1})$$

with $E N N^{\ell(\beta)-1} \tilde{\pi} \in \text{ED}^{\ell(\alpha)+1}$ and $\tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}$, so that

$$(54) \quad d(E N N^{\ell(\beta)-1} \tilde{\pi}) = \frac{[d_-, d_+]}{q-1} d_-^{\ell(\beta)-1} d(\tilde{\pi}).$$

Notice that if $(\pi, dc) \in D'(\alpha\beta)^{\bullet k-1}$ and we apply γ_r^\bullet , then we obtain the exact same identity.

If $a = 1$ instead, in the first case, i.e. when we apply γ_r , we have $\beta \vDash 0$, so

$$(55) \quad \gamma_r(\pi, dc) = (N^{\ell(\alpha)+1} E \tilde{\pi}, dc^{+1}),$$

with $E \tilde{\pi} \in \text{ED}^{\ell(\alpha)+1}$ and $\tilde{\pi} \in \text{ED}^{\ell(\alpha)}$, from which we get

$$(56) \quad d(E \tilde{\pi}) = d_+ d(\tilde{\pi})$$

In the other case, i.e. when we apply γ_r^\bullet , we have $\beta \vDash 1$ instead, and it works as before.

Now we look at what happens to the bounce. By Lemma ??, we have

$$\begin{aligned} \text{bounce}(\gamma_r(\pi, dc)) &= \text{bounce}(\gamma_{r-1}(\pi, dc)) - \alpha_r \\ &= \text{bounce}(\gamma_{r-2}(\pi, dc)) - \alpha_{r-1} - \alpha_r \\ &\vdots \\ &= \text{bounce}(\gamma_0(\pi, dc)) - \sum_{i=1}^r \alpha_i \\ &= \text{bounce}(\gamma_0(\pi, dc)) - |\alpha| \end{aligned}$$

The same identity holds when replacing γ_r with γ_r^\bullet and γ_0 with γ_0^\bullet .

Since $\text{bounce}(\gamma_0(\pi, dc)) = \text{bounce}(\pi, dc) + |\alpha\beta|$, we get for $(\pi, dc) \in D'(\alpha\beta)^{\bullet k}$

$$(57) \quad \text{bounce}(\gamma_r(\pi, dc)) = \text{bounce}(\pi, dc) + |\beta| = \text{bounce}(\pi, dc) + a - 1$$

Similarly, we have $\text{bounce}(\gamma_0^\bullet(\pi, dc)) = \text{bounce}(\pi, dc) + |\alpha\beta| - 1$ and so for $(\pi, dc) \in D'(\alpha\beta)^{\bullet k-1}$ we get

$$(58) \quad \text{bounce}(\gamma_r^\bullet(\pi, dc)) = \text{bounce}(\pi, dc) - 1 + |\beta| = \text{bounce}(\pi, dc) + a - 1.$$

We can conclude that when $a > 1$, by applying (??), we get

$$\begin{aligned}
M_{(a)\alpha}^{*k} &= \frac{t^{a-1}}{q-1} [d_-, d_+] \left(\sum_{\beta=a-1} d_-^{\ell(\beta)-1} M_{\alpha\beta}^{*k} + \sum_{\beta=a} d_-^{\ell(\beta)-1} M_{\alpha\beta}^{*k-1} \right) \\
(\text{using (??), (??)}) &= \sum_{\beta=a-1} \sum_{\substack{(\pi, dc) \in \mathcal{D}'(\alpha\beta)^{\bullet k} \\ \pi = N^{\ell(\alpha\beta)} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}}} t^{a-1} t^{\text{bounce}(\pi, dc)} \frac{[d_-, d_+]}{q-1} d_-^{\ell(\beta)-1} d(\tilde{\pi}) \\
&\quad + \sum_{\beta=a} \sum_{\substack{(\pi, dc) \in \mathcal{D}'(\alpha\beta)^{\bullet k-1} \\ \pi = N^{\ell(\alpha\beta)} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}}} t^{a-1} t^{\text{bounce}(\pi, dc)} \frac{[d_-, d_+]}{q-1} d_-^{\ell(\beta)-1} d(\tilde{\pi}) \\
(\text{using (??), (??) and (??)}) &= \sum_{\beta=a-1} \sum_{\substack{(\pi, dc) \in \mathcal{D}'(\alpha\beta)^{\bullet k} \\ \pi = N^{\ell(\alpha\beta)} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}}} t^{\text{bounce}(\gamma_r(\pi, dc))} d(EN N^{\ell(\beta)-1} \tilde{\pi}) \\
&\quad + \sum_{\beta=a} \sum_{\substack{(\pi, dc) \in \mathcal{D}'(\alpha\beta)^{\bullet k-1} \\ \pi = N^{\ell(\alpha\beta)} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha\beta)}}} t^{\text{bounce}(\gamma_r^{\bullet}(\pi, dc))} d(EN N^{\ell(\beta)-1} \tilde{\pi}) \\
(\text{using (??) and (??)}) &= \sum_{\substack{(\pi, dc) \in \mathcal{D}'((a)\alpha)^{\bullet k} \\ \pi = N^{\ell(\alpha)+1} \tilde{\pi}: \tilde{\pi} \in \text{ED}^{\ell(\alpha)+1}}} t^{\text{bounce}(\pi, dc)} d(\tilde{\pi})
\end{aligned}$$

which is precisely (??).

The argument for $a = 1$ is very similar but it uses (??), (??) and (??) instead of (??), (??) and (??). This completes the proof of Theorem ??.

6.2.5. Operator version of the compositional Delta conjecture. We can state our *operator version* of the compositional Delta conjecture.

Conjecture 6.29 (Operator Delta conjecture). *If α is a composition of length ℓ , then*

$$(59) \quad \Theta_k \nabla C_\alpha = d_-^\ell M_\alpha^{*k},$$

with M_α^{*k} defined as in Theorem ??.

The following proposition is an immediate consequence of Theorem ?? and (??).

Proposition 6.30. *The compositional Delta conjecture, i.e. Conjecture ??, is equivalent to the operator Delta conjecture, i.e. to Conjecture ??.*

7. ABOUT THE TOUCHING VERSION

7.1. Proof of the generalized shuffle conjecture. In this section we prove the *generalized shuffle conjecture*, i.e. the special case $k = 0$ of Conjecture ??, which is also Conjecture 7.5 in [?Haglund-Remmel-Wilson-2015].

Theorem 7.1 (Touching generalized shuffle). *Given $n, m, r \in \mathbb{N}$ with $n \geq r \geq 1$,*

$$(60) \quad \Delta_{h_m} \nabla E_{n,r} = \sum_{\substack{P \in \text{LD}(m,n)^{*0} \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

This theorem extends the shuffle theorem of Carlsson and Mellit [?Carlsson-Mellit-ShuffleConj-2015].

The rest of this subsection is devoted to the proof of this result. We start by introducing some convenient combinatorial objects.

Definition 7.2. An *augmented Dyck path* is a (partially) labelled Dyck path with labels in $\mathbb{N} \cup \{\infty\}$, such that all the labels ∞ lie on the main diagonal (so they are the only labels in their own columns), they cannot be immediately followed by a 0 label, and the leftmost label on the diagonal that is not a ∞ cannot be a 0. See Figure ?? for two examples.

We denote by $\text{LD}(m, n, s)$ the set of augmented Dyck paths of size $m + n + s$ with m labels 0 and s labels ∞ . We compute the statistics as follows: the area is not influenced by the labels. The definition of the dinv stays the same, using the following conventions: $\ell < \infty$ for each positive label ℓ , but $0 \not< \infty$ and $0 \not> \infty$. Finally, to compute the relevant monomials, we set $x_\infty = 1$. For example, the

augmented Dyck path to the right of Figure ?? has area 5, dinv 16, and the corresponding monomial equals $x_1x_2x_3x_4x_5x_6x_7$.

We have the following combinatorial identity.

Proposition 7.3. *Given $m, n, s \in \mathbb{N}$ with $n \geq 1$*

$$(61) \quad \sum_{P \in \text{LD}(m, n, s)} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P = \sum_{r=1}^n \begin{bmatrix} r+s \\ r \end{bmatrix}_q \sum_{\substack{P \in \text{LD}(m, n) \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

Proof. Given any path $P \in \text{LD}(m, n)$ with $\text{touch}(P) = r$, we can insert s labels ∞ on the diagonal, and the contribution to the dinv given by those labels is exactly $\begin{bmatrix} r+s \\ r \end{bmatrix}_q$: choose an interlacing between the s labels ∞ and the r nonzero labels on the diagonal; each time one of the latter precedes one of the formers, a unit of dinv is created. Summing over all the possible values of r we get the desired identity. \square

Applying nabla to (??), we get

$$(62) \quad \nabla e_n [X[s+1]_q] = \sum_{r=1}^n \begin{bmatrix} r+s \\ r \end{bmatrix}_q \nabla E_{n,r}.$$

Recall that Carlsson and Mellit in [Carlsson-Mellit-ShuffleConj-2015] showed in particular that

$$(63) \quad \nabla E_{n,r} = \sum_{\substack{P \in \text{LD}(0, n) \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

Combining all this with (??) we get immediately the following corollary.

Corollary 7.4. *For $n, s \in \mathbb{N}$ with $n \geq 1$, we have*

$$(64) \quad \nabla e_n [X[s+1]_q] = \sum_{P \in \text{LD}(0, n, s)} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

Our goal is to show the following more general theorem.

Theorem 7.5. *For $m, n, s \in \mathbb{N}$ with $n \geq 1$, we have*

$$(65) \quad \Delta_{h_m} \nabla e_n [X[s+1]_q] = \sum_{P \in \text{LD}(m, n, s)} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

On the symmetric function side, the key identity is provided by the following theorem, that we prove in Section ??.

Theorem 7.6. *For $n, j, s \in \mathbb{N}$ with $n \geq 1$, we have*

$$(66) \quad h_j^\perp \nabla e_n [X[s+1]_q] = \sum_{p=0}^j t^{j-p} \begin{bmatrix} s+p \\ p \end{bmatrix}_q \Delta_{h_{j-p}} \nabla e_{n-j} [X[s+p+1]_q].$$

Proof of Theorem ??. We proceed by induction on m . The base case $m = 0$ is exactly (??), which is proved. Now suppose that the thesis is true for $m < j$. To show that the thesis holds for j , we need a combinatorial interpretation of (??). Recall that, as by definition $\langle h_j^\perp f, h_\mu \rangle = \langle f, h_j h_\mu \rangle$, a combinatorial interpretation of $h_j^\perp \nabla e_n [X[s+1]_q]$ is given by the q, t, \underline{x} -enumerator of the elements in $P \in \text{LD}(0, n, s)$ whose reading word is a $(j, \mu_1, \dots, \mu_{\ell(\mu)})$ -shuffle, where the j biggest labels (which we call *big labels*) do not contribute to the monomial (see [Haglund-Book-2008, Chapter 6] for the theory of shuffle).

Notice that each occurrence of a big label must be a peak, since all the ∞ labels lie on the diagonal. Suppose that exactly p of these labels lie on the diagonal, while $j-p$ do not. First we replace the p big labels on the diagonal by ∞ labels: this gives a contribution of $\begin{bmatrix} s+p \\ p \end{bmatrix}_q$ to the dinv depending on the interlacing of these p labels with the s many ∞ labels that were already on the diagonal. Then we “push in the peaks”, namely we move the vertical step in the rows containing a big label right after the following horizontal step, and we replace the label with a 0: see Figure ?? for an example.

It is easy to check that this operation decreases the area by $j-p$ (taken into account by the factor t^{j-p}) and does not change the dinv (as it sends the primary inversions involving these labels to secondary inversions and vice versa). This way we get a path in $\text{LD}(j-p, n, s+p)$. By inductive hypothesis, for

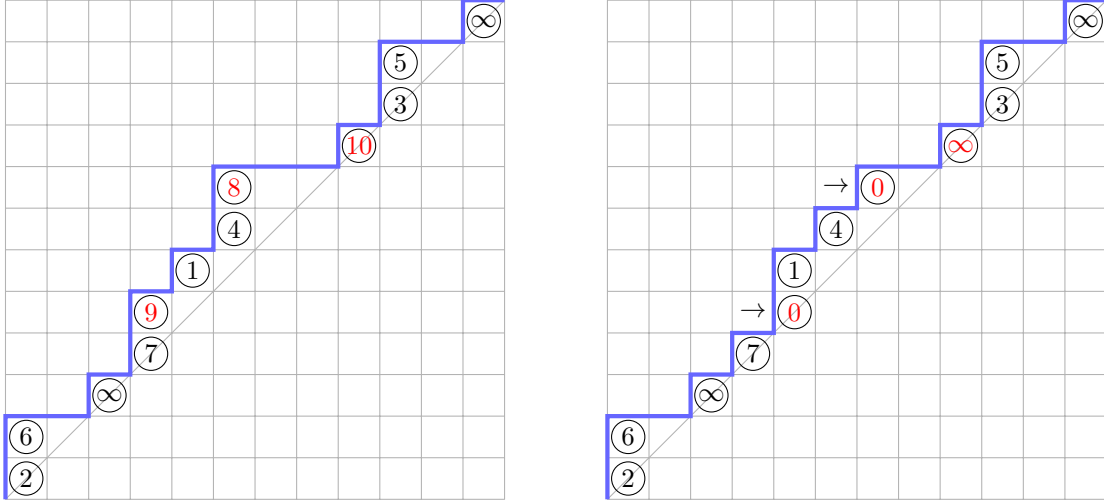


FIGURE 9. The pushing algorithm for a path of size 12 with 3 big labels and 2 ∞ labels. On the left, the big labels appear in red.

$p > 0$ we have that $\Delta_{h_{j-p}} \nabla e_{n-j} [X[s+p+1]_q]$ is the q, t, \underline{x} -enumerator of this set. Replacing this in (??), and taking into account the combinatorial interpretation of the left hand side, we get

$$\begin{aligned} \sum_{p=0}^j t^{j-p} \begin{bmatrix} s+p \\ p \end{bmatrix}_q \text{LD}_{q,t,\underline{x}}(j-p, n, s+p) &= \\ &= t^j \Delta_{h_j} \nabla e_n [X[s+1]_q] + \sum_{p=1}^j t^{j-p} \begin{bmatrix} s+p \\ p \end{bmatrix}_q \text{LD}_{q,t,\underline{x}}(j-p, n, s+p) \end{aligned}$$

which simplifies to

$$\text{LD}_{q,t,\underline{x}}(j, n, s) = \Delta_{h_j} \nabla e_n [X[s+1]_q]$$

which is what we wanted to prove. \square

Applying Δ_{h_m} to (??) we get the symmetric function counter part of (??), i.e.

$$(67) \quad \Delta_{h_m} \nabla e_n [X[s+1]_q] = \sum_{r=1}^n \begin{bmatrix} r+s \\ r \end{bmatrix}_q \Delta_{h_m} \nabla E_{n,r}.$$

Now to see that (??) implies our touching generalized shuffle conjecture, observe that (??) for $s = 0, 1, \dots, n-1$ gives a system of linear equations relating the symmetric functions

$$\{\Delta_{h_m} \nabla e_n [X[s+1]_q]\}_{0 \leq s \leq n-1}$$

with $\{\Delta_{h_m} \nabla E_{n,r}\}_{1 \leq r \leq n}$, whose coefficient matrix is $\left\| \begin{bmatrix} r+s \\ r \end{bmatrix}_q \right\|_{0 < r \leq n, 0 \leq s < n} \in M_{n \times n}(\mathbb{Q}(q))$. Clearly the determinant of this matrix is in $\mathbb{Z}[q]$, and in fact setting $q = 1$ in it we get $\det \left\| \begin{bmatrix} r+s \\ r \end{bmatrix} \right\|_{0 < r \leq n, 0 \leq s < n}$. The following lemma shows that this determinant is 1.

Lemma 7.7. For $n \geq 1$,

$$\det \left\| \begin{bmatrix} r+s \\ r \end{bmatrix} \right\|_{0 < r \leq n, 0 \leq s < n} = 1.$$

Proof. Set $A_n := \left\| \begin{bmatrix} r+s \\ r \end{bmatrix} \right\|_{0 < r \leq n, 0 \leq s < n}$, and call $R(i, j)$ (resp. $C(i, j)$) the row (resp. column) operator that replaces the i -th row (resp. column) of a matrix with itself minus the j -th one. Notice that these operators do not change the determinant.

Let us consider the matrix

$$B_n := C(2, 1)C(3, 2) \cdots C(n-1, n-2)C(n, n-1)R(2, 1)R(3, 2) \cdots R(n-1, n-2)R(n, n-1)A_n.$$

By construction, it is immediate to check that in the first column of B_n all coefficients are 0, except the top entry, which is equal to 1.

On the other hand, using the well-known identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we can check that first for $r' > 0$ the row operators changed the coefficient $\binom{r'+s}{r'}$ into $\binom{r'+s-1}{r'}$, and then for $s > 0$ the column operators changed the latter into $\binom{r'-1+s-1}{r'-1}$. So the submatrix of B_n corresponding to the indices $1 \leq r', s > 0$ is actually equal to A_{n-1} . By induction on n (the case $n = 1$ being trivial), we conclude that $\det A_n = \det B_n = \det A_{n-1} = 1$, as we wanted. \square

From this it follows that the system (??) for $s = 0, 1, \dots, n-1$ has a unique solution, so that the $\{\Delta_{h_m} \nabla E_{n,r}\}_{1 \leq r \leq n}$ are uniquely determined by the $\{\Delta_{h_m} \nabla e_n[X[s+1]_q]\}_{0 \leq s \leq n-1}$ and by the system.

Now the combinatorial counterpart of (??) is (??), so that also the combinatorial interpretations of $\{\Delta_{h_m} \nabla e_n[X[s+1]_q]\}_{0 \leq s \leq n-1}$ and $\{\Delta_{h_m} \nabla E_{n,r}\}_{1 \leq r \leq n}$ are in the same relation. From this we can conclude that the proof of the combinatorial interpretation for $\Delta_{h_m} \nabla e_n[X[s+1]_q]$ implies the one for $\Delta_{h_m} \nabla E_{n,r}$, which is what we wanted. This completes the proof of Theorem ??.

7.2. Proof of the generalized square conjecture. In this section we show that the touching generalized shuffle conjecture (Theorem ??) implies the touching generalized square conjecture:

Theorem 7.8 (Touching generalized square). *Given $n, m, r \in \mathbb{N}$ with $n \geq r \geq 1$,*

$$(68) \quad \frac{[n]_q}{[r]_q} \Delta_{h_m} \nabla E_{n,r} = \sum_{\substack{P \in \text{LSQ}(m,n)^{*0} \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

The rest of this subsection is dedicated to the proof of this theorem.

We begin by giving equivalent well-known formulations of Theorem ?? and Theorem ?. In order to do this, we need some definitions.

Definition 7.9. A *partial preference function* is a partially labelled square path in $\text{LSQ}(m, n)^{*0}$ whose nonzero labels are exactly the numbers $1, 2, \dots, n$. The subset of these paths is denoted by $\text{Pref}(m, n)$.

A *partial parking function* is a partial preference function whose shift is 0. The set of such paths is denoted by $\text{Park}(m, n)$.

Definition 7.10. For every $S \subseteq \{1, 2, \dots, n-1\}$, let $Q_{S,n}$ denote the *Gessel fundamental quasisymmetric function* of degree n indexed by S , i.e.

$$(69) \quad Q_{S,n} := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Definition 7.11. The *descent set* of a given permutation τ is the set of indices i such that $\tau_i > \tau_{i+1}$, denoted by $\text{Des}(\tau)$. We define the *inverse descent set of a permutation τ* as

$$\text{ides}(\tau) := \text{Des}(\tau^{-1}).$$

Take $P \in \text{Pref}(m, n)$. We set $r(P)$ to be the reverse reading word of P (see Definition ??). The *inverse descent set of a preference function P* is defined by

$$\text{ides}(P) := \text{ides}(r(P))$$

Theorem ?? and Theorem ?? are equivalent to the following equations, respectively

$$(70) \quad \Delta_{h_m} \nabla E_{n,r} = \sum_{\substack{P \in \text{Park}(m,n) \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P),n}$$

$$(71) \quad \frac{[n]_q}{[r]_q} \Delta_{h_m} \nabla E_{n,r} = \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P),n}.$$

The argument is essentially identical to the one given for the shuffle conjecture in [?Haglund-Book-2008, Chapter 6], so we omit it.

We will show here that the first of these equations implies the second one. We will use the same strategy that Sergel used in [?Leven-2016], where she proved that the shuffle conjecture implies the square conjecture. Notice that our notation will be slightly different from the one in [?Leven-2016].

Definition 7.12. Take $P \in \text{Pref}(m, n)$ with shift s and set $l := \max\{a_i(P) \mid 1 \leq i \leq m + n\}$. For $i = l, l-1, \dots, -s$ define ρ_i to be the word formed by the labels of P contained in the diagonal $y = x + i$, in increasing order. Define the *diagonal word* of P as

$$\text{diagword}(P) := \rho_l \rho_{l-1} \cdots \rho_0 \rho_{-1} \cdots \rho_{-s}.$$

It is clear from the definition of a preference function that the ρ_i are the maximal substrings of consecutive weakly increasing numbers of $\text{diagword}(P)$: we will call them *runs* of $\text{diagword}(P)$. We set $|\rho_i|$ to be the number of elements in ρ_i . A run ρ_i will be referred to as *positive*, *zero* or *negative*, referring to the sign of its index i .

Remark 7.13. Notice that a run of a diagonal word never consists of only zeros, as the first label after a touching point must be nonzero (see Definition ??).

Definition 7.14. Let $P \in \text{Pref}(m, n)$ be a preference function with shift s and diagonal word $\tau = \text{diagword}(P) = \rho_l \rho_{l-1} \cdots \rho_0 \rho_{-1} \cdots \rho_{-s}$, where the ρ_i 's are the runs of τ . Let c be a nonzero element of τ . We define the *schedule number* $w^s(c)$ of c as follows:

- (1) if c is in a positive run, then $w^s(c)$ equals the number of elements bigger than c in its run plus the number of elements smaller than c in the next run;
- (2) if c is the zero run, then $w^s(c)$ equals 1 plus the number of elements bigger than c in its own run;
- (3) if c is in a negative run, then $w^s(c)$ equals the number of elements smaller than c in its run plus the number of elements bigger than c in the previous run.

Note that these schedule numbers do depend on the shift s to determine which is the zero run.

Definition 7.15. For a $P \in \text{Pref}(m, n)$ whose diagonal word is τ we define its *reduced diagonal word*, $\text{rdiagword}(P)$ or $\tilde{\tau}$ to be the word obtained from τ by deleting all the zeros.

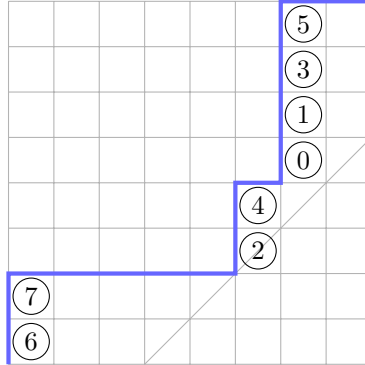


FIGURE 10. Example of an element in $\text{Pref}(1, 7)$.

Example 7.16. Consider the path P represented in Figure ???. Its shift is 3 and its diagonal word is 57 36 1 04 2. There is one positive run which is 57, the zero run is 36 and there are three negative runs: 1, 04 and 2. The schedule numbers are

$$\begin{array}{ccccc} w^3(5) = 2 & w^3(3) = 2 & w^3(1) = 2 & w^3(4) = 1 & w^3(2) = 1 \\ w^3(7) = 2 & w^3(6) = 1 & & & \end{array}$$

Its reduced diagonal word is 57 36 1 4 2

The following theorem extends [Leven-2016, Theorem 2.5].

Theorem 7.17. Let $S \subseteq \text{Pref}(m, n)$ be the set of preference functions P such that

$$\text{shift}(P) = s \quad \text{and} \quad \text{diagword}(P) = \tau = \rho_l \cdots \rho_0 \cdots \rho_{-s},$$

where the ρ_i 's are the runs of τ . Let $\text{rdiagword}(P) = \tilde{\tau} = \pi_l \cdots \pi_0 \cdots \pi_{-s}$ where we obtain π_i by deleting the zeros from ρ_i . Set $r_i = |\rho_i|$ and $p_i = |\pi_i|$. Then

$$\sum_{P \in S} q^{\text{dinv}(P)} t^{\text{area}(P)} = t^{\text{maj}(\tau)} q^{p_{-1} + \cdots + p_{-s}} \left(\prod_{c \in \tilde{\tau}} [w^s(c)]_q \right) \left(\prod_{i=0}^l \begin{bmatrix} r_i - 1 \\ r_i - p_i \end{bmatrix}_q \right) \times \\ \times \left(\prod_{i=-s}^{-1} \begin{bmatrix} p_{i+1} + r_i - p_i - 1 \\ r_i - p_i \end{bmatrix}_q \right).$$

Remark 7.18. Notice that the π_i are not necessarily the runs of $\tilde{\tau}$, as the last element of π_i might be smaller than the first element of π_{i-1} , if there were zeros in between in τ .

Proof. First of all, observe that the area of a path in S is given by $\text{maj}(\tau)$. Indeed, its area is clearly given by

$$0 \cdot r_{-s} + 1 \cdot r_{-s+1} + \cdots + (l + s) \cdot r_l.$$

Let us now consider the dinv . We obtain the formula by constructing every possible path in S , while keeping track of the dinv . See Figure ?? for an example.

We start with an empty path. For $i = 0, \dots, l$ we do the following.

- Reading ρ_i from *right to left*, we insert first its nonzero elements one by one, into the diagonal $y = x + i$ of the grid, in a way that each step defines a parking function. For each of these elements $c \neq 0$ there are exactly $w^s(c)$ ways to do this: indeed, when $i = 0$, the newly inserted label may be inserted in any position relative to the labels that are already present, so the number of elements bigger than c plus one gives the number of possible insertions. When $i > 0$, the new label may either be placed on top of a smaller label in a lower diagonal (i.e. an element smaller than c in the next run) or right after an already present element of its own diagonal (i.e. an element bigger than c in its run). Furthermore, the dinv added to the resulting path by each of these choices gives all the values from 0 to $w^s(c) - 1$, hence the factor $[w^s(c)]_q$.
- Next, we insert the $r_i - p_i$ zeros of ρ_i , also into the diagonal $y = x + i$. A zero label can occur directly after another label in its diagonal, provided that the first label in the diagonal is not zero. Each time a zero label precedes a nonzero label, one unit of dinv is created: indeed, since we are always inserting into the highest diagonal, the zeros only create primary dinv . It follows that the dinv that is created is q -counted by the factor $\begin{bmatrix} r_i - 1 \\ r_i - p_i \end{bmatrix}_q$.

Then, for $i = -1, -2, \dots, -s$ we proceed as follows.

- We insert $r_i - p_i$ zeros into the diagonal $y = x + i$, so as to obtain a labelled square path. We must insert every such zero label directly underneath a nonzero label of the diagonal $y = x + i + 1$, of which there are p_{i+1} , or directly before another zero. To give a preference function, the last zero in the diagonal must always be of the first kind. Each time a nonzero label in the diagonal $y = x + i + 1$ precedes a zero label in the diagonal $y = x + i$, one unit of secondary dinv is created. This explains the factor $\begin{bmatrix} p_{i+1} + r_i - p_i - 1 \\ r_i - p_i \end{bmatrix}_q$.
- Next, we insert the nonzero labels of ρ_i , one by one, from *left to right*. For such a $c \neq 0$ there are exactly $w^s(c)$ ways to insert it: indeed such a label may either be inserted right underneath a bigger label of the diagonal above it (i.e. a bigger label of the previous run), or right before an already present label in its own diagonal (i.e. a smaller label in its run). It is not hard to see that the dinv of the different options is q -counted by $[w^s(c)]_q$.

Finally, the factor $q^{p_{-1} + \cdots + p_{-s}}$ accounts for the bonus dinv , i.e. for the number of nonzero labels in negative runs. \square

Definition 7.19. Set $\mathfrak{S}_{m,n}$ to be the set of permutations of $\mathcal{O} \cup \{1, \dots, n\}$ where \mathcal{O} is the multiset containing m zeros.

Proposition 7.20. Consider $\tau \in \mathfrak{S}_{m,n}$. Let $\tau = \rho_l \dots \rho_0$ be its runs and set $\tilde{\tau} = \pi_l \cdots \pi_0 \in \mathfrak{S}_n$, where π_i is obtained from ρ_i by deleting its zeros. Set $r_i = |\rho_i|$ and $p_i = |\pi_i|$. We have

$$\frac{[p_s]_q}{[p_0]_q} q^{p_{s-1} + \cdots + p_0} \sum_{\substack{D \in \text{Park}(m,n) \\ \text{diagword}(D) = \tau}} q^{\text{dinv}(D)} t^{\text{area}(D)} = \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{shift}(P) = s \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)}$$

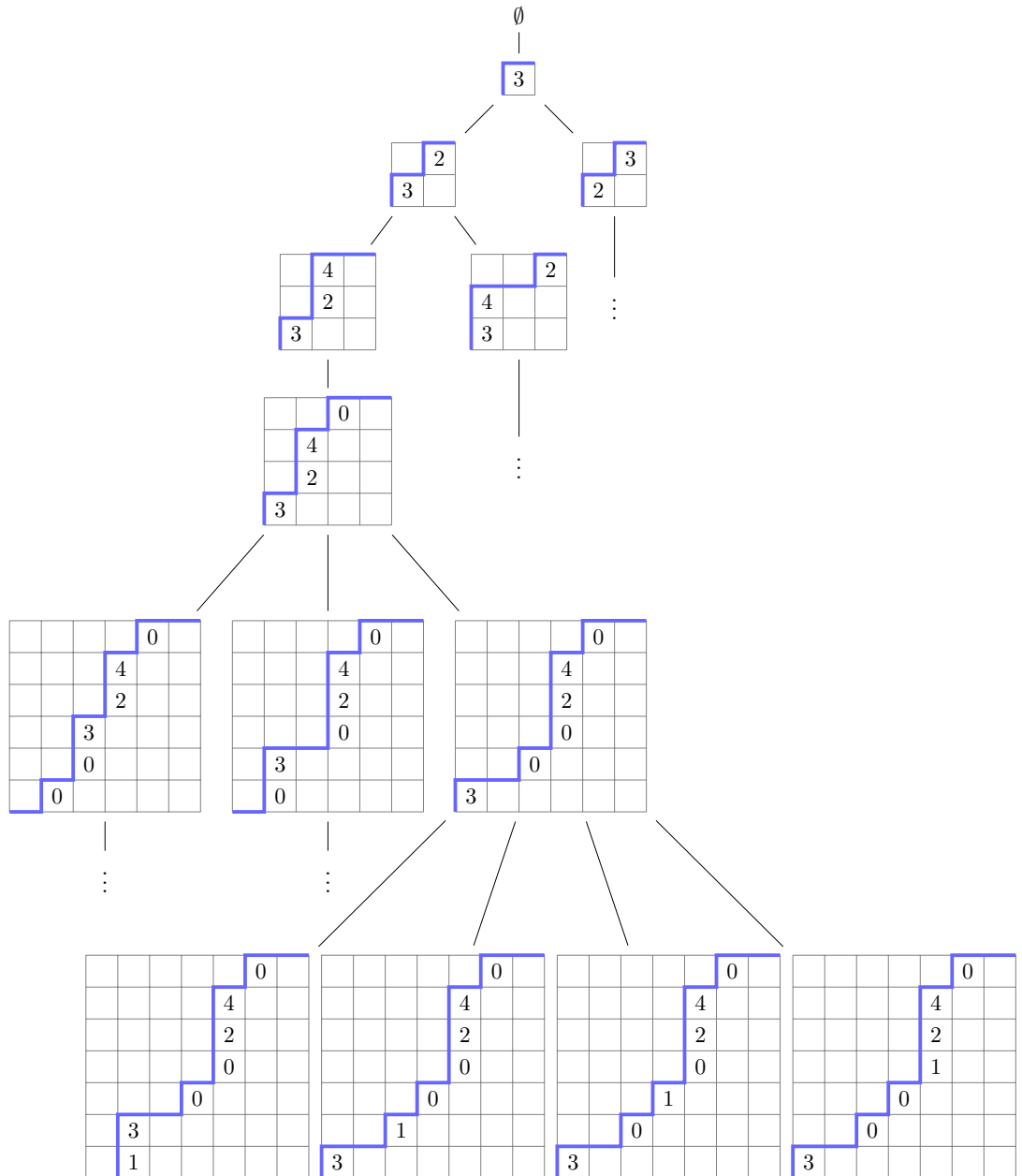


FIGURE 11. Partial tree of construction of paths with diagonal word 0423001 and shift 1.

Proof. In fact we will show that

$$(72) \quad \frac{[p_s]_q}{[p_{s-1}]_q} q^{p_{s-1}} \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{shift}(P)=s-1 \\ \text{diagword}(P)=\tau}} q^{\text{div}(P)} t^{\text{area}(P)} = \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{shift}(P)=s \\ \text{diagword}(P)=\tau}} q^{\text{div}(P)} t^{\text{area}(P)}.$$

This easily implies the thesis: just divide by the sum on the left hand side, and multiply these identities starting from $s = 1$, in order to get the identity in the statement.

We will make use of Theorem ??.

Notice that, with the exception of ρ_s and ρ_{s-1} , the sign of any other run ρ_i (i.e. it being positive, negative or zero) is the same for shift s and shift $s - 1$. It follows that, after replacing in (??) the corresponding formulae from Theorem ?? (notice the difference in the numbering of the runs in the statement), the terms concerning these runs are the same for the right hand side and left hand side.

Therefore, after the obvious cancellations, we are left to prove that

$$\begin{aligned} & \frac{[p_s]_q}{[p_{s-1}]_q} q^{p_{s-1}} \left(\begin{bmatrix} r_{s-1} - 1 \\ r_{s-1} - p_{s-1} \end{bmatrix}_q \begin{bmatrix} r_s - 1 \\ r_s - p_s \end{bmatrix}_q q^{p_{s-2} + \dots + p_0} \prod_{c \in \pi_s \cup \pi_{s-1}} [w^{s-1}(c)]_q \right) \\ &= \begin{bmatrix} r_s - 1 \\ r_s - p_s \end{bmatrix}_q \begin{bmatrix} p_s + r_{s-1} - p_{s-1} - 1 \\ r_{s-1} - p_{s-1} \end{bmatrix}_q q^{p_{s-1} + \dots + p_0} \prod_{c \in \pi_s \cup \pi_{s-1}} [w^s(c)]_q, \end{aligned}$$

which is equivalent, after further cancellations, to

$$\frac{[p_s]_q}{[p_{s-1}]_q} \left(\begin{bmatrix} r_{s-1} - 1 \\ r_{s-1} - p_{s-1} \end{bmatrix}_q \prod_{c \in \pi_s \cup \pi_{s-1}} [w^{s-1}(c)]_q \right) = \begin{bmatrix} p_s + r_{s-1} - p_{s-1} - 1 \\ r_{s-1} - p_{s-1} \end{bmatrix}_q \prod_{c \in \pi_s \cup \pi_{s-1}} [w^s(c)]_q.$$

By the definition of the schedule numbers we know that

$$\prod_{c \in \pi_{s-1}} [w^{s-1}(c)]_q = [p_{s-1}]_q! \quad \text{and} \quad \prod_{c \in \pi_s} [w^s(c)]_q = [p_s]_q!.$$

So the above condition reduces to showing that

$$\frac{[p_s]_q}{[p_{s-1}]_q} [p_{s-1}]_q! \frac{[r_{s-1} - 1]_q!}{[p_{s-1} - 1]_q!} \prod_{c \in \pi_s} [w^{s-1}(c)]_q = [p_s]_q! \frac{[p_s + r_{s-1} - p_{s-1} - 1]_q!}{[p_s - 1]_q!} \prod_{c \in \pi_{s-1}} [w^s(c)]_q,$$

which is equivalent, after trivial cancellations, to

$$[r_{s-1} - 1]_q! \prod_{c \in \pi_s} [w^{s-1}(c)]_q = [p_s + r_{s-1} - p_{s-1} - 1]_q! \prod_{c \in \pi_{s-1}} [w^s(c)]_q.$$

We will prove this identity in Lemma ??, concluding the proof of this proposition. \square

Lemma 7.21. *Let $\tau = \alpha\beta$ and $\tilde{\tau} = \tilde{\alpha}\tilde{\beta}$ be the diagonal word and reduced diagonal word of a given preference function, respectively, where α and β are the runs of τ and $\tilde{\alpha}$ and $\tilde{\beta}$ are obtained by deleting the zeros of α and β , respectively. Set $a = |\alpha|$, $b = |\beta|$, $\tilde{a} = |\tilde{\alpha}|$ and $\tilde{b} = |\tilde{\beta}|$. Then*

$$[b - 1]_q! \prod_{c \in \tilde{\alpha}} [w^0(c)]_q = [\tilde{a} + b - \tilde{b} - 1]_q! \prod_{c \in \tilde{\beta}} [w^1(c)]_q$$

Proof. The general argument is better understood with the help of a specific example.

Consider $\tau = 01468\ 00023579$, so that

$$\begin{array}{llll} \alpha = 01468 & \tilde{\alpha} = 1468 & a = 5 & \tilde{a} = 4 \\ \beta = 00023579 & \tilde{\beta} = 23579 & b = 8 & \tilde{b} = 5. \end{array}$$

Let us define a partition λ by setting

$$\lambda_i = |\{c \in \tilde{\beta} \mid c < \text{the } (\tilde{a} + 1 - i)\text{-th element of } \tilde{\alpha}\}|.$$

In Figure ?? we construct the Young diagram of λ as follows: draw a $b \times a$ grid. Label its rows, bottom to top with the elements of $\tilde{\alpha}$ and its columns, left to right with the elements of $\tilde{\beta}$. Then λ_i is the number of cells in the i -th row from the top such that the label of its column is smaller than the label of its row. We coloured all such cells blue.

It is now clear from Figure ?? that the conjugate partition λ' of λ is such that

$$\lambda'_i = |\{c \in \tilde{\alpha} \mid c > \text{the } i\text{-th element of } \tilde{\beta}\}|.$$

These partitions are useful because they encode essential information about the schedule numbers of elements that appear in the thesis. Recall that α is a positive run for shift 0 and so $w^0(c)$ equals the number of elements bigger than c in α plus the number of elements smaller than c in β . Similarly, β is a negative run for shift 1, so $w^1(c)$ equals the number of elements smaller than c in β plus the number of elements bigger than c in α .

So if c is the $(\tilde{a} + 1 - i)$ -th element of $\tilde{\alpha}$, we must have

$$w^0(c) = (i - 1) + \lambda_i + (b - \tilde{b})$$

where the first term is the number of elements of α that are bigger than c , the second term accounts for the number of elements in β that are smaller than c and different from 0 and the last term counts the number of zeros in the next run (that must necessarily be smaller than $c \neq 0$).

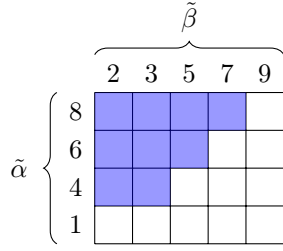


FIGURE 12. Construction of the Young diagram of λ .

Similarly, if c is the i -th element of $\tilde{\beta}$, we have

$$w^1(c) = (i - 1) + (b - \tilde{b}) + \lambda'_i$$

where the first term is the number of elements in β smaller than c and different from 0, the second term the number of zeros in β (which are smaller than c) and the third term the number of elements in α bigger than c .

So for our running example, the schedule numbers are computed in Figure ??.

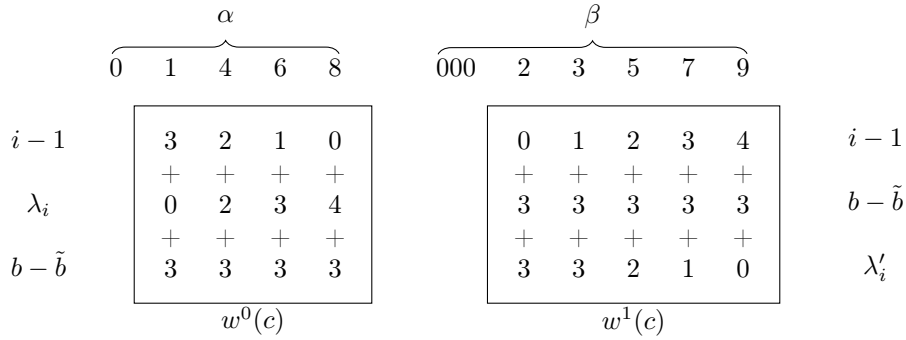


FIGURE 13. Schedule numbers $w^0(c)$ for $c \in \tilde{\alpha}$ and $w^1(c)$ for $c \in \tilde{\beta}$.

Using this decomposition of the schedule numbers, we can write

$$(73) \quad [b - 1]_q! \prod_{c \in \tilde{\alpha}} [w^0(c)]_q = [b - 1]_q! \prod_{i=1}^{\tilde{a}} [\lambda_i + (i - 1) + b - \tilde{b}]_q$$

$$(74) \quad [\tilde{a} + b - \tilde{b} - 1]_q! \prod_{c \in \tilde{\beta}} [w^1(c)]_q = [\tilde{a} + b - \tilde{b} - 1]_q! \prod_{i=1}^{\tilde{b}} [\lambda'_i + (i - 1) + b - \tilde{b}]_q.$$

The fact that these two equations are equal turns out to be a consequence of a general fact about partitions.

Consider the Ferrers diagram of $\delta_{b+\tilde{a}-1} := (b + \tilde{a} - 1, b + \tilde{a} - 2, \dots, 2, 1)$ whose parts are justified to the right (see Figure ??). Next, delete all the cells in the bottom right $\tilde{b} \times \tilde{a}$ rectangle that are not elements of λ , when λ is placed in the top left corner of this rectangle: see Figure ?. Call the resulting skew diagram Γ . Next, label the bottom \tilde{a} rows with the elements of $\tilde{\alpha}$, starting from the bottom; and the \tilde{b} rightmost columns with the elements of $\tilde{\beta}$, starting from the left.

- *The rows of Γ .* It follows from the discussion above and the construction of Γ , that if $c \in \tilde{\alpha}$ the number of squares in the *row* labelled c is exactly $w^0(c)$. The remaining $(b - \tilde{b}) + (\tilde{b} - 1) = b - 1$ rows form a staircase.
- *The columns of Γ .* Similarly, for $c \in \tilde{\beta}$ the number of squares in the *column* labelled c is exactly $w^1(c)$. The remaining $(b - \tilde{b}) + (\tilde{a} - 1)$ columns form a staircase.

It follows that taking the product of the q -analogues of the multiset recording the length of the rows (respectively columns) yields (??) (respectively (??)).

To conclude, we only need the following general observation, which also appears in [Leven-2016].

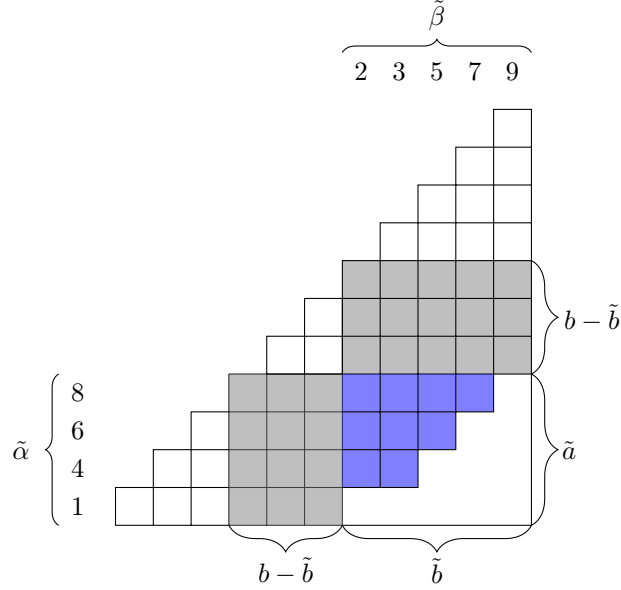


FIGURE 14. Construction and interpretation of Γ .

Lemma 7.22. Let $\delta = (n, n-1, \dots, 2, 1)$ be the staircase partition, and let $\lambda \subseteq \delta$. Then the multiset of the lengths of the rows of δ/λ equals the multiset of the lengths of its columns.

Proof. It follows immediately from the observation that any internal corner of a skew shape of the form δ/μ with $\mu \subseteq \delta$ (i.e. a cell in δ/μ adjacent to μ that once removed leaves a skew shape) belongs to exactly one row and exactly one column of δ/μ and these must have the same length. Then removing an internal corner does not change the equality between the multisets that we are considering, so that we can remove the cells from $\lambda \subseteq \delta$ one at the time to get δ/λ . \square

This concludes the proof of the theorem. \square

Corollary 7.23. Take $\tau \in \mathfrak{S}_{m,n}$. Let r be the number of nonzero elements in the rightmost run of τ . Then

$$\sum_{\substack{P \in \text{Pref}(m,n) \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} = \frac{[n]_q}{[r]_q} \sum_{\substack{D \in \text{Park}(m,n) \\ \text{diagword}(D) = \tau}} q^{\text{dinv}(D)} t^{\text{area}(D)}.$$

Proof. Let $\tau = \rho_l \dots \rho_0$ with the ρ_i 's its runs and $\tilde{\tau} = \pi_l \dots \pi_0$, where π_i is obtained from ρ_i by deleting its zeros. Set $p_i = |\pi_i|$. Applying Proposition ??, we get

$$\begin{aligned} \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} &= \sum_{s=0}^l \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{diagword}(P) = \tau \\ \text{shift}(P) = s}} q^{\text{dinv}(P)} t^{\text{area}(P)} \\ &= \sum_{s=0}^l \frac{[p_s]_q}{[p_0]_q} q^{p_{s-1} + \dots + p_0} \sum_{\substack{D \in \text{Park}(m,n) \\ \text{diagword}(D) = \tau}} q^{\text{dinv}(D)} t^{\text{area}(D)}. \end{aligned}$$

But

$$\begin{aligned} \sum_{s=0}^l \frac{[p_s]_q}{[p_0]_q} q^{p_0 + \dots + p_{s-1}} &= \frac{1}{[p_0]_q} ([p_0]_q + [p_1]_q q^{p_0} + \dots + [p_l]_q q^{p_0 + \dots + p_{l-1}}) \\ &= \frac{[p_0 + \dots + p_l]_q}{[p_0]_q} = \frac{[n]_q}{[r]_q} \end{aligned}$$

concluding the proof. \square

This result gives a link between the q, t -enumerators of preference functions and parking functions. Next, we need to deal with the Gessel quasisymmetric functions in (??).

In her thesis [?Hicks-Thesis, Corollary 73], Hicks found a way to factor the q, t -enumerator

$$\sum_{P \in S} q^{\text{dinv}(P)} t^{\text{area}(P)}$$

out of the expression

$$\sum_{P \in S} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n},$$

where S is the set of parking functions of size n with a fixed diagonal word. In [?Leven-2016, Lemma 4.2], Sergel showed that Hicks' argument generalizes in a straightforward way to the case where S is the set of preference functions of a given diagonal word and shift. We notice here that in fact again the same argument works in the case where S is the set of partially labelled preference functions of a given diagonal word and shift. The proof will be exactly the same as the ones given in the two aforementioned references, so we omit it.

Definition 7.24. Consider $\tau \in \mathfrak{S}_{m,n}$. A *consecutive block* of τ is a substring of τ of the form $i, i + 1, \dots, i + k$ with $i \neq 0$. Define $\text{Yconsec}(\tau)$ to be the Young subgroup of \mathfrak{S}_n which permutes only elements within the same consecutive block of τ .

This definition coincides with the one in [?Hicks-Thesis] when $m = 0$.

Example 7.25. If $\tau = 00412506703$ then $\text{Yconsec}(\tau) = \mathfrak{S}_{\{1,2\}} \times \mathfrak{S}_{\{3\}} \times \mathfrak{S}_{\{4\}} \times \mathfrak{S}_{\{5\}} \times \mathfrak{S}_{\{6,7\}}$.

Definition 7.26. If $\tau \in \mathfrak{S}_{m,n}$ and $\tilde{\tau} \in \mathfrak{S}_n$ is obtained from τ by deleting its zeros, we set $\text{ides}(\tau) := \text{ides}(\tilde{\tau})$.

For an argument to prove the following proposition, see [?Hicks-Thesis, Corollary 73] or [?Leven-2016, lemma 4.2].

Proposition 7.27. Given $\tau \in \mathfrak{S}_{m,n}$, if

$$S := \{P \in \text{Pref}(m, n) \mid \text{shift}(P) = s, \text{diagword}(P) = \tau\},$$

then

$$\sum_{P \in S} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} = \left(\sum_{P \in S} q^{\text{dinv}(P)} t^{\text{area}(P)} \right) \times \left(\frac{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)} Q_{\text{ides}(\tau) \cup \text{ides}(\pi), n}}{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)}} \right).$$

We combine this proposition with Corollary ??.

Corollary 7.28. Take $\tau \in \mathfrak{S}_{m,n}$. Let r be the number of nonzero elements in the rightmost run of τ . Then

$$\sum_{\substack{P \in \text{Pref}(m, n) \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} = \frac{[n]_q}{[r]_q} \sum_{\substack{D \in \text{Park}(m, n) \\ \text{diagword}(D) = \tau}} q^{\text{dinv}(D)} t^{\text{area}(D)} Q_{\text{ides}(D), n}$$

Proof. Take $l + 1$ to be the number of runs of τ . Then

$$\begin{aligned} & \sum_{\substack{P \in \text{Pref}(m, n) \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} \\ &= \sum_{s=0}^l \sum_{\substack{P \in \text{Pref}(m, n) \\ \text{shift}(P) = s \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} \\ & \text{(by Proposition ??)} = \sum_{s=0}^l \left(\sum_{\substack{P \in \text{Pref}(m, n) \\ \text{shift}(P) = s \\ \text{diagword}(P) = \tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} \right) \times \left(\frac{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)} Q_{\text{ides}(\tau) \cup \text{ides}(\pi), n}}{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)}} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{\substack{P \in \text{Pref}(m,n) \\ \text{diagword}(P)=\tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} \right) \times \left(\frac{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)} Q_{\text{ides}(\tau) \cup \text{ides}(\pi), n}}{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)}} \right) \\
(\text{by Corollary ??}) &= \left(\frac{[n]_q}{[r]_q} \sum_{\substack{D \in \text{Park}(m,n) \\ \text{diagword}(D)=\tau}} q^{\text{dinv}(D)} t^{\text{area}(D)} \right) \times \left(\frac{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)} Q_{\text{ides}(\tau) \cup \text{ides}(\pi), n}}{\sum_{\pi \in \text{Yconsec}(\tau)} q^{\text{inv}(\pi)}} \right) \\
(\text{by Proposition ??}) &= \frac{[n]_q}{[r]_q} \sum_{\substack{D \in \text{Park}(m,n) \\ \text{diagword}(D)=\tau}} q^{\text{dinv}(D)} t^{\text{area}(D)} Q_{\text{ides}(D), n}
\end{aligned}$$

□

We are now ready to prove the announced result.

Proof of Theorem ??. Using Theorem ??, we have

$$\begin{aligned}
\frac{[n]_q}{[r]_q} \Delta_{h_m} \nabla E_{n,r} &= \frac{[n]_q}{[r]_q} \sum_{\substack{P \in \text{Park}(m,n) \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} \\
&= \sum_{\substack{\tau \in \mathfrak{S}_{m,n} \\ \text{last run of } \tau \text{ has} \\ r \text{ nonzero elements}}} \frac{[n]_q}{[r]_q} \sum_{\substack{P \in \text{Park}(m,n) \\ \text{diagword}(P)=\tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} \\
(\text{using Corollary ??}) &= \sum_{\substack{\tau \in \mathfrak{S}_{m,n} \\ \text{last run of } \tau \text{ has} \\ r \text{ nonzero elements}}} \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{diagword}(P)=\tau}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n} \\
&= \sum_{\substack{P \in \text{Pref}(m,n) \\ \text{touch}(P)=r}} q^{\text{dinv}(P)} t^{\text{area}(P)} Q_{\text{ides}(P), n}.
\end{aligned}$$

□

8. SUPER-DIAGONAL COINVARIANTS AND THETA OPERATORS

In this section we extend a conjecture of Zabrocki in [?Zabrocki_Delta_Module].

Let $n, r \in \mathbb{N}$, $n, r \geq 1$, and consider the algebra of polynomials in $2 + r$ sets of n variables

$$R_n^{(r)} := \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1^{(1)}, \dots, \theta_n^{(1)}, \dots, \theta_1^{(r)}, \dots, \theta_n^{(r)}],$$

where the x_i and y_j are commuting variables, while the $\theta_i^{(k)}$ are Grassmannian variables, i.e. $\theta_i^{(k)} \theta_j^{(k)} = -\theta_j^{(k)} \theta_i^{(k)}$ for $1 \leq i \neq j \leq n$ and $\theta_i^{(k)} \theta_i^{(k)} = 0$ (notice that in $R_n^{(r)}$ variables from different sets commute with each other).

Consider the diagonal action of the symmetric group \mathfrak{S}_n , i.e. each element of \mathfrak{S}_n permutes simultaneously the $2 + r$ sets of variables acting on their indices, and let $I_n^{(r)}$ be the ideal of $R_n^{(r)}$ generated by the homogeneous invariants of positive degree. Following [?Zabrocki_Delta_Module], we call the quotient $M_n^{(r)} := R_n^{(r)} / I_n^{(r)}$ the space of *super-diagonal coinvariants* (in [?Zabrocki_Delta_Module] Zabrocki calls in this way only the case $r = 1$).

Clearly $M_n^{(r)}$ is an \mathfrak{S}_n -module, naturally $(2 + r)$ -graded by the degree in the $2 + r$ sets of variables. For $a, b \in \mathbb{N}$ and $\alpha := (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$, denote by $M_n^{(r)}(a, b, \alpha)$ the homogeneous submodule of $M_n^{(r)}$ of multidegree $(a, b, \alpha_1, \dots, \alpha_r)$ in the respective variables x_i 's, y_i 's, $\theta_i^{(1)}$'s, \dots , $\theta_i^{(r)}$'s.

For a partition μ of n , let $\chi_{M_n^{(r)}(a, b, \alpha)}(\mu)$ be the value of the character of the \mathfrak{S}_n -module $M_n^{(r)}(a, b, \alpha)$ on a permutation of cycle type μ . Define the q, t, \underline{z} -Frobenius image for super-diagonal coinvariants as

$$\mathcal{F}_{q, t, \underline{z}}(M_n^{(r)}) := \sum_{a, b \in \mathbb{N}} \sum_{\alpha \in (\mathbb{N})^r} q^a t^b z^\alpha \chi_{M_n^{(r)}(a, b, \alpha)}(\mu) \frac{p(\mu)}{z_\mu} \in \Lambda_{\mathbb{Q}(q, t, \underline{z})},$$

where $\underline{z} := z_1, \dots, z_r$, $z^\alpha := z_1^{\alpha_1} \cdots z_r^{\alpha_r}$, and $z_\mu := \prod_{i=1}^{\mu_1} m_i! i^{m_i}$ with m_i equal to the number of parts of size i in μ .

Using Theorem ??, Zabrocki's conjecture in [?Zabrocki_Delta_Module] can be restated in the following way.

Conjecture 8.1 (Zabrocki). *For $n \geq 1$*

$$(75) \quad \mathcal{F}_{q,t,\underline{z}}(M_n^{(1)}) = \sum_{i=0}^{n-1} z_1^i \Theta_i \nabla e_{n-i}.$$

Now we state our conjecture for $r = 2$.

Conjecture 8.2. *For $n \geq 1$*

$$(76) \quad \mathcal{F}_{q,t,\underline{z}}(M_n^{(2)}) = \sum_{\substack{i,j \geq 0 \\ 1 \leq i+j < n}} z_1^i z_2^j \Theta_i \Theta_j \nabla e_{n-(i+j)}.$$

More generally, we risk the following conjecture.

Conjecture 8.3. *For $n \geq 1$, $r \geq 3$ and $\alpha \in \mathbb{N}^r$ with $|\alpha| := \sum_{i=1}^r \alpha_i < n$*

$$(77) \quad \mathcal{F}_{q,t,\underline{z}}(M_n^{(r)}(\alpha)) = z^\alpha \Theta_{\alpha_1} \cdots \Theta_{\alpha_r} \nabla e_{n-|\alpha|},$$

where the left hand side is the q, t, \underline{z} -Frobenius image of the \mathfrak{S}_n -module $M_n^{(r)}(\alpha) := \oplus_{a,b \geq 0} M_n^{(r)}(a, b, \alpha)$.

Notice that this last conjecture does not wholly cover the module $M_n^{(r)}$. Indeed it seems that there are nonzero submodules of total degree $\geq n$ in the $\theta_i^{(k)}$ variables for $r \geq 3$ that are not present for $r = 2$: we do not have a formula for those.

Remark 8.4. It is a classical result that the Frobenius characteristic of the classical harmonics \mathcal{H}_n , i.e. the coinvariants of the action of \mathfrak{S}_n on $\mathbb{C}[x_1, \dots, x_n]$ is given by

$$\mathcal{F}_q(\mathcal{H}_n) = \tilde{H}_{(n)}[X] = \prod_{i=1}^n (1 - q^i) h_n \left[\frac{X}{1 - q} \right],$$

while, thanks to a theorem of Haiman [?Haiman-Vanishing-2002], the Frobenius characteristic of the *diagonal harmonics* $\mathcal{M}_n^{(0)}$, i.e. the coinvariants of the diagonal action of \mathfrak{S}_n on $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ is given by

$$\mathcal{F}_{q,t}(\mathcal{M}_n^{(0)}) = \nabla e_n.$$

It is probably worth mentioning that Haglund's [?Haglund-Schroeder-2004, Theorem 2.5] translates into

$$\nabla E_{n,r} = t^{n-r} \Theta_{h_{n-r}} \tilde{H}_{(r)}[X],$$

so that

$$\nabla e_n = \sum_{r=1}^n t^{n-r} \Theta_{h_{n-r}} \tilde{H}_{(r)}[X],$$

and hence

$$\mathcal{F}_{q,t}(\mathcal{M}_n^{(0)}) = \sum_{r=1}^n t^{n-r} \Theta_{h_{n-r}} \mathcal{F}_q(\mathcal{H}_r).$$

Though the obvious generalization of this formula does not seem to hold, it is at least conceivable that the Theta operators (or possibly some variation/extension of them) will give the universal formula for super-diagonal coinvariants with r sets of n commuting variables and r' sets of n Grassmannian variables conjectured by Bergeron and Zabrocki (see [?Bergeron_Multivariate_Coinvariants] for these explicitly computed formulas up to $n = 5$).

9. A THETA CONJECTURE (?)

Let $\text{LD}(0, n)^{*k, or}$ be the set of partially labelled Dyck paths $P \in \text{LD}(0, n)^{*k}$ with r contractible valleys decorated with a \circ , where a *contractible valley* (see [?Haglund-Remmel-Wilson-2015]) is a vertical step which is either preceded by two horizontal steps or by one horizontal step and the label in its row is bigger than the label in the row immediately below. We define the area of a path $P \in \text{LD}(0, n)^{*k, or}$ as the area of the underlying decorated Dyck path in $\text{D}(n)^{*k}$, and its monomial x^P in the usual way (i.e. the decorations on the valleys do not affect the area nor the monomial).

Experimentally, we observed the following formula.

Conjecture 9.1. For $n, k, r \in \mathbb{N}$ with $n \geq 1$ and $r + k < n$,

$$(78) \quad \Theta_r \Theta_k \nabla e_{n-r-k} \Big|_{q=1} = \sum_{P \in \text{LD}(0, n)^{*k, or}} t^{\text{area}(P)} x^P.$$

It is a very interesting problem to find a dinv statistic on $\text{LD}(0, n)^{*k, or}$ which would give the full $\Theta_r \Theta_k \nabla e_{n-r-k}$. Notice that this might be a statistic that “interpolates” the *rise version* of the Delta conjecture (i.e. the one that we stated in this article) and its *valley version*: see [?Haglund-Remmel-Wilson-2015] for details. Such a conjecture, which should better be called *Theta conjecture*, would bring the whole framework of the Delta conjecture to this new more general setting.

10. MORE ON THETA OPERATORS

10.1. Relation to the D_k operators. Recall the definition of the operators D_k from [?Bergeron-Garsia-Haiman-Tesler] for $k \in \mathbb{Z}$ and any $F[X] \in \Lambda$

$$(79) \quad D_k F[X] := \left(F \left[X + \frac{M}{z} \right] \sum_{r \geq 0} (-z)^r e_r[X] \right) \Big|_{z^k}.$$

Recall the following well-known identities [?Bergeron-Garsia-Haiman-Tesler-Positivity-1999, Equation I.12 (i) and (ii)]

$$(80) \quad D_0 \tilde{H}_\mu[X] = -D_\mu \tilde{H}_\mu[X]$$

$$(81) \quad D_k \underline{e}_1 - \underline{e}_1 D_k = M D_{k+1}$$

where \underline{e}_1 denotes the multiplication by e_1 .

The next proposition is proved in Section ??.

Proposition 10.1. *We have*

$$(82) \quad \Theta_1 = \frac{1}{M} (\underline{e}_1 + D_1).$$

Remark 10.2. It has been shown in [?Garsia-Haiman-Tesler-Explicit-1999] that Λ is spanned by the symmetric functions obtained by applying D_1 and e_1 to the constant 1.

We use the standard commutator notation for associative algebras, i.e. $[a, b] := ab - ba$.

Conjecture 10.3. *For any $k \geq 0$,*

$$(83) \quad [\Theta_k, \underline{e}_1] = \sum_{i=1}^k (-1)^{i+1} D_{i+1} \Theta_{k-i}$$

and

$$(84) \quad [\Theta_k, D_1] = - \sum_{i=1}^k (-1)^{i+1} D_{i+1} \Theta_{k-i}.$$

Remark 10.4. It is clear from their definition that the operators Θ_k commute with each other, i.e. for all $a, b \geq 0$ we have $[\Theta_a, \Theta_b] = 0$. Using (??), it is now clear that the identities (??) and (??) are equivalent.

From Remark ?? it follows that the relations (??) and (??) determine the operators Θ_k uniquely.

Finally, notice that in terms of the operators of Carlsson and Mellit (cf. [?Carlsson-Mellit-ShuffleConj-2015]), the identity (??) translates to

$$(85) \quad \Theta_1 = \frac{1}{M} d_- (d_+ - d_+^*) \quad \text{on } V_0 = \Lambda.$$

It would be interesting to see if the Θ_k 's have a similar formula, i.e. a formula in terms of the operators of the Dyck path algebra. This could be a step towards a proof of the operator Delta conjecture (and hence of the compositional Delta conjecture).

10.2. Schur positivity conjectures. As it is natural to do in these situations, after we discovered the Theta operators we looked for Schur positivity. Sure enough, we found out experimentally that anything that comes naturally from the nabla operator and is Schur positive remains Schur positive after applying Θ_{s_λ} for any λ .

With our limited experimental evidence, we risk the following conjecture.

Conjecture 10.5. *For $n \in \mathbb{N}$, $n \geq 1$, $\mu \vdash n$, $\alpha \vDash n$ and any partition λ we have*

$$\begin{aligned} (-1)^{|\mu|-\ell(\mu)} \langle \Theta_{s_\lambda} \nabla m_\mu, s_\nu \rangle &\in \mathbb{N}[q, t] && \text{for every } \nu \vdash n + |\lambda|, \\ (-1)^{\text{spin}(\mu)} \langle \Theta_{s_\lambda} \nabla s_\mu, s_\nu \rangle &\in \mathbb{N}[q, t] && \text{for every } \nu \vdash n + |\lambda|, \\ \langle \Theta_{s_\lambda} \nabla C_\alpha, s_\nu \rangle &\in \mathbb{N}[q, t] && \text{for every } \nu \vdash n + |\lambda|, \end{aligned}$$

where $\text{spin}(\mu)$ is a nonnegative integer that depends on μ , but not on λ nor on ν (cf. [?Haglund-Book-2008, Appendix B]).

11. TECHNICAL PROOFS

11.1. Symmetric functions: tools. In this subsection we introduce more tools from symmetric function theory, that we are going to use in our proofs.

11.1.1. More symmetric function notation. It is useful to introduce the so called *star scalar product* on Λ given by

$$\langle p_\lambda, p_\mu \rangle_* = (-1)^{|\mu|-\ell(\mu)} \prod_{i=1}^{\ell(\mu)} (1 - q^{\mu_i})(1 - t^{\mu_i}) z_\mu \delta_{\lambda, \mu}.$$

For every symmetric function $f[X]$ and $g[X]$ we have (see [?Garsia-Haiman-Tesler-Explicit-1999, Proposition 1.8])

$$(86) \quad \langle f, g \rangle_* = \langle \omega \phi f, g \rangle = \langle \phi \omega f, g \rangle$$

where

$$(87) \quad \phi f[X] := f[MX] \quad \text{for all } f[X] \in \Lambda.$$

Observe that

$$(88) \quad f^* = f^*[X] = f \left[\frac{X}{M} \right] = \phi^{-1} f[X].$$

For all symmetric functions f, g, h we have

$$(89) \quad \langle h^\perp f, g \rangle_* = \langle h^\perp f, \omega \phi g \rangle = \langle f, h \omega \phi g \rangle = \langle f, \omega \phi((\omega h)^* \cdot g) \rangle = \langle f, (\omega h)^* \cdot g \rangle_*,$$

so the operator h^\perp is the adjoint of the multiplication by $(\omega h)^*$ with respect to the star scalar product.

We record here the addition formulas

$$(90) \quad p_k[X + Y] = p_k[X] + p_k[Y] \quad \text{and} \quad p_k[X - Y] = p_k[X] - p_k[Y],$$

and

$$(91) \quad e_n[X + Y] = \sum_{i=0}^n e_{n-i}[X] e_i[Y] \quad \text{and} \quad h_n[X + Y] = \sum_{i=0}^n h_{n-i}[X] h_i[Y].$$

Notice in particular that $p_k[-X]$ equals $-p_k[X]$ and not $(-1)^k p_k[X]$. As the latter sort of negative sign can be also useful, it is customary to use the notation ϵ to express it: we will have $p_k[\epsilon X] = (-1)^k p_k[X]$, so that, in general,

$$(92) \quad f[-\epsilon X] = \omega f[X]$$

for any symmetric function f .

We refer to [?Haglund-Book-2008] for more informations on this topic.

11.1.2. *Macdonald symmetric functions fundamental identities.* It turns out that the Macdonald polynomials are orthogonal with respect to the star scalar product: more precisely

$$(93) \quad \langle \tilde{H}_\lambda, \tilde{H}_\mu \rangle_* = w_\mu(q, t) \delta_{\lambda, \mu}.$$

These orthogonality relations give the following Cauchy identities

$$(94) \quad e_n \left[\frac{XY}{M} \right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X] \tilde{H}_\mu[Y]}{w_\mu} \quad \text{for all } n.$$

We will use the following form of *Macdonald-Koornwinder reciprocity* (see [?Macdonald-Book-1995] p. 332 or [?Garsia-Haiman-Tesler-Explicit-1999]): for all nonempty partitions α and β

$$(95) \quad \frac{\tilde{H}_\alpha[MB_\beta]}{\Pi_\alpha} = \frac{\tilde{H}_\beta[MB_\alpha]}{\Pi_\beta}.$$

11.1.3. *Pieri rules and summation formulae.* For a given $k \geq 1$, we define the Pieri coefficients $c_{\mu\nu}^{(k)}$ and $d_{\mu\nu}^{(k)}$ by setting

$$(96) \quad h_k^\perp \tilde{H}_\mu[X] = \sum_{\nu \subset_k \mu} c_{\mu\nu}^{(k)} \tilde{H}_\nu[X],$$

$$(97) \quad e_k \left[\frac{X}{M} \right] \tilde{H}_\nu[X] = \sum_{\mu \supset_k \nu} d_{\mu\nu}^{(k)} \tilde{H}_\mu[X].$$

The following identity is Proposition 5 in [?Bergeron-Haiman-2013], written in the notation of [?Garsia-Haglund-Xin-Zabrocki-Pieri-2016], which is coherent with ours:

$$(98) \quad c_{\mu\nu}^{(k+1)} = \frac{1}{B_{\mu/\nu}} \sum_{\nu \subset_1 \alpha \subset_k \mu} c_{\mu\alpha}^{(k)} c_{\alpha\nu}^{(1)} \frac{T_\alpha}{T_\nu} \quad \text{with } B_{\mu/\nu} := B_\mu - B_\nu,$$

where $\nu \subset_k \mu$ means that ν is contained in μ (as Ferrers diagrams) and μ/ν has k lattice cells, while the symbol $\mu \supset_k \nu$ is analogously defined. It follows from (??) that

$$(99) \quad c_{\mu\nu}^{(k)} = \frac{w_\mu}{w_\nu} d_{\mu\nu}^{(k)}.$$

11.1.4. *Other useful identities.* In this section we collect some results from the literature that we are going to use later in the text.

The following identity is Proposition 2.2 in [?Garsia-Haglund-qtCatalan-2002]:

$$(100) \quad \tilde{H}_\mu[(1-t)(1-q^j)] = (1-q^j) \Pi_\mu h_j[(1-t)B_\mu].$$

So, using (??) with $Y = [j]_q = \frac{1-q^j}{1-q}$, we get

$$(101) \quad e_n \left[X \frac{1-q^j}{1-q} \right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X] \tilde{H}_\mu[(1-t)(1-q^j)]}{w_\mu} \\ = (1-q^j) \sum_{\mu \vdash n} \frac{\Pi_\mu \tilde{H}_\mu[X] h_j[(1-t)B_\mu]}{w_\mu}.$$

For $\mu \vdash n$, Macdonald proved (see [?Macdonald-Book-1995] p. 362) that

$$(102) \quad \langle \tilde{H}_\mu, s_{(n-r, 1^r)} \rangle = e_r[B_\mu - 1],$$

so that, since by Pieri rule $e_r h_{n-r} = s_{(n-r, 1^r)} + s_{(n-r+1, 1^{r-1})}$,

$$(103) \quad \langle \tilde{H}_\mu, e_r h_{n-r} \rangle = e_r[B_\mu].$$

We need the following well-known proposition.

Proposition 11.1. For $n \in \mathbb{N}$ we have

$$(104) \quad e_n[X] = e_n \left[\frac{XM}{M} \right] = \sum_{\mu \vdash n} \frac{MB_\mu \Pi_\mu \tilde{H}_\mu[X]}{w_\mu}.$$

Moreover, for all $k \in \mathbb{N}$ with $0 \leq k \leq n$, we have

$$(105) \quad h_k \left[\frac{X}{M} \right] e_{n-k} \left[\frac{X}{M} \right] = \sum_{\mu \vdash n} \frac{e_k[B_\mu] \tilde{H}_\mu[X]}{w_\mu},$$

and

$$(106) \quad \omega(p_n[X]) = [n]_q [n]_t \sum_{\mu \vdash n} \frac{M \Pi_\mu \tilde{H}_\mu[X]}{w_\mu}.$$

We will need two more identities: Lemma 5.2 in [DAdderio-VandenWyngaerd-2017], i.e. for $\beta \vdash n > k \geq 1$

$$(107) \quad e_{n-k-1}[B_\beta - 1]B_\beta = \sum_{\gamma \subset_k \beta} c_{\beta\gamma}^{(k)} B_\gamma T_\gamma,$$

and

$$(108) \quad \begin{aligned} e_{n-k}[B_\beta] &= T_\beta e_k[B_\beta(1/q, 1/t)] \\ &\text{(using [Zabrocki-4Catalan-2016, Lemma 13])} = \sum_{\gamma \subset_k \beta} c_{\beta\gamma}^{e_k^\perp}(1/q, 1/t) T_\beta \\ &\text{(using [Garsia-Haglund-qtCatalan-2002, Equation (3.16)])} = \sum_{\gamma \subset_k \beta} c_{\beta\gamma}^{(k)} T_\gamma, \end{aligned}$$

where $c_{\mu\nu}^{e_n^\perp-k}$ is the generalized Pieri coefficient defined by

$$(109) \quad \sum_{\nu \subset_{n-k} \mu} c_{\mu\nu}^{e_n^\perp-k} \tilde{H}_\nu = e_{n-k}^\perp \tilde{H}_\mu.$$

We will use also [Haglund-Schroeder-2004, Theorem 2.6], i.e. for any $A, F \in \Lambda$ homogeneous and $\nu \neq \emptyset$

$$(110) \quad \sum_{\mu \vdash n} \Pi_\mu F[MB_\mu] d_{\mu\nu}^A = \Pi_\nu (\Delta_{A[MX]} F[X]) [MB_\nu],$$

where $d_{\mu\nu}^A$ is the generalized Pieri coefficient defined by

$$(111) \quad \sum_{\mu \supset \nu} d_{\mu\nu}^A \tilde{H}_\mu = A \tilde{H}_\nu.$$

Another identity that we will need is [Garsia-Hicks-Stout-2011, Proposition 2.6]:

$$(112) \quad h_k \left[\frac{X}{1-q} \right] e_{n-k} \left[\frac{X}{M} \right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X]}{w_\mu} \sum_{r=1}^k \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q q^{\binom{r}{2} + r - kr} (-1)^{k-r} h_r[(1-t)B_\mu].$$

Finally, we will use the following theorem from [DAdderio-VandenWyngaerd-2017].

Theorem 11.2 ([DAdderio-VandenWyngaerd-2017, Theorem 3.1]). For $m, k \geq 1$ and $\ell \geq 0$, we have

$$(113) \quad \sum_{\gamma \vdash m} \frac{\tilde{H}_\gamma[X]}{w_\gamma} h_k[(1-t)B_\gamma] e_\ell[B_\gamma] = \sum_{j=0}^{\ell} t^{\ell-j} \sum_{s=0}^k q^{\binom{s}{2}} \begin{bmatrix} s+j \\ s \end{bmatrix}_q \begin{bmatrix} k+j-1 \\ s+j-1 \end{bmatrix}_q \\ \times h_{s+j} \left[\frac{X}{1-q} \right] h_{\ell-j} \left[\frac{X}{M} \right] e_{m-s-\ell} \left[\frac{X}{M} \right].$$

11.2. **Proof of Theorem ??.** Using (??), we have

$$\begin{aligned}
\Pi e_k^* \Pi^{-1} \nabla e_{n-k} &= \Pi e_k^* \Pi^{-1} \sum_{\mu \vdash n-k} \frac{M B_\mu \Pi_\mu T_\mu}{w_\mu} \tilde{H}_\mu[X] \\
(\text{using (??)}) &= \Pi \sum_{\mu \vdash n-k} \frac{M B_\mu T_\mu}{w_\mu} \sum_{\lambda \supset_k \mu} d_{\lambda\mu}^{(k)} \tilde{H}_\lambda[X] \\
(\text{using (??)}) &= \Pi \sum_{\lambda \vdash n} M \sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{(k)} B_\mu T_\mu \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
(\text{using (??)}) &= \Pi \sum_{\lambda \vdash n} M e_{n-k-1} [B_\lambda - 1] B_\lambda \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
&= \sum_{\lambda \vdash n} M e_{n-k-1} [B_\lambda - 1] \Pi_\lambda B_\lambda \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
(\text{using (??)}) &= \Delta'_{e_{n-k-1}} e_n.
\end{aligned}$$

11.3. **Proof of Theorem ??.** Using (??), we have

$$\begin{aligned}
\Pi e_k^* \Pi^{-1} \nabla \frac{[n]_q}{[n-k]_q} \omega(p_{n-k}) &= [n]_q [n-k]_t \Pi e_k^* \Pi^{-1} \sum_{\mu \vdash n-k} \frac{M \Pi_\mu T_\mu}{w_\mu} \tilde{H}_\mu[X] \\
(\text{using (??)}) &= [n]_q [n-k]_t \Pi \sum_{\mu \vdash n-k} \frac{M T_\mu}{w_\mu} \sum_{\lambda \supset_k \mu} d_{\lambda\mu}^{(k)} \tilde{H}_\lambda[X] \\
(\text{using (??)}) &= [n]_q [n-k]_t \Pi \sum_{\lambda \vdash n} M \sum_{\mu \subset_k \lambda} c_{\lambda\mu}^{(k)} T_\mu \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
(\text{using (??)}) &= [n]_q [n-k]_t \Pi \sum_{\lambda \vdash n} M e_{n-k} [B_\lambda] \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
&= [n]_q [n-k]_t \sum_{\lambda \vdash n} M e_{n-k} [B_\lambda] \Pi_\lambda \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
(\text{using (??)}) &= \frac{[n-k]_t}{[n]_t} \Delta_{e_{n-k}} \omega(p_n).
\end{aligned}$$

11.4. **Proof of Proposition ??.** It is enough to check (??) on the Macdonald basis: for $\mu \vdash n$

$$\begin{aligned}
\Theta_1 \tilde{H}_\mu[X] &= \Pi e_1^* \Pi^{-1} \tilde{H}_\mu[X] \\
(\text{using (??)}) &= \Pi \sum_{\lambda \supset_1 \mu} d_{\lambda\mu}^{(1)} \frac{1}{\Pi_\mu} \tilde{H}_\lambda[X] \\
&= \sum_{\lambda \supset_1 \mu} d_{\lambda\mu}^{(1)} \frac{\Pi_\lambda}{\Pi_\mu} \tilde{H}_\lambda[X] \\
&= \sum_{\lambda \supset_1 \mu} d_{\lambda\mu}^{(1)} \left(1 - \frac{D_\lambda - D_\mu}{M} \right) \tilde{H}_\lambda[X] \\
&= \sum_{\lambda \supset_1 \mu} d_{\lambda\mu}^{(1)} \tilde{H}_\lambda[X] - \frac{1}{M} \sum_{\lambda \supset_1 \mu} d_{\lambda\mu}^{(1)} D_\lambda \tilde{H}_\lambda[X] + \frac{1}{M} \sum_{\lambda \supset_1 \mu} d_{\lambda\mu}^{(1)} D_\mu \tilde{H}_\lambda[X] \\
(\text{using (??) and (??)}) &= \frac{1}{M} \underline{e}_1 \tilde{H}_\mu[X] + \frac{1}{M} D_0 \frac{1}{M} \underline{e}_1 \tilde{H}_\lambda[X] - \frac{1}{M} \frac{1}{M} \underline{e}_1 D_0 \tilde{H}_\mu[X] \\
&= \frac{1}{M} \left(\underline{e}_1 + \frac{1}{M} (D_0 \underline{e}_1 - \underline{e}_1 D_0) \right) \tilde{H}_\lambda[X] \\
(\text{using (??)}) &= \frac{1}{M} (\underline{e}_1 + D_1) \tilde{H}_\lambda[X].
\end{aligned}$$

11.5. **Proof of Theorem ??.** For $j < n$ we have

$$\begin{aligned}
& h_j^\perp \nabla e_n[X[s+1]_q] = \\
& \text{(using (??))} = \sum_{\lambda \vdash n} (1 - q^{s+1}) h_{s+1}[(1-t)B_\lambda] \Pi_\lambda T_\lambda h_j^\perp \frac{\tilde{H}_\lambda[X]}{w_\lambda} \\
& \text{(using (??))} = \sum_{\mu \vdash n-j} \tilde{H}_\mu[X] (1 - q^{s+1}) \sum_{\lambda \vdash n} \frac{c_{\lambda\mu}^{(j)}}{w_\lambda} h_{s+1}[(1-t)B_\lambda] T_\lambda \Pi_\lambda \\
& \text{(using (??))} = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \sum_{\lambda \vdash n} d_{\lambda\mu}^{(j)} h_{s+1}[(1-t)B_\lambda] T_\lambda \Pi_\lambda \\
& \text{(using (??))} = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \Pi_\mu \left(\Delta_{e_j} h_{s+1}[Z/(1-q)] e_n[Z/M] \right) \Big|_{Z=MB_\mu} \\
& \text{(using (??))} = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \Pi_\mu \sum_{\beta \vdash n+s+1} e_j[B_\beta] \frac{\tilde{H}_\beta[MB_\mu]}{w_\beta} \\
& \quad \times \sum_{i=1}^{s+1} \begin{bmatrix} s \\ i-1 \end{bmatrix}_q q^{\binom{i}{2} + i - i(s+1)} (-1)^{s+1-i} h_i[(1-t)B_\beta] \\
& = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \Pi_\mu \sum_{i=1}^{s+1} \begin{bmatrix} s \\ i-1 \end{bmatrix}_q q^{\binom{i}{2} + i - i(s+1)} (-1)^{s+1-i} \\
& \quad \times \sum_{\beta \vdash n+s+1} \frac{\tilde{H}_\beta[MB_\mu]}{w_\beta} h_i[(1-t)B_\beta] e_j[B_\beta] \\
& \text{(using (??))} = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \Pi_\mu \sum_{i=1}^{s+1} \begin{bmatrix} s \\ i-1 \end{bmatrix}_q q^{\binom{i}{2} + i - i(s+1)} (-1)^{s+1-i} \\
& \quad \times \sum_{p=0}^j t^{j-p} \sum_{b=0}^i q^{\binom{b}{2}} \begin{bmatrix} b+p \\ p \end{bmatrix}_q \begin{bmatrix} i+p-1 \\ b+p-1 \end{bmatrix}_q h_{b+p}[(1-t)B_\mu] h_{j-p}[B_\mu] e_{n+s+1-b-j}[B_\mu] \\
& (e_r[B_\mu] = 0 \text{ for } r > |\mu|) = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \Pi_\mu q^{\binom{s+1}{2} + s+1 - (s+1)^2} \times \\
& \quad \times \sum_{p=0}^j t^{j-p} q^{\binom{s+1}{2}} \begin{bmatrix} s+1+p \\ p \end{bmatrix}_q h_{s+1+p}[(1-t)B_\mu] h_{j-p}[B_\mu] e_{n-j}[B_\mu] \\
& \text{(using (??))} = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} (1 - q^{s+1}) \Pi_\mu \sum_{p=0}^j t^{j-p} \begin{bmatrix} s+1+p \\ p \end{bmatrix}_q h_{s+1+p}[(1-t)B_\mu] h_{j-p}[B_\mu] T_\mu \\
& = \sum_{\mu \vdash n-j} \frac{\tilde{H}_\mu[X]}{w_\mu} \Pi_\mu \sum_{p=0}^j t^{j-p} \begin{bmatrix} s+p \\ p \end{bmatrix}_q (1 - q^{s+1+p}) h_{s+1+p}[(1-t)B_\mu] h_{j-p}[B_\mu] T_\mu \\
& \text{(using (??))} = \sum_{p=0}^j t^{j-p} \begin{bmatrix} s+p \\ p \end{bmatrix}_q \Delta_{h_{j-p}} \nabla e_{n-j}[X[s+1+p]_q]
\end{aligned}$$

where we used

$$\binom{s+1}{2} + s+1 - (s+1)^2 = \frac{s^2 + s + 2s + 2 - 2s^2 - 4s - 2}{2} = -\binom{s+1}{2}.$$

For $j = n$ we get

$$h_n^\perp \nabla e_n[X[s+1]_q] = \langle \nabla e_n \left[\frac{XM[s+1]_q}{M} \right], h_n \rangle$$

$$\begin{aligned}
(\text{using } (??)) &= \sum_{\lambda \vdash n} \frac{\tilde{H}_\lambda[M[s+1]_q]}{w_\lambda} \langle \nabla \tilde{H}_\lambda[X], h_n \rangle \\
(\text{using } (??)) &= \sum_{\lambda \vdash n} \frac{T_\lambda \tilde{H}_\lambda[M[s+1]_q]}{w_\lambda} \\
(\text{using } (??)) &= h_n \left[\frac{M[s+1]_q}{M} \right] \\
&= h_n[[s+1]_q] \\
&= \begin{bmatrix} s+n \\ n \end{bmatrix}_q,
\end{aligned}$$

where the last equality follows from the well-known (cf. [Stanley-Book-1999, Theorem 7.21.2])

$$(114) \quad h_k[[n]_q] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \quad \text{for } n \geq 1 \text{ and } k \geq 0.$$

This proves the case $j = n$. The case $j > n$ is trivial.

11.6. Proof of Lemma ??. We have

$$\begin{aligned}
&\langle \Delta_{h_\ell} f, h_k e_{n-\ell-k} \rangle = \\
(\text{using } (??)) &= \langle \Delta_{h_\ell} f, e_k^* h_{n-\ell-k}^* \rangle_* \\
&= \langle f, \Delta_{h_\ell} e_k^* h_{n-\ell-k}^* \rangle_* \\
(\text{using } (??)) &= \sum_{\beta \vdash n-\ell} h_\ell[B_\beta] e_{n-\ell-k}[B_\beta] \langle f, \frac{\tilde{H}_\beta}{w_\beta} \rangle_* \\
&= \sum_{\beta \vdash n-\ell} \Pi_\beta (h_\ell^* e_{n-\ell-k}^*)|_{X=MB_\beta} \langle f, \Pi^{-1} \frac{\tilde{H}_\beta}{w_\beta} \rangle_* \\
(\text{using } (??)) &= \sum_{\beta \vdash n-\ell} \Pi_\beta (\Delta_{e_\ell} e_{n-k}^*)|_{X=MB_\beta} \langle \Pi^{-1} f, \frac{\tilde{H}_\beta}{w_\beta} \rangle_* \\
(\text{using } (??)) &= \sum_{\beta \vdash n-\ell} \sum_{\mu \supseteq_\ell \beta} \Pi_\mu d_{\mu\beta}^{e_\ell^*} e_{n-k}[B_\mu] \langle \Pi^{-1} f, \frac{\tilde{H}_\beta}{w_\beta} \rangle_* \\
(\text{using } (??)) &= \sum_{\mu \vdash n} \frac{\Pi_\mu}{w_\mu} e_{n-k}[B_\mu] \left\langle \Pi^{-1} f, \sum_{\beta \subset_\ell \mu} c_{\mu\beta}^{h_\ell^\perp} \tilde{H}_\beta \right\rangle_* \\
&(\text{using } (??)) = \sum_{\mu \vdash n} \frac{\Pi_\mu}{w_\mu} e_{n-k}[B_\mu] \left\langle \Pi^{-1} f, h_\ell^\perp \tilde{H}_\mu \right\rangle_* \\
&(\text{using } (??)) = \sum_{\mu \vdash n} \frac{1}{w_\mu} e_{n-k}[B_\mu] \left\langle \Pi e_\ell^* \Pi^{-1} f, \tilde{H}_\mu \right\rangle_* \\
&(\text{using } (??)) = \langle \Pi e_\ell^* \Pi^{-1} f, e_k^* h_{n-k}^* \rangle_* \\
&(\text{using } (??)) = \langle \Pi e_\ell^* \Pi^{-1} f, h_k e_{n-k} \rangle \\
&= \langle \Theta_\ell f, h_k e_{n-k} \rangle.
\end{aligned}$$

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