

Approximability and exact resolution of the Multidimensional Binary Vector Assignment problem

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Abstract. In this paper we consider the multidimensional binary vector assignment problem. An input of this problem is defined by m disjoint multisets V^1, V^2, \dots, V^m , each composed of n binary vectors of size p . An output is a set of n disjoint m -tuples of vectors, where each m -tuple is obtained by picking one vector from each multiset V^i . To each m -tuple we associate a p dimensional vector by applying the bit-wise AND operation on the m vectors of the tuple. The objective is to minimize the total number of zeros in these n vectors. We denote this problem by $\min \sum 0$, and the restriction of this problem where every vector has at most c zeros by $(\min \sum 0)_{\#0 \leq c}$. $(\min \sum 0)_{\#0 \leq 2}$ was only known to be **APX**-complete, even for $m = 3$ [5]. We show that, assuming the unique games conjecture, it is **NP**-hard to $(n - \varepsilon)$ -approximate $(\min \sum 0)_{\#0 \leq 1}$ for any fixed n and ε . This result is tight as any solution is a n -approximation. We also prove without assuming UGC that $(\min \sum 0)_{\#0 \leq 1}$ is **APX**-complete even for $n = 2$, and we provide an example of $n - f(n, m)$ -approximation algorithm for $\min \sum 0$. Finally, we show that $(\min \sum 0)_{\#0 \leq 1}$ is polynomial-time solvable for fixed m (which cannot be extended to $(\min \sum 0)_{\#0 \leq 2}$ according to [5]).

1 Introduction

1.1 Problem definition

In this paper we consider the multidimensional binary vector assignment problem denoted by $\min \sum 0$. An input of this problem (see Figure 1) is described by m multisets V^1, \dots, V^m , each multiset V^i containing n binary p -dimensional vectors. For any $j \in [n]$ ¹, and any $i \in [m]$, the j^{th} vector of multiset V^i is denoted v_j^i , and for any $k \in [p]$, the k^{th} coordinate of v_j^i is denoted $v_j^i[k]$.

The objective of this problem is to create a set S of n stacks. A stack $s = (v_1^s, \dots, v_m^s)$ is an m -tuple of vectors such that $v_i^s \in V^i$, for any $i \in [m]$. Furthermore, S has to be such that every vector of the input appears in exactly one created stack.

¹ Note that $[n]$ stands for $\{1, 2, \dots, n\}$.

We now introduce the operator \wedge which assigns to a pair of vectors (u, v) the vector given by $u \wedge v = (u[1] \wedge v[1], u[2] \wedge v[2], \dots, u[p] \wedge v[p])$. We associate to each stack s a unique vector given by $v_s = \bigwedge_{i \in [m]} v_i^s$.

The cost of a vector v is defined as the number of zeros in it. More formally if v is p -dimensional, $c(v) = p - \sum_{k \in [p]} v[k]$. We extend this definition to a set of stacks $S = \{s_1, \dots, s_n\}$ as follows : $c(S) = \sum_{s \in S} c(v_s)$.

The objective is then to find a set S of n disjoint stacks minimizing the total number of zeros. This leads us to the following definition of the problem:

Optimization Problem 1 $\min \sum 0$

Input m multisets of n p -dimensional binary vectors.

Output A set S of n disjoint stacks minimizing $c(S)$.

Throughout this paper, we denote $(\min \sum 0)_{\#0 \leq c}$ the restriction of $\min \sum 0$ where the number of zeros per vector is upper bounded by c .

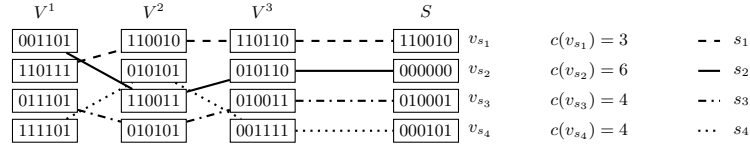


Fig. 1: Example of $\min \sum 0$ instance with $m = 3, n = 4, p = 6$ and of a feasible solution S of cost $c(S) = 17$.

1.2 Related work

The dual version of the problem called $\max \sum 1$ (where the objective is to maximize the total number of 1 in the created stacks) has been introduced by Reda et al. in [8] as the “yield maximization problem in Wafer-to-Wafer 3-D Integration technology”. They prove the **NP**-completeness of $\max \sum 1$ and provide heuristics without approximation guarantee. In [6] we proved that, even for $n = 2$, for any $\varepsilon > 0$, $\max \sum 1$ is $\mathcal{O}(m^{1-\varepsilon})$ and $\mathcal{O}(p^{1-\varepsilon})$ inapproximable unless **P** = **NP**. We also provide an ILP formulation proving that $\max \sum 1$ (and thus $\min \sum 0$) is **FPT**² when parameterized by p .

We introduced $\min \sum 0$ in [4] where we provide in particular $\frac{4}{3}$ -approximation algorithm for $m = 3$. In [5], authors focus on a generalization of $\min \sum 0$, called **MULTI DIMENSIONAL VECTOR ASSIGNMENT**, where vectors are not necessary binary vectors. They extend the approximation algorithm of [4] to get a $f(m)$ -approximation algorithm for arbitrary m . They also prove the **APX**-completeness of the $(\min \sum 0)_{\#0 \leq 2}$ for $m = 3$. This result was the only known inapproximability result for $\min \sum 0$.

² *i.e.* admits an algorithm in $f(p)poly(|I|)$ for an arbitrary function f .

1.3 Contribution

In section 2 we study the approximability of $\min \sum 0$. Our main result in this section is to prove that assuming UGC, it is **NP**-hard to $(n - \varepsilon)$ -approximate $(\min \sum 0)_{\#0 \leq 1}$ (and thus $\min \sum 0$) for any fixed $n \geq 2$, $\forall \varepsilon > 0$. This result is tight as any solution is a n -approximation.

Notice that this improves the only existing negative result for $\min \sum 0$, which was the **APX**-hardness of [5] (implying only no-**PTAS**).

We also show how this reduction can be used to obtain the **APX**-hardness for $(\min \sum 0)_{\#0 \leq 1}$ for $n = 2$ unless $\mathbf{P} = \mathbf{NP}$, which is weaker negative result, but does not require UGC. We then give an example $n - f(n, m)$ approximation algorithm for the general problem $\min \sum 0$.

In section 3, we consider the exact resolution of $\min \sum 0$. We focus on *sparse* instances, *i.e.* instances of $(\min \sum 0)_{\#0 \leq 1}$. Indeed, recall that authors of [5] show that $(\min \sum 0)_{\#0 \leq 2}$ is **APX**-complete even for $m = 3$, implying that $(\min \sum 0)_{\#0 \leq 2}$ cannot be polynomial-time solvable for fixed m unless $\mathbf{P} = \mathbf{NP}$. Thus, it is natural to ask if $(\min \sum 0)_{\#0 \leq 1}$ is polynomial-time solvable for fixed m . Section 3 is devoted to answer positively to this question. Notice that the question of determining if $(\min \sum 0)_{\#0 \leq 1}$ is **FPT** when parameterized by m remains open.

2 Approximability of $\min \sum 0$

We refer the reader to [1] and [7] for the definitions of Gap and L -reductions.

2.1 Inapproximability results for $(\min \sum 0)_{\#0 \leq 1}$

From now we suppose that $\forall k \in [p]$, $\exists i$, $\exists j$ such that $v_j^i[k] = 0$. In other words, for any solution S and $\forall k$, there exists a stack s such that $v_s[k] = 0$. Otherwise, we simply remove such a coordinate from every vector of every set, and decrease p by one. Since this coordinate would be set to 1 in all the stacks of all solutions, such a preprocessing preserves approximation ratios and exact results.

In a first time, we define the following polynomial-time computable function f which associates an instance of $(\min \sum 0)_{\#0 \leq 1}$ to any k -uniform hypergraph, *i.e.* an hypergraph $G = (U, E)$ such that every hyperedges of E contains exactly k distinct elements of U .

Definition of f We consider a k -uniform hypergraph $G = (U, E)$. We call f the polynomial-time computable function that creates an instance of $(\min \sum 0)_{\#0 \leq 1}$ from a G as follows.

1. We set $m = |E|$, $n = k$ and $p = |U|$.
2. For each hyperedge $e = \{u_1, u_2, \dots, u_k\} \in E$, we create the set V^e containing k vectors $\{v_j^e, j \in [k]\}$, where for all $j \in [k]$, $v_j^e[u_j] = 0$ and $v_j^e[l] = 1$ for $l \neq u_j$. We say that a vector v **represents** $u \in U$ iff $v[u] = 0$ and $v[l \neq u] = 1$ (and thus vector v_j^e represents u_j).

An example of this construction is given in Figure 2.

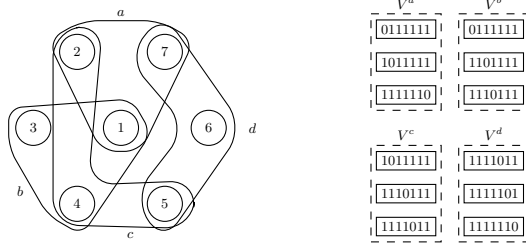


Fig. 2: Illustration of the reduction from a hypergraph $G = (U = \{1, 2, 3, 4, 5, 6, 7\}, E = \{\{1, 2, 7\}, \{1, 3, 4\}, \{2, 4, 5\}, \{5, 6, 7\}\})$ to an instance $(\min \sum 0)_{\#0 \leq 1}$

Negative results assuming UGC We consider the following problem. Notice that what we call a vertex cover in a k -regular hypergraph $G = (U, E)$ is a set $U' \subseteq U$ such that for any hyperedge $e \in E$, $U' \cap e \neq \emptyset$.

Decision Problem 1 ALMOST Ek VERTEX COVER

Input We are given an integer $k \geq 2$, two arbitrary positive constants ε and δ and a k -uniform hypergraph $G = (U, E)$.

Output Distinguish between the following cases:

YES Case there exist k disjoint subsets $U^1, U^2, \dots, U^k \subseteq U$, satisfying $|U^i| \geq \frac{1-\varepsilon}{k}|U|$ and such that every hyperedge contains at most one vertex from each U^i .

NO Case every vertex cover has size at least $(1 - \delta)|U|$.

It is shown in [2] that, assuming UGC, this problem is **NP**-complete.

Theorem 1. For any fixed $n \geq 2$, for any constants $\varepsilon, \delta > 0$, there exists a $\frac{n-n\delta}{1+n\varepsilon}$ -Gap reduction from ALMOST Ek VERTEX COVER to $(\min \sum 0)_{\#0 \leq 1}$. Consequently, under UGC, for any fixed n $(\min \sum 0)_{\#0 \leq 1}$ is **NP**-hard to approximate within a factor $(n - \varepsilon')$ for any $\varepsilon' > 0$.

Proof. We consider an instance I of ALMOST Ek VERTEX COVER defined by two positive constants δ and ε , an integer k and a k -regular hypergraph $G = (U, E)$.

We use the function f previously defined to construct an instance $f(I)$ of $\min \sum 0$. Let us now prove that if I is a positive instance, $f(I)$ admits a solution S of cost $c(S) < (1 + n\varepsilon)|U|$, and otherwise any solution S of $f(I)$ has cost $c(S) \geq n(1 - \delta)|U|$.

NO Case Let S be a solution of $f(I)$. Let us first remark that for any stack $s \in S$, the set $\{k : v_s[k] = 0\}$ defines a vertex cover in G . Indeed, s contains exactly one vector per set, and thus by construction s selects one vertex per hyperedge in G . Remark also that the cost of s is equal to the size of the corresponding vertex cover.

Now, suppose that I is a negative instance. Hence each vertex cover has a size at least equal to $(1 - \delta)|U|$, and any solution S of $f(I)$, composed of exactly n stacks, verifies $c(S) \geq n(1 - \delta)|U|$.

YES Case If I is a positive instance, there exists k disjoint sets $U^1, U^2, \dots, U^k \subseteq U$ such that $\forall i = 1, \dots, k, |U^i| \geq \frac{1-\varepsilon}{k}|U|$ and such that every hyperedge contains at most one vertex from each U^i .

We introduce the subset $X = U \setminus \bigcup_{i=1}^k U^i$. By definition $\{U^1, U^2, \dots, U^k, X\}$ is a partition of U and $|X| \leq \varepsilon|U|$. Furthermore, $U^i \cup X$ is a vertex cover $\forall i = 1, \dots, k$. Indeed, each hyperedge $e \in E$ that contains no vertex of U^i , contains at least one vertex of X since e contains k vertices.

We now construct a solution S of $f(I)$. Our objective is to construct stacks $\{s_i\}$ such that for any i , the zeros of s_i are included in $U_i \cup X$ (*i.e.* $\{l : v_{s_i}[l] = 0\} \subseteq U_i \cup X$). For each $e = \{u_1, \dots, u_k\} \in E$, we show how to assign exactly one vector of V^e to each stack s_1, \dots, s_k . For all $i \in [k]$, if v_j^e represents a vertex u with $u \in U^i$, then we assign v_j^e to s_i . W.l.o.g, let $S'_e = \{s_1, \dots, s_{k'}\}$ (for $k' \leq k$) be the set of stacks that received a vertex during this process. Notice that as every hyperedge contains at most one vertex from each U^i , we only assigned one vector to each stack of S'_e . After this, every unassigned vector $v \in V^e$ represents a vertex of X (otherwise, such a vector v would belong to a set $U^i, i \in [k]$, a contradiction). We assign arbitrarily these vectors to the remaining stacks that are not in S'_e . As by construction $\forall i \in [k], v_{s_i}$ contains only vectors representing vertices from $U^i \cup X$, we get $c(s_i) \leq |U^i| + |X|$.

Thus, we obtain a feasible solution S of cost $c(S) = \sum_{i=1}^k c(s_i) \leq k|X| + \sum_{i=1}^k |U^i|$. As by definition we have $|X| + \sum_{i=1}^k |U^i| = |U|$, it follows that $c(S) \leq |U| + (k - 1)\varepsilon|U|$ and since $k = n$, $c(S) < |U|(1 + n\varepsilon)$.

If we define $a(n) = (1 + n\varepsilon)|U|$ and $r(n) = \frac{n(1-\delta)}{(1+n\varepsilon)}$, the previous reduction is a $r(n)$ -Gap reduction. Furthermore, $\lim_{\delta, \varepsilon \rightarrow 0} r(n) = n$, thus it is **NP**-hard to approximate $(\min \sum 0)_{\#0 \leq 1}$ within a ratio $(n - \varepsilon')$ for any $\varepsilon' > 0$.

□

Notice that, as a function of n , this inapproximability result is optimal. Indeed, we observe that any feasible solution S is an n -approximation as, for any instance I of $\min \sum 0^3$, $Opt(I) \geq p$ and for any solution S , $c(S) \leq pn$.

Negative results without assuming UGC Let us now study the negative results we can get when only assuming $\mathbf{P} \neq \mathbf{NP}$. Our objective is to prove that $(\min \sum 0)_{\#0 \leq 1}$ is **APX**-hard, even for $n = 2$. To do so, we present a reduction from **ODD CYCLE TRANSVERSAL**, which is defined as follows. Given an input graph $G = (U, E)$, the objective is to find an odd cycle transversal of minimum size, *i.e.* a subset $T \subseteq U$ of minimum size such that $G[U \setminus T]$ is bipartite.

For any integer $\gamma \geq 2$, we denote \mathcal{G}_γ the class of graphs $G = (U, E)$ such that any optimal odd cycle transversal T has size $|T| \geq \frac{|U|}{\gamma}$. Given \mathcal{G} a class of

³ Recall that we assume $\forall k \in [p], \exists i, \exists j$ such that $v_j^i[k] = 0$

graphs, we denote $OCT_{\mathcal{G}}$ the ODD CYCLE TRANSVERSAL problem restricted to \mathcal{G} .

Lemma 1. *For any constant $\gamma \geq 2$, there exists an L-reduction from $OCT_{\mathcal{G}_\gamma}$ to $(\min \sum 0)_{\#0 \leq 1}$ with $n = 2$.*

Proof. Let us consider an integer γ , an instance I of $OCT_{\mathcal{G}_\gamma}$, defined by a graph $G = (V, E)$ such that $G \in \mathcal{G}_\gamma$. W.l.o.g., we can consider that G contains no isolated vertex.

Remark that any graph can be seen as a 2-uniform hypergraph. Thus, we use the function f previously defined to construct an instance $f(I)$ of $(\min \sum 0)_{\#0 \leq 1}$ such that $n = 2$. Since, G contains no isolated vertex, $f(I)$ contains no position k such that $\forall i \in [m], \forall j \in [n], v_j^i[k] = 1$.

Let us now prove that I admits an odd cycle transversal of size t if and only if $f(I)$ admits a solution of cost $p + t$.

\Leftarrow We consider an instance $f(I)$ of $(\min \sum 0)_{\#0 \leq 1}$ with $n = 2$ admitting a solution $S = \{s_A, s_B\}$ with cost $c(S) = p + t$. Let us specify a function g which produces from S a solution $T = g(I, S)$ of $OCT_{\mathcal{G}_\gamma}$, i.e. a set of vertices of U such that $G[U \setminus T]$ is bipartite.

We define $T = \{u \in U : v_{s_A}[u] = v_{s_B}[u] = 0\}$, the set of coordinates equal to zero in both s_A and s_B . We also define $A = \{u \in V : v_{s_A}[u] = 0 \text{ and } v_{s_B}[u] = 1\}$ (resp. $B = \{u \in V : v_{s_B}[u] = 0 \text{ and } v_{s_A}[u] = 1\}$), the set of coordinates set to zero only in s_A (resp. s_B). Notice that $\{T, A, B\}$ is a partition of U .

Remark that A and B are independent sets. Indeed, suppose that $\exists \{u, v\} \in E$ such that $u, v \in A$. As $\{u, v\} \in E$ there exists a set $V^{(u,v)}$ containing a vector that represents u and another vector that represents v , and thus these vectors are assigned to different stacks. This leads to a contradiction. It follows that $G[U \setminus T]$ is bipartite and T is an odd cycle transversal.

Since $c(S) = |A| + |B| + 2|T| = p + |T| = p + t$, we get $|T| = t$.

\Rightarrow We consider an instance I of $OCT_{\mathcal{G}_\gamma}$ and a solution T of size t . We now construct a solution $S = \{s_A, s_B\}$ of $f(I)$ from T .

By definition, $G[U \setminus T]$ is a bipartite graph, thus the vertices in $U \setminus T$ may be split into two disjoint independent sets A and B . For each edge $e \in E$, the following cases can occur:

- if $\exists u \in e$ such that $u \in A$, then the vector corresponding to u is assigned to s_A , and the vector corresponding to $e \setminus \{u\}$ is assigned to s_B (and the same rule holds by exchanging A and B)
- otherwise, u and $v \in T$, and we assign arbitrarily v_u^e to s_A and the other to s_B .

We claim that the stacks s_A and s_B describe a feasible solution S of cost at most $p + t$.

Since, for each set, only one vector is assigned to s_A and the other to s_B , the two stacks s_A and s_B are disjoint and contain exactly m vectors. S is therefore a feasible solution.

Remark that v_{s_A} (resp. v_{s_B}) contains only vectors v such that $v[k] = 0 \implies k \in A \cup T$ (resp. $k \in B \cup T$), and thus $c(v_A) \leq |A| + |T|$ (resp. $c(v_B) \leq |B| + |T|$). Hence $c(S) \leq |A| + |B| + 2|T| = p + t$.

Let us now prove that this reduction is an L -reduction.

1. By definition, any instance I of $OCT_{\mathcal{G}_\gamma}$ verifies $|Opt(I)| \geq |U|/\gamma$. Thus,

$$Opt(f(I)) \leq |U| + Opt(I) \leq (\gamma + 1)Opt(I)$$

2. We consider an arbitrary instance I of $OCT_{\mathcal{G}_\gamma}$, $f(I)$ the corresponding instance of $(\min \sum 0)_{\#0 \leq 1}$, S a solution of $f(I)$ and $T = g(I), S$ the corresponding solution of I .

We proved $|T| - Opt(I) = c(S) - |U| - (Opt(f(I)) - |U|) = c(S) - Opt(f(I))$.

Therefore, we get an L -reduction for $\alpha = \gamma + 1$ and $\beta = 1$. \square

Lemma 2 ([3]). *There exist a constant γ and $\mathcal{G} \subset \mathcal{G}_\gamma$ such that $OCT_{\mathcal{G}}$ is APX-hard.*

The following result is now immediate.

Theorem 2. $(\min \sum 0)_{\#0 \leq 1}$ is APX-hard, even for $n = 2$.

2.2 Approximation algorithm for $\min \sum 0$

Let us now show an example of algorithm achieving a $n - f(n, m)$ ratio. Notice that the $(n - \epsilon)$ inapproximability result holds for fixed n and $\#0 = 1$, while the following algorithm is polynomial-time computable when n is part of the input and $\#0$ is arbitrary.

Proposition 1. *There is a polynomial-time $n - \frac{n-1}{n\rho(n,m)}$ approximation algorithm for $\min \sum 0$, where $\rho(n, m) > 1$ is the approximation ratio for independent set in graphs that are the union of m complete n -partite graphs.*

Proof. Let I be an instance of $\min \sum 0$. Let us now consider an optimal solution $S^* = \{s_1^*, \dots, s_n^*\}$ of I . For any $i \in [n]$, let $Z_i^* = \{l \in [p] : v_{s_i^*}[l] = 0 \text{ and } v_{s_t^*}[l] = 1, \forall t \neq i\}$ be the set of coordinates equal to zero only in stack s_i^* . Let $\Delta = \sum_{i=1}^n |Z_i^*|$. Notice that we have $c(S^*) \geq \Delta + 2(p - \Delta)$, as for any coordinate l outside $\bigcup_i Z_i^*$, there are at least two stacks with a zero at coordinate l . W.l.o.g., let us suppose that Z_1^* is the largest set among $\{Z_i^*\}$, implying $|Z_1^*| \geq \frac{\Delta}{n}$.

Given a subset $Z \subset [p]$, we will construct a solution $S = \{s_1, \dots, s_n\}$ such that for any $l \in Z$, $v_{s_1}[l] = 0$, and for any $i \neq 1$, $v_{s_i}[l] = 1$. Informally, the zero at coordinates Z will appear only in s_1 , which behaves as a "trash" stack. The cost of such a solution is $c(S) \leq c(s_1) + \sum_{i=2}^n c(s_i) \leq p + (n - 1)(p - |Z|)$. Our objective is now to compute such a set Z , and to lower bound $|Z|$ according to $|Z_1^*|$.

Let us now define how we compute Z . Let $P = \{l \in [p] : \forall i \in [m], |\{j : v_j^i[l] = 0\}| \leq 1\}$ be the subset of coordinates that are never nullified in two

different vectors of the same set. We will construct a simple undirected graph $G = (P, E)$, and thus it remains to define E . For vector v_j^i , let $Z_j^i = Z(v_j^i) \cap P$, where $Z(v) \subseteq [p]$ denotes the set of null coordinates of vector v . For any $i \in [m]$, we add to G the edges of the complete n -partite graph $G^i = (\{Z_1^i \times \cdots \times Z_n^i\})$ (i.e. for any $j_1, j_2, v_1 \in Z_{j_1}^i, v_2 \in Z_{j_2}^i$, we add edge $\{v_1, v_2\}$ to G). This concludes the description of G , which can be seen as the union of m complete n -partite graphs.

Let us now see the link between independent set in G and our problem. Let us first see why Z_1^* is a independent set in G . Recall that by definition of Z_1^* , for any $l \in Z_1^*, v_{s_1^*}[k] = 0$, but $v_{s_j^*}[k] = 1, j \geq 2$. Thus, it is immediate that $Z_1^* \subseteq P$. Moreover, assume by contradiction that there exists an edge in G between to vertices l_1 and l_2 of Z_1^* . This implies that there exists $i \in [m], j_1$ and $j_2 \neq j_1$ such that $v_{j_1}^i[l_1] = 0$ and $v_{j_2}^i[l_2] = 0$. As by definition of Z_1^* we must have $v_{s_j^*}[k_1] = 1$ and $v_{s_j^*}[k_2] = 1$ for $j \geq 2$, this implies that s_1^* must contains both $v_{j_1}^i$ and $v_{j_2}^i$, a contradiction. Thus, we get $Opt(G) \geq |Z_1^*|$, where $Opt(G)$ is the size of a maximum independent set in G .

Now, let us check that for any independent set $Z \subseteq P$ in G , we can construct a solution $S = \{s_1, \dots, s_n\}$ such that for any $l \in Z, v_{s_1}[l] = 0$, and for any $i \neq 1, v_{s_i}[l] = 1$. To construct such a solution, we have to prove that we can add in s_1 all the vectors v such that $\exists l \in Z$ such that $v[l] = 0$. However, this last statement is clearly true as for any $i \in [m]$, there is at most one vector v_j^i with $Z(v_j^i) \subseteq Z$.

Thus, any $\rho(n, m)$ approximation algorithm gives us a set Z with $|Z| \geq \frac{|Z_1^*|}{\rho(n, m)} \geq \frac{\Delta}{n\rho(n, m)}$, and we get a ratio of $\frac{p+(n-1)(p-\frac{\Delta}{n\rho(n, m)})}{2p-\Delta} \leq n - \frac{n-1}{n\rho(n, m)}$ for $\Delta = p$.

□

Remark 1. We can get, for example, $\rho(n, m) = mn^{m-1}$ using the following algorithm. For any $i \in [m]$, let $G^i = (A_1^i, \dots, A_n^i)$ be the i -th complete n -partite graph. W.l.o.g., suppose that A_1^1 is the largest set among $\{A_j^i\}$. Notice that $|A_1^1| \geq \frac{Opt^1}{m}$. The algorithm starts by setting $S_1 = A_1^1$ (S_1 may not be an independent set). Then, for any i from 2 to m , the algorithm set $S_i = S_{i-1} \setminus (\cup_{j \neq j_0} A_j^i)$, where $j_0 = \arg \max_j \{|S_{i-1} \cap A_j^i|\}$. Thus, for any i we have $|S_i| \geq \frac{|S_{i-1}|}{n}$, and S_i is an independent set when considering only edges from $\cup_{l=1}^i G^l$. Finally, we get an independent set of G of size $|S_m| \geq \frac{S_1}{n^{m-1}} \geq \frac{Opt^1}{mn^{m-1}}$.

3 Exact resolution of sparse instances

The section is devoted to the exact resolution of $\min \sum 0$ for sparse instances where each vector has at most one zero ($\#0 \leq 1$). As we have seen in Section 2, $(\min \sum 0)_{\#0 \leq 1}$ remains **NP-hard** (even for $n = 2$). Thus it is natural to ask if $(\min \sum 0)_{\#0 \leq 1}$ is polynomial-time solvable for fixed m (for general n). This section is devoted to answer positively to this question. Notice that we cannot extend this result to a more general notion of sparsity as $(\min \sum 0)_{\#0 \leq 2}$ is

APX-complete for $m = 3$ [5]. However, the question if $(\min \sum 0)_{\#0 \leq 1}$ is fixed parameter tractable when parameterized by m is left open.

We first need some definitions, and refer the reader to Figure 3 where an example is depicted.

Definition 1.

- For any $l \in [p], i \in [m]$, we define $B^{(l,i)} = \{v_j^i : v_j^i[l] = 0\}$ to be the set of vectors of set i that have their (unique) zero at position l . For the sake of homogeneous notation, we define $B^{(p+1,i)} = \{v_j^i : v_j^i \text{ is a 1 vector}\}$. Notice that the $B^{(l,i)}$ form a partition of all the vectors of the input, and thus an input of $(\min \sum 0)_{\#0 \leq 1}$ is completely characterized by the $B^{(l,i)}$.
- For any $l \in [p + 1]$, the **block** $B^l = \bigcup_{i \in [m]} B^{(l,i)}$.

Informally, the idea to solve $(\min \sum 0)_{\#0 \leq 1}$ in polynomial time for fixed m is to parse the input block after block using a dynamic programming algorithm. When arriving at block B^l we only need to remember for each $c \subseteq [m]$ the number x_c of “partial stacks” that have only one vector for each $V^i, i \in c$. Indeed, we do not need to remember what is “inside” these partial stacks as all the remaining vectors from $B^{l'}, l' \geq l$ cannot “match” (i.e. have their zero in the same position) the vectors in these partial stacks.

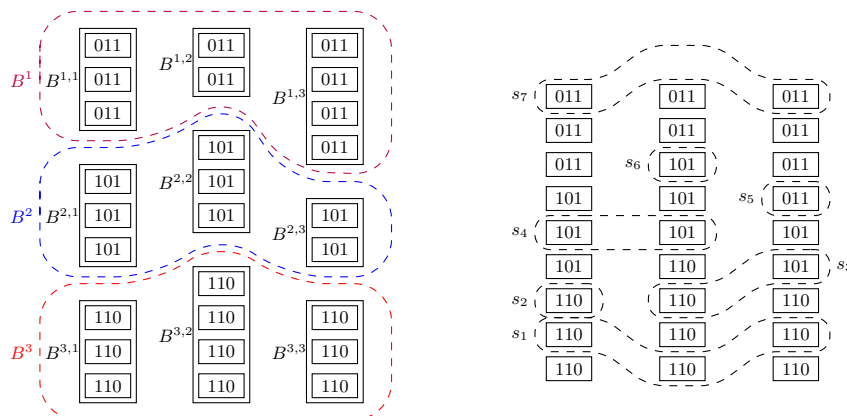


Fig. 3: Left: instance I of $(\min \sum 0)_{\#0 \leq 1}$ partitioned into blocks. Right: A profile $P = \{x_{\{0\}} = 2, x_{\{1\}} = 1, x_{\{2\}} = 1, x_{\{3\}} = 1, x_{\{1,2\}} = 1, x_{\{1,3\}} = 1, x_{\{2,3\}} = 1, x_{\{1,2,3\}} = 1\}$ encoding a set S of partial stacks of I containing two empty stacks. The support of s_7 is $\text{supp}(s_7) = \{1, 3\}$ and has cost $c(s_7) = 1$.

Definition 2.

- A **partial stack** $s = \{v_{i_1}^s, \dots, v_{i_k}^s\}$ of I is such that $\{i_x \in [m], x \in [k]\}$ are pairwise disjoint, and for any $x \in [k]$, $v_{i_x}^s \in V^{i_x}$. The **support** of a partial stack s is $\text{sup}(s) = \{i_x, x \in [k]\}$. Notice that a stack s (i.e. non partial) has $\text{sup}(s) = [m]$.
- The cost is extended in the natural way: the cost of a partial stack $c(s) = c(\bigwedge_{x \in [k]} v_{i_x}^s)$ is the number of zeros of the bitwise AND of the vectors of s .

We define the notion of profile as follows:

Definition 3. A **profile** $P = \{x_c, c \subseteq [m]\}$ is a set of 2^m positive integers such that $\sum_{c \subseteq [m]} x_c = n$.

In the following, a profile will be used to encode a set S of n partial stacks by keeping a record of their support. In other words, $x_c, c \subseteq [m]$ will denote the number of partial stacks in S of support c . This leads us to introduce the notion of reachable profile as follows:

Definition 4. Given two profiles $P = \{x_c : c \subseteq [m]\}$ and $P' = \{x_{c'} : c' \subseteq [m]\}$ and a set $S = \{s_1, \dots, s_n\}$ of n partial stacks, P' is said reachable from P through S iff there exist n couples $(s_1, c_1), (s_2, c_2), \dots, (s_n, c_n)$ such that:

- For each couple (s, c) , $\text{sup}(s) \cap c = \emptyset$.
- For each $c \subseteq [m]$, $|\{(s_j, c_j) : c_j = c, j = 1, \dots, n\}| = x_c$. Intuitively, the configuration c appears in exactly x_c couples.
- For each $c' \subseteq [m]$, $|\{(s_j, c_j) : \text{sup}(s_j) \cup c_j = c', j = 1, \dots, n\}| = x_{c'}$. Intuitively, there exist exactly $x_{c'}$ couples that, when associated, create a partial of profile c' .

Given two profiles P and P' , P' is said reachable from P , if there exists a set S of n partial stacks such that P' is reachable from P through S .

Intuitively, a profile P' is reachable from P through S if every partial stack of the set encoded by P can be assigned to a unique partial stack from S to obtain a set of new partial stacks encoded by P' .

Remark that, given a set of partial stacks S only their profile is used to determine whether a profile is reachable or not. An example of a reachable profile is given on Figure 4.

We introduce now the following problem II . We then show that this problem can be used to solve $(\min \sum_0)_{\#0 \leq 1}$ problem, and we present a dynamic programming algorithm that solves II in polynomial time when m is fixed.

Optimization Problem 2 II

Input (l, P) with $l \in [p + 1]$, P a profile.

Output A set of n partial stacks $S = \{s_1, s_2, \dots, s_n\}$ such that S is a partition of $\mathcal{B} = \bigcup_{l' \geq l} B^{l'}$ and for every $c \subseteq [m]$, $|\{s \in S | \text{sup}(s) = [m] \setminus c\}| = x_c$ and such that $c(S) = \sum_{j=1}^n c(s_j)$ is minimum.

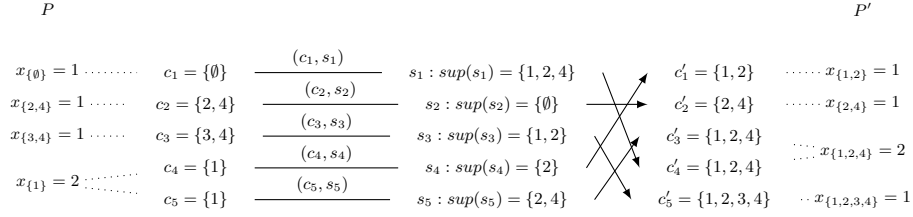


Fig. 4: Example of a profile $P' = \{x_{\{1,2\}} = 1, x_{\{2,4\}} = 1, x_{\{1,2,4\}} = 2, x_{\{1,2,3,4\}} = 1\}$ reachable from $P = \{x_{\{\emptyset\}} = 1, x_1 = 2, x_{\{2,4\}} = 1, x_{\{3,4\}} = 1\}$ through $S = \{s_1 : \text{sup}(s_1) = \{1, 2, 4\}, s_2 : \text{sup}(s_2) = \{\emptyset\}, s_3 : \text{sup}(s_3) = \{1, 2\}, s_4 : \text{sup}(s_4) = \{2\}, s_5 : \text{sup}(s_5) = \{2, 4\}\}$.

Remark that an instance I of $(\min \sum_0)_{\#0 \leq 1}$ can be solved optimally by solving optimally the instance $I' = (1, P = \{x_{\emptyset} = n, x_c = 0, \forall c \neq \emptyset\})$ of Π . The optimal solution of I' is indeed a set of n partial disjoint stacks of support $[m]$ of minimum cost.

We are now ready to define the following dynamic programming algorithm that solves any instance (l, P) of Π by parsing the instance block after block and branching for each of these blocks on every reachable profile.

Function $\text{MinSumZeroDP}(l, P)$

if $k == p + 1$ **then**

return 0;

return $\min(c(S') + \text{MinSumZeroDP}(l + 1, P'))$, with P' reachable from P through S' , where S' partition of B^l ;

Note that this dynamic programming assumes the existence of a procedure that enumerates *efficiently* all the profiles P' that are reachable from P . The existence of such a procedure will be shown thereafter.

Lemma 3. *For any instance of Π (l, P) , $\text{MinSumZeroDP}(l, P) = \text{Opt}(l, P)$.*

Proof. Lemma 3 is true as in a given block l , the algorithm tries every reachable profile, and the zeros of vectors in blocks $\mathcal{B} = \bigcup_{l' < l} B^{l'}$ cannot be matched with those of vectors in block $\mathcal{B}' = \bigcup_{l' \geq l} B^{l'}$. This is the reason why the support of the already created partial stacks (stored in profile P) is sufficient to keep a record of what have been done (the positions of the zeros in the partial stacks corresponding to P is not relevant). \square

Let us focus now on the procedure in charge of the enumeration of the reachable profile. A first and intuitive way to perform this operation is by guessing, for all $c, c' \subseteq [m]$, $y_{c, c'}$ the number of partial stacks in configuration c that will

be turned into configuration c' with vectors of current block B^l . For each such guess it is possible to greedily verify that each $y_{c,c'}$ can be satisfied with the vectors of the current block. As each of the $y_{c,c'}$ can take values from 0 to n and c and c' can be both enumerated in $\mathcal{O}^*(n^{2^m})$, the previous algorithm runs in $\mathcal{O}^*(n^{2^m})$.

This complexity can be improved as follows. The idea is to enumerate every possible profile P' and to verify using another dynamic programming algorithm if such a P' is reachable from P . We define $Aux_{P'}(P, X)$, that verifies if P' is reachable from P by using all vectors of X . If $X = \emptyset$, then the algorithm returns whether P is equal to P' or not. Otherwise, we consider the first vector v of X (we fix any arbitrary order) for which a branching is done on every possible assignment of v . More formally, the algorithm returns $\bigvee_{c \subseteq [m], x_c > 0, c \cap \text{sup}(v) = \emptyset} Aux_{P'}(P_2 = \{x'_l\}, X \setminus \{v\})$, where $x'_l = x_l - 1$ if $l = c$, $x'_l = x_l + 1$ if $l = c \cup \text{sup}(v)$, and $x'_l = x_l$ otherwise.

Using Aux in MinSumZeroDP , we get the following theorem.

Theorem 3. $(\min \sum 0)_{\#0 \leq 1}$ can be solved in $\mathcal{O}^*(n^{2^{m+2}})$.

We compute the overall complexity as follows: for each of the pn^{2^m} possible values of the parameters of MinSumZeroDP , the algorithm tries the n^{2^m} profiles P' , and run for each one $Aux_{P'}$ in $\mathcal{O}^*(n^{2^m} nm)$ (the first parameter of Aux can take n^{2^m} values, and the second nm as we just encode how many vectors left in X).

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