FACULTÉ
DES SCIENCES

UNIVERSITÉ LIBRE DE BRUXELLES

On the Various Extensions of the BMS Group

Thesis submitted by Romain RUZZICONI
in fulfilment of the requirements of the PhD Degree in Sciences
("Docteur en Sciences")
Academic year 2019-2020

Supervisor: Professor Glenn BARNICH
Theoretical and Mathematical Physics

Thesis jury:

Prof. Riccardo ARGURIO (Université libre de Bruxelles, Chair)
Prof. Geoffrey COMPÈRE (Université libre de Bruxelles, Secretary)
Prof. Daniel GRUMILLER (Technische Universität Wien)
Prof. Marios PETROPOULOS (École Polytechnique de Paris)

LA LIBERTÉ DE CHERCHER

Abstract

The Bondi-Metzner-Sachs-van der Burg (BMS) group is the asymptotic symmetry group of radiating asymptotically flat spacetimes. It has recently received renewed interest in the context of the flat holography and the infrared structure of gravity. In this thesis, we investigate the consequences of considering extensions of the BMS group in four dimensions with superrotations. In particular, we apply the covariant phase space methods on a class of first order gauge theories that includes the Cartan formulation of general relativity and specify this analysis to gravity in asymptotically flat spacetime. Furthermore, we renormalize the symplectic structure at null infinity to obtain the generalized BMS charge algebra associated with smooth superrotations. We then study the vacuum structure of the gravitational field, which allows us to relate the so-called superboost transformations to the velocity kick/refraction memory effect. Afterward, we propose a new set of boundary conditions in asymptotically locally (A)dS spacetime that leads to a version of the BMS group in the presence of a non-vanishing cosmological constant, called the Λ-BMS asymptotic symmetry group. Using the holographic renormalization procedure and a diffeomorphism between Bondi and Fefferman-Graham gauges, we construct the phase space of Λ-BMS and show that it reduces to the one of the generalized BMS group in the flat limit.

Contents

Acknowledgements

This thesis could not have been completed without the help and support of several people. First, I am indebted to my supervisor, Glenn Barnich, for having guided my first steps in the marvellous world of asymptotic symmetries, from my Master's studies to my PhD. I am grateful to him for having shared his extensive knowledge on the subject and his enthusiasm about research. He has been an example of rigour and independence that will definitely influence the future of my research. I would also like to thank him for the complete freedom that I had in my research, which allowed me to start new collaborations and complete side-projects.

Furthermore, I am grateful to Geoffrey Compère for the many projects and ideas he shared with me. I also thank him for his availability for discussions at all stages of the projects.

I would also like to thank Adrien Fiorucci for the numerous hours of stimulating discussions that were the origin of many of the original ideas presented in this thesis. Our collaboration and friendship are among the best memories I will keep from this PhD experience.

I thank my other collaborators, Luca Ciambelli, Pujian Mao, Charles Marteau, Marios Petropoulos, for a fruitful exchange of ideas. Collaborating with them was a real enlightenment.

I would like to thank Francesco Alessio, Laura Donnay, Daniel Grumiller, Yannick Herfray and Céline Zwikel for interesting discussions that will set the basis for future collaborations.

Moreover, I thank all the members of the Mathematical Physics research group for the nice and emulating atmosphere.

Once again, I would like to thank the members of my PhD jury Riccardo Argurio, Geoffrey Compère, Daniel Grumiller and Marios Petropoulos for agreeing to be my examiners and for their time dedicated to read this thesis.

From a personal perspective, I thank my family and friends for their support throughout the whole process.

This work was supported by the FRIA, F.R.S.-FNRS Belgium (2016-2020).

Chapter 1

Introduction

To specify most of the physical theories, one has to consider two ingredients: the kinematics which defines the allowed states and observables of the system, and the dynamics which dictates the evolution of the state through some equations of motion. An essential piece to define the kinematics is the set of boundary conditions that selects, using the equations of motion, the allowed solutions of the theory. Depending on the context, this set of boundary conditions should enable one to determine the exact initial conditions/characteristic initial value problem that one has to provide to select a particular solution in the allowed space. The choice of boundary conditions is dictated by the physical situation one wants to describe. As broadly illustrated in this thesis, several sets of boundary conditions may be relevant to specify the kinematics for the same dynamical part of the theory.

In this work, we are mainly interested by the study of boundary conditions in gauge theories, and especially in general relativity. Indeed, gauge theories are of major importance in physics since they are involved in the description of the four fundamental interactions through the standard model of particle physics and the general relativity theory. Furthermore, as their name suggests, gauge theories exhibit some symmetries of the dynamics called gauge symmetries. Among the gauge symmetries preserving the chosen boundary conditions, several will be trivial and seen as redundancies of the theory, while others will change the physical state of the system by their actions. The latter are called asymptotic symmetries and form a group (or, more generally, a groupoid) known as the asymptotic symmetry group. In particular, different sets of boundary conditions lead to different asymptotic symmetry groups.

In a series of seminal papers [1–3], Bondi, Metzner, Sachs and van der Burg have shown that the asymptotic symmetry group of four-dimensional general relativity in asymptotically flat spacetimes at null infinity is an infinite-dimensional group enhancing the Poincaré group and is called the (global) BMS group. It is given by the the semi-direct product between the Lorentz group and an infinite-dimensional enhancement of the translations, called the supertranslations. This result was very surprising since one could have naively expected to find the symmetry group of flat space by studying the behaviour of the gravitational field in asymptotic regions. However, this infinite-dimensional enhancement was necessary to allow for some radiative spacetime solutions. Furthermore, this analysis led to the Bondi mass loss formula, which states that the the flux of energy-momentum at null infinity is positive. This argument served to resolve the then-controversial debate of whether gravitational waves are physical or a pure gauge artifact of the linearized theory [4].

An extension of the global BMS algebra has recently been proposed, called the extended BMS algebra [5–7]. More precisely, the Lorentz part of the semi-direct sum defining the BMS algebra has been enhanced into the infinite-dimensional algebra of conformal transformations in two dimensions. These new symmetries are called superrotations (or super-Lorentz transformations [8]). At the level of the group, these superrotations are singular when considering the topology of the sphere as sections of null infinity. Therefore, only the global subgroup of the extended BMS group is globally well defined, which justifies the epithet "global". As discussed in the following, this singular extension has been shown to be of major importance when considered as symmetries of the $\mathcal{S}$-matrix of quantum gravity [9–11]. Even more recently, an alternative extension of the BMS group has been considered by replacing the singular superrotations with smooth Diff(S^2) superrotations [12, 13]. This new extension, called the generalized BMS group, is made possible by relaxing the definition of asymptotic flatness and allowing a fluctuating induced boundary metric.

It should be noted that the analysis of asymptotic symmetries in general relativity has been purused for other types of asymptotics and other spacetime dimensions, including three- and four-dimensional asymptotically anti-de Sitter (AdS) and asymptotically de Sitter (dS) spacetimes (see e.g. [14–19]). Furthermore, it has been performed on other types of gauge theories including Maxwell, Yang-Mills and Chern-Simons theories (see e.g. [20–26]). The interests of these investigations are various and depend on the main research question. In section 1.1, following [27], we relate the study of asymptotic symmetries in gauge theories with major research directions in theoretical physics.

1.1 Use of asymptotic symmetries

The study of asymptotic symmetries in gauge theories is an old subject that has recently received renewed interest. A first direction is motivated by the AdS/CFT correspondence where the asymptotic symmetries of the gravity theory in the bulk spacetime correspond to the global symmetries of the dual quantum field theory through the holographic dictionary [28–32]. A strong control of asymptotic symmetries allows us to investigate new holographic dualities. A second direction is driven by the recently-established connections among asymptotic symmetries, soft theorems and memory effects [10]. These connections furnish crucial information about the infrared structure of quantized gauge theories. In gravity, they may be relevant to solve the long-standing problem of black hole information paradox [33–37].

1.1.1 Holography

The *holographic principle* states that quantum gravity can be described in terms of lower-dimensional dual quantum field theories [28, 29]. A concrete realization of the holographic principle asserts that the type IIB string theory living in the bulk spacetime $\mathrm{AdS}_5 \times S^5$ is dual to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory living on the four-dimensional spacetime boundary [30]. The gravitational theory is effectively living in the five-dimensional spacetime AdS_5, the five dimensions of the factor S^5 being compactified. A first extension of this original holographic duality is the *AdS/CFT correspondence*, which tells us that the gravitational theory living in the $(d+1)$-dimensional asymptotically AdS spacetime is dual to a CFT living on the d-dimensional boundary. Other holographic dualities with different types of asymptotics have also been studied. A holographic dictionary enables one to interpret properties of the bulk theory in terms of the dual boundary theory. For example, the dictionary poses the following relationship between the symmetries of the two theories:

$$\left[\begin{array}{c} \text{Gauge symmetries in the bulk theory} \\ \Longleftrightarrow \\ \text{Global symmetries in the boundary theory.} \end{array} \right] \tag{1.1.1}$$

More specifically for us, consider a given bulk solution space with asymptotic symmetries. The correspondence tells us that a set of quantum field theories exist that are associated with the bulk solutions, such that in the UV regime, the *global symmetries* of these theories are exactly the *asymptotic symmetries* of the bulk solution space. Even if the AdS/CFT correspondence has not been proven yet, it has been verified in a number of situations and extended in various directions.

We now mention a famous hint in favour of this correspondence using the relation (1.1.1). Brown and Henneaux have shown that the asymptotic symmetry group for asymptotically AdS_3 spacetime with Dirichlet boundary conditions is given by the infinite-dimensional group of conformal transformations in two dimensions. Furthermore, they have revealed that the associated surface charges are finite, are integrable, and exhibit a non-trivial central extension in their algebra. This *Brown-Henneaux central charge* is given by

$$c = \frac{3\ell}{2G}, \tag{1.1.2}$$

where ℓ is the AdS_3 radius ($\Lambda = -1/\ell^2$) and G is the gravitational constant. The AdS/CFT correspondence indicates that there is a set of two-dimensional dual conformal field theories. The remarkable fact is that, when inserting the central charge (1.1.2) into the Cardy entropy formula valid for $2d$ CFT [38], this reproduces exactly the entropy of three-dimensional BTZ black hole solutions [39, 40].

The holographic principle is believed to hold in all types of asymptotics. In particular, in asymptotically flat spacetimes, from the correspondence (1.1.1), the dual theory would have BMS as the global symmetry. Important steps have been taken in this direction in three and four dimensions (see e.g. [41–48] and references therein). Furthermore, in four-dimensional asymptotically flat spacetimes, traces of two-dimensional CFT seem to appear, enabling the use of well-known techniques of the AdS/CFT correspondence [6, 49–57]. The global BMS symmetry can be seen as a conformal Carroll symmetry [58–60], which is especially relevant in the context of the fluid/gravity correspondence [61–66].

1.1.2 Infrared structure of gauge theories

A connection has recently been established among various areas of gauge theories that are *a priori* unrelated, namely *asymptotic symmetries*, *soft theorems* and *memory effects* (see [10] for a review). These fields of research are often referred to as the three corners of the infrared triangle of gauge theories (see figure 1.1).

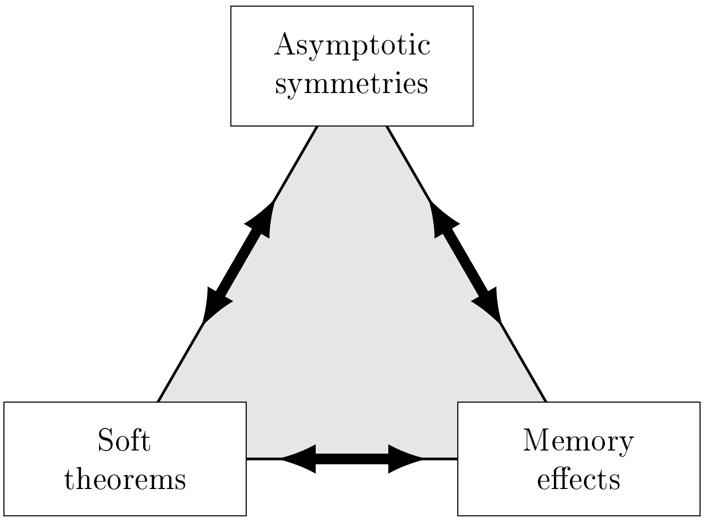


Figure 1.1: Infrared sector of gauge theories.

The first corner is the area of *asymptotic symmetries*, which is extensively studied in this thesis. The second corner is the topic of *soft theorems* [67–71]. These theorems state that any $(n+1)$-particles scattering amplitude involving a massless soft particle, namely a particle with momentum $q \to 0$ (that may be a photon, a gluon or a graviton), is equal to the n-particles scattering amplitude without the soft particle, multiplied by the soft factor, plus corrections of order q^0. We have

$$\mathcal{M}_{n+1}(q, p_1, \dots p_n) = S^{(0)} \mathcal{M}_n(p_1, \dots p_n) + \mathcal{O}(q^0), \tag{1.1.3}$$

where $S^{(0)} \sim q^{-1}$ is the soft factor whose precise form depends on the nature of the soft particle involved. Taking as soft particle a photon, gluon or graviton will respectively lead to the soft photon theorem, soft gluon theorem and soft graviton theorem. A remarkable property is that the soft factor is independent of the spin of the n particles involved in the process. Furthermore, some so-called *subleading soft theorems* have been established for the different types of soft particles and they provide some information about the subleading terms in q [72–76]. They take the form

$$\mathcal{M}_{n+1}(q, p_1, \dots p_n) = (S^{(0)} + S^{(1)}) \mathcal{M}_n(p_1, \dots p_n) + \mathcal{O}(q), \tag{1.1.4}$$

where $S^{(1)} \sim q^0$ is the subleading soft factor. Proposals for sub-subleading soft theorems can also be found [77–79].

The third corner of the triangle is the topic of memory effects [80–89]. In gravity, the *displacement memory effect* occurs, for example, in the passage of gravitational waves. It can be shown that this produces a permanent shift in the relative positions of a pair of inertial detectors. This shift is controlled by a field in the metric, called the memory field, that is turned on when the gravitational wave is passing through the spacetime region of interest. The analogous memory effects can also be established in electrodynamics (electromagnetic memory effect) [90, 91] and in Yang-Mills theory (color memory effect) [89], where a field is turned on as a result of a burst of energy passing through the region of interest, leading to an observable phenomenon. Notice that other memory effects have been identified in gravity [8, 92, 93, 93–98], including the *spin memory effect* and the *refraction memory effect*.

We now briefly discuss the relation between these different topics. It has been shown that if the quantum gravity $\mathcal{S}$-matrix is invariant under the BMS symmetry [99], then the Ward identity associated with the supertranslations is equivalent to the soft graviton theorem [100]. Furthermore, the displacement memory effect is equivalent to performing a supertranslation [101]. More precisely, the action of the supertranslation on the memory field has the same effect as a burst of gravitational waves passing through the region of interest. This can be understood as a vacuum transition process [102–106]. Finally, a Fourier transform enables us to relate the soft theorem with the memory effect, which closes the triangle. This triangle controlling the infrared structure of the theory has also been constructed for other gauge theories [89, 107, 108]. Moreover, subleading infrared triangles have been uncovered and discussed [8, 9, 11, 93, 107, 109, 110]. In particular, the Ward identities of superrotations have been shown to be equivalent to the subleading soft graviton theorem. Furthermore, the spin memory effect and the center-of-mass memory effect have been related to the superrotations.

Finally, let us mention that this understanding of the infrared structure of quantum gravity is relevant to address the *black hole information paradox* [33]. Indeed, an infinite number of soft gravitons are produced in the process of black hole evaporation. Through the above correspondence, these soft gravitons are related with surface charges, called *soft hairs*, that have to be taken into account in the information storage [34–37, 111, 112].

1.2 Original results

The aim of this thesis is to investigate some aspects of the BMS group and its various extensions, including the associated phase spaces, vacuum structures and memory effects. In doing so, we elaborate on the covariant phase space methods for first order gauge theories. In addition, a new version of the BMS symmetry for asymptotically (A)dS$_4$ will be presented. The original results discussed in this thesis are based on the following works:

- [A] Conserved currents in the Cartan formulation of general relativity
  Glenn Barnich, Pujian Mao, Romain Ruzziconi
  Proceedings of the workshop "About various kinds of interactions" (2016)
  arXiv:1611.01777

- [B] Superboost transitions, refraction memory and super-Lorentz charge algebra
  Geoffrey Compère, Adrien Fiorucci, Romain Ruzziconi
  Journal of High Energy Physics (2018)
  arXiv:1810.00377

- [C] The Λ-BMS$_4$ group of dS$_4$ and new boundary conditions for AdS$_4$
  Geoffrey Compère, Adrien Fiorucci, Romain Ruzziconi
  Classical and Quantum Gravity (2019)
  arXiv:1905.00971

- [D] Asymptotic symmetries in the gauge fixing approach and the BMS group
  Romain Ruzziconi
  Proceedings of Science (2020)
  arXiv:1910.08367

- [E] BMS current algebra in the context of the Newman-Penrose formalism
  Glenn Barnich, Pujian Mao, Romain Ruzziconi

Classical and Quantum Gravity (2020)
arXiv:1910.14588

- [F] The Λ-BMS$_4$ charge algebra
Geoffrey Compère, Adrien Fiorucci, Romain Ruzziconi
Submitted in Journal of High Energy Physics
arXiv:2004.10769

- [G] Conserved currents in the Palatini formulation of general relativity
Glenn Barnich, Pujian Mao, Romain Ruzziconi
Proceedings of Science (2020)
arXiv:2004.15002

- [H] Gauges in three-dimensional gravity and holographic fluids
Luca Ciambelli, Charles Marteau, Marios Petropoulos, Romain Ruzziconi
To be published

- [I] Fefferman–Graham and Bondi gauges in the fluid/gravity correspondence
Luca Ciambelli, Charles Marteau, Marios Petropoulos, Romain Ruzziconi
To be published

The next subsection briefly summarizes some of the main results and research guidelines of this thesis.

1.2.1 First order program

The covariant phase space methods allowing for constructing meaningful surface charges in gauge theories are based on jet bundles and homotopy operators. This powerful machinery can quickly become complicated for theories of second order derivative or higher. However, for first order theories, namely theories involving at most first order derivatives in the equations of motion and in the transformation of the fields, the computations simplify drastically and do not even require the technology of homotopy operators. Furthermore, for first order theories, the different procedures to construct surface charges, namely Barnich-Brandt [113–115] and Iyer-Wald [116–118] procedures, give the same results.

Another interesting feature is that most of the known gauge theories, including Maxwell, Yang-Mills, general relativity and Chern-Simons, are first order gauge theories, or own a formulation that is of first order using auxiliary fields. For example, Maxwell theory can be formulated as a first order gauge theory by considering the field strength as an auxiliary field (see e.g. [119]). Similarly, the Cartan formulation of general relativity is a first order theory.

In this thesis, we discuss covariant phase space formalism in the context of first order gauge theories. More specifically, we consider a class of theories that we call *covariantized Hamiltonian theories*, which includes all the examples cited above. In particular, we investigate the breaking in the conservation of the charges for these theories. We then specify the general results obtained in this context to write the expressions of the surface charges in different first order formulations of gravity, including Cartan formulation and a new Newman-Penrose-type formulation. Finally, we apply these results to the case of four-dimensional gravity in asymptotically flat spacetimes at null infinity and obtain the currents associated with extended BMS. The results of [120] are reproduced in a self-consistent way, and enlarged to allow for an arbitrary time-dependent conformal factor for the transverse boundary metric. This discussion is based on the works [A], [E] and [G].

1.2.2 Extended and generalized BMS

As discussed in the introduction, two extensions of the global BMS group have been considered. The first is called the extended BMS group and involves singular superrotations (and, consequently, singular supertranslations) on the celestial sphere [5–7]. The second one is called the generalized BMS group and involves smooth superrotations on the celestial sphere [12, 13]. In this thesis, we investigate both the phase spaces of the first and second extensions, using covariant phase space methods. A common feature between these analyses is that the associated charges are non-integrable but still satisfy an algebra, provided one modifies the Dirac bracket following the prescription of [121]. As reviewed in section 2.3, the non-integrability of the charges is related to their non-conservation due to the radiation at null infinity.

As explained in subsection 1.1.2, the BMS symmetries are related to gravitational memory effects. This relation shows up by investigating the vacuum orbit of the theory. Indeed, the vacuum structure of gravity in asymptotically flat spacetime is degenerate. The fields in the metric parametrizing the different vacua, which are turned on when acting with BMS transformations on the Minkowski space, are precisely the memory fields discussed above. In this thesis, we extend the work of [102–104] by studying the orbit of Minkowski under generalized BMS transformations. Furthermore, we relate a field that is turned on in the metric under superboost transformations, to the so-called refraction memory/velocity kick [95, 96, 122]. We show that this superboost field satisfies a Liouville equation. These results are based on [B]

1.2.3 BMS in asymptotically locally (A)dS$_4$ spacetimes

The BMS symmetry and its extensions are symmetries of asymptotically flat spacetimes. A legitimate question to ask is if the analogue symmetry also exists in asymptotically locally (A)dS$_4$ spacetimes. Such a generalization would be relevant for the two research guidelines discussed in subsection 1.1. Indeed, studying the BMS symmetry in AdS spacetimes, where holography is well controlled, would shed some light on holography in flat space. Furthermore, if the BMS symmetry exists in spacetimes with non-vanishing cosmological constant, it may be related to memory effects and soft theorems in this context [123–130].

In this thesis, we propose a version of the BMS symmetry in presence of a non-vanishing cosmological constant. This new asymptotic symmetry group, called the Λ-BMS$_4$ group, is obtained by imposing some partial Dirichlet boundary conditions in asymptotically locally (A)dS$_4$ spacetimes. We show that this proposal reduces to the generalized BMS group in the flat limit. Furthermore, we prove that the flat limit also works at the level of the associated phase spaces. This analysis is based on a diffeomorphism between Bondi and Fefferman-Graham gauges that we have explicitly constructed. In particular, the charge algebra of the most general asymptotically locally (A)dS$_4$ spacetime is worked out in the Fefferman-Graham gauge using the covariant phase space methods and the holographic renormalization procedure. Then, imposing the boundary conditions that lead to Λ-BMS$_4$ symmetry, we translate the symplectic structure into the Bondi gauge, where the flat limit is well defined. Taking the flat limit leads to the symplectic structure of generalized BMS discussed above. This presentation is based on [C], [D] and [F].

1.3 Plan

The rest of this thesis is organized as follows. In Chapter 2, we present in a self-consistent way some methods to study asymptotic symmetries in gauge theories and how to construct meaningful surface charges. This chapter is essentially a review of the existing literature, with an attempt to present the different concepts in both a unified and more abstract way. Examples illustrating the general definitions are provided. Some of those are based on results obtained in the framework of this thesis and explained in more detail in the subsequent chapters.

In chapter 3, we restrict our study to a particular class of first order gauge theories that we call covariantized Hamiltonian theories. We show that the covariant phase space methods simply drastically in this framework and do not require the technology of jet bundles and homotopy operators discussed in Chapter 2 (see also appendix A). Furthermore, we provide a discussion on vielbeins and connection by including torsion and non-metricity into the standard discussion. Then we investigate the Cartan formulation of general relativity and its different avatars and derive the expressions of the surface charges. These are particular examples of covariantized Hamiltonian theories. Starting from a Newman-Penrose-adapted variational principle, we derive the BMS current algebra in a self-consistent way for an arbitrary u-dependent conformal factor.

In chapter 4, we study the phase space associated with the generalized BMS symmetry. In particular, we show that a renormalization procedure using Iyer-Wald ambiguity is needed to obtain finite symplectic structure. The associated charges are finite, but non-integrable. They satisfy an algebra when using the modified Dirac bracket.

In chapter 5, we act on the Minkowski space with (both extended and generalized) BMS transformations and obtain the orbit of vacua. We then relate the superboosts transformations, which are part of the BMS symmetries, to the velocity kick/refraction memory. Finally, we propose a Wald-Zoupas-like prescription to isolate meaningful finite charges from the infinitesimal non-integrable expressions. Applying this prescription to the generalized BMS charges leads to the finite charges that are used in the Ward identities to establish the equivalence with soft theorems.

In chapter 6, we study the most general solution spaces and the residual gauge diffeomorphisms of Fefferman-Graham and Bondi gauges in three and four dimensions. We relate the results obtained in the two gauges by constructing a diffeomorphism that maps one gauge to the other. We then focus on the four-dimensional case and propose new boundary conditions in asymptotically locally (A)dS$_4$ spacetimes. We show that the associated asymptotic symmetry group, called the Λ-BMS$_4$ group, is infinite-dimensional and reduces to the generalized BMS group in the flat limit. Then, using the holographic renormalization procedure, we construct the phase space associated with the most general asymptotically locally AdS$_4$ spacetimes in Fefferman-Graham gauge. That allows us to derive the associated charge algebra that we specify for Λ-BMS$_4$ symmetry. Transforming the Λ-BMS$_4$ symplectic structure through the diffeomorphism between Fefferman-Graham and Bondi gauges, and taking the flat limit, we prove that it reduces to the generalized BMS symplectic structure. Finally, we study new mixed boundary conditions in asymptotically locally AdS$_4$ spacetime that allow us to have a well-defined Cauchy problem. The associated asymptotic symmetry group is an infinite-dimensional subgroup of Λ-BMS$_4$ consisting of the area preserving diffeomorphisms and the time translations.

This thesis also contains several appendices that are referenced in the core of the text.

Chapter 2

Asymptotic symmetries and surface charges

This chapter is an introduction to asymptotic symmetries in gauge theories, with a focus on general relativity in four dimensions. We explain how to impose consistent sets of boundary conditions in the gauge fixing approach and how to derive the asymptotic symmetry parameters. The different procedures to obtain the associated charges are presented. As an illustration of these general concepts, the examples of four-dimensional general relativity in asymptotically locally (A)dS$_4$ and asymptotically flat spacetimes are covered. This enables us to discuss the different extensions of the BMS group that will be investigated with more details in the subsequent chapters.

This chapter essentially reproduces the lecture notes [27].

2.1 Definitions of asymptotics

Several frameworks exist to impose boundary conditions in gauge theories. Some of them are mentioned next.

2.1.1 Geometric approach

The geometric approach of boundary conditions was initiated by Penrose, who introduced the techniques of conformal compactification to study general relativity in asymptotically flat spacetimes at null infinity [131, 132]. According to this perspective, the boundary conditions are defined by requiring that certain data on a fixed boundary be preserved. The asymptotic symmetry group G is then defined as the quotient:

$$G = \frac{\text{Gauge transformation preserving the boundary conditions}}{\text{Trivial gauge transformations}}, \tag{2.1.1}$$

where the trivial gauge transformations are the gauge transformations that reduce to the identity on the boundary. In other words, the asymptotic symmetry group is isomorphic to the group of gauge transformations induced on the boundary which preserve the given data. This is the *weak* definition of the asymptotic symmetry group. A *stronger* definition of the asymptotic symmetry group is given by the quotient (2.1.1), where the trivial gauge transformations are now the gauge transformations that have associated vanishing charges.

The geometric approach was essentially used in gravity theory and led to much progress in the study of symmetries and symplectic structures for asymptotically flat spacetimes at null infinity [4,133–136] and spatial infinity [137,138]. It was also considered to study asymptotically (A)dS spacetimes [19,139–141]. Moreover, this framework was recently applied to study boundary conditions and associated phase spaces on null hypersurfaces [142].

The advantage of this approach is that it is manifestly gauge invariant, since we do not refer to any particular coordinate system to impose the boundary conditions. Furthermore, the geometric interpretation of the symmetries is transparent. The weak point is that the definition of boundary conditions is rigid. It is a non-trivial task to modify a given set of boundary conditions in this framework to highlight new asymptotic symmetries. It is often *a posteriori* that boundary conditions are defined in this framework, after having obtained the results in coordinates.

2.1.2 Gauge fixing approach

A gauge theory has redundant degrees of freedom. The gauge fixing approach consists in using the gauge freedom of the theory to impose some constraints on the fields. This enables one to quotient the field space to eliminate some of the unphysical or pure gauge redundancies in the theory. For a given gauge theory, an *appropriate gauge fixing* (where appropriate will be defined below) still allows some redundancy. For example, in electrodynamics, the gauge field A_μ transforms as $A_\mu \to A_\mu + \partial_\mu \alpha$ (α is a function of the spacetime coordinates) under a gauge transformation. The Lorenz gauge is defined by setting $\partial_\mu A^\mu = 0$. This gauge can always be reached using the gauge redundancy, since $\partial_\mu \partial^\mu \alpha = -\partial_\nu A^\nu$ always admits a solution for α, regardless of the exact form of A_μ. However, residual gauge transformations remain that preserve the Lorenz gauge. These are given by $A_\mu \to A_\mu + \partial_\mu \beta$, where β is a function of the spacetime coordinates satisfying $\partial_\mu \partial^\mu \beta = 0$ (see, e.g., [143]). The same phenomenon occurs in general relativity where spacetime diffeomorphisms can be performed to reach a particular gauge defined by some conditions imposed on the metric $g_{\mu\nu}$. Some explicit examples are discussed below.

Then, the boundary conditions are imposed on the fields of the theory written in the chosen gauge. The *weak* version of the definition of the asymptotic symmetry group is given by

$$G_{\text{weak}} = \left[\begin{array}{l} \text{Residual gauge diffeomorphisms} \\ \text{preserving the boundary conditions.} \end{array} \right] \tag{2.1.2}$$

Intuitively, the gauge fixing procedure eliminates part of the pure gauge degrees of freedom, namely, the trivial gauge transformations defined under (2.1.1). Therefore, fixing the gauge is similar to taking the quotient as in equation (2.1.1), and the two definitions of asymptotic symmetry groups coincide in most of the practical situations. As in the geometric approach, a stronger version of the asymptotic symmetry group exists and is given by

$$G_{\text{strong}} = \left[\begin{array}{l} \text{Residual gauge diffeomorphisms preserving the boundary} \\ \text{conditions with associated non-vanishing charges.} \end{array} \right] \tag{2.1.3}$$

Notice that $G_{\text{strong}} \subseteq G_{\text{weak}}$[1].

The advantage of the gauge fixing approach is that it is highly flexible to impose boundary conditions, since we are working with explicit expressions in coordinates. For example, the BMS group in four dimensions was first discovered in this framework [1–3]. Furthermore, a gauge fixing is a local consideration (i.e. it holds in a coordinate patch of the spacetime). Therefore, the global considerations related to the topology are not directly relevant in this analysis, thereby allowing further flexibility. For example, as we will discuss in subsection 2.2.4, this allowed to consider singular extensions of the BMS group: the Witt $\times$ Witt superrotations [5,7]. These new asymptotic symmetries are well-defined locally; however, they have poles on the celestial sphere. In the geometric approach, one would have to modify the topology of the spacetime boundary to allow these superrotations by considering some punctured celestial sphere [145,146]. The weakness of this approach is that it is not manifestly gauge invariant. Hence, even if the gauge fixing approach is often preferred to unveil new boundary conditions and symmetries, the geometric approach is complementary and necessary to make the gauge invariance of the results manifest. In section 2.2, we study the gauge fixing approach and provide some examples related to gravity in asymptotically flat and asymptotically (A)dS spacetimes.

[1] One of the most striking examples of the difference between the weak and the strong definitions of the asymptotic symmetry group is given by considering Neumann boundary conditions in asymptotically AdS_{d+1} spacetimes. Indeed, in this situation, we have $G_{\text{weak}} = \text{Diff}(\mathbb{R} \times S^{d-1})$, and G_{strong} is trivial [144].

2.1.3 Hamiltonian approach

Some alternative approaches exist that are also powerful in practice. For example, in the Hamiltonian formalism, asymptotically flat [147] and AdS [14, 15] spacetimes have been studied at spatial infinity. Furthermore, the global BMS group was recently identified at spatial infinity using twisted parity conditions [148–150]. In this framework, the computations are done in a coordinate system making the split between space and time explicit, without performing any gauge fixing. Then, the asymptotic symmetry group is defined as the quotient between the gauge transformations preserving the boundary conditions and the trivial gauge transformations, where trivial means that the associated charges are identically vanishing on the phase space. This definition of the asymptotic symmetry group corresponds to the strong definition in the two first approaches.

2.2 Asymptotic symmetries in the gauge fixing approach

We now focus on the aforementioned gauge fixing approach of asymptotic symmetries in gauge theories. We illustrate the different definitions and concepts using examples, with a specific focus on asymptotically flat and asymptotically (A)dS spacetimes in four-dimensional general relativity.

2.2.1 Gauge fixing procedure

Definition [Gauge symmetry] Let us start with a Lagrangian theory in a n-dimensional spacetime M

$$S[\phi] = \int_M \mathbf{L}[\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \ldots], \tag{2.2.1}$$

where $\mathbf{L} = L \mathrm{d}^n x$ is the Lagrangian and $\phi = (\phi^i)$ are the fields of the theory. A *gauge transformation* is a transformation acting on the fields, and which depends on parameters $f = (f^\alpha)$ that are taken to be arbitrary functions of the spacetime coordinates. We write

$$\begin{aligned} \delta_f \phi^i &= R^i[f] \\ &= R^i_\alpha f^\alpha + R^{i\mu}_\alpha \partial_\mu f^\alpha + R^{i(\mu\nu)}_\alpha \partial_\mu \partial_\nu f^\alpha + \ldots \\ &= \sum_{k \geqslant 0} R^{i(\mu_1 \ldots \mu_k)}_\alpha \partial_{\mu_1} \ldots \partial_{\mu_k} f^\alpha \end{aligned} \tag{2.2.2}$$

the infinitesimal gauge transformation of the fields. In this expression, $R^{i(\mu_1 ... \mu_k)}_\alpha$ are *local functions*, namely functions of the coordinates, the fields, and their derivatives. The gauge transformation is a *symmetry* of the theory if, under (2.2.2), the Lagrangian transforms as

$$\delta_f \mathbf{L} = \mathrm{d}\mathbf{B}_f, \tag{2.2.3}$$

where $\mathbf{B}_f = B^\mu_f (\mathrm{d}^{n-1}x)_\mu$.

Examples We illustrate this definition by providing some examples. First, consider classical vacuum electrodynamics

$$S[A] = \int_M \mathbf{F} \wedge \star\mathbf{F}, \tag{2.2.4}$$

where $\mathbf{F} = \mathrm{d}\mathbf{A}$ and $\mathbf{A}$ is a 1-form. It is straightforward to check that the gauge transformation $\delta_\alpha \mathbf{A} = \mathrm{d}\alpha$, where α is an arbitrary function of the coordinates, is a symmetry of the theory.

Now, consider the general relativity theory

$$S[g] = \frac{1}{16\pi G} \int_M (R - 2\Lambda)\sqrt{-g}\,\mathrm{d}^n x, \tag{2.2.5}$$

where R and $\sqrt{-g}$ are the scalar curvature and the square root of minus the determinant associated with the metric $g_{\mu\nu}$ respectively, and Λ is the cosmological constant. It can be checked that the gauge transformation $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\rho\nu}\partial_\mu \xi^\rho$, where ξ^μ is a vector field generating a diffeomorphism, is a symmetry of the theory.

Notice that in these examples, the transformation of the fields (2.2.2) is of the form

$$\delta_f \phi^i = R^i_\alpha f^\alpha + R^{i\mu}_\alpha \partial_\mu f^\alpha, \tag{2.2.6}$$

namely they involve at most first order derivatives of the parameters.

Definition [Gauge fixing] Starting from a Lagrangian theory (2.2.1) with gauge symmetry (2.2.2), the *gauge fixing* procedure involves imposing some algebraic or differential constraints on the fields in order to eliminate (part of) the redundancy in the description of the theory. We write

$$G[\phi] = 0 \tag{2.2.7}$$

a generic gauge fixing condition. This gauge has to satisfy two conditions:

- It has to be reachable by a gauge transformation, which means that the number of independent conditions in (2.2.7) is inferior or equal to the number of independent parameters $f = (f^\alpha)$ generating the gauge transformation.

- It has to use all of the available freedom of the arbitrary functions parametrizing the gauge transformations to reach the gauge[2], which means that the number of independent conditions in (2.2.7) is superior or equal to the number of independent parameters $f = (f^\alpha)$ generating the gauge transformations.

Considering these two requirements together tells us that the number of independent gauge fixing conditions in (2.2.7) has to be equal to the number of independent gauge parameters $f = (f^\alpha)$ involved in the fields transformation (2.2.2).

Examples In electrodynamics, several gauge fixings are commonly used. Let us mention the Lorenz gauge $\partial^\mu A_\mu = 0$, the Coulomb gauge $\partial^i A_i = 0$, the temporal gauge $A_0 = 0$, and the axial gauge $A_3 = 0$. As previously discussed, the Lorenz gauge can always be reached by performing a gauge transformation. We can check that the same statement holds for all the other gauge fixings. Notice that these gauge fixing conditions involve only one constraint, as there is only one free parameter α in the gauge transformation.

In gravity, many gauge fixings are also used in practice. For example, the *De Donder (or harmonic) gauge* requires that the coordinates x^μ be harmonic functions, namely, $\Box x^\mu = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\partial^\nu x^\mu) = 0$. Notice that the number of constraints, n, is equal to the number of independent gauge parameters ξ^μ. This gauge condition is suitable for studying gravitational waves in perturbation theory (see, e.g., [151]).

Another important gauge fixing in configurations where $\Lambda \neq 0$ is the *Fefferman-Graham gauge* [152–156]. We write the coordinates as $x^\mu = (\rho, x^a)$, where $a = 1, \ldots, n-1$ and ρ is an expansion parameter ($\rho = 0$ is at the spacetime boundary, and $\rho > 0$ is in the bulk). It is defined by the following conditions:

$$g_{\rho\rho} = -\frac{(n-1)(n-2)}{2\Lambda\rho^2}, \quad g_{\rho a} = 0 \tag{2.2.8}$$

(n conditions). The coordinate ρ is spacelike for $\Lambda < 0$ and timelike for $\Lambda > 0$. The most general metric takes the form

$$ds^2 = -\frac{(n-1)(n-2)}{2\Lambda}\frac{\mathrm{d}\rho^2}{\rho^2} + \gamma_{ab}(\rho, x^c)\mathrm{d}x^a\mathrm{d}x^b. \tag{2.2.9}$$

Finally, the *Bondi gauge* will be relevant for us in the following [1–3]. This gauge fixing is valid for both $\Lambda = 0$ and $\Lambda \neq 0$ configurations. Writing the coordinates as (u, r, x^A), where $x^A = (\theta_1, \ldots, \theta_{n-2})$ are the transverse angular coordinates on the

[2] If the available freedom is not used, we talk about *partial gauge fixing*. In this configuration, there are still some arbitrary functions of the coordinates in the parameters of the residual gauge transformations.

$(n-2)$-celestial sphere, the Bondi gauge is defined by the following conditions[3]:

$$g_{rr} = 0, \quad g_{rA} = 0, \quad \partial_r \left(\frac{\det g_{AB}}{r^{2(n-2)}} \right) = 0 \tag{2.2.10}$$

(n conditions). These conditions tell us that, geometrically, u labels null hypersurfaces in the spacetime, x^A labels null geodesics inside a null hypersurface, and r is the luminosity distance along the null geodesics. The most general metric takes the form

$$ds^2 = e^{2\beta} \frac{V}{r} \mathrm{d}u^2 - 2e^{2\beta} \mathrm{d}u \mathrm{d}r + g_{AB}(\mathrm{d}x^A - U^A \mathrm{d}u)(dx^B - U^B \mathrm{d}u) \tag{2.2.11}$$

where β, U^A and $\frac{V}{r}$ are arbitrary functions of the coordinates, and the $(n-2)$-dimensional metric g_{AB} satisfies the determinant condition in the third equation of (2.2.10). Let us mention that the Bondi gauge is closely related to the *Newman-Unti gauge* [157, 158] involving only algebraic conditions:

$$g_{rr} = 0, \quad g_{rA} = 0, \quad g_{ru} = -1 \tag{2.2.12}$$

(n conditions).

Definition [Residual gauge transformation] After having imposed a gauge fixing as in equation (2.2.7), there usually remain some *residual gauge transformations*, namely gauge transformations preserving the gauge fixing condition. Formally, the residual gauge transformations with generators F have to satisfy $\delta_f G[\phi] = 0$. They are local functions parametrized as $f = f(a)$, where the parameters a are arbitrary functions of $(n-1)$ coordinates.

Examples Consider the Lorenz gauge $\partial^\mu A_\mu = 0$ in electrodynamics. As we discussed earlier, the residual gauge transformations for the Lorenz gauge are the gauge transformations $\delta_\alpha A_\mu = \partial_\mu \alpha$, where $\partial^\mu \partial_\mu \alpha = 0$.

Similarly, consider the Fefferman-Graham gauge (2.2.8) in general relativity with $\Lambda \neq 0$. The residual gauge transformations generated by ξ^μ have to satisfy $\mathcal{L}_\xi g_{\rho\rho} = 0$ and $\mathcal{L}_\xi g_{\rho a} = 0$. The solutions to these equations are given by

$$\xi^\rho = \sigma(x^a)\rho, \quad \xi^a = \xi^a_0(x^b) + \frac{(n-1)(n-2)}{2\Lambda} \partial_b \sigma \int_0^\rho \frac{\mathrm{d}\rho'}{\rho'} \gamma^{ab}(\rho', x^c). \tag{2.2.13}$$

These solutions are parametrized by n arbitrary functions σ and ξ^a_0 of $(n-1)$ coordinates x^a.

[3]Notice that the determinant condition in (2.2.10) is weaker than the historical one considered in [1–3]. We refer to appendix B for more details on this condition.

In the Bondi gauge (2.2.10), the residual gauge transformations generated by ξ^μ have to satisfy $\mathcal{L}_\xi g_{rr} = 0$, $\mathcal{L}_\xi g_{rA} = 0$ and $g^{AB}\mathcal{L}_\xi g_{AB} = 2(n-2)\omega$, where ω is an arbitrary function of (u, x^A) (see appendix B). The solutions to these equations are given by

$$\begin{aligned}
\xi^u &= f, \\
\xi^A &= Y^A + I^A, \quad I^A = -\partial_B f \int_r^\infty \mathrm{d}r' (e^{2\beta} g^{AB}), \\
\xi^r &= -\frac{r}{n-2}(\mathcal{D}_A Y^A - (n-2)\omega + \mathcal{D}_A I^A - \partial_B f U^B + \frac{1}{2} f g^{-1}\partial_u g),
\end{aligned} \tag{2.2.14}$$

where $\partial_r f = 0 = \partial_r Y^A$, and $g = \det(g_{AB})$ [24]. The covariant derivative $\mathcal{D}_A$ is associated with the $(n-2)$-dimensional metric g_{AB}. The residual gauge transformations are parametrized by the n functions ω, f and Y^A of $(n-1)$ coordinates (u, x^A).

2.2.2 Boundary conditions

Definition [Boundary conditions] Once a gauge condition (2.2.7) has been fixed, we can impose *boundary conditions* for the theory by requiering some constraints on the fields in a neighbourhood of a given spacetime region. Most of those boundary conditions are fall-off conditions on the fields in the considered asymptotic region[4], or conditions on the leading functions in the expansion. This choice of boundary conditions is motivated by the physical model that we want to consider. A set of boundary conditions is usually considered to be interesting if it provides non-trivial asymptotic symmetry group and solution space, exhibiting interesting properties for the associated charges (finite, generically non-vanishing, integrable and conserved; see below). If the boundary conditions are too strong, the asymptotic symmetry group will be trivial, with vanishing surface charges. Furthermore, the solution space will not contain any solution of interest. If they are too weak, the associated surface charges will be divergent. Consistent and interesting boundary conditions should therefore be located between these two extreme situations.

Examples Let us give some examples of boundary conditions in general relativity theory. Many examples of boundary conditions for other gauge theories can be found in the literature (see e.g. [20–26]).

Let us consider the *Bondi gauge* defined in equation (2.2.10) in dimension $n \geqslant 3$. There exist several definitions of *asymptotic flatness at null infinity* ($r \to \infty$) in the literature. For all of them, we require the following preliminary boundary conditions

[4]Notice that the asymptotic region could be taken not only at (spacelike, null or timelike) infinity, but also in other spacetime regions, such as near a black hole horizon [34–37, 159–163].

on the functions of the metric (2.2.11) in the asymptotic region $r \to \infty$:

$$\beta = o(1), \quad \frac{V}{r} = o(r^2), \quad U^A = o(1), \quad g_{AB} = r^2 q_{AB} + r C_{AB} + D_{AB} + \mathcal{O}(r^{-1}), \tag{2.2.15}$$

where q_{AB}, C_{AB} and D_{AB} are $(n-2)$-dimensional symmetric tensors, which are functions of (u, x^A). Notice in particular that q_{AB} is kept free at this stage.

A first definition of asymptotic flatness at null infinity (AF1) is a sub-case of (2.2.15). In addition to all these fall-off conditions, we require the transverse boundary metric q_{AB} to have a fixed determinant, namely,

$$\sqrt{q} = \sqrt{\bar{q}}, \tag{2.2.16}$$

where $\bar{q}$ is a fixed volume element (which may possibly depend on time) on the $(n-2)$-dimensional transverse space [8, 12, 13, 164].

A second definition of asymptotic flatness at null infinity (AF2) is another sub-case of the definition (2.2.15). All the conditions are the same, except that we require that the transverse boundary metric q_{AB} be conformally related to the unit $(n-2)$-sphere metric, namely,

$$q_{AB} = e^{2\varphi} \mathring{q}_{AB}, \tag{2.2.17}$$

where $\mathring{q}_{AB}$ is the unit $(n-2)$-sphere metric [6]. Note that for $n = 4$, this condition can always be reached by a coordinate transformation, since every metric on a two dimensional surface is conformally flat (but even in this case, as we will see below, this restricts the form of the symmetries).

A third definition of asymptotic flatness at null infinity (AF3), which is the historical one [1–3], is a sub-case of the second definition (2.2.17). We require (2.2.15) and we demand that the transverse boundary metric q_{AB} be the unit $(n-2)$-sphere metric, namely,

$$q_{AB} = \mathring{q}_{AB}. \tag{2.2.18}$$

Note that this definition of asymptotic flatness is the only one that has the property to be *asymptotically Minkowskian*, that is, for $r \to \infty$, the leading orders of the spacetime metric (2.2.11) tend to the Minkowski line element $\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2 \mathring{q}_{AB} \mathrm{d}x^A \mathrm{d}x^B$.

Let us now present several definitions of *asymptotically (A)dS spacetimes* in both the *Fefferman Graham gauge* (2.2.8) and *Bondi gauge* (2.2.10). A preliminary boundary condition, usually called the asymptotically locally (A)dS condition, requires the following conditions on the functions of the Fefferman-Graham metric (2.2.9):

$$\gamma_{ab} = \mathcal{O}(\rho^{-2}) \tag{2.2.19}$$

or, equivalently, $\gamma_{ab} = \rho^{-2} g^{(0)}_{ab} + o(\rho^{-2})$. Notice that the $(n-1)$-dimensional boundary metric $g^{(0)}_{ab}$ is kept free in this preliminary set of boundary conditions, thus justifying the adjective "locally" [165]. In the Bondi gauge, as we will see below, these fall-off conditions are (on-shell) equivalent to demand that

$$g_{AB} = \mathcal{O}(r^2) \tag{2.2.20}$$

or, equivalently, $g_{AB} = r^2 q_{AB} + o(r^2)$.

A first definition of asymptotically (A)dS spacetime (AAdS1) is a sub-case of the definition (2.2.19). In addition to these fall-off conditions, we demand the following constraints on the $(n-1)$-dimensional boundary metric $g^{(0)}_{ab}$:

$$g^{(0)}_{tt} = \frac{2\Lambda}{(n-1)(n-2)}, \quad g^{(0)}_{tA} = 0, \quad \det(g^{(0)}_{ab}) = \frac{2\Lambda}{(n-1)(n-2)} \bar{q}, \tag{2.2.21}$$

where $\bar{q}$ is a fixed volume form for the transverse $(n-2)$-dimensional space (which may possibly depend on t) [166]. In the Bondi gauge, the boundary conditions (2.2.21) translate into

$$\beta = o(1), \quad \frac{V}{r} = \frac{2r^2\Lambda}{(n-1)(n-2)} + o(r^2), \quad U^A = o(1), \quad \sqrt{q} = \sqrt{\bar{q}}. \tag{2.2.22}$$

Notice the similarity of these conditions to the definition (AF1) (equations (2.2.15) and (2.2.16)) of asymptotically flat spacetime.

A second definition of asymptotically AdS spacetime[5] (AAdS2) is a sub-case of the definition (2.2.19). We require the same conditions as in the preliminary boundary condition (2.2.19), except that we demand that the $(n-1)$-dimensional boundary metric $g^{(0)}_{ab}$ be fixed [15]. These conditions are called *Dirichlet boundary conditions*. One usually chooses the cylinder metric as the boundary metric, namely,

$$g^{(0)}_{ab} \mathrm{d}x^a \mathrm{d}x^b = \frac{2\Lambda}{(n-1)(n-2)} \mathrm{d}t^2 + \mathring{q}_{AB} \mathrm{d}x^A \mathrm{d}x^B, \tag{2.2.23}$$

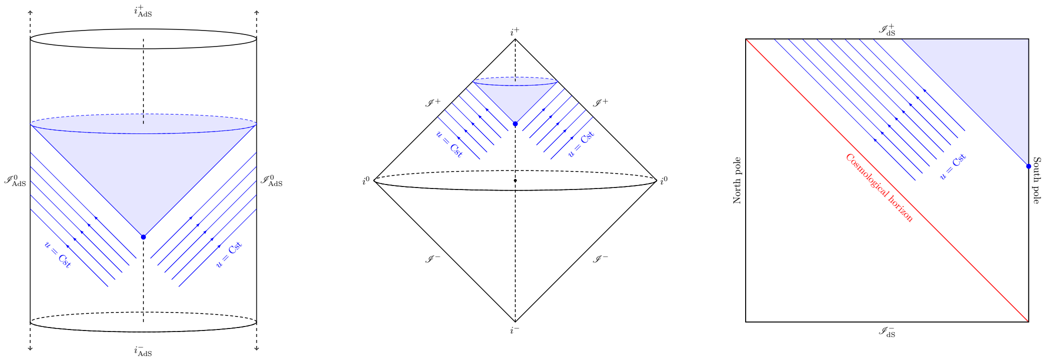
where $\mathring{q}_{AB}$ are the components of the unit $(n-2)$-sphere metric (as in the Bondi gauge, the upper case indices $A, B, \dots$ run from 3 to n, and $x^a = (t, x^A)$). In the Bondi gauge, the boundary conditions (2.2.23) translate into

$$\beta = o(1), \quad \frac{V}{r} = \frac{2r^2\Lambda}{(n-1)(n-2)} + o(r^2), \quad U^A = o(1), \quad q_{AB} = \mathring{q}_{AB}. \tag{2.2.24}$$

Notice the similarity of these conditions to the definition (AF3) (equations (2.2.15) and (2.2.18)) of asymptotically flat spacetime.

As we see it, the Bondi gauge is well-adapted for each type of asymptotics (see figure 2.1), while the Fefferman-Graham gauge is only defined in asymptotically (A)dS spacetimes.

[5] This choice is less relevant for asymptotically dS spacetimes, since it strongly restricts the Cauchy problem and the bulk spacetime dynamics [19, 139].



AdS case ($\Lambda < 0$). Flat case ($\Lambda = 0$). dS case ($\Lambda > 0$).

Figure 2.1: Bondi gauge for any Λ.

2.2.3 Solution space

Definition [Solution space] Given a gauge fixing (2.2.7) and boundary conditions, a *solution* of the theory is a field configuration $\tilde{\phi}$ satisfying $G[\tilde{\phi}] = 0$, the boundary conditions, and the *Euler Lagrange-equations*

$$\left.\frac{\delta \mathbf{L}}{\delta \phi^i}\right|_{\tilde{\phi}} = 0, \tag{2.2.25}$$

where the *Euler-Lagrange derivative* is defined in equation (A.2.1). The set of all solutions of the theory is called the *solution space*. It is parametrized as $\tilde{\phi} = \tilde{\phi}(b)$, where the parameters b are arbitrary functions of $(n-1)$ coordinates.

Examples We now provide some examples of solution spaces of four-dimensional general relativity in different gauge fixings. These examples will be re-discussed in details in the remaining of the text (see e.g. subsections 6.2.1 and 6.2.2). We first consider the *Fefferman-Graham gauge* in asymptotically (A)dS$_4$ spacetimes with preliminary boundary conditions (2.2.19). Solving the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{2.2.26}$$

we obtain the following analytic fall-offs:

$$\gamma_{ab} = \rho^{-2} g^{(0)}_{ab} + \rho^{-1} g^{(1)}_{ab} + g^{(2)}_{ab} + \rho g^{(3)}_{ab} + \mathcal{O}(\rho^2), \tag{2.2.27}$$

where $g^{(i)}_{ab}$ are functions of x^a [152–156]. The only free data in this expansion are $g^{(0)}_{ab}$ and $g^{(3)}_{ab}$. All the other coefficients are determined in terms of these free data. Following the holographic dictionary, we call $g^{(0)}_{ab}$ the boundary metric and we define

$$T_{ab} = \frac{\sqrt{3|\Lambda|}}{16\pi G} g^{(3)}_{ab} \tag{2.2.28}$$

as the stress energy tensor. From the Einstein equations, we have

$$g^{(0)}_{ab} T^{ab} = 0, \quad D^{(0)}_a T^{ab} = 0, \tag{2.2.29}$$

where $D^{(0)}_a$ is the covariant derivative with respect to $g^{(0)}_{ab}$. In summary, the solution space of general relativity in the Fefferman-Graham gauge with the preliminary boundary condition (2.2.19) is parametrized by the set of functions

$$\{g^{(0)}_{ab}, T_{ab}\}_{\Lambda\neq 0}, \tag{2.2.30}$$

where T_{ab} satisfies (2.2.29) (11 functions).

Now, for the restricted set of boundary conditions (2.2.21), that is, (AAdS1), the solution space reduces to

$$\{g^{(0)}_{AB}, T_{ab}\}_{\Lambda\neq 0}, \tag{2.2.31}$$

where $g^{(0)}_{AB}$ has a fixed determinant and T_{ab} satisfies (2.2.29) (7 functions). Finally, for Dirichlet boundary conditions (2.2.23) (AAdS2), the solution space reduces to

$$\{T_{ab}\}_{\Lambda\neq 0}, \tag{2.2.32}$$

where T_{ab} satisfies (2.2.29) (5 functions).

Let us now consider the *Bondi gauge* in asymptotically (A)dS$_4$ spacetimes with preliminary boundary condition (2.2.20). From the Fefferman-Graham theorem and the gauge matching between Bondi and Fefferman-Graham that is described in appendix D (see also [166, 167]), we know that the functions appearing in the metric admit an analytic expansion in powers of r. In particular, we can write

$$g_{AB} = r^2 q_{AB} + r C_{AB} + D_{AB} + \frac{1}{r} E_{AB} + \frac{1}{r^2} F_{AB} + \mathcal{O}(r^{-3}), \tag{2.2.33}$$

where q_{AB}, C_{AB}, D_{AB}, E_{AB}, F_{AB}, ... are functions of (u, x^A). The determinant condition defining the Bondi gauge and appearing in the third equation of (2.2.10) implies $g^{AB}\partial_r g_{AB} = 4/r$, which imposes successively that $\det(g_{AB}) = r^4 \det(q_{AB})$, $q^{AB} C_{AB} = 0$ and

$$\begin{aligned}
D_{AB} &= \frac{1}{4} q_{AB} C^{CD} C_{CD} + \mathcal{D}_{AB}(u, x^C), \\
E_{AB} &= \frac{1}{2} q_{AB} \mathcal{D}_{CD} C^{CD} + \mathcal{E}_{AB}(u, x^C), \\
F_{AB} &= \frac{1}{2} q_{AB} \Big[C^{CD} \mathcal{E}_{CD} + \frac{1}{2} \mathcal{D}^{CD} \mathcal{D}_{CD} - \frac{1}{32} (C^{CD} C_{CD})^2 \Big] + \mathcal{F}_{AB}(u, x^C),
\end{aligned} \tag{2.2.34}$$

with $q^{AB}\mathcal{D}_{AB} = q^{AB}\mathcal{E}_{AB} = q^{AB}\mathcal{F}_{AB} = 0$ (indices are lowered and raised with the metric q_{AB} and its inverse). We now sketch the results obtained by solving the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.2.35}$$

for $\Lambda \neq 0$ (we follow [166,167]; see also [168] for the Newman-Penrose version). The component (rr) gives the following radial constraints on the Bondi functions:

$$\begin{aligned}\beta(u,r,x^A) = \beta_0(u,x^A) &+ \frac{1}{r^2}\Big[-\frac{1}{32}C^{AB}C_{AB}\Big] + \frac{1}{r^3}\Big[-\frac{1}{12}C^{AB}\mathcal{D}_{AB}\Big] \\ &+ \frac{1}{r^4}\Big[-\frac{3}{32}C^{AB}\mathcal{E}_{AB} - \frac{1}{16}\mathcal{D}^{AB}\mathcal{D}_{AB} + \frac{1}{128}(C^{AB}C_{AB})^2\Big] + \mathcal{O}(r^{-5}).\end{aligned} \tag{2.2.36}$$

where $\beta_0(u,x^A)$ is an arbitrary function. The component (rA) yields

$$\begin{aligned}U^A = U_0^A(u,x^B) &+ \overset{(1)}{U}{}^A(u,x^B)\frac{1}{r} + \overset{(2)}{U}{}^A(u,x^B)\frac{1}{r^2} \\ &+ \overset{(3)}{U}{}^A(u,x^B)\frac{1}{r^3} + \overset{(L3)}{U}{}^A(u,x^B)\frac{\ln r}{r^3} + o(r^{-3})\end{aligned} \tag{2.2.37}$$

with

$$\begin{aligned}\overset{(1)}{U}{}^A(u,x^B) &= 2e^{2\beta_0}\partial^A\beta_0, \\ \overset{(2)}{U}{}^A(u,x^B) &= -e^{2\beta_0}\Big[C^{AB}\partial_B\beta_0 + \frac{1}{2}D_BC^{AB}\Big], \\ \overset{(3)}{U}{}^A(u,x^B) &= -\frac{2}{3}e^{2\beta_0}\Big[N^A - \frac{1}{2}C^{AB}D^CC_{BC} + (\partial_B\beta_0 - \frac{1}{3}D_B)\mathcal{D}^{AB} - \frac{3}{16}C_{CD}C^{CD}\partial^A\beta_0\Big], \\ \overset{(L3)}{U}{}^A(u,x^B) &= -\frac{2}{3}e^{2\beta_0}D_B\mathcal{D}^{AB}.\end{aligned} \tag{2.2.38}$$

In these expressions, $U_0^A(u,x^B)$ and $N^A(u,x^B)$ are arbitrary functions. We call N^A the *angular momentum aspect*. Notice that, at this stage, logarithmic terms are appearing in the expansion (2.2.37). However, we will see below that these terms vanish for $\Lambda \neq 0$. The component (ru) leads to

$$\begin{aligned}\frac{V}{r} =& \frac{\Lambda}{3}e^{2\beta_0}r^2 - r(l + D_AU_0^A) \\ &- e^{2\beta_0}\Big[\frac{1}{2}\Big(R[q] + \frac{\Lambda}{8}C_{AB}C^{AB}\Big) + 2D_A\partial^A\beta_0 + 4\partial_A\beta_0\partial^A\beta_0\Big] - \frac{2M}{r} + o(r^{-1}),\end{aligned} \tag{2.2.39}$$

where $l = \partial_u \ln\sqrt{q}$, $R[q]$ is the scalar curvature associated with the metric q_{AB} and $M(u,x^A)$ is an arbitrary function called the *Bondi mass aspect*. Afterwards, we solve the components (AB) of the Einstein equations order by order, thereby providing us with the constraints imposed on each order of g_{AB}. The leading order $\mathcal{O}(r^{-1})$ of that equation yields to

$$\frac{\Lambda}{3}C_{AB} = e^{-2\beta_0}\Big[(\partial_u - l)q_{AB} + 2D_{(A}U^0_{B)} - D^C U^0_C q_{AB}\Big]. \tag{2.2.40}$$

Going to $\mathcal{O}(r^{-2})$, we get

$$\frac{\Lambda}{3}\mathcal{D}_{AB} = 0, \tag{2.2.41}$$

which removes the logarithmic term in (2.2.37) for $\Lambda \neq 0$ (but not for $\Lambda = 0$). The condition at the next order $\mathcal{O}(r^{-3})$

$$\partial_u \mathcal{D}_{AB} + U^C_0 D_C \mathcal{D}_{AB} + 2\mathcal{D}_{C(A}D_{B)}U^C_0 = 0 \tag{2.2.42}$$

is trivial for $\Lambda \neq 0$. Using an iterative argument as in [167], we now make the following observation. If we decompose $g_{AB} = r^2 \sum_{n\geqslant 0} g^{(n)}_{AB} r^{-n}$, we see that the iterative solution of the components (AB) of the Einstein equations organizes itself as $\Lambda g^{(n)}_{AB} = \partial_u g^{(n-1)}_{AB} + (...)$ at order $\mathcal{O}(r^{-n})$, $n \in \mathbb{N}_0$. Accordingly, the form of $\mathcal{E}_{AB}$ should have been fixed by the equation found at $\mathcal{O}(r^{-3})$; however, this is not the case, since both contributions of $\mathcal{E}_{AB}$ cancel between G_{AB} and Λg_{AB}. Moreover, the equation $\Lambda g^{(4)}_{AB} = \partial_u g^{(3)}_{AB} + (...)$ at the next order turns out to be a constraint for $g^{(4)}_{AB} \sim \mathcal{F}_{AB}$, determined with other subleading data such as C_{AB} or $\partial_u g^{(3)}_{AB} \sim \partial_u \mathcal{E}_{AB}$. It shows that $\mathcal{E}_{AB}$ is a set of two free data on the boundary, built up from two arbitrary functions of (u, x^A). Morover, it indicates that no more data exist to be uncovered for $\Lambda \neq 0$. Finally, the components (uu) and (uA) of the Einstein equations give some evolution constraints with respect to the u coordinate for the Bondi mass aspect M and the angular momentum aspect N^A. We will not describe these equations explicitly here (see [166, 167] or subsection 6.2.2).

In summary, the solution space for general relativity in the Bondi gauge with the preliminary boundary condition (2.2.33) and $\Lambda \neq 0$ is parametrized by the set of functions

$$\{\beta_0, U^A_0, q_{AB}, \mathcal{E}_{AB}, M, N^A\}_{\Lambda\neq 0} \tag{2.2.43}$$

(11 functions), where M and N^A have constrained evolutions with respect to the u coordinate. Therefore, the characteristic initial value problem is well-defined when the following data are given: $\beta_0(u, x^C)$, $U^A_0(u, x^C)$, $\mathcal{E}_{AB}(u, x^C)$, $q_{AB}(u, x^C)$, $M(u_0, x^C)$ and $N^A(u_0, x^C)$, where u_0 is a fixed value of u.

Notice that for the boundary conditions (2.2.22) (AAdS1), the solution space reduces to

$$\{q_{AB}, \mathcal{E}_{AB}, M, N^A\}_{\Lambda\neq 0}, \tag{2.2.44}$$

where M and N^A have constrained evolutions with respect to the u coordinate, and q_{AB} has a fixed determinant [166] (7 functions). Finally, for the Dirichlet boundary conditions (2.2.24) (AAdS2), the solution space finally reduces to

$$\{\mathcal{E}_{AB}, M, N^A\}_{\Lambda\neq 0}, \tag{2.2.45}$$

where M and N^A have constrained evolutions with respect to the u coordinate (5 functions).

Let us finally discuss the *Bondi gauge* in asymptotically flat spacetimes [1–3, 6, 8, 166, 169]. We first consider the preliminary boundary conditions (2.2.15). From the previous analysis of solution space with $\Lambda \neq 0$, we can readily obtain the solution space with $\Lambda = 0$, that is, the solution of

$$G_{\mu\nu} = 0, \tag{2.2.46}$$

by taking the flat limit $\Lambda \to 0$. The radial constraints (2.2.36), (2.2.38) and (2.2.39) are still valid by setting to zero β_0, U_0^A (see equation (2.2.15)) and all the terms proportional to Λ. Furthermore, by the same procedure, the constraint equation (2.2.40) becomes

$$(\partial_u - l)q_{AB} = 0. \tag{2.2.47}$$

Therefore, the asymptotic shear C_{AB} becomes unconstrained, and the metric q_{AB} gets a time evolution constraint. Similarly, the equation (2.2.41) becomes trivial and $\mathcal{D}_{AB}$ is not constrained at this order. In particular, this allows for the existence of logarithmic terms in the Bondi expansion (see equation (2.2.37)). One has to impose the additional condition $D^A\mathcal{D}_{AB} = 0$ to make these logarithmic terms disappear. Finally, one can see that for $\Lambda = 0$, the subleading orders of the components (AB) of the Einstein equations impose time evolution constraints on $\mathcal{D}_{AB}$, $\mathcal{E}_{AB}$, ... , but this infinite tower of functions is otherwise unconstrained and they become free parameters of the solution space. Finally, as for the case $\Lambda \neq 0$, the components (uu) and (uA) of the Einstein equations yield time evolution constraints for the Bondi mass aspect M and the angular momentum aspect N^A.

In summary, the solution space for general relativity in the Bondi gauge with the preliminary boundary condition (2.2.15) is parametrized by the set of functions

$$\{q_{AB}, C_{AB}, M, N^A, \mathcal{D}_{AB}, \mathcal{E}_{AB}, \mathcal{F}_{AB}, \ldots\}_{\Lambda=0}, \tag{2.2.48}$$

where q_{AB}, M, N^A, $\mathcal{D}_{AB}$, $\mathcal{E}_{AB}$, $\mathcal{F}_{AB}$, ... have constrained time evolutions (infinite tower of independent functions). Therefore, the characteristic initial value problem is well-defined when the following data are given: $C_{AB}(u, c^C)$, $q_{AB}(u_0, x^C)$, $M(u_0, x^C)$, $N^A(u_0, x^C)$, $\mathcal{D}_{AB}(u_0, x^C)$, $\mathcal{E}_{AB}(u_0, x^C)$, $\mathcal{F}_{AB}(u_0, x^C)$, ... where u_0 is a fixed value of u. Notice a subtle point here: by taking the flat limit of the solution space with $\Lambda \neq 0$, we assumed that g_{AB} is analytic in r and can be expanded as (2.2.33) (this condition was not restrictive for $\Lambda \neq 0$). This condition is slightly more restrictive than (2.2.15) where analyticity is assumed only up to order r^{-1}. Therefore, by this flat limit procedure, we only obtain a subsector of the most general solution space. Writing $g_{AB}(u,r,x^C) = r^2 q_{AB}(u,x^C) + rC_{AB}(u,x^C) + D_{AB}(u,x^C) + \tilde{E}_{AB}(u,r,x^C)$, where $\tilde{E}_{AB}$ is a function of all the coordinates of order $\mathcal{O}(r^{-1})$ in r, the most general solution space can be written as

$$\{q_{AB}, C_{AB}, M, N^A, \mathcal{D}_{AB}, \tilde{\mathcal{E}}_{AB}\}_{\Lambda=0}, \tag{2.2.49}$$

where $\tilde{\mathcal{E}}_{AB}$ is the trace-free part of $\tilde{E}_{AB}$, and q_{AB}, M, N^A, $\mathcal{D}_{AB}$, $\tilde{\mathcal{E}}_{AB}$ obey time evolution constraints. Now, the characteristic initial value problem is well-defined when the following data are given: $C_{AB}(u,c^C)$, $q_{AB}(u_0,x^C)$, $M(u_0,x^C)$, $N^A(u_0,x^C)$, $\mathcal{D}_{AB}(u_0,x^C)$ and $\tilde{\mathcal{E}}_{AB}(u_0,r,x^C)$.

We complete this set of examples by mentioning the restricted solution spaces in the different definitions of asymptotic flatness introduced above. For boundary conditions (AF1) (equations (2.2.15) with (2.2.16)), we obtain

$$\{q_{AB}, C_{AB}, M, N^A, \mathcal{D}_{AB}, \tilde{\mathcal{E}}_{AB}\}_{\Lambda=0}, \tag{2.2.50}$$

where q_{AB}, M, N^A, $\mathcal{D}_{AB}$ and $\tilde{\mathcal{E}}_{AB}$ obey time evolution constraints, and $\sqrt{q}$ is fixed. In particular, if we choose a branch where $\sqrt{q}$ is time-independent, from (2.2.47), we immediately see that $\partial_u q_{AB} = 0$. For boundary conditions (AF2) (equations (2.2.15) with (2.2.17)), the solution space reduces to

$$\{\varphi, C_{AB}, M, N^A, \mathcal{D}_{AB}, \tilde{\mathcal{E}}_{AB}\}_{\Lambda=0}, \tag{2.2.51}$$

where M, N^A, $\mathcal{D}_{AB}$ and $\tilde{\mathcal{E}}_{AB}$ obey time evolution equations. Notice that the metric q_{AB} of the form (2.2.17) automatically satisfies (2.2.47). This agrees with results of [6]. Finally, taking the boundary conditions (AF3) (equations (2.2.15) with (2.2.18)) yields the solution space

$$\{C_{AB}, M, N^A, \mathcal{D}_{AB}, \tilde{\mathcal{E}}_{AB}\}_{\Lambda=0}, \tag{2.2.52}$$

where M, N^A, $\mathcal{D}_{AB}$ and $\tilde{\mathcal{E}}_{AB}$ obey time evolution equations. This agrees with the historical results of [1–3].

2.2.4 Asymptotic symmetry algebra

Definition [Asymptotic symmetry] Given boundary conditions imposed in a chosen gauge, the *asymptotic symmetries* are defined as the residual gauge transformations preserving the boundary conditions[6]. In other words, the asymptotic symmetries considered on-shell are the gauge transformations $R[f]$ tangent to the solution

[6]This is the weak definition of asymptotic symmetry, in the sense of (2.1.2).

space. In practice, the requirement to preserve the boundary conditions gives some constraints on the functions parametrizing the residual gauge transformations. In gravity, the generators of asymptotic symmetries are often called *asymptotic Killing vectors*.

Definition [Asymptotic symmetry algebra] Once the asymptotic symmetries are known, we have

$$\begin{aligned}[R[f_1], R[f_2]] &= \delta_{f_1} R[f_2] - \delta_{f_2} R[f_1] \\ &\approx R[[f_1, f_2]_A],\end{aligned} \tag{2.2.53}$$

where $\approx$ means that this equality holds on-shell, i.e. on the solution space. In this expression, the bracket $[f_1, f_2]_A$ of gauge symmetry generators is given by

$$[f_1, f_2]_A = C(f_1, f_2) - \delta_{f_1} f_2 + \delta_{f_2} f_1, \tag{2.2.54}$$

where $C(f_1, f_2)$ is a skew-symmetric bi-differential operator [170, 171]

$$C(f_1, f_2) = \sum_{k,l\geqslant 0} C^{(\mu_1\cdots\mu_k)(\nu_1\cdots\nu_l)}_{[\alpha\beta]} \partial_{\mu_1} \ldots \partial_{\mu_k} f_1^\alpha \partial_{\nu_1} \ldots \partial_{\nu_l} f_2^\beta. \tag{2.2.55}$$

The presence of the terms $-\delta_{f_1} f_2 + \delta_{f_2} f_1$ in (2.2.53) is due to the possible field-dependence of the asymptotic symmetry generators. We can verify that (2.2.54) satisfies the Jacobi identity, i.e. the asymptotic symmetry generators form a (solution space-dependent) Lie algebra for this bracket. It is called the *asymptotic symmetry algebra*. The statement (2.2.53) means that the infinitesimal action of the gauge symmetries on the fields forms a representation of the Lie algebra of asymptotic symmetry generators: $[\delta_{f_1}, \delta_{f_2}]\phi = \delta_{[f_1,f_2]_A}\phi$. Let us mention that a Lie algebroid structure is showing up at this stage [146, 170, 172]. The base manifold is given by the solution space, the field-dependent Lie algebra is the Lie algebra of asymptotic symmetry generators introduced above and the anchor is the map $f \to R[f]$.

Examples The examples that we present here will be re-discussed in much details in the remaining of the text. Let us start by considering *asymptotically AdS_4 spacetimes* in the *Fefferman-Graham* and *Bondi gauge*. The preliminary boundary condition (2.2.19) does not impose any constraint on the generators of the residual gauge diffeomorphisms of the Fefferman-Graham gauge given in (2.2.13). Similarly, the generators of the residual gauge diffeomorphisms in Bondi gauge given in (2.2.14) do not get further constraints with (2.2.20).

Now, let us consider the boundary conditions (AAdS1) (equation (2.2.19) together with (2.2.21)) in the Fefferman-Graham gauge. The asymptotic symmetries are generated by the vectors fields ξ^μ given in (2.2.13) preserving the boundary conditions, namely, satisfying $\mathcal{L}_\xi g^{(0)}_{tt} = 0$, $\mathcal{L}_\xi g^{(0)}_{tA} = 0$ and $g^{ab}_{(0)} \mathcal{L}_\xi g^{(0)}_{ab} = 0$. This leads to the following constraints on the parameters:

$$\left(\partial_u - \frac{1}{2} l\right) \xi^t_0 = \frac{1}{2} D^{(0)}_A \xi^A_0, \quad \partial_u Y^A = -\frac{\Lambda}{3} g^{AB}_{(0)} \partial_B \xi^t_0, \quad \sigma = \frac{1}{2}(D^{(0)}_A \xi^A_0 + \xi^t_0 l), \tag{2.2.56}$$

where $l = \partial_u \ln \sqrt{q}$. In this case, the Lie bracket (2.2.54) is given by

$$[\xi_1, \xi_2]_A = \mathcal{L}_{\xi_1} \xi_2 - \delta_{\xi_1} \xi_2 + \delta_{\xi_2} \xi_1 \tag{2.2.57}$$

and is referred as the *modified Lie bracket* [6]. Therefore, the asymptotic symmetry algebra can be worked out and is given explicitly by $[\xi(\xi^t_{0,1}, \xi^A_{0,1}), \xi(\xi^t_{0,2}, \xi^A_{0,2})]_A = \xi(\hat{\xi}^t_0, \hat{\xi}^A_0)$, where[7]

$$\begin{aligned}
\hat{\xi}^t_0 &= \xi^A_{0,1} \partial_A \xi^t_{0,2} + \frac{1}{2} \xi^t_{0,1} D^{(0)}_A \xi^A_{0,2} - \delta_{\xi(\xi^t_{0,1}, \xi^A_{0,1})} \xi^t_{0,2} - (1 \leftrightarrow 2), \\
\hat{\xi}^A_0 &= \xi^B_{0,1} \partial_B \xi^A_{0,2} - \frac{\Lambda}{3} \xi^t_{0,1} g^{AB}_{(0)} \partial_B \xi^t_{0,2} - \delta_{\xi(\xi^t_{0,1}, \xi^A_{0,1})} \xi^A_{0,2} - (1 \leftrightarrow 2).
\end{aligned} \tag{2.2.58}$$

In the Bondi gauge with corresponding boundary conditions (2.2.22), the constraints on the parameters are given by

$$\left(\partial_u - \frac{1}{2} l\right) f = \frac{1}{2} D_A Y^A, \quad \partial_u Y^A = -\frac{\Lambda}{3} q^{AB} \partial_B f, \quad \omega = 0 \tag{2.2.59}$$

and the asymptotic symmetry algebra is written as $[\xi(f_1, Y^A_1), \xi(f_2, Y^A_2)]_A = \xi(\hat{f}, \hat{Y}^A)$, where

$$\begin{aligned}
\hat{f} &= Y^A_1 \partial_A f_2 + \frac{1}{2} f_1 D_A Y^A_2 - \delta_{\xi(f_1, Y^A_1)} f_2 - (1 \leftrightarrow 2), \\
\hat{Y}^A &= Y^B_1 \partial_B Y^A_2 - \frac{\Lambda}{3} f_1 q^{AB} \partial_B f_2 - \delta_{\xi(f_1, Y^A_1)} Y^A_2 - (1 \leftrightarrow 2).
\end{aligned} \tag{2.2.60}$$

This asymptotic symmetry algebra is infinite-dimensional (in particular, it contains the area-preserving diffeomorphisms as a subgroup) and field-dependent. It is called the Λ-BMS$_4$ algebra [166] and is denoted as $\mathfrak{bms}^\Lambda_4$. The parameters f are called the supertranslation generators, while the parameters Y^A are called the superrotation generators. These names will be justified below when studying the flat limit of this asymptotic symmetry algebra $\mathfrak{bms}^\Lambda_4$. The computation of the modified Lie bracket (2.2.57) in the Bondi gauge for these boundary conditions[8] follows closely [6].

---

[7] The terms $\delta_{\xi(\xi^t_{0,1}, \xi^A_{0,1})} \xi^t_{0,2}$ and $\delta_{\xi(\xi^t_{0,1}, \xi^A_{0,1})} \xi^A_{0,2}$ in (2.2.58) take into account the possible field-dependence of the parameters $(\xi^t_{0,2}, \xi^A_{0,2})$.

[8] This completes the results obtained in [166] where the asymptotic symmetry algebra was obtained by pullback methods.

Let us consider the Fefferman-Graham gauge with Dirichlet boundary conditions (AAdS2), that is, (2.2.19) together with (2.2.23). Compared to the above situation, the equations (2.2.56) reduce to

$$\partial_u \xi_0^t = \frac{1}{2} D_A^{(0)} \xi_0^A, \quad \partial_u Y^A = -\frac{\Lambda}{3} \mathring{q}_{(0)}^{AB} \partial_B \xi_0^t, \quad \sigma = \frac{1}{2} D_A^{(0)} \xi_0^A, \tag{2.2.61}$$

where $D_A^{(0)}$ is the covariant derivative associated with the fixed unit sphere metric $\mathring{q}_{AB}$. Furthermore, there is an additional constraint: $\mathcal{L}_\xi g_{AB}^{(0)} = o(\rho^{-2})$, which indicates that ξ_0^A is a conformal Killing vector of $\mathring{q}_{AB}$, namely,

$$D_A^{(0)} \xi_B^0 + D_B^{(0)} \xi_A^0 = D_C^{(0)} \xi_0^C \mathring{q}_{AB}. \tag{2.2.62}$$

The asymptotic symmetry algebra remains of the same form as (2.2.58), with $\delta_{\xi(\xi_{0,1}^t, \xi_{0,1}^A)} \xi_{0,2}^t = 0 = \delta_{\xi(\xi_{0,1}^t, \xi_{0,1}^A)} \xi_{0,2}^A$. In the Bondi gauge, Dirichlet boundary conditions are given by (2.2.20) together with (2.2.24). The equations (2.2.59) become

$$\partial_u f = \frac{1}{2} D_A Y^A, \quad \partial_u Y^A = -\frac{\Lambda}{3} \mathring{q}^{AB} \partial_B \xi_0^t, \quad \omega = 0, \tag{2.2.63}$$

where D_A is the covariant derivative with respect to $\mathring{q}_{AB}$, while the additional constraint $\mathcal{L}_\xi g_{AB} = o(r^2)$ gives

$$D_A Y_B + D_B Y_A = D_C Y^C \mathring{q}_{AB}. \tag{2.2.64}$$

This means that Y^A is a conformal Killing vector of $\mathring{q}_{AB}$. The asymptotic symmetry algebra (2.2.60) remains of the same form, with $\delta_{\xi(f_1, Y_1^A)} f_2 = 0 = \delta_{\xi(f_1, Y_1^A)} Y_2^A$. It can be shown that the asymptotic symmetry algebra corresponds to $\mathfrak{so}(3,2)$ algebra for $\Lambda < 0$ and $\mathfrak{so}(1,4)$ algebra for $\Lambda > 0$ [24] (see also appendix A of [166]). Therefore, we see how the infinite-dimensional asymptotic symmetry algebra $\mathfrak{bms}_4^\Lambda$ reduces to these finite-dimensional algebras, which are the symmetry algebras of global AdS_4 and global dS_4, respectively.

Let us now consider *four-dimensional asymptotically flat spacetimes* in the *Bondi gauge*. The asymptotic Killing vectors can be derived in a similar way to that in the previous examples. Another way in which to proceed is to take the flat limit of the previous results obtained in the Bondi gauge. We sketch the expressions obtained by following these two equivalent procedures. First, consider the preliminary boundary conditions (2.2.15). The *asymptotic Killing vectors* ξ^μ are the residual gauge diffeomorphisms (2.2.14) with the following constraints on the parameters:

$$\left(\partial_u - \frac{1}{2} l\right) f = \frac{1}{2} D_A Y^A - \omega, \quad \partial_u Y^A = 0, \tag{2.2.65}$$

where $l = \partial_u \ln \sqrt{q}$. These equations can be readily solved and the solutions are given by

$$f = q^{\frac{1}{4}} \left[T(x^A) + \frac{1}{2} \int_0^u \mathrm{d}u' [q^{-\frac{1}{4}} (D_A Y^A - 2\omega)] \right], \quad Y^A = Y^A(x^B), \tag{2.2.66}$$

where T are called supertranslation generators and Y^A superrotation generators. Notice that there is no additional constraint on Y^A at this stage. Computing the modified Lie bracket (2.2.57), we obtain $[\xi(f_1, Y_1^A, \omega_1), \xi(f_2, Y_2^A, \omega_2)]_A = \xi(\hat{f}, \hat{Y}^A, \hat{\omega})$ where

$$\begin{aligned} \hat{f} &= Y_1^A \partial_A f_2 + \frac{1}{2} f_1 D_A Y_2^A - (1 \leftrightarrow 2), \\ \hat{Y}^A &= Y_1^B \partial_B Y_2^A - (1 \leftrightarrow 2), \\ \hat{\omega} &= 0. \end{aligned} \tag{2.2.67}$$

Now, we discuss the two relevant sub-cases of boundary conditions in asymptotically flat spacetimes. Adding the condition (2.2.16) to the preliminary condition (2.2.15), i.e. considering (AF1), gives the additional constraint

$$\omega = 0 \tag{2.2.68}$$

Note that this case corresponds exactly to the flat limit of the (AAdS1) case (equations (2.2.19) and (2.2.21)). The asymptotic symmetry algebra reduces to the semi-direct product

$$\mathfrak{bms}_4^{\text{gen}} = \mathfrak{diff}(S^2) \loplus \mathfrak{s}, \tag{2.2.69}$$

where $\mathfrak{diff}(S^2)$ are the smooth superrotations generated by Y^A and $\mathfrak{s}$ are the smooth supertranslations generated by T. This extension of the original global BMS_4 algebra (see below) is called the *generalized BMS_4 algebra* [8, 12, 13, 164]. Therefore, the Λ-BMS_4 algebra reduces in the flat limit to the smooth extension (2.2.69) of the BMS_4 algebra.

The other sub-case of boundary conditions for asymptotically flat spacetimes (AF2) is given by adding condition (2.2.17) to the preliminary boundary condition (2.2.15). The additional constraint on the parameters is now given by

$$D_A Y_B + D_B Y_A = D_C Y^C \mathring{q}_{AB}, \tag{2.2.70}$$

i.e. Y^A is a conformal Killing vector of the unit round sphere metric $\mathring{q}_{AB}$. If we allow Y^A to not be globally well-defined on the 2-sphere, then the asymptotic symmetry algebra has the structure

$$[(\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)) \loplus \mathfrak{s}^*] \oplus \mathbb{R}. \tag{2.2.71}$$

Here, $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ is the direct product of two copies of the Witt algebra, parametrized by Y^A. Furthermore, $\mathfrak{s}^*$ are the supertranslations, parametrized by T, and $\mathbb{R}$ are the abelian Weyl rescalings of $\mathring{q}_{AB}$, parametrized by ω. Note that the supetranslations also contain singular elements since they are related to the singular superrotations through the algebra (2.2.67). This extension of the global BMS_4 algebra is called the *extended BMS_4 algebra* [6] and is denoted as $\mathfrak{bms}_4^{\text{ext}} \oplus \mathbb{R}$. Finally, as a sub-case of this one, considering the more restrictive constraints (2.2.18), i.e. (AF3), and allowing only globally well-defined Y^A, we recover the *global BMS_4 algebra* [1–3], which is given by

$$\mathfrak{bms}_4^{\text{glob}} = \mathfrak{so}(3,1) \loplus \mathfrak{s}, \tag{2.2.72}$$

where $\mathfrak{s}$ are the supertranslations and $\mathfrak{so}(3,1)$ is the algebra of the globally well-defined conformal Killing vectors of the unit 2-sphere metric, which is isomorphic to the proper orthocronous Lorentz group in four dimensions.

Definition [Action on the solution space] Given boundary conditions imposed in a chosen gauge, there is a natural *action of the asymptotic symmetry algebra*, with generators $f = f(a)$, *on the solution space* $\tilde{\phi} = \tilde{\phi}(b)$. The form of this action can be deduced from (2.2.2) by inserting the solution space and the explicit form of the asymptotic symmetry generators[9].

Examples Again, the examples given here will be discussed in more details in the text. In the *Fefferman-Graham gauge* with Dirichlet boundary conditions for *asymptotically AdS_4 spacetimes* (AAdS2) ((2.2.19) with (2.2.23)), the asymptotic symmetry algebra $\mathfrak{so}(3,2)$ acts on the solution space (2.2.32) as

$$\delta_{\xi_0^c} T_{ab} = \left(\mathcal{L}_{\xi_0^c} + \frac{1}{3} D_c^{(0)} \xi_0^c \right) T_{ab}. \tag{2.2.73}$$

In the *Bondi gauge* with definition (AF3) ((2.2.15) with (2.2.18)) of *asymptotically flat spacetime*, the global BMS_4 algebra $\mathfrak{bms}_4^{\text{glob}}$ acts on the leading functions of the

[9]This action is usually not linear. However, in three-dimensional general relativity, this action is precisely the coadjoint representation of the asymptotic symmetry algebra [173–177].

solution space (2.2.52) as

$$\begin{aligned}
\delta_{(f,Y)} C_{AB} &= [f\partial_u + \mathcal{L}_Y - \frac{1}{2} D_C Y^C] C_{AB} - 2 D_A D_B f + \mathring{q}_{AB} D_C D^C f, \\
\delta_{(f,Y)} M &= [f\partial_u + \mathcal{L}_Y + \frac{3}{2} D_C Y^C] M + \frac{1}{4} N^{AB} D_A D_B f \\
&\quad + \frac{1}{2} D_A f D_B N^{AB} + \frac{1}{8} D_C D_B D_A Y^A C^{BC}, \\
\delta_{(f,Y)} N_A &= [f\partial_u + \mathcal{L}_Y + D_C Y^C] N_A + 3 M D_A f - \frac{3}{16} D_A f N_{BC} C^{BC} \\
&\quad - \frac{1}{32} D_A D_B Y^B C_{CD} C^{CD} + \frac{1}{4} (2 D^B f + D^B D_C D^C f) C_{AB} \\
&\quad - \frac{3}{4} D_B f (D^B D^C C_{AC} - D_A D_C C^{BC}) + \frac{3}{8} D_A (D_C D_B f C^{BC}) \\
&\quad + \frac{1}{2} (D_A D_B f - \frac{1}{2} D_C D^C f \mathring{q}_{AB}) D_C C^{BC} + \frac{1}{2} D_B f N^{BC} C_{AC},
\end{aligned} \tag{2.2.74}$$

where $N_{AB} = \partial_u C_{AB}$ [6]. For the action of the associated asymptotic symmetry group on these solution spaces, see [178].

2.3 Surface charges

In this section, we review how to construct the surface charges associated with gauge symmetries. After recalling some results about global symmetries and Noether currents, the Barnich-Brandt prescription to obtain the surface charges in the context of asymptotic symmetries is discussed. We illustrate this construction with the example of general relativity in asymptotically (A)dS and asymptotically flat spacetimes. The relation between this prescription and the Iyer-Wald construction is established.

2.3.1 Global symmetries and Noether's first theorem

Definition [Global symmetry] Let us consider a Lagrangian theory with Lagrangian density $\mathbf{L}[\phi, \partial_\mu \phi, \dots]$ and a transformation $\delta_Q \phi = Q$ of the fields, where Q is a local function. In agreement with the above definition (2.2.3), this transformation is said to be a *symmetry* of the theory if

$$\delta_Q \mathbf{L} = \mathrm{d}\mathbf{B}_Q, \tag{2.3.1}$$

where $\mathbf{B}_Q = B^\mu_Q (\mathrm{d}^{n-1}x)_\mu$. Then, as defined in (2.2.2), a *gauge symmetry* is just a symmetry that depends on arbitrary spacetime functions $f = (f^\alpha)$, i.e. $Q = R[f]$. We define an on-shell equivalence relation $\sim$ between the symmetries of the theory as

$$Q \sim Q + R[f], \tag{2.3.2}$$

i.e. two symmetries are equivalent if they differ, on-shell, by a gauge transformation $R[f]$. The equivalence classes $[Q]$ for this equivalence relation are called the *global symmetries*. In particular, a gauge symmetry is a trivial global symmetry.

Definition [Noether current] A *conserved current* $\mathbf{j}$ is an on-shell closed $(n-1)$-form, i.e. $\mathrm{d}\mathbf{j} \approx 0$. We define an on-shell equivalence relation $\sim$ between the currents as

$$\mathbf{j} \sim \mathbf{j} + \mathrm{d}\mathbf{K}, \tag{2.3.3}$$

where $\mathbf{K}$ is a $(n-2)$-form. A *Noether current* is an equivalence class $[\mathbf{j}]$ for this equivalence relation.

Theorem [Noether's first theorem] A one-to-one correspondence exists between *global symmetries* Q and *Noether currents* $[\mathbf{j}]$, which can be written as

$$[Q] \xleftrightarrow{\text{1-1}} [\mathbf{j}]. \tag{2.3.4}$$

In particular, Noether currents associated with gauge symmetries are trivial. Recent demonstrations of this theorem can for example be found in [113, 171].

Remark This theorem also enables us to construct explicit representatives of the Noether current for a given global symmetry. We have

$$\delta_Q \mathbf{L} = \mathrm{d}\mathbf{B}_Q = (\partial_\mu B^\mu_Q)\mathrm{d}^n x. \tag{2.3.5}$$

Furthermore, writing $\mathbf{L} = L\mathrm{d}^n x$, we obtain

$$\begin{aligned}
\delta_Q L &= \delta_Q \phi^i \frac{\partial L}{\partial \phi^i} + \delta_Q \partial_\mu \phi^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \\
&= Q^i \frac{\partial L}{\partial \phi^i} + \partial_\mu Q^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \\
&= Q^i \left(\frac{\partial L}{\partial \phi^i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \right) + \partial_\mu \left(Q^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \right) \\
&= Q^i \frac{\delta L}{\delta \phi^i} + \partial_\mu \left(Q^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \right),
\end{aligned} \tag{2.3.6}$$

where, in the second line, we used

$$[\delta_Q, \partial_\mu] = 0 \tag{2.3.7}$$

and, in the last equality, we used (A.2.1). Putting (2.3.5) and (2.3.6) together, we obtain

$$Q^i \frac{\delta L}{\delta \phi^i} = \partial_\mu \left(B^\mu_Q - Q^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \right) \equiv \partial_\mu j^\mu_Q \tag{2.3.8}$$

or, equivalently

$$Q^i \frac{\delta \mathbf{L}}{\delta \phi^i} = \mathrm{d}\mathbf{j}_Q, \tag{2.3.9}$$

where $\mathbf{j}_Q = j^\mu_Q (\mathrm{d}^{n-1}x)_\mu$. In particular, $\mathrm{d}\mathbf{j}_Q \approx 0$ holds on-shell. Hence, we have obtained a representative of the Noether current associated with the global symmetry Q through the correspondence (2.3.4).

Theorem [Noether representation theorem] Defining the bracket as

$$\{\mathbf{j}_{Q_1}, \mathbf{j}_{Q_2}\} = \delta_{Q_1} \mathbf{j}_{Q_2}, \tag{2.3.10}$$

we have

$$\{\mathbf{j}_{Q_1}, \mathbf{j}_{Q_2}\} \approx \mathbf{j}_{[Q_1,Q_2]} \tag{2.3.11}$$

($n > 1$), where $[Q_1, Q_2] = \delta_{Q_1} Q_2 - \delta_{Q_2} Q_1$. In other words, the Noether currents form a representation of the symmetries.

To prove this theorem, we apply δ_{Q_1} on the left-hand side and the right-hand side of (2.3.9), where Q is replaced by Q_2. On the right-hand side, using the first equation of (A.2.4), we obtain

$$\delta_{Q_1} \mathrm{d}\mathbf{j}_{Q_2} \approx \mathrm{d}\delta_{Q_1} \mathbf{j}_{Q_2}. \tag{2.3.12}$$

On the left-hand side, we have

$$\begin{aligned} \delta_{Q_1}\left(Q_2^i \frac{\delta \mathbf{L}}{\delta \phi^i}\right) &= \delta_{Q_1} Q_2^i \frac{\delta \mathbf{L}}{\delta \phi^i} + Q_2^i \delta_{Q_1} \frac{\delta \mathbf{L}}{\delta \phi^i} \\ &= \delta_{Q_1} Q_2^i \frac{\delta \mathbf{L}}{\delta \phi^i} + Q_2^i \frac{\delta}{\delta \phi^i}(\delta_{Q_1}\mathbf{L}) - Q_2^i \sum_{k\geqslant 0} (-1)^k \partial_{\mu_1} \dots \partial_{\mu_k} \left(\frac{\partial Q_1^j}{\partial \phi^i_{\mu_1 \dots \mu_k}} \frac{\delta \mathbf{L}}{\delta \Phi^j} \right) \\ &= \delta_{Q_1} Q_2^i \frac{\delta \mathbf{L}}{\delta \phi^i} - Q_2^i \sum_{k\geqslant 0} (-1)^k \partial_{\mu_1} \dots \partial_{\mu_k} \left(\frac{\partial Q_1^j}{\partial \phi^i_{\mu_1 \dots \mu_k}} \frac{\delta \mathbf{L}}{\delta \phi^j} \right), \end{aligned} \tag{2.3.13}$$

where, to obtain the second equality, we used (A.2.5). In the last equality, we used (2.3.1) together with (A.2.2). Now, using Leibniz rules in the second term of the right-hand side, we get

$$
\begin{aligned}
\delta_{Q_1}\left(Q_2^i\frac{\delta \mathbf{L}}{\delta\phi^i}\right) &= \delta_{Q_1}Q_2^i\frac{\delta \mathbf{L}}{\delta\phi^i} - \sum_{k\geqslant 0}\partial_{\mu_1}\dots\partial_{\mu_k}Q_2^j\left(\frac{\partial Q_1^i}{\partial\phi^j_{\mu_1\dots\mu_k}}\frac{\delta \mathbf{L}}{\delta\phi^i}\right) + \partial_\mu T^\mu_{Q_1}\left(Q_2,\frac{\delta L}{\delta\phi}\right)\mathrm{d}^n x \\
&= (\delta_{Q_1}Q_2^i - \delta_{Q_2}Q_1^i)\frac{\delta \mathbf{L}}{\delta\phi^i} + \partial_\mu T^\mu_{Q_1}\left(Q_2,\frac{\delta L}{\delta\phi}\right)\mathrm{d}^n x \\
&= [Q_1,Q_2]^i\frac{\delta \mathbf{L}}{\delta\phi^i} + \partial_\mu T^\mu_{Q_1}\left(Q_2,\frac{\delta L}{\delta\phi}\right) d^n x \\
&= \mathrm{d}\mathbf{j}_{[Q_1,Q_2]} + \mathrm{d}\mathbf{T}_{Q_1}\left(Q_2,\frac{\delta L}{\delta\phi}\right),
\end{aligned}
\tag{2.3.14}
$$

where $T^\mu_{Q_1}\left(Q_2,\frac{\delta L}{\delta\phi}\right)$ is an expression vanishing on-shell. In the second equality, we used (A.2.3), and in the last equality, we used (2.3.9). Putting (2.3.12) and (2.3.14) together results in

$$
\mathrm{d}\left[\delta_{Q_1}\mathbf{j}_{Q_2} - \mathbf{j}_{[Q_1,Q_2]} - \mathbf{T}_{Q_1}\left(Q_2,\frac{\delta L}{\delta\phi}\right)\right] = 0. \tag{2.3.15}
$$

We know from Poincaré lemma that locally, every closed form is exact[10]. However, this cannot be the case in Lagrangian field theories. In fact, this would imply that every n-form is exact, and therefore, there would not be any possibility of non-trivial dynamics. Let us remark that the operator d that we are using is not the standard exterior derivative, but a horizontal derivative in the jet bundle (see definition (A.1.3)) that takes into account the field-dependence. In this context, we have to use the *algebraic Poincaré lemma*.

Lemma [Algebraic Poincaré lemma] The cohomology class $H^p(\mathrm{d})$ for the operator d defined in (A.1.3) is given by

$$
H^p(\mathrm{d}) = \begin{cases} [\boldsymbol{\alpha}^n] & \text{if } p = n \\ 0 & \text{if } 0 < p < n \\ \mathbb{R} & \text{if } p = 0 \end{cases} \tag{2.3.16}
$$

where $[\boldsymbol{\alpha}^n]$ designates the equivalence classes of n-forms for the equivalence relation $\boldsymbol{\alpha}^n \sim \boldsymbol{\alpha}'^n$ if $\boldsymbol{\alpha}^n = \boldsymbol{\alpha}'^n + \mathrm{d}\boldsymbol{\beta}^{n-1}$ [171].

[10] The Poincaré lemma states that in a star-shaped open subset, the de Rham cohomology class H^p_{dR} is given by

$$
H^p_{dR} = \begin{cases} 0 & \text{if } 0 < p \leqslant n \\ \mathbb{R} & \text{if } p = 0 \end{cases}.
$$

Le us go back to the proof of (2.3.11). Applying the algebraic Poincaré lemma to (2.3.15) yields

$$\delta_{Q_1}\mathbf{j}_{Q_2} = \mathbf{j}_{[Q_1,Q_2]} + \mathbf{T}_{Q_1}\left(Q_2, \frac{\delta L}{\delta\phi}\right) + \mathrm{d}\boldsymbol{\eta}, \tag{2.3.17}$$

where $\boldsymbol{\eta}$ is a $(n-2)$-form. Therefore, on-shell, since $\mathbf{T}_{Q_1}\left(Q_2, \frac{\delta L}{\delta\phi}\right) \approx 0$ and because Noether currents are defined up to exact $(n-1)$-forms, we obtain the result (2.3.11). Notice that in classical mechanics (i.e. $n = 1$), from (2.3.16), constant central extensions may appear in the current algebra.

Definition [Noether charge] Given a Noether current $[\mathbf{j}]$, we can construct a *Noether charge* by integrating it on a $(n-1)$-dimensional spacelike surface Σ, with boundary $\partial\Sigma$, as

$$H_Q[\phi] = \int_\Sigma \mathbf{j}. \tag{2.3.18}$$

If we assume that the currents and their ambiguities vanish at infinity, this definition does not depend on the representative of the Noether current. Indeed,

$$H'_Q[\phi] = \int_\Sigma (\mathbf{j} + \mathrm{d}\mathbf{K}) = H_Q[\phi] + \int_{\partial\Sigma} \mathbf{K}, \tag{2.3.19}$$

where we used the Stokes theorem. Since $\int_{\partial\Sigma}\mathbf{K} = 0$, we have $H'_Q[\phi] = H_Q[\phi]$.

Remark [Conservation and algebra of Noether charges] The Noether charge (2.3.18) is conserved in time, that is,

$$\frac{d}{dt}H_Q[\phi] \approx 0. \tag{2.3.20}$$

In fact, consider two spacelike hypersurfaces $\Sigma_1 \equiv t_1 = 0$ and $\Sigma_2 \equiv t_2 = 0$. We have

$$H_Q^{t_2}[\phi] - H_Q^{t_1}[\phi] = \int_{\Sigma_2} \mathbf{j}_Q - \int_{\Sigma_1} \mathbf{j}_Q = \int_{\Sigma_2 - \Sigma_1} \mathrm{d}\mathbf{j}_Q \approx 0, \tag{2.3.21}$$

where $\Sigma_2 - \Sigma_1$ is the spacetime volume encompassed between Σ_1 and Σ_2. In the second equality, we used the hypothesis that currents vanish at infinity and the Stokes theorem.

The Noether charges (2.3.18) form a representation of the algebra of global symmetries, i.e.

$$\{H_{Q_1}, H_{Q_2}\} \approx H_{[Q_1,Q_2]}, \tag{2.3.22}$$

where the bracket of Noether charges is defined as

$$\{H_{Q_1}, H_{Q_2}\} = \delta_{Q_1} H_{Q_2} = \int_\Sigma \delta_{Q_1}\mathbf{j}_{Q_2}. \tag{2.3.23}$$

This is a direct consequence of (2.3.11).

2.3.2 Gauge symmetries and lower degree conservation law

Definition [Noether identities] Consider the relation (2.3.9) for a gauge symmetry:

$$R^i[f]\frac{\delta L}{\delta \phi^i} = \partial_\mu j^\mu_f. \tag{2.3.24}$$

The left-hand side can be worked out as

$$\begin{aligned} R^i[f]\frac{\delta L}{\delta \phi^i} &= \left(R^i_\alpha f^\alpha + R^{i\mu}_\alpha \partial_\mu f^\alpha + R^{i(\mu\nu)}_\alpha \partial_\mu\partial_\nu f^\alpha + \dots\right)\frac{\delta L}{\delta \phi^i} \\ &= f^\alpha\left[R^i_\alpha \frac{\delta L}{\delta \phi^i} - \partial_\mu\left(R^{i\mu}_\alpha \frac{\delta L}{\delta \phi^i}\right) + \partial_\mu\partial_\nu\left(R^{i(\mu\nu)}_\alpha \frac{\delta L}{\delta \phi^i}\right) + \dots\right] \\ &\quad + \partial_\mu \underbrace{\left[R^{i\mu}_\alpha f^\alpha \frac{\delta L}{\delta \phi^i} - f^\alpha \partial_\nu\left(R^{i(\mu\nu)}_\alpha \frac{\delta L}{\delta \phi^i}\right) + \dots\right]}_{\equiv S^\mu_F}. \end{aligned} \tag{2.3.25}$$

Therefore, the equation (2.3.24) can be rewritten as

$$f^\alpha R^\dagger_\alpha\left(\frac{\delta L}{\delta \phi}\right) = \partial_\mu(j^\mu_F - S^\mu_F), \tag{2.3.26}$$

where $R^\dagger_\alpha\left(\frac{\delta L}{\delta \phi^i}\right) = R^i_\alpha \frac{\delta L}{\delta \phi^i} - \partial_\mu\left(R^{i\mu}_\alpha \frac{\delta L}{\delta \phi^i}\right) + \partial_\mu\partial_\nu\left(R^{i(\mu\nu)}_\alpha \frac{\delta L}{\delta \phi^i}\right) + \dots$ Since f is a set of arbitrary functions, we can apply the Euler-Lagrange derivative (A.2.1) with respect to f^α on this equation. Since the right-hand side is a total derivative, it vanishes under the action of the Euler-Lagrange derivative (see (A.2.2)) and we obtain

$$R^\dagger_\alpha\left(\frac{\delta \mathbf{L}}{\delta \phi}\right) = 0. \tag{2.3.27}$$

This identity is called a *Noether identity*. There is one identity for each independent generator f^α. Notice that these identities are satisfied off-shell.

Theorem [Noether's second theorem] We have

$$R^i[f]\frac{\delta \mathbf{L}}{\delta \phi^i} = \mathrm{d}\mathbf{S}_f\left[\frac{\delta L}{\delta \phi}\right], \tag{2.3.28}$$

where $\mathbf{S}_f = S^\mu_f(\mathrm{d}^{n-1}x)_\mu$ is the *weakly vanishing Noether current* (i.e. $\mathbf{S}_f \approx 0$) that was defined in the last line of (2.3.25). This is a direct consequence of (2.3.25), taking the Noether identity (2.3.27) into account.

Example Consider the theory of general relativity $\mathbf{L} = (16\pi G)^{-1}(R-2\Lambda)\sqrt{-g}\mathrm{d}^n x$. The Euler-Lagrange derivative of the Lagrangian is given by

$$\frac{\delta \mathbf{L}}{\delta g_{\mu\nu}} = -(16\pi G)^{-1}(G^{\mu\nu} + g^{\mu\nu}\Lambda)\sqrt{-g}\mathrm{d}^n x. \tag{2.3.29}$$

The Noether identity associated with the diffeomorphism generated by ξ^μ is obtained by following the lines of (2.3.25):

$$\begin{aligned}(16\pi G)\delta_\xi g_{\mu\nu}\frac{\delta \mathbf{L}}{\delta g_{\mu\nu}} &= -2\nabla_\mu \xi_\nu (G^{\mu\nu} + g^{\mu\nu}\Lambda)\sqrt{-g}\mathrm{d}^n x \\ &= 2\xi_\nu \nabla_\mu G^{\mu\nu}\sqrt{-g}\mathrm{d}^n x - \partial_\mu[2\xi_\nu(G^{\mu\nu} + g^{\mu\nu}\Lambda)\sqrt{-g}]\mathrm{d}^n x.\end{aligned} \tag{2.3.30}$$

Therefore, the Noether identity is the Bianchi identity for the Einstein tensor

$$\nabla_\mu G^{\mu\nu} = 0 \tag{2.3.31}$$

and the weakly vanishing Noether current of Noether's second theorem (2.3.28) is given by

$$\mathbf{S}_\xi = -\frac{\sqrt{-g}}{8\pi G}\xi_\nu (G^{\mu\nu} + g^{\mu\nu}\Lambda)(\mathrm{d}^{n-1}x)_\mu. \tag{2.3.32}$$

Remark From (2.3.24) and (2.3.28), we have $\mathrm{d}(\mathbf{j}_f - \mathbf{S}_f) = 0$, and hence, from the algebraic Poincaré lemma (2.3.16),

$$\mathbf{j}_f = \mathbf{S}_f + \mathrm{d}\mathbf{K}_f, \tag{2.3.33}$$

where $\mathbf{K}_f$ is a $(n-2)$-form. Therefore, as already stated in Noether's first theorem (2.3.4), the Noether current associated with a gauge symmetry is trivial, i.e. vanishing on-shell, up to an exact $(n-1)$-form. A natural question arises at this stage: is it possible to define a notion of conserved quantity for gauge symmetries? Naively, following the definition (2.3.18), one may propose the following definition for conserved charge:

$$H_f = \int_\Sigma \mathbf{j}_f \approx \int_{\partial\Sigma} \mathbf{K}_f \tag{2.3.34}$$

where, in the second equality, we used (2.3.33) and Stokes' theorem. This charge will be conserved on-shell since $\mathrm{d}\mathbf{j}_f \approx 0$. The problem is that the $(n-2)$-form $\mathbf{K}_f$ appearing in (2.3.34) is completely arbitrary. Indeed, the Noether currents are equivalence classes of currents (see equation (2.3.3)). Therefore, we have to find an appropriate procedure to isolate a particular $\mathbf{K}_f$.

Definition [Reducibility parameter] *Reducibility parameters* $\bar{f}$ are parameters of gauge transformations satisfying

$$R[\bar{f}] \approx 0. \tag{2.3.35}$$

Two reducibility parameters $\bar{f}$ and $\bar{f}'$ are said to be equivalent, i.e. $\bar{f} \sim \bar{f}'$, if $\bar{f} \approx \bar{f}'$. Note that for a large class of gauge theories (including electrodynamics, Yang-Mills and general relativity in dimensions superior or equal to three [113, 171]), these equivalence classes of asymptotic reducibility parameters are determined by field-independent ordinary functions $\bar{f}(x)$ satisfying the off-shell condition

$$R[\bar{f}] = 0. \tag{2.3.36}$$

We will call them *exact reducibility parameters*.

Theorem [Generalized Noether's theorem] A one-to-one correspondence exists between *equivalence classes of reducibility parameters* and *equivalence classes of on-shell conserved $(n-2)$-forms* $[\mathbf{K}]$, which can be written as

$$[\bar{f}] \overset{\text{1-1}}{\longleftrightarrow} [\mathbf{K}]. \tag{2.3.37}$$

In this statement, two conserved $(n-2)$-forms $\mathbf{K}$ and $\mathbf{K}'$ are said to be equivalent, i.e. $\mathbf{K} \sim \mathbf{K}'$, if $\mathbf{K} \approx \mathbf{K}' + \mathrm{d}\mathbf{l}$ where $\mathbf{l}$ is a $(n-3)$-form [179, 180].

Remark The Barnich-Brandt procedure allows for the construction of explicit representatives of the conserved $(n-2)$-forms for given exact reducibility parameters $\bar{f}$ [113, 114]. From Noether's second theorem (2.3.28) and (2.3.36), we have

$$\mathrm{d}\mathbf{S}_{\bar{f}} = 0. \tag{2.3.38}$$

From the algebraic Poincaré Lemma (2.3.16), we get[11]

$$-\,\mathrm{d}\mathbf{K}_{\bar{f}} = \mathbf{S}_{\bar{f}} \approx 0. \tag{2.3.39}$$

Using the homotopy operator (A.2.12), we define

$$\mathbf{k}_{\bar{f}}[\phi;\delta\phi] = -I^{n-1}_{\delta\phi}\mathbf{S}_{\bar{f}}. \tag{2.3.40}$$

This $\mathbf{k}_{\bar{f}}[\phi;\delta\phi]$ is an element of $\Omega^{n-2,1}$ (see appendix A) and is defined up to an exact $(n-2)$-form. This enables us to find an explicit expression for the conserved $(n-2)$-form $\mathbf{K}_{\bar{f}}[\phi]$ as

$$\mathbf{K}_{\bar{f}}[\phi] = \int_\gamma \mathbf{k}_{\bar{f}}[\phi;\delta\phi], \tag{2.3.41}$$

[11] The minus sign on the left-hand side of (2.3.39) is a matter of convention.

where γ is a path on the solution space relating $\bar{\phi}$ such that $S_{\bar{f}}[\bar{\phi}] = 0$ to the solution ϕ of interest. Applying the operator d on (2.3.41) gives back (2.3.39), using the property (A.2.14) of the homotopy operator. Notice that the expression (2.3.41) of $\mathbf{K}_{\bar{f}}[\phi]$ generically depends on the chosen path γ. Therefore, in practice, we consider the $(n-2)$-form $\mathbf{k}_{\bar{f}}[\phi; \delta\phi]$ defined in (2.3.40) as the fundamental object, rather than $\mathbf{K}_{\bar{f}}[\phi]$.

Example Let us return to our example of general relativity. The exact reducibility parameters of the theory are the diffeomorphism generators $\bar{\xi}$, which satisfy

$$\delta_{\bar{\xi}} g_{\mu\nu} = \mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0, \tag{2.3.42}$$

i.e. they are the Killing vectors of $g_{\mu\nu}$. Note that for a generic metric, this equation does not admit any solution. Hence, the previous construction is irrelevant for this general case. Now, consider linearized general relativity around a background $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. We can show that

$$\delta_{\bar{\xi}} h_{\mu\nu} = \mathcal{L}_{\bar{\xi}} \bar{g}_{\mu\nu} = 0, \tag{2.3.43}$$

i.e. the exact reducibility parameters of the linearized theory are the Killing vectors of the background $\bar{g}_{\mu\nu}$. If $\bar{g}_{\mu\nu}$ is taken to be the Minkowski metric, then the solutions of (2.3.43) are the generators of the Poincaré transformations. The $(n-2)$-form (2.3.41) can be constructed explicitly and integrated on a $(n-2)$-sphere at infinity. This gives the ADM charges of linearized gravity [113].

2.3.3 Asymptotic symmetries and surface charges

We now come to the case of main interest, where we are dealing with asymptotic symmetries in the sense of the definition in subsection 2.2.4. The prescription to construct the $(n-2)$-form $\mathbf{k}_f[\phi, \delta\phi]$ associated with generators of asymptotic symmetries f is essentially the same as the one introduced above for exact reducibility parameters. However, this $(n-2)$-form will not be conserved on-shell. Indeed, for a generic asymptotic symmetry, (2.3.38) does not hold; therefore, the equation (2.3.39) is not valid anymore. Nonetheless, as we will see below, we still have a control on the *breaking* in the conservation law.

Definition [Barnich-Brandt $(n-2)$-form for asymptotic symmetries] The $(n-2)$-form $\mathbf{k}_f$ associated with asymptotic symmetries generated by f is defined as

$$\mathbf{k}_f[\phi; \delta\phi] = -I^{n-1}_{\delta\phi} \mathbf{S}_f, \tag{2.3.44}$$

where $I^{n-1}_{\delta\phi}$ is the homotopy operator (A.2.12) and $\mathbf{S}_f$ is the weakly vanishing Noether current defined in the last line of (2.3.25). For a *first order gauge theory,* namely a gauge theory involving only first order derivatives of the gauge parameters $f = (f^\alpha)$ and the fields $\phi = (\phi^i)$ in the gauge transformations, and first order equations of motion for the fields, the $(n-2)$-form (2.3.44) becomes

$$\mathbf{k}_f[\phi; \delta\phi] = -\frac{1}{2}\delta\phi^i \frac{\partial}{\partial(\partial_\mu \phi^i)} \left(\frac{\partial}{\partial \mathrm{d}x^\mu} \mathbf{S}_f \right), \tag{2.3.45}$$

where

$$\mathbf{S}_f = R^{i\mu}_\alpha f^\alpha \frac{\delta L}{\delta \phi^i} (\mathrm{d}^{n-1}x)_\mu. \tag{2.3.46}$$

The simplicity of these expressions motivates the study of first order formulations of gauge theories in this context [181–184] (see chapter 3).

Example Let us consider the theory of general relativity. Applying the homotopy operator on the weakly vanishing Noether current $\mathbf{S}_\xi$ obtained in equation (2.3.32), we deduce the explicit expression

$$\begin{aligned} \mathbf{k}_\xi[g; h] = \frac{\sqrt{-g}}{8\pi G}(\mathrm{d}^{n-2}x)_{\mu\nu}[\xi^\nu \nabla^\mu h + \xi^\mu \nabla_\sigma h^{\sigma\nu} + \xi_\sigma \nabla^\nu h^{\sigma\mu} \\ + \frac{1}{2}(h \nabla^\nu \xi^\mu + h^{\mu\sigma}\nabla_\sigma \xi^\nu + h^{\nu\sigma}\nabla^\mu \xi_\sigma)], \end{aligned} \tag{2.3.47}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$. Indices are lowered and raised by $g_{\mu\nu}$ and its inverse, and $h = h^\mu{}_\mu$ [113]. Notice that this expression has also been derived both in the first order Cartan formulation and in the Palatini formulation of general relativity [181, 185].

Theorem [Conservation law] Define the *invariant presymplectic current as*

$$\mathbf{W}[\phi; \delta\phi, \delta\phi] = \frac{1}{2} I^n_{\delta\phi} \left(\delta\phi^i \frac{\delta \mathbf{L}}{\delta \phi^i} \right). \tag{2.3.48}$$

We have the following *conservation law*

$$\mathrm{d}\mathbf{k}_f[\phi; \delta\phi] \approx \mathbf{W}[\phi; R[f], \delta\phi], \tag{2.3.49}$$

where, in the equality $\approx$, it is implied that ϕ is a solution of the Euler-Lagrange equations and $\delta\phi$ is a solution of the linearized Euler-Lagrange equations. Furthermore, we use the notation $\mathbf{W}[\phi; R[f], \delta\phi] = i_{R[f]}\mathbf{W}[\phi; \delta\phi, \delta\phi]$.

The proof of this proposition involves the properties of the operators introduced in appendix A. We have

$$
\begin{aligned}
\mathrm{d}\mathbf{k}_f[\phi;\delta\phi] &= -\mathrm{d}I^{n-1}_{\delta\phi}\mathbf{S}_f \\
&= \delta\mathbf{S}_f - I^n_{\delta\phi}\mathrm{d}\mathbf{S}_f \\
&\approx -I^n_{\delta\phi}\mathrm{d}\mathbf{S}_f \\
&\approx -I^n_{\delta\phi}\left(R^i[f]\frac{\delta\mathbf{L}}{\delta\phi^i}\right) \\
&\approx \frac{1}{2}i_{R[f]}I^n_{\delta\phi}\left(\delta\phi^i\frac{\delta\mathbf{L}}{\delta\phi^i}\right) \\
&\approx i_{R[f]}\mathbf{W}[\phi;\delta\phi,\delta\phi] \\
&\approx \mathbf{W}[\phi;R[f],\delta\phi].
\end{aligned}
\tag{2.3.50}
$$

In the second equality, we used (A.2.14). In the third equality, we used the fact that $\delta\mathbf{S}_f \approx 0$, since $\delta\phi$ is a solution of the linearized Euler-Lagrange equations. In the fourth equality, we used Noether's second theorem (2.3.28). In the fifth equality, we used

$$
i_{R[f]}\mathbf{W}[\phi;\delta\phi,\delta\phi] = I^n_{R[f]}\left(\delta\phi^i\frac{\delta\mathbf{L}}{\delta\phi^i}\right) = -I^n_{\delta\phi}\left(R^i[f]\frac{\delta\mathbf{L}}{\delta\phi^i}\right). \tag{2.3.51}
$$

The proof of this statement can be found in appendix A.5 of [115]. Finally, in the sixth equality, we used the definition (2.3.48).

Definition [Surface charges] Let Σ be a $(n-1)$-surface and $\partial\Sigma$ its $(n-2)$-dimensional boundary. We define the *infinitesimal surface charge* $\delta\!\!\!/ H_f[\phi]$ as

$$
\delta\!\!\!/ H_f[\phi] = \int_{\partial\Sigma}\mathbf{k}_f[\phi;\delta\phi] \approx \int_{\Sigma}\mathbf{W}[\phi;R[f],\delta\phi]. \tag{2.3.52}
$$

The infinitesimal surface charge $\delta\!\!\!/ H_f[\phi]$ is said to be *integrable* if it is δ-exact, i.e. if $\delta\!\!\!/ H_f[\phi] = \delta H_f[\phi]$. The symbol $\delta\!\!\!/$ in (2.3.52) emphasizes that the infinitesimal surface charge is not necessarily integrable. If it is actually integrable, then we can define the *integrated surface charge* $H_f[\phi]$ as

$$
H_f[\phi] = \int_\gamma \delta H_f[\phi] + N[\bar{\phi}] = \int_\gamma\int_{\partial\Sigma}\mathbf{k}_f[\phi;\delta\phi] + N[\bar{\phi}], \tag{2.3.53}
$$

where γ is a path in the solution space, going from a reference solution $\bar{\phi}$ to the solution ϕ. $N[\bar{\phi}]$ is a chosen value of the charge for this reference solution, which is not fixed by the formalism. Notice that for integrable infinitesimal charge, the integrated charge $H_f[\phi]$ is independent from the chosen path γ [186].

Theorem [Charge representation theorem] Assuming integrability[12], the integrated surface charges satisfy the algebra

$$\{H_{f_1}, H_{f_2}\} \approx H_{[f_1,f_2]_A} + K_{f_1;f_2}[\bar{\phi}]. \tag{2.3.54}$$

In this expression, the integrated charges bracket is defined as

$$\{H_{f_1}, H_{f_2}\} = \delta_{f_2} H_{f_1} = \int_{\partial\Sigma} \mathbf{k}_{f_1}[\phi; \delta_{f_2}\phi]. \tag{2.3.55}$$

Furthermore, the central extension $K_{f_1;f_2}[\bar{\phi}]$, which depends only on the reference solution $\bar{\phi}$, is antisymmetric with respect to f_1 and f_2, i.e. $K_{f_1;f_2}[\bar{\phi}] = K_{f_2;f_1}[\bar{\phi}]$. It satisfies the 2-cocycle condition

$$K_{[f_1,f_2]_A;f_3}[\bar{\phi}] + K_{[f_2,f_3]_A;f_1}[\bar{\phi}] + K_{[f_3,f_1]_A;f_2}[\bar{\phi}] \approx 0. \tag{2.3.56}$$

Therefore, the integrated charges form a representation of the asymptotic symmetry algebra, up to a central extension [113, 115].

For the proof of this theorem, see e.g. section 1.4 of [186].

Remark In the literature, there are several criteria based on properties of the surface charges, that make a choice of boundary conditions interesting. The main properties are the following:

- The charges are usually required to be *finite*. Two types of divergences may occur: divergences in the expansion parameter defining asymptotics, say r, and divergences when performing the integration on the $(n-2)$-surface $\partial\Sigma$.

- The charges have to be *integrable*. As explained above, this criterion enables us to define integrated surface charges as in (2.3.53). Integrability implies that the charges form a representation of the asymptotic symmetry algebra, up to a central extension (see (2.3.54)).

- The charges have to be *generically non-vanishing*. Indeed, the asymptotic symmetries for which associated integrated charges identically vanish are considered as trivial in the strong definition of asymptotic symmetry group (2.1.3).

- The charges have to be *conserved* in time when the integration is performed on a spacelike $(n-2)$-dimensional surface $\partial\Sigma$ at infinity. This statement is not guaranteed a priori because of the breaking in the conservation law (2.3.49).

[12]One also has to assume $\mathbf{E}[\phi; \delta\phi, \delta\phi] = 0$, where $\mathbf{E}[\phi; \delta\phi, \delta\phi]$ is defined in (2.3.85).

However, even if these requirements seem reasonable, in practice, some of them may not be satisfied. Indeed, as we will see below, the BMS charges in four dimensions are not always finite, neither integrable, nor conserved [121]. We now discuss the violation of some of the above requirements:

- The fact that the charges may *not be finite* in terms of the expansion parameter r can be expected when the asymptotic region is taken to be at infinity. Indeed, consider r as a cut-off. It makes sense to integrate on a surface $\partial\Sigma$ at a constant finite value of r, encircling a finite volume. Then, taking the limit $r \to \infty$ leads to an infinite volume; therefore, it does not come as a surprise that quantities diverge. Furthermore, it has recently been shown in [187] that subleading orders in r in the $(n-2)$-form $\mathbf{k}_f[\phi; \delta\phi]$ contain some interesting physical information, such as the 10 conserved Newman-Penrose charges [188]. Therefore, it seems reasonable to think that overleading orders in r may also contain relevant information (see e.g. [189–191]).

- The *non-integrability* of the charges may be circumvented by different procedures to isolate an integrable part in the expression of the charges (see e.g. [118] and [8]). However, the final integrated surface charges obtained by these procedures do not have all the properties that integrable charges would have. In particular, the representation theorem does not generically hold. Another philosophy is to keep working with non-integrable expressions, without making any specific choice for the integrable part of the charges. In some situations, it is still possible to define a modified bracket for the charges, leading to a representation of the asymptotic symmetry algebra, up to a 2-cocycle which may depend on fields [8, 121]. However, no general representation theorem exists in this context, even if some progress has been made [192].

- Finally, the *non-conservation* of the charges contains some important information on the physics. For example, in asymptotically flat spacetimes at null infinity, the non-conservation in time of the charges associated with time translations is known as the *Bondi mass loss*. This tells us that the mass decreases in time at future null infinity because of a flux of radiation through the boundary. Hence, the non-conservation of the charges contains important information on the dynamics of the system.

Even if the charges have these pathologies, they still offer important insights on the system. They could be seen as interesting combinations of the elements of the solution space that enjoy some properties in their transformation (see e.g. [175,178]).

Examples We now provide explicit examples of surface charge constructions in four-dimensional general relativity. First, consider asymptotically AdS_4 spacetimes with Dirichlet boundary conditions (AAdS2) (condition (2.2.19) together with (2.2.23)), the associated solution derived in subsection 2.2.3 (equation (2.2.32)), and the associated asymptotic Killing vectors derived in subsection 2.2.4. Inserting this solution space and these asymptotic Killing vectors into the $(n-2)$-form (2.3.47) results in an integrable expression at order ρ^0. Therefore, we can construct an integrated surface charge (2.3.53) where the 2-surface $\partial\Sigma$ is taken to be the 2-sphere at infinity, written S^2_∞. We have the explicit expression

$$H_\xi[g] = \int_{S^2_\infty} \mathrm{d}^2\Omega \; (\xi^a_0 T_a{}^t), \tag{2.3.57}$$

where $\mathrm{d}^2\Omega$ is the integration measure on the 2-sphere (see e.g. [144]). These charges are finite and generically non-vanishing. Furthermore, we can easily show that they are conserved in time, i.e.

$$\frac{d}{dt} H_\xi[g] \approx 0. \tag{2.3.58}$$

Now, we consider definition (2.2.15) with (2.2.18) of asymptotically flat spacetimes in four dimensions (AF3). The surface charges are obtained by inserting the corresponding solution space derived in subsection 2.2.3 (see equation (2.2.52)) and the asymptotic Killing vectors discussed in subsection 2.2.4 into the expression (2.3.47), and then integrating over S^2_∞. The result is given by

$$\delta\!\!\!/ H_\xi[g;\delta g] \approx \delta J_\xi[g] + \Theta_\xi[g;\delta g], \tag{2.3.59}$$

where

$$\begin{aligned} J_\xi[g] &= \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \; \left[4fM + Y^A(2N_A + \frac{1}{16}\partial_A(C^{CB}C_{CB}))\right] \\ \Theta_\xi[g;\delta g] &= \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \; \left[\frac{f}{2} N_{AB}\delta C^{AB}\right] \end{aligned} \tag{2.3.60}$$

and where $N_{AB} = \partial_u C_{AB}$ [121]. As mentioned above, the infinitesimal surface charges are not integrable. Therefore, we cannot unambiguously define an integrated surface charge as in (2.3.53) (see, however, [8,118]). In particular, the representation theorem (2.3.54) does not hold. Nevertheless, we can define the following modified bracket [121]:

$$\{J_{\xi_1}, J_{\xi_2}\}_* = \delta_{\xi_2} J_{\xi_1}[g] + \Theta_{\xi_2}[g;\delta_{\xi_1} g]. \tag{2.3.61}$$

We can show that

$$\{J_{\xi_1}, J_{\xi_2}\}_* \approx J_{[\xi_1,\xi_2]_A}[g] + K_{\xi_1;\xi_2}[g], \tag{2.3.62}$$

where $K_{\xi_1;\xi_2}[g]$ is a *field-dependent* 2-cocycle given explicitly by[13]

$$K_{\xi_1;\xi_2}[g] = \frac{1}{32\pi G} \int_{S^2_\infty} \mathrm{d}^2\Omega \; [C^{BC}(f_1 D_B D_C D_A Y_2^A - f_2 D_B D_C D_A Y_1^A)]. \tag{2.3.63}$$

It satisfies the generalized 2-cocycle condition

$$K_{[\xi_1,\xi_2]_A,\xi_3} + \delta_{\xi_3} K_{\xi_1,\xi_2} + \text{cyclic (1,2,3)} \approx 0. \tag{2.3.64}$$

For the algebra (2.3.62) to make sense, its form should not depend on the particular choice of integrable part $J_\xi[g]$ in (2.3.60). In particular, defining $J' = J - N$ and $\Theta' = \Theta + \delta N$ for some $N = N_\xi[g]$, we obtain

$$\{J'_{\xi_1}, J'_{\xi_2}\}_* = J'_{[\xi_1,\xi_2]_A}[g] + K'_{\xi_1;\xi_2}[g], \tag{2.3.65}$$

where $\{J'_{\xi_1}, J'_{\xi_2}\}_* = \delta_{\xi_2} J'_{\xi_1}[g] + \Theta'_{\xi_2}[g;\delta_{\xi_1} g]$ and

$$K'_{\xi_1;\xi_2} = K_{\xi_1,\xi_2} - \delta_{\xi_2} N_{\xi_1} + \delta_{\xi_1} N_{\xi_2} + N_{[\xi_1,\xi_2]_A}. \tag{2.3.66}$$

Notice that $-\delta_{\xi_2} N_{\xi_1} + \delta_{\xi_1} N_{\xi_2} + N_{[\xi_1,\xi_2]_A}$ automatically satisfies the generalized 2-cocycle condition (2.3.64) [121]. Another property of the surface charges (2.3.59) and (2.3.60) is that they are not conserved. Indeed,

$$\frac{d}{du} \delta\!\!\!/ H_\xi[g] = \int_{S^2_\infty} \mathbf{W}[g;\delta_\xi g, \delta g], \tag{2.3.67}$$

where $\mathbf{W}[g;\delta g,\delta g]$ was computed[14] in [8]. We have

$$\int_{S^2_\infty} \mathbf{W}[g;\delta g,\delta g] = -\frac{1}{32\pi G} \int_{S^2_\infty} \mathrm{d}^2\Omega \; [\delta N^{AB} \wedge \delta C_{AB}]. \tag{2.3.68}$$

Notice that taking $f = 1$ and $Y^A = 0$ in (2.3.67), we recover the famous *Bondi mass loss formula* [1–3]. This formula indicates that the mass is decreasing in time because of the leak of radiation through $\mathscr{I}^+$. This was a striking argument for the existence of gravitational waves at the non-linear level of the theory. Finally, despite the BMS charges (2.3.59) and (2.3.60) not being divergent in r[15], we can show that some of the supertranslation charges diverge for the Kerr solution [121].

[13] Notice that this 2-cocycle is zero for globally well-defined conformal transformations on the 2-sphere. It becomes non-trivial when considering the extended BMS$_4$ group with $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ superrotations.

[14] More precisely, in [8], we computed the presymplectic form $\boldsymbol{\omega}[g;\delta g,\delta g]$ introduced below. However, as we will see, this is equal to the invariant presymplectic current in the Bondi gauge.

[15] As explained in [8,57,193], when taking into account the contact terms due to the meromorphic poles on the celestial sphere, divergences in r actually appear in the expressions (2.3.60).

Remark A non-trivial relation seems to exist between conservation and integrability of the surface charges. For example, in the case of Dirichlet boundary conditions in asymptotically AdS_4 spacetimes (AAdS2) considered above, we see that the surface charges are both integrable and conserved. Reciprocally, there is a relation between non-conservation and non-integrability of the surface charges. For example, in the asymptotically flat case (AF3), we see that the source of non-integrability is contained in the asymptotic shear C_{AB} and the news function $N_{AB} = \partial_u C_{AB}$. These are precisely the functions involved in the right-hand side of (2.3.68). We can consider many other examples where this phenomenon appears. Therefore, non-integrability is related to non-conservation of the charges. We will see below that for diffeomorphism-invariant theories, the relation between non-conservation and integrability is transparent in the covariant phase space formalism.

2.3.4 Relation between Barnich-Brandt and Iyer-Wald procedures

In this subsection, we briefly discuss the covariant phase space formalism leading to the Iyer-Wald prescription for surface charges [116–118, 194]. Notice that this method is valid only for diffeomorphism-invariant theories (including general relativity), and not for any gauge theories. In practice, this means that the parameters of the asymptotic symmetries are diffeomorphisms generators, i.e. $f \equiv \xi$ and $\delta_f \phi \equiv \mathcal{L}_\xi \phi$. Finally, we relate this prescription to the Barnich-Brandt prescription presented in detail in the previous section.

Definition [Presymplectic form] Consider a diffeomorphism-invariant theory with Lagrangian $\mathbf{L} = L\mathrm{d}^n x$. Let us perform an arbitrary variation of the Lagrangian. Using a similar procedure as in (2.3.6), we obtain

$$\begin{aligned} \delta L &= \delta\phi^i \frac{\partial L}{\partial \phi^i} + \delta \partial_\mu \phi^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \\ &= \delta\phi^i \frac{\delta L}{\delta \phi^i} + \partial_\mu \left(\delta\phi^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \right) \\ &= \delta\phi^i \frac{\delta L}{\delta \phi^i} + \partial_\mu \theta^\mu[\phi; \delta\phi], \end{aligned} \tag{2.3.69}$$

where

$$\boldsymbol{\theta}[\phi; \delta\phi] = \theta^\mu[\phi; \delta\phi](\mathrm{d}^{n-1}x)_\mu = \left(\delta\phi^i \frac{\partial L}{\partial(\partial_\mu \phi^i)} + \dots \right) (\mathrm{d}^{n-1}x)_\mu = I^n_{\delta\phi} \mathbf{L} \tag{2.3.70}$$

is the *presymplectic potential*. Taking into account that δ is Grassmann odd, the equation (2.3.69) can be rewritten as

$$\delta \mathbf{L} = \delta\phi^i \frac{\delta \mathbf{L}}{\delta \phi^i} - \mathrm{d}\boldsymbol{\theta}[\phi;\delta\phi]. \tag{2.3.71}$$

Now, the *presymplectic form* $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega}[\phi;\delta\phi,\delta\phi] = \delta\boldsymbol{\theta}[\phi,\delta\phi]. \tag{2.3.72}$$

Definition [Iyer-Wald $(n-2)$-form for asymptotic symmetries] The *Iyer-Wald $(n-2)$-form* $\mathbf{k}^{IW}_\xi$ associated with asymptotic symmetries generated by ξ is defined as

$$\mathbf{k}^{IW}_\xi[\phi;\delta\phi] = -\delta \mathbf{Q}_\xi[\phi] + \iota_\xi \boldsymbol{\theta}[\phi;\delta\phi], \tag{2.3.73}$$

up to an exact $(n-2)$-form[16]. In this expression, $\mathbf{Q}_\xi[\phi] = -I^{n-1}_\xi \boldsymbol{\theta}[\phi;\mathcal{L}_\xi\phi]$ is called the *Noether-Wald* surface charge.

Example For general relativity theory, the (canonical) presymplectic potential (2.3.70) is given by

$$\boldsymbol{\theta}[g;h] = \frac{\sqrt{-g}}{16\pi G}(\nabla_\nu h^{\mu\nu} - \nabla^\mu h)(\mathrm{d}^{n-1}x)_\mu, \tag{2.3.74}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$. Indices are lowered and raised by $g_{\mu\nu}$ and its inverse, and $h = h^\mu{}_\mu$. From this expression, the Noether-Wald charge can be computed; we obtain

$$\mathbf{Q}_\xi[g] = -I^{n-1}_\xi \boldsymbol{\theta}[g;\mathcal{L}_\xi g] = \frac{\sqrt{-g}}{8\pi G}\nabla^\mu\xi^\nu (\mathrm{d}^{n-2}x)_{\mu\nu} \tag{2.3.75}$$

and we recognize the *Komar charge*. Finally, inserting these expression into (2.3.73) yields

$$\mathbf{k}^{IW}_\xi[g;h] = \frac{\sqrt{-g}}{8\pi G}\left(\xi^\mu\nabla_\sigma h^{\nu\sigma} - \xi^\mu\nabla^\nu h + \xi_\sigma\nabla^\nu h^{\mu\sigma} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\rho\nu}\nabla_\rho\xi^\mu\right)(\mathrm{d}^{n-2}x)_{\mu\nu}. \tag{2.3.76}$$

[16] In the definition (2.3.73), we assumed that the variational operator δ in front of the Noether-Wald charge does not see the possible field-dependence of the asymptotic Killing vectors ξ^μ. Strictly speaking, one should write $\mathbf{k}^{IW}_\xi[\phi;\delta\phi] = -\delta\mathbf{Q}_\xi[\phi] + \mathbf{Q}_{\delta\xi}[\phi] + \iota_\xi\boldsymbol{\theta}[\phi;\delta\phi]$.

Theorem [Conservation law] We have the following *conservation law*:

$$\mathrm{d}\mathbf{k}_\xi^{IW}[\phi;\delta\phi] \approx \boldsymbol{\omega}[\phi;\mathcal{L}_\xi\phi,\delta\phi], \tag{2.3.77}$$

where, in the equality $\approx$, it is implied that ϕ is a solution of the Euler-Lagrange equations and $\delta\phi$ is a solution of the linearized Euler-Lagrange equations. Furthermore, $\boldsymbol{\omega}[\phi;\mathcal{L}_\xi\phi,\delta\phi] = i_{\mathcal{L}_\xi\phi}\boldsymbol{\omega}[\phi;\delta\phi,\delta\phi] = -\boldsymbol{\omega}[\phi;\delta\phi,\mathcal{L}_\xi\phi]$.

This can be proved using Noether's second theorem (2.3.28) (see e.g. [186] for a detailed proof).

Remark In the covariant phase space formalism, the relation between non-integrability and non-conservation mentioned in the previous subsection is clear. Indeed,

$$\begin{aligned}
\delta\oint H_\xi[\phi] &= \int_{\partial\Sigma} \delta\mathbf{k}_\xi^{IW}[\phi,\delta\phi] \\
&= \int_{\partial\Sigma} +\delta\iota_\xi\boldsymbol{\theta}[g,\delta g] \\
&= -\int_{\partial\Sigma} \iota_\xi\delta\boldsymbol{\theta}[g,\delta g] \\
&= -\int_{\partial\Sigma} \iota_\xi\boldsymbol{\omega}[g;\delta g,\delta g],
\end{aligned} \tag{2.3.78}$$

where we used (2.3.73) and (2.3.72) in the second and the fourth equality, respectively. The surface charge $\oint H_\xi[\phi]$ is integrable only if $\delta\oint H_\xi[\phi] = 0$, if and only if

$$\int_{\partial\Sigma} \iota_\xi\boldsymbol{\omega}[g;\delta g,\delta g] = 0 \tag{2.3.79}$$

Therefore, from

$$\mathrm{d}\oint H_\xi[\phi] = \int_{\partial\Sigma} \mathrm{d}\mathbf{k}_\xi^{IW}[g,\delta g] \approx \int_{\partial\Sigma} \boldsymbol{\omega}[\phi;\mathcal{L}_\xi\phi,\delta\phi], \tag{2.3.80}$$

the non-conservation is controlled by $\boldsymbol{\omega}[g,\delta g,\delta g]$ and is an obstruction for the integrability.

Remark As in the Barnich-Brandt procedure, the Iyer-Wald $(n-2)$-form (2.3.73) is defined up to an exact $(n-2)$-form. However, there is another source of ambiguity here coming from the definition of the presymplectic potential (2.3.70). In fact, we have the freedom to shift $\boldsymbol{\theta}$ by an exact $(n-1)$-form as

$$\boldsymbol{\theta}[\phi;\delta\phi] \to \boldsymbol{\theta}[\phi;\delta\phi] - \mathrm{d}\mathbf{Y}[\phi;\delta\phi], \tag{2.3.81}$$

where $\mathbf{Y}[\phi;\delta\phi]$ is a $(n-2)$-form. This implies that the presymplectic form (2.3.72) is modified as

$$\boldsymbol{\omega}[\phi;\delta\phi,\delta\phi] \to \boldsymbol{\omega}[\phi;\delta\phi,\delta\phi] + \mathrm{d}\delta\mathbf{Y}[\phi;\delta\phi], \tag{2.3.82}$$

where we used the fact that both d and δ are Grassmann odd. The Noether-Wald charge becomes

$$\mathbf{Q}_\xi[\phi] \to \mathbf{Q}_\xi[\phi] + \mathbf{Y}[\phi;\mathcal{L}_\xi\phi], \tag{2.3.83}$$

up to an exact $(n-2)$-form which can be reabsorbed in the $(n-2)$-form ambiguity for $\mathbf{k}^{IW}_\xi$ discussed above. Therefore, this ambiguity modifies $\mathbf{k}^{IW}_F$ given in (2.3.73) by

$$\mathbf{k}^{IW}_\xi[\phi;\delta\phi] \to \mathbf{k}^{IW}_\xi[\phi;\delta\phi] - \delta\mathbf{Y}[\phi;\mathcal{L}_\xi\phi] - \iota_\xi\mathrm{d}\mathbf{Y}[\phi;\delta\phi]. \tag{2.3.84}$$

Definition Let us introduce an important $(n-2)$-form which is involved in the relation between the Barnich-Brandt and Iyer-Wald prescriptions discussed in the remark below. We define

$$\mathbf{E}[\phi;\delta\phi,\delta\phi] = -\frac{1}{2}I^{n-1}_{\delta\phi}\boldsymbol{\theta} = -\frac{1}{2}I^{n-1}_{\delta\phi}I^{n}_{\delta\phi}\mathbf{L}. \tag{2.3.85}$$

Remark We now relate the Barnich-Brandt and the Iyer-Wald prescriptions to construct the $(n-2)$-form. Let us start from the expression (2.3.71) of the variation of the Lagrangian. We apply the homotopy operator on each side of the equality. We have

$$\begin{aligned} I^n_{\delta\phi}\delta\mathbf{L} &= I^n_{\delta\phi}\left(\delta\phi\frac{\delta\mathbf{L}}{\delta\phi}\right) - I^n_{\delta\phi}\mathrm{d}\boldsymbol{\theta} \\ &= I^n_{\delta\phi}\left(\delta\phi\frac{\delta\mathbf{L}}{\delta\phi}\right) - \delta\boldsymbol{\theta} - \mathrm{d}I^{n-1}_{\delta\phi}\boldsymbol{\theta}. \end{aligned} \tag{2.3.86}$$

Therefore,

$$I^n_{\delta\phi}\delta\mathbf{L} + \delta\boldsymbol{\theta} = I^n_{\delta\phi}\left(\delta\phi\frac{\delta\mathbf{L}}{\delta\phi}\right) - \mathrm{d}I^{n-1}_{\delta\phi}\boldsymbol{\theta}. \tag{2.3.87}$$

Since $[\delta, I^n_{\delta\phi}] = 0$ because $\delta^2 = 0$, the left-hand side of the last equality can be rewritten as $\delta I^n_{\delta\phi}\mathbf{L} + \delta\boldsymbol{\theta} = 2\delta\boldsymbol{\theta} = 2\boldsymbol{\omega}$ where we used (2.3.70). Now, using (2.3.48) and (2.3.85), we obtain the relation between the presymplectic form $\boldsymbol{\omega}$ and the invariant presymplectic current $\mathbf{W}$ as

$$\boldsymbol{\omega}[\phi;\delta\phi,\delta\phi] = \mathbf{W}[\phi;\delta\phi,\delta\phi] + \mathrm{d}\mathbf{E}[\phi;\delta\phi,\delta\phi]. \tag{2.3.88}$$

Contracting this relation with $i_{\mathcal{L}_\xi\phi}$ results in

$$\boldsymbol{\omega}[\phi;\mathcal{L}_\xi\phi,\delta\phi] = \mathbf{W}[\phi;\mathcal{L}_\xi\phi,\delta\phi] + \mathrm{d}\mathbf{E}[\phi;\delta\phi,\mathcal{L}_\xi\phi]. \tag{2.3.89}$$

Finally, using the on-shell conservation laws (2.3.49) and (2.3.77), we obtain

$$\mathbf{k}^{IW}_\xi[\phi;\delta\phi] \approx \mathbf{k}_\xi[\phi;\delta\phi] + \mathbf{E}[\phi;\delta\phi,\mathcal{L}_\xi\phi], \tag{2.3.90}$$

up to an exact $(n-2)$-form. Therefore, the Barnich-Brandt $(n-2)$-form $\mathbf{k}_\xi[\phi;\delta\phi]$ differs from the Iyer-Wald $(n-2)$-form $\mathbf{k}^{IW}_\xi[\phi;\delta\phi]$ by the term $\mathbf{E}[\phi;\delta\phi,\mathcal{L}_\xi\phi]$.

Examples We illustrate these concepts with the case of general relativity. The $(n-2)$-form $\mathbf{E}[\phi;\delta\phi,\delta\phi]$ can be computed using (2.3.85). We obtain

$$\mathbf{E}[g;\delta g,\delta g] = \frac{\sqrt{-g}}{32\pi G}(\delta g)^\mu{}_\sigma \wedge (\delta g)^{\sigma\nu}(\mathrm{d}^{n-2}x)_{\mu\nu}. \tag{2.3.91}$$

When contracted with $i_{\mathcal{L}_\xi g}$, this leads to

$$\mathbf{E}[g;\delta g,\mathcal{L}_\xi g] = -\frac{\sqrt{-g}}{16\pi G}(\nabla^\mu\xi_\sigma + \nabla_\sigma\xi^\mu)(\delta g)^{\sigma\nu}(\mathrm{d}^{n-2}x)_{\mu\nu}, \tag{2.3.92}$$

up to an exact $(n-2)$-form. This expression can also be obtained from (2.3.90) by comparing the explicit expressions (2.3.47) and (2.3.76). Notice that the difference between the Barnich-Brandt and the Iyer-Wald definitions (2.3.92) vanishes for a Killing vectors ξ^μ. Furthermore, a simple computation shows that the relevant components of the $(n-2)$-form (2.3.91) involved in the computation of the surface charges vanish in both the Fefferman-Graham gauge (2.2.8) and the Bondi gauge (2.2.10). Therefore, the Barnich-Brandt and the Iyer-Wald prescriptions lead to the same surface charges in these gauges. For an example where the two prescriptions do not coincide, see for instance, [195].

Chapter 3

First order formulations and surface charges

As mentioned in the previous chapter, the formalism to construct the co-dimension 2 forms containing the information on the surface charges is particularly well-adapted for first order gauge theories. In this chapter, we review some first order formulations of general relativity and apply the techniques of the covariant phase space formalism in this context.

In section 3.1, we study a class of theories that encompasses most of the first order gauge theories, including general relativity in Cartan and Newman-Penrose formulations, first order Maxwell theory, first order Yang-Mills theory and Chern-Simons theory. We also discuss vielbeins and connections in presence torsion and non-metricity. In section 3.2, we review important first order formulations of general relativity, Cartan and Newman-Penrose formulations, and apply the surface charges formalism. For each case, we relate the obtained results to the standard second order metric formulation of general relativity discussed in the examples in section 2.3.

This chapter essentially reproduces [181, 182, 185].

3.1 Generalities

3.1.1 Covariantized Hamiltonian formulations

In this subsection, we study an important class of first order gauge theories that is particularly well-adapted to the application of the surface charges formalism presented in section 2.3. Let us consider a first order theory that depends at most linearly on the derivatives of the fields,

$$L = a^\mu_j \partial_\mu \phi^j - h, \tag{3.1.1}$$

with a generating set of gauge transformations that depends at most on first order derivatives of the gauge parameters,

$$\delta_f \phi^i = R^i[f] = R^i_\alpha f^\alpha + R^{i\mu}_\alpha \partial_\mu f^\alpha, \tag{3.1.2}$$

and where the derivatives of the fields occur at most linearly in the term that does not contain derivatives of gauge parameters,

$$R^i_\alpha = R^{i0}_\alpha + R^{i\nu}_{j\alpha} \partial_\nu \phi^j. \tag{3.1.3}$$

We thus assume that $a^\mu_j[x,\phi], h[x,\phi], R^{i0}_\alpha[x,\phi], R^{i\nu}_{j\alpha}[x,\phi], R^{i\mu}_\alpha[x,\phi]$ do not depend on derivatives of the fields.

As the notation indicates, this is a covariantized version of first order Hamiltonian actions, where ϕ^i contains both the canonical variables and the Lagrange multipliers, while h includes both the canonical Hamiltonian and the constraints. For instance, for a first class Hamiltonian system, we have

$$L[z,u] = a_A(z)\dot{z}^A - H(z) - u^a \gamma_a(z). \tag{3.1.4}$$

Here z^A are the phase-space variables and $a_A(z)$ are the components of the symplectic potential. In the case of Darboux coordinates for instance, $z^A = (q^i, p_j)$ and $a_A = (p_1, \dots p_n, 0 \dots, 0)$. Furthermore, H is the Hamiltonian, γ_a are the first-class constraints and u^a are the associated Lagrange multipliers. The symplectic 2-form $\sigma_{AB} = \partial_A a_B - \partial_B a_A$ is assumed to be invertible, $\sigma^{CA}\sigma_{AB} = \delta^C_B$ with associated Poisson bracket $\{F,G\} = \frac{\partial F}{\partial z^A}\sigma^{AB}\frac{\partial G}{\partial z^B}$ and

$$\{\gamma_a, \gamma_b\} = C^c_{ab}(z)\gamma_c, \quad \{H, \gamma_a\} = V^b_a(z)\gamma_b. \tag{3.1.5}$$

For such systems, a generating set of gauge symmetries is given by

$$\delta_f z^A = \{z^A, \gamma_a\} f^a, \quad \delta_f u^a = \dot{f}^a - C^a_{bc} u^b f^c - V^a_b f^b, \tag{3.1.6}$$

see e.g. [196] for more details.

By using suitable sets of *auxiliary fields*, namely fields whose equations of motion can be solved algebraically in terms of the other fields and their derivatives [196], the class of theories (3.1.1) is relevant for gravity in the standard Cartan formulation or the one adapted to the Newman-Penrose formalism, as discussed below. Chern-Simons theory is directly of this type, while Yang-Mills theories are of this type when using the curvatures as auxiliary fields (see e.g. [119] for the case of Maxwell's theory).

For a Lagrangian of the form (3.1.1), the Euler-Lagrange derivative of the Lagrangian reduces to $\frac{\delta L}{\delta \phi^i} = \frac{\partial L}{\partial \phi^i} - \partial_\mu \left(\frac{\partial L}{\partial \partial_\mu \phi^i} \right)$ and is explicitly given by

$$\frac{\delta L}{\delta \phi^i} = \sigma^\mu_{ij}\partial_\mu \phi^j - \partial_i h - \frac{\partial}{\partial x^\mu} a^\mu_i, \quad \sigma^\mu_{ij} = \partial_i a^\mu_j - \partial_j a^\mu_i \Longrightarrow \partial_{[i}\sigma^\mu_{jk]} = 0, \tag{3.1.7}$$

where $\partial_i = \dfrac{\partial}{\partial\phi^i}$.

It is instructive to repeat the reasoning of section 2.3 leading to (2.3.27) and (2.3.28) for the class of theories at hand. Starting from

$$\delta_f\phi^i\frac{\delta L}{\delta\phi^i} = \partial_\mu j^\mu_f, \tag{3.1.8}$$

where j^μ_f is a representative of the Noether current (see equation (2.3.24)), and integrating by parts on the left hand side so as to make the undifferentiated gauge parameters appear, one obtains

$$f^\alpha[R^i_\alpha\frac{\delta L}{\delta\phi^i} - \partial_\mu(R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i})] = \partial_\mu(j^\mu_f - S^\mu_f), \quad S^\mu_f = f^\alpha R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i}. \tag{3.1.9}$$

Since this is an off-shell identity that has to hold for all $f^\alpha[x]$, one concludes not only that the Noether identities

$$R^i_\alpha\frac{\delta L}{\delta\phi^i} - \partial_\mu(R^{i\mu}_\alpha\frac{\delta L}{\delta\phi^i}) = 0 \tag{3.1.10}$$

hold, but also that $\partial_\mu(j^\mu_f - S^\mu_f) = 0$, which is the second Noether theorem. This implies in particular that S^μ_f is a representative for the Noether current associated with gauge symmetries that is trivial in the sense that it vanishes on-shell. Furthermore, every other representative j^μ_f differs from S^μ_f at most by the divergence of an arbitrary superpotential $\partial_\nu\eta^{[\mu\nu]}_f$.

Since the fields and their derivatives can be seen as independent coordinates on the jet space (see appendix A.1), the Noether identities (3.1.10) can be separated into terms involving $\partial_\mu\partial_\nu\phi^j$, $\partial_\mu\phi^k\partial_\nu\phi^j$, $\partial_\mu\phi^j$ or no derivatives. The vanishing of the coefficients of these terms yields

$$\begin{aligned}
&R^{i(\mu}_\alpha\sigma^{\nu)}_{ij} = 0,\\
&\partial_k(R^{i\mu}_\alpha\sigma^\nu_{ij}) + \partial_j(R^{i\nu}_\alpha\sigma^\mu_{ik}) - R^{i\mu}_{k\alpha}\sigma^\nu_{ij} - R^{i\nu}_{j\alpha}\sigma^\mu_{ik} = 0,\\
&R^{i0}_\alpha\sigma^\mu_{ij} + \partial_j[R^{i\mu}_\alpha(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)] - R^{k\mu}_{j\alpha}(\partial_k h + \frac{\partial}{\partial x^\nu}a^\nu_k) - \frac{\partial}{\partial x^\nu}(R^{i\nu}_\alpha\sigma^\mu_{ij}) = 0,\\
&R^{i0}_\alpha(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i) - \frac{\partial}{\partial x^\mu}[R^{i\mu}_\alpha(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)] = 0.
\end{aligned} \tag{3.1.11}$$

The construction of the co-dimension 2 form deeply relies on the linearized theory. Writing φ^i the Grassmann even variation of ϕ^i, the Lagrangian $L^{(2)}[\phi;\varphi]$ of the linearized theory is obtained by collecting the quadratic terms in the expansion of $L[\phi+\varphi]$ in φ^i and their derivatives around a solution ϕ. We obtain

$$L^{(2)}[\phi;\varphi] = \partial_i a^\mu_j\varphi^i\partial_\mu\varphi^j + \frac{1}{2}\partial_i\partial_j a^\mu_k\varphi^i\varphi^j\partial_\mu\phi^k - \frac{1}{2}\partial_i\partial_j h\varphi^i\varphi^j. \tag{3.1.12}$$

The linearized equations of motion are then given by

$$\frac{\delta L^{(2)}[\phi;\varphi]}{\delta\varphi^i} = [\sigma^\mu_{ij}\partial_\mu + \partial_j\sigma^\mu_{ik}\partial_\mu\phi^k - \partial_j(\partial_i h + \frac{\partial}{\partial x^\nu}a^\nu_i)]\varphi^j = 0. \tag{3.1.13}$$

Consider now the co-dimension 2 form,

$$k_f^{[\mu\nu]}[\phi;\delta\phi] = R^{i[\mu}_\alpha\sigma^{\nu]}_{ij}\delta\phi^j f^\alpha, \tag{3.1.14}$$

and the invariant presymplectic current

$$W^\mu[\phi;\delta\phi_1,\delta\phi_2] = \frac{1}{2}\sigma^\mu_{ij}\delta\phi^i_1\wedge\delta\phi^j_2. \tag{3.1.15}$$

By using the equations of motion, the linearized equations of motion and the Noether identities in the form of (3.1.11), one may then check by a direct computation that

$$\partial_\nu k_f^{[\mu\nu]}[\phi;\delta\phi] = -W^\mu[\phi;R[f],\delta\phi] \quad \text{when} \quad \frac{\delta L}{\delta\phi^i} = 0 = \frac{\delta L^{(2)}[\phi;\varphi]}{\delta\varphi^i}. \tag{3.1.16}$$

This means that this co-dimension 2 form is conserved on all solutions of the linearized equations of motion around a given background solution ϕ when using reducibility parameters $\bar{f}^\alpha$, which satisfy

$$R[\bar{f}] = R^i_\alpha\bar{f}^\alpha + R^{i\mu}_\alpha\partial_\mu\bar{f}^\alpha = 0. \tag{3.1.17}$$

In terms of forms, we can write

$$\mathbf{k}_f[\phi;\delta\phi] = R^{i\mu}_\alpha\sigma^\nu_{ij}\delta\phi^j f^\alpha(\mathrm{d}^{n-2}x)_{\mu\nu}, \tag{3.1.18}$$

where

$$\mathrm{d}\mathbf{k}_f[\phi;\delta\phi] = \mathbf{W}[\phi;R[f],\delta\phi] \quad \text{when} \quad \frac{\delta L}{\delta\phi^i} = 0 = \frac{\delta L^{(2)}[\phi;\varphi]}{\delta\varphi^i}, \tag{3.1.19}$$

with

$$\mathbf{W}[\phi;\delta\phi,\delta\phi] = \frac{1}{2}\sigma^\mu_{ij}\delta\phi^i\wedge\delta\phi^j(\mathrm{d}^{n-1}x)_\mu. \tag{3.1.20}$$

We see that we have re-derived some key results of section 2.3 (in particular, the conservation law (2.3.49)) without using the properties of the homotopy operator (A.2.12). Therefore, the class of covariantized Hamiltonian theories considered here drastically simplifies the computations in the Barnich-Brandt formalism.

These results can be related to the Iyer-Wald formalism introduced in subsection 2.3.4. The presymplectic potential is given by

$$\boldsymbol{\theta}[\phi;\delta\phi] = a^\mu_i\delta\phi^i(\mathrm{d}^{n-1}x)_\mu, \tag{3.1.21}$$

and its associated presymplectic form reads as

$$\boldsymbol{\omega}[\phi;\delta\phi,\delta\phi] = \delta\boldsymbol{\theta}[\phi;\delta\phi] = \mathbf{W}[\phi;\delta\phi,\delta\phi]. \tag{3.1.22}$$

In particular, we see that the $(n-2)$-form defined in (2.3.85) and that controls the difference between Barnich-Brandt and Iyer-Wald procedures vanishes for covariantized Hamiltonian theories, i.e. $\mathbf{E}[\phi;\delta\phi,\delta\phi] = 0$. Finally, equation (3.1.18) can be expressed in terms of $\boldsymbol{\theta}[\phi;\delta\phi]$ as

$$\mathbf{k}_f[\phi;\delta\phi] = -\frac{1}{2}\left(f^\alpha R^{i\mu}_\alpha \frac{\partial}{\partial\delta\phi^i}\right)\frac{\partial}{\partial \mathrm{d}x^\mu}\delta\boldsymbol{\theta}[\phi;\delta\phi]. \tag{3.1.23}$$

3.1.2 Vielbeins and connection

Now, we recall several notions of vielbeins and connections by including torsion and non-metricity into the standard discussion. This formalism is useful in section 3.2 when discussing the first order formulations of general relativity.

General case

Consider an n-dimensional spacetime with a *moving frame* (or *vielbein*)

$$e_a = e_a{}^\mu \frac{\partial}{\partial x^\mu}, \quad e^a = e^a{}_\mu \mathrm{d}x^\mu, \tag{3.1.24}$$

where $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$, $e_a{}^\mu e^b{}_\mu = \delta^b_a$, and $\partial_a f = e_a(f)$. Under a combined frame and coordinate transformation, we have $e'_a{}^\mu(x') = \Lambda_a{}^b(x) e_b{}^\nu(x)\Lambda^\mu{}_\nu(x)$, where $\Lambda^a{}_b(x)$ denotes a local $GL(n,\mathbb{R})$ element with $\Lambda_a{}^b = (\Lambda^{-1})^b{}_a$ while $\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ is the Jacobian matrix of the coordinate transformation, with $\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$. The Lie algebra generators of $\mathfrak{gl}(n,\mathbb{R})$ are denoted by $\Delta_a{}^b$, $(\Delta_a{}^b)^c{}_d = \delta^c_a\delta^b_d$, with generators for the vector representation denoted by t_a,

$$[\Delta_a{}^b, \Delta_c{}^d] = \delta^b_c \Delta_a{}^d - \delta^d_a \Delta_c{}^b, \quad \Delta_a{}^b t_c = \delta^b_c t_a. \tag{3.1.25}$$

The structure functions are defined by

$$[e_a, e_b] = D^c{}_{ab} e_c \iff \mathrm{d}e^a = -\frac{1}{2} D^a{}_{bc} e^b e^c. \tag{3.1.26}$$

For further use, note that if $\mathbf{e} = \det e^a{}_\mu$, then

$$\partial_\mu(\mathbf{e}\, e^\mu{}_a) = \mathbf{e}\, D^b{}_{ba}, \tag{3.1.27}$$

and, if we define,

$$d^a{}_{bc} = e^a{}_\lambda \partial_b e_c{}^\lambda, \tag{3.1.28}$$

then

$$d^\sigma{}_{\rho\mu} = -e_d{}^\sigma \partial_\rho e^d{}_\mu, \quad D^a{}_{bc} = 2d^a{}_{[bc]}, \tag{3.1.29}$$

where it is understood that tangent space indices $a, b, \ldots$ and world-indices $\mu, \nu, \ldots$ are transformed into each other by using the vielbeins and their inverse.

In addition, we assume that there is an *affine connection*

$$D_a e_b = \Gamma^c{}_{ba} e_c \iff D_b v^a = \partial_b v^a + \Gamma^a{}_{cb} v^c. \tag{3.1.30}$$

If $\Gamma^a{}_b = \Gamma^a{}_{bc} e^c$, $\Gamma = \Gamma^a{}_b \Delta_a{}^b$, and $e = e^a t_a$, the *torsion* tensor and *curvature* tensors are defined by

$$\mathcal{T} = T^a t_a = \mathrm{d}e + \Gamma \wedge e, \quad \mathcal{R} = R^a{}_b \Delta_a{}^b = \mathrm{d}\Gamma + \frac{1}{2}[\Gamma, \Gamma], \tag{3.1.31}$$

where the bracket is the graded commutator. More explicitly, $T^a = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c = \mathrm{d}e^a + \Gamma^a{}_b \wedge e^b$, so that

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \Gamma^a{}_{b\mu} e^b{}_\nu - \Gamma^a{}_{b\nu} e^b{}_\mu, \tag{3.1.32}$$

$$T^c{}_{ab} = 2\Gamma^c{}_{[ba]} + D^c{}_{ba} = 2(\Gamma^c{}_{[ba]} + d^c{}_{[ba]}), \tag{3.1.33}$$

and $R^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d = \mathrm{d}\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b$, so that

$$R^f{}_{c\mu\nu} = \partial_\mu \Gamma^f{}_{c\nu} - \partial_\nu \Gamma^f{}_{c\mu} + \Gamma^f{}_{d\mu} \Gamma^d{}_{c\nu} - \Gamma^f{}_{d\nu} \Gamma^d{}_{c\mu}, \tag{3.1.34}$$

$$R^f{}_{cab} = \partial_a \Gamma^f{}_{cb} - \partial_b \Gamma^f{}_{ca} + \Gamma^f{}_{da} \Gamma^d{}_{cb} - \Gamma^f{}_{db} \Gamma^d{}_{ca} - D^d{}_{ab} \Gamma^f{}_{cd}. \tag{3.1.35}$$

Furthermore,

$$[D_a, D_b] v_c = -R^d{}_{cab} v_d - T^d{}_{ab} D_d v_c. \tag{3.1.36}$$

Under a local frame transformation, we have

$$e' = \Lambda e, \quad \Gamma' = \Lambda \Gamma \Lambda^{-1} + \Lambda \mathrm{d}\Lambda^{-1}, \tag{3.1.37}$$

so that

$$\mathcal{T}' = \Lambda \mathcal{T}, \quad \mathcal{R}' = \Lambda \mathcal{R} \Lambda^{-1}. \tag{3.1.38}$$

Defining $\Lambda = \mathbf{1} + \omega + O(\omega^2)$, with $\omega = \omega^a{}_b \Delta_a{}^b$ and also $\omega_b{}^a = -\omega^a{}_b$, we have

$$\delta_\omega \Gamma = -(\mathrm{d}\omega + [\Gamma, \omega]) \iff \delta_\omega \Gamma^a{}_b = \mathrm{d}\omega_b{}^a + \Gamma^a{}_c \omega_b{}^c - \Gamma^c{}_b \omega_c{}^a, \tag{3.1.39}$$

and also

$$\delta_\omega e = [\omega, e] \iff \delta_\omega e^a = \omega^a{}_b e^b. \tag{3.1.40}$$

Under a coordinate transformation, we have

$$e'^a{}_\mu = \Lambda_\mu{}^\nu e^a{}_\nu, \quad \Gamma'^a{}_{b\mu} = \Lambda_\mu{}^\nu \Gamma^a{}_{b\nu}, \tag{3.1.41}$$

and for $x'^\mu = x^\mu - \xi^\mu + O(\xi^2)$, $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \partial_\nu \xi^\mu + O(\xi^2)$, so that $\omega_\nu{}^\mu = \partial_\nu \xi^\mu$ and

$$\delta_\xi e^a{}_\mu = \mathcal{L}_\xi e^a_\mu, \quad \delta_\xi \Gamma^a{}_{b\mu} = \mathcal{L}_\xi \Gamma^a{}_{b\mu}, \tag{3.1.42}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative.

The *Bianchi identities* are

$$\mathrm{d}\mathcal{T} + \Gamma \wedge \mathcal{T} = \mathcal{R} \wedge e, \quad \mathrm{d}\mathcal{R} + [\Gamma, \mathcal{R}] = 0. \tag{3.1.43}$$

Explicitly,

$$R^a{}_{[bcd]} = D_{[b} T^a{}_{cd]} + T^a{}_{f[b} T^f{}_{cd]}, \quad D_{[f} R^a{}_{|b|cd]} = -R^a{}_{bg[f} T^g{}_{cd]}, \tag{3.1.44}$$

where a bar encloses indices that are not involved in the (anti) symmetrization. The Ricci tensor is defined by $R_{ab} = R^c{}_{acb}$, while $S_{ab} = R^c{}_{cab}$. Contracting the Bianchi identities gives

$$R_{ab} - R_{ba} = S_{ab} - D_c T^c{}_{ab} - 2D_{[a} T^c{}_{b]c} - T^c{}_{dc} T^d{}_{ab}, \tag{3.1.45}$$

$$2D_{[f} R_{|b|d]} + D_c R^c{}_{bdf} = R_{bg} T^g{}_{df} - 2R^c{}_{b[f|g|} T^g{}_{d]c}, \tag{3.1.46}$$

$$D_{[f} S_{cd]} = -S_{g[f} T^g{}_{cd]}. \tag{3.1.47}$$

Assume now that there is a *pseudo-Riemannian metric*,

$$g_{\mu\nu} = e^a{}_\mu g_{ab} e^b{}_\nu, \tag{3.1.48}$$

i.e., a symmetric, non-degenerate 2-tensor . As usual, tangent space indices $a, b, \ldots$ and world indices $\mu, \nu, \ldots$ are lowered and raised with g_{ab}, $g_{\mu\nu}$, and their inverses.

The *non-metricity tensor* is defined as $\Xi^{ab} = dg^{ab} + 2\Gamma^{(ab)}$. The associated Bianchi identities are given by $d\Xi^{ab} + \Gamma^a{}_c \Xi^{cb} + \Gamma^b{}_c \Xi^{ac} = 2R^{(ab)}$. More explicitly,

$$\Xi^{ab}{}_c = D_c g^{ab}, \quad 2D_{[c} \Xi^{ab}{}_{d]} = -\Xi^{ab}{}_f T^f{}_{cd} + 2R^{(ab)}{}_{cd}. \tag{3.1.49}$$

It should also be noted that from $g^{ab} g_{bc} = \delta^a_c$, it follows

$$D_c g_{ab} = -\Xi_{abc}. \tag{3.1.50}$$

Contracting the last of (3.1.49) with g_{ab} gives

$$S_{cd} = g_{ab} D_{[c} \Xi^{ab}{}_{d]} + \frac{1}{2} \Xi^a{}_{af} T^f{}_{cd}, \tag{3.1.51}$$

while (3.1.46) with g^{bf} gives

$$\begin{aligned} D^b R_{ba} - \frac{1}{2} D_a R = {} & \frac{1}{2} R^{bc}{}_{da} T^d{}_{bc} + R^b{}_c T^c{}_{ab} \\ & - \frac{1}{2} (\Xi^{bc}{}_c R_{ba} + \Xi^{cd}{}_b R^b{}_{cda} + \Xi^{bc}{}_a R_{bc}) \\ & + D_c (D_{[b} \Xi^{bc}{}_{a]} + \frac{1}{2} \Xi^{bc}{}_d T^d{}_{ba}) + (D_{[b} \Xi^{bc}{}_{d]} + \frac{1}{2} \Xi^{bc}{}_d T^d{}_{bd}) T^d{}_{ac}. \end{aligned} \tag{3.1.52}$$

The curvature scalar is defined by $R = g^{ab}R_{ab}$, the Einstein tensor by

$$G_{ab} = R_{(ab)} - \frac{1}{2}g_{ab}R. \tag{3.1.53}$$

When combining with (3.1.45), the contracted Bianchi identity (3.1.52) written in terms of the Einstein tensor is

$$\begin{aligned} D^b G_{ba} = {} & \frac{1}{2}R^{bc}{}_{da}T^d{}_{bc} + R^b{}_c T^c{}_{ab} - \frac{1}{2}\Xi_{ab}{}^b R \\ & + \frac{1}{2}D^b(S_{ab} - D_c T^c{}_{ab} - 2D_{[a}T^c{}_{b]c} - T^c{}_{dc}T^d{}_{ab}) \\ & - \frac{1}{2}(\Xi^{bc}{}_c R_{ba} + \Xi^{cd}{}_b R^b{}_{cda} + \Xi^{bc}{}_a R_{bc}) \\ & + D_c(D_{[b}\Xi^{bc}{}_{a]} + \frac{1}{2}\Xi^{bc}{}_d T^d{}_{ba}) + (D_{[b}\Xi^{bc}{}_{d]} + \frac{1}{2}\Xi^{bc}{}_d T^d{}_{bd})T^d{}_{ac}. \end{aligned} \tag{3.1.54}$$

By the usual manipulations, one may show that the existence of the metric implies that the most general connection can be written as

$$\Gamma_{abc} = \{_{abc}\} + M_{abc} + K_{abc} + r_{abc} \tag{3.1.55}$$

where the *Christoffel symbols*, the *conmetricity*, the *contorsion tensor*, and the *co-structure functions* are given by

$$\{_{abc}\} = \frac{1}{2}(g_{ab,c} + g_{ac,b} - g_{bc,a}) = \{_{acb}\}, \tag{3.1.56}$$

$$M_{abc} = \frac{1}{2}(\Xi_{abc} + \Xi_{acb} - \Xi_{bca}) = M_{acb}, \tag{3.1.57}$$

$$K_{abc} = \frac{1}{2}(T_{bac} + T_{cab} - T_{abc}) = -K_{bac}, \tag{3.1.58}$$

$$r_{abc} = \frac{1}{2}(D_{bac} + D_{cab} - D_{abc}) = -r_{bac}. \tag{3.1.59}$$

Furthermore, one can directly show that

$$\Gamma^a{}_{b\mu} = e^a{}_\nu(\partial_\mu e_b{}^\nu + \Gamma^\nu{}_{\rho\mu}e^\rho{}_b) \iff \Gamma_{abc} = e_{a\nu}\partial_c e_b{}^\nu + e_a{}^\mu e_b{}^\nu e_c{}^\rho \Gamma_{\mu\nu\rho}. \tag{3.1.60}$$

Finally, we need the following variations,

$$\delta R^a{}_{b\mu\nu} = D_\mu \delta\Gamma^a{}_{b\nu} - D_\nu \delta\Gamma^a{}_{b\mu}, \tag{3.1.61}$$

$$\begin{aligned} \delta R^a{}_{bcd} = {} & D_c\delta\Gamma^a{}_{bd} - D_d\delta\Gamma^a{}_{bc} + T^f{}_{cd}\delta\Gamma^a{}_{bf} \\ & + e^f{}_\sigma\big[\Gamma^a{}_{bf}\mathcal{D}_d + \partial_f\Gamma^a{}_{bd} - D^g{}_{fd}\Gamma^a{}_{bg}\big]\delta e_c{}^\sigma \\ & - e^f{}_\sigma\big[\Gamma^a{}_{bf}\mathcal{D}_c + \partial_f\Gamma^a{}_{bc} - D^g{}_{fc}\Gamma^a{}_{bg}\big]\delta e_d{}^\sigma. \end{aligned} \tag{3.1.62}$$

To write the latter variation, we introduced the derivative operator $\mathcal{D}$ defined through

$$\mathcal{D}_\mu \delta e_b{}^\sigma = \partial_\mu \delta e_b{}^\sigma + d^\sigma{}_{\mu\rho}\delta e_b{}^\rho - d^c{}_{\mu b}\delta e_c{}^\sigma. \tag{3.1.63}$$

Metricity and Lorentz metric

When requiring metricity, $D_a g^{bc} = 0 = \Xi^{bc}{}_a$, the connection is given by

$$\Gamma_{abc} = \{_{abc}\} + K_{abc} + r_{abc}, \tag{3.1.64}$$

From $D_{[a}D_{b]}g_{cd} = 0$ it also follows that

$$R_{abcd} = -R_{bacd}, \quad S_{ab} = 0. \tag{3.1.65}$$

This can be used to show that

$$\begin{aligned} R_{abcd} - R_{cdab} = \frac{3}{2}(D_{[b}T_{|a|cd]} + T_{af[b}T^f{}_{cd]} - D_{[a}T_{|b|cd]} - T_{bf[a}T^f{}_{cd]} \\ - D_{[d}T_{|c|ab]} - T_{cf[d}T^f{}_{ab]} + D_{[c}T_{|d|ab]} + T_{df[c}T^f{}_{ab]}), \end{aligned} \tag{3.1.66}$$

while the contracted Bianchi identities (3.1.54) become

$$D^b G_{ba} = \frac{1}{2}R^{bc}{}_{da}T^d{}_{bc} + R^b{}_c T^c{}_{ab} - \frac{1}{2}D^b(D_c T^c{}_{ab} + 2D_{[a}T^c{}_{b]c} + T^c{}_{dc}T^d{}_{ab}). \tag{3.1.67}$$

If metricity holds and we assume a constant Lorentz metric $g_{ab} = \eta_{ab}$, we have $\{abc\} = 0$ and

$$\Gamma_{abc} = -\Gamma_{bac}. \tag{3.1.68}$$

Local Lorentz transformations are denoted by $\Lambda_a{}^b(x)$ with $\Lambda_a{}^b\eta_{bc}\Lambda_d{}^c = \eta_{ad}$, or equivalently, $\Lambda^d{}_b\Lambda_a{}^b = \delta^d_a$. In terms of the Poincaré algebra,

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \quad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \tag{3.1.69}$$

one defines the Lorentz connection $\Gamma = \frac{1}{2}\Gamma^{ab}J_{ab}$, $e = e^a P_a$, $R = \frac{1}{2}R^{ab}J_{ab}$, $T = T^a P_a$, so that $R = \mathrm{d}\Gamma + \frac{1}{2}[\Gamma,\Gamma]$, $T = \mathrm{d}e + [\Gamma, e]$. In this case,

$$\partial_\mu(\mathbf{e}\, v^\mu) = \mathbf{e}\,(D_\mu + e_b{}^\nu\partial_\mu e^b{}_\nu)v^\mu = D_\mu(\mathbf{e}v^\mu), \tag{3.1.70}$$

with $D_\mu v^\mu = \partial_\mu v^\mu$ for the Lorentz connection and the definition

$$D_\mu\mathbf{e} = \mathbf{e}\,(e_b{}^\nu\partial_\mu e^b{}_\nu). \tag{3.1.71}$$

In particular, this implies that

$$D_\mu(\mathbf{e}\, e^\mu{}_a) = \mathbf{e}\, T^b{}_{ab}. \tag{3.1.72}$$

The connection reduces to

$$\Gamma_{abc} = K_{abc} + r_{abc}. \tag{3.1.73}$$

Finally, if one imposes, in addition, vanishing of torsion, the connection is reduced further to

$$\Gamma_{abc} = r_{abc}, \tag{3.1.74}$$

and the contracted Bianchi identities (3.1.67) reduce to

$$D_b G^b{}_a = 0. \tag{3.1.75}$$

Coordinate basis, torsionless connection

In a coordinate basis, $e_a{}^\mu = \delta_a{}^\mu$, $D^\lambda{}_{\mu\nu} = 0$ and $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}$. Imposing vanishing of torsion, $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$, (3.1.45) implies $S_{\mu\nu} = R_{\mu\nu} - R_{\nu\mu}$ and the contracted Bianchi identities (3.1.54) become

$$D^\nu G_{\nu\mu} = D^\nu R_{[\mu\nu]} + D_\lambda R^{(\lambda\nu)}{}_{\nu\mu} \\ - \frac{1}{2}(D_\nu g^{\nu\lambda} R_{\lambda\mu} + D_\nu g^{\lambda\rho} R^\nu{}_{\lambda\rho\mu} + D_\mu g^{\nu\lambda} R_{\nu\lambda} + D^\nu g_{\nu\mu} R), \quad (3.1.76)$$

while the variation (3.1.61) simplifies to

$$\delta R^\alpha{}_{\beta\mu\nu} = D_\mu \delta\Gamma^\alpha{}_{\beta\nu} - D_\nu \delta\Gamma^\alpha{}_{\beta\mu}. \quad (3.1.77)$$

We also have

$$\partial_\mu(\sqrt{|g|}v^\mu) = \sqrt{|g|}(D_\mu - \Gamma^\nu{}_{\mu\nu} + \frac{1}{2}g^{\nu\lambda}\partial_\mu g_{\nu\lambda})v^\mu = D_\mu(\sqrt{|g|}v^\mu), \quad (3.1.78)$$

where we defined

$$D_\mu\sqrt{|g|} = \sqrt{|g|}(\frac{1}{2}g^{\nu\lambda}\partial_\mu g_{\nu\lambda} - \Gamma^\nu{}_{\mu\nu}) \quad (3.1.79)$$

to write the last equality. Under an infinitesimal coordinate transformation, besides $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$, we have

$$\delta_\xi \Gamma^\mu{}_{\nu\rho} = \partial_\rho\partial_\nu\xi^\mu + \xi^\sigma\partial_\sigma\Gamma^\mu{}_{\nu\rho} - \partial_\sigma\xi^\mu\Gamma^\sigma{}_{\nu\rho} + \partial_\nu\xi^\sigma\Gamma^\mu{}_{\sigma\rho} + \partial_\rho\xi^\sigma\Gamma^\mu{}_{\nu\sigma}. \quad (3.1.80)$$

Requiring in addition metricity, this leads to the Levi-Civita connection ($D_\mu \equiv \nabla_\mu$)

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu}), \quad (3.1.81)$$

while the contracted Bianchi identities (3.1.76) reduce to

$$\nabla^\nu G_{\nu\mu} = 0, \quad (3.1.82)$$

and (3.1.78) to

$$\partial_\mu(\sqrt{|g|}v^\mu) = \sqrt{|g|}\nabla_\mu v^\mu. \quad (3.1.83)$$

3.2 First order formulations of general relativity

In this section, we review some first order formulations of general relativity including Cartan and Newman-Penrose formulations, and apply the surface charges formalism. In particular, we see that these are first order formulation in the sense of the covariantized Hamiltonian theories of subsection 3.1.1. For each case, we relate the obtained results to the standard second order metric formulation of general relativity discussed in the examples in section 2.3. We use the compact notation

$$\kappa = \frac{1}{16\pi G}. \quad (3.2.1)$$

3.2.1 Cartan formulation I

Variational principle

In the standard Cartan formulation, the variables of the variational principle are the components of the vielbein $e_a{}^\mu$ and a Lorentz connection 1-form in the coordinate basis, $\Gamma^a{}_{b\mu}$ in terms of which the action is

$$S^C[e_a{}^\mu, \Gamma^b{}_{c\nu}] = \kappa \int \mathrm{d}^n x\, L^C = \kappa \int \mathrm{d}^n x\, \mathbf{e}\,(R^{ab}{}_{\mu\nu} e_a{}^\mu e_b{}^\nu - 2\Lambda). \tag{3.2.2}$$

Using (3.1.61), the variation of the action is given by

$$\delta S^C = \kappa \int \mathrm{d}^n x\, \mathbf{e}\left[2(G^a{}_\mu + \Lambda e^a{}_\mu)\delta e_a{}^\mu + e_a{}^\mu e_b{}^\nu (D_\mu \delta\Gamma^{ab}{}_\nu - D_\nu \delta\Gamma^{ab}_\mu)\right]. \tag{3.2.3}$$

Now, using (3.1.70) and neglecting boundary terms, this gives

$$\delta S^C = \kappa \int \mathrm{d}^n x \left[2\mathbf{e}\,(G^a{}_\mu + \Lambda e^a{}_\mu)\delta e_a{}^\mu + 2D_\nu(\mathbf{e}\, e_a{}^\mu e_b{}^\nu)\delta\Gamma^{ab}{}_\mu\right], \tag{3.2.4}$$

so that

$$\frac{\delta L^C}{\delta e_a{}^\mu} = 2\mathbf{e}\,(G^a{}_\mu + \Lambda e^a{}_\mu), \tag{3.2.5}$$

$$\frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} = 2D_\nu(\mathbf{e}\, e_{[a}{}^\mu e_{b]}{}^\nu) = \mathbf{e}\,(T^\mu{}_{ab} + 2e^\mu_{[a} T^c{}_{b]c}). \tag{3.2.6}$$

Contracting the equations of motions associated to (3.2.6) with $e_\mu{}^b$ gives $T^b{}_{ab} = 0$. When re-injecting, this implies $T^a{}_{bc} = 0$. It follows that when the equations of motion for $\Gamma^{ab}{}_\mu$ hold, the connection is torsionless and thus given by $\Gamma_{abc} = r_{abc}$. The fields $\Gamma^{ab}{}_\mu$ are thus entirely determined by $e_a{}^\mu$ so that $\Gamma^{ab}{}_\mu$ are auxiliary fields.

Symmetries

The gauge symmetries of the action (3.2.2) are the diffeomorphisms and the Lorentz gauge transformations. The infinitesimal transformations of the fields under these symmetries are given by

$$\begin{aligned} \delta_{\xi,\omega} e_a{}^\mu &= \xi^\nu \partial_\nu e_a{}^\mu - \partial_\nu \xi^\mu e_a{}^\nu + \omega_a{}^b e_b{}^\mu, \\ \delta_{\xi,\omega} \Gamma^{ab}{}_\mu &= \xi^\nu \partial_\nu \Gamma^{ab}{}_\mu + \partial_\mu \xi^\nu \Gamma^{ab}{}_\nu - D_\mu \omega^{ab}. \end{aligned} \tag{3.2.7}$$

Following the lines of (2.3.25), we consider

$$\frac{\delta L^C}{\delta e_a{}^\mu} \delta_{\xi,\omega} e_a{}^\mu + \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} \delta_{\xi,\omega} \Gamma^{ab}{}_\mu \tag{3.2.8}$$

and integrate by parts in order to isolate the undifferentiated gauge parameters ω^{ab} and ξ^ρ. Discarding the boundary terms, this leads to the Noether identities

$$\frac{\delta L^C}{\delta e^{[a|\mu|}} e_{b]}{}^\mu + D_\mu \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} = 0, \tag{3.2.9}$$

$$\frac{\delta L^C}{\delta e_a{}^\mu} \partial_\rho e_a{}^\mu + \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} \partial_\rho \Gamma^{ab}_\mu + \partial_\mu \left(\frac{\delta L^C}{\delta e_a{}^\rho} e_a{}^\mu - \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} \Gamma^{ab}{}_\rho \right) = 0. \tag{3.2.10}$$

Identity (3.2.9), associated with Lorentz gauge symmetries, can be shown to be equivalent to (3.1.45). Using (3.2.9), identity (3.2.10), associated with diffeomorphisms, can be written as

$$\partial_\mu \left(\frac{\delta L^C}{\delta e_a{}^\rho} e_a{}^\mu \right) + \frac{\delta L^C}{\delta e_a{}^\mu} D_\rho e_a{}^\mu + \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} R^{ab}{}_{\rho\mu} = 0, \tag{3.2.11}$$

and can then be shown to be equivalent to (3.1.67).

Construction of the co-dimension 2 form

When keeping the boundary terms, one finds the weakly vanishing Noether current associated with the gauge symmetries as

$$\kappa^{-1} S^\mu_{\xi,\omega} = \frac{\delta L^C}{\delta \Gamma^{ab}{}_\mu} (-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho) - \frac{\delta L^C}{\delta e_a{}^\rho} e_a{}^\mu \xi^\rho. \tag{3.2.12}$$

The associated co-dimension 2 form $k_{\xi,\omega} = k^{\mu\nu}_{\xi,\omega} (\mathrm{d}^{n-2}x)_{\mu\nu}$ computed through (3.1.18) is given by

$$\begin{aligned} \kappa^{-1} k^{\mu\nu}_{\xi,\omega} = \mathbf{e} \big[&(2\delta e_a{}^\mu e_b{}^\nu + e^c{}_\lambda \delta e_c{}^\lambda e_a{}^\nu e_b{}^\mu)(-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho) \\ &+ \delta\Gamma^{ab}{}_\rho (\xi^\rho e_a{}^\mu e_b{}^\nu + 2\xi^\mu e_a{}^\nu e_b{}^\rho) - (\mu \longleftrightarrow \nu) \big]. \end{aligned} \tag{3.2.13}$$

This can also be written as

$$\mathbf{k}_{\xi,\omega} = -\delta \mathbf{K}^K_{\xi,\omega} + \mathbf{K}^K_{\delta\xi,\delta\omega} - \xi^\nu \frac{\partial}{\partial \mathrm{d}x^\nu} \boldsymbol{\Theta}, \tag{3.2.14}$$

where

$$\mathbf{K}^K_{\xi,\omega} = 2\kappa \mathbf{e}\, e_a{}^\nu e_b{}^\mu (-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho)(\mathrm{d}^{n-2}x)_{\mu\nu}, \quad \boldsymbol{\Theta} = 2\kappa \mathbf{e}\, \delta\Gamma^{ab}{}_\rho e_a{}^\mu e_b{}^\rho (\mathrm{d}^{n-1}x)_\mu. \tag{3.2.15}$$

According to the general results reviewed in section 2.3, the co-dimension 2 form is closed, $\mathrm{d}\mathbf{k}_{\xi,\omega} = 0$, or, equivalently, $\partial_\nu k^{\mu\nu}_{\xi,\omega} = 0$, if $e_a{}^\mu, \Gamma^{ab}{}_\mu$ are solutions to the Euler-Lagrange equations of motion, and thus to the Einstein equations, $\delta e_a{}^\mu, \delta\Gamma^{ab}{}_\mu$ solutions to the linearized equations and ω^{ab}, ξ^ρ satisfy

$$\mathcal{L}_\xi e_a{}^\mu + \omega_a{}^b e_b{}^\mu \approx 0, \quad \mathcal{L}_\xi \Gamma^{ab}{}_\mu \approx D_\mu \omega^{ab}, \tag{3.2.16}$$

where $\approx$ now denotes on-shell for the background solution and is relevant in case the parameters ω^{ab}, ξ^ρ explicitly depend on the background solution $e_a{}^\mu, \Gamma^{ab}{}_\mu$ around which one linearizes. The first equation also implies in particular that ξ^ρ is a possibly field dependent Killing vector of the background solution $g_{\mu\nu}$,

$$\mathcal{L}_\xi g_{\mu\nu} \approx 0, \tag{3.2.17}$$

and that

$$\omega^{ab} \approx -e^b{}_\mu \mathcal{L}_\xi e^{a\mu} \approx -e^{[b}{}_\mu \mathcal{L}_\xi e^{a]\mu}. \tag{3.2.18}$$

Reduction to the metric formulation

To compare with the results in the metric formulation, let us go on-shell for the auxiliary fields $\Gamma^{ab}{}_\mu$ and eliminate ω^{ab} using (3.2.18). The former implies that we are in the torsionless case with the Lorentz connection simplified to $\Gamma^{ab}{}_\mu = r^{ab}{}_\mu$, while (3.1.60) reduces to

$$\Gamma^{ab}{}_\mu = e^a{}_\nu \nabla_\mu e^{b\nu} = e^{[a}{}_\nu \nabla_\mu e^{b]\nu}, \tag{3.2.19}$$

with $\nabla_\mu v^\nu = \partial_\mu v^\nu + \{^\nu{}_{\rho\mu}\} v^\rho$. Note also that the Killing equation can be written as $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \approx 0$. Together with (3.2.19), we have

$$-\omega^{ab} + \Gamma^{ab}{}_\rho \xi^\rho \approx -e^{[a}{}_\rho e^{b]}{}_\sigma \nabla^\rho \xi^\sigma, \tag{3.2.20}$$

$$\delta\Gamma^{ab}{}_\rho = \delta e^{[a}{}_\sigma \nabla_\rho e^{b]\sigma} + e^{[a}{}_\sigma \delta\{^\sigma{}_{\tau\rho}\} e^{b]\tau} + e^{[a}{}_\sigma \nabla_\rho \delta e^{b]\sigma}, \tag{3.2.21}$$

with

$$\delta\{^\sigma{}_{\tau\rho}\} = \frac{1}{2} g^{\sigma\delta}(\nabla_\rho \delta g_{\delta\tau} + \nabla_\tau \delta g_{\delta\rho} - \nabla_\delta \delta g_{\tau\rho}). \tag{3.2.22}$$

Using that

$$\delta e^a{}_\mu e_{a\nu} = \frac{1}{2} h_{\mu\nu} + \delta e^a{}_{[\mu} e_{|a|\nu]}, \tag{3.2.23}$$

with $h_{\mu\nu} = \delta g_{\mu\nu}$, indices being lowered and raised with $g_{\mu\nu}$ and its inverse, and $h = h^\mu_\mu$, substitution into (3.2.13) gives

$$6\sqrt{|g|}\nabla_\rho(\delta e_a{}^{[\mu} e^{|a|\nu} \xi^{\rho]}) + k_\xi^{\mu\nu}, \tag{3.2.24}$$

where the first term can be dropped since it is trivial in the sense that it corresponds to the exterior derivative of an $n-3$ form, while

$$k_\xi^{\mu\nu} = \sqrt{|g|}\Big[\xi^\nu \nabla^\mu h + \xi^\mu \nabla_\sigma h^{\sigma\nu} + \xi_\sigma \nabla^\nu h^{\sigma\mu} \\ + \frac{1}{2} h \nabla^\nu \xi^\mu + \frac{1}{2} h^{\mu\sigma} \nabla_\sigma \xi^\nu + \frac{1}{2} h^{\nu\sigma} \nabla^\mu \xi_\sigma - (\mu \longleftrightarrow \nu)\Big]. \tag{3.2.25}$$

We have thus recovered the results of the metric formulation since the last expression agrees with the one given in (2.3.47).

3.2.2 Cartan formulation II

Variational principle

This version of the Cartan formulation is an intermediate between the Cartan formulation of subsection 3.2.1 and the Newman-Penrose formulation discussed in subsection 3.2.3. Here, the variables of the variational principle are the components $\Gamma_{abc} = \Gamma_{[ab]c}$ of a Lorentz connection in the non-holonomic frame and the vielbein components $e_a{}^\mu$. In term of these variables, the action reads

$$S^{CNH}[\Gamma_{abc}, e_a{}^\mu] = \kappa \int \mathrm{d}^n x L^{CNH} = \kappa \int \mathrm{d}^n x \mathbf{e}(R_{abcd}\eta^{ac}\eta^{bd} - 2\Lambda). \tag{3.2.26}$$

Varying the action by using (3.1.62) and dropping the boundary terms, one obtains

$$\frac{\delta L^{CNH}}{\delta e^\tau_h} = 2\mathbf{e}(G^h{}_c e^c_\tau + \Lambda e^h_\tau) + \mathbf{e}(2T^g{}_{cg}\Gamma^{ch}{}_\tau - T^h{}_{cd}\Gamma^{cd}{}_\tau), \tag{3.2.27}$$

$$\begin{aligned} \frac{\delta L^{CNH}}{\delta \Gamma_{abf}} &= \mathbf{e}T^f{}_{cd}\eta^{ac}\eta^{bd} + 2D_\mu[\mathbf{e}e^\mu_c(\eta^{[a|f|}\eta^{b]c})] \\ &= \mathbf{e}(T^{fab} + 2\eta^{f[a}T^{|c|b]}{}_c). \end{aligned} \tag{3.2.28}$$

Contracting the equations of motion associated with (3.2.28) with η_{fa} gives $T^{cb}{}_c = 0$. When re-injecting, this implies $T^f{}_{ab} = 0$. This torsionless condition for on-shell connection is the analogue of the one encountered in (3.2.6) above. The fields Γ_{abf} are thus auxiliary fields in this formulation. Taking these fields on-shell in (3.2.27), one obtains the Einstein equations, as expected.

Symmetries

The gauge symmetries of the action (3.2.26) are the diffeomorphisms and the Lorentz gauge transformations. The infinitesimal transformations of the fields under these symmetries are given by

$$\delta_{\xi,\omega} e_a{}^\mu = \xi^\nu \partial_\nu e_a{}^\mu - \partial_\nu \xi^\mu e_a{}^\nu + \omega_a{}^b e_b{}^\mu, \tag{3.2.29}$$

$$\delta_{\xi,\omega}\Gamma_{abc} = \xi^\rho \partial_\rho \Gamma_{abc} - D_c\omega_{ab} + \Gamma_{abd}\omega_c^{\ d} \tag{3.2.30}$$

Following the lines of (2.3.25), we consider

$$\frac{\delta L^C}{\delta e_a{}^\mu}\delta_{\xi,\omega} e_a{}^\mu + \frac{\delta L^C}{\delta \Gamma_{abc}}\delta_{\xi,\omega}\Gamma_{abc} \tag{3.2.31}$$

and integrate by parts to isolate the undifferentiated gauge parameters ω^{ab} and ξ^ρ. Discarding the boundary terms, this leads to the Noether identities

$$\frac{\delta L^{CNH}}{\delta e^{[a|\tau}} e^\tau_{b]} + D_\mu\left(e^\mu_f \frac{\delta L^{CNH}}{\delta \Gamma^{ab}{}_f}\right) + \frac{\delta L^{CNH}}{\delta \Gamma_{cf}{}^{[a}}\Gamma_{|cf|b]} = 0 \tag{3.2.32}$$

$$\frac{\delta L^{CNH}}{\delta e^\tau_h}\partial_\rho e^\tau_h + \frac{\delta L^{CNH}}{\delta \Gamma_{abf}}\partial_\rho \Gamma_{abf} + \partial_\mu\left(\frac{\delta L^{CNH}}{\delta e^\rho_h} e^\mu_h\right) = 0 \tag{3.2.33}$$

The first identity corresponds to Lorentz gauge symmetry and can be shown to be exactly the same as the one found in (3.2.9) in the first Cartan formulation. The second identity corresponds to diffeomorphism symmetry and can be shown to be the same as (3.2.10). Then, as above, the Noether identities are equivalent to the Bianchi identities (3.1.45) and (3.1.67).

Co-dimension 2 form and equivalence with the other formulations

When keeping the boundary terms, one finds the weakly vanishing Noether current

$$\kappa^{-1} S^\mu_{\xi,\omega} = -\frac{\delta L^{CNH}}{\delta \Gamma_{abf}} e^\mu_f \omega_{ab} - \frac{\delta L^{CNH}}{\delta e^\tau_h} e^\mu_h \xi^\tau. \tag{3.2.34}$$

Then the co-dimension 2 form is given by

$$\begin{aligned}\kappa^{-1} k^{\mu\nu}_{\xi,\omega} = \mathbf{e}\big[&(2\delta e_a{}^\mu e_b{}^\nu + e^c{}_\lambda \delta e_c{}^\lambda e_a{}^\nu e_b{}^\mu)(-\omega^{ab} + \Gamma^{ab}{}_d e^d_\rho \xi^\rho) \\ &+ \delta(\Gamma^{ab}{}_d e^d_\rho)(\xi^\rho e_a{}^\mu e_b{}^\nu + 2\xi^\mu e_a{}^\nu e_b{}^\rho) - (\mu \longleftrightarrow \nu)\big],\end{aligned} \tag{3.2.35}$$

which is obviously the same as (3.2.13) by performing the field redefinition $\Gamma^{ab}{}_\rho = \Gamma^{ab}{}_c e^c_\rho$.

3.2.3 Newman-Penrose formulation

Variational principle

The Newman-Penrose (NP) equations are a set of first order equations involving the spin coefficients, the vielbein and the curvature components at the same footage [197, 198]. The NP formulation that we introduce here leads to Euler-Lagrange equations that impose vanishing of torsion together with all NP equations. This is achieved by introducing additional auxiliary fields in the Cartan formulation II (3.2.26). It involves as dynamical variables the vielbein components $e_a{}^\mu$, the Lorentz connection components in the non-holonomic frame Γ_{abc}, and a suitable set of auxiliary fields $\mathbf{R}_{abcd} = \mathbf{R}_{[ab][cd]}, \lambda^{abcd} = \lambda^{[ab][cd]}$,

$$\begin{aligned}S[\Gamma_{abc}, e_a{}^\mu, \mathbf{R}_{abcd}, \lambda^{abcd}] &= \kappa \int \mathrm{d}^n x L^{NP} \\ &= \kappa \int \mathrm{d}^n x \mathbf{e}[\mathbf{R}_{abcd}(\eta^{ac}\eta^{bd} - \lambda^{abcd}) + \lambda^{abcd} R_{abcd} - 2\Lambda],\end{aligned} \tag{3.2.36}$$

where $R_{abcd} = \eta_{ae} R^e{}_{bcd}$ is explicitly given in (3.1.35) as a function of the variables $e_a{}^\mu, \Gamma_{abc}$ and their first order derivatives.

The equations of motion for the auxiliary fields follow from equating to zero the Euler-Lagrange derivatives of L^{NP}

$$\begin{aligned} \frac{\delta L^{NP}}{\delta \mathbf{R}_{abcd}} &= -\mathbf{e}\left[\lambda^{abcd} - \frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc})\right], \\ \frac{\delta L^{NP}}{\delta \lambda^{abcd}} &= -\mathbf{e}\left[\mathbf{R}_{abcd} - R_{abcd}\right]. \end{aligned} \tag{3.2.37}$$

They thus fix the auxiliary λ fields in terms of the Minkowski metric,

$$\lambda^{abcd} = \frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) \equiv \lambda^{abcd}_\eta, \tag{3.2.38}$$

and impose the definition of the Riemann tensor in terms of vielbein and connection components as on-shell relations, $\mathbf{R}_{abcd} = R_{abcd}$, which is desirable from the view-point of the NP formalism. They can be eliminated by solving inside the action. The resulting reduced action coincides with the action associated with the Cartan formulation II (3.2.26).

The next equations of motion follow from the vanishing of

$$\frac{\delta L^{NP}}{\delta \Gamma_{abc}} = 2\mathbf{e}\left[D_f \lambda^{abcf} + \lambda^{abdf}(T^h{}_{fh}\delta^c_d + \frac{1}{2}T^c{}_{df})\right]. \tag{3.2.39}$$

When putting λ^{abcd} on-shell, they are equivalent to vanishing of torsion, $T^a{}_{bc} = 0$. It follows that $\Gamma_{abc} = r_{abc}$ or, equivalently, that $\Gamma^a{}_{bc} = e^a{}_\nu e_c{}^\mu \nabla_\mu e_b{}^\nu$, where ∇_μ denotes the Christoffel connection. In other words, the connection components are also auxiliary fields that can be expressed in terms of vielbein components and eliminated by their own equations of motion.

The last equations of motion follow from the vanishing of

$$\frac{\delta L^{NP}}{\delta e_a{}^\mu} = e^b{}_\mu\left[2\mathbf{e}(\lambda^{cdfa}R_{cdfb}) - \frac{\delta L^{NP}}{\delta \Gamma_{cda}}\Gamma_{cdb}\right] - e^a{}_\mu\left[\mathbf{e}(\mathbf{R} - 2\Lambda) + \lambda^{bcdf}\frac{\delta L^{NP}}{\delta \lambda^{bcdf}}\right]. \tag{3.2.40}$$

On-shell for the auxiliary fields, we have

$$\frac{\delta \mathcal{L}}{\delta e_a{}^\mu}|_{\text{aux on-shell}} = 2\mathbf{e} e^b{}_\mu(G^a{}_b + \Lambda\delta^a_b), \tag{3.2.41}$$

which imply the standard Einstein equations.

Finally, it should be noted that the equations of motion associated with (3.2.39) and (3.2.40) consistently reduce to (3.2.27) and (3.2.28) when taking the auxiliary fields λ^{abcd} and $\mathbf{R}_{abcd}$ on-shell.

Improved gauge transformations and Noether identities

Diffeomorphisms and local Lorentz transformations are extended in a natural way to the auxiliary fields. If $\xi^\mu, \omega'^a{}_b = -\omega_b{}^a$ denote parameters for the infinitesimal transformations, they act on the fields as

$$\begin{aligned}
\delta_{\xi,\omega} e_a{}^\mu &= \xi^\nu \partial_\nu e_a{}^\mu - \partial_\nu \xi^\mu e_a{}^\nu + \omega_a{}^b e_b{}^\mu, \\
\delta_{\xi,\omega} \Gamma_{abc} &= \xi^\nu \partial_\nu \Gamma_{abc} - D_c \omega_{ab} + \omega_c{}^d \Gamma_{abd}, \\
\delta_{\xi,\omega} \mathbf{R}_{abcd} &= \xi^\nu \partial_\nu \mathbf{R}_{abcd} + \omega_a{}^f \mathbf{R}_{fbcd} + \omega_b{}^f \mathbf{R}_{afcd} + \omega_c{}^f \mathbf{R}_{abfd} + \omega_d{}^f \mathbf{R}_{abcf}, \\
\delta_{\xi,\omega} \lambda^{abcd} &= \xi^\nu \partial_\nu \lambda^{abcd} + \omega^a{}_f \lambda^{fbcd} + \omega^b{}_f \lambda^{afcd} + \omega^c{}_f \lambda^{abfd} + \omega^d{}_f \lambda^{abcf}.
\end{aligned} \tag{3.2.42}$$

In terms of the redefined gauge parameters, which are spacetime scalars and thus in agreement with the general strategy of the NP approach,

$$\xi'^a = e^a{}_\mu \xi^\mu, \quad \omega'_a{}^b = \omega_a{}^b + \xi^\mu \Gamma^b{}_{ac} e^c{}_\mu, \tag{3.2.43}$$

these gauge transformations become

$$\begin{aligned}
\delta_{\xi',\omega'} e_a{}^\mu &= (\xi'^c T^b{}_{ac} - D_a \xi'^b + \omega'_a{}^b) e_b{}^\mu, \\
\delta_{\xi',\omega'} \Gamma_{abc} &= -\xi'^d R_{abcd} + (\xi'^f T^d{}_{cf} - D_c \xi'^d + \omega'_c{}^d) \Gamma_{abd} - D_c \omega'_{ab}, \\
\delta_{\xi',\omega'} \mathbf{R}_{abcd} &= \xi'^f D_f \mathbf{R}_{abcd} + \omega'_a{}^f \mathbf{R}_{fbcd} + \omega'_b{}^f \mathbf{R}_{afcd} + \omega'_c{}^f \mathbf{R}_{abfd} + \omega'_d{}^f \mathbf{R}_{abcf}, \\
\delta_{\xi',\omega'} \lambda^{abcd} &= \xi'^f D_f \lambda^{abcd} + \omega'^a{}_f \lambda^{fbcd} + \omega'^b{}_f \lambda^{afcd} + \omega'^c{}_f \lambda^{abfd} + \omega'^d{}_f \lambda^{abcf}.
\end{aligned} \tag{3.2.44}$$

Isolating the undifferentiated gauge parameters by dropping the exterior derivative of an $n-1$ form, the invariance of action (3.2.36) under these transformations leads to the Noether identities. Since the change of gauge parameters is invertible, the identities associated with both sets are equivalent. We can thus concentrate on this second set. For later use, note that

$$\delta_{\xi',\omega'} \Gamma_{abc} - (\delta_{\xi',\omega'} e_c{}^\mu) e^d{}_\mu \Gamma_{abd} = -\xi'^d R_{abcd} - D_c \omega'_{ab}. \tag{3.2.45}$$

When using (3.1.71), the Noether identities associated with the Lorentz parameters ω'_{ab} become

$$\begin{aligned}
2\frac{\delta L^{NP}}{\delta \mathbf{R}_{[a|cdf|}} \mathbf{R}^{b]}{}_{cdf} + 2\frac{\delta L^{NP}}{\delta \mathbf{R}_{cd[a|f|}} \mathbf{R}_{cd}{}^{b]}{}_f + 2\frac{\delta L^{NP}}{\delta \lambda^{fhcd}} \eta^{f[a} \lambda^{b]hcd} + 2\frac{\delta L^{NP}}{\delta \lambda^{cdfh}} \eta^{f[a} \lambda^{|cd|b]h} \\
+ \frac{\delta L^{NP}}{\delta e_{[a}{}^\mu} e^{b]\mu} + \frac{\delta L^{NP}}{\delta \Gamma_{cd[a}} \Gamma_{cd}{}^{b]} + \mathbf{e}\big[(D_c + T^c{}_{cf})(\mathbf{e}^{-1} \frac{\delta L^{NP}}{\delta \Gamma_{abc}})\big] = 0.
\end{aligned} \tag{3.2.46}$$

while the Noether identities for the vector fields ξ'^f read

$$\begin{aligned}
\frac{\delta L^{NP}}{\delta \mathbf{R}_{abcd}} D_f \mathbf{R}_{abcd} + \frac{\delta L^{NP}}{\delta \lambda^{abcd}} D_f \lambda^{abcd} + \frac{\delta L^{NP}}{\delta e_a{}^\mu} T^b{}_{af} e_b{}^\mu + \frac{\delta L^{NP}}{\delta \Gamma_{abc}} (T^d{}_{cf} \Gamma_{abd} - R_{abcf}) \\
+ \mathbf{e}\big[(D_c + T^h{}_{ch}) \mathbf{e}^{-1} (\frac{\delta L^{NP}}{\delta e_c{}^\mu} e_f{}^\mu + \frac{\delta L^{NP}}{\delta \Gamma_{abc}} \Gamma_{abf})\big] = 0.
\end{aligned} \tag{3.2.47}$$

It follows from general results on auxiliary fields (see e.g. [196]) that these Noether identities are equivalent to those of the Cartan formulation II (see equations (3.2.32) and (3.2.33)), which have been investigated and related to the Bianchi identities. More explicitly, we have $L^{NP} = L^{CNH} + A$ with $A = [(\mathbf{R}_{abcd} - R_{abcd})(\eta^{ac}\eta^{bd} - \lambda^{abcd})]$. Identity (3.2.46) for L^{NP} replaced by A is equivalent to (3.1.45). This then implies that (3.2.46) reduces to

$$\frac{\delta L^{CNH}}{\delta e_{[a}{}^{\mu}} e^{b]\mu} + \frac{\delta L^{CNH}}{\delta \Gamma_{cd[a}} \Gamma_{cd}{}^{b]} + \mathbf{e}\big[(D_c + T^f{}_{cf})(\mathbf{e}^{-1}\frac{\delta L^{CNH}}{\delta \Gamma_{abc}})\big] = 0, \tag{3.2.48}$$

which in turn is also equivalent to (3.2.32) and so to (3.1.45).

Identity (3.2.47) for L^{NP} replaced by A is equivalent to the second identity of (3.1.44). This then implies that (3.2.47) reduces to

$$\frac{\delta L^{CNH}}{\delta e_a{}^{\mu}} T^b{}_{af} e_b{}^{\mu} + \frac{\delta L^{CNH}}{\delta \Gamma_{abc}} (T^d{}_{fc}\Gamma_{abd} - R_{abcf}) \\ + \mathbf{e}\big[(D_c + T^h{}_{ch})\mathbf{e}^{-1}(\frac{\delta L^{CNH}}{\delta e_c{}^{\mu}} e_f{}^{\mu} + \frac{\delta L^{CNH}}{\delta \Gamma_{abc}} \Gamma_{abf})\big] = 0, \tag{3.2.49}$$

which is equivalent to (3.2.33) and so to (3.1.67).

Co-dimension 2 form and breaking

Writing $\phi^i = (\mathbf{R}_{abcd}, \lambda^{abcd}, \Gamma_{abc}, e_a{}^{\mu})$, the presymplectic potential associated with the action (3.2.36) is given by

$$\boldsymbol{\theta}[\phi; \delta\phi] = 2\kappa \mathbf{e} \lambda^{abcd} e_c{}^{\mu} \delta\Gamma_{ab\nu} e^{\nu}{}_d e_c{}^{\mu} (\mathrm{d}^{n-1}x)_{\mu}, \tag{3.2.50}$$

where $\delta\Gamma_{ab\nu} e^{\nu}{}_d = \delta\Gamma_{abd} - \Gamma_{abf} e^f{}_{\nu} \delta e_d{}^{\nu}$.

Writing the gauge parameters as $f^{\alpha} = (\omega_{ab}, \xi^a)$, the weakly vanishing Noether current is given by

$$\mathbf{S}_{\xi',\omega'}[\phi] = -\kappa\Big[\frac{\delta L^{NP}}{\delta \Gamma_{abc}}(\omega'_{ab} + \Gamma_{abf}\xi'^f) + \frac{\delta L^{NP}}{\delta e^{\tau}_c} e^{\tau}_f \xi'^f\Big] e^{\mu}_c (\mathrm{d}^{n-1}x)_{\mu}. \tag{3.2.51}$$

The co-dimension 2 form can be obtained from (3.1.18) or alternatively from (3.1.23). Using the Euler-Lagrange equations for the auxiliary fields, we obtain

$$\mathbf{k}_{\xi',\omega'}[\phi; \delta\phi] = 2\kappa \mathbf{e}\big[-(2\delta e_a{}^{\mu} e_b{}^{\nu} + e^c{}_{\lambda}\delta e_c{}^{\lambda} e_a{}^{\nu} e_b{}^{\mu})\omega'^{ab} \\ + \delta(\Gamma^{ab}{}_d e^d_{\rho})(\xi'^c e_c{}^{\rho} e_a{}^{\mu} e_b{}^{\nu} + 2\xi'^c e_c{}^{\mu} e_a{}^{\nu} e_b{}^{\rho})\big](\mathrm{d}^{n-2}x)_{\mu\nu}. \tag{3.2.52}$$

Notice that this last expression obtained from the NP formulation is exactly the same as (3.2.35) obtained from the Cartan formulation II, up to the parameters redefinition (3.2.43).

The breaking in the conservation law of $\mathbf{k}[\phi, \delta\phi]$ (see equation (2.3.49)) is given by the invariant presymplectic current (3.1.20) with an evaluated variation. For the present case, using the equations of motion for the auxiliary fields, we explicitly find

$$\mathbf{W}[\phi; \delta_{\xi',\omega'}\phi, \delta\phi] = 2\kappa\mathbf{e}\big[\delta_{\xi',\omega'} e_b{}^\mu \delta\Gamma^{ab}{}_\nu e_a{}^\nu e^c{}_\mu + \delta_{\xi',\omega'} e_a{}^\mu \delta\Gamma^{ac}{}_\mu \\ - \delta_{\xi',\omega'} \ln \mathbf{e}\, \delta\Gamma^{cb}{}_\nu e_b{}^\nu - (\delta_{\xi',\omega'} \longleftrightarrow \delta)\big] e^\rho_c (\mathrm{d}^{n-1}x)_\rho. \quad (3.2.53)$$

Exact reducibility parameters General considerations on auxiliary fields imply that, on-shell, reducibility parameters should be given by Killing vectors $\bar{\xi}'^a$ of the metric (see e.g. [196]). Let us see how this comes about here.

Merely the first of (3.2.44) encodes gauge transformations of fields that are not auxiliary. The associated equation $\delta_{\bar{\omega}',\bar{\xi}'} e_a{}^\mu \approx 0$ is equivalent to

$$D_{(a}\bar{\xi}'_{b)} - \bar{\xi}'^c T_{(ba)c} \approx 0, \quad \bar{\omega}'_{ab} \approx D_{[a}\bar{\xi}'_{b]} - \bar{\xi}'^c T_{[ba]c}. \quad (3.2.54)$$

On-shell when torsion vanishes, the first indeed requires $\bar{\xi}'^a$ to be a Killing vector, while the second uniquely fixes the Lorentz parameters in terms of it. In particular,

$$\bar{\omega}'_{ab} \approx D_a\bar{\xi}'_b \approx -D_b\bar{\xi}'_a. \quad (3.2.55)$$

The other equations impose no additional constraints. Indeed, $\delta_{\bar{\omega}',\bar{\xi}'}\lambda^{abcd} \approx 0$ is satisfied identically on account of the skew-symmetry of $\bar{\omega}'^{ab}$. Instead of $\delta_{\bar{\xi}',\bar{\omega}'}\Gamma_{abc} \approx 0$ we can consider the combination (3.2.45). Requiring this to vanish on-shell amounts to

$$D_c\omega'_{ab} \approx -\bar{\xi}'^d R_{abcd}, \quad (3.2.56)$$

which holds as a consequence of the second equation in (3.2.54), when using that

$$D_a D_b \bar{\xi}'_c \approx R^d{}_{abc}\bar{\xi}'_d, \quad (3.2.57)$$

which can be shown as in [199] appendix C.3, and when also using (3.1.66). Finally, $\delta_{\bar{\xi}',\bar{\omega}'}\mathbf{R}_{abcd} \approx 0$, reduces on-shell to

$$\bar{\xi}'^f D_f R_{abcd} + \bar{\omega}'_a{}^f R_{fbcd} + \bar{\omega}'_b{}^f R_{afcd} + \bar{\omega}'_c{}^f R_{abfd} + \bar{\omega}'_d{}^f R_{abcf} \approx 0. \quad (3.2.58)$$

This equation holds because one can show that, on-shell, the left-hand side is equal to its opposite when using the previous relations (3.2.55), (3.2.6) together with the Bianchi identities (3.1.44) and the on-shell symmetries of the Riemann tensor.

3.3 Application to asymptotically flat 4d gravity

In this section, we illustrate the general results obtained in the first order formulations presented above into a concrete situation. More precisely, starting from the solution space of general relativity in asymptotically flat spacetime in NP formalism, we compute the asymptotic symmetries, the currents and the breaking in their conservation laws. This derivation is done in a self-consistent way, without resorting to the metric formulation. This contrasts with the approach used in [120, 158], where the expressions of the currents in the NP formulation were obtained by translation from the metric formalism. Our direct road allows us to extend the previous results for a time-dependent (but non-dynamical) conformal factor $P = P(u, \zeta, \bar{\zeta})$.

In this section (and this section only), we follow the conventions of [178] and work with a metric of signature $(+ - - -)$, which is more adapted to the NP literature. In particular, one has to adapt some signs in the equations established in the previous section before applying them here. We refer to our article [182] where the convention $(+ - - -)$ was chosen from the very beginning.

3.3.1 Newman-Penrose notations

Following the literature on NP formalism [197, 198, 200], we assign some notations to the fields of the NP formulation introduced in subsection 3.2.3. In four spacetime dimensions, the tetrads $e_1 = l$, $e_2 = n$, $e_3 = m$, $e_4 = \bar{m}$ are chosen as null vectors, $e_a \cdot e_b = \eta_{ab}$ with

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.3.1}$$

The components of the Lorentz connection are traded for the spin coefficients,

$$\begin{aligned} \kappa &= \Gamma_{311}, \quad \pi = -\Gamma_{421}, \quad \epsilon = \frac{1}{2}(\Gamma_{211} - \Gamma_{431}), \\ \tau &= \Gamma_{312}, \quad \nu = -\Gamma_{422}, \quad \gamma = \frac{1}{2}(\Gamma_{212} - \Gamma_{432}), \\ \sigma &= \Gamma_{313}, \quad \mu = -\Gamma_{423}, \quad \beta = \frac{1}{2}(\Gamma_{213} - \Gamma_{433}), \\ \rho &= \Gamma_{314}, \quad \lambda = -\Gamma_{424}, \quad \alpha = \frac{1}{2}(\Gamma_{214} - \Gamma_{434}). \end{aligned} \tag{3.3.2}$$

The other half of the spin coefficients are denoted with a bar on the symbols in the left-hand sides and obtained by exchanging the index 3 and 4 on the right-hand sides. The Weyl tensor C_{abcd} is encoded in terms of

$$\Psi_0 = -C_{1313}, \quad \Psi_1 = -C_{1213}, \quad \Psi_2 = -C_{1342}, \quad \Psi_3 = -C_{1242}, \quad \Psi_4 = -C_{2324}, \tag{3.3.3}$$

with the same rule as above for $\bar{\Psi}_i$, $i = 0, \dots, 4$, while the Ricci tensor is organized as

$$\begin{array}{lll} \Phi_{00} = -\frac{1}{2}R_{11}, & \Phi_{11} = -\dfrac{1}{4}(R_{12}+R_{34}), & \Phi_{22} = -\frac{1}{2}R_{22}, \\ \Phi_{02} = -\frac{1}{2}R_{33}, & \Phi_{01} = -\frac{1}{2}R_{13}, & \Phi_{12} = -\frac{1}{2}R_{23}, \\ \Phi_{20} = -\frac{1}{2}R_{44}, & \Phi_{10} = -\frac{1}{2}R_{14}, & \Phi_{21} = -\frac{1}{2}R_{24}, \end{array} \quad (3.3.4)$$
$$\tilde{\Lambda} = \frac{1}{24}R = \frac{1}{12}(R_{12}-R_{34}).$$

There is no torsion in the NP approach, $T^a{}_{bc} = 0$. In this case, the vacuum Einstein equations in flat space are equivalent to the vanishing of the Φ's. The equations governing the NP quantities can then be interpreted as follows: (i) The metric equations express commutators of tetrads in terms of spin coefficients. This is the first of (3.1.26) when taking into account that $D^a{}_{bc} = 2\Gamma^a{}_{[cb]}$ in the absence of torsion. (ii) The spin coefficient equations express directional derivatives of spin coefficients in terms of spin coefficients and the Weyl and Ricci tensors. In the torsion-free case, they are equivalent to the definition of R_{abcd} in (3.1.35). (iii) The Bianchi identities express directional derivatives of the $\Psi's$ and Φ's in terms of spin coefficents and Ψ's and Φ's. They are equivalent to the second of (3.1.44) in the absence of torsion.

3.3.2 Solution space

Four-dimensional asymptotically flat spacetimes at null infinity in the NP formalism have been studied in [157,197,201] (see [178] for a summary and conventions appropriate to the current context). In terms of coordinates $x^\mu = (u, r, x^A)$, $x^A = (\zeta, \bar{\zeta})$ and using the notations of section 3.3.1, the Newman-Unti solution space is entirely determined by the conditions

$$\begin{aligned} &\kappa = \epsilon = \pi = 0, \quad \rho = \overline{\rho}, \quad \tau = \bar{\alpha} + \beta, \\ &l = \frac{\partial}{\partial r}, \quad n = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^A\frac{\partial}{\partial x^A}, \quad m = \omega\frac{\partial}{\partial r} + L^A\frac{\partial}{\partial x^A}, \end{aligned} \quad (3.3.5)$$

where U, X^A, ω and L^A are arbitrary functions of the coordinates, together with the fall-off conditions

$$\begin{aligned} &X^A = \mathcal{O}(r^{-1}), \quad \Psi_0 = \Psi_0^0 r^{-5} + \mathcal{O}(r^{-6}), \quad \rho = -\frac{1}{r} + \mathcal{O}(r^{-3}), \quad \tau = \mathcal{O}(r^{-2}), \\ &g_{AB}\mathrm{d}x^A\mathrm{d}x^B = -2r^2\frac{\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{P\bar{P}} + \mathcal{O}(r). \end{aligned} \quad (3.3.6)$$

Here, Ψ_0^0 and P are arbitrary complex functions of $(u, \zeta, \bar{\zeta})$. Below we also use the real function $\varphi(u, \zeta, \bar{\zeta})$ defined by $P\bar{P} = 2e^{-2\varphi}$. The associated asymptotic expansion of the solution space in terms of $\Psi_0(u_0, r, \zeta, \bar{\zeta})$, $(\Psi_2^0 + \bar{\Psi}_2^0)(u_0, \zeta, \bar{\zeta})$, $\Psi_1^0(u_0, \zeta, \bar{\zeta})$

at fixed u_0 and of the asymptotic shear $\sigma^0(u,\zeta,\bar{\zeta})$ and the conformal factor $P(u,\zeta,\bar{\zeta})$ is summarized in appendix C.1. These data characterizing the solution space are collectively denoted by χ.

On a space-like cut of $\mathscr{I}^+$, we use coordinates $\zeta,\bar{\zeta}$, and the (rescaled) induced metric

$$d\bar{s}^2 = -\bar{\gamma}_{AB}dx^A dx^B = -2(P\bar{P})^{-1}d\zeta d\bar{\zeta}, \tag{3.3.7}$$

with $\bar{P}P > 0$. For the unit sphere, we have $\zeta = \cot\frac{\theta}{2}e^{i\phi}$ in terms of standard spherical coordinates and

$$P_S(\zeta,\bar{\zeta}) = \frac{1}{\sqrt{2}}(1+\zeta\bar{\zeta}). \tag{3.3.8}$$

The covariant derivative on the 2 surface is encoded in the operators

$$\begin{aligned} \eth\eta^s &= P\bar{P}^{-s}\bar{\partial}(\bar{P}^s\eta^s) = P\bar{\partial}\eta^s + sP\bar{\partial}\ln\bar{P}\eta^s = P\bar{\partial}\eta^s + 2s\overline{\alpha}^0\eta^s, \\ \bar{\eth}\eta^s &= \bar{P}P^s\partial(P^{-s}\eta^s) = \bar{P}\partial\eta^s - s\bar{P}\partial\ln P\eta^s = \bar{P}\partial\eta^s - 2s\alpha^0\eta^s, \end{aligned} \tag{3.3.9}$$

where s is the spin weight of the field η and $\partial = \partial_\zeta, \bar{\partial} = \partial_{\bar{\zeta}}$. The spin and conformal weights of relevant fields are listed in Table 3.1. Complex conjugation transforms

Table 3.1: Spin and conformal weights

	$\eth$	∂_u	γ^0	ν^0	μ^0	σ^0	λ^0	Ψ^0_4	Ψ^0_3	Ψ^0_2	Ψ^0_1	Ψ^0_0	$\mathcal{Y}$
s	1	0	0	-1	0	2	-2	-2	-1	0	1	2	-1
w	-1	-1	-1	-2	-2	-1	-2	-3	-3	-3	-3	-3	1

the spin weight into its opposite and leaves the conformal weight unchanged. The operators $\eth$, $\bar{\eth}$ respectively raise and lower the spin weight by one unit. The Laplacian is $\bar{\Delta} = 4e^{-2\varphi}\partial\bar{\partial} = 2\bar{\eth}\eth$. Note that P is of spin weight 1 and "holomorphic", $\bar{\eth}P = 0$ and that

$$[\bar{\eth},\eth]\eta^s = \frac{s}{2}R\eta^s, \tag{3.3.10}$$

with $R = -4\mu^0 = \bar{\Delta}\ln(P\bar{P})$, $R_S = 2$. We also have

$$[\partial_u,\eth]\eta^s = -2(\overline{\gamma}^0\eth + s\eth\gamma^0)\eta^s, \quad [\partial_u,\bar{\eth}]\eta^s = -2(\gamma^0\bar{\eth} - s\bar{\eth}\overline{\gamma}^0)\eta^s. \tag{3.3.11}$$

The components of the inverse metric associated with the tetrad given in (3.3.5) is

$$g^{0\mu} = \delta^\mu_1,\ g^{rr} = 2(U-\omega\overline{\omega}),\ g^{rA} = X^A - (\overline{\omega}L^A + \omega\overline{L}^A),\ g^{AB} = -(L^A\overline{L}^B + L^B\overline{L}^A).$$

Note furthermore that if $L_A = g_{AB}L^B$ with g_{AB} the two dimensional metric inverse to g^{AB}, then $L^A\overline{L}_A = -1$, $L^AL_A = 0 = \overline{L}^A\overline{L}_A$. The co-tetrad is given by

$$\begin{aligned} e^1 &= -[U + X^A(\omega\overline{L}_A + \overline{\omega}L_A)]du + dr + (\omega\overline{L}_A + \overline{\omega}L_A)dx^A, \\ e^2 &= du, \quad e^3 = X^A\overline{L}_A du - \overline{L}_A dx^A, \quad e^4 = X^AL_A du - L_A dx^A. \end{aligned} \tag{3.3.12}$$

3.3.3 Residual gauge transformations

The parameters of residual gauge transformations that preserve the solution space are entirely determined by asking that conditions (3.3.5) and (3.3.6) be preserved on-shell. This is worked out in detail in appendix C.2, where it is shown that these parameters are given by

$$f(u,\zeta,\bar{\zeta}), \quad Y^{\zeta}=Y(\zeta), \quad Y^{\bar{\zeta}}=\bar{Y}(\bar{\zeta}), \quad \omega_0^{34}(u,\zeta,\bar{\zeta}). \tag{3.3.13}$$

The associated residual gauge transformations are explicitly determined by the gauge parameters,

$$\begin{aligned}\xi^u = f(u,\zeta,\bar{\zeta}), \quad \xi^A = Y^A(x^A) - \partial_B f\int_r^{+\infty}\mathrm{d}r[L^A\bar{L}^B+\bar{L}^AL^B],\\ \xi^r = -\partial_u f r + \frac{1}{2}\bar{\Delta}f - \partial_A f\int_r^{+\infty}\mathrm{d}r[\omega\bar{L}^A+\bar{\omega}L^A+X^A],\end{aligned} \tag{3.3.14}$$

and

$$\begin{aligned}\omega^{12} =& \partial_u f + X^A\partial_A f, \quad \omega^{23} = \bar{L}^A\partial_A f, \quad \omega^{24} = L^A\partial_A f,\\ \omega^{13} =& (\gamma^0+\bar{\gamma}^0)\bar{P}\eth f - \bar{P}\partial_u\eth f + \partial_A f\int_r^{+\infty}\mathrm{d}r[\lambda L^A+\mu\bar{L}^A],\\ \omega^{14} =& (\gamma^0+\bar{\gamma}^0)P\bar{\eth} f - P\partial_u\bar{\eth} f + \partial_A f\int_r^{+\infty}\mathrm{d}r[\bar{\lambda}\bar{L}^A+\bar{\mu}L^A],\\ \omega^{34} =& \omega_0^{34}(u,\zeta,\bar{\zeta}) - \partial_A f\int_r^{+\infty}\mathrm{d}r[(\bar{\alpha}-\beta)\bar{L}^A+(\bar{\beta}-\alpha)L^A].\end{aligned} \tag{3.3.15}$$

For the computations below, the leading orders of their asymptotic on-shell expansions are also useful,

$$\begin{aligned}\xi^u =& f, \quad \xi^{\zeta} = Y - \frac{\bar{P}\eth f}{r} + \frac{\sigma^0\bar{P}\bar{\eth} f}{r^2} + O(r^{-3}), \quad \xi^{\bar{\zeta}} = \overline{\xi^{\zeta}},\\ \xi^r =& -r\partial_u f + \frac{1}{2}\bar{\Delta}f - \frac{\bar{\eth}\sigma^0\bar{\eth} f + \eth\bar{\sigma}^0\eth f}{r} + O(r^{-2}),\end{aligned} \tag{3.3.16}$$

and

$$
\begin{aligned}
\omega^{12} =& \partial_u f + O(r^{-3}), \quad \omega^{23} = \frac{\bar{\eth} f}{r} - \frac{\bar{\sigma}^0 \eth f}{r^2} + \frac{\sigma^0 \bar{\sigma}^0 \bar{\eth} f}{r^3} + O(r^{-4}),\\
\omega^{13} =& (\gamma^0 + \bar{\gamma}^0)\bar{\eth} f - \bar{\eth}\partial_u f + \frac{\lambda^0 \eth f + \mu^0 \bar{\eth} f}{r}\\
&\qquad - \frac{\bar{\sigma}^0 \mu^0 \eth f + \sigma^0 \lambda^0 \bar{\eth} f}{r^2} - \frac{\Psi_2^0 \bar{\eth} f}{2r^2} + O(r^{-3}),\\
\omega^{34} =& \omega_0^{34} + \frac{\bar{P}\partial \ln P \eth f - P \bar{\partial} \ln \bar{P} \bar{\eth} f}{r}\\
&\qquad + \frac{P \bar{\partial} \ln \bar{P} \bar{\sigma}^0 \eth f - \bar{P} \partial \ln P \sigma^0 \bar{\eth} f}{r^2} + O(r^{-3}),
\end{aligned}
\tag{3.3.17}
$$

with $\omega^{24} = \overline{\omega^{23}}$, $\omega^{14} = \overline{\omega^{13}}$, $\omega^{34} = -\overline{\omega^{34}}$.

3.3.4 Residual symmetry algebra

The variation of the free data parametrizing the solution space under residual gauge transformation in terms of the parametrization provided by (3.3.13) is given by

$$
\delta_{f,Y,\omega_0} P = P\partial_u f + f\partial_u P + Y\partial P + \bar{Y}\bar{\partial} P - P\bar{\partial}\bar{Y} + P\omega_0^{34},
\tag{3.3.18}
$$

together with the variation of the rest of the free data and derived quantities that is written in appendix C.3.

To make these variations more transparent, it is useful to re-parametrize residual gauge symmetries through *field-dependent redefinitions*. In a first step, one trades the real function $\partial_u f(u,\zeta,\bar{\zeta})$ and the imaginary $\omega_0^{34}(u,\zeta,\bar{\zeta})$ for a complex $\Omega(u,\zeta,\bar{\zeta})$ according to

$$
\begin{aligned}
\partial_u f &= \frac{1}{2}[\bar{\partial}\bar{Y} - \bar{Y}\bar{\partial}\ln(P\bar{P}) + \partial Y - Y\partial \ln(P\bar{P})] + f(\gamma^0 + \bar{\gamma}^0) + \frac{1}{2}(\Omega + \bar{\Omega}),\\
\omega_0^{34} &= \frac{1}{2}[\bar{\partial}\bar{Y} - \bar{Y}\bar{\partial}\ln P + \bar{Y}\bar{\partial}\ln\bar{P} - \partial Y + Y\partial\ln\bar{P} - Y\partial\ln P]\\
&\qquad\qquad\qquad\qquad + f(\bar{\gamma}^0 - \gamma^0) + \frac{1}{2}(\Omega - \bar{\Omega}).
\end{aligned}
\tag{3.3.19}
$$

It then follows that the first of (3.3.19) can be solved for f in terms of an integration function $T_R(\zeta,\bar{\zeta})$, (called $\tilde{T}$ in [6,121,158])

$$
f(u,\zeta,\bar{\zeta}) = \frac{1}{\sqrt{P\bar{P}}}[T_R(\zeta,\bar{\zeta}) + \frac{\tilde{u}}{2}(\partial Y + \bar{\partial}\bar{Y}) - Y\partial\tilde{u} - \bar{Y}\bar{\partial}\tilde{u} + \frac{1}{2}(\tilde{\Omega} + \bar{\tilde{\Omega}})],
\tag{3.3.20}
$$

where

$$\tilde{u} = \int_{u_0}^{u} du\sqrt{P\bar{P}}, \quad \tilde{\Omega} = \int_{u_0}^{u} du\sqrt{P\bar{P}}\,\Omega. \tag{3.3.21}$$

This redefinition of parameters is such that

$$\delta_{Y,T_R,\Omega}P = \Omega P, \tag{3.3.22}$$

together with the complex conjugate relation $\delta_{Y,T_R,\Omega}\bar{P} = \bar{\Omega}\bar{P}$.

Denoting by ϕ^α the fields $(e_a{}^\mu, \Gamma_{abc})$ (together with the auxiliary fields $\mathbf{R}_{abcd}, \lambda^{abcd}$ if useful), it follows from (3.2.42) that

$$\begin{aligned}
&[\delta_{\xi_1,\omega_1}, \delta_{\xi_2,\omega_2}]\phi^\alpha = \delta_{\hat{\xi},\hat{\omega}}\phi^\alpha,\\
&\hat{\xi}^\mu = [\xi_1, \xi_2]^\mu, \quad (\hat{\omega})_a{}^b = \xi_1{}^\rho\partial_\rho\omega_{2a}{}^b + \omega_{1a}{}^c\omega_{2c}{}^b - (1 \leftrightarrow 2),
\end{aligned} \tag{3.3.23}$$

when the gauge parameters ξ, ω are field-independent. In case gauge parameters do depend on the fields, one finds instead

$$\begin{aligned}
&[\delta_{\xi_1,\omega_1}, \delta_{\xi_2,\omega_2}]\phi^\alpha = \delta_{\hat{\xi}_M,\hat{\omega}_M}\phi^\alpha,\\
&\hat{\xi}^\mu_M = [\xi_1, \xi_2]^\mu - \delta_{\xi_1,\omega_1}\xi^\mu_2 + \delta_{\xi_2,\omega_2}\xi^\mu_1,\\
&(\hat{\omega}_M)_a{}^b = \xi_1{}^\rho\partial_\rho\omega_{2a}{}^b + \omega_{1a}{}^c\omega_{2c}{}^b - \delta_{\xi_1,\omega_1}\omega_{2a}{}^b - (1 \leftrightarrow 2).
\end{aligned} \tag{3.3.24}$$

We now have the following result:

The gauge parameters $(\xi[Y, T_R, \Omega], \omega[Y, T_R\Omega])$ equipped with the modified commutator for field dependent gauge transformations realize the direct sum of the abelian ideal of complex Weyl rescalings with the (extended) BMS algebra, $\mathfrak{bms}_4^{\text{ext}}$, everywhere in the bulk spacetime,

$$\begin{aligned}
&\hat{\xi}_M = \xi[\hat{Y}, \hat{T}_R, \hat{\Omega}], \quad \hat{\omega}_M = \omega[\hat{Y}, \hat{T}_R, \hat{\Omega}],\\
&\hat{Y}^A = Y_1^B\partial_B Y_2^A - Y_2^B\partial_B Y_1^A,\\
&\hat{T}_R = Y_1^A\partial_A T_{R2} + \frac{1}{2}T_{R1}\partial_A Y_2^A - (1 \leftrightarrow 2),\\
&\hat{\Omega} = 0.
\end{aligned} \tag{3.3.25}$$

The proof follows by adapting the ones provided in [6, 24, 158] to the current set-up.

3.3.5 Action of symmetries on solutions

A further field-dependent redefinition consists in

$$Y = \bar{P}\overline{\mathcal{Y}}, \quad \bar{Y} = P\mathcal{Y}. \tag{3.3.26}$$

The transformations (C.1) then become

$$
\begin{aligned}
\delta_s\sigma^0 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\overline{\eth\mathcal{Y}} + \frac{3}{2}\Omega - \frac{1}{2}\bar{\Omega}]\sigma^0 + f\bar{\lambda}^0 - \eth^2 f,\\
\delta_s\Psi^0_0 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + \frac{5}{2}\eth\mathcal{Y} + \frac{1}{2}\overline{\eth\mathcal{Y}} + \frac{5}{2}\Omega + \frac{1}{2}\bar{\Omega}]\Psi^0_0 + f\eth\Psi^0_1 + 3f\sigma^0\Psi^0_2 + 4\Psi^0_1\eth f,\\
\delta_s\Psi^0_1 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + 2\eth\mathcal{Y} + \overline{\eth\mathcal{Y}} + 2\Omega + \bar{\Omega}]\Psi^0_1 + f\eth\Psi^0_2 + 2f\sigma^0\Psi^0_3 + 3\Psi^0_2\eth f,\\
\delta_s\left(\frac{\Psi^0_2+\bar{\Psi}^0_2}{2}\right) &= [\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\overline{\eth\mathcal{Y}} + \frac{3}{2}\Omega + \frac{3}{2}\bar{\Omega}]\left(\frac{\Psi^0_2+\bar{\Psi}^0_2}{2}\right)\\
&\quad + \frac{1}{2}(f\eth\Psi^0_3 + f\sigma^0\Psi^0_4 + 2\Psi^0_3\eth f + (c.c.)),
\end{aligned}
\tag{3.3.27}
$$

while (C.2)-(C.4) read as

$$
\begin{aligned}
\delta_s\Psi^1_0 = \big[\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + 3\eth\mathcal{Y} + \overline{\eth\mathcal{Y}} + 3\Omega + \bar{\Omega}\big]\Psi^1_0&\\
- \bar{\eth}\big[5\eth f\Psi^0_0 + f\eth\Psi^0_0 + 4f\Psi^0_1\sigma^0\big],&
\end{aligned}
\tag{3.3.28}
$$

$$
\begin{aligned}
\delta_s\Psi^2_0 &= \big[\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + \frac{7}{2}\eth\mathcal{Y} + \frac{3}{2}\overline{\eth\mathcal{Y}} + \frac{7}{2}\Omega + \frac{3}{2}\bar{\Omega}\big]\Psi^2_0\\
&\quad + \big[-3\bar{\Delta}f - \bar{\eth}f\eth - 3\eth f\bar{\eth} - \frac{1}{2}f\eth\bar{\eth} - \frac{5}{4}fR\big]\Psi^1_0\\
&+ \big[-5f\Psi^0_2 - \frac{5}{2}f\bar{\Psi}^0_2 + \frac{5}{2}f\sigma^0\bar{\eth}^2 + 5f\bar{\eth}\sigma^0\bar{\eth} + 3f\eth\bar{\sigma}^0\eth + \frac{1}{2}f\bar{\sigma}^0\eth^2 + \frac{5}{2}f\eth^2\bar{\sigma}^0 + \frac{5}{2}f\sigma^0\lambda^0\\
&\quad + 5\bar{\eth}\sigma^0\bar{\eth}f + 15\eth\bar{\sigma}^0\eth f + 5\sigma^0\bar{\eth}f\bar{\eth} + 3\bar{\sigma}^0\eth f\eth\big]\Psi^0_0\\
&\quad + \big[5f\Psi^0_1 + 12\sigma^0\bar{\sigma}^0\eth f + 12f\sigma^0\eth\bar{\sigma}^0 + 2f\eth\sigma^0\bar{\sigma}^0 + \frac{9}{2}f\sigma^0\bar{\sigma}^0\eth\big]\Psi^0_1\\
&\quad + \frac{15}{2}f(\sigma^0)^2\bar{\sigma}^0\Psi^0_2,
\end{aligned}
\tag{3.3.29}
$$

$$
\begin{aligned}
\delta_s\Psi^n_0 = \big[\mathcal{Y}\eth + \overline{\mathcal{Y}\eth} + \frac{5+n}{2}\eth\mathcal{Y} + \frac{1+n}{2}\overline{\eth\mathcal{Y}} + \frac{5+n}{2}\Omega + \frac{1+n}{2}\bar{\Omega}\big]\Psi^n_0&\\
+ (\text{inhomogeneous terms}).&
\end{aligned}
\tag{3.3.30}
$$

Finally, the variations (C.5) are given by

$$
\begin{aligned}
\delta_s\lambda^0 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\bar{\eth}\overline{\mathcal{Y}} + 2\bar{\Omega}]\lambda^0 - f\Psi_4^0 - \frac{1}{2}\bar{\eth}^2(\eth\mathcal{Y} + \bar{\eth}\overline{\mathcal{Y}}), \\
\delta_s\Psi_2^0 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\overline{\mathcal{Y}} + \frac{3}{2}\Omega + \frac{3}{2}\bar{\Omega}]\Psi_2^0 \\
&\qquad + f\eth\Psi_3^0 + f\sigma^0\Psi_4^0 + 2\Psi_3^0\eth f, \\
\delta_s\Psi_3^0 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \eth\mathcal{Y} + 2\bar{\eth}\overline{\mathcal{Y}} + \Omega + 2\bar{\Omega}]\Psi_3^0 + f\eth\Psi_4^0 + \Psi_4^0\eth f, \\
\delta_s\Psi_4^0 &= [\mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{1}{2}\eth\mathcal{Y} + \frac{5}{2}\bar{\eth}\overline{\mathcal{Y}} + \frac{1}{2}\Omega + \frac{5}{2}\bar{\Omega}]\Psi_4^0 \\
&\qquad + f\partial_u\Psi_4^0 + 2(2\gamma^0 + \bar{\gamma}^0)\Psi_4^0.
\end{aligned}
\tag{3.3.31}
$$

3.3.6 Reduction of solution space

Besides conditions (3.3.5) and (3.3.6), additional constraints may be imposed on solution space. A standard choice is to fix the conformal factor P to be equal to P_S given in (3.3.8). We also fix P here, without committing to a specific value. In other words, we consider P to be part of the background structure. As a consequence, infinitesimal complex Weyl rescalings (whose finite counterparts have been discussed in [178]) are frozen and $\Omega = 0$ in the formulas above, while in the formulas below, s stands for $(\mathcal{Y}, \overline{\mathcal{Y}}, T_R, 0)$. The main reason we do not perform the analysis below while keeping $P(u, \zeta, \bar{\zeta})$ arbitrary is computational simplicity. We plan to return elsewhere to a detailed discussion of the current algebra and its interpretation when complex Weyl rescalings are allowed.

3.3.7 Breaking and co-dimension 2 form

Under this additional constraint on the solution space, the invariant presymplectic current can be computed using equation (3.2.53),

$$
\mathbf{W}[\phi, \delta_s\phi, \phi] = -W^r_{s(0)}\mathrm{d}u\mathrm{d}\zeta\mathrm{d}\bar{\zeta} + \mathcal{O}(r^{-1}),
\tag{3.3.32}
$$

where

$$
W^r_{s(0)} = \frac{1}{8\pi G P\bar{P}}\left(\delta\sigma^0\delta_s\lambda^0 + \delta\bar{\sigma}^0\delta_s\bar{\lambda}^0 - \delta\lambda^0\delta_s\sigma^0 - \delta\bar{\lambda}^0\delta_s\bar{\sigma}^0\right).
\tag{3.3.33}
$$

The expression containing the information about the non-conservation of the currents, it should not come as a surprise that it involves the news functions encoded in λ^0 and $\bar{\lambda}^0$.

Furthermore, the co-dimension 2 form (3.2.52) takes the form

$$
\mathbf{k}_s[\phi; \delta\phi] = k^{ur}_{s(0)}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} - k^{\zeta r}_{s(0)}\mathrm{d}u\mathrm{d}\bar{\zeta} + k^{\bar{\zeta}r}_{s(0)}\mathrm{d}u\mathrm{d}\zeta + \mathcal{O}(r^{-1}),
\tag{3.3.34}
$$

where

$$k^{ur}_{s(0)} = -\frac{1}{P\bar{P}8\pi G}\Big(\delta\big[f(\Psi^0_2+\sigma^0\lambda^0)+\mathcal{Y}(\sigma^0\eth\bar{\sigma}^0+\frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)+\Psi^0_1) \\ -\frac{1}{2}\eth(\mathcal{Y}\sigma^0\bar{\sigma}^0)-r\eth(\overline{\mathcal{Y}}\bar{\sigma}^0)\big]-f\lambda^0\delta\sigma^0+c.c.\Big), \quad (3.3.35)$$

$$k^{\zeta r}_{s(0)} = -\frac{1}{P8\pi G}\Big(\delta\big[\overline{\mathcal{Y}}(\bar{\lambda}^0\bar{\sigma}^0-\bar{\Psi}^0_2)-f\bar{\Psi}^0_3+\frac{1}{2}\bar{\eth}\sigma^0(\eth\mathcal{Y}-\overline{\eth\mathcal{Y}})+\frac{1}{2}\sigma^0\bar{\eth}(\overline{\eth\mathcal{Y}}-\eth\mathcal{Y}) \\ -\bar{\lambda}^0\bar{\eth}f+r\mathcal{Y}(\bar{\lambda}^0+\sigma^0(\gamma^0+\bar{\sigma}^0))\big]-\overline{\mathcal{Y}}(\bar{\lambda}^0\delta\bar{\sigma}^0+\lambda^0\delta\sigma^0)\Big), \quad (3.3.36)$$

and $k^{\bar{\zeta}r}_{s(0)}$ given by the complex conjugate. By construction

$$\partial_u k^{ur}_{s(0)}+\partial_\zeta k^{\zeta r}_{s(0)}+\partial_{\bar{\zeta}}k^{\bar{\zeta}r}_{s(0)} = -W^r_{s(0)}, \quad (3.3.37)$$

which may also be checked by direct computation. Note that $k^{ur}_{s(0)}, k^{\zeta r}_{s(0)}, k^{\bar{\zeta}r}_{s(0)}$ also contain, in addition to a finite contribution, linearly divergent terms when $r\to\infty$. Following [120], the latter can be removed through an exact 2-form $\partial_\rho\eta^{\mu\nu\rho}_s$. Defining

$$\bar{P}\eta^{[ur\bar{\zeta}]}_s = \mathcal{N}^u_s = -r\overline{\mathcal{Y}}\bar{\sigma}^0-\frac{1}{2}\mathcal{Y}\sigma^0\bar{\sigma}^0, \quad \eta^{[\zeta r\bar{\zeta}]}_s = \mathcal{N}^\zeta_s = 0, \quad (3.3.38)$$

and splitting into an integrable part

$$\mathcal{J}^u_s = -\frac{1}{8\pi G}\big[f(\Psi^0_2+\sigma^0\lambda^0)+\mathcal{Y}[\sigma^0\eth\bar{\sigma}^0+\Psi^0_1+\frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)]+\text{c.c.}\big], \quad (3.3.39)$$

$$\mathcal{J}^\zeta_s = \frac{1}{8\pi G}\Big[\overline{\mathcal{Y}}\bar{\Psi}^0_2+f\bar{\Psi}^0_3+\frac{1}{2}\overline{\mathcal{Y}}(\lambda^0\sigma^0-\bar{\lambda}^0\bar{\sigma}^0)+\frac{1}{2}\bar{\eth}\sigma^0(\overline{\eth\mathcal{Y}}-\eth\mathcal{Y}) \\ -\frac{1}{2}\sigma^0\bar{\eth}(\overline{\eth\mathcal{Y}}-\eth\mathcal{Y})+\bar{\lambda}^0\bar{\eth}f\Big], \quad (3.3.40)$$

and a non-integrable one

$$\Theta^u_s(\delta\chi) = \frac{1}{8\pi G}(f\lambda^0\delta\sigma^0+\text{c.c.}), \quad \Theta^\zeta_s(\delta\chi) = \frac{1}{8\pi G}\overline{\mathcal{Y}}(\lambda^0\delta\sigma^0+\bar{\lambda}^0\delta\bar{\sigma}^0), \quad (3.3.41)$$

one finally arrives at

$$\begin{aligned} \delta\mathcal{J}^u_s &= P\bar{P}[k^{ur}_{s(0)}-\bar{\partial}\eta^{[ur\bar{\zeta}]}_s-\partial\eta^{[ur\zeta]}_s]-\Theta^u_s, \\ \delta\mathcal{J}^\zeta_s &= P[k^{\zeta r}_{s(0)}+\partial_u\eta^{[ur\zeta]}_s+\bar{\partial}\eta^{[\bar{\zeta}r\zeta]}_s]-\Theta^\zeta_s, \end{aligned} \quad (3.3.42)$$

where $\mathcal{J}^{\bar{\zeta}}_s, \Theta^{\bar{\zeta}}_s$ are the complex conjugates of $\mathcal{J}^\zeta_s, \Theta^\zeta_s$. The results of [120] are recovered when taking P to be u-independent, which implies $\gamma^0=\bar{\gamma}^0=0$ and $\lambda^0=\dot{\bar{\sigma}}^0$. The associated forms are given by

$$\begin{aligned} J_s &= (P\bar{P})^{-1}\mathcal{J}^u_s\mathrm{d}\zeta\mathrm{d}\bar{\zeta}-P^{-1}\mathcal{J}^\zeta_s\mathrm{d}u\mathrm{d}\bar{\zeta}+\bar{P}^{-1}\mathcal{J}^{\bar{\zeta}}_s\mathrm{d}u\mathrm{d}\zeta, \\ \theta_s &= (P\bar{P})^{-1}\Theta^u_s\mathrm{d}\zeta\mathrm{d}\bar{\zeta}-P^{-1}\Theta^\zeta_s\mathrm{d}u\mathrm{d}\bar{\zeta}+\bar{P}^{-1}\Theta^{\bar{\zeta}}_s\mathrm{d}u\mathrm{d}\zeta. \end{aligned} \quad (3.3.43)$$

3.3.8 Current algebra

Using the relations of appendix C.4, the first independent component of the current algebra can be written as

$$\delta_{s_2}\mathcal{J}^u_{s_1} + \Theta^u_{s_2}(\delta_{s_1}\chi) \approx \mathcal{J}^u_{[s_1,s_2]} + \mathcal{K}^u_{s_1,s_2} + \eth\mathcal{L}_{s_1,s_2} + \overline{\eth\mathcal{L}_{s_1,s_2}}, \tag{3.3.44}$$

where

$$\mathcal{K}^u_{s_1,s_2} = \frac{1}{8\pi G}\Big[\Big(\frac{1}{2}\overline{\sigma}^0\left[f_1\eth^2(\eth\mathcal{Y}_2+\overline{\eth\mathcal{Y}}_2)\right] - f_1\eth f_2\overline{\eth}\mu^0 - (1\leftrightarrow 2)\Big) + \text{c.c.}\Big], \tag{3.3.45}$$

and

$$\begin{aligned}\mathcal{L}_{s_1,s_2} = \mathcal{Y}_2\mathcal{J}^u_{s_1} - f_2\mathcal{J}^{\bar\zeta}_{s_1} \\ &- \frac{1}{8\pi G}\Big[\big(\frac{1}{2}(\eth\mathcal{Y}_1+\overline{\eth\mathcal{Y}}_1)\eth f_2 - \frac{1}{2}\mathcal{Y}_1\eth^2 f_2 - \overline{\mathcal{Y}}_1\eth\overline{\eth} f_2\big)\overline{\sigma}^0 \\ &- \frac{1}{2}\mathcal{Y}_1\overline{\eth}^2 f_2\sigma^0 - \mathcal{Y}_1\eth f_2\eth\overline{\sigma}^0 + \overline{\mathcal{Y}}_1\overline{\eth} f_2\eth\overline{\sigma}^0 - f_1\eth f_2\lambda^0\Big].\end{aligned} \tag{3.3.46}$$

The second independent component of the current algebra is

$$\delta_{s_2}\mathcal{J}^{\bar\zeta}_{s_1} + \Theta^{\bar\zeta}_{s_2}(\delta_{s_1}\chi) \approx \mathcal{J}^{\bar\zeta}_{[s_1,s_2]} + \mathcal{K}^{\bar\zeta}_{s_1,s_2} - \partial_u\mathcal{L}_{s_1,s_2} - 2\gamma^0\mathcal{L}_{s_1,s_2} + \overline{\eth\mathcal{M}_{s_1,s_2}}, \tag{3.3.47}$$

where

$$\begin{aligned}\mathcal{K}^{\bar\zeta}_{s_1,s_2} = -\frac{1}{8\pi G}\Big[& f_2\eth f_1\overline{\eth}\nu^0 + \frac{1}{2}\eth f_1\overline{\eth}^3\overline{\mathcal{Y}}_2 + \mathcal{Y}_1\overline{\eth} f_2\eth\mu^0 + f_1\mathcal{Y}_2(\sigma^0\overline{\eth}\nu^0 + \overline{\sigma}^0\eth\overline{\nu}^0) \\ &+ \frac{1}{2}\mathcal{Y}_2\overline{\eth}^2(\eth\mathcal{Y}_1+\overline{\eth\mathcal{Y}}_1)\sigma^0 + \frac{1}{2}\mathcal{Y}_2\eth^2(\eth\mathcal{Y}_1+\overline{\eth\mathcal{Y}}_1)\overline{\sigma}^0 - (1\leftrightarrow 2)\Big],\end{aligned} \tag{3.3.48}$$

and

$$\overline{\mathcal{M}_{s_1,s_2}} = \overline{\mathcal{Y}}_2\mathcal{J}^{\bar\zeta}_{s_1} - \frac{1}{8\pi G}\Big[\frac{1}{2}\overline{\eth}(\overline{\eth\mathcal{Y}}_1 - \eth\mathcal{Y}_1)\eth f_2 + \frac{1}{2}\eth\mathcal{Y}_1\eth\overline{\eth} f_2\Big] - \text{c.c.}. \tag{3.3.49}$$

3.3.9 Cocycle condition

The components of $\mathcal{K}_{s_1,s_2}$ satisfy the 2-cocycle conditions

$$\mathcal{K}^u_{[s_1,s_2],s_3} + \delta_{s_3}\mathcal{K}^u_{s_1,s_2} + \text{cyclic}(1,2,3) = \eth\mathcal{N}_{s_1,s_2,s_3} + \overline{\eth\mathcal{N}}_{s_1,s_2,s_3}, \tag{3.3.50}$$

where

$$\mathcal{N}_{s_1,s_2,s_3} = -f_3\mathcal{K}^{\bar\zeta}_{s_1,s_2} + \text{cyclic}(1,2,3), \tag{3.3.51}$$

and

$$\mathcal{K}^{\bar\zeta}_{[s_1,s_2],s_3} + \delta_{s_3}\mathcal{K}^{\bar\zeta}_{s_1,s_2} + \text{cyclic}(1,2,3) = -\partial_u\mathcal{N}_{s_1,s_2,s_3} - 2\gamma^0\mathcal{N}_{s_1,s_2,s_3} + \overline{\eth\mathcal{O}_{s_1,s_2,s_3}}, \tag{3.3.52}$$

where

$$\overline{\mathcal{O}_{s_1,s_2,s_3}} = -\frac{1}{8\pi G}\overline{\mathcal{Y}}_3\Big[(f_1\mathcal{Y}_2 - f_2\mathcal{Y}_1)\sigma^0\bar{\eth}\nu^0 + \frac{1}{2}\sigma^0(\mathcal{Y}_2\bar{\eth}^3\overline{\mathcal{Y}}_1 - \mathcal{Y}_1\bar{\eth}^3\overline{\mathcal{Y}}_2)$$
$$+\frac{1}{2}(\eth f_1\bar{\eth}^3\overline{\mathcal{Y}}_2 - \eth f_2\bar{\eth}^3\overline{\mathcal{Y}}_1) + (f_2\eth f_1 - f_1\eth f_2)\bar{\eth}\nu^0\Big] - \text{c.c.} + \text{cyclic}(1,2,3). \quad (3.3.53)$$

A situation where this 2-cocycle is relevant is discussed in [146].

3.3.10 Discussion

The results obtained in this section generalize the results discussed in one of the examples of subsection 2.3.3 for an arbitrary time-dependent non-dynamical conformal factor $P = P(u,\zeta,\bar{\zeta})$ and in the Newman-Penrose formalism. In particular, equations (3.3.33), (3.3.39), (3.3.41) and (3.3.44) can be compared with (2.3.68), (2.3.60) and (2.3.62).

Let us now give some comments about the results. The BMS current algebra discussed in the previous section not only involves a consistent mathematical structure, but also contains some physical information on the system that would have been lost by considering only an integrable piece in the currents. To illustrate this claim, let us restrict ourselves to globally well-defined quantities on the sphere, with $P = P_S = \frac{1}{\sqrt{2}}(1+\zeta\bar{\zeta})$, there are no superrotations and $\mathcal{K}^u_{s_1,s_2} = 0 = \mathcal{K}^{\bar{\zeta}}_{s_1,s_2}$. In this case, BMS charges are defined by integrating the forms (3.3.43) at fixed retarded time over the celestial sphere,

$$Q_s = \int_{u=\text{cte}} J_s = \int_{u=\text{cte}} (P_S\bar{P}_S)^{-1}\mathcal{J}^u_s \mathrm{d}\zeta\mathrm{d}\bar{\zeta} \quad (3.3.54)$$

(see one of the example in subsection 2.3.3). If one also defines

$$\Theta_s = \int_{u=\text{cte}} \theta_s = \int_{u=\text{cte}} (P_S\bar{P}_S)^{-1}\Theta^u_s \mathrm{d}\zeta\mathrm{d}\bar{\zeta}, \quad (3.3.55)$$

and the bracket

$$\{Q_{s_1}, Q_{s_2}\}_* = \delta_{s_2}Q_{s_1} + \Theta_{s_2}[\delta_{s_1}\chi], \quad (3.3.56)$$

the integrated version of equation (3.3.44), becomes

$$\{Q_{s_1}, Q_{s_2}\}_* = Q_{[s_1,s_2]}, \quad (3.3.57)$$

This charge algebra contains for instance information on non-conservation of BMS charges. Indeed, let us take $s_2 = \partial_u$, by which we mean that $T_R = \sqrt{P_S\bar{P}_S}, Y = 0 = \bar{Y}$, so that $f = 1, \mathcal{Y} = 0 = \bar{\mathcal{Y}}$. In this case, equation (3.3.57) together with the definition on the left-hand side in (3.3.56) becomes

$$\delta_{\partial_u}Q_s + \Theta_{\partial_u}[\delta_s\chi] = Q_{[s,\partial_u]}. \quad (3.3.58)$$

When using that

$$\frac{\mathrm{d}}{\mathrm{d}u}Q_s = \delta_{\partial_u}Q_s + \frac{\partial}{\partial u}Q_s, \tag{3.3.59}$$

and $\frac{\partial}{\partial u}Q_s = Q_{\partial s/\partial u} = -Q_{[s,\partial_u]}$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}u}Q_s = -\Theta_{\partial_u}[\delta_s\chi]. \tag{3.3.60}$$

If one now chooses $s = \partial_u$, one recovers the Bondi mass loss formula.

More generally, equation (3.3.44) is the local version of (3.3.57), where superrotations and arbitrary fixed $P(u,\zeta,\bar{\zeta})$ are allowed. When choosing $s_2 = \partial_u$ in that equation, it encodes the non-conservation of BMS currents (cf. equation (4.22) in [120]). For particular choices of s_1, it controls the time evolution of the Bondi mass and angular momentum aspects.

Even though we concentrated here on the case of standard Einstein gravity, all the kinematics is in place to generalize the constructions to gravitational theories with higher derivatives and/or dynamical torsion.

For the most part of section 3.3, the standard discussion has been extended to include an arbitrary u-dependent conformal factor P. This has been done so as to manifestly include the Robinson-Trautman solution [202, 203] in the solution space. The application of the current set-up to these solutions requires the inclusion of a dynamical conformal factor in the derivation of the current algebra.

Chapter 4

Generalized BMS$_4$ and renormalized phase space

In section 3.3, we discussed the solution space of four-dimensional general relativity in asymptotically flat spacetime (AF2) (see equations (2.2.15) and (2.2.17) for these boundary conditions in metric formalism). Furthermore, we investigated the associated phase space, assuming the conformal factor to be non-dynamical. As stated in subsection 2.2.4, the asymptotic symmetry algebra is given by the extended BMS algebra written as $\mathfrak{bms}_4^{\text{ext}}$, namely the semi-direct sum between the superrotations $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ and the supertranslations $\mathfrak{s}^*$ [5, 6, 121].

In this chapter, we consider another set of boundary conditions (AF1) (see equations (2.2.15) and (2.2.16)) corresponding to a new definition of asymptotic flatness. We also study the associated asymptotic symmetry algebra and phase space. The former is given by a new extension of the global BMS algebra, called the generalized BMS algebra. This is given by $\mathfrak{bms}_4^{\text{gen}} = \mathfrak{diff}(S^2) \oplus\!\!\!\!+ \mathfrak{s}$ [8, 12, 13, 164, 204]. This alternative extension of the global BMS algebra is motivated by two points: (i) it is essential to establish the full equivalence between Ward identities for superrotations and subleading soft graviton theorem, and (ii) $\mathfrak{bms}_4^{\text{gen}}$ is obtained in the flat limit of $\mathfrak{bms}_4^{\Lambda}$, the latter being a version of BMS in asymptotically locally (A)dS$_4$ spacetime.

In section 4.1, we recall the set of boundary conditions that leads to the generalized BMS group and we discuss the associated solution space. In section 4.2, we compute the corresponding symplectic structure and notice the presence of divergences in $\sim r$. We renormalize these divergences by using the Iyer-Wald ambiguity and obtain a finite symplectic structure, from which we derive the charge algebra.

This chapter has strong intersections with [8].

4.1 Generalized BMS$_4$ group and solution space

4.1.1 Solution space

We recall that the Bondi gauge (2.2.10) leads to coordinates (u, r, x^A) where u labels null outgoing geodesic congruences, r is a parameter along these geodesics, and x^A are two coordinates on the 2-sphere. The Bondi metric is parametrized as

$$\mathrm{d}s^2 = \frac{V}{r} e^{2\beta} \mathrm{d}u^2 - 2e^{2\beta} \mathrm{d}u\mathrm{d}r + g_{AB}(\mathrm{d}x^A - U^A \mathrm{d}u)(\mathrm{d}x^B - U^B \mathrm{d}u), \tag{4.1.1}$$

where g_{AB} satisfies the determinant condition

$$\partial_r \left(\frac{\det(g_{AB})}{r^4} \right) = 0. \tag{4.1.2}$$

We choose the definition (AF1) of asymptotic flatness (equations (2.2.15) and (2.2.16)) and repeat it here:

$$\begin{gathered} \beta = o(1), \quad \frac{V}{r} = o(r^2), \quad U^A = o(1), \\ g_{AB} = r^2 q_{AB} + r C_{AB} + D_{AB} + \mathcal{O}(r^{-1}), \quad \sqrt{q} = \sqrt{\bar{q}} \end{gathered} \tag{4.1.3}$$

Using these gauge and fall-off conditions, the Einstein equations entirely determine the solution space (see equations (2.2.48) for a parametrization of the solution space). Furthermore, we assume two additional constraints:

$$\mathcal{D}_{AB} = 0, \qquad \sqrt{\bar{q}} = \sqrt{\mathring{q}} \tag{4.1.4}$$

where $\mathcal{D}_{AB}$ is the trace-free part of D_{AB} and $\mathring{q}$ is the determinant of the unit 2-sphere metric. The first condition in (4.1.4) guarantees that there is no logarithmic term in the expansion of β and U^A. The second condition ensures that $l = \partial_u \ln \sqrt{\bar{q}} = 0$. The solution space is then explicitly given by

$$\begin{aligned} \frac{V}{r} &= -\frac{R}{2} + \frac{2M}{r} + \mathcal{O}(r^{-2}), \\ \beta &= -\frac{1}{32 r^2} C^{AB} C_{AB} + \mathcal{O}(r^{-3}), \\ g_{AB} &= r^2 q_{AB} + r C_{AB} + \frac{1}{4} q_{AB} C^{CD} C_{CD} + \mathcal{O}(r^{-1}), \\ U^A &= -\frac{1}{2r^2} D_B C^{AB} - \frac{2}{3} \frac{1}{r^3} \left[N^A - \frac{1}{2} C^{AB} D^C C_{BC} \right] + \mathcal{O}(r^{-4}) \end{aligned} \tag{4.1.5}$$

where all functions appearing in the expansions of $\frac{1}{r}$ depend upon u and x^A. All 2-sphere indices in (4.1.5) are raised and lowered with q_{AB}, and D_A is the Levi-Civita connection associated with q_{AB}. The determinant condition (4.1.2) imposes in particular that $q_{AB}C^{AB} = 0$. C_{AB} is otherwise completely arbitrary, and its time derivative $N_{AB} = \partial_u C_{AB}$ is the Bondi news tensor which describes gravitational radiation.

Let us recall that the Einstein equations impose the following time evolution equations:

$$\partial_u q_{AB} = 0, \tag{4.1.6}$$

$$\partial_u M = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}D_A D_B N^{AB} + \frac{1}{8}D_A D^A R, \tag{4.1.7}$$

$$\begin{aligned}\partial_u N_A = {} & D_A M + \frac{1}{16}D_A(N_{BC}C^{BC}) - \frac{1}{4}N^{BC}D_A C_{BC} \\ & - \frac{1}{4}D_B(C^{BC}N_{AC} - N^{BC}C_{AC}) - \frac{1}{4}D_B D^B D^C C_{AC} \\ & + \frac{1}{4}D_B D_A D_C C^{BC} + \frac{1}{4}C_{AB}D^B \mathring{R}.\end{aligned} \tag{4.1.8}$$

Here $M(u, x^A)$ is the Bondi mass aspect, $N_A(u, x^B)$ is the angular momentum aspect. Concerning this quantity, our conventions are those of Barnich-Troessaert [6, 121] (also followed by [205]), but differ from those of Flanagan-Nichols (FN) [169] and Hawking-Perry-Strominger (HPS) [35]. Here is the dictionary to match the different conventions:

$$N_A^{(FN)} = N_A + \frac{1}{4}C_{AB}D_C C^{BC} + \frac{3}{32}\partial_A(C_{BC}C^{BC}), \tag{4.1.9}$$

$$N_A^{(HPS)} = N_A^{(FN)} - uD_A M. \tag{4.1.10}$$

4.1.2 Asymptotic Killing vectors

The asymptotic Killing vectors ξ^μ are obtained by imposing the preservation of the Bondi gauge (equations (4.1.1) and (4.1.2)) and the asymptotic flatness conditions (equations (4.1.3) and (4.1.4)). They are explicitly given by

$$\begin{aligned}\xi^u &= f(u, x^A), \\ \xi^A &= Y^A(u, x^A) + I^A, \quad I^A = -D_B f \int_r^\infty \mathrm{d}r'(e^{2\beta}g^{AB}), \\ \xi^r &= -\frac{1}{2}r(D_A Y^A + D_A I^A - U^B D_B f),\end{aligned} \tag{4.1.11}$$

with $\partial_r f = \partial_r Y^A = 0$. Furthermore, the parameters satisfy

$$\begin{aligned}\partial_u Y^A = 0 &\Longleftrightarrow Y^A = Y^A(x^B), \\ \partial_u f = \frac{1}{2}D_A Y^A &\Longleftrightarrow f = T(x^B) + \frac{u}{2}D_A Y^A.\end{aligned} \tag{4.1.12}$$

We can perform the radial integration in (4.1.11) to get a perturbative expression of the infinitesimal residual diffeomorphisms using the explicit solution space (4.1.5):

$$\xi^u = f, \tag{4.1.13}$$

$$\xi^A = Y^A - \frac{1}{r}D^A f + \frac{1}{r^2}\left(\frac{1}{2}C^{AB}D_B f\right) + \frac{1}{r^3}\left(-\frac{1}{16}C_{BC}C^{BC}D^A f\right) + \mathcal{O}(r^{-4}), \tag{4.1.14}$$

$$\xi^r = -\frac{1}{2}rD_A Y^A + \frac{1}{2}D_A D^A f + \frac{1}{r}\left(-\frac{1}{2}D_A C^{AB}D_B f - \frac{1}{4}C^{AB}D_A D_B f\right) + \mathcal{O}(r^{-2}). \tag{4.1.15}$$

The asymptotic Killing vectors are spanned by $\mathfrak{diff}(S^2)$ *super-Lorentz* transformations generated by $Y^A(x^B)$ and by (smooth) *supertranslations* generated by $T(x^A)$. We therefore denote them as $\xi(T, Y)$. Notice that *in chapters 4 and 5* (and only in these chapters), we find it convenient to call the extension of the Lorentz transformations as the super-Lorentz transformations instead of superrotations. Any 2-vector on the sphere can be decomposed into a divergence-free part and a rotational-free part. A super-Lorentz transformation whose pullback on the celestial sphere is divergence-free is a superrotation. This generalizes the rotations. A super-Lorentz transformation whose pullback on the celestial sphere is rotational-free is a superboost. This generalizes the boosts.

4.1.3 Asymptotic symmetry algebra

As discussed in subsection 2.2.4, to obtain the asymptotic symmetry algebra, one has to consider the modified Lie bracket

$$[\xi_1, \xi_2]_A = [\xi_1, \xi_2] - \left(\delta^g_{\xi_1}\xi_2 - \delta^g_{\xi_2}\xi_1\right), \tag{4.1.16}$$

where $\delta^g_{\xi_1}\xi_2$ denotes the variation of ξ_2 caused by the Lie dragging along ξ_1 of the metric contained in the definition of ξ_2. We find

$$[\xi(T_1, Y_1^A), \xi(T_2, Y_2^A)]_A = \xi(\hat{T}, \hat{Y}), \tag{4.1.17}$$

where

$$\begin{aligned} \hat{T} &= Y_1^A D_A T_2 + \frac{1}{2}T_1 D_A Y_2^A - (1 \leftrightarrow 2), \\ \hat{Y}^A &= Y_1^B D_B Y_2^A - (1 \leftrightarrow 2). \end{aligned} \tag{4.1.18}$$

This defines the *generalized BMS algebra* $\mathfrak{bms}_4^{\text{gen}}$. It consists of the semi-direct sum of the diffeomorphism algebra on the celestial 2-sphere $\mathfrak{diff}(S^2)$ and the abelian ideal $\mathfrak{s}$ of supertranslations, consisting of arbitrary smooth densities (of weight $-1/2$) on the 2-sphere:

$$\mathfrak{bms}_4^{\text{gen}} = \mathfrak{diff}(S^2) \inplus \mathfrak{s}. \tag{4.1.19}$$

4.1.4 Action on the solution space

The vectors (4.1.11) preserve the solution space in the sense that infinitesimally

$$\mathcal{L}_{\xi(T,Y)} g_{\mu\nu}[\phi^i] = g_{\mu\nu}[\phi^i + \delta_{(T,Y)}\phi^i] - g_{\mu\nu}[\phi^i] \tag{4.1.20}$$

where $\phi^i = \{q_{AB}, C_{AB}, M, N_A\}$ denotes the collection of relevant fields that describe the metric in Bondi gauge. The action of the vectors preserve the form of the metric but modify the fields ϕ^i, in such a way that the above equation is verified. We can show that

$$\delta_{(T,Y)} q_{AB} = 2D_{(A}Y_{B)} - D_C Y^C q_{AB}, \tag{4.1.21}$$

$$\delta_{(T,Y)} C_{AB} = [f\partial_u + \mathcal{L}_Y - \frac{1}{2} D_C Y^C] C_{AB} - 2D_A D_B f + q_{AB} D_C D^C f, \tag{4.1.22}$$

$$\delta_{(T,Y)} N_{AB} = [f\partial_u + \mathcal{L}_Y] N_{AB} - (D_A D_B D_C Y^C - \frac{1}{2} q_{AB} D_C D^C D_D Y^D), \tag{4.1.23}$$

$$\delta_{(T,Y)} M = [f\partial_u + \mathcal{L}_Y + \frac{3}{2} D_C Y^C] M + \frac{1}{4} D_A f D^A R \tag{4.1.24}$$

$$+ \frac{1}{8} D_C D_B D_A Y^A C^{BC} + \frac{1}{4} N^{AB} D_A D_B f + \frac{1}{2} D_A f D_B N^{AB}, \tag{4.1.25}$$

$$\begin{aligned}\delta_{(T,Y)} N_A = {} & [f\partial_u + \mathcal{L}_Y + D_C Y^C] N_A + 3M D_A f - \frac{3}{16} D_A f N_{BC} C^{BC} \\ & - \frac{1}{32} D_A D_B Y^B C_{CD} C^{CD} + \frac{1}{4}(D^B f \mathring{R} + D^B D_C D^C f) C_{AB} \\ & - \frac{3}{4} D_B f (D^B D^C C_{AC} - D_A D_C C^{BC}) + \frac{3}{8} D_A (D_C D_B f C^{BC}) \\ & + \frac{1}{2}(D_A D_B f - \frac{1}{2} D_C D^C f q_{AB}) D_C C^{BC} + \frac{1}{2} D_B f N^{BC} C_{AC}.\end{aligned} \tag{4.1.26}$$

Note that the boundary Ricci scalar R transforms as

$$\delta_{(T,Y)} R = Y^A D_A R + D_A Y^A R + D^2 D_B Y^B. \tag{4.1.27}$$

4.2 Renormalized phase space

In this section, we define an extended phase space invariant under the action of Diff(S^2) super-Lorentz transformations and supertranslations. Super-Lorentz transformations are overleading in the sense that they change the boundary metric, which is usually fixed in standard asymptotically flat spacetimes. We can therefore expect that a renormalization procedure will be required. In this context, we use the Iyer-Wald procedure described in subsection 2.3.4 that allows us to remove the divergences by utilizing the ambiguity in the definition of the presymplectic potential.

4.2.1 Presymplectic potential

In the metric formalism, the canonical presymplectic potential is given in equation (2.3.74). Plugging the solution space (4.1.5) into this expression, we obtain

$$\theta^u[g;\delta g] \quad = \quad r\,\theta^u_{(div)} + \theta^u_{(0)} + r^{-1}\theta^u_{(1)} + \mathcal{O}(r^{-2}), \tag{4.2.1}$$
$$\theta^r[g;\delta g] \quad = \quad r\,\theta^r_{(div)} + \theta^r_{(0)} + \mathcal{O}(r^{-1}). \tag{4.2.2}$$

We have $\theta^u_{(div)} \propto \delta\sqrt{q}$ and therefore $\theta^u_{(div)} = 0$ as a result of the boundary condition (4.1.3). Furthermore, we find $\theta^u_{(1)} = 0$. The other components are

$$\theta^u_{(0)} = \frac{\sqrt{q}}{16\pi G}\frac{1}{2}C_{AB}\delta q^{AB}, \tag{4.2.3}$$
$$\theta^r_{(div)} = -\frac{\sqrt{q}}{16\pi G}\delta R - \frac{1}{2}\frac{\sqrt{q}}{16\pi G}N_{AB}\delta q^{AB}, \tag{4.2.4}$$
$$\begin{aligned}\theta^r_{(0)} &= -\frac{\sqrt{q}}{16\pi G}\,\delta\left[\frac{1}{8}N^{AB}C_{AB} - 2M + \frac{1}{2}D_A D_B C^{AB}\right] \\ &\quad + \bar{\theta}_{flux} + \frac{\sqrt{q}}{16\pi G}D_A\left[\frac{1}{2}D^C C_{BC}\delta q^{AB}\right],\end{aligned} \tag{4.2.5}$$

where we define with hindsight the important quantity

$$\bar{\theta}_{flux} \equiv \frac{\sqrt{q}}{16\pi G}\left[\frac{1}{2}N_{AB}\delta C^{AB} - \frac{1}{4}RC_{AB}\delta q^{AB} - \frac{1}{2}D^C C_{BC} D_A \delta q^{AB}\right]. \tag{4.2.6}$$

We note that one can isolate a total derivative and a total variation as

$$\theta^u_{(0)} = -\partial_r Y^{ur}, \tag{4.2.7}$$
$$r\theta^r_{(div)} = -\partial_u Y^{ru} - \delta(\sqrt{q}R\, r) = -\partial_u Y^{ru} - \partial_A Y^{rA} \tag{4.2.8}$$

where $Y^{ur} = -Y^{ru} = -r\frac{1}{2}\frac{\sqrt{q}}{16\pi G}C_{AB}\delta q^{AB}$ and $Y^{rA} = r\,\frac{1}{16\pi G}\,\theta^A_{2d}(q;\delta q)$ is r times the presymplectic potential of the two-dimensional Einstein-Hilbert action, $\partial_A\theta^A_{2d} = \delta(\sqrt{q}R)$.

4.2.2 Presymplectic form

The bare Lee-Wald presymplectic form is given by $\boldsymbol{\omega}[g;\delta g,\delta g] = \delta\boldsymbol{\theta}[g;\delta g]$ (see equation (2.3.72)). We have already obtained that the bare presymplectic potential $\boldsymbol{\theta}[g;\delta g]$ is divergent (see (4.2.1) and (4.2.2)). However, it is ambiguous under the change $\boldsymbol{\theta}[g;\delta g] \to \boldsymbol{\theta}[g;\delta g] - \mathrm{d}\mathbf{Y}[g;\delta g]$, where $\mathbf{Y}[g;\delta g]$ is a co-dimension 2 form (see (2.3.81) and the associated discussion). We have already identified in (4.2.7) and (4.2.8) the counterterms required to make the presymplectic potential finite.

Let us discuss this point in detail, starting from the bare presymplectic form. We have

$$\omega^u = \frac{\sqrt{q}}{16\pi G}\left(\frac{1}{2}\delta q_{AB} \wedge \delta C^{AB}\right) + \mathcal{O}(r^{-2}), \tag{4.2.9}$$

$$\begin{aligned}\omega^r &= -r\frac{\sqrt{q}}{16\pi G}\left(\frac{1}{2}\delta N_{AB} \wedge \delta q^{AB}\right) \\ &\quad + \frac{\sqrt{q}}{16\pi G}\left[\frac{1}{2}\delta\left(N^{AB} + \frac{1}{2}Rq^{AB}\right) \wedge \delta C_{AB} + \frac{1}{2}\delta(D_A D^C C_{BC}) \wedge \delta q^{AB}\right] + \mathcal{O}(r^{-1}).\end{aligned} \tag{4.2.10}$$

Clearly, such a presymplectic form is divergent. After choosing the boundary term $\mathbf{Y}[g;\delta g]$ as in (4.2.7) and (4.2.8), the presymplectic form becomes well-defined,

$$\omega^u_{\text{ren}} = \mathcal{O}(r^{-2}), \tag{4.2.11}$$

$$\omega^r_{\text{ren}} = \frac{\sqrt{q}}{16\pi G}\left[\frac{1}{2}\delta\left(N^{AB} + \frac{1}{2}Rq^{AB}\right) \wedge \delta C_{AB} + \frac{1}{2}\delta(D_A D^C C_{BC}) \wedge \delta q^{AB}\right] + \mathcal{O}(r^{-1}). \tag{4.2.12}$$

Since Y^{Ar} is exact, it does not contribute here. This defines the presymplectic structure at $\mathscr{I}^+$

$$\begin{aligned}&\Omega_{\text{ren}}[g;\delta g,\delta g] \\ &\quad = \frac{1}{16\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\,\mathrm{d}^2\Omega\left[\frac{1}{2}\delta\left(N^{AB} + \frac{1}{2}Rq^{AB}\right) \wedge \delta C_{AB} + \frac{1}{2}\delta(D_A D_C C^C_B) \wedge \delta q^{AB}\right] \\ &\quad = \int_{\mathscr{I}^+} \mathrm{d}u\,\mathrm{d}^2x\left(\delta\bar{\theta}_{flux}[g;\delta g]\right),\end{aligned} \tag{4.2.13}$$

where $\bar{\theta}_{flux}$ is defined in (4.2.6), after discarding a boundary term.

4.2.3 Infinitesimal surface charges

The (ur)-component of the bare Iyer-Wald co-dimension 2 form is

$$k^{ur}_\xi[g;\delta g] = -\delta Q^{ur}_\xi[g] + Q^{ur}_{\delta\xi}[g] + \xi^u\theta^r[g;\delta g] - \xi^r\theta^u[g;\delta g] \tag{4.2.14}$$

(we note the presence of the term $Q^{ur}_{\delta\xi}[g]$ compared to the expression (2.3.73), as already discussed in the footnote 16 of subsection 2.3.4). Expanding in powers of $1/r$, we get

$$k^{ur}_\xi[g;\delta g] = r k^{ur}_{(div)} + k^{ur}_{(0)} + \mathcal{O}(r^{-1}). \tag{4.2.15}$$

We define $\delta\!\!\!/ \bar{H}_\xi = \int_{S^2_\infty} \mathbf{k}_\xi[g;\delta g]$. The divergent term is

$$\delta\!\!\!/ \bar{H}^{(div)}_\xi = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\left[\delta(Y^A D_B C^B_A) - f\delta R - \frac{1}{2}fN_{AB}\delta q^{AB} + \frac{1}{4}D_C Y^C q_{AB}\delta C^{AB}\right] \tag{4.2.16}$$

while the finite term is

$$\begin{aligned}\oint \bar{H}_\xi^{(0)} &= \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\delta\left(4fM + 2Y^A N_A + \frac{1}{16}Y^A D_A(C_{BC}C^{BC})\right)\right] \\ &+ \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\frac{1}{2}f\left(N^{AB} + \frac{1}{2}q^{AB}R\right)\delta C_{AB}\right. \\ &\left. + D_A f D^C C_{BC}\delta q^{AB} + \frac{1}{2}f D_A D^C C_{BC}\delta q^{AB} - \frac{1}{4}D^2 f q_{AB}\delta C^{AB}\right].\end{aligned} \tag{4.2.17}$$

Clearly, the charges are neither finite nor integrable.

For covariant counter-term $\mathbf{Y}[g;\delta g]$, the modification at the level of the co-dimension 2 form is

$$k_\xi^{ur} \to k_\xi^{ur} - \delta Y^{ur}[g;\mathcal{L}_\xi g] + Y^{ur}[g;\mathcal{L}_{\delta\xi} g] + \xi^u \Delta\theta^r - \xi^r\Delta\theta^u. \tag{4.2.18}$$

More specifically, for the $\mathbf{Y}[g;\delta g]$ given in (4.2.7) and (4.2.8), we have

$$-\delta Y^{ur}[g;\mathcal{L}_\xi g] = \frac{1}{2}r\frac{\sqrt{q}}{16\pi G}\delta\left(C_{AB}\delta_{(T,Y)}q^{AB}\right); \tag{4.2.19}$$

$$Y^{ur}[g;\mathcal{L}_{\delta\xi} g] = 0, \tag{4.2.20}$$

where the last equation follows from the fact that the fields are not modified by $\delta\xi$ at leading order in r. Moreover,

$$\xi^u\Delta\theta^r = f(\partial_u Y^{ru} + \partial_A Y^{rA}) = \frac{1}{2}r\frac{\sqrt{q}}{16\pi G}fN_{AB}\delta q^{AB} + \frac{\sqrt{q}}{16\pi G}r\left(f\delta R\right); \tag{4.2.21}$$

$$-\xi^r\Delta\theta^u = -\xi^r(\partial_r Y^{ur}) = -\frac{1}{4}\frac{\sqrt{q}}{16\pi G}rD_C Y^C C_{AB}\delta q^{AB} + \frac{1}{4}\frac{\sqrt{q}}{16\pi G}D_C D^C f C_{AB}\delta q^{AB} + \mathcal{O}(r^{-1}). \tag{4.2.22}$$

Here, the $\mathcal{O}(1)$ part in (4.2.22) is due to the $\mathcal{O}(1)$ contribution from ξ^r, and exactly cancels the last non-integrable term at $\mathcal{O}(1)$ in the charges (4.2.17).

We see that any divergent term will disappear due to this choice of Y^{ur}, and the infinitesimal surface charges reduce to

$$\begin{aligned}\oint H_\xi^{\text{intermediate}} &= \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\delta\left(4fM + 2Y^A N_A + \frac{1}{16}Y^A D_A(C_{BC}C^{BC})\right)\right] \\ &+ \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\frac{1}{2}f\left(N^{AB} + \frac{1}{2}q^{AB}R\right)\delta C_{AB} + \frac{1}{2}D_A(fD_C C^C_B)\delta q^{AB}\right].\end{aligned} \tag{4.2.23}$$

Before proceeding, let us notice a very subtle point. Using the Iyer-Wald ambiguity, we saw that $\mathbf{Y}[g;\delta g]$ allowed us to cancel the divergent part of the presymplectic form, but did not affect the finite part. However, when reporting this Iyer-Wald modification at the level of the co-dimension 2 form in (4.2.18), we showed that it affected the finite part of the charges. This is in tension with the fact that the co-dimension 2 form is completely determined by the presymplectic potential through the fundamental relation (2.3.77). This apparent discrepancy in the formalism is due to an inappropriate use of the algorithm that brings the renormalization at the level of the charge. More precisely, the formula (4.2.18) is valid only for $\mathbf{Y}[g;\delta g]$ covariant with respect to the bulk and thus does not hold in our current context. We therefore used an alternative way to obtain the surface charges and checked that the finite part (4.2.17) of the bare charge satisfies the fundamental relation

$$\partial_u \delta\!\!\!/ \bar{H}^{(0)}_\xi = -\Omega_{\text{ren}}[g;\delta_\xi g,\delta g], \tag{4.2.24}$$

where $\Omega_{\text{ren}}[g;\delta_\xi g, g]$ is the renormalized symplectic form given in (4.2.13). The final renormalized infinitesimal charge is finite and given explicitly by

$$\begin{aligned}\delta\!\!\!/ H_\xi = &\frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\delta\left(4fM + 2Y^A N_A + \frac{1}{16}Y^A D_A(C_{BC}C^{BC})\right)\right] \\ &+ \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\frac{1}{2}f\left(N^{AB}+\frac{1}{2}q^{AB}R\right)\delta C_{AB}\right. \\ &\left. + D_A f D^C C_{BC}\delta q^{AB} + \frac{1}{2}f D_A D^C C_{BC}\delta q^{AB} - \frac{1}{4}D^2 f q_{AB}\delta C^{AB}\right].\end{aligned} \tag{4.2.25}$$

When q_{AB} is the fixed unit metric on the sphere, it reproduces the expression of Barnich-Troessaert [121] (see equations (2.3.59) and (2.3.60)).

The infinitesimal surface charges can be written as

$$\delta\!\!\!/ H_\xi[g] = \delta H_\xi[g] + \Xi_\xi[g;\delta g], \tag{4.2.26}$$

where

$$H_\xi[g] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left(4fM + 2Y^A N_A + \frac{1}{16}Y^A D_A(C_{BC}C^{BC})\right) \tag{4.2.27}$$

and

$$\begin{aligned}\Xi_\xi[g;\delta g] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega &\left[\frac{1}{2}f\left(N^{AB}+\frac{1}{2}q^{AB}R\right)\delta C_{AB} + D_A f D^C C_{BC}\delta q^{AB}\right. \\ &\left. + \frac{1}{2}f D_A D^C C_{BC}\delta q^{AB} - \frac{1}{4}D^2 f q_{AB}\delta C^{AB}\right].\end{aligned} \tag{4.2.28}$$

Of course, the canonical Hamiltonian (4.2.27) cannot be deduced solely from the relation (4.2.26) since one can shift H_ξ as

$$\delta\!\!\!/ H_\xi[g] = \delta(H_\xi[g] + \Delta H_\xi[g]) + \Xi_\xi[g;\delta g] - \delta\Delta H_\xi[g]. \tag{4.2.29}$$

We therefore need additional input to fix the finite Hamiltonian. This is discussed in subsection 5.3.

4.2.4 Charge algebra

After an involved computation, we get the following charge algebra

$$\delta_{\xi_2} H_{\xi_1}[g] + \Xi_{\xi_2}[g,\delta_{\xi_1} g] = H_{[\xi_1,\xi_2]_A}[g] + \mathcal{K}_{\xi_1,\xi_2}[g]. \tag{4.2.30}$$

In this relation,

$$\begin{aligned}\Xi_{\xi_2}[g;\delta_{\xi_1} g] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \Bigg[\frac{1}{2} f_2 \left(N^{AB} + \frac{1}{2} q^{AB} R\right) \delta_{\xi_1} C_{AB} + D_A f_2 D^C C_{BC} \delta_{\xi_1} q^{AB} \\ + \frac{1}{2} f_2 D_A D^C C_{BC} \delta_{\xi_1} q^{AB} - \frac{1}{4} D^2 f_2 q_{AB} \delta_{\xi_1} C^{AB}\Bigg].\end{aligned} \tag{4.2.31}$$

Furthermore, the 2-cocycle $\mathcal{K}_{\xi_1,\xi_2}[g]$ is antisymmetric and satisfies

$$\mathcal{K}_{[\xi_1,\xi_2],\xi_3} + \delta_{\xi_3}\mathcal{K}_{\xi_1,\xi_2} + \text{cyclic}(1,2,3) = 0. \tag{4.2.32}$$

It is given explicitly by

$$\mathcal{K}_{\xi_1,\xi_2}[g] = \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega \left[\frac{1}{2} f_1 D_A f_2 D^A \mathring{R} + \frac{1}{2} C^{BC} f_1 D_B D_C D_D Y_2^D - (1 \leftrightarrow 2)\right]. \tag{4.2.33}$$

The result (4.2.30) is the analogue of (2.3.62) (or (3.3.44) in NP formalism), but for $\mathfrak{diff}(S^2)$ superrotations.

Chapter 5

Vacuum structure, superboost transitions and refraction memory

In section 5.1, we study the vacuum structure of the gravitational field in asymptotically flat spacetimes for both extensions of the BMS group. This analysis shows that one field in the metric that is turned on after acting with superboost transformations exhibits the properties of a Liouville field. Furthermore, in section 5.2, we argue that this field is precisely the memory field associated with the velocity kick/refraction memory effects. In section 5.3, using some definitions introduced in the previous sections, we propose a prescription to obtain meaningful finite charges out of (4.2.25). Applying this procedure, we find precisely the charges needed to establish the equivalence between Ward identities and soft theorems.

5.1 Vacuum structure

The orbit of Minkowski spacetime under the BMS group is defined as the class of Riemann-flat metrics obtained by exponentiating a general BMS transformation starting from Minkowski spacetime as a seed. The subset of this orbit where only supertranslations act are the non-equivalent vacua of asymptotically flat spacetimes, which are characterized, contrary to Minkowski spacetime, by non-vanishing super-Lorentz charges, while all Poincaré charges remain zero [102]. In the global BMS case, the exponentiation leads to a single fundamental field labeling inequivalent vacua: the supertranslation field $C(x^A)$. The displacement memory effect is a transition among vacua mediated by gravitational or other null radiation, which effectively induces a supertranslation of C [101].

For the extended BMS asymptotic symmetry group, this exponentiation leads to two fundamental fields: the supertranslation field and what we will call the superboost or Liouville field Φ. The corresponding solution in Bondi and Newman-

Unti gauges was constructed in [102]. Here, we extend the construction to finite Diff(S^2) super-Lorentz transformations following methods similar to the appendix of [103]. The corresponding boundary fields will also be the supertranslation C and superboost Φ fields, complemented by an additional superrotation field Ψ. To understand the memory effects associated with super-Lorentz transformations, we start by deriving the structure of the vacua.

5.1.1 Generation of the vacua

In this subsection, we construct the finite diffeomorphisms associated with the asymptotic Killing vectors (4.1.13-4.1.15) acting on the Minkowski space. We start from the Minkowski metric written in complex plane coordinates[1]

$$ds^2 = -2\mathrm{d}u_c \mathrm{d}r_c + 2r_c^2 dz_c d\bar{z}_c. \tag{5.1.1}$$

We define the background structures

$$\gamma_{ab} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \epsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.1.2}$$

with inverse $\gamma^{ab} = \gamma_{ab}$, $\epsilon^{ab} = \epsilon_{ab}$. The goal is to introduce a diffeomorphism to Bondi gauge $(u_c, r_c, z_c, \bar{z}_c) \to (u, r, z, \bar{z})$ that exponentiates Diff(S^2) super-Lorentz transformations and supertranslations. Requiring that $(u, r, z, \bar{z})$ are Bondi coordinates leads to two sets of conditions: (i) the algebraic conditions $g_{rr} = 0 = g_{rA}$, and (ii) the determinant condition $\partial_r(r^{-4} \det g_{AB}) = 0$.

The first set of conditions yields

$$r_c = r_c(r, u, z, \bar{z}), \tag{5.1.3}$$
$$u_c = W(u, z, \bar{z}) - r_c^{-1}\gamma_{ab}H^a(u, z, \bar{z})H^b(u, z, \bar{z}), \tag{5.1.4}$$
$$z_c^a = G^a(z, \bar{z}) - r_c^{-1}H^a(u, z, \bar{z}), \quad H^a(u, z, \bar{z}) = -D_G^{-1}\epsilon^{ab}\gamma_{bc}\epsilon^{AB}\partial_A W \partial_B G^c \tag{5.1.5}$$

where $D_G = \det(\partial_A G^b) = \frac{1}{2!}\epsilon_{ab}\epsilon^{AB}\partial_A G^a \partial_B G^b$. The second condition fixes the functional dependence of r_c as

$$r_c(r, u, z, \bar{z}) = R_0(u, z, \bar{z}) + \sqrt{\frac{r^2}{(\partial_u W)^2} + R_1(u, z, \bar{z})}. \tag{5.1.6}$$

---

[1] The metric (5.1.1) can be related to the standard Minkowski metric $ds^2 = -du_s^2 - 2du_s dr_s + \frac{4r_s^2}{(1+z_s\bar{z}_s)^2} dz_s d\bar{z}_s$ by performing the following coordinate transformation (see appendix A of [103]): $r_c = \frac{\sqrt{2}r_s}{1+z_s\bar{z}_s} + \frac{u_s}{\sqrt{2}}$, $u_c = \frac{1+z_s\bar{z}_s}{\sqrt{2}}u_s - \frac{z_s\bar{z}_s u_s^2}{2r_c}$, $z_c = z_s - \frac{z_s u_s}{\sqrt{2}r_c}$, $\bar{z}_c = \bar{z}_s - \frac{\bar{z}_s u_s}{\sqrt{2}r_c}$.

For $l = \partial_u \ln \sqrt{q} = 0$ to be satisfied (see (4.1.4)), we have to impose that $\partial_u^2 W = 0$, so W is at most linear in u. Moreover, regularity implies that $\partial_u W$ is nowhere vanishing. Therefore,

$$W(u, z^c) = \exp\left[\frac{1}{2}\Phi(z, \bar{z})\right](u + C(z, \bar{z})). \tag{5.1.7}$$

Expanding g_{AB} in powers of r as in (4.1.5), we can read the boundary metric as

$$q_{AB} = q_{AB}^{\text{vac}} \equiv e^{-\Phi}\partial_A G^a \partial_B G^b \gamma_{ab}. \tag{5.1.8}$$

It is indeed the result of a large diffeomorphism and a Weyl transformation. If one is restricted to the transformations that lead to $\sqrt{q} = \sqrt{\mathring{q}}$ (see equation (4.1.4)), we have the relation

$$|\det(\partial_A G^a)| = \frac{2}{(1 + z\bar{z})^2} e^{\Phi}. \tag{5.1.9}$$

The shear C_{AB} is found to be the trace-free (TF) part of the following tensor

$$C_{AB} = C_{AB}^{\text{vac}} \equiv \left[\frac{2}{(\partial_u W)^2}\partial_u\left(D_A W D_B W\right) - \frac{2}{\partial_u W} D_A D_B W\right]^{\text{TF}}. \tag{5.1.10}$$

Introducing (5.1.7), it becomes

$$C_{AB}^{\text{vac}}[\Phi, C] = (u + C)N_{AB}^{\text{vac}} + C_{AB}^{(0)}, \qquad \begin{cases} N_{AB}^{\text{vac}} &= \left[\frac{1}{2}D_A\Phi D_B\Phi - D_A D_B\Phi\right]^{\text{TF}}, \\ C_{AB}^{(0)} &= -2D_A D_B C + q_{AB}D^2 C. \end{cases} \tag{5.1.11}$$

We find that all explicit reference on γ_{ab} or G^a disappeared. Moreover, the news tensor of the vacua N_{AB}^{vac} is only built up with Φ. It can be checked that the boundary Ricci scalar is given in terms of Φ as

$$R = D^2\Phi, \tag{5.1.12}$$

which implies

$$D_A N_{\text{vac}}^{AB} = -\frac{1}{2}D^B R. \tag{5.1.13}$$

We can therefore add a trace to N_{AB}^{vac} to form the conserved stress-tensor

$$T_{AB}[\Phi] = \frac{1}{2}D_A\Phi D_B\Phi - D_A D_B\Phi + \frac{1}{2}q_{AB}\left(2D^2\Phi - \frac{1}{2}D^C\Phi D_C\Phi\right). \tag{5.1.14}$$

Its trace is equal to $D^2\Phi$. The tensor T_{AB} is precisely the stress-tensor of Euclidean Liouville theory

$$L[\Phi; q_{AB}] = \sqrt{q}\left(\frac{1}{2}D^A\Phi D_A\Phi + \Lambda e^{\Phi} + R[q]\Phi\right), \tag{5.1.15}$$

where the parameter Λ is zero in order to satisfy (5.1.12). To derive the stress-tensor from the Lagrangian, one must set the Liouville field off-shell by not imposing the equation (5.1.12) but considering the metric as a background field. Under a super-Lorentz transformation

$$\delta_Y(D^2\Phi - R) = (\mathcal{L}_Y + D_A Y^A)(D^2\Phi - R). \tag{5.1.16}$$

Therefore, imposing the Liouville equation is consistent with the action of super-Lorentz transformations.

Using this boundary metric and shear, one can work out the covariant expressions for R_0 and R_1 in (5.1.6). They are given by

$$R_0 = \frac{1}{2}e^{-\Phi}D^2W \qquad \text{and} \qquad R_1 = \frac{1}{8}e^{-\Phi}C_{AB}C^{AB}. \tag{5.1.17}$$

Finally, after some algebra, one can write the full metric as

$$\begin{aligned} ds^2 = &-\frac{R}{2}\mathrm{d}u^2 - 2\mathrm{d}\rho\mathrm{d}u + (\rho^2 q_{AB} + \rho C^{\text{vac}}_{AB} + \frac{1}{8}C^{\text{vac}}_{CD}C^{CD}_{\text{vac}}q_{AB})\mathrm{d}x^A\mathrm{d}x^B \\ &+ D^B C^{\text{vac}}_{AB}\mathrm{d}x^A\mathrm{d}u \end{aligned} \tag{5.1.18}$$

where $\rho = \sqrt{r^2 + \frac{1}{8}C^{\text{vac}}_{CD}C^{CD}_{\text{vac}}}$ is a derived quantity in terms of the Bondi radius r. The metric is more natural in Newman-Unti gauge (u, ρ, z^A) where $g_{\rho\mu} = -\delta^\rho_\mu$ (see (2.2.12)).

Let us also comment on the meromorphic extension of the Lorentz group instead of Diff(S^2). When super-Lorentz transformations reduce to local conformal Killing vectors on S^2, i.e. $G^z = G(z)$ and $G^{\bar{z}} \equiv \bar{G}(\bar{z})$, the boundary metric after a diffeomorphism is the unit round metric on the sphere

$$\mathring{q}_{AB}\mathrm{d}z^A\mathrm{d}z^B = 2\gamma_s\mathrm{d}z\mathrm{d}\bar{z}, \qquad \gamma_s = \frac{2}{(1+z\bar{z})^2} \tag{5.1.19}$$

(and $\mathring{R} = 2$) *except at the singular points of* $G(z)$. The Liouville field reduces to the sum of a meromorphic and an anti-meromorphic part minus the unit sphere factor

$$\Phi = \phi(z) + \bar{\phi}(\bar{z}) - \log\gamma_s. \tag{5.1.20}$$

The metric (5.1.18) then exactly reproduces the expression of [102] with the substitution $T^{(\text{there})}_{AB} = 1/2N^{\text{vac}}_{AB}$. We have therefore found the generalization of the metric of the vacua for arbitrary Diff(S^2) super-Lorentz transformations together with arbitrary supertranslations.

5.1.2 The superboost, superrotation and supertranslation fields

A general vacuum metric is parametrized by a boundary metric q^{vac}_{AB}, the field C that we call the *supertranslation field* and Φ that we will call either the *Liouville field* or the *superboost field*. Under a BMS transformation, the bulk metric transforms into itself, with the following transformation law of its boundary fields:

$$\begin{aligned} \delta_{T,Y} q^{\text{vac}}_{AB} &= D_A Y_B + D_B Y_A - q^{\text{vac}}_{AB} D_C Y^C, && (5.1.21)\\ \delta_{T,Y}\Phi &= Y^A \partial_A \Phi + D_A Y^A, && (5.1.22)\\ \delta_{T,Y} C &= T + Y^A \partial_A C - \frac{1}{2} C D_A Y^A. && (5.1.23) \end{aligned}$$

Only the divergence of a general super-Lorentz transformation sources the Liouville field. Since rotations are divergence-free but boosts are not, we call Φ the *superboost field*. In general, one can decompose a vector on the 2-sphere as a divergence and a rotational part. For a generic superrotation, there should be a field that is sourced by the rotational of Y^A. We call this field the *superrotation field* Ψ and we postulate its transformation law

$$\delta_{T,Y}\Psi = Y^A \partial_A \Psi + \varepsilon^{AB} D_A Y_B. \tag{5.1.24}$$

Where is that field in (5.1.18)? In fact, the boundary metric q^{vac}_{AB} is not a fundamental field. It depends upon the Liouville field Φ and the background metric γ_{ab}. Since it transforms under superrotations, the metric (5.1.8) should also depend upon the superrotation field Ψ. The explicit form $q^{\text{vac}}_{AB}[\gamma_{ab}, \Phi, \Psi]$ is not known to us. We call the set of boundary fields (Φ, Ψ) the super-Lorentz fields.

Under a BMS transformation, the news of the vacua N^{vac}_{AB} and the tensor $C^{(0)}_{AB}$ transform inhomogeneously as

$$\begin{aligned} \delta_{T,Y} N^{\text{vac}}_{AB} &= \mathcal{L}_Y N^{\text{vac}}_{AB} - D_A D_B D_C Y^C + \frac{1}{2} q_{AB} D^2 D_C Y^C, && (5.1.25)\\ \delta_{T,Y} C^{(0)}_{AB} &= \mathcal{L}_Y C^{(0)}_{AB} - \frac{1}{2} D_C Y^C C^{(0)}_{AB} - 2 D_A D_B T + q_{AB} D^2 T. && (5.1.26) \end{aligned}$$

From (5.1.18), one can read off the explicit expressions of the Bondi mass and angular momentum aspects of the vacua

$$\begin{aligned} M &= -\frac{1}{8} N^{\text{vac}}_{AB} C^{AB}_{\text{vac}}, \\ N_A &= -\frac{3}{32} D_A (C^{\text{vac}}_{BC} C^{BC}_{\text{vac}}) - \frac{1}{4} C^{\text{vac}}_{AB} D_C C^{BC}_{\text{vac}}. \end{aligned} \tag{5.1.27}$$

The Bondi mass is time-dependent and its spectrum is not bounded from below because $\partial_u M = -\frac{1}{8} N^{\text{vac}}_{AB} N^{AB}_{\text{vac}}$ as observed in [102]. Nonetheless, the Weyl tensor is identically zero, so the standard Newtonian potential vanishes. This indicates that the mass is identically zero. The relationship between the Bondi mass and the mass is given below in subsection 5.3.3.

5.2 Superboost transitions

The main interest of the non-trivial vacua lies in the dynamical processes that allow transitions from one vacuum to another. In what follows, we focus on the processes associated with both $\mathrm{Diff}(S^1)\times\mathrm{Diff}(S^1)$ and $\mathrm{Diff}(S^2)$ super-Lorentz extensions of the BMS_4 group. We study several examples of transitions and investigate the related memory effects at null infinity.

5.2.1 The impulsive Robinson-Trautman metric as a vacuum transition

The impulsive limit of the Robinson-Trautman type N of positive 2-curvature ($M = 0$, $R = 2$) can be rewritten after a coordinate transformation as the metric of the impulsive gravitational waves of Penrose [206, 207], as shown in [208, 209][2]

$$\begin{aligned} ds^2 =& -\mathrm{d}u^2 - 2\mathrm{d}\rho\mathrm{d}u \\ &+ (\rho^2 q_{AB} + u\rho\Theta(u)N^{\mathrm{vac}}_{AB} + \frac{u^2}{8}\Theta(u)N^{\mathrm{vac}}_{CD}N^{CD}_{\mathrm{vac}}q_{AB})\mathrm{d}x^A\mathrm{d}x^B, \end{aligned} \tag{5.2.1}$$

where $N^{\mathrm{vac}}_{AB} = \left[\frac{1}{2}D_A\phi_f D_B\phi_f - D_A D_B\phi_f\right]^{\mathrm{TF}}$. The vacuum news coincides with (5.1.11) after substituting $\Phi = -\log\gamma_s + \phi_f$ as in (5.1.20). This metric is in Newman-Unti gauge, not in Bondi gauge. It represents the transition between two vacua labelled by distinct meromorphic superboost fields[3] (initial $\phi_i = 0$ for $u < 0$ and final $\phi_f = \phi(z) + \bar{\phi}(\bar{z})$ for $u > 0$). The metric q_{AB} is the unit sphere metric globally for $u < 0$ and locally for $u > 0$ but it contains singularities at isolated points for $u > 0$. These singularities can be understood as cosmic string decays [145, 209, 210].

5.2.2 General impulsive gravitational wave transitions

In general, both the supertranslation field C and the superboost field Φ can change with hard (finite energy) processes involving null radiation reaching $\mathcal{I}^+$. Such processes induce vacuum transitions among initial (C_-, Φ_-) and final (C_+, Φ_+) boundary fields. The difference between these fields can be expressed in terms of components of the matter stress-tensor and metric potentials reaching $\mathcal{I}^+$. The simplest possible transition between vacua are shockwaves that carry a matter stress-tensor proportional to a $\delta(u)$ function, as in the original Penrose construction [206]. A

[2] It is exactly the solution (2.10) of [95] with $\varepsilon = +1$ upon substituting $U \to u/\sqrt{2}$, $V \to -\sqrt{2}\rho$, $H \to -1/2N^{\mathrm{vac}}_{zz}$. Strictly speaking, $g_{uu} = -1 - \frac{D^2\phi}{2}$ at the poles of the meromorphic function $\phi(z)$, but $g_{uu} = -1$ otherwise.

[3] The singular impulsive limit requires considering singular diffeomorphisms transitions which turn out to reduce to meromorphic superboost transitions.

distinct vacuum lies on each side of the shockwave and the transition between the boundary fields is dictated by the matter stress-tensor. Such a general shockwave takes the form

$$\begin{aligned} ds^2 = &-\frac{\mathring{R}}{2}\mathrm{d}u^2 - 2\mathrm{d}\rho\mathrm{d}u + (\rho^2 q_{AB} + \rho C_{AB} + \frac{1}{8}C_{CD}C^{CD}q_{AB})\mathrm{d}x^A\mathrm{d}x^B \\ &+ D^B C_{AB}\mathrm{d}x^A\mathrm{d}u, \end{aligned} \tag{5.2.2}$$

where

$$q_{AB} = \Theta(-u)q^{\mathrm{vac}}_{AB}[\Phi_-] + \Theta(u)q^{\mathrm{vac}}_{AB}[\Phi_+], \tag{5.2.3}$$

$$C_{AB} = \Theta(-u)C^{\mathrm{vac}}_{AB}[\Phi_-, C_-] + \Theta(u)C^{\mathrm{vac}}_{AB}[\Phi_+, C_+], \tag{5.2.4}$$

where $q^{\mathrm{vac}}_{AB}[\Phi]$ and $C^{\mathrm{vac}}_{AB}[\Phi, C]$ are given in (5.1.8) and (5.1.11). The metric (5.2.1) is recovered for $\Phi_- = -\log\gamma_s$, $\Phi_+ = -\log\gamma_s + \phi(z) + \bar{\phi}(\bar{z})$ as in (5.1.20) and $C_+ = C_- = 0$.

5.2.3 Conservation of the Bondi mass aspect and the center-of-mass

In the absence of superboost transitions and for the standard case of the unit round celestial sphere, the integral between initial u_i and final retarded times u_f of the conservation equation for the Bondi mass aspect (4.1.7) can be reexpressed as the differential equation determining the difference of supertranslation field $\Delta C = C_+ - C_-$ between initial and final retarded times after assuming suitable fall-off conditions [101]

$$-\frac{1}{4}D^2(D^2+2)\Delta C = \Delta M + \int_{u_-}^{u_+} \mathrm{d}u\, T_{uu}, \tag{5.2.5}$$

where $T_{uu} = \frac{1}{8}N_{AB}N^{AB}$ and ΔM is the difference between the Bondi mass aspects after and before the burst. The four lowest spherical harmonics $\ell = 0, 1$ are zero modes of the differential operator appearing on the left-hand side of (5.2.5). Recall that translations precisely shift the supertranslation field as (5.1.23). The four lowest harmonics of C can thus be interpreted as the center-of-mass of the asymptotically flat system. This center-of-mass is not constrained by the conservation law (5.2.5).

A new feature arises in the presence of a superboost transition. The four zero modes of the supertranslation field C are now determined by the conservation equation. This can be seen in the context of impulsive transitions (5.2.2). For simplicity, we take $C_- = 0$ and $\Phi_- = -\log\gamma_s$ ($q_{AB}[\Phi_-] = \mathring{q}_{AB}$ the unit round sphere metric). Given that the Bondi mass aspect and the Bondi news of the vacua are non-zero

(5.1.27), we first define the renormalized Bondi mass aspect and Bondi news as

$$\hat{M} = M + \frac{1}{8} C_{AB} N^{AB}_{\text{vac}}[\Phi_+], \tag{5.2.6}$$

$$\hat{N}_{AB} = N_{AB} - \Theta(u) N^{\text{vac}}_{AB}[\Phi_+], \tag{5.2.7}$$

which are zero for the vacua (5.1.18). This mass is obtained in section 5.3 in (5.3.9).

After integration over u of (4.1.7) and using the corollary of the Liouville equation (5.1.13) we obtain

$$-\frac{1}{4} D^2(D^2 + R) C_+ + \frac{1}{4} N^{AB}_{\text{vac}}[\Phi_+] D_A D_B C_+ + \frac{1}{8} C_+ D^2 R = \Delta \hat{M} + \int_{u_-}^{u_+} \mathrm{d}u \, T_{uu}, \tag{5.2.8}$$

where $T_{uu} = \frac{1}{8} \hat{N}_{AB} \hat{N}^{AB}$ and $\Delta \hat{M}$ act as sources for C_+ and all quantities are evaluated on the final metric $q_{AB}[\Phi_+]$. We have that $\Delta \hat{M} = 0$ for transitions between vacua but we included it for making the comparison with (5.2.5) more manifest.

The lowest $\ell = 0, 1$ spherical harmonics of C are not zero modes of the quartic differential operator on the left-hand side of (5.2.8) for any inhomogeously curved boundary metric. Therefore, the center-of-mass is also determined by the conservation law of the Bondi mass aspect.

5.2.4 Refraction/Velocity kick memory

We now consider the simplified case where the change of the boundary metric is localized at individual points. This happens for impulsive gravitational wave transitions that relate the initial and final boundary metric by a meromorphic super-Lorentz transformation (which is a combination of superboosts and superrotations). One example is the original Penrose construction [206]. In these cases we consider observers away from these singular points so that we can ignore these singularities.

We can consider either timelike or null geodesics leading respectively to the velocity kick and refraction memory. Let us first discuss a congruence of timelike geodesics that evolve at finite large radius r in the impulsive gravitational wave spacetime (5.2.1). Such observers have a velocity $v^\mu \partial_\mu = \partial_u + \mathcal{O}(\rho^{-1})$. The deviation vector s^μ between two neighboring geodesics obeys $\nabla_v \nabla_v s^\mu = R^\mu{}_{\alpha\beta\gamma} v^\alpha v^\beta s^\gamma$, where the directional derivative is defined as $\nabla_v = v^\mu \nabla_\mu$. We have $R_{uAuB} = -\frac{\rho}{2} \partial_u^2 C_{AB} + \mathcal{O}(\rho^0)$ where $C_{AB} = u \Theta(u) N^{\text{vac}}_{AB}$ and therefore

$$q_{AB} \partial_u^2 s^B = \frac{1}{2\rho} \delta(u) N^{\text{vac}}_{AB} s^B + \mathcal{O}(\rho^{-2}). \tag{5.2.9}$$

We deduce that $s^A = s^A_{lead}(x^A) + \frac{1}{\rho} s^A_{sub}(u, x^A) + \mathcal{O}(\rho^{-2})$ and after two integrations in u,

$$s^A_{sub} = \frac{u}{2}\Theta(u) q^{AB} N^{\text{vac}}_{BC} s^C_{lead}. \tag{5.2.10}$$

Before the shockwave, there is no relative angular velocity between observers. After the shockwave, there will be a relative angular velocity at order $\propto \rho^{-1}$. This is the velocity kick between two such neighboring geodesics due to the shockwave [95, 96, 122]. This is a qualitatively distinct effect from the displacement memory effect [83, 84, 211–213] and the spin memory effect [11, 92, 214].

Analogously, one can consider a congruence of null geodesics that admits a constant leading angular velocity $\Omega^A(x^B)\partial_A$, with total 4-velocity

$$v^\mu \partial_\mu = (\sqrt{\Omega^A q_{AB} \Omega^B} + \mathcal{O}(\rho^{-1}))\partial_u + \mathcal{O}(\rho^{-1})\partial_\rho + \frac{1}{\rho}(\Omega^A + \mathcal{O}(\rho^{-1}))\partial_A. \tag{5.2.11}$$

We consider again a deviation vector of the form $s^A = s^A_{lead}(x^A) + \frac{1}{\rho} s^A_{sub}(u, x^A) + \mathcal{O}(\rho^{-2})$. The deviation vector obeys again (5.2.10). Null geodesics are refracted by the shockwave. This is the refraction memory effect usually described in the bulk of spacetime [95, 96, 122]. We identified here the class of null geodesics that displays the refraction memory effect close to null infinity.

Let us now shortly discuss the case where the change of boundary metric is not localized at individual points. The main point is that timelike geodesics will now admit non-trivial deviation vector already at leading order $\propto \rho^0$, $s^A = s^A_{lead}(u, x^A) + \mathcal{O}(\rho^{-1})$, with

$$\frac{1}{2} q_{AB} \partial_u^2 s^B_{lead} + \frac{1}{2}\partial_u^2 (q_{AB} s^B_{lead}) \quad = \quad -\frac{1}{2}\partial_u^2 q_{AB} s^B_{lead}. \tag{5.2.12}$$

A velocity kick will therefore already occur at order ρ^0.

5.2.5 A new non-linear displacement memory

It should also be mentioned that there is a non-linear displacement memory induced by a superboost transition, when it is accompanied by a supertranslation transition. This case was not considered in [95, 96, 122], where all supertranslation transitions were vanishing. In order to describe the effect, we can consider either timelike or null geodesics. For definiteness, we consider a congruence of timelike geodesics that evolve at finite large radius ρ in the general impulsive gravitational wave spacetime (5.2.2). For simplicity we assume global Minkowski in the far past and we only consider the simplified case where the change of the boundary metric is localized at individual points. In other words, we assume $\Phi_- = -\log \gamma_s$ ($q^{\text{vac}}_{AB}[\Phi_-]$ is the unit sphere metric), $C_- = 0$, $\Phi_+ = -\log\gamma_s + \phi(z) + \bar{\phi}(\bar{z})$ and $C_+ = C_+(z,\bar{z})$ arbitrary. The velocity is now $v^\mu\partial_\mu = \sqrt{\frac{2}{R}}\partial_u + \mathcal{O}(\rho^{-1})$. We have $R_{uAuB} = -\frac{\rho}{2}\partial_u^2 C_{AB} + \mathcal{O}(\rho^0)$. Following the same procedure as above, we obtain $s^A = s^A_{lead}(x^A) + \frac{1}{\rho}s^A_{sub}(u,x^A) + \mathcal{O}(\rho^{-2})$ and away from the singular points on the sphere,

$$s^A_{sub} = \frac{1}{2}q^{AB}C_{BC}s^C_{lead}. \tag{5.2.13}$$

$$= \frac{1}{2}\Theta(u)q^{AB}C^{\text{vac}}_{BC}s^C_{lead}. \tag{5.2.14}$$

$$= \frac{1}{2}q^{AB}(u\Theta(u)N^{\text{vac}}_{BC} + \Theta(u)C^{(0)}_{BC} + \Theta(u)CN^{\text{vac}}_{BC})s^C_{lead}. \tag{5.2.15}$$

The first term $\propto u\Theta(u)$ leads to the velocity kick memory effect. The second term $\propto \Theta(u)C^{(0)}_{BC}$ leads to the displacement memory effect due to a change of supertranslation field C between the final and initial states [101]. The third and last term $\propto \Theta(u)CN^{\text{vac}}_{BC}$ is a new type of non-linear displacement memory effect due to changes of both the superboost field Φ and the supertranslation field C. The four lowest spherical harmonics $\ell = 0, 1$ of C, interpreted as the center-of-mass, do not contribute to the standard displacement memory effect because they are zero modes of the differential operator $C^{(0)}_{AB}$. Here, they do contribute to the non-linear displacement memory effect. The transition of the supertranslation field and in particular of the center-of-mass are determined by (5.2.8), as discussed earlier.

5.3 Finite charges and soft theorems

In this section, we present a prescription to extract a meaningful integrable charge from the non-integrable infinitesimal charge expression obtained in equation (4.2.25) of the previous chapter. This procedure is based on ingredients introduced in section 5.1 and is inspired by the Wald-Zoupas procedure[4] [118]. We then relate our associated finite charge to the existing literature and show that this is consistent with the soft graviton theorems, the action of asymptotic symmetries and the vanishing energy of the vacua.

5.3.1 Finite surface charges

In this analysis, we assume that the Liouville equation

$$R = D^2\Phi, \tag{5.3.1}$$

[4] Notice that the Wald-Zoupas procedure discussed in [118] could not be readily applied to our case due to the non-vanishing contribution of the term $Q^{ur}_{\delta\xi}[g]$ in (4.2.14) to the non-integrable part.

which was derived for the vacua orbit in (5.1.12), holds in our phase space. This is not guaranteed a priori for the general phase space studied in chapter 4 and is therefore an additional restriction. In particular, equation (5.1.13) will be satisfied.

Starting from (4.2.25), we want to define the finite charges H_ξ associated with ξ. Following the Wald-Zoupas procedure [118], it would be natural to request that the flux $\partial_u H_\xi[g]$ is identically zero in the absence of news. However, the news tensor transforms inhomogeneously under (both $\mathrm{Diff}(S^1)\times\mathrm{Diff}(S^1)$ and $\mathrm{Diff}(S^2)$) super-Lorentz transformations so this condition is not invariant under the action of the asymptotic symmetry group. Instead, we request that the flux $\partial_u H_\xi[g]$ is identically zero in the absence of shifted news $\hat{N}_{AB}$,

$$\hat{N}_{AB} = N_{AB} - N^{\text{vac}}_{AB}. \tag{5.3.2}$$

Since the latter transforms homogeneously under super-Lorentz transformations, this prescription is *invariant* under the action of all asymptotic symmetries. For future use, we define the shifted $\hat{C}_{AB}$ tensor

$$\hat{C}_{AB} = C_{AB} - uN^{\text{vac}}_{AB}. \tag{5.3.3}$$

such that $\partial_u \hat{C}_{AB} = \hat{N}_{AB}$. To obtain our ansatz, let us start with the charge (4.2.27). The flux associated with (4.2.27) reads as

$$\begin{aligned}\partial_u H^{int}_\xi[g] = -\frac{1}{32\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\,\big[&fN_{AB}N^{AB} - 2fD_AD_BN^{AB} - fD_AD^AR - Y^A\mathcal{H}_A(N,C)\\ &+Y^AD_BD^BD^CC_{AC} - Y^AD_BD_AD_CC^{BC} - Y^AC_{AB}D^BR\big]\end{aligned} \tag{5.3.4}$$

Here, we defined for later convenience the bilinear operator on rank-2 spherical traceless tensors P_{AB} and Q_{AB}:

$$\mathcal{H}_A(P,Q) \equiv \frac{1}{2}\partial_A(P_{BC}Q^{BC}) - P^{BC}D_AQ_{BC} + D_B(P^{BC}Q_{AC} - Q^{BC}P_{AC}) \tag{5.3.5}$$

which enjoys the property $\mathcal{H}_A(P,P) = 0$. When $\hat{N}_{AB} = 0$, we are left with

$$\begin{aligned}\partial_u H^{int}_\xi|_{\hat{N}_{AB}=0} = -\frac{1}{32\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\,\big[&fN^{\text{vac}}_{AB}N^{AB}_{\text{vac}} - Y^A\mathcal{H}_A(N^{\text{vac}},C) - Y^AC_{AB}D^BR\\ &+Y^AD_BD^BD^CC_{AC} - Y^AD_BD_AD_CC^{BC}\big]\end{aligned} \tag{5.3.6}$$

after using the relation (5.1.13), which follows from our additional boundary condition (5.3.1). We now want to define a counterterm that is only built out of the fields at $\mathscr{I}^+$ (q_{AB}, C_{AB}, N_{AB}) and out of N^{vac}_{AB}, which is the only boundary field that appears in the condition (5.3.2). Our prescription that cancels the right-hand side of (5.3.6) is

$$\begin{aligned}\Delta H_\xi[g;\delta g] \equiv{}& \frac{1}{16\pi G}\int_{S^2_\infty} \mathrm{d}^2\Omega\Big[\frac{u}{2}Y^A D_B D^B D^C C_{AC} - \frac{u}{2}Y^A D_B D_A D_C C^{BC} \\ &-\frac{u}{2}Y^A C_{AB}D^B R + \frac{1}{2}T C_{AB}N^{AB}_{\text{vac}} - \frac{u}{2}Y^A \mathcal{H}_A(N^{\text{vac}},C) \\ &+\frac{u^2}{8}D_C Y^C N^{\text{vac}}_{AB}N^{AB}_{\text{vac}} + \frac{u^2}{4}Y^A N^{\text{vac}}_{AB}D^B R\Big]. \end{aligned} \tag{5.3.7}$$

This is the minimal ansatz that cancels the right-hand side of (5.3.6). Of course, there is considerable ambiguity in defining $\Delta H_\xi[g;\delta g]$. We will justify our minimal choice in (5.3.7) by showing consistency with the leading and subleading soft theorems, and for defining the charges of the vacua.

Our final prescription for the canonical charges is $H_\xi[g] = H^{int}_\xi[g] + \Delta H_\xi[g]$. The charges are conveniently written as

$$H_\xi[g] = \frac{1}{16\pi G}\int_{S^2_\infty}\mathrm{d}^2\Omega\left[4T\hat{M} + 2Y^A\hat{N}_A\right] \tag{5.3.8}$$

where the final mass and angular momentum aspects are given by

$$\hat{M} = M + \frac{1}{8}C_{AB}N^{AB}_{\text{vac}}; \tag{5.3.9}$$

$$\begin{aligned}\hat{N}_A ={}& N_A - u\partial_A M + \frac{1}{32}\partial_A(\hat{C}_{CD}\hat{C}^{CD}) + \frac{u}{16}\partial_A(\hat{C}^{CD}N^{\text{vac}}_{CD}) - \frac{1}{32}u^2\partial_A(N^{\text{vac}}_{BC}N^{BC}_{\text{vac}}) \\ &-\frac{u}{4}\mathcal{H}_A(N^{\text{vac}},\hat{C}) - \frac{u}{4}\hat{C}^B_A D_B\mathring{R} + \frac{u}{4}D_B D^B D_C\hat{C}^C_A - \frac{u}{4}D_B D_A D^C\hat{C}^B_C - \frac{u^2}{8}N^{\text{vac}}_{AB}D^B\mathring{R}.\end{aligned} \tag{5.3.10}$$

This is a new prescription for the charges. In the standard asymptotically flat spacetimes where the boundary metric is the round sphere ($q_{AB} = \mathring{q}_{AB}$ with $\mathring{R} = 2$), our expressions reduce to

$$\begin{aligned}&\hat{M} = M; \\ &\hat{N}_A = N_A - u\partial_A M + \frac{1}{32}\partial_A(C_{CD}C^{CD}) + \frac{u}{4}D_B D^B D^C C_{AC} - \frac{u}{4}D_B D_A D_C C^{BC}.\end{aligned} \tag{5.3.11}$$

The Lorentz charges differ from the existing prescriptions [35, 121, 169] since the angular momentum aspect is now enhanced with the two soft terms linear in u. We will show that our prescription correctly reproduces the fluxes needed for the subleading soft theorem. Furthermore, our expressions are exactly those needed for the BMS flux balance laws discussed in [215].

Let us mention that another interesting prescription for $\Delta H_\xi[g;\delta g]$ has been proposed in [193]. The finite charges considered in that reference have two interesting properties: (i) the charges of the vacua are all vanishing, and (ii) the charges represent $\mathfrak{bms}_4^{\text{gen}}$ without central extension at the corners of null infinity under the standard Dirac bracket. We refer to [193] for more details about this prescription.

5.3.2 Flux formulae for the soft Ward identities

Let us show that our expressions for the fluxes reproduce the expressions of the literature used in the Ward identities displaying the equivalence to the leading [70] and subleading [77] soft graviton theorems. The final flux can be decomposed in soft and hard parts, where the soft terms (resp. hard terms) are linear (resp. quadratic) in $\hat{C}_{AB}$ or its time variation $\hat{N}_{AB}$. We have

$$\int_{\mathscr{I}^+} \mathrm{d}u\, \partial_u H_\xi[g] = Q_S[T] + Q_H[T] + Q_S[Y] + Q_H[Y] \tag{5.3.12}$$

where

$$Q_S[T] = \frac{1}{16\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega\; \partial_u \left(T D_A D_B \hat{C}^{AB}\right), \tag{5.3.13}$$

$$Q_H[T] = \frac{1}{16\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega \left(-\frac{1}{2} T \hat{N}_{AB} N^{AB}\right), \tag{5.3.14}$$

$$Q_S[Y] = \frac{1}{16\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega\; u\, \partial_u \left(\hat{C}^{AB} s_{AB}\right), \tag{5.3.15}$$

$$Q_H[Y] = \frac{1}{16\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega \left(\frac{1}{2} Y^A \mathcal{H}_A(\hat{N},\hat{C}) + \frac{u}{2} Y^A N^D_C D_A \hat{N}^C_D + \frac{u}{2} N^{CD}_{\text{vac}} Y^A D_A \hat{N}_{CD}\right) \tag{5.3.16}$$

and

$$s_{AB} = \left[D_A D_B D_C Y^C + \frac{\mathring{R}}{2} D_{(A} Y_{B)} - \frac{1}{2} D_{(A}(D^2 + \frac{\mathring{R}}{2}) Y_{B)}\right]^{\text{TF}} \tag{5.3.17}$$

after integrations by parts on the sphere.

In the standard case where $N^{\text{vac}}_{AB} = 0$, the flux of supermomenta reproduces (2.11) of [100] up to a conventional overall sign, which itself agrees with previous results [137]. After one imposes the antipodal matching condition on $\hat{M}$ at spatial infinity, one can equate the flux on $\mathscr{I}^+$ with the antipodally related flux on $\mathscr{I}^-$. The result of [100] is precisely that the quantum version of this identity is the Ward identity of the leading soft graviton theorem. We have now obtained a generalization in the presence of superboost background flux.

We now consider the hard terms for super-Lorentz transformations. Using the identity

$$\frac{1}{2}(\hat{N}^{AC}\hat{C}_{BC} + \hat{C}^{AB}\hat{N}_{BC}) = \frac{1}{2}(\hat{N}^{BC}\hat{C}_{BC})\delta^A_B \tag{5.3.18}$$

and integrating by parts, it can be shown that (5.3.16) can be rewritten as

$$\begin{aligned} Q_H[Y] &= \frac{1}{16\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega \left[-\frac{1}{2}\hat{N}^{AB}\left(\mathcal{L}_Y\hat{C}_{AB} - \frac{1}{2}D_C Y^C \hat{C}_{AB} + \frac{u}{2}D_C Y^C \hat{N}_{AB}\right) \right. \\ &\qquad\qquad \left. + u N^{AB}_{\text{vac}} Y^C D_C \hat{N}_{AB} \right] \\ &= -\frac{1}{32\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega \left[\hat{N}^{AB}\delta^H_Y \hat{C}_{AB} - 2u N^{AB}_{\text{vac}} Y^C D_C \hat{N}_{AB} \right] \end{aligned} \tag{5.3.19}$$

where δ^H_Y is the homogeneous part of the transformation of $\hat{C}_{AB}$. After restricting to standard configurations where $N^{\text{vac}}_{AB} = 0$, the expression matches (up to the overall conventional sign) with equation (40) of [13].

Next, we consider the soft terms for super-Lorentz transformations. Noting that $D^C \delta_Y q_{AC} = D_C D^C Y_A + \frac{R}{2} Y_A$ we can rewrite (5.3.17) as

$$s_{AB} = \left[D_A D_B D_C Y^C + \frac{R}{2} D_{(A} Y_{B)} - \frac{1}{2} D_{(A} D^C \delta_Y q_{B)C} \right]^{\text{TF}}. \tag{5.3.20}$$

The tensor s_{AB} is recognized as the generalization of equation (47) of [13] in the presence of non-trivial boundary curvature. After some algebra, we can rewrite it in terms of the inhomogeneous part $\delta^I_Y C_{AB}$ of the transformation law of C_{AB} (4.1.22):

$$-u s_{AB} = \delta^I_Y C_{AB} \equiv -u(D_A D_B D_C Y^C + \frac{1}{2} q_{AB} D_C D^C D_E Y^E). \tag{5.3.21}$$

Now that we identified our expressions with the ones of [13], we can use their results. After imposing the antipodal matching condition on $\hat{N}_A$ at spatial infinity, one can equate the flux of super-Lorentz charge on $\mathscr{I}^+$ with the antipodally related flux on $\mathscr{I}^-$ as originally proposed in [35] (but where the expression for $\hat{N}_A$ should be modified to (5.3.11)). The result of [13] is precisely that this identity is the Ward identity of the subleading soft graviton theorem [77].

We end up with two further comments. Note that the soft charges for super-Lorentz transformations agree with equation (41) of [13] (up to an overall conventional sign) after an integration by parts on u and after using the restrictive boundary condition $\hat{C}_{AB} = o(u^{-1})$,

$$\begin{aligned} \frac{\sqrt{q}}{16\pi G}\int \mathrm{d}u\, \hat{C}^{AB} s_{AB} &= \frac{\sqrt{q}}{16\pi G}\left[u\hat{C}^{AB} s_{AB} \right]^{\mathcal{I}^+_+}_{\mathcal{I}^+_-} - \frac{\sqrt{q}}{16\pi G}\int \mathrm{d}u\, (u\hat{N}^{AB} s_{AB}) \\ &= -\frac{\sqrt{q}}{16\pi G}\int \mathrm{d}u\, (u\hat{N}^{AB} s_{AB}) = -Q_S[Y]. \end{aligned} \tag{5.3.22}$$

However, the boundary condition $\hat{C}_{AB} = o(u^{-1})$ is not justified since displacement memory effects lead to a shift of C, e.g. in binary black hole mergers. Therefore, using more general boundary conditions, the valid expression for the soft charge is only given by (5.3.15).

Considering only the background Minkowski spacetime ($q_{AB} = \mathring{q}_{AB}$ the unit metric on the 2-sphere and $N_{AB}^{\text{vac}} = 0$), one can check that in stereographic coordinates one has $s_{zz} = \partial_z^3 Y^z = D_z^3 Y^z$. The soft charge then reads as

$$Q_S[Y] = \frac{1}{16\pi G} \int_{\mathscr{I}^+} \mathrm{d}u \mathrm{d}^2 z\, \gamma_s \, (u\hat{N}^{zz} D_z^3 Y^z + u\hat{N}^{\bar{z}\bar{z}} D_{\bar{z}}^3 Y^{\bar{z}}) \tag{5.3.23}$$

where we keep $Y^A\partial_A = Y^z(z,\bar{z})\partial_z + Y^{\bar{z}}(z,\bar{z})\partial_{\bar{z}}$ arbitrary. In the case of meromorphic super-Lorentz transformations, this reproduces equation (5.3.17) of [10] (up to a conventional global sign). It shows that the Ward identities of supertranslations and super-Lorentz transformations are equivalent to the leading and subleading soft graviton theorems following the arguments of [12, 100].

5.3.3 Charges of the vacua

Using the values (5.1.27) in our prescription (5.3.8) we deduce the mass and angular momenta of the vacua

$$H_\xi^{\text{vac}}[\Phi, C] = \frac{1}{8\pi G} \int_{S^2_\infty} \mathrm{d}^2\Omega [2T\hat{M}^{\text{vac}} + Y^A \hat{N}_A^{\text{vac}}] \tag{5.3.24}$$

where

$$\begin{aligned} \hat{M}^{\text{vac}} &= 0, \\ \hat{N}_A^{\text{vac}} &= -\frac{1}{4}\hat{C}_{AB} D_C \hat{C}^{BC} - \frac{1}{16}\partial_A(\hat{C}_{CD}\hat{C}^{CD}), \end{aligned} \tag{5.3.25}$$

and $\hat{C}_{AB} = CN_{AB}^{\text{vac}} - 2D_A D_B C + q_{AB} D^C D_C C$ in this case.

The supermomenta are all identically vanishing. Remember that the Lorentz generators are uniquely defined as the six global solutions Y^A to the conformal Killing equation $D_A Y_B + D_B Y_A = q_{AB} D_C Y^C$. In general, the Lorentz charges as well as the super-Lorentz charges are non-vanishing.

For the round sphere metric $q_{AB} = \mathring{q}_{AB}$ ($\Phi = -\log\gamma_s$), we have $\mathring{R} = 2$, $N_{AB}^{\text{vac}} = 0$ and $D^B \hat{C}_{AB} = D^B C_{AB}^{(0)} = -D_A(D^2 + 2)C$. The charges then reduce to

$$H_\xi^{\text{vac}}[C] = \frac{1}{8\pi G} \int_{S^2_\infty} \mathrm{d}^2\Omega \left[T \times 0 + Y^A \Big(-\frac{1}{4} C_{AB}^{(0)} D_C C_{(0)}^{BC} - \frac{1}{16}\partial_A (C_{CD}^{(0)} C_{(0)}^{CD}) \Big) \right]. \tag{5.3.26}$$

As shown in appendix A.3 of [102], the Lorentz charges are identically zero. The difference of charges between our prescription and the one of [102] are the last two terms of (5.3.11), which exactly cancel for the vacua with a round sphere boundary metric. Therefore, we confirm that the vacua with only the supertranslation field turned on do not carry Lorentz charges. The super-Lorentz charges are conserved and non-vanishing in general, which allows us to distinguish the vacua.

5.4 Discussion

Supertranslation BMS symmetry, the leading soft graviton theorem and the displacement memory effect form three corners of a triangle describing the leading infrared structure of asymptotically flat spacetimes at null infinity [10]. The three edges of the triangle can be described in the language of vacuum transitions, Ward identities and Fourier transforms. In the case of super-Lorentz BMS symmetry, it seems that this network of relations is more subtle. Indeed, while the connection among super-Lorentz symmetry, subleading soft theorem and spin memory effect has been established [9,11–13,92], we have shown in this chapter that another memory effect associated with superboosts appeared at the leading order metric at null infinity. More precisely, we clarified how the superboost transitions lead to the refraction or velocity kick memory effect at null infinity. We also described a nonlinear displacement memory effect that occurs in the case of combined superboost and supertranslation transitions. Finally, we obtained a new definition of the angular momentum for standard asymptotically flat spacetimes that is consistent with the fluxes required for the subleading soft graviton theorem.

Chapter 6

Λ-BMS$_4$ group

In this chapter, we investigate a new set of boundary conditions in asymptotically locally (A)dS$_4$ spacetime which is such that the associated asymptotic symmetry algebra is infinite-dimensional and reduces to the generalized BMS algebra $\mathfrak{bms}_4^{\text{gen}}$ in the flat limit ($\Lambda \to 0$). For this reason, we call this new asymptotic symmetry algebra the Λ-BMS$_4$ algebra[1] and we write it as $\mathfrak{bms}_4^\Lambda$.

In section 6.1, we investigate the most general solution spaces of three-dimensional general relativity in both Fefferman-Graham and Bondi gauges in asymptotically locally (A)dS$_3$ spacetime. We construct the explicit diffeomorphism between the two gauges and identify their solution space. Imposing the Dirichlet boundary conditions, we show how the associated asymptotic symmetry group $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ reduces to $\mathfrak{bms}_3 = \mathfrak{diff}(S^1) \inplus_{\text{ad}} \mathfrak{vect}(S^1)$ in the flat limit.

After this warm-up, in section 6.2, we repeat the analysis in four-dimensional asymptotically locally (A)dS$_4$ spacetime. We derive the most general solution spaces in Fefferman-Graham and Bondi gauges and construct the explicit diffeomorphism that maps one to the other. Then we propose a new set of boundary conditions that leads to the $\mathfrak{bms}_4^\Lambda$ asymptotic symmetry algebra and we show how it reduces to $\mathfrak{bms}_4^{\text{gen}}$ in the flat limit.

In section 6.3, repeating the holographic renormalization procedure, we construct the phase space and derive the associated co-dimension 2 form for the most general solution space in Fefferman-Graham gauge. In section 6.4, imposing the new set of partial Dirichlet boundary conditions, we find the $\mathfrak{bms}_4^\Lambda$ phase space. In the flat limit, we exactly recover the regularized phase space of section 4.2.

Finally, in section 6.5, we restrict the analysis to the case $\Lambda < 0$ and require that the symplectic flux vanishes at infinity to have a globally hyperbolic spacetime. This is done by imposing Neumann-type boundary conditions, in addition to the partial Dirichlet boundary conditions already imposed. The associated asymptotic symme-

[1] As we discuss in this chapter, $\mathfrak{bms}_4^\Lambda$ is not strictly speaking a Lie algebra, but a Lie algebroid.

try algebra is an infinite-dimensional subalgebra of $\mathfrak{bms}_4^\Lambda$ formed by the direct sum between the area preserving diffeomorphisms and the abelian time translations. We show that the phase space contains interesting solutions, including a new stationary and axisymmetric solution different from Kerr-AdS$_4$.

This chapter has strong intersections with [166, 193], except for section 6.1, which partially reproduces [216, 217].

6.1 Bondi and Fefferman-Graham gauges in three dimensions

In this section, we present the Fefferman-Graham and Bondi gauges in three dimensions following this general pattern: off-shell definition of the gauge, residual gauge diffeomorphisms, solution space and on-shell variation of the solution space. This analysis follows the logic discussed in section 2.2 and particularizes it to the three-dimensional case. In Bondi gauge, the results that we obtain generalize previous considerations by allowing an arbitrary boundary structure encoding the notion of asymptotically locally (A)dS$_3$ spacetime. Furthermore, we construct the explicit diffeomorphism that maps one gauge to the other. We finish this section with a discussion on the asymptotic symmetries aspects and investigate the flat limit in the Bondi gauge.

6.1.1 Fefferman-Graham gauge in 3d

Definition

In the Fefferman-Graham gauge (2.2.8), the metric is given by

$$\mathrm{d}s^2 = -\frac{1}{\Lambda\rho^2}\mathrm{d}\rho^2 + \gamma_{ab}(\rho, x)\mathrm{d}x^a\mathrm{d}x^b, \tag{6.1.1}$$

with coordinates (ρ, x^a), $x^a = (t, \phi)$, and the boundary located at $\rho = 0$. The three gauge fixing conditions are

$$g_{\rho\rho} = -\frac{1}{\Lambda\rho^2}, \quad g_{\rho a} = 0. \tag{6.1.2}$$

Residual gauge diffeomorphisms ξ, namely diffeomorphisms that preserve the gauge fixing conditions (6.1.2), satisfy

$$\mathcal{L}_\xi g_{\rho\rho} = 0, \quad \mathcal{L}_\xi g_{\rho a} = 0. \tag{6.1.3}$$

The explicit solution of these equations is given by

$$\xi^\rho = \rho\sigma(x), \quad \xi^a = \xi^a_0(x) + \frac{1}{\Lambda}\partial_b\sigma\int_0^\rho \frac{\mathrm{d}\rho'}{\rho'}\gamma^{ab}(\rho', x), \tag{6.1.4}$$

where $\sigma(x)$ and $\xi^a_0(x)$ are arbitrary functions of x^a.

Solution space

We impose the preliminary boundary condition $\gamma_{ab} = \mathcal{O}(\rho^{-2})$. Solving the three-dimensional Einstein equations leads to the analytic finite expansion

$$\gamma_{ab}(\rho, x) = \rho^{-2} g^{(0)}_{ab}(x) + g^{(2)}_{ab}(x) + \rho^2 g^{(4)}_{ab}(x), \tag{6.1.5}$$

where $g^{(4)}_{ab}$ is determined by $g^{(0)}_{ab}$ and $g^{(2)}_{ab}$ as

$$g^{(4)}_{ab} = \frac{1}{4} g^{(2)}_{ac} g^{cd}_{(0)} g^{(2)}_{db}. \tag{6.1.6}$$

On the other hand, the Einstein equations leave $g^{(2)}_{ab}$ unspecified up to its trace $\mathrm{Tr}\left[g^{(2)}\right] = \frac{1}{2\Lambda}R^{(0)}$ and the dynamical constraint $D^a_{(0)} g^{(2)}_{ab} = \frac{1}{2\Lambda} g^{(0)}_{ab} D^a_{(0)} R^{(0)}$. Here, $D^a_{(0)}$ is the covariant derivative with respect to $g^{(0)}_{ab}$ and indices are lowered and raised by $g^{(0)}_{ab}$ and its inverse. Motivated by the holographic dictionary [218,219], we define the holographic energy-momentum tensor

$$\begin{aligned} T_{ab} &= \frac{\sqrt{|\Lambda|}}{8\pi G}\left(g^{(2)}_{ab} - g^{(0)}_{ab}\mathrm{Tr}\left[g^{(2)}\right]\right) \\ &= \frac{\sqrt{|\Lambda|}}{8\pi G}\left(g^{(2)}_{ab} - \frac{1}{2\Lambda} g^{(0)}_{ab} R^{(0)}\right). \end{aligned} \tag{6.1.7}$$

Therefore the Einstein equations infer

$$T_a{}^a = \eta\frac{c}{24\pi}R^{(0)}, \quad D^a_{(0)}T_{ab} = 0, \tag{6.1.8}$$

where $\eta = -\mathrm{sgn}(\Lambda)$ and $c = \frac{3}{2G|\Lambda|}$ is the three-dimensional Brown–Henneaux central charge [14, 220].

The solution space is thus characterized by five arbitrary functions of x^a. Three are in the symmetric tensor $g^{(0)}_{ab}$ and two in the symmetric tensor T_{ab} with constrained trace. These data are subject to two dynamical equations given by $D^a_{(0)}T_{ab} = 0$.

Variation of the solution space

The residual gauge diffeomorphisms (6.1.4) evaluated on-shell are given by

$$\xi^\rho = \sigma\rho, \quad \xi^a = \xi^a_0 + \frac{\rho^2}{2\Lambda} g^{ab}_{(0)}\partial_b\sigma - \frac{\rho^4}{4\Lambda} g^{ac}_{(0)} g^{(2)}_{cd} g^{db}_{(0)}\partial_b\sigma + \mathcal{O}(\rho^6). \tag{6.1.9}$$

Under these residual gauge diffeomorphisms, the unconstrained part of the solution space transforms as

$$\delta_\xi g^{(0)}_{ab} = \mathcal{L}_{\xi_0} g^{(0)}_{ab} - 2\sigma g^{(0)}_{ab} \tag{6.1.10}$$

while the constrained part transforms as

$$\delta_\xi g^{(2)}_{ab} = \mathcal{L}_{\xi_0} g^{(2)}_{ab} + \frac{1}{2\Lambda}\mathcal{L}_{\partial\sigma} g^{(0)}_{ab}, \tag{6.1.11}$$

from which one can extract the variation of T_{ab}.

6.1.2 Bondi gauge in 3d

We now repeat this analysis for the Bondi gauge and extend the results of [6, 17] to asymptotically locally (A)dS$_3$ space-times by including the boundary metric in the solution space.

Definition

In the Bondi gauge (2.2.10), the metric is given by

$$\mathrm{d}s^2 = \frac{V}{r} e^{2\beta}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + r^2 e^{2\varphi}(\mathrm{d}\phi - U\mathrm{d}u)^2, \tag{6.1.12}$$

with coordinates (u, r, ϕ). In this expression, V, β and U are functions of (u, r, ϕ), and φ is a function of (u, ϕ). The three gauge fixing conditions are

$$g_{rr} = 0, \quad g_{r\phi} = 0, \quad g_{\phi\phi} = r^2 e^{2\varphi}. \tag{6.1.13}$$

Note that $g_{\phi\phi} = r^2 e^{2\varphi}$ is the unique solution of the determinant condition

$$\partial_r\left(\frac{g_{\phi\phi}}{r^2}\right) = 0, \tag{6.1.14}$$

which can be generalized to define the Bondi gauge in higher dimensions (see (2.2.10) and appendix B).

The residual gauge diffeomorphisms ξ preserving the Bondi gauge fixing (6.1.13) have to satisfy the three conditions

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{r\phi} = 0, \quad g^{\phi\phi}\mathcal{L}_\xi g_{\phi\phi} = 2\omega(u, \phi). \tag{6.1.15}$$

The explicit solution of these equations is given by

$$\xi^u = f, \tag{6.1.16}$$

$$\xi^\phi = Y - \partial_\phi f \, e^{-2\varphi} \int_r^{+\infty} \frac{\mathrm{d}r'}{r'^2} e^{2\beta}, \tag{6.1.17}$$

$$\xi^r = -r[\partial_\phi \xi^\phi - \omega - U\partial_\phi f + \xi^\phi \partial_\phi \varphi + f\partial_u \varphi], \tag{6.1.18}$$

where $f(u,\phi)$, $Y(u,\phi)$ and $\omega(u,\phi)$ are arbitrary functions of (u,ϕ).

Solution space

In this section, we discuss the most general solution space for three-dimensional general relativity in Bondi gauge. This analysis is new and generalizes the results of [6]. Interestingly, we do not have to impose any preliminary boundary condition here. This is in contrast with the procedure followed in the Fefferman-Graham gauge. Therefore, in three dimensions, the gauge conditions (6.1.13) are to some extent stronger than those imposed to define the Fefferman-Graham gauge (6.1.2).

First we impose the Einstein equations leading to the metric radial constraints. Solving $G_{rr} + \Lambda g_{rr} = R_{rr} = 0$ gives

$$\beta = \beta_0(u,\phi). \tag{6.1.19}$$

The equation $G_{r\phi} + \Lambda g_{r\phi} = R_{r\phi} = 0$ leads to

$$U = U_0(u,\phi) + \frac{1}{r} 2e^{2\beta_0} e^{-2\varphi} \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N(u,\phi). \tag{6.1.20}$$

Eventually, $G_{ur} + \Lambda g_{ur} = 0$ gives

$$\frac{V}{r} = \Lambda r^2 e^{2\beta_0} - 2r(\partial_u \varphi + D_\phi U_0) + M(u,\phi) + \frac{1}{r} 4e^{2\beta_0} e^{-2\varphi} N \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N^2, \tag{6.1.21}$$

where $D_\phi U_0 = \partial_\phi U_0 + \partial_\phi \varphi U_0$. Taking into account the previous results, the Einstein equation $G_{\phi\phi} + \Lambda g_{\phi\phi} = 0$ is automatically satisfied at all orders.

We now solve the Einstein equations to get the time evolution constraints on M and N. The equation $G_{u\phi} + \Lambda g_{u\phi} = 0$ returns

$$\begin{aligned}(\partial_u + \partial_u \varphi) N = {}& \left(\frac{1}{2} \partial_\phi + \partial_\phi \beta_0 \right) M - 2N \partial_\phi U_0 - U_0 (\partial_\phi N + N \partial_\phi \varphi) \\ &+ 4e^{2\beta_0 - 2\varphi} [2(\partial_\phi \beta_0)^3 - (\partial_\phi \varphi)(\partial_\phi \beta_0)^2 + (\partial_\phi \beta_0)(\partial_\phi^2 \beta_0)]. \end{aligned} \tag{6.1.22}$$

Moreover, $G_{uu} + \Lambda g_{uu} = 0$ imposes

$$\begin{aligned}\partial_u M = &(-2\partial_u\varphi + 2\partial_u\beta_0 - 2\partial_\phi U_0 + U_0 2\partial_\phi\beta_0 - U_0 2\partial_\phi\varphi - U_0\partial_\phi)M \\ &- 2\Lambda e^{4\beta_0-2\varphi}[\partial_\phi N + N(4\partial_\phi\beta_0 - \partial_\phi\varphi)] \\ &- 2e^{2\beta_0-2\varphi}\{\partial_\phi U_0[8(\partial_\phi\beta_0)^2 - 4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 4\partial_\phi^2\beta_0 - 2\partial_\phi^2\varphi] - \partial_\phi^3 U_0 \\ &+ U_0[\partial_\phi\beta_0(8\partial_\phi^2\beta_0 - 2\partial_\phi^2\varphi) + \partial_\phi\varphi(-2\partial_\phi^2\beta_0 + \partial_\phi^2\varphi) + 2\partial_\phi^3\beta_0 - \partial_\phi^3\varphi] \\ &+ 2\partial_u\partial_\phi\beta_0(4\partial_\phi\beta_0 - \partial_\phi\varphi) + \partial_u\partial_\phi\varphi(-2\partial_\phi\beta_0 + \partial_\phi\varphi) + 2\partial_u\partial_\phi^2\beta_0 - \partial_u\partial_\phi^2\varphi\}\end{aligned} \tag{6.1.23}$$

The solution space is thus characterized by five arbitrary functions of (u, ϕ), given by β_0, U_0, M, N, φ, with two dynamical constraints expressing the time evolution of M and N. This counting argument is in agreement with the results obtained by solving the Einstein equations in the Fefferman-Graham gauge. The precise matching between the two solution spaces is established in section 6.1.3.

Variation of the solution space

The residual gauge diffeomorphisms (6.1.16–6.1.18) evaluated on-shell are given by

$$\xi^u = f, \tag{6.1.24}$$

$$\xi^\phi = Y - \frac{1}{r}\partial_\phi f\, e^{2\beta_0-2\varphi}, \tag{6.1.25}$$

$$\begin{aligned}\xi^r = &- r[\partial_\phi Y - \omega - U_0\partial_\phi f + Y\partial_\phi\varphi + f\partial_u\varphi] \\ &+ e^{2\beta_0-2\varphi}(\partial_\phi^2 f - \partial_\phi f\partial_\phi\varphi + 4\partial_\phi f\partial_\phi\beta_0) - \frac{1}{r}e^{2\beta_0-2\varphi}\partial_\phi f\, N.\end{aligned} \tag{6.1.26}$$

Under these residual gauge diffeomorphisms, the unconstrained part of the solution space transforms as

$$\delta_\xi\varphi = \omega, \tag{6.1.27}$$

$$\delta_\xi\beta_0 = (f\partial_u + Y\partial_\phi)\beta_0 + \left(\frac{1}{2}\partial_u - \frac{1}{2}\partial_u\varphi + U_0\partial_\phi\right)f - \frac{1}{2}(\partial_\phi Y + Y\partial_\phi\varphi - \omega), \tag{6.1.28}$$

$$\delta_\xi U_0 = (f\partial_u + Y\partial_\phi - \partial_\phi Y)U_0 - \left(\partial_u Y + \Lambda e^{4\beta_0}e^{-2\varphi}\partial_\phi f\right) + U_0(\partial_u f + U_0\partial_\phi f), \tag{6.1.29}$$

while the constrained part transforms as

$$\begin{aligned}
\delta_\xi N = {} & (f\partial_u + Y\partial_\phi + 2\partial_\phi Y + f\partial_u\varphi + Y\partial_\phi\varphi - \omega - 2U_0\partial_\phi f)N \\
& + M\partial_\phi f - e^{2\beta_0 - 2\varphi}[3\partial_\phi^2 f(2\partial_\phi\beta_0 - \partial_\phi\varphi) + \partial_\phi^3 f \\
& + \partial_\phi f(4(\partial_\phi\beta_0)^2 - 8\partial_\phi\beta_0\partial_\phi\varphi + 2(\partial_\phi\varphi)^2 + 2\partial_\phi^2\beta_0 - \partial_\phi^2\varphi)], \qquad (6.1.30) \\
\delta_\xi M = {} & -4\Lambda\partial_\phi f e^{4\beta_0 - 2\varphi}N + (f\partial_u + \partial_u f + f\partial_u\varphi + Y\partial_\phi + \partial_\phi Y + Y\partial_\phi\varphi - \omega)M \\
& - 2e^{2\beta_0 - 2\varphi}\Big[2\partial_\phi^2 f\partial_u\beta_0 + 4\partial_u\partial_\phi f\partial_\phi\beta_0 + \partial_u\partial_\phi^2 f + \partial_\phi^2 f\partial_\phi U_0 + 8\partial_\phi^2 f\partial_\phi\beta_0 U_0 \\
& + \partial_\phi f\Big((4\partial_\phi\beta_0 - \partial_\phi\varphi)(2\partial_u\beta_0 - \partial_u\varphi) + 4\partial_u\partial_\phi\beta_0 + \partial_\phi U_0(8\partial_\phi\beta_0 - 2\partial_\phi\varphi) \\
& - \partial_\phi^2 U_0 - 2\partial_u\partial_\phi\varphi + U_0(-4\partial_\phi\beta_0\partial_\phi\varphi + 8(\partial_\phi\beta_0)^2 + 4\partial_\phi^2\beta_0 + (\partial_\phi\varphi)^2 - 2\partial_\phi^2\varphi)\Big) \\
& - 2\partial_\phi^2 f U_0\partial_\phi\varphi + \partial_\phi^3 f U_0 - \partial_u\partial_\phi f\partial_\phi\varphi - \partial_\phi^2 f\partial_u\varphi - 2f\partial_\phi\beta_0\partial_u\partial_\phi\varphi \\
& + 2\partial_\phi\beta_0\partial_\phi\omega - 2\partial_\phi\beta_0\partial_\phi Y\partial_\phi\varphi - 2\partial_\phi\beta_0\partial_\phi^2 Y - 2\partial_\phi\beta_0 Y\partial_\phi^2\varphi\Big]. \qquad (6.1.31)
\end{aligned}$$

These are the most general variations of the solution space in the Bondi gauge. They are key ingredients in the computation of the asymptotic charge algebra.

6.1.3 Gauge matching

In this section, we perform an explicit diffeomorphism to go from the Bondi gauge to the Fefferman-Graham gauge (we refer to appendix D for the four-dimensional analogous discussion). This will enable us to identify the most general solution spaces obtained separately in the two gauges. We proceed in two stages.

First, we pass from Bondi to tortoise coordinates $(u, r, \phi) \to (t_\star, r_\star, \phi_\star)$, where

$$\begin{aligned}
& u \to t_\star - r_\star, \quad \phi \to \phi_\star, \\
& r \to \begin{cases} -\frac{1}{\sqrt{-\Lambda}}\cot\left(r_\star\sqrt{-\Lambda}\right) & \text{if } \Lambda < 0 \\ \frac{1}{\sqrt{\Lambda}}\coth\left(r_\star\sqrt{\Lambda}\right) & \text{if } \Lambda > 0 \end{cases}.
\end{aligned} \qquad (6.1.32)$$

Second, we go from tortoise to Fefferman-Graham performing the coordinates transformation $(t_\star, r_\star, \phi_\star) \to (\rho, t, \phi)$, with

$$t_\star = t + T_1(t,\phi)\rho + T_2(t,\phi)\rho^2 + T_3(t,\phi)\rho^3 + \mathcal{O}(\rho^4), \qquad (6.1.33)$$

$$r_\star = R_1(t,\phi)\rho + R_2(t,\phi)\rho^2 + R_3(t,\phi)\rho^3 + \mathcal{O}(\rho^4), \qquad (6.1.34)$$

$$\phi_\star = \phi + Z_1(t,\phi)\rho + Z_2(t,\phi)\rho^2 + Z_3(t,\phi)\rho^3 + \mathcal{O}(\rho^4). \qquad (6.1.35)$$

The explicit form of the functions $T_i(t,\phi)$, $R_i(t,\phi)$ and $Z_i(t,\phi)$ $(i = 1, 2, 3)$ can be worked out explicitly. For the sake of brevity, we report here only the leading orders

$$R_1(t,\phi) = \frac{1}{\Lambda}, \tag{6.1.36}$$

$$R_2(t,\phi) = -e^{-2\beta_0}\frac{1}{\Lambda^2}(\partial_\phi U_0 + U_0\partial_\phi\varphi + \partial_t\varphi), \tag{6.1.37}$$

$$T_1(t,\phi) = -\frac{1}{\Lambda}(1-e^{-2\beta_0}), \tag{6.1.38}$$

$$T_2(t,\phi) = -e^{-4\beta_0}\frac{1}{\Lambda^2}[e^{2\beta_0}\partial_\phi U_0 + U_0(\partial_\phi\beta_0 + e^{2\beta_0}\partial_\phi\varphi) + \partial_t\beta_0 + e^{2\beta_0}\partial_t\varphi], \tag{6.1.39}$$

$$Z_1(t,\phi) = \frac{1}{\Lambda}e^{-2\beta_0}U_0, \tag{6.1.40}$$

$$\begin{aligned} Z_2(t,\phi) = &-\frac{1}{2\Lambda}\Big[2e^{-2\varphi}\partial_\phi\beta_0 + \frac{2}{\Lambda}e^{-4\beta_0}U_0^2\partial_\phi\beta_0 \\ &- \frac{1}{\Lambda}e^{-4\beta_0}\partial_t U_0 - \frac{1}{\Lambda}e^{-4\beta_0}U_0(\partial_\phi U_0 - 2\partial_t\beta_0)\Big]. \end{aligned} \tag{6.1.41}$$

Solution space matching

In this subsection, we use the notation $\Lambda = -1/\ell^2$ for compactness of the expressions ($\ell \in \mathbb{R}$ if $\Lambda < 0$ and $i\ell \in \mathbb{R}$ if $\Lambda > 0$). Using the diffeomorphism (6.1.33-6.1.35), the solution space of the Fefferman-Graham gauge (left-hand side) is related to the solution space of the Bondi gauge (right-hand side) through

$$g^{(0)}_{ab} = \begin{pmatrix} -\frac{e^{4\beta_0}}{\ell^2} + e^{2\varphi}U_0^2 & -e^{2\varphi}U_0 \\ -e^{2\varphi}U_0 & e^{2\varphi} \end{pmatrix} \tag{6.1.42}$$

and

$$\begin{aligned} T_{tt} = &\frac{1}{16\pi G\ell}e^{-4\beta_0-2\varphi}\{4e^{8\beta_0}[2(\partial_\phi\beta_0)^2 - \partial_\phi\beta_0\partial_\phi\varphi + \partial_\phi^2\beta_0] + e^{4\beta_0+2\varphi}[e^{2\beta_0}(M-4NU_0) \\ &- \ell^2((\partial_\phi U_0)^2 + U_0^2(-8\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 4\partial_\phi^2\varphi) + (\partial_t\varphi)^2 \\ &+ 2\partial_\phi U_0(U_0(-4\partial_\phi\beta_0 + 3\partial_\phi\varphi) + \partial_t\varphi) + 2U_0(2\partial_\phi^2 U_0 + (-4\partial_\phi\beta_0 + \partial_\phi\varphi)\partial_t\varphi \\ &+ 2\partial_t\partial_\phi\varphi))] + e^{4\varphi}\ell^2U_0^2[e^{2\beta_0}M + \ell^2((\partial_\phi U_0)^2 + U_0^2(-4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi) \\ &+ 2\partial_\phi\varphi\partial_t U_0 + \partial_t\varphi(-4\partial_t\beta_0 + \partial_t\varphi) + 2\partial_\phi U_0(2U_0(-\partial_\phi\beta_0 + \partial_\phi\varphi) \\ &- 2\partial_t\beta_0 + \partial_t\varphi) + 2U_0(\partial_\phi^2 U_0 - 2\partial_\phi\beta_0\partial_t\varphi + \partial_\phi\varphi(-2\partial_t\beta_0 + \partial_t\varphi) + 2\partial_t\partial_\phi\varphi) \\ &+ 2(\partial_t\partial_\phi U_0 + \partial_t^2\varphi))]\}, \end{aligned} \tag{6.1.43}$$

$$T_{t\phi} = \frac{1}{16\pi G\ell}e^{-4\beta_0}\{2e^{6\beta_0}N - 2e^{4\beta_0}\ell^2[\partial_\phi U_0(2\partial_\phi\beta_0 - \partial_\phi\varphi) - \partial_\phi^2 U_0$$

$$
\begin{aligned}
&+ U_0(2\partial_\phi\beta_0\partial_\phi\varphi - \partial_\phi^2\varphi) + 2\partial_\phi\beta_0\partial_t\varphi - \partial_t\partial_\phi\varphi] \\
&+ e^{2\varphi}\ell^2 U_0[-e^{2\beta_0}M - \ell^2((\partial_\phi U_0)^2 + U_0^2(-4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi) \\
&+ 2\partial_\phi\varphi\partial_t U_0 + \partial_t\varphi(-4\partial_t\beta_0 + \partial_t\varphi) + 2\partial_\phi U_0(2U_0(-\partial_\phi\beta_0 + \partial_\phi\varphi) - 2\partial_t\beta_0 + \partial_t\varphi) \\
&+ 2U_0(\partial_\phi^2 U_0 - 2\partial_\phi\beta_0\partial_t\varphi + \partial_\phi\varphi(-2\partial_t\beta_0 + \partial_t\varphi) + 2\partial_t\partial_\phi\varphi) + 2(\partial_t\partial_\phi U_0 + \partial_t^2\varphi))]\},
\end{aligned} \tag{6.1.44}
$$

$$
\begin{aligned}
T_{\phi\phi} = \frac{1}{16\pi G\ell}&e^{-4\beta_0+2\varphi}\{e^{2\beta_0}\ell^2 M + \ell^4[(\partial_\phi U_0)^2 + U_0^2(-4\partial_\phi\beta_0\partial_\phi\varphi + (\partial_\phi\varphi)^2 + 2\partial_\phi^2\varphi) \\
&+ 2\partial_\phi\varphi\partial_t U_0 + \partial_t\varphi(-4\partial_t\beta_0 + \partial_t\varphi) + 2\partial_\phi U_0(2U_0(-\partial_\phi\beta_0 + \partial_\phi\varphi) - 2\partial_t\beta_0 + \partial_t\varphi) \\
&+ 2U_0(\partial_\phi^2 U_0 - 2\partial_\phi\beta_0\partial_t\varphi + \partial_\phi\varphi(-2\partial_t\beta_0 + \partial_t\varphi) + 2\partial_t\partial_\phi\varphi) + 2(\partial_t\partial_\phi U_0 + \partial_t^2\varphi)]\}.
\end{aligned} \tag{6.1.45}
$$

One can check on the right-hand side expressions that the trace condition given by the first equation of (6.1.8) is satisfied.

Taking $U_0 = 0$, $\beta_0 = 0$ and $\varphi = \bar\varphi$ (three-dimensional analogue of the boundary gauge fixing of [166]), T_{ab} reduces to[2]

$$
T_{ab} = \frac{1}{16\pi G\ell}\begin{pmatrix} M - \ell^2(\partial_t\bar\varphi)^2 & 2N + 2\ell^2\partial_t\partial_\phi\bar\varphi \\ 2N + 2\ell^2\partial_t\partial_\phi\bar\varphi & e^{2\bar\varphi}\ell^2[M + \ell^2((\partial_t\bar\varphi)^2 + 2\partial_t^2\bar\varphi)] \end{pmatrix}. \tag{6.1.46}
$$

Residual gauge parameters matching

Using the diffeomorphism (6.1.33-6.1.35), the parameters of the residual gauge diffeomorphisms of the Fefferman-Graham gauge (6.1.4) are related to those of the Bondi gauge (6.1.16–6.1.18) as

$$
\xi_0^t = f, \tag{6.1.47}
$$

$$
\xi_0^\phi = Y, \tag{6.1.48}
$$

$$
\sigma = \partial_\phi Y - \omega - U_0\partial_\phi f + Y\partial_\phi\varphi + f\partial_t\varphi. \tag{6.1.49}
$$

6.1.4 Flat limit in the Bondi gauge

In this subsection, we investigate the flat limit of the above results. Notice that the flat limit is well defined in the Bondi gauge, but not in the Fefferman-Graham gauge. This is an illustration of our general statement: the Fefferman-Graham gauge is well adapted for computations in asymptotically (locally) (A)dS spacetime due to its covariance with respect to the boundary structure. However, to consider the flat limit, the results have to be translated into the Bondi gauge, where the computations are more involved but the limit is perfectly defined.

[2]In term of Λ, the pre-factor in the right-hand side of (6.1.46) is given by $\sqrt{|\Lambda|}/16\pi G$.

Let us first study the solution space in the Bondi gauge for vanishing cosmological constant. In three dimensions, the full solution space in the Bondi gauge for vanishing cosmological constant can be readily obtained by taking the flat limit of the solution space obtained in section 6.1.2 for non-vanishing cosmological constant. This contrasts with the four-dimensional case where only the analytic part of the solution space is recovered (see the difference between (2.2.48) and (2.2.49) and the associated discussion). In practice, we take $\Lambda \to 0$ in the equations. The equation $G_{rr} = 0$ gives

$$\beta = \beta_0(u, \phi) \tag{6.1.50}$$

Solving $G_{r\phi} = 0$ leads to

$$U = U_0(u, \phi) + \frac{1}{r} 2 e^{2\beta_0} e^{-2\varphi} \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N(u, \phi). \tag{6.1.51}$$

Solving $G_{ur} = 0$ gives

$$\frac{V}{r} = -2r(\partial_u \varphi + D_\phi U_0) + M(u, \phi) + \frac{1}{r} 4 e^{2\beta_0} e^{-2\varphi} N \partial_\phi \beta_0 - \frac{1}{r^2} e^{2\beta_0} e^{-2\varphi} N^2, \tag{6.1.52}$$

where $D_\phi U_0 = \partial_\phi U_0 + \partial_\phi \varphi U_0$. Taking the previous results into account, the Einstein equation $G_{\phi\phi} = 0$ is satisfied at all orders. Finally, we solve the Einstein equations giving the time evolution constraints on M and N. The equation $G_{u\phi} = 0$ gives

$$\begin{aligned}(\partial_u + \partial_u \varphi) N = &\left(\frac{1}{2} \partial_\phi + \partial_\phi \beta_0 \right) M - 2N \partial_\phi U_0 - U_0 (\partial_\phi N + N \partial_\phi \varphi) \\ &+ 4 e^{2\beta_0} e^{-2\varphi} [2(\partial_\phi \beta_0)^3 - (\partial_\phi \varphi)(\partial_\phi \beta_0)^2 + (\partial_\phi \beta_0)(\partial_\phi^2 \beta_0)],\end{aligned} \tag{6.1.53}$$

whereas $G_{uu} = 0$ results in

$$\begin{aligned}\partial_u M = &(-2\partial_u \varphi + 2\partial_u \beta_0 - 2\partial_\phi U_0 + U_0 2 \partial_\phi \beta_0 - U_0 2 \partial_\phi \varphi - U_0 \partial_\phi) M \\ &- 2 e^{2\beta_0 - 2\varphi} \{ \partial_\phi U_0 [8(\partial_\phi \beta_0)^2 - 4 \partial_\phi \beta_0 \partial_\phi \varphi + (\partial_\phi \varphi)^2 + 4 \partial_\phi^2 \beta_0 - 2 \partial_\phi^2 \varphi] - \partial_\phi^3 U_0 \\ &+ U_0 [\partial_\phi \beta_0 (8 \partial_\phi^2 \beta_0 - 2 \partial_\phi^2 \varphi) + \partial_\phi \varphi (-2 \partial_\phi^2 \beta_0 + \partial_\phi^2 \varphi) + 2 \partial_\phi^3 \beta_0 - \partial_\phi^3 \varphi] \\ &+ 2 \partial_u \partial_\phi \beta_0 (4 \partial_\phi \beta_0 - \partial_\phi \varphi) + \partial_u \partial_\phi \varphi (-2 \partial_\phi \beta_0 + \partial_\phi \varphi) + 2 \partial_u \partial_\phi^2 \beta_0 - \partial_u \partial_\phi^2 \varphi \}\end{aligned} \tag{6.1.54}$$

The solution space is thus characterized by five arbitrary functions of (u, ϕ), given by β_0, U_0, M, N, φ, with two dynamical constraints given by the time evolution equations of M and N.

Through a similar procedure, the on-shell residual gauge diffeomorphisms and the variations of the solution space are obtained by taking $\Lambda \to 0$ in the expressions (6.1.24-6.1.26) and (6.1.27-6.1.31), respectively. The on-shell residual gauge diffeomorphisms are given by

$$\xi^u = f, \tag{6.1.55}$$

$$\xi^\phi = Y - \frac{1}{r}\partial_\phi f\, e^{2\beta_0 - 2\varphi}, \tag{6.1.56}$$

$$\begin{aligned}\xi^r = & -r[\partial_\phi Y - \omega - U_0\partial_\phi f + Y\partial_\phi\varphi + f\partial_u\varphi] \\ & + e^{2\beta_0-2\varphi}(\partial_\phi^2 f - \partial_\phi f\partial_\phi\varphi + 4\partial_\phi f\partial_\phi\beta_0) - \frac{1}{r}e^{2\beta_0-2\varphi}\partial_\phi f\, N.\end{aligned} \tag{6.1.57}$$

Under these residual gauge diffeomorphisms, the unconstrained part of the solution space transforms as

$$\delta_\xi\varphi = \omega, \tag{6.1.58}$$

$$\delta_\xi\beta_0 = (f\partial_u + Y\partial_\phi)\beta_0 + \left(\frac{1}{2}\partial_u - \frac{1}{2}\partial_u\varphi + U_0\partial_\phi\right)f - \frac{1}{2}(\partial_\phi Y + Y\partial_\phi\varphi - \omega), \tag{6.1.59}$$

$$\delta_\xi U_0 = (f\partial_u + Y\partial_\phi - \partial_\phi Y)U_0 - \left(\partial_u Y + \Lambda e^{4\beta_0}e^{-2\varphi}\partial_\phi f\right) + U_0(\partial_u f + U_0\partial_\phi f), \tag{6.1.60}$$

while the constrained part transforms as

$$\begin{aligned}\delta_\xi N = & (f\partial_u + Y\partial_\phi + 2\partial_\phi Y + f\partial_u\varphi + Y\partial_\phi\varphi - \omega - 2U_0\partial_\phi f)N \\ & + M\partial_\phi f - e^{2\beta_0-2\varphi}[3\partial_\phi^2 f(2\partial_\phi\beta_0 - \partial_\phi\varphi) + \partial_\phi^3 f \\ & + \partial_\phi f(4(\partial_\phi\beta_0)^2 - 8\partial_\phi\beta_0\partial_\phi\varphi + 2(\partial_\phi\varphi)^2 + 2\partial_\phi^2\beta_0 - \partial_\phi^2\varphi)],\end{aligned} \tag{6.1.61}$$

$$\begin{aligned}\delta_\xi M = & (f\partial_u + \partial_u f + f\partial_u\varphi + Y\partial_\phi + \partial_\phi Y + Y\partial_\phi\varphi - \omega)M \\ & - 2e^{2\beta_0-2\varphi}\Big[2\partial_\phi^2 f\partial_u\beta_0 + 4\partial_u\partial_\phi f\partial_\phi\beta_0 + \partial_u\partial_\phi^2 f + \partial_\phi^2 f\partial_\phi U_0 + 8\partial_\phi^2 f\partial_\phi\beta_0 U_0 \\ & + \partial_\phi f\Big((4\partial_\phi\beta_0 - \partial_\phi\varphi)(2\partial_u\beta_0 - \partial_u\varphi) + 4\partial_u\partial_\phi\beta_0 + \partial_\phi U_0(8\partial_\phi\beta_0 - 2\partial_\phi\varphi) \\ & - \partial_\phi^2 U_0 - 2\partial_u\partial_\phi\varphi + U_0(-4\partial_\phi\beta_0\partial_\phi\varphi + 8(\partial_\phi\beta_0)^2 + 4\partial_\phi^2\beta_0 + (\partial_\phi\varphi)^2 - 2\partial_\phi^2\varphi)\Big) \\ & - 2\partial_\phi^2 f U_0\partial_\phi\varphi + \partial_\phi^3 f U_0 - \partial_u\partial_\phi f\partial_\phi\varphi - \partial_\phi^2 f\partial_u\varphi - 2f\partial_\phi\beta_0\partial_u\partial_\phi\varphi \\ & + 2\partial_\phi\beta_0\partial_\phi\omega - 2\partial_\phi\beta_0\partial_\phi Y\partial_\phi\varphi - 2\partial_\phi\beta_0\partial_\phi^2 Y - 2\partial_\phi\beta_0 Y\partial_\phi^2\varphi\Big].\end{aligned} \tag{6.1.62}$$

6.1.5 From asymptotically AdS$_3$ to asymptotically flat spacetime

Here, we discuss how the $\mathfrak{bms}_3$ algebra can be obtained by taking the flat limit of $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ in asymptotically AdS$_3$ spacetime.

In asymptotically locally AdS$_3$ spacetime, the Dirichlet boundary conditions are obtained in the Fefferman-Graham gauge by imposing

$$g^{(0)}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \Lambda\mathrm{d}t^2 + \mathrm{d}\phi^2 \tag{6.1.63}$$

(see definition (AAdS2) given in equation (2.2.23)). It is a well-known result that the asymptotic symmetry algebra associated with Dirichlet boundary conditions is given by the direct sum between two copies of the Witt algebra, $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ [14]. The corresponding surface charges are finite, integrable and form a representation of $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$ with a central extension involving the Brown-Henneaux central charge $c = \frac{3}{2G|\Lambda|}$.

The analogous boundary conditions of (6.1.63) in Bondi gauge can be readily obtained from (6.1.42) and are explicitly given by

$$\beta_0 = 0, \qquad U_0 = 0, \qquad \varphi = 0 \tag{6.1.64}$$

(see equation (2.2.24)). The residual gauge diffeomorphisms preserving these constraints are given by (6.1.24-6.1.26), where the parameters satisfy

$$\partial_u f = \partial_\phi Y, \qquad \partial_u Y = -\Lambda \partial_\phi f, \qquad \omega = 0. \tag{6.1.65}$$

We express f and Y as $f = \frac{1}{\Lambda}(Y^+ + Y^-)$, $Y = \frac{1}{2}(Y^+ - Y^-)$, where $x^\pm = -\Lambda u \pm \phi$ and $Y^\pm = Y^\pm(x^\pm)$ [221]. Using the modified Lie bracket (2.2.57), one can show that $[\xi(Y_1^\pm), \xi(Y_2^\pm)]_A = \xi(\hat{Y}^\pm)$, where

$$\hat{Y}^\pm = Y_1^\pm \partial_\pm Y_2^\pm - Y_2^\pm \partial_\pm Y_1^\pm, \tag{6.1.66}$$

which corresponds to $\mathfrak{diff}(S^1) \oplus \mathfrak{diff}(S^1)$, as it should. Furthermore, on the cylinder, we can expand $Y^\pm$ as $Y^\pm(x^\pm) = \sum_{m\in\mathbb{Z}} Y_m^\pm e^{imx^\pm}$, with $\bar{Y}_m^\pm = Y_{-m}^\pm$. Writing $l_m^+ = \xi(Y^+ = e^{imx^+}, Y^- = 0)$ and $l_m^- = \xi(Y^+ = 0, Y^- = e^{imx^-})$, the commutation relations (6.1.66) become

$$i[l_m^\pm, l_n^\pm]_A = (m-n) l_{m+n}^\pm, \qquad [l_m^\pm, l_n^\mp]_A = 0. \tag{6.1.67}$$

The flat limit $\Lambda \to 0$ can be readily taken in the Bondi gauge. In this context, the boundary conditions (6.1.64) become asymptotically flat boundary conditions (AF3) (equation (2.2.15) together with (2.2.18)). The constraint equations (6.1.65) reduce to

$$\partial_u f = \partial_\phi Y, \qquad \partial_u Y = 0 \qquad \omega = 0. \tag{6.1.68}$$

These equations can be solved as

$$f = T + u\partial_\phi Y, \qquad Y = Y(\phi) \tag{6.1.69}$$

where $T = T(\phi)$ and $Y = Y(\phi)$ are the parameters of supertranslations and superrotations, respectively. Using the modified Lie bracket (2.2.57), one can show that $[\xi(T_1, Y_1), \xi(T_2, Y_2)]_A = \xi(\hat{T}, \hat{Y})$, where

$$\begin{aligned} \hat{T} &= Y_1 \partial_\phi T_2 + T_1 \partial_\phi Y_2 - Y_2 \partial_\phi T_1 - T_2 \partial_\phi Y_1, \\ \hat{Y} &= Y_1 \partial_\phi Y_2 - Y_2 \partial_\phi Y_1, \end{aligned} \tag{6.1.70}$$

which corresponds to the algebra $\mathfrak{bms}_3 = \mathfrak{diff}(S^1) \loplus_{\text{ad}} \mathfrak{vect}(S^1)$. Expanding T and Y on the circle as $T(\phi) = \sum_{m\in\mathbb{Z}} T_m e^{im\phi}$ and $Y(\phi) = \sum_{m\in\mathbb{Z}} Y_m e^{im\phi}$, and writing $P_m = \xi(T = e^{im\phi}, Y = 0)$ and $J_m = \xi(T = 0, Y = e^{im\phi})$, the commutation relations (6.1.70) reduce to

$$i[J_m, J_n]_A = (m-n)J_{m+n}, \qquad [P_m, P_n]_A = 0, \qquad i[J_m, P_n]_A = (m-n)P_{m+n}. \tag{6.1.71}$$

6.2 Bondi and Fefferman-Graham gauges in four dimensions

In this section, we repeat the analysis performed in the previous section but for the four-dimensional case. After briefly introducing the Fefferman-Graham gauge in the four-dimensional case, we focus on the Bondi gauge where we derive the most general solution space in asymptotically locally (A)dS$_4$ spacetime (these results were already summarized in an example of subsection 2.2.3). As in the three-dimensional case, the results that we obtain here in four dimensions generalize previous considerations (see e.g. [24, 221]) by allowing an arbitrary boundary structure encoding the notion of asymptotically locally (A)dS$_4$ spacetime. We also briefly discuss the flat limit of these new results, which allows to find the solution space described in (2.2.48). Notice that the flat limit process in four dimensions is more subtle and requires a prescription in order to get the right results. Furthermore, we construct the explicit diffeomorphism that maps the Bondi to the Fefferman-Graham gauge, which is the second main result of this section. We finish this section by defining a new set of boundary conditions in asymptotically locally (A)dS$_4$ spacetime whose asymptotic symmetry algebra is the Λ-BMS$_4$ algebra, written $\mathfrak{bms}_4^\Lambda$. The latter is infinite-dimensional and reduces to $\mathfrak{bms}_4^{\text{gen}}$ in the flat limit, which was studied in chapter 4. This is the third main result of this section.

6.2.1 Fefferman-Graham gauge in 4d

Definition

We particularize to the four-dimensional case the results discussed in section 2.2. The Fefferman-Graham metric is given by

$$ds^2 = -\frac{3}{\Lambda}\frac{\mathrm{d}\rho^2}{\rho^2} + \gamma_{ab}(\rho, x^c)\mathrm{d}x^a \mathrm{d}x^b \tag{6.2.1}$$

(see (2.2.8)). The infinitesimal diffeomorphisms preserving the Fefferman-Graham gauge are generated by vector fields ξ^μ satisfying $\mathcal{L}_\xi g_{\rho\rho} = 0$, $\mathcal{L}_\xi g_{\rho a} = 0$. The first condition leads to the equation $\partial_\rho \xi^\rho = \frac{1}{\rho}\xi^\rho$, which can be solved for ξ^ρ as

$$\xi^\rho = \sigma(x^a)\rho. \tag{6.2.2}$$

The second condition leads to the equation $\rho^2 \gamma_{ab}\partial_\rho \xi^b - \frac{3}{\Lambda}\partial_a \xi^\rho = 0$, which can be solved for ξ^a as

$$\xi^a = \xi^a_0(x^b) + \frac{3}{\Lambda}\partial_b \sigma \int_0^\rho \frac{\mathrm{d}\rho'}{\rho'}\gamma^{ab}(\rho', x^c). \tag{6.2.3}$$

Solution space

Assuming $\gamma_{ab} = \mathcal{O}(\rho^{-2})$ (see (6.2.4)), the general asymptotic expansion that solves Einstein's equations is analytic,

$$\gamma_{ab} = \frac{1}{\rho^2}\, g^{(0)}_{ab} + \frac{1}{\rho}\, g^{(1)}_{ab} + g^{(2)}_{ab} + \rho\, g^{(3)}_{ab} + \mathcal{O}(\rho^2) \tag{6.2.4}$$

where $g^{(i)}_{ab}$ are arbitrary functions of $x^a = (t, x^A)$. Following the standard holographic dictionary (see e.g. [222]), we call $g^{(0)}_{ab}$ the boundary metric and

$$T_{ab} = \frac{\sqrt{3|\Lambda|}}{16\pi G} g^{(3)}_{ab} \tag{6.2.5}$$

the energy-momentum tensor. Einstein's equations fix $g^{(1)}_{ab} = 0$ and $g^{(2)}_{ab}$ in terms of $g^{(0)}_{ab}$ while all subleading terms in (6.2.4) are determined in terms of the free data $g^{(0)}_{ab}$ and T_{ab} satisfying

$$D^{(0)}_a T^{ab} = 0, \quad g^{(0)}_{ab} T^{ab} = 0. \tag{6.2.6}$$

Here, $D^{(0)}_a$ is the covariant derivative with respect to $g^{(0)}_{ab}$ and indices are raised with the inverse metric $g^{ab}_{(0)}$.

Variation of the solution space

Expanding the residual gauge diffeomorphisms (6.2.2) and (6.2.3) in power of ρ yields

$$\begin{aligned} \xi^\rho &= \rho\, \sigma(x^a) \\ \xi^a &= \xi^a_0 + \frac{3}{2\Lambda}\rho^2 g^{ab}_{(0)}\partial_b \sigma - \frac{3}{4\Lambda}\rho^4 g^{ab}_{(2)}\partial_b\sigma - \frac{3}{5\Lambda}\rho^5 g^{ab}_{(3)}\partial_b \sigma + \mathcal{O}(\rho^6). \end{aligned} \tag{6.2.7}$$

The variations of the data parametrizing the solution space under the residual gauge transformations are given by (see also [156])

$$\delta_\xi g^{(0)}_{ab} = \mathcal{L}_{\xi^c_0} g^{(0)}_{ab} - 2\sigma\, g^{(0)}_{ab}, \tag{6.2.8}$$

$$\delta_\xi T_{ab} = \mathcal{L}_{\xi^c_0} T_{ab} + \sigma\, T_{ab}. \tag{6.2.9}$$

Useful expressions

Here, we collect some useful formulae needed in the process of holographic renormalization below (see section 6.3), especially the coefficients of the Levi-Civita connection in the Fefferman-Graham gauge.

The inverse γ^{ab} of the $3d$ metric

$$\gamma_{ab} = \frac{1}{\rho^2}\Big(g^{(0)}_{ab} + \rho^2\, g^{(2)}_{ab} + \rho^3\, g^{(3)}_{ab} + \rho^4\, g^{(4)}_{ab} + \mathcal{O}(\rho^5)\Big) \tag{6.2.10}$$

is given by

$$\gamma^{ab} = \rho^2\Big[g^{ab}_{(0)} - \rho^2\, g^{ab}_{(2)} - \rho^3\, g^{ab}_{(3)} + \rho^4\,(-g^{ab}_{(4)} + g^{ac}_{(2)} g^{(2)}_{cd}\, g^{db}_{(0)}) + \mathcal{O}(\rho^5)\Big] \tag{6.2.11}$$

We denote $g^{ab}_{(i)} \equiv g^{ac}_{(0)} g^{bd}_{(0)} g^{(i)}_{cd}$. Only $g^{ab}_{(0)}$ is the true inverse matrix of $g^{(0)}_{ab}$; the indices of other fields are simply raised with respect to the boundary metric.

The volume form is given by

$$\sqrt{-g} = \sqrt{\frac{3}{|\Lambda|}}\,\frac{1}{\rho}\,\sqrt{|\gamma|} \tag{6.2.12}$$

with

$$\begin{aligned} \sqrt{|\gamma|} &= \sqrt{|g_{(0)}|}\,\frac{1}{\rho^3}\Big(1 + \frac{1}{2} g^{ab}_{(0)} g^{(2)}_{ab}\,\rho^2 + \frac{1}{2} g^{ab}_{(0)} g^{(3)}_{ab}\,\rho^3 + \mathcal{O}(\rho^4)\Big) \\ &= \sqrt{|g_{(0)}|}\,\frac{1}{\rho^3}\Big(1 + \rho^2 \frac{3}{8\Lambda} R_{(0)} + \mathcal{O}(\rho^4)\Big). \end{aligned} \tag{6.2.13}$$

We compute the Christoffel symbols for (6.2.1). Gauge conditions imply directly that

$$\Gamma^\rho_{\rho\rho} = \frac{1}{2} g^{\rho\rho}\partial_\rho g_{\rho\rho} = -\frac{1}{\rho}, \qquad \Gamma^\rho_{\rho a} = 0, \qquad \Gamma^a_{\rho\rho} = 0. \tag{6.2.14}$$

Using the power series for γ_{ab} and its inverse, we get

$$\begin{aligned} \Gamma^\rho_{ab} &= -\frac{1}{2} g^{\rho\rho}\partial_\rho \gamma_{ab} = \frac{\Lambda}{6}\rho^2 \partial_\rho \gamma_{ab} \\ &= -\frac{\Lambda}{3}\frac{1}{\rho} g^{(0)}_{ab} + \frac{\Lambda}{6}\rho^2 g^{(3)}_{ab} + \mathcal{O}(\rho^3), \end{aligned} \tag{6.2.15}$$

and

$$\begin{aligned} \Gamma^a_{\rho b} &= \frac{1}{2}\gamma^{ac}\partial_\rho \gamma_{bc} \\ &= -\frac{1}{\rho}\delta^a_b + \rho\, g^{ac}_{(0)} g^{(2)}_{bc} + \frac{3}{2}\,\rho^2\, g^{ac}_{(0)} g^{(3)}_{bc} + \mathcal{O}(\rho^3). \end{aligned} \tag{6.2.16}$$

Finally,

$$
\begin{aligned}
\Gamma^a_{bc} &= \Gamma^a_{bc}[\gamma] \\
&= \Gamma^a_{bc}[g_{(0)}] + \frac{1}{2}\rho^2 g^{ad}_{(0)}(D^{(0)}_b g^{(2)}_{dc} + D^{(0)}_c g^{(2)}_{db} - D^{(0)}_d g^{(2)}_{bc}) \\
&\quad + \frac{1}{2}\rho^3 g^{ad}_{(0)}(D^{(0)}_b g^{(3)}_{dc} + D^{(0)}_c g^{(3)}_{db} - D^{(0)}_d g^{(3)}_{bc}) + \mathcal{O}(\rho^4).
\end{aligned}
\tag{6.2.17}
$$

6.2.2 Bondi gauge in 4d

We now briefly discuss again the Bondi gauge in four dimensions (see section 2.2). Then we provide a full derivation of the most general solution space in four-dimensional asymtotically locally (A)dS$_4$ spacetime. Throughout this analysis, we discuss the flat limit of these results and relate them to those considered in chapter 4.

Definition and residual transformations

The four-dimensional Bondi metric is given by

$$
ds^2 = e^{2\beta}\frac{V}{r}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + g_{AB}(\mathrm{d}x^A - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u) \tag{6.2.18}
$$

where β, U^A, g_{AB} and V are arbitrary functions of the coordinates. The 2-dimensional metric g_{AB} satisfies the determinant condition

$$
\partial_r\left(\frac{\det(g_{AB})}{r^4}\right) = 0. \tag{6.2.19}
$$

Any metric can be written in this gauge. For example, global (A)dS$_4$ is obtained by choosing $\beta = 0$, $U^A = 0$, $V/r = (\Lambda r^2/3) - 1$, $g_{AB} = r^2 \mathring{q}_{AB}$, where $\mathring{q}_{AB}$ is the unit round-sphere metric.

Infinitesimal diffeomorphisms preserving the Bondi gauge are generated by vector fields ξ^μ satisfying

$$
\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad g^{AB}\mathcal{L}_\xi g_{AB} = 4\omega(u, x^A). \tag{6.2.20}
$$

The prefactor of 4 is introduced for convenience. As discussed in appendix B, the last condition is equivalent to the determinant condition (6.2.19). From (6.2.20), we deduce

$$
\begin{aligned}
\xi^u &= f, \\
\xi^A &= Y^A + I^A, \quad I^A = -\partial_B f \int_r^\infty \mathrm{d}r' (e^{2\beta} g^{AB}), \\
\xi^r &= -\frac{r}{2}(\mathcal{D}_A Y^A - 2\omega + \mathcal{D}_A I^A - \partial_B f U^B + \frac{1}{2} f g^{-1}\partial_u g),
\end{aligned}
\tag{6.2.21}
$$

where $\partial_r f = 0 = \partial_r Y^A$, and $g = \det(g_{AB})$. The covariant derivative $\mathcal{D}_A$ is associated with the 2-dimensional metric g_{AB}. The residual gauge transformations are parametrized by the 4 functions ω, f and Y^A of (u, x^A).

Procedure to resolve Einstein's equations

We solve the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ in Bondi gauge. We follow the integration scheme and notations of [6]. In particular, we use the Christoffel symbols that have been derived in this reference.

Minimal fall-off requirements: We impose the preliminary boundary condition $g_{AB} = \mathcal{O}(r^2)$ (see (2.2.33)) and assume an analytic expansion for g_{AB}, namely

$$g_{AB} = r^2\, q_{AB} + r\, C_{AB} + D_{AB} + \frac{1}{r} E_{AB} + \frac{1}{r^2} F_{AB} + \mathcal{O}(r^{-3}), \tag{6.2.22}$$

where each term involves a symmetric tensor whose components are arbitrary functions of (u, x^C). For $\Lambda \neq 0$, the Fefferman-Graham theorem [152–156] together with the map between the Fefferman-Graham gauge and Bondi gauge, derived in appendix D, ensures that the expansion (6.2.22) leads to the most general solution to the vacuum Einstein equations. For $\Lambda = 0$, the analytic expansion (6.2.22) is a hypothesis since additional logarithmic branches might occur [223–225].

This fall-off condition does not impose any constraints on the generators of residual diffeomorphisms (6.2.21). In the following, upper-case Latin indices are lowered and raised by the 2-dimensional metric q_{AB} and its inverse. The gauge condition (6.2.19) implies $g^{AB}\partial_r g_{AB} = 4/r$ which imposes successively that $\det(g_{AB}) = r^4 \det(q_{AB})$, $q^{AB}C_{AB} = 0$ and

$$\begin{aligned}
D_{AB} &= \frac{1}{4} q_{AB} C^{CD} C_{CD} + \mathcal{D}_{AB}(u, x^C),\\
E_{AB} &= \frac{1}{2} q_{AB} \mathcal{D}_{CD} C^{CD} + \mathcal{E}_{AB}(u, x^C),\\
F_{AB} &= \frac{1}{2} q_{AB} \Big[C^{CD} \mathcal{E}_{CD} + \frac{1}{2} \mathcal{D}^{CD} \mathcal{D}_{CD} - \frac{1}{32} (C^{CD} C_{CD})^2 \Big] + \mathcal{F}_{AB}(u, x^C),
\end{aligned} \tag{6.2.23}$$

with $q^{AB}\mathcal{D}_{AB} = q^{AB}\mathcal{E}_{AB} = q^{AB}\mathcal{F}_{AB} = 0$.

Organization of Einstein's equations: We organize the equations of motion as follows. First, we solve the equations that do not involve the cosmological constant. The radial constraint $G_{rr} = R_{rr} = 0$ fixes the r-dependence of β, while the cross-term constraint $G_{rA} = R_{rA} = 0$ fixes the r-dependence of U^A.

Next, we treat the equations that do depend on Λ. The equation $G_{ur} + \Lambda g_{ur} = 0$ determines the r-dependence of V/r in terms of the previous variables. Noticing that $R = g^{\mu\nu} R_{\mu\nu} = 2 g^{ur} R_{ur} + g^{rr} R_{rr} + 2 g^{rA} R_{rA} + g^{AB} R_{AB}$, and taking into account that $R_{rr} = 0 = R_{rA}$, one gets $G_{ur} + \Lambda g_{ur} = R_{ur} + \frac{1}{2} g_{ur} R + \Lambda g_{ur} = \frac{1}{2} g_{ur} g^{AB} R_{AB} = 0$ so that we can solve equivalently $g^{AB} R_{AB} = 0$.

Next, we concentrate on the pure angular equation, $G_{AB} + \Lambda g_{AB} = 0$, which can be split into a trace-free part

$$G_{AB} - \frac{1}{2} g_{AB}\, g^{CD} G_{CD} = 0 \tag{6.2.24}$$

and a pure-trace part

$$g^{CD} G_{CD} + 2\Lambda = 0. \tag{6.2.25}$$

Consider the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$ which can be rewritten as

$$2\sqrt{-g}\nabla_\mu G^\mu_\nu = 2\partial_\mu(\sqrt{-g}G^\mu_\nu) - \sqrt{-g}G^{\mu\lambda}\partial_\nu g_{\mu\lambda} = 0. \tag{6.2.26}$$

Since $\partial_\nu g_{\mu\lambda} = -g_{\mu\alpha} g_{\lambda\beta} \partial_\nu g^{\alpha\beta}$, we have

$$2\partial_\mu(\sqrt{-g}G^\mu_\nu) + \sqrt{-g}G_{\mu\lambda}\partial_\nu g^{\mu\lambda} = 0. \tag{6.2.27}$$

Taking $\nu = r$ and noting that $G_{r\alpha} + \Lambda g_{r\alpha} = 0$ have already been solved, one gets

$$G_{AB}\partial_r g^{AB} = \frac{4\Lambda}{r}. \tag{6.2.28}$$

Recalling that (6.2.24) holds, and that the determinant condition implies that $g^{AB}\partial_r g_{AB} = 4/r$, we see that (6.2.28) is equivalent to (6.2.25). As a consequence, the equation $G_{AB} + \Lambda g_{AB} = 0$ is completely obeyed if (6.2.24) is solved. Indeed, once the trace-free part (6.2.24) has been set to zero, the tracefull part (6.2.25) is automatically constrained by the Bianchi identity. Another way to see this is as follows: imposing that $G_{r\alpha} + \Lambda g_{r\alpha} = 0$ holds, (6.2.24) is equivalent to

$$(M^{TF})^A_{\ B} \equiv M^A_{\ B} - \frac{1}{2}\delta^A_B M^C_{\ C} = 0, \quad M^A_{\ B} \equiv g^{AC} R_{CB}, \tag{6.2.29}$$

since the trace part of $M^A_{\ B}$ has already been set to zero to fix the radial dependence of V/r.

At this stage, Einstein's equations (r,r), (r,A), (r,u) and (A,B) have been solved. The (u,u) and (u,A) components remain to be solved. In doing so, we will derive the evolution equations for the Bondi mass and angular momentum aspects (see section 6.2.2 below). Expressing the A component of the contracted Bianchi identities (6.2.26) yields

$$\partial_r\Big[r^2\Big(G_{uA} + \Lambda g_{uA}\Big)\Big] = \partial_r\Big[r^2\Big(R_{uA} - \Lambda g_{uA}\Big)\Big] = 0. \tag{6.2.30}$$

Therefore, we can isolate the only non-trivial equation to be the $1/r^2$ part of $G_{uA} + \Lambda g_{uA} = 0$. This will determine the evolution of $N_A^{(\Lambda)}(u, x^B)$ related to the Bondi angular momentum aspect. Assuming that $G_{uA} + \Lambda g_{uA} = 0$ is solved, the last Bianchi identity (6.2.26) for $\nu = u$ becomes

$$\partial_r\Big[r^2\Big(G_{uu} + \Lambda g_{uu}\Big)\Big] = \partial_r\Big[r^2\Big(R_{uu} - \Lambda g_{uu}\Big)\Big] = 0, \tag{6.2.31}$$

and the reasoning is similar. We will solve the r-independent part of $r^2(R_{uu} - \Lambda g_{uu})$, which will uncover the equation governing the time evolution of $M^{(\Lambda)}(u, x^A)$ related to the Bondi mass aspect.

Solution to the algebraic equations

We define several auxiliary fields as in [6]. Starting from (6.2.22), we can build $k_{AB} = \frac{1}{2}\partial_r g_{AB}$, $l_{AB} = \frac{1}{2}\partial_u g_{AB}$, and $n_A = \frac{1}{2}e^{-2\beta} g_{AB}\partial_r U^B$. The determinant condition (6.2.19) allows us to split the tensors k_{AB} and l_{AB} in leading trace-full parts and subleading trace-free parts as

$$\begin{aligned} k^A_B &\equiv g^{AC}k_{BC} = \frac{1}{r}\delta^A_B + \frac{1}{r^2}K^A_B, \qquad K^A_A = 0, \\ l^A_B &\equiv g^{AC}l_{BC} = \frac{1}{2}q^{AC}\partial_u q_{BC} + \frac{1}{r}L^A_B, \qquad L^A_A = 0. \end{aligned} \tag{6.2.32}$$

Note that

$$l = l^A_A = \frac{1}{2}q^{AB}\partial_u q_{AB} = \partial_u \ln\sqrt{q}. \tag{6.2.33}$$

Let us start by solving $R_{rr} = 0$, which leads to

$$\partial_r\beta = -\frac{1}{2r} + \frac{r}{4}k^A_B k^B_A = \frac{1}{4r^3}K^A_B K^B_A. \tag{6.2.34}$$

Expanding K^A_B in powers of $1/r$, we get

$$\begin{aligned} \beta(u, r, x^A) = \beta_0(u, x^A) &+ \frac{1}{r^2}\Big[-\frac{1}{32}C^{AB}C_{AB}\Big] + \frac{1}{r^3}\Big[-\frac{1}{12}C^{AB}\mathcal{D}_{AB}\Big] \\ &+ \frac{1}{r^4}\Big[-\frac{3}{32}C^{AB}\mathcal{E}_{AB} - \frac{1}{16}\mathcal{D}^{AB}\mathcal{D}_{AB} + \frac{1}{128}(C^{AB}C_{AB})^2\Big] + \mathcal{O}(r^{-5}). \end{aligned} \tag{6.2.35}$$

Up to the integration "constant" $\beta_0(u, x^A)$, the condition (6.2.22) uniquely determines β. In particular, the $1/r$ order is always zero on-shell. This equation also holds for $\Lambda = 0$ but standard asymptotic flatness conditions set $\beta_0 = 0$ (see equation (2.2.15)). We keep it arbitrary here.

Next, we develop $R_{rA} = 0$, which gives

$$\partial_r(r^2 n_A) = r^2\Big(\partial_r - \frac{2}{r}\Big)\partial_A\beta - \mathcal{D}_B K^B_A. \tag{6.2.36}$$

We now expand the transverse covariant derivative $\mathcal{D}_A$

$$\Gamma^B_{AC}[g_{AB}] = \Gamma^B_{AC}[q_{AB}] + \frac{1}{r}\Big[\frac{1}{2}(D_A C^B_C + D_C C^B_A - D^B C_{AC})\Big] + \mathcal{O}(r^{-2}), \tag{6.2.37}$$

in terms of the transverse covariant derivative D_A defined with respect to the leading transverse metric q_{AB}. This implies in particular that

$$\mathcal{D}_B K^B_A = -\frac{1}{2}D^B C_{AB} + \frac{1}{r}\Big[- D^B \mathcal{D}_{AB} + \frac{1}{8}\partial_A (C_{BC}C^{BC})\Big] + \mathcal{O}(r^{-2}). \tag{6.2.38}$$

Explicitly using (6.2.35), we find

$$n_A = -\partial_A \beta_0 + \frac{1}{r}\Big[\frac{1}{2}D^B C_{AB}\Big] + \frac{1}{r^2}\Big[\ln r\, D^B \mathcal{D}_{AB} + N_A\Big] + o(r^{-2}) \tag{6.2.39}$$

where N_A is a second integration "constant" (i.e. $\partial_r N_A = 0$), which corresponds to the Bondi angular momentum aspect in the asymptotically flat case. After inverting the definition of n_A, integrating one time further on r and raising the index A, we end up with

$$\begin{aligned} U^A = {}& U^A_0(u,x^B) + \overset{(1)}{U^A}(u,x^B)\frac{1}{r} + \overset{(2)}{U^A}(u,x^B)\frac{1}{r^2} \\ & + \overset{(3)}{U^A}(u,x^B)\frac{1}{r^3} + \overset{(L3)}{U^A}(u,x^B)\frac{\ln r}{r^3} + o(r^{-3}) \end{aligned} \tag{6.2.40}$$

with

$$\begin{aligned} \overset{(1)}{U^A}(u,x^B) &= 2e^{2\beta_0}\partial^A\beta_0, \\ \overset{(2)}{U^A}(u,x^B) &= -e^{2\beta_0}\Big[C^{AB}\partial_B\beta_0 + \frac{1}{2}D_B C^{AB}\Big], \\ \overset{(3)}{U^A}(u,x^B) &= -\frac{2}{3}e^{2\beta_0}\Big[N^A - \frac{1}{2}C^{AB}D^C C_{BC} + (\partial_B\beta_0 - \frac{1}{3}D_B)\mathcal{D}^{AB} - \frac{3}{16}C_{CD}C^{CD}\partial^A\beta_0\Big], \\ \overset{(L3)}{U^A}(u,x^B) &= -\frac{2}{3}e^{2\beta_0}D_B\mathcal{D}^{AB}, \end{aligned} \tag{6.2.41}$$

where $U^A_0(u,x^B)$ is a new integration "constant". Again, this equation also holds if Λ is absent, but standard asymptotic flatness sets this additional parameter to zero. As known in standard flat case analysis, the presence of $\mathcal{D}_{AB}$ is responsible for logarithmic terms in the expansion of U^A. We will shortly derive that for $\Lambda \neq 0$, $\mathcal{D}_{AB}$ vanishes on-shell.

Given that

$$\begin{aligned} M^A_B = e^{-2\beta}\Big[(\partial_r + \frac{2}{r})(l^A_B + k^A_B\frac{V}{r} + \frac{1}{2}\mathcal{D}_B U^A + \frac{1}{2}\mathcal{D}^A U_B) \\ + k^A_C\mathcal{D}_B U^C - k^C_B\mathcal{D}_C U^A + (\partial_u + l)k^A_B + \mathcal{D}_C(U^C k^A_B)\Big] \\ + R^A_B[g_{CD}] - 2(\mathcal{D}_B\partial^A\beta + \partial^A\beta\partial_B\beta + n^A n_B), \end{aligned} \tag{6.2.42}$$

we extract the r-dependence of V/r thanks to $M^A_A = 0$, which reads as

$$\begin{aligned} \partial_r V = & -2r(l + D_A U^A) + \\ & e^{2\beta}r^2\Big[D_A D^A\beta + (n^A - \partial^A\beta)(n_A - \partial_A\beta) - D_A n^A - \frac{1}{2}R[g_{AB}] + \Lambda\Big]. \end{aligned} \tag{6.2.43}$$

Considering (6.2.22), (6.2.35) and (6.2.40), we get, after integration on r

$$\begin{aligned} \frac{V}{r} = & \frac{\Lambda}{3}e^{2\beta_0}r^2 - r(l + D_A U_0^A) \\ & - e^{2\beta_0}\Big[\frac{1}{2}\Big(R[q] + \frac{\Lambda}{8}C_{AB}C^{AB}\Big) + 2D_A\partial^A\beta_0 + 4\partial_A\beta_0\partial^A\beta_0\Big] - \frac{2M}{r} + o(r^{-1}) \end{aligned} \tag{6.2.44}$$

where $M(u, x^A)$ is an integration "constant" which, in flat asymptotics, is recognized as the Bondi mass aspect.

Afterwards, we solve (6.2.29) order by order, which provides us with the constraints imposed on each independent order of g_{AB}. The leading $\mathcal{O}(r^{-1})$ order of that equation yields

$$\frac{\Lambda}{3}C_{AB} = e^{-2\beta_0}\Big[(\partial_u - l)q_{AB} + 2D_{(A}U^0_{B)} - D^C U^0_C q_{AB}\Big]. \tag{6.2.45}$$

This result shows that there is a discrete bifurcation between the asymptotically flat case and the case $\Lambda \neq 0$. Indeed, when $\Lambda = 0$, the left-hand side vanishes, which leads to a constraint on the time-dependence of q_{AB}. Consequently, the field q_{AB} is constrained while C_{AB} is completely free and interpreted as the shear. For (A)dS$_4$ asymptotics, C_{AB} is entirely determined by q_{AB}, β_0 and U_0^A, while the boundary metric $q_{AB} = q_{AB}(u, x^A)$ is left completely undetermined by the equations of motion. This is consistent with previous analyses [19, 167, 226–228].

Going to $\mathcal{O}(r^{-2})$, we get

$$\frac{\Lambda}{3}\mathcal{D}_{AB} = 0, \tag{6.2.46}$$

which removes the logarithmic term in (6.2.40) for $\Lambda \neq 0$, but not for $\Lambda = 0$. The condition at next $\mathcal{O}(r^{-3})$ order

$$\partial_u\mathcal{D}_{AB} + U_0^C D_C\mathcal{D}_{AB} + 2\mathcal{D}_{C(A}D_{B)}U_0^C = 0, \tag{6.2.47}$$

is thus trivial for $\Lambda \neq 0$, but reduces to $\partial_u \mathcal{D}_{AB} = 0$ in the flat limit, consistently with previous results.

Using an iterative argument as in [167], we now make the following observation. If we decompose $g_{AB} = r^2 \sum_{n\geqslant 0} \overset{(n)}{g}_{AB} r^{-n}$, we see that the iterative solution of (6.2.29) organizes itself as $\Lambda \overset{(n)}{g}_{AB} = \partial_u \overset{(n-1)}{g}_{AB} + (...)$ at order $\mathcal{O}(r^{-n})$, $n \in \mathbb{N}_0$. Accordingly, the form of $\mathcal{E}_{AB}$ should have been fixed by the equation found at $\mathcal{O}(r^{-3})$, but it is not the case, since both contributions of $\mathcal{E}_{AB}$ cancel between G_{AB} and Λg_{AB}. Moreover, the equation $\Lambda \overset{(4)}{g}_{AB} = \partial_u \overset{(3)}{g}_{AB} + (...)$ at next order turns out to be a constraint for $\overset{(4)}{g}_{AB} \sim \mathcal{F}_{AB}$, determined with other subleading data such as C_{AB} or $\partial_u \overset{(3)}{g}_{AB} \sim \partial_u \mathcal{E}_{AB}$. It shows that $\mathcal{E}_{AB}$ is a set of two free data on the boundary, built up from two arbitrary functions of (u, x^A). It shows moreover that there is no more data to be uncovered for $\Lambda \neq 0$. This matches with the number of free data of the solution space in the Fefferman-Graham gauge, as discussed in subsection 6.2.3.

As a conclusion, Einstein's equations (r,r), (r,A), (r,u) and (A,B) can be solved iteratively in the asymptotic expansion for $\Lambda \neq 0$. We identified 11 independent functions $\{\beta_0(u,x^A), U_0^A(u,x^B), q_{AB}(u,x^C), M(u,x^C), N_A(u,x^C), \mathcal{E}_{AB}(u,x^C)\}$ that determine the asymptotic solution. We see in the following subsection that the remaining equations are equivalent to evolution equations for $M(u,x^A)$ and $N_A(u,x^B)$. This contrasts with the asymptotically flat case $\Lambda = 0$ where an infinite series of functions appear in the radial expansion, see e.g. [6].

Boundary gauge fixing

In this section, we simplify our analysis by imposing a (co-dimension 1) boundary gauge fixing. The latter can also be interpreted as a partial Dirichlet boundary condition with respect to the bulk spacetime. Let us consider the pullback of the most general Bondi metric satisfying (6.2.22) to the boundary $\mathscr{I} \equiv \{r \to \infty\}$,

$$ds^2\big|_{\mathscr{I}} = \left[\frac{\Lambda}{3} e^{4\beta_0} + U_0^A U_A^0\right] du^2 - 2U_A^0 du dx^A + q_{AB} dx^A dx^B. \tag{6.2.48}$$

We use the boundary gauge freedom to reach the gauge

$$\beta_0 = 0, \quad U_0^A = 0, \quad \sqrt{q} = \sqrt{\bar{q}} \tag{6.2.49}$$

where $\sqrt{\bar{q}}$ is a fixed area of the two-dimensional transverse space spanned by x^A. This gauge is a temporal boundary gauge for $\Lambda < 0$, a radial boundary gauge for $\Lambda > 0$ and a null boundary gauge for $\Lambda = 0$ with $g_{ur} = -1 + \mathcal{O}(r^{-1})$ in (6.2.18).

Intuitively, this amounts to using the gauge freedom at the boundary $\mathscr{I}$, to eliminate three pure-gauge degrees of freedom thanks to a diffeomorphism defined intrinsically on $\mathscr{I}$ and lifted to the bulk in order to preserve the Bondi gauge. Such a transformation also involves a Weyl rescaling of the boundary metric, as can be seen from (6.2.21), which consists in a redefinition of the coordinate r by an arbitrary factor depending on (u, x^A). We can use this Weyl rescaling to gauge-fix one further quantity in the boundary metric, namely the area of the transverse space. The details are provided below.

Computing the Lie derivative on the Bondi metric on-shell and retaining only the leading $\mathcal{O}(r^2)$ terms, we get the transformation laws of the boundary fields q_{AB}, β_0 and U_0^A under the set of residual gauge transformations (6.2.21):

$$\begin{aligned} \delta_\xi q_{AB} &= f(\partial_u - l)q_{AB} + (\mathcal{L}_Y - D_C Y^C + 2\omega)q_{AB} \\ &\quad - 2(U^0_{(A}\partial_{B)} f - \frac{1}{2} q_{AB} U_0^C \partial_C f), \end{aligned} \tag{6.2.50}$$

$$\delta_\xi \beta_0 = (f\partial_u + \mathcal{L}_Y)\beta_0 + \frac{1}{2}\Big[\partial_u - \frac{1}{2}l + \frac{3}{2}U_0^A\partial_A\Big]f - \frac{1}{4}(D_A Y^A - 2\omega), \tag{6.2.51}$$

$$\delta_\xi U_0^A = (f\partial_u + \mathcal{L}_Y)U_0^A - \Big[\partial_u Y^A - \frac{1}{\ell^2}e^{4\beta_0}q^{AB}\partial_B f\Big] + U_0^A(\partial_u f + U_0^B\partial_B f). \tag{6.2.52}$$

The first equation implies that $q^{AB}\delta_\xi q_{AB} = 4\omega$. We can therefore adjust the Weyl generator ω in order to reach the gauge $\sqrt{q} = \sqrt{\bar{q}}$. The form of the infinitesimal transformations (6.2.51)-(6.2.52) involves $\partial_u f$ and $\partial_u Y^A$. This ensures that a finite gauge transformation labelled by f, Y^A can be found by integration over u to reach $\beta_0 = 0$, $U_0^A = 0$, at least in a local patch. As a result, the conditions (6.2.49) can be reached by gauge fixing, at least locally. The vanishing of the inhomogeneous contributions in the transformation laws (6.2.51)-(6.2.52) constrains parameters f, Y^A and reduces the set of allowed vectors among (6.2.21). The remaining residual transformations are studied in subsection 6.2.4.

Constraint equations as Bondi evolution equations

Assuming the gauge fixing conditions (6.2.49), we are now ready to present the evolution equations that follow from the remaining Einstein equations. Moreover, we suppose that $\mathcal{D}_{AB} = 0$ in the case $\Lambda = 0$ to simplify our computation. As justified before, the $\mathcal{O}(r^0)$ part of $r^2(R_{uA} - \Lambda g_{uA}) = 0$ will fix the temporal evolution of N_A. From the Christoffel symbols, we can develop the first term as

$$\begin{aligned} R_{uA} = &- (\partial_u - l)\partial_A\beta - \partial_A l - (\partial_u + l)n_A \\ &+ n_B\mathcal{D}^B U_A - \partial_B\beta\mathcal{D}_A U^B + 2U^B(\partial_A\beta\partial_B\beta + n_A n_B) \\ &+ \mathcal{D}_B\Big[l_A^B + \frac{1}{2}(\mathcal{D}^B U_A - \mathcal{D}_A U^B) + U^B(\partial_A\beta - n_A)\Big] + 2n_B l_A^B \\ &- \frac{1}{2}(\partial_r + 2\partial_r\beta + \frac{2}{r})\partial_A\frac{V}{r} - \frac{V}{r}(\partial_r + \frac{2}{r})n_A + k_A^B(\partial_B\frac{V}{r} + 2\frac{V}{r}n_B) \\ &- e^{-2\beta}(\partial_r + \frac{2}{r})\Big[U^B(l_{AB} + \frac{V}{r}k_{AB} + \mathcal{D}_{(A}U_{B)})\Big] \end{aligned} \tag{6.2.53}$$

$$- e^{-2\beta}U^B\Big[(\partial_u + l)k_{AB} - 4l^C_{(A}k_{B)C} - 2k^C_A k_{BC}\frac{V}{r} + \mathcal{D}_C(k_{AB}U^C) - 2k_{C(A}\mathcal{D}^C U_{B)}\Big].$$

Let us emphasize that the r-dependence of the fields is not yet explicit in this expression, so the upper case Latin indices are lowered and raised by the full metric g_{AB} and its inverse. Expanding all the fields in power series of $1/r$ in R_{uA} and Λg_{uA} and selecting the $1/r^2$ terms yields

$$(\partial_u + l)N_A^{(\Lambda)} - \partial_A M^{(\Lambda)} - \frac{\Lambda}{2}D^B J_{AB} = 0. \tag{6.2.54}$$

Here, we defined with hindsight the Bondi mass and angular momentum aspects for $\Lambda \neq 0$ as

$$M^{(\Lambda)} = M + \frac{1}{16}(\partial_u + l)(C_{CD}C^{CD}), \tag{6.2.55}$$

$$N_A^{(\Lambda)} = N_A - \frac{3}{2\Lambda}D^B(N_{AB} - \frac{1}{2}lC_{AB}) - \frac{3}{4}\partial_A(\frac{1}{\Lambda}R[q] - \frac{3}{8}C_{CD}C^{CD}), \tag{6.2.56}$$

and the traceless symmetric tensor J_{AB} ($q^{AB}J_{AB} = 0$) as

$$\begin{aligned} J_{AB} = &- \mathcal{E}_{AB} - \frac{3}{\Lambda^2}\Big[\partial_u(N_{AB} - \frac{1}{2}lC_{AB}) - \frac{\Lambda}{2}q_{AB}C^{CD}(N_{CD} - \frac{1}{2}lC_{CD})\Big] \\ &+ \frac{3}{\Lambda^2}(D_A D_B l - \frac{1}{2}q_{AB}D_C D^C l) \\ &- \frac{1}{\Lambda}(D_{(A}D^C C_{B)C} - \frac{1}{2}q_{AB}D^C D^D C_{CD}) \\ &+ C_{AB}\Big[\frac{5}{16}C_{CD}C^{CD} + \frac{1}{2\Lambda}R[q]\Big]. \end{aligned} \tag{6.2.57}$$

We used the notation $N_{AB} \equiv \partial_u C_{AB}$. This tensor is symmetric and obeys $q^{AB}N_{AB} = \frac{\Lambda}{3}C^{AB}C_{AB}$. When $\Lambda = 0$, N_{AB} is thus traceless and represents the Bondi news tensor.

We will justify the definitions of Bondi mass and angular momentum aspects in section 6.2.3. Note that $\partial_u q_{AB}$ has been eliminated using (6.2.45). The transformations of these fields under the residual gauge symmetries ξ preserving the Bondi gauge (6.2.18) and the boundary gauge (6.2.49) are given by

$$\delta_\xi M^{(\Lambda)} = [f\partial_u + \mathcal{L}_Y + \frac{3}{2}(D_A Y^A + fl - 2\omega)]M^{(\Lambda)} - \frac{\Lambda}{3}N_A^{(\Lambda)}\partial^A f, \tag{6.2.58}$$

$$\delta_\xi N_A^{(\Lambda)} = [f\partial_u + \mathcal{L}_Y + D_B Y^B + fl - 2\omega]N_A^{(\Lambda)} + 3M^{(\Lambda)}\partial_A f + \frac{\Lambda}{2}J_{AB}\partial^B f, \tag{6.2.59}$$

$$\begin{aligned} \delta_\xi J_{AB} = &[f\partial_u + \mathcal{L}_Y + \frac{1}{2}(D_C Y^C + fl - 2\omega)]J_{AB} \\ &- \frac{4}{3}(N^{(\Lambda)}_{(A}\partial_{B)}f - \frac{1}{2}N_C^{(\Lambda)}\partial^C f q_{AB}). \end{aligned} \tag{6.2.60}$$

The asymptotically flat limit is not trivial in the equation (6.2.54) due to terms $\sim \Lambda^{-1}$ above which we collect here:

$$\begin{aligned}
-\frac{3}{2\Lambda}\Big[&(\partial_u+l)D^B(\partial_u C_{AB}-\frac{1}{2}lC_{AB})-D^B\partial_u(\partial_u C_{AB}-\frac{1}{2}lC_{AB})\\
&+\frac{1}{2}(\partial_u+l)\partial_A R[q]+D^B(D_AD_Bl-\frac{1}{2}D_CD^Clq_{AB})\Big].
\end{aligned} \tag{6.2.61}$$

There are two subtle steps here needed to massage the evolution equation before taking the limit $\Lambda\to 0$. First, we must develop the remaining u-derivatives acting on covariant derivatives and taking the constraint (6.2.45) into account to highlight Λ factors. Next, we can extract the trace of N_{AB}, which also contains a residual contribution $\sim \Lambda$. We end up with

$$\begin{aligned}
(6.2.61)=&\frac{1}{2}D_C(N^{TF}_{AB}C^{BC})+\frac{1}{4}N^{TF}_{BC}D_AC^{BC}-\frac{1}{4}D_AD_BD_CC^{BC}\\
&+\frac{1}{8}C^B_CC^C_B\partial_Al-\frac{3}{16}l\partial_A(C^B_CC^C_B)
\end{aligned} \tag{6.2.62}$$

where N^{TF}_{AB} denotes the trace-free part of N_{AB}. The following identities turn out to be useful for the computation:

$$\begin{aligned}
(\partial_u+l)H^{AB}&=q^{AC}\partial_uH_{CD}q^{BD}-lH^{AB}-\frac{2\Lambda}{3}C^{C(A}H^{B)}_C,\\
(\partial_u+l)(D^BH_{AB})&=D^B\partial_uH_{AB}-\frac{1}{2}q^{CD}H_{CD}\partial_Al\\
&\quad-\frac{\Lambda}{3}\Big[D_C(H_{AB}C^{BC})+\frac{1}{2}H^{BC}D_AC_{BC}\Big],\\
(\partial_u+l)C^{AB}C_{AB}&=2N^{AB}C_{AB}-lC_{AB}C^{AB},\\
(\partial_u+l)\partial_AR[q]&=-(D^BD_B+\frac{1}{2}R[q])\partial_Al+\frac{\Lambda}{3}D_AD_BD_CC^{BC}
\end{aligned} \tag{6.2.63}$$

where $H_{AB}(u,x^C)$ is any symmetric rank 2 transverse tensor. We note that $N^{TF}_{AB}C^{BC}+C_{AB}N^{BC}_{TF}=\delta^C_AC_{BD}N^{BD}_{TF}$, thanks to which the first term of (6.2.62) can be rewritten as

$$\frac{1}{2}D_C(N^{TF}_{AB}C^{BC})=\frac{1}{4}D_B(N^{TF}_{AC}C^{BC}-C_{AC}N^{BC}_{TF})+\frac{1}{4}\partial_A(C_{BD}N^{BD}_{TF}). \tag{6.2.64}$$

We can now present (6.2.54) in a way that makes terms in Λ explicit:

$$\begin{aligned}
(\partial_u+l)N_A&-\partial_AM-\frac{1}{4}C_{AB}\partial^BR[q]-\frac{1}{16}\partial_A(N^{TF}_{BC}C^{BC})\\
&-\frac{1}{32}l\partial_A(C_{BC}C^{BC})+\frac{1}{4}N^{TF}_{BC}D_AC^{BC}+\frac{1}{4}D_B(C^{BC}N^{TF}_{AC}-N^{BC}_{TF}C_{AC})\\
&+\frac{1}{4}D_B(D^BD^CC_{AC}-D_AD_CC^{BC})+\frac{\Lambda}{2}D^B(\mathcal{E}_{AB}-\frac{7}{96}C^C_DC^D_CC_{AB})=0.
\end{aligned} \tag{6.2.65}$$

As a result, the asymptotically flat limit can be safely taken and (6.2.65) reduces to

$$\begin{aligned}(\partial_u + l)N_A - \partial_A M - \frac{1}{4}C_{AB}\partial^B R[q] - \frac{1}{16}\partial_A(N_{BC}C^{BC}) \\ - \frac{1}{32}l\partial_A(C_{BC}C^{BC}) + \frac{1}{4}N_{BC}D_A C^{BC} + \frac{1}{4}D_B(C^{BC}N_{AC} - N^{BC}C_{AC}) \\ + \frac{1}{4}D_B(D^B D^C C_{AC} - D_A D_C C^{BC}) = 0,\end{aligned} \tag{6.2.66}$$

which fully agrees with (4.49) of [6] after a change of conventions[3]. It must be mentioned that $N_{AB} = N^{TF}_{AB}$ when $\Lambda = 0$.

We now derive the temporal evolution of M, encoded in the r-independent part of $r^2(R_{uu} - \Lambda g_{uu}) = 0$. The first term is worked out to be

$$\begin{aligned}R_{uu} = \; & (\partial_u + 2\partial_u\beta + l)\Gamma^u_{uu} + (\partial_r + 2\partial_r\beta + \frac{2}{r})\Gamma^r_{uu} + (\mathcal{D}_A + 2\partial_A\beta)\Gamma^A_{uu} \\ & - 2\partial_u^2\beta - \partial_u l - (\Gamma^u_{uu})^2 - 2\Gamma^u_{uA}\Gamma^A_{uu} - (\Gamma^r_{ur})^2 - 2\Gamma^r_{uA}\Gamma^A_{ur} - \Gamma^A_{uB}\Gamma^B_{uA}\end{aligned} \tag{6.2.67}$$

where all Christoffel symbols can be found on page 26 of [6]. We finally get

$$(\partial_u + \frac{3}{2}l)M^{(\Lambda)} + \frac{\Lambda}{6}D^A N_A^{(\Lambda)} + \frac{\Lambda^2}{24}C_{AB}J^{AB} = 0. \tag{6.2.68}$$

Here, the asymptotically flat limit is straightforward and gives

$$\begin{aligned}(\partial_u + \frac{3}{2}l)M + \frac{1}{8}N_{AB}N^{AB} - \frac{1}{8}lN_{AB}C^{AB} + \frac{1}{32}l^2 C_{AB}C^{AB} - \frac{1}{8}D_A D^A R[q] \\ -\frac{1}{4}D_A D_B N^{AB} + \frac{1}{4}C^{AB}D_A D_B l + \frac{1}{4}\partial_{(A}l D_{B)}C^{AB} + \frac{1}{8}l D_A D_B C^{AB} = 0,\end{aligned} \tag{6.2.69}$$

in agreement with (4.50) of [6]. As a conclusion, in Bondi gauge (6.2.18) with fall-off condition (6.2.22) and boundary gauge fixing (6.2.49), the general solution to Einstein's equations is entirely determined by the seven free functions of (u, x^A) for the case $\Lambda \neq 0$: q_{AB} with fixed area $\sqrt{\bar{q}}$, M, N_A and trace-free J_{AB} where M and N_A are constrained by the evolution equations (6.2.68) and (6.2.54). This contrasts with the asymptotically flat case $\Lambda = 0$ where an infinite series of functions appearing in the radial expansion of g_{AB} have to be specified to parametrize the solution (see e.g. [6]).

6.2.3 Dictionary between Fefferman-Graham and Bondi gauges

In appendix D, we establish a coordinate transformation between Fefferman-Graham and Bondi gauges, which extends the procedure used in [167] to a generic spacetime

[3] The Bondi news tensor is defined in [6] as $N^{\text{there}}_{AB} = \partial_u C_{AB} - lC_{AB}$ while we define $N^{\text{here}}_{AB} = \partial_u C_{AB}$.

metric. The boundary metric in the Fefferman-Graham gauge is related to the functions in the Bondi gauge through

$$g^{(0)}_{tt} = \frac{\Lambda}{3} e^{4\beta_0} + U^C_0 U^0_C, \qquad g^{(0)}_{tA} = -U^0_A, \qquad g^{(0)}_{AB} = q_{AB}, \tag{6.2.70}$$

where all functions on the right-hand sides are now evaluated as functions of (t, x^A).

The parameters $\{\sigma, \xi^t_0, \xi^A_0\}$ of the residual gauge diffeomorphisms in the Fefferman-Graham gauge (6.2.2) and (6.2.3) can be related to those of the Bondi gauge appearing in (6.2.21) through

$$\begin{aligned} \xi^t_0 &= f, \\ \xi^A_0 &= Y^A, \\ \sigma &= \frac{1}{2}(D_A Y^A + f l - U^A_0 \partial_A f - 2\omega), \end{aligned} \tag{6.2.71}$$

where all functions on the right-hand sides are also evaluated as functions of (t, x^A).

The boundary gauge fixing (6.2.49) described in section 6.2.2 can now be understood as a gauge fixation of the boundary metric to

$$g^{(0)}_{tt} = \frac{\Lambda}{3}, \qquad g^{(0)}_{tA} = 0, \qquad \det(g_{(0)}) = \frac{\Lambda}{3}\bar{q}. \tag{6.2.72}$$

For $\Lambda < 0$ (resp. $\Lambda > 0$), this is exactly the temporal (resp. radial) gauge for the boundary metric, with a fixed area form for the 2-dimensional transverse space.

Let us develop the constraint equations (6.2.6) after boundary gauge fixing. First, the tracelessness condition determines the trace of T_{AB} to be

$$q^{AB} T_{AB} = -\frac{3}{\Lambda} T_{tt}. \tag{6.2.73}$$

We define T^{TF}_{AB} as the trace-free part of T_{AB}, i.e. $T_{AB} = T^{TF}_{AB} - \frac{3}{2\Lambda} T_{tt} q_{AB}$. The conservation equation $D^{(0)}_a T^{ab} = 0$ reads as

$$\begin{aligned} (\partial_t + \frac{3}{2} l) T_{tt} + \frac{\Lambda}{3} D^A T_{tA} - \frac{\Lambda}{6} \partial_t q_{AB} T^{AB}_{TF} &= 0, \\ (\partial_t + l) T_{tA} - \frac{1}{2} \partial_A T_{tt} + \frac{\Lambda}{3} D^B T^{TF}_{AB} &= 0. \end{aligned} \tag{6.2.74}$$

Pursuing the change of coordinates to the Fefferman-Graham gauge up to fourth order in ρ, it can be shown that the stress tensor is given, in terms of Bondi variables, by

$$T_{ab} = \frac{\sqrt{3|\Lambda|}}{16\pi G} \begin{bmatrix} -\frac{4}{3} M^{(\Lambda)} & -\frac{2}{3} N^{(\Lambda)}_B \\ -\frac{2}{3} N^{(\Lambda)}_A & J_{AB} + \frac{2}{\Lambda} M^{(\Lambda)} q_{AB} \end{bmatrix}, \tag{6.2.75}$$

where $M^{(\Lambda)}(t, x^A)$ and $N_A^{(\Lambda)}(t, x^B)$ are the boundary fields defined as (6.2.55)-(6.2.56) and J_{AB} is precisely the tensor (6.2.57), all evaluated as functions of t instead of u. The conservation equations (6.2.74) are, in fact, equivalent to (6.2.68) and (6.2.54) after using the dictionary (6.2.75) and solving $\partial_t q_{AB}$ in terms of C_{AB} using (6.2.45). Moreover, we checked that the transformation laws (6.2.58)-(6.2.60) are equivalent to (6.2.9). We therefore identified the Bondi mass aspect $M^{(\Lambda)}$ and the Bondi angular momentum aspect $N_A^{(\Lambda)}$ as the components T_{tt} and T_{tA} of the holographic stress-tensor, up to a normalization constant.

6.2.4 Symmetries and flat limit

In contrast to the three-dimensional case discussed in subsection 6.1.5, the BMS group in four dimensions is not readily obtained by taking the flat limit of the asymptotic symmetry group associated with Dirichlet boundary conditions in AdS. In the following, we discuss the technical issue of finding a version of BMS in AdS, which reduces to the BMS group in the flat limit. Then, we present our new set of boundary conditions in asymptotically locally (A)dS$_4$ spacetime that leads to the Λ-BMS$_4$ algebra $\mathfrak{bms}_4^\Lambda$. We show that in the flat limit, this reduces to $\mathfrak{bms}_4^{\text{gen}}$.

The problem to obtain BMS in the flat limit

In this subsection, mimicking the three-dimensional case discussed in subsection 6.1.5, we consider Dirichlet boundary conditions defining asymptotically AdS$_4$ spacetimes in the Fefferman-Graham gauge:

$$g_{ab}^{(0)} \mathrm{d}x^a \mathrm{d}x^b = \frac{\Lambda}{3} \mathrm{d}t^2 + \mathring{q}_{AB} \mathrm{d}x^A \mathrm{d}x^B, \tag{6.2.76}$$

where $\mathring{q}_{AB}$ is the unit sphere metric (see definition (AAdS2) given in equation (2.2.23)). It has been shown in [15] that the asymptotic symmetry algebra is given by $\mathfrak{so}(3,2)$ and the associated charges are finite, integrable, and form a representation of $\mathfrak{so}(3,2)$ without central extension. Therefore, we obtain a finite-dimensional algebra, which already ends any hope of obtaining BMS in the flat limit.

Using (6.2.70), we can translate the Dirichlet boundary condition (6.2.76) into the Bondi gauge as

$$\beta_0 = 0, \qquad U_A^0 = 0, \qquad q_{AB} = \mathring{q}_{AB} \tag{6.2.77}$$

(this is the four-dimensional analogue of (6.1.64)). The residual gauge diffeomorphisms preserving these constraints are given by (6.2.21), where the parameters satisfy

$$\begin{aligned} &\partial_u f = \frac{1}{2} D_A Y^A, \qquad \partial_u Y^A = -\frac{\Lambda}{3} \mathring{q}^{AB} \partial_B f, \qquad \omega = 0, \\ &\mathcal{L}_Y \mathring{q}_{AB} = (D_C Y^C) \mathring{q}_{AB}, \end{aligned} \tag{6.2.78}$$

where the last equation tells us that Y^A is a conformal Killing vector of the unit 2-sphere metric. One can show that these conditions imply that f and Y^A are the parameters of the $\mathfrak{so}(3,2)$ asymptotic symmetry algebra, as it should be [24, 166]. Therefore, we conclude that, despite the conditions being interpreted as asymptotically flat boundary conditions (AF3) (see equation (2.2.15) together with (2.2.18)) in the flat limit $\Lambda \to 0$, we recover the Poincaré group instead of the BMS group [24, 166]. In particular, the supertranslations cannot be recovered through this process.

The Λ-BMS group and its flat limit

We now circumvent this issue by proposing a new set of boundary condition in asymptotically locally (A)dS$_4$. We require that

$$\beta_0 = 0, \qquad U^0_A = 0, \qquad \sqrt{q} = \sqrt{\mathring{q}}, \tag{6.2.79}$$

where $\mathring{q}$ is the determinant of the unit sphere metric (this last condition leads to $\delta\sqrt{q} = 0$). Several comments can be made about these boundary conditions. They are very similar to (6.2.77), except that we allow some fluctuations of the two-dimensional boundary metric q_{AB} with fixed determinant. These boundary conditions are inspired by those investigated in the asymptotically flat context in chapter 4 (see equation (4.1.3)). Furthermore, the boundary conditions (6.2.79) are precisely the conditions imposed in the boundary gauge fixing (6.2.49) (with $\bar{q} = \mathring{q}$) to write the evolution equations of the Bondi mass and the angular momentum aspect with respect to the u coordinate. Finally, we notice that the boundary conditions (6.2.79) are valid for both $\Lambda > 0$ and $\Lambda < 0$. Indeed, as discussed around equation (6.2.49), every solution written in the Bondi gauge and satisfying the preliminary boundary conditions $g_{AB} = \mathcal{O}(r^2)$, can be transformed through a diffeomorphism to satisfy (6.2.79). Therefore, this does not constrain the Cauchy problem in dS$_4$. This contrasts with the Dirichlet boundary conditions (6.2.77) that do not make sense to impose in dS$_4$ since they would strongly constrain the Cauchy problem.

The residual gauge diffeomorphisms (6.2.21) preserving the boundary conditions (6.2.79) have the following constraints on their parameters:

$$\partial_u f = \frac{1}{2} D_A Y^A, \qquad \partial_u Y^A = -\frac{\Lambda}{3} q^{AB} \partial_B f, \qquad \omega = 0. \tag{6.2.80}$$

Note that the solutions of these equations admit three integration "constants" $S(x^A)$, $V^A(x^B)$, though these are difficult to solve explicitly for an arbitrary transverse metric q_{AB} in terms of these functions (see appendix C of [193] for an explicit solution in the case $q_{AB} = \mathring{q}_{AB}$). We call the vectors generated by $S(x^A)$ and $V^A(x^B)$ the supertranslation and superrotation generators, respectively. The use of this terminology will be justified below. In the Fefferman-Graham notation, the equations in (6.2.80) are equivalent to

$$\begin{aligned} \sigma &= \frac{1}{2} D_A^{(0)} \xi_0^A, \\ \partial_t \xi_0^t &= \frac{1}{2} D_A^{(0)} \xi_0^A, \quad \partial_t \xi_0^A = -\frac{\Lambda}{3} g_{(0)}^{AB} \partial_B \xi_{(0)}^t. \end{aligned} \tag{6.2.81}$$

As already discussed in one of the examples in subsection 2.2.4, the asymptotic Killing vectors satisfy the following commutation relations with the modified Lie bracket (2.2.57):

$$[\xi(f_1, Y_1^A), \xi(f_2, Y_2^A)]_A = \xi(\hat{f}, \hat{Y}^A), \tag{6.2.82}$$

where

$$\hat{f} = Y_1^A \partial_A f_2 + \frac{1}{2} f_1 D_A Y_2^A - \delta_{\xi(f_1, Y_1^A)} f_2 - (1 \leftrightarrow 2), \tag{6.2.83}$$

$$\hat{Y}^A = Y_1^B \partial_B Y_2^A - \frac{\Lambda}{3} f_1 q^{AB} \partial_B f_2 - \delta_{\xi(f_1, Y_1^A)} Y_2^A - (1 \leftrightarrow 2). \tag{6.2.84}$$

In the asymptotically flat limit $\Lambda = 0$, the functions Y^A, f reduce to $Y^A = Y^A(x^B)$, $f = T(x^A) + \frac{u}{2} D_A V^A$ and the structure constants reduce to the ones of the generalized BMS$_4$ algebra $\mathfrak{bms}_4^{\text{gen}}$ (see equations (4.1.12) and (4.1.18)). For $\Lambda \neq 0$, the supertranslations do not commute and the structure constants depend explicitly on q_{AB}. We therefore find the structure of a Lie algebroid [146, 170, 172, 229][4]. We call it the Λ-BMS$_4$ algebra and we write it $\mathfrak{bms}_4^\Lambda$. This algebroid gives an infinite-dimensional algebra at each point of the solution space. Indeed, it always contains the area preserving diffeomorphisms given by $\xi = \xi(f = 0, Y^A = Y^A(x^B))$, where $D_A Y^A = 0$[5].

When the transverse metric q_{AB} is equal to the unit round sphere metric $\mathring{q}_{AB}$, we are back to the Dirchlet boundary conditions (6.2.77) and, therefore, $\mathfrak{bms}_4^\Lambda$ reduces to $\mathfrak{so}(3,2)$ for $\Lambda < 0$ and the $\mathfrak{so}(1,4)$ algebra for $\Lambda > 0$ (see [24] and appendix A of [166]).

6.3 Holographic renormalization and surface charges

In this section, we reproduce the holographic renormalization in asymptotically locally (A)dS$_4$ spacetime in the Fefferman-Graham gauge [144, 154, 222]. In this pro-

[4]The existence of the Λ-BMS$_4$ Lie algebroid is not in contradiction with recent no-go results [230] that were obtained for Lie algebra deformations. Here, we have a field-dependent Lie algebroid deformation of the BMS Lie algebra in asymptotically locally (A)dS$_4$ spacetimes.

[5]These vectors are called area preserving diffeomorphisms since, for a diffeomorphism on a two-dimensional Riemannian manifold with metric q_{AB}, the determinant transforms infinitesimally as $\delta_Y \sqrt{q} = D_A Y^A$. Therefore, the divergence-free vectors fields Y^A generate diffeomorphisms that preserve the area.

cess, we assume only the preliminary boundary condition $\gamma_{ab} = \mathcal{O}(\rho^{-2})$ (see (6.2.4)). This allows us to obtain the renormalized presymplectic form, from which we extract the charges for the most general solution space of asymptotically locally (A)dS$_4$ spacetime in the Fefferman-Graham gauge. Then, we compute the charge algebra and show that it closes under the modified bracket, without central extension.

6.3.1 Presymplectic structure and its ambiguities

In what follows, we will see that the counter-terms brought to the presymplectic form by the holographic renormalization process can be interpreted as ambiguities from the point of view of the Iyer-Wald procedure discussed in subsection 2.3.4. Let us mention two possible sources of ambiguities in the procedure that will appear in this process.

A first ambiguity is the one discussed in subsection 2.3.4 (see equation (2.3.81) and the discussion that follows) and allows us to shift the presymplectic potential $\boldsymbol{\theta}[g;\delta g]$ by an exact $(n-1)$-form as

$$\boldsymbol{\theta}[g;\delta g] \to \boldsymbol{\theta}[g;\delta g] - \mathrm{d}\mathbf{Y}[g;\delta g]. \tag{6.3.1}$$

This leads to the following shift in the presymplectic form

$$\boldsymbol{\omega}[g;\delta g,\delta g] \to \boldsymbol{\omega}[g.\delta g,\delta g] - \delta\mathrm{d}\mathbf{Y}[g;\delta g]. \tag{6.3.2}$$

In particular, this ambiguity has already been used in section 4.2 to renormalize the symplectic structure in asymptotically flat spacetime.

Another freedom that we have is to modify the Lagrangian $\mathbf{L}[g]$ of the theory by boundary terms,

$$\boldsymbol{L}_{EH}[g] \to \boldsymbol{L}_{EH}[g] + \mathrm{d}\boldsymbol{A}[g]. \tag{6.3.3}$$

This shifts the presymplectic potential by an exact term

$$\boldsymbol{\theta}[g;\delta g] \to \boldsymbol{\theta}[g;\delta g] + \delta\boldsymbol{A}[g] \tag{6.3.4}$$

but leaves the presymplectic form invariant ($\delta^2 = 0$). Therefore, this freedom does not lead to further ambiguity in the symplectic structure.

For a non-vanishing cosmological constant, the Einstein-Hilbert Lagrangian is

$$\boldsymbol{L}_{EH}[g] = \frac{1}{16\pi G}\left(R[g] - 2\Lambda\right)\sqrt{-g}\,\mathrm{d}^4x. \tag{6.3.5}$$

The associated canonical presymplectic potential is given by

$$\boldsymbol{\theta}_{EH}[g;\delta g] = \frac{\sqrt{-g}}{16\pi G}\Big(\nabla_\nu(\delta g)^{\mu\nu} - \nabla^\mu(\delta g)^\nu{}_\nu\Big)(\mathrm{d}^3x)_\mu \tag{6.3.6}$$

where $(\delta g)^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$ (see equation (2.3.74)).

The radial component of the presymplectic potential can be computed as follows:

$$\begin{aligned}\Theta^\rho_{EH}[g;\delta g] &= \frac{\sqrt{-g}}{16\pi G}\Big(\nabla_\alpha(\delta g)^{\rho\alpha} - g^{\rho\rho}\partial_\rho(\delta g)^\alpha{}_\alpha\Big) \\ &= \frac{\sqrt{-g}}{16\pi G}\Big(\Gamma^\rho_{ab}\gamma^{ac}\gamma^{bd}\delta\gamma_{cd} - g^{\rho\rho}\partial_\rho(\gamma^{ab}\delta\gamma_{ab})\Big).\end{aligned} \tag{6.3.7}$$

Expanding the metric $\gamma_{ab}(\rho, x^c)$ in powers of ρ, we get

$$\begin{aligned}\Theta^\rho_{EH}[g;\delta g] = \sqrt{\frac{3}{|\Lambda|}}\left[-\frac{1}{\rho^3}\frac{2\Lambda}{3}\frac{\delta\sqrt{|g_{(0)}|}}{16\pi G} + \frac{1}{\rho}\left(-\frac{3}{4}\delta L_{EH,(0)} + \partial_a\Theta^a_{EH,(0)}\right)\right] \\ + \frac{1}{2}\mathrm{sgn}(\Lambda)\sqrt{|g_{(0)}|}\,T^{ab}\delta g^{(0)}_{ab} + \mathcal{O}(\rho).\end{aligned} \tag{6.3.8}$$

We denoted by $L_{EH,(0)} = \frac{1}{16\pi G}R_{(0)}\sqrt{|g_{(0)}|}$ the Einstein-Hilbert Lagrangian density for the boundary metric field $g^{(0)}_{ab}$ and $\Theta^a_{EH,(0)}$ the canonical boundary term in the variation $\delta L_{EH,(0)}$. We observe that the presymplectic potential is radially divergent as we approach the boundary $\mathscr{I} \equiv \{\rho = 0\}$, so we need a renormalization procedure to obtain a well-defined symplectic structure at $\mathscr{I}$, allowing us to compute the surface charges. The precise form of the divergence suggests that there is a boundary-covariant way to subtract these divergences by refining the action principle of pure gravity in asymptotically locally (A)dS$_4$ spacetimes: this is the point of the *holographic renormalization* [144, 154, 222] that we review in the next section.

6.3.2 Holographic renormalization

The action for general relativity in asymptotically locally (A)dS$_4$ spacetimes is given by [144, 154, 222]

$$S = \frac{1}{16\pi G}\int_{\mathscr{M}} \mathrm{d}^4x\,\sqrt{|g|}\,(R[g] - 2\Lambda) + \int_{\mathscr{I}} \mathrm{d}^3x\, L_{GHY} + \int_{\mathscr{I}} \mathrm{d}^3x\, L_{ct}. \tag{6.3.9}$$

Here, $\mathscr{M}$ denotes the bulk spacetime and $\mathscr{I} = \partial\mathscr{M}$ its boundary. We impose that $\int_{\mathscr{M}} \mathrm{d}^4x = \int_0^\infty \mathrm{d}\rho' \int_{\rho=\rho'} \mathrm{d}^3x$. Remark that this convention sets the *lower* bound of the radial integral to be the boundary. The integration measure d^3x should be understood as a measure on the hypersurface at fixed $\rho = \rho'$. In particular, consistently with the notations of appendix A, we have

$$(\mathrm{d}^3x)^{n=3} = \frac{1}{3!}\epsilon^{n=3}_{abc}\mathrm{d}x^a\wedge\mathrm{d}x^b\wedge\mathrm{d}x^c \equiv \frac{1}{1!3!}\epsilon^{n=4}_{\rho abc}\mathrm{d}x^a\wedge\mathrm{d}x^b\wedge\mathrm{d}x^c = (\mathrm{d}^3x)^{n=4}_\rho. \tag{6.3.10}$$

This allows us to interpret the top form on the hypersurface $\rho = \rho'$ as co-dimension 1 with respect to the four-dimensional spacetime (for example, $L_{GHY}(\mathrm{d}^3x)^{n=3}|_{\rho=\rho'} \equiv$

$L^\rho_{GHY}(\mathrm{d}^3x)^{n=4}_\rho|_{\rho=\rho'})$. The first term in (6.3.9) is the bare Einstein-Hilbert action S_{EH}, the second term is the Gibbons-Hawking-York term S_{GHY} and the third term is the counter-term action S_{ct}.

Let us describe the additional boundary terms in (6.3.9) and justify their presence. To have a well-defined variational principle for Dirichlet boundary conditions, i.e. when all induced fields at the boundary are kept fixed ($\delta\gamma_{ab}|_{\mathscr{I}} = 0$), the action must be completed by the usual Gibbons-Hawking-York boundary term $S_{GHY} = \int_{\partial\mathscr{M}} \mathrm{d}^3x\, L_{GHY}$. Let us denote the outward normal unit vector by $\boldsymbol{n} = n^\mu\partial_\mu$, such that $n^\mu n_\mu = \eta$, where $\eta = -\mathrm{sgn}(\Lambda)$. Here, "outward" means that the vector points from the inside of the enclosed region to the outside. Recall that $\mathscr{I}$ is defined as the set of roots of the scalar field $f(\rho, x^a) = \rho$. Hence n_μ is collinear to $\partial_\mu f$ and differs only by a normalization factor and a relative sign. Since the coordinate ρ increases inwards, n^μ must point in the direction of decreasing f, such that $n^\mu\partial_\mu f < 0$, independently of η. We get $n_\mu = -\eta\sqrt{|g_{\rho\rho}|}\delta^\rho_\mu$. The knowledge of this unit normal vector allows us to define the extrinsic curvature as the trace of the second fundamental form $K = \gamma^{ab}K_{ab} = \frac{1}{2}\gamma^{ab}\mathcal{L}_{\boldsymbol{n}}\gamma_{ab}$, and build the Gibbons-Hawking-York piece

$$S_{GHY} = \frac{1}{8\pi G}\eta\int_{\mathscr{I}} \mathrm{d}^3x\sqrt{|\gamma|}K = \frac{1}{8\pi G}\sqrt{\frac{3}{|\Lambda|}}\int_{\mathscr{I}} \mathrm{d}^3x\,\frac{\Lambda}{3}\,\rho\partial_\rho\sqrt{|\gamma|}. \tag{6.3.11}$$

An important observation is that the on-shell action $S_{EH} + S_{GHY}$ is divergent. In order to deal with these divergences, we introduce an infrared cut-off $\varepsilon > 0$ (called the *regulator*) sufficiently small so that the Fefferman-Graham expansion is still valid around $\{\rho = \varepsilon\}$. The regulated variational principle

$$S^{\text{reg}} = \frac{1}{16\pi G}\int_\varepsilon^\infty \mathrm{d}\rho'\int_{\rho=\rho'} \mathrm{d}^3x(R[g] - 2\Lambda)\sqrt{|g|} + \frac{1}{8\pi G}\,\eta\int_{\rho=\varepsilon} \mathrm{d}^3x\sqrt{|\gamma|}K \tag{6.3.12}$$

possesses two divergent pieces on-shell

$$S^{\text{reg}} = \frac{1}{16\pi G}\sqrt{\frac{3}{|\Lambda|}}\int_{\rho=\varepsilon} \mathrm{d}^3x\left[-\frac{4\Lambda}{3}\sqrt{|g_{(0)}|}\frac{1}{\varepsilon^3} + \frac{1}{2}R_{(0)}\sqrt{|g_{(0)}|}\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)\right]. \tag{6.3.13}$$

The holographic renormalization procedure amounts to supplying the regulated variation principle with a second counterterm $S_{ct} = \int_{\rho=\varepsilon} \mathrm{d}^3x\, L_{ct}$ which must obey several requirements: S_{ct} is a boundary action constructed from a Lagrangian L_{ct} considered as a top-form living on the regulated hypersurface $\{\rho = \varepsilon\}$. The latter is built up from covariant objects living on $\{\rho = \varepsilon\}$, but is not required to be covariant with respect to the bulk geometry. In particular, it will involve the metric $\gamma_{ab}(\varepsilon, x^c)$ only. The renormalization requirement imposes that $S^{\text{reg}} + S_{ct} = \mathcal{O}(\varepsilon^0)$ after expanding in power series of ε. The working counterterm has been prescribed in [154, 222] and is given by

$$S_{ct} = \int_{\rho=\varepsilon} \mathrm{d}^3x \, L_{ct}[\gamma], \quad L_{ct}[\gamma] = \frac{1}{16\pi G}\sqrt{\frac{3}{\Lambda}}\left[\frac{4\Lambda}{3}\sqrt{|\gamma|} - R[\gamma]\sqrt{|\gamma|}\right]. \tag{6.3.14}$$

Indeed, it evidently satisfies the first two requirements, and we also check the last one by expanding L_{ct} in ε,

$$L_{ct} = \frac{1}{16\pi G}\sqrt{\frac{3}{|\Lambda|}}\left[\frac{4\Lambda}{3}\sqrt{|g_{(0)}|}\frac{1}{\varepsilon^3} - \frac{1}{2}R_{(0)}\sqrt{|g_{(0)}|}\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)\right], \tag{6.3.15}$$

hence $S^{\text{reg}} + S_{ct} = \mathcal{O}(\varepsilon)$. For later purposes, we define the presymplectic potential associated with L_{ct} as $\delta L_{ct} = \frac{\delta L_{ct}}{\delta\gamma^{ab}}\delta\gamma^{ab} + \partial_a \theta^a_{ct}[\gamma;\delta\gamma]$. It is given explicitly by

$$\begin{aligned}
\theta^a_{ct}[\gamma;\delta\gamma] &= -\frac{1}{16\pi G}\sqrt{\frac{3}{|\Lambda|}}\sqrt{|\gamma|}\left[D_b(\delta\gamma)^{ab} - \gamma^{ab}D_b(\delta\gamma)^c{}_c\right] \\
&= -\frac{1}{16\pi G}\rho\sqrt{|g|}\left[D_b(\delta\gamma)^{ab} - \gamma^{ab}D_b(\delta\gamma)^c{}_c\right] \\
&= -\rho\theta^a_{EH}[\gamma;\delta\gamma],
\end{aligned} \tag{6.3.16}$$

where D_a denotes the Levi-Civita connection with respect to γ_{ab}. Therefore,

$$\theta^a_{ct}[\gamma;\delta\gamma]\Big|_{\rho=\varepsilon} = -\sqrt{\frac{3}{\Lambda}}\Theta^a_{EH,(0)}[g_{(0)};\delta g_{(0)}]\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon). \tag{6.3.17}$$

Now we can concentrate on the renormalization of the presyplectic potential. On-shell, we have

$$\delta S^{\text{reg}}_{EH} = \int_{\rho\geqslant\varepsilon} \mathrm{d}^4x \, \partial_\mu \theta^\mu_{EH}[g;\delta g] = -\int_{\rho=\varepsilon} \mathrm{d}^3x \, \theta^\rho_{EH}[g;\delta g], \tag{6.3.18}$$

where the minus sign in the last equality is due to the fact that we integrate on ρ from the boundary to the bulk, which gives the negative orientation to the Stokes formula. The resulting integrand is only the ρ component of $\boldsymbol{\theta}_{EH}$ since the outward normal to the regulating surface is collinear to ∂_ρ. Therefore, we can prove by a straightforward computation that the renormalization of the presymplectic potential works as follows [144]:

$$\theta^\rho_{\text{ren}}[g;\delta g] = \theta^\rho_{EH} - \delta L_{GHY} - \delta L_{ct} + \partial_a\theta^a_{ct} = -\frac{1}{2}\eta\sqrt{|g_{(0)}|}T^{ab}\delta g^{(0)}_{ab} + \mathcal{O}(\rho). \tag{6.3.19}$$

We deduce from (6.3.19) that the modification brought to the presymplectic potential by the holographic renormalization is two-fold. The contribution of the exact terms with respect to δ are top-forms on the regularized boundary that can be promoted as bulk co-dimension 1-forms, collectively denoted by $\boldsymbol{A} = A^\rho (\mathrm{d}^4 x)_\rho$, with $A^\rho \equiv -(L_{GHY} + L_{ct})$, such that $\boldsymbol{\theta}_{EH} \to \boldsymbol{\theta}' = \boldsymbol{\theta}_{EH} + \delta \boldsymbol{A}$ after renormalization (see (6.3.4)). The contribution of the Iyer-Wald ambiguity appears here thanks to θ^a_{ct}, which is a co-dimension 1-form on the regularized boundary. Again we can promote it as a co-dimension 2-form on the bulk geometry $\boldsymbol{Y} = Y^{\rho a}(\mathrm{d}^2 x)_{\rho a}$, and $Y^{\rho a}[g;\delta g] \equiv \theta^a_{ct}[\gamma;\delta\gamma]$. The potential will be modified as $\boldsymbol{\theta}' \to \boldsymbol{\theta}' - \mathrm{d}\boldsymbol{Y}$ (see (6.3.3)), or in components,

$$\theta'^\rho \to \theta'^\rho + \partial_a Y^{\rho a} = \theta'^\rho + \partial_a \theta^a_{ct}, \tag{6.3.20}$$
$$\theta'^a \to \theta'^a + \partial_\rho Y^{a\rho} = \theta'^a - \partial_\rho \theta^a_{ct}, \tag{6.3.21}$$

In particular, (6.3.20) is consistent with (6.3.19), and it can be shown that (6.3.21) renormalizes the tangent components of the presymplectic potential as well. Finally, the renormalized presymplectic current is given by

$$\omega^\rho_{\text{ren}}[g;\delta g, \delta g] = -\frac{1}{2}\eta \delta \left(\sqrt{|g_{(0)}|} T^{ab}\right) \wedge \delta g^{(0)}_{ab} + \mathcal{O}(\rho). \tag{6.3.22}$$

6.3.3 Surface charges

Once the expression for the renormalized presymplectic potential $\boldsymbol{\theta}_{\text{ren}}[g;\delta g]$ is established, one can compute the Iyer-Wald co-dimension 2 form as $\boldsymbol{k}_{\xi,\text{ren}}[g;\delta g] = -\delta \boldsymbol{Q}_{\xi,\text{ren}}[g] + \boldsymbol{Q}_{\delta\xi,\text{ren}}[g] + i_\xi \boldsymbol{\theta}_{\text{ren}}[g;\delta g]$ (see equation (2.3.73)). In the present context, instead of directly computing this expression, we propose an ansatz for the co-dimension 2 form inspired by the results of [144] that were obtained for a subcase ($\sigma = 0$, $\delta\xi^a = 0$, $\Lambda < 0$). Our ansatz for the components ρa of the co-dimension 2 form associated with the most general asymptotically locally (A)dS$_4$ spacetime in Fefferman-Graham gauge is given by

$$k^{\rho a}_{\xi,\text{ren}}[g;\delta g] = \eta \delta \left(\sqrt{|g_{(0)}|} T^a{}_b\right) \xi^b_0 - \frac{1}{2}\eta \sqrt{|g_{(0)}|}\, \xi^a_0\, T^{bc} \delta g^{(0)}_{bc} + \mathcal{O}(\rho). \tag{6.3.23}$$

To confirm that this proposal is the correct one, we check that it satisfies the conservation law (2.3.77), namely

$$\mathrm{d}\boldsymbol{k}_{\xi,\text{ren}}[g;\delta g] = \boldsymbol{\omega}_{\text{ren}}[g;\mathcal{L}_\xi g, \delta g] \quad \Rightarrow \quad \partial_a k^{\rho a}_{\xi,\text{ren}}[g;\delta g] = \omega^\rho_{\text{ren}}[g;\mathcal{L}_\xi g, \delta g]. \tag{6.3.24}$$

The detailed computation can be found in appendix E.1. Since the co-dimension 2 form $k_{\xi,\text{ren}}[g;\delta g]$ is defined up to an exact co-dimension 2 form, we are certain that the proposal (6.3.23) is the right one for the renormalized presymplectic form (6.3.22).

6.3.4 Charge algebra

Modified Lie bracket for residual gauge diffeomorphisms

Let us denote by ξ and χ two arbitrary residual gauge diffeomorphisms of the Fefferman-Graham expansion which are of the form (6.2.2) and (6.2.3). We recall that the modified Lie bracket is given by

$$[\xi, \chi]_A = [\xi, \chi] - \delta_\xi \chi + \delta_\chi \xi, \tag{6.3.25}$$

with $\delta_\xi g_{\mu\nu} \equiv \mathcal{L}_\xi g_{\mu\nu}$ (see (2.2.57)). We now provide an explicit computation of this bracket for the present case. Since ξ and χ preserve the Fefferman-Graham gauge, they satisfy

$$\begin{cases} \xi^\rho = \rho\sigma_\xi(x^a), \quad \partial_\rho \xi^a = \dfrac{3}{\Lambda}\dfrac{1}{\rho}\gamma^{ab}\partial_b\sigma_\xi, \quad \lim\limits_{\rho\to 0}\xi^a = \xi^a_0(x^b), \\ \chi^\rho = \rho\sigma_\xi(x^a), \quad \partial_\rho \chi^a = \dfrac{3}{\Lambda}\dfrac{1}{\rho}\gamma^{ab}\partial_b\sigma_\chi, \quad \lim\limits_{\rho\to 0}\chi^a = \chi^a_0(x^b). \end{cases} \tag{6.3.26}$$

As a result, the computation of $[\xi, \chi]^\rho_A$ is straightforward and gives

$$\frac{1}{\rho}[\xi, \chi]^\rho_A = (\xi^a\partial_a\sigma_\chi - \chi^a\partial_a\sigma_\xi) - \delta_\xi\sigma_\chi + \delta_\chi\sigma_\xi. \tag{6.3.27}$$

Taking a derivative with respect to ρ, and again using $\rho^2\gamma_{ab}\partial_\rho\xi^b - \frac{3}{\Lambda}\partial_a\xi^\rho = 0$, we get

$$\partial_\rho\left(\frac{1}{\rho}[\xi, \chi]^\rho_A\right) = \partial_\rho\xi^a\partial_a\sigma_\chi - \partial_\rho\chi^a\partial_a\sigma_\xi = 0, \tag{6.3.28}$$

which shows that $[\xi, \chi]^\rho_\star = \rho\hat{\sigma}$, and

$$\hat{\sigma} = \frac{1}{\rho}[\xi, \chi]^\rho_A\Big|_{\rho=0} = \xi^a_0\partial_a\sigma_\chi - \chi^a_0\partial_a\sigma_\xi - \delta_\xi\sigma_\chi + \delta_\chi\sigma_\xi. \tag{6.3.29}$$

Let us now consider the transverse components. By evaluating the commutator at leading order, we derive that

$$\hat{\xi}^a_0 = \lim_{\rho\to 0}[\xi, \chi]^a_A = [\xi_0, \chi_0]^a - \delta_\xi\chi^a_0 + \delta_\chi\xi^a_0. \tag{6.3.30}$$

Recalling that $\delta_\xi\gamma^{ab} = \mathcal{L}_\xi\gamma^{ab} = \rho\sigma_\xi\partial_\rho\gamma^{ab} + \xi^c\partial_c\gamma^{ab} - 2\gamma^{c(a}\partial_c\xi^{b)}$ and explicitly using (6.3.26) to express $\partial_\rho\xi^a$ and $\partial_\rho\chi^b$ in terms of σ_ξ and σ_χ, respectively, a direct computation yields

$$\partial_\rho\left([\xi, \chi]^a_A\right) = \frac{3}{\Lambda}\frac{1}{\rho}\gamma^{ab}\partial_b\hat{\sigma}. \tag{6.3.31}$$

We have just proven that residual gauge diffeomorphisms ξ and χ of the Fefferman-Graham gauge satisfy

$$[\xi(\sigma_\xi, \xi_0), \chi(\sigma_\chi, \chi_0)]_A = \xi(\hat{\sigma}, \hat{\zeta}_0), \tag{6.3.32}$$

where

$$\begin{aligned} \hat{\sigma} &= \xi_0^a \partial_a \sigma_\chi - \chi_0^a \partial_a \sigma_\xi - \delta_\xi \sigma_\chi + \delta_\chi \sigma_\xi, \\ \hat{\zeta}^a &= \xi_0^b \partial_b \chi_0^a - \chi_0^b \partial_b \xi_0^a - \delta_\xi \chi_0^a + \delta_\chi \xi_0^a. \end{aligned} \tag{6.3.33}$$

Charge algebra with modified bracket

The co-dimension 2 form derived in (6.3.23) is generically non-integrable for the most general asymptotically locally (A)dS$_4$ boundary conditions in the Fefferman-Graham gauge. Therefore, the representation theorem (2.3.54) does not hold and one has to consider the modified bracket for the charges.

The leading term of (6.3.23) ($\sim \rho^0$) can be written as

$$k^{\rho a}_{\xi,\text{ren}}[g;\delta g]\Big|_{\rho=0} = \delta J^a_\xi[g] + \Xi^a_\xi[g;\delta g], \tag{6.3.34}$$

where the integrable part is taken as

$$J^a_\xi[g] = \eta\sqrt{|g_{(0)}|}\, g^{ac}_{(0)} T_{bc} \xi^b_0 \tag{6.3.35}$$

and the corresponding non-integrable part

$$\Xi^a_\xi[g;\delta g] = -\frac{1}{2}\eta\sqrt{|g_{(0)}|}\xi^a_0 \left(T^{bc}\delta g^{(0)}_{bc}\right) - J^a_{\delta\xi}. \tag{6.3.36}$$

Defining the modified bracket as

$$\{J_\xi[g], J_\chi[g]\}^a_* = \delta_\chi J^a_\xi[g] + \Xi^a_\chi[g;\delta_\xi g] \tag{6.3.37}$$

(this is the analogue of (2.3.61)), we show in appendix E.2 that

$$\{J_\xi[g], J_\chi[g]\}^a_* = J^a_{[\xi,\chi]_A}[g] + \partial_b \ell^{ab}[g]. \tag{6.3.38}$$

where $\partial_b \ell^{ab}[g]$ (see equation (E.2.4)) is a term that will disappear once integrated on S^2_∞. Therefore, when considering the modified bracket (6.3.37), we conclude that the currents satisfy a consistent algebra without central extension. This contrasts with the asymptotically flat case where a 2-cocycle appears in the right-hand side (compare (6.3.38) with (2.3.62)). Of course, as discussed around equation (2.3.66), modifying the split (6.3.35)-(6.3.36) between integrable and non-integrable parts will bring a trivial 2-cocycle in the algebra (6.3.38). This situation is similar to the case studied in [231] where the 2-cocycle could be absorbed by choosing an appropriate split.

6.4 Λ-BMS$_4$ phase space and its flat limit

In the previous section, we obtained through the holographic renormalization process the renormalized co-dimension 2 form associated with the most general asymptotically locally (A)dS$_4$ boundary conditions in the Fefferman-Graham gauge. We saw that the associated currents satisfy an algebra by using the modified bracket. In subsection 6.4.1, we particularize the analysis for the boundary conditions (6.2.79) and obtain the symplectic structure and the surface charges associated with the $\mathfrak{bms}_4^\Lambda$ asymptotic symmetry algebra. In subsection 6.4.2, we express the symplectic structure in terms of the Bondi gauge variables and perform the diffeomorphism discussed in appendix D. As discussed in section 6.2.2, the solution space with boundary conditions (6.2.79) reduces to the solution space considered in chapter 4. Similarly, in section 6.2.4, we showed that $\mathfrak{bms}_4^\Lambda$ reduces to $\mathfrak{bms}_4^{\text{gen}}$ in the flat limit. In subsection 6.4.3, after renormalization of $\sim 1/\Lambda$ divergences, we prove that this limit also holds at the level of the phase space.

6.4.1 Λ-BMS$_4$ phase space in Fefferman-Graham gauge

Using (6.2.70), the boundary conditions (6.2.79) can be expressed in the Fefferman-Graham gauge as

$$g^{(0)}_{tt} = \frac{\Lambda}{3}, \quad g^{(0)}_{tA} = 0, \quad \sqrt{|g_{(0)}|} = \sqrt{\frac{|\Lambda|}{3}}\sqrt{\mathring{q}}, \tag{6.4.1}$$

where $\mathring{q}$ is the determinant of the unit sphere metric. The asymptotic Killing vectors are the residual gauge diffeomorphisms given in (6.2.2) and (6.2.3), whose parameters satisfy (6.2.81). Inserting the conditions (6.4.1) into the renormalized presymplectic potential (6.3.19), we obtain

$$\theta^\rho_{\Lambda\text{-BMS}}[g;\delta g] = -\frac{\sqrt{\mathring{q}}}{2}\eta\sqrt{\frac{|\Lambda|}{3}}\, g^{AB}_{(0)}\delta T^{TF}_{AB} + \mathcal{O}(\rho), \tag{6.4.2}$$

where $T^{TF}_{AB} = T_{AB} - \frac{1}{2} g^{(0)}_{AB}(g^{CD}_{(0)}T_{CD})$. Similarly, the presymplectic form (6.3.22) reduces to

$$\omega^\rho_{\Lambda\text{-BMS}}[g;\delta g] = -\frac{\sqrt{\mathring{q}}}{2}\eta\sqrt{\frac{|\Lambda|}{3}}\,\delta g^{AB}_{(0)} \wedge \delta T^{TF}_{AB} + \mathcal{O}(\rho) \tag{6.4.3}$$

From (6.3.23), we deduce that the Λ-BMS$_4$ surface charges are given by

$$\delta\!\!\!/ H^{\Lambda\text{-BMS}}_\xi[g] = \int_{S^2_\infty} 2(\mathrm{d}^2x)_{\rho t} k^{\rho t}_{\xi,\Lambda\text{-BMS}}[g;\delta g] = \delta H^{\Lambda\text{-BMS}}_\xi[g] + \Theta^{\Lambda\text{-BMS}}_\xi[g;\delta g], \tag{6.4.4}$$

where we performed a split between integrable and non-integrable parts as in (6.3.35) and (6.3.36):

$$
\begin{aligned}
H_\xi^{\Lambda\text{-BMS}}[g] &= -\sqrt{\frac{3}{|\Lambda|}} \int_{S^2_\infty} \mathrm{d}^2\Omega\ \left[\xi_0^t T_{tt} + \xi_0^B T_{tB}\right], \\
\Theta_\xi^{\Lambda\text{-BMS}}[g;\delta g] &= \sqrt{\frac{3}{|\Lambda|}} \int_{S^2_\infty} \mathrm{d}^2\Omega\ \left[\frac{\Lambda}{6}\xi_0^t g_{(0)}^{AB}\delta T_{AB}\right] - H_{\delta\xi}^{\Lambda\text{-BMS}}[g].
\end{aligned}
\tag{6.4.5}
$$

Here, $\mathrm{d}^2\Omega = 2\sqrt{\mathring{q}}(\mathrm{d}^2x)_{\rho t}$ denotes the measure on S^2_∞. As a corollary of (6.3.38), they satisfy an algebra for the modified bracket

$$
\{H_\xi^{\Lambda\text{-BMS}}[g], H_\chi^{\Lambda\text{-BMS}}[g]\}_\star = H_{[\xi,\chi]_A}^{\Lambda\text{-BMS}}[g],
\tag{6.4.6}
$$

with $\{H_\xi^{\Lambda\text{-BMS}}[g], H_\chi^{\Lambda\text{-BMS}}[g]\}_\star = \delta_\chi H_\xi^{\Lambda\text{-BMS}}[g] + \Theta_\chi^{\Lambda\text{-BMS}}[g;\delta_\xi g]$, and $[\xi,\chi]_A$ given by

$$
[\xi(\xi_0^t,\xi_0^A),\chi(\chi_0^t,\chi_0^A)]_A = \hat{\xi}(\hat{\xi}_0^t,\hat{\xi}_0^A),
\tag{6.4.7}
$$

where

$$
\left\{
\begin{aligned}
\hat{\xi}_0^t &= \xi_0^A\partial_A\chi_0^t + \frac{1}{2}\xi_0^t D_A^{(0)}\chi_0^A - \delta_{\xi(\xi_0^t,\xi_0^A)}\chi_0^t - (\xi\leftrightarrow\chi), \\
\hat{\xi}_0^A &= \xi_0^B\partial_B\chi_0^A - \frac{\Lambda}{3}\xi_0^t g_{(0)}^{AB}\partial_B\chi_0^t - \delta_{\xi(\xi_0^t,\xi_0^A)}\chi_0^A - (\xi\leftrightarrow\chi).
\end{aligned}
\right.
\tag{6.4.8}
$$

This is a corollary of (6.3.33) and (6.3.32). Alternatively, one can obtain the commutation relations (6.4.8) from those written in the Bondi variables (6.2.84), using the dictionary (6.2.71).

6.4.2 Translation into the Bondi gauge

The next step is to perform the change of coordinates described in appendix D between Fefferman-Graham and Bondi gauges and deduce the transformation of the presymplectic potential. Starting from Fefferman-Graham coordinates, we first go to tortoise coordinates $(r_\star, x^a_\star)$. The presymplectic potential reads as $\boldsymbol{\theta}_{\Lambda\text{-BMS}} = \theta^\rho_{\Lambda\text{-BMS}}(\mathrm{d}^3x)_\rho + \theta^a_{\Lambda\text{-BMS}}(\mathrm{d}^3x)_a$. At leading order, $\rho = -\frac{\Lambda}{3}r_\star + \mathcal{O}(r_\star^2)$ and $x^a = x^a_\star + \mathcal{O}(r_\star)$, hence $\theta^{r_\star}_{\Lambda\text{-BMS}} = \theta^\rho_{\Lambda\text{-BMS}} + \mathcal{O}(r_\star)$. Therefore, the leading order of the radial component of the presymplectic potential is not affected. Now we can reach the Bondi gauge (u,r,x^A) by a second change of coordinates

$$
t_\star = u + \frac{3}{\Lambda}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad r_\star = \frac{3}{\Lambda}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad x^A_\star = x^A.
\tag{6.4.9}
$$

We obtain $\theta^r_{\Lambda\text{-BMS}} = \theta^{r\star}_{\Lambda\text{-BMS}} + \mathcal{O}(r)$[6]. Therefore, using (6.2.75), the renormalized presymplectic potential (6.4.2) is expressed in Bondi gauge as

$$\theta^r_{\Lambda\text{-BMS}}[g;\delta g] = \frac{\Lambda}{2}\frac{\sqrt{\mathring{q}}}{16\pi G}\, q^{AB}\delta J_{AB} + \mathcal{O}(r^{-1}). \tag{6.4.10}$$

The associated presymplectic form is readily obtained

$$\omega^r_{\Lambda\text{-BMS}}[g;\delta g] = \frac{\Lambda}{2}\frac{\sqrt{\mathring{q}}}{16\pi G}\,\delta q^{AB}\wedge\delta J_{AB} + \mathcal{O}(r^{-1}). \tag{6.4.11}$$

As discussed in [166], by analogy with the flat case (see e.g. (2.3.68)), this last expression allows us to identify the Bondi news functions in (A)dS$_4$ as the symplectic couple (q^{AB}, J_{AB}).

6.4.3 Renormalization in Λ and flat limit

The Bondi gauge admitting a well-defined flat limit, nothing prevents us from considering the flat limit of the above symplectic structure. Let us recall the prescription followed in subsection 6.2.2 when discussing the flat limit of the solution space:

1. In the Bondi gauge, we explicitly identify the dependence in Λ in the different expressions by using the relations in section 6.2.2 (see e.g. (6.2.55), (6.2.56), (6.2.57)), until obtaining only the functions C_{AB}, q_{AB}, N^{TF}_{AB}, M, N_A and their derivatives with respect to the angles, where N^{TF}_{AB} designates the trace-free part of $N_{AB} = \partial_u C_{AB}$. These functions are those that admit a well-defined interpretation in the flat limit, which contrasts with J_{AB}, $M^{(\Lambda)}$ and $N^{(\Lambda)}_A$. Furthermore, the relation (6.2.45) is extensively used in order to exchange terms $\sim \partial_u q_{AB}$ for terms $\sim \Lambda C_{AB}$. The following identities turn out to be useful for the computation:

$$\begin{aligned}
\partial_u H^{AB} &= q^{AC}\partial_u H_{CD} q^{BD} - \frac{2\Lambda}{3} C^{C(A}H^{B)}_C, \\
\partial_u (D^B H_{AB}) &= D^B \partial_u H_{AB} - \frac{\Lambda}{3}\Big[D_C(H_{AB}C^{BC}) + \frac{1}{2}H^{BC}D_A C_{BC}\Big], \\
\partial_u C^{AB} C_{AB} &= 2N^{AB}C_{AB}, \\
\partial_u \partial_A R[q] &= \frac{\Lambda}{3} D_A D_B D_C C^{BC},
\end{aligned} \tag{6.4.12}$$

   where $H_{AB}(u, x^C)$ is any symmetric rank 2 transverse tensor.

[6] Since the field-dependence involved in the diffeomorphism (6.4.9) appears only at subleading orders in r, we assumed that the variational operator δ is not affected at leading order and therefore does not bring any contribution to the leading order in the transformation of $\theta^\rho_{\Lambda\text{-BMS}}$.

2. Once this procedure is achieved, we take the flat limit by putting $\Lambda \to 0$. As we later explain, this limit may require a renormalization procedure to remove the divergences $\sim \frac{1}{\Lambda}$.

Let us apply the first step of the procedure to the presymplectic potential. Starting from its expression in the Bondi gauge (6.4.10) and using (6.2.57), we get

$$\begin{aligned}\theta^r_{\Lambda\text{-BMS}}[g;\delta g] = \frac{\sqrt{\mathring{q}}}{16\pi G}\Big[\frac{3}{2\Lambda}\partial_u(N^{TF}_{AB}\delta q^{AB}) + \frac{1}{2}\Big(N^{AB}_{TF} + \frac{1}{2}R[q]q^{AB}\Big)\delta C_{AB} \\ + \frac{1}{2}D_A D^C C_{BC}\delta q^{AB}\Big] + \mathcal{O}(\Lambda; r^{-1}),\end{aligned} \tag{6.4.13}$$

where the notation $\mathcal{O}(\Lambda, r^{-1})$ designates terms $\mathcal{O}(r^{-1})$ and/or $\mathcal{O}(\Lambda)$. A striking observation is that we have a term $\sim \Lambda^{-1}$ in this expression, which does not allow us to go to the second stage of the flat limit procedure. We have to suppress this divergence before taking $\Lambda \to 0$. A way to proceed is to use the ambiguity allowed by the covariant phase space formalism (2.3.81). Indeed, noticing that the term $\sim \Lambda^{-1}$ can be expressed as

$$\frac{\sqrt{\mathring{q}}}{16\pi G}\frac{3}{2\Lambda}\partial_u(N^{TF}_{AB}\delta q^{AB}) = \frac{1}{16\pi G}\frac{3}{\Lambda}\partial_u\left[\frac{1}{2}\partial_u(\sqrt{\mathring{q}}\,C_{AB}\delta q^{AB})\right] + \frac{\sqrt{\mathring{q}}}{16\pi G}\,\delta(C^{AB}N^{TF}_{AB}), \tag{6.4.14}$$

we define

$$Y^{ru}_{(\Lambda)}[g;\delta g] = -\frac{1}{16\pi G}\frac{3}{\Lambda}\frac{1}{2}\partial_u(\sqrt{\mathring{q}}\,C_{AB}\delta q^{AB}), \tag{6.4.15}$$

and $Y^{rA} = 0$. As discussed in [193], the presence of this Iyer-Wald ambiguity can be justified by adding corner terms in the variational principle (6.3.9). The presymplectic potential is renormalized as

$$\begin{aligned}\theta^r_{\text{ren}(\Lambda)}[g;\delta g] &= \theta^r_{\Lambda\text{-BMS}}[g;\delta g] + \partial_u Y^{ru}_{(\Lambda)}[g;\delta g] \\ &= \frac{\sqrt{\mathring{q}}}{16\pi G}\Big[\frac{1}{2}\Big(N^{AB}_{TF} + \frac{1}{2}R[q]q^{AB}\Big)\delta C_{AB} + \frac{1}{2}\Big(D_A D^C C_{BC}\Big)\delta q^{AB} + \delta(C^{AB}N^{TF}_{AB})\Big] \\ &\quad + \mathcal{O}(\Lambda; r^{-1})\end{aligned} \tag{6.4.16}$$

and is finite in the limit $\Lambda \to 0$. The associated symplectic potential $\omega^r_{\text{ren}(\Lambda)}$ is explicitly given by

$$\begin{aligned}&\omega^r_{\text{ren}(\Lambda)}[g;\delta_1 g,\delta_2 g] \\ &\quad = \frac{\sqrt{\mathring{q}}}{16\pi G}\left[\frac{1}{2}\delta_1\left(N^{AB}_{TF} + \frac{1}{2}R[q]q^{AB}\right)\wedge\delta_2 C_{AB} + \frac{1}{2}\delta_1\left(D_A D^C C_{BC}\right)\wedge\delta_2 q^{AB}\right] \\ &\qquad + \mathcal{O}(\Lambda; r^{-1}).\end{aligned} \tag{6.4.17}$$

Finally, taking the flat limit $\Lambda \to 0$, we obtain

$$\begin{aligned}
&\omega^r_{\text{ren}(\Lambda)}[g;\delta_1 g,\delta_2 g] \\
&\quad = \frac{\sqrt{\mathring{q}}}{16\pi G}\left[\frac{1}{2}\delta_1\left(N^{AB}_{TF}+\frac{1}{2}R[q]q^{AB}\right)\wedge\delta_2 C_{AB}+\frac{1}{2}\delta_1\left(D_A D^C C_{BC}\right)\wedge\delta_2 q^{AB}\right] \\
&\quad + \mathcal{O}(r^{-1}).
\end{aligned} \tag{6.4.18}$$

This result precisely corresponds to the presymplectic form (4.2.12) obtained in asymptotically flat spacetime. Therefore, we have shown that through an appropriate renormalization process, the flat limit of the $\mathfrak{bms}_4^\Lambda$ symplectic structure yields the $\mathfrak{bms}_4^{\text{gen}}$ symplectic structure in asymptotically flat spacetime.

An interesting observation is that, to obtain (4.2.12), we had to renormalize the symplectic structure using the Iyer-Wald ambiguity (2.3.81) with a term $Y^{ru} = -r\frac{1}{2}\frac{\sqrt{\mathring{q}}}{16\pi G}C_{AB}\delta q^{AB}$ to remove the $\sim r$ divergences. To take the flat limit in the present context, we also had to renormalize the symplectic structure with a term $Y^{ru}_{(\Lambda)} = -\frac{1}{16\pi G}\frac{3}{\Lambda}\frac{1}{2}\partial_u(\sqrt{\mathring{q}}\,C_{AB}\delta q^{AB})$ to remove the $\sim \Lambda$ divergences. Therefore, even if the nature of the divergences is different in both contexts, the expressions are astonishingly very similar and may rely on deeper reasons.

6.5 New boundary conditions for asymptotically locally AdS$_4$ spacetime

We now particularize our discussion to the case $\Lambda < 0$. The presymplectic form (6.3.22) obtained through the holographic renormalization procedure is generically non-vanishing for asymptotically locally AdS$_4$ spacetimes. Allowing some flux at infinity leads to an ill-defined Cauchy problem [232]. Depending on the physical context, one may be interested in studying open systems allowing flux at infinity (see e.g. [233]) or isolated systems with a well-defined dynamics (see e.g. [232]). In this section, we propose a new set of boundary conditions for which the symplectic flux vanishes. The associated phase space admits the Schwarzschild-AdS$_4$ black hole and a stationary rotating solution distinct from the Kerr-AdS$_4$ black hole. The asymptotic symmetry algebra is shown to be a subalgebra of $\mathfrak{bms}_4^\Lambda$ consisting of time translations and area-preserving diffeomorphisms.

6.5.1 Mixed boundary conditions

We start from the expression of the presymplectic form (6.3.22) that we repeat here for $\Lambda < 0$ ($\eta = 1$)

$$\omega^{\rho}_{\text{ren}}[g;\delta g,\delta g] = -\frac{1}{2}\delta\left(\sqrt{|g_{(0)}|}T^{ab}\right)\wedge\delta g^{(0)}_{ab} + \mathcal{O}(\rho). \tag{6.5.1}$$

In the literature, both Dirichlet and Neumann boundary conditions have been studied to set this presymplectic form to zero. On the one hand, Dirichlet boundary conditions [15] amount to freezing the components of the boundary metric $g^{(0)}_{ab}$ to the ones of the unit cylinder while leaving the holographic stress-tensor T^{ab} free. The resulting asymptotic symmetry algebra is the algebra of exact symmetries of global AdS_4, namely $\mathfrak{so}(3,2)$. On the other hand, Neumann boundary conditions [144] freeze the components of T^{ab} while leaving the boundary metric $g^{(0)}_{ab}$ free. The resulting asymptotic symmetry group is empty: all residual gauge transformations have vanishing charges.

We now present new mixed Dirichlet-Neumann boundary conditions. We first impose the boundary conditions (6.4.1), which leads to $\mathfrak{bms}_4^\Lambda$. This is a Dirichlet boundary condition on a part of the boundary metric, which is reachable locally by a choice of boundary gauge. The symplectic flux at the spatial boundary is then given by (6.4.11), which we repeat here:

$$\omega^{\rho}_{\Lambda\text{-BMS}}[g;\delta g] = \frac{\Lambda}{2}\frac{\sqrt{\mathring{q}}}{16\pi G}\,\delta q^{AB}\wedge\delta J_{AB} + \mathcal{O}(r^{-1}). \tag{6.5.2}$$

We now further impose the Neumann boundary conditions

$$J_{AB} = 0. \tag{6.5.3}$$

This cancels the symplectic flux, as required. The boundary condition (6.5.3) restricts the solution space.

6.5.2 Asymptotic symmetry algebra

Let us now derive both the asymptotic symmetries preserving the boundary conditions and the associated charge algebra.

The boundary gauge fixing (6.4.1) is preserved by the $\mathfrak{bms}_4^\Lambda$ asymptotic symmetry algebra of residual gauge transformations as derived in section 6.2.4 (see equation (6.2.81)). We now show that the boundary condition (6.5.3) further reduces $\mathfrak{bms}_4^\Lambda$ to the direct sum $\mathbb{R}\oplus\mathcal{A}$, where $\mathbb{R}$ are the time translations and $\mathcal{A}$ is the algebra of two-dimensional area-preserving diffeomorphisms. We further show that the charges associated with this asymptotic symmetry algebra are finite, integrable, conserved and generically non-vanishing on the phase space.

The variation of J_{AB} is given by

$$\begin{aligned}\delta_\xi J_{AB} &= (\xi_0^t \partial_t + \mathcal{L}_{\xi_0^C} + \sigma) J_{AB} - \frac{4}{3}\Big[N_{(A}\partial_{B)}\xi_0^t - \frac{1}{2}N_C \partial^C \xi_0^t q_{AB}\Big] \\ &\overset{(6.2.81)}{=} \Big[\xi_0^t\partial_t + \mathcal{L}_{\xi_0^C} + \frac{1}{2}D_A\xi_0^A\Big] J_{AB} - \frac{4}{3}\Big[N_{(A}\partial_{B)}\xi_0^t - \frac{1}{2}N_C\partial^C\xi_0^t q_{AB}\Big].\end{aligned} \tag{6.5.4}$$

We recall that D_A is the covariant derivative with respect to the transverse metric $g^{(0)}_{AB} = q_{AB}$. Imposing $\delta_\xi J_{AB} = 0$ leads to the following constraint on the $\mathfrak{bms}_4^\Lambda$ asymptotic Killing vectors:

$$\partial_A \xi_0^t = 0. \tag{6.5.5}$$

Therefore, the asymptotic symmetry generators satisfy the relations

$$\partial_t \xi_0^t = \frac{1}{2} D_A \xi_0^A, \qquad \partial_t \xi_0^A = 0. \tag{6.5.6}$$

The second equation implies $\xi_0^A = V^A(x^B)$, while the first gives

$$\xi_0^t = S + \frac{t}{2} D_A V^A \tag{6.5.7}$$

where S is a constant by virtue of (6.5.5), and $D_A V^A \equiv c$, where c is also a constant. Using Helmholtz's theorem, the vector V^A can be decomposed into a divergence-free and a curl-free part as $V^A = \epsilon^{AB}\partial_B \Phi + q^{AB}\partial_B\Psi$, where Ψ and Φ are functions of x^C. Injecting this expression for V^A into this equation gives $D_A D^A \Psi = c$. This equation admits a solution if and only if $c = 0$, which is given by $\Psi = 0$. Therefore, the asymptotic symmetry generators are given by

$$\xi_0^t = S, \qquad \xi_0^A = \epsilon^{AB}\partial_B \Phi(x^C) \tag{6.5.8}$$

where S is a constant and $\Phi(x^C)$ is arbitrary. Writing $\xi = \xi(S, \Phi)$, the residual gauge diffeomorphisms, the commutation relations (6.4.7) and (6.4.8) reduce to $[\xi(S_1, \Phi_1), \xi(S_2, \Phi_2)]_A = \xi(\hat{S}, \hat{\Phi})$, where

$$\hat{S} = 0, \qquad \hat{\Phi} = \epsilon^{AB}\partial_A\Phi_2\partial_B\Phi_1. \tag{6.5.9}$$

Hence, after imposing the boundary condition (6.5.3), $\mathfrak{bms}_4^\Lambda$ reduces to the $\mathbb{R}\oplus\mathcal{A}$ algebra, where $\mathbb{R}$ denotes the abelian time translations and $\mathcal{A}$ is the algebra of two-dimensional area-preserving diffeomorphisms. The latter symmetries are an infinite-dimensional extension of the $\mathfrak{so}(3)$ rotations.

Let us now study the associated surface charges. Starting from the $\mathfrak{bms}_4^\Lambda$ surface charges given in (6.4.4) and (6.4.5), and imposing the boundary condition (6.5.3), one sees that the charges are integrable. The integrated charges reduce to

$$H_\xi^{\text{AdS}}[g] = -\sqrt{\frac{3}{|\Lambda|}} \int_{S^2_\infty} d^2\Omega \, [S T_{tt} + T_{tA}\epsilon^{AB}\partial_B\Phi]. \tag{6.5.10}$$

From this expression, we see that the charges associated with the symmetry $\mathbb{R} \oplus \mathcal{A}$ are generically non-vanishing. Taking $S = 1$ and $\Phi = 0$ gives the energy. The first harmonic modes of Φ give the angular momenta, while the higher modes give an infinite tower of charges. Using (6.5.9) and (6.2.74), a simple computation shows that the charges (6.5.10) satisfy the algebra

$$\delta_{\xi(S_2,\Phi_2)} H^{\text{AdS}}_{\xi(S_1,\Phi_1)} = H^{\text{AdS}}_{\xi(\hat{S},\hat{\Phi})}. \tag{6.5.11}$$

The charges form a representation of $\mathbb{R} \oplus \mathcal{A}$ without central extension. This result is also a direct consequence of (6.4.6) when taking (6.5.3) into account.

6.5.3 Stationary solutions

Here, we study the stationary sector of the phase space associated with the boundary conditions. In this subsection, we write $\Lambda = -3/\ell^2$. The Schwarzschild-AdS$_4$ solution is included in the phase space. Indeed, Schwarzschild-AdS$_4$ can be set in the Fefferman-Graham gauge, which allows to identify $q_{AB} = \mathring{q}_{AB}$ the unit metric on the sphere, as well as $T_{tt} = -\frac{M}{4\pi G \ell}$, $T_{tA} = 0$ and $J_{AB} = 0$.

The boundary metric and holographic stress-tensor of Kerr-AdS$_4$ are given in the conformally flat frame by [234–236]

$$\begin{aligned} g^{(0)}_{ab} dx^a dx^b &= -\ell^{-2} dt^2 + d\theta^2 + \sin^2\theta d\phi^2, & (6.5.12)\\ T^{ab} &= T^{ab}_{\text{Kerr}} \equiv -\frac{m\gamma^3 \ell}{8\pi}(3u^a u^b + g^{ab}_{(0)}), & (6.5.13)\end{aligned}$$

where $\Xi = 1 - a^2\ell^{-2}$ and

$$u^a \partial_a = \gamma\ell(\partial_t + \frac{a}{\ell^2}\partial_\phi), \qquad \gamma^{-1} \equiv \sqrt{1 - \frac{a^2}{\ell^2}\sin^2\theta}. \tag{6.5.14}$$

The mass and angular momentum are $M = -\ell \int \mathrm{d}^2\Omega\, T_{tt} = \frac{m}{\Xi^2}$, $J = Ma = -\ell \int \mathrm{d}^2 T_{t\phi} = \frac{ma}{\Xi^2}$. We observe that $J_{AB} \neq 0$. Therefore, the Kerr-AdS$_4$ solution is not included in the phase space. However, it is possible to obtain a stationary axisymmetric solution with $J_{AB} = 0$ as follows. The most general diagonal, traceless, divergence-free, stationary and axisymmetric T^{ab} is given by

$$\begin{aligned} &T^{tt}_{\text{corr}} = \ell^2[2T^{\theta\theta}(\theta) + \tan\theta\, T^{\theta\theta\prime}(\theta)], \quad T^{\theta\theta}_{\text{corr}} = T^{\theta\theta}(\theta), \\ &T^{\phi\phi}_{\text{corr}} = \frac{1}{\sin^2(\theta)}[T^{\theta\theta}(\theta) + \tan\theta\, T^{\theta\theta\prime}(\theta)] \end{aligned} \tag{6.5.15}$$

and the other components are set to zero. We consider the sum of $T_{\text{Kerr}} + T_{\text{corr}}$. We solve for $T^{\theta\theta}(\theta)$ to set $J^{AB} = 0$. The regular solution at $\mathscr{I}$ is unique and given by

$$T^{tt} = -\frac{m\ell^3}{4\pi}, \qquad T^{t\phi} = -\frac{3am\ell\gamma^5}{8\pi}, \qquad T^{AB} = -\frac{m\ell}{8\pi} q^{AB}. \tag{6.5.16}$$

The mass and angular momentum are $M = -\ell \int \mathrm{d}^2\Omega\, T_{tt} = m$, $J = -\ell \int \mathrm{d}^2\Omega T_{t\phi} = \frac{ma}{\Xi^2}$. It would be interesting to know whether this solution is regular in the bulk of spacetime.

From the conservation of the stress-energy tensor T^{ab} given by the first equation of (6.2.6), the most general stationary solution with flat boundary metric (6.5.12) is only constrained by the following conditions:

$$D_A N^A = 0 \Leftrightarrow N^A = \epsilon^{AB} D_B \alpha(x^C), \quad \partial_A M = 0, \tag{6.5.17}$$

where $\alpha(x^C)$ is an arbitrary function of x^C. To obtain these expressions, we also used equations (6.2.75) and (6.5.3). Therefore, even for stationary solutions, we see that the charges associated with the area-preserving diffeomorphisms are generically non-vanishing. It would be interesting to study the regularity of the general solutions (6.5.17) in the bulk of spacetime.

Chapter 7

Conclusion

The discovery of the global BMS symmetry group at null infinity came as a surprise in 1962. This infinite-dimensional endowing of the Poincaré group was, however, necessary to include radiative spacetimes in the four-dimensional analysis. Since then, the extensions of the BMS group have highlighted the richness of the asymptotic structure of the gravitational field.

In this thesis, we have explored the extensions of the BMS group and their implications for the phase space of the theory. Furthermore, we have established new relations between those symmetries and the gravitational memory effects. We have also elaborated on the covariant phase space methods, allowing us to compute the gravitational surface charges in a first order framework.

Before concluding this manuscript, we would like to mention some current or future research directions that are suggested by the present work.

As discussed in chapter 2, we have always adopted the gauge fixing approach throughout this thesis [27]. This approach allows us to eliminate the arbitrary functions of the gauge transformations and therefore fix the dependence of the residual gauge diffeomorphisms at all orders in the expansion parameter. However, even if one can always reach a gauge by definition, the gauge transformations that are necessary to reach a particular gauge might be large gauge transformations, namely, they could be associated with non-vanishing surface charges [18, 237]. Therefore, it would be interesting to study asymptotic symmetries by considering only partial gauge fixings. For example, we showed in [216, 217] that the Bondi gauge can be embedded in the Derivative expansion, which is a partial gauge fixing admitting additional parameters in its solution space. Another example is the extended Fefferman-Graham gauge considered, for example, in [18, 238], where the Weyl transformations preserving the radial foliation are well defined.

Furthermore, it has been noted that the solution space of three-dimensional general relativity transforms in the coadjoint representation of the asymptotic symmetry group [175]. It would be insightful to investigate how the coadjoint patterns appear in the transformation of the solution space of four-dimensional gravity. Based on the results established in section 3.3, we have already found that only a subsector of the solution space transforms in the coadjoint representation of BMS_4. This subsector couples with the radiation to form the full four-dimensional gravitational theory. Furthermore, it has been shown that three-dimensional gravity could be described by a geometric action defined on coadjoint orbits [176]. It would be fascinating to have the same construction for the coadjoint subsector of four-dimensional gravity. These questions are part of our current research.

Moreover, we saw in this thesis that some gravitational memory effects could be related to the BMS symmetries in asymptotically flat spacetimes. Since, in chapter 6, we found the analogue of the BMS group in asymptotically locally $(\text{A})\text{dS}_4$ spacetimes, it would be worth investigating if similar relations exist with memory effects in these kinds of asymptotics (see e.g. [125, 126]).

The AdS/CFT correspondence and the associated holographic dictionary are now clearly stated and have been checked in many situations. Surprisingly, though, the analogous holographic correspondence in asymptotically flat spacetimes is poorly understood. However, all the ingredients needed to clearly state the holographic duality in flat space and its associated dictionary are now present. Indeed, from the point of view of the bulk theory, the Bondi expansion of the metric enables us to approach the spacetime boundary in the flat case, as the Fefferman-Graham does in the AdS case. Furthermore, as discussed in much detail in chapter 6, the Bondi gauge also exists in asymptotically AdS spacetimes and has been related to the Fefferman-Graham gauge [166, 167]. Therefore, many results and interpretations of the bulk spacetime metric obtained in AdS can be directly imported into flat space by taking the well-defined flat limit in the Bondi gauge. For example, the process of holographic renormalization could be adapted for asymptotically flat spacetimes at null infinity. From the point of view of the dual boundary theory, using the geometric action construction mentioned above which is based on coadjoint methods, one could construct a boundary action invariant under the BMS_4 symmetry. This would be the effective action of the theory dual to the coadjoint subsector of four-dimensional asymptotically flat gravity. Adding Hamiltonians and source terms to this action would lead to an effective dual description of the full asymptotically flat gravity theory.

Appendix A

Useful results and conventions

In this appendix, we establish some important frameworks and conventions. The aim of this formalism is to manipulate some local expressions, as this is convenient in field theory. We closely follow [27, 186, 239].

A.1 Jet bundles

Let M be the n-dimensional spacetime with local coordinates x^μ ($\mu = 0, \dots, n-1$). The fields, written as $\phi = (\phi^i)$, are supposed to be Grassmann even. The *jet space* J consists in the fields and the symmetrized derivatives of the fields $(\phi, \phi_\mu, \phi_{\mu\nu}, \dots)$, where $\phi^i_{\mu_1 \dots \mu_k} = \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_k}} \phi^i$. The *symmetrized derivative* is defined as

$$\frac{\partial \phi^i_{\nu_1 \dots \nu_k}}{\partial \phi^j_{\mu_1 \dots \mu_k}} = \delta^{\nu_1}_{(\mu_1} \dots \delta^{\nu_k}_{\mu_k)} \delta^i_j. \tag{A.1.1}$$

In the jet space, the cotangent space at a point is generated by the variations of the fields and their derivatives at that point, namely $(\delta\phi, \delta\phi_\mu, \delta\phi_{\mu\nu}, \dots)$. The *variational operator* is defined as

$$\delta = \sum_{k \geqslant 0} \delta \phi^i_{\mu_1 \dots \mu_k} \frac{\partial}{\partial \phi^i_{\mu_1 \dots \mu_k}}. \tag{A.1.2}$$

We choose all the $\delta\phi$, $\delta\phi_\mu$, $\delta\phi_{\mu\nu}, \dots$ to be Grassmann odd, which implies that $\delta^2 = 0$. Hence, δ is seen as an exterior derivative on the jet space.

Now, we define the *jet bundle* as the fiber bundle with local trivialization $(x^\mu, \phi, \phi_\mu, \phi_{\mu\nu}, \dots)$. Locally, the total space of the jet bundle looks like $M \times J$. A section of this fiber bundle is a map $x \to (\phi(x), \phi_\mu(x), \phi_{\mu\nu}(x), \dots)$. The *horizontal derivative* is defined as

$$\mathrm{d} = \mathrm{d}x^\mu \partial_\mu, \quad \text{where} \quad \partial_\mu = \frac{\partial}{\partial x^\mu} + \sum_{k \geqslant 0} \left(\phi^i_{\mu\nu_1 \dots \nu_k} \frac{\partial}{\partial \phi^i_{\nu_1 \dots \nu_k}} + \delta \phi^j_{\mu\nu_1 \dots \nu_k} \frac{\partial}{\partial \delta \phi^j_{\nu_1 \dots \nu_k}} \right). \tag{A.1.3}$$

In this perspective, the variational operator can also be seen as the *vertical derivative*, i.e. the derivative along the fibers. The exterior derivative on the total space can be defined as $\mathrm{d}_{Tot} = \mathrm{d} + \delta$. Notice that both d and δ are Grassmann odd and they anti-commute, namely

$$\mathrm{d}\delta = -\delta\mathrm{d}. \tag{A.1.4}$$

On the jet bundle, we write $\Omega^{p,q}$ for the set of functions that are p-forms with respect to the spacetime and q-forms with respect to the jet space[1].

A.2 Some operators

In this subsection, we introduce additional operators used in the text and discuss their properties.

The *Euler-Lagrange* derivative of a *local function* f, i.e. a function on the total space of the jet bundle $f = f[x, \phi, \phi_\mu, \phi_{\mu\nu}, \ldots]$, is defined as

$$\frac{\delta f}{\delta \phi^i} = \sum_{k\geqslant 0} (-1)^k \partial_{\mu_1} \ldots \partial_{\mu_k} \left(\frac{\partial f}{\partial \phi^i_{\mu_1 \ldots \mu_k}} \right). \tag{A.2.1}$$

This operator satisfies

$$\frac{\delta f}{\delta \phi^i} = 0 \quad \Leftrightarrow \quad f = \partial_\mu j^\mu, \tag{A.2.2}$$

where j^μ is a local function (for a proof, see e.g. section 1.2 of [171]).

The *variation under a transformation of characteristic* Q (i.e. $\delta_Q \phi^i = Q^i$) is given by

$$\delta_Q f = \sum_{k\geqslant 0} (\partial_{\mu_1} \ldots \partial_{\mu_k} Q^i) \frac{\partial f}{\partial \phi^i_{\mu_1 \ldots \mu_k}} + (\partial_{\mu_1} \ldots \partial_{\mu_k} \delta Q^j) \frac{\partial f}{\partial \delta \phi^j_{\mu_1 \ldots \mu_k}}. \tag{A.2.3}$$

The Lie bracket of characteristics is defined by $[Q_1, Q_2] = \delta_{Q_1} Q_2 - \delta_{Q_2} Q_1$ and satisfies $[\delta_{Q_1}, \delta_{Q_2}] = \delta_{[Q_1, Q_2]}$. A contracted variation of this type is Grassmann even and we have

$$\delta_Q \mathrm{d} = \mathrm{d}\delta_Q, \quad \delta\delta_Q = \delta_Q \delta. \tag{A.2.4}$$

We also have the following relation between the variation under a transformation of characteristic Q and the Euler-Lagrange derivative:

$$\delta_Q \frac{\delta f}{\delta \phi^i} = \frac{\delta}{\delta \phi^i}(\delta_Q f) - \sum_{k\geqslant 0} (-1)^k \partial_{\mu_1} \ldots \partial_{\mu_k} \left(\frac{\partial Q^j}{\partial \phi^i_{\mu_1 \ldots \mu_k}} \frac{\delta f}{\delta \phi^j} \right). \tag{A.2.5}$$

[1] One often refers to a *variational bicomplex* structure

Let $\boldsymbol{\alpha} \in \Omega^{n-k,q}$. We use the notation

$$\boldsymbol{\alpha} = \alpha^{\mu_1\ldots\mu_k}(\mathrm{d}^{n-k}x)_{\mu_1\ldots\mu_k}, \tag{A.2.6}$$

where

$$(\mathrm{d}^{n-k}x)_{\mu_1\ldots\mu_k} = \frac{1}{k!(n-k)!}\epsilon_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{n-k}}\mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_{n-k}} \tag{A.2.7}$$

and where $\epsilon_{\mu_1\ldots\mu_n}$ is completely antisymmetric and $\epsilon_{01\ldots n-1} = 1$. We can check that

$$\mathrm{d}\boldsymbol{\alpha} = (-1)^q \partial_\sigma \alpha^{[\mu_1\ldots\mu_{k-1}\sigma]}(\mathrm{d}^{n-k+1}x)_{\mu_1\ldots\mu_{k-1}}. \tag{A.2.8}$$

The *interior product of a spacetime form* with respect to a vector field ξ is defined as

$$\iota_\xi = \xi^\mu \frac{\partial}{\partial \mathrm{d}x^\mu}. \tag{A.2.9}$$

We can also define the *interior product of a jet space form* with respect to a characteristic Q as

$$i_Q = \sum_{k\geqslant 0}(\partial_{\mu_1}\ldots\partial_{\mu_k}Q^i)\frac{\partial}{\partial\delta\phi^i_{\mu_1\ldots\mu_k}}. \tag{A.2.10}$$

It satisfies

$$i_Q\delta + \delta i_Q = \delta_Q, \quad i_{Q_1}\delta_{Q_2} - \delta_{Q_2}i_{Q_1} = i_{[Q_1,Q_2]}. \tag{A.2.11}$$

The *homotopy operator* $I^p_{\delta\phi} : \Omega^{p,q} \mapsto \Omega^{p-1,q+1}$ is defined as

$$I^p_{\delta\phi}\boldsymbol{\alpha} = \sum_{k\geqslant 0}\frac{k+1}{n-p+k+1}\partial_{\mu_1}\ldots\partial_{\mu_k}\left(\delta\phi^i\frac{\delta}{\delta\phi^i_{\mu_1\ldots\mu_k\nu}}\frac{\partial\boldsymbol{\alpha}}{\partial\mathrm{d}x^\nu}\right) \tag{A.2.12}$$

for $\boldsymbol{\alpha} \in \Omega^{p,q}$. This operator satisfies the following relations

$$\delta = \delta\phi^i\frac{\delta}{\delta\phi^i} - \mathrm{d}I^n_{\delta\phi} \quad \text{when acting on spacetime } n\text{-forms}, \tag{A.2.13}$$

$$\delta = I^{p+1}_{\delta\phi}\mathrm{d} - \mathrm{d}I^p_{\delta\phi} \quad \text{when acting on spacetime } p\text{-forms } (p<n). \tag{A.2.14}$$

Furthermore,

$$\delta I^p_{\delta\phi} = I^p_{\delta\phi}\delta. \tag{A.2.15}$$

Notice that the homotopy operator is used to prove the algebraic Poincaré lemma (2.3.16).

Similarly, the *homotopy operator with respect to gauge parameters* $f = (f^\alpha)$ is defined as $I^p_f : \Omega^{p,q} \mapsto \Omega^{p-1,q}$, where

$$I^p_f\boldsymbol{\alpha} = \sum_{k\geqslant 0}\frac{k+1}{n-p+k+1}\partial_{\mu_1}\ldots\partial_{\mu_k}\left(f^\alpha\frac{\delta}{\delta f^\alpha_{\mu_1\ldots\mu_k\nu}}\frac{\partial\boldsymbol{\alpha}}{\partial\mathrm{d}x^\nu}\right). \tag{A.2.16}$$

It satisfies

$$I^{p+1}_f\mathrm{d} + \mathrm{d}I^p_f = 1. \tag{A.2.17}$$

Appendix B

Determinant condition in Bondi gauge

In this appendix, we discuss the determinant condition used to define the Bondi gauge in equation (2.2.10) and repeated here:

$$\partial_r \left(\frac{\det g_{AB}}{r^{2(n-2)}} \right) = 0. \tag{B.1}$$

Let us emphasize that this condition is weaker than the historical one given by $\det(g_{AB}) = r^{2(n-2)} \det(\mathring{q}_{AB})$, where $\mathring{q}_{AB}$ is the unit sphere metric [1–3]. The relaxed determinant condition (B.1) is inspired by [6] and is essential if one wants to consider Weyl rescalings of the transverse boundary metric.

To illustrate this claim, we derive the implication of the determinant condition in the derivation of the residual gauge diffeomorphisms. The equation (B.1) is equivalent to

$$\det(g_{AB}) = r^{2(n-2)} \chi(u, x^C), \tag{B.2}$$

where χ is an arbitrary function of (u, x^C). When the preliminary boundary condition $g_{AB} = \mathcal{O}(r^2) \Leftrightarrow g_{AB} = r^2 q_{AB} + o(r^2)$ is imposed, we have $\chi(u, x^C) = \det(q_{AB})$. From (B.2), we obtain

$$\delta_\xi \ln[\det(g_{AB})] = g^{AB} \mathcal{L}_\xi g_{AB} = \delta_\xi \ln \chi \equiv 2(n-2)\omega, \tag{B.3}$$

where we introduced the parameter $\omega(u, x^A)$. We deduce

$$\xi^r = -\frac{r}{(n-2)} \left[\mathcal{D}_A \xi^A - U^A \partial_A \xi^u + \frac{1}{2} \xi^u \partial_u \ln g - (n-2)\omega \right], \tag{B.4}$$

where $\mathcal{D}_A$ is the covariant derivative with respect to g_{AB} and $g = \det(g_{AB})$. Indices are lowered and raised by g_{AB} and its inverse.

In this derivation, the introduction of the parameter ω in (B.3) is somewhat peculiar and may seem artificial. This way of introducing the parameters of the residual gauge diffeomorphisms *by hand* was also used in [216, 217] to define the additional parameters in the Derivative expansion compared to the Bondi gauge. Let us show here that (B.4) can be deduced from the determinant condition without forcing ω in (B.3). The condition (B.3) tells us that

$$\delta_\xi \ln[\det(g_{AB})] \sim \text{order } r^0. \tag{B.5}$$

Working out the left-hand side yields

$$\frac{(n-2)}{r}\xi^r + \mathcal{D}_A \xi^A + \frac{1}{2}\xi^u \partial_u \ln g - U^A \partial_A \xi^u \sim \text{order } r^0 \tag{B.6}$$

Taking into account the other gauge conditions in (2.2.10), the preliminary boundary conditions $g_{AB} = \mathcal{O}(r^2)$ and the associated fall-offs imposed by the Einstein equations, we obtain that ξ^r is determined at all orders, except at leading order $\sim r$. In other words, writing $\xi^r = rR(u, x^A) + o(r)$, the remaining free parameter is $R(u, x^A)$ and

$$\xi^r = rR(u, x^A) + \left[-\frac{r}{(n-2)} \left(\mathcal{D}_A I^A - U^A \partial_A f \right) \right] \Big|_{\mathcal{O}(r^{n<1})}, \tag{B.7}$$

where I^A is defined in (2.2.14) and the notation $\mathcal{O}(r^{n<1})$ means that the expression inside the brackets is truncated for terms of order $\sim r$ or higher. Finally, doing the following field-dependent redefinition of the free-parameter:

$$R(u, x^A) = \omega(u, x^A) - \frac{1}{(n-2)} \left[\mathcal{D}_A Y^A + \frac{1}{2}\xi^u \partial_u \ln g - U_0^A \partial_A f \right], \tag{B.8}$$

we recover the original result (B.4).

Appendix C

Further results in Newman-Penrose formalism

C.1 Newman-Unti solution space in NP formalism

When conditions (3.3.5) supplemented by the fall-off conditions (3.3.6) are imposed, the asymptotic expansion of on-shell spin coefficients, tetrads and the associated components of the Weyl tensor can be determined. All the coefficients in the expansions are functions of the three coordinates $u, \zeta, \bar{\zeta}$. In this approach to the characteristic initial value problem, freely specifiable initial data at fixed u_0 is given by $\Psi_0(u_0, r, \zeta, \bar{\zeta})$ in the bulk with the fall-offs given below and by $(\Psi_2^0 + \bar{\Psi}_2^0)(u_0, \zeta, \bar{\zeta})$, $\Psi_1^0(u_0, \zeta, \bar{\zeta})$ at $\mathscr{I}^+$. The asymptotic shear $\sigma^0(u, \zeta, \bar{\zeta})$ and the conformal factor $P(u, \zeta, \bar{\zeta})$ are free data at $\mathscr{I}^+$ for all u.

Explicitly,

$$
\begin{aligned}
\Psi_0 &= \frac{\Psi_0^0}{r^5} + \frac{\Psi_0^1}{r^6} + \frac{\Psi_0^2}{r^7} + \mathcal{O}(r^{-8}), \\
\Psi_1 &= \frac{\Psi_1^0}{r^4} - \frac{\bar{\eth}\Psi_0^0}{r^5} + \frac{2\sigma^0\bar{\sigma}^0\Psi_1^0 + \frac{5}{2}\eth\bar{\sigma}^0\Psi_0^0 + \frac{1}{2}\bar{\sigma}^0\eth\Psi_0^0 - \frac{1}{2}\bar{\eth}\Psi_0^1}{r^6} + \mathcal{O}(r^{-7}), \\
\Psi_2 &= \frac{\Psi_2^0}{r^3} - \frac{\bar{\eth}\Psi_1^0}{r^4} + \frac{2\eth\bar{\sigma}^0 + \frac{1}{2}\lambda^0\Psi_0^0 + \frac{3}{2}\sigma^0\bar{\sigma}^0\Psi_2^0 + \frac{1}{2}\bar{\sigma}^0\eth\Psi_1^9 + \frac{1}{2}\bar{\eth}^2\Psi_0^0}{r^5} + \mathcal{O}(r^{-6}), \\
\Psi_3 &= \frac{\Psi_3^0}{r^2} - \frac{\bar{\eth}\Psi_2^0}{r^3} + \mathcal{O}(r^{-4}), \quad \Psi_4 = \frac{\Psi_4^0}{r} - \frac{\bar{\eth}\Psi_3^0}{r^2} + \mathcal{O}(r^{-3}),
\end{aligned}
$$

$$\rho = -\frac{1}{r} - \frac{\sigma^0\overline{\sigma}^0}{r^3} + \mathcal{O}(r^{-5}),\ \sigma = \frac{\sigma^0}{r^2} + \frac{\bar{\sigma}^0\sigma^0\sigma^0 - \frac{1}{2}\Psi_0^0}{r^4} + \mathcal{O}(r^{-5}),$$
$$\tau = -\frac{\Psi_1^0}{2r^3} + \frac{\frac{1}{2}\sigma^0\bar{\Psi}_1^0 + \bar{\eth}\Psi_0^0}{3r^4} + \mathcal{O}(r^{-5}),\ \alpha = \frac{\alpha^0}{r} + \frac{\overline{\sigma}^0\overline{\alpha}^0}{r^2} + \frac{\sigma^0\overline{\sigma}^0\alpha^0}{r^3} + \mathcal{O}(r^{-4}),$$
$$\beta = -\frac{\overline{\alpha}^0}{r} - \frac{\sigma^0\alpha^0}{r^2} - \frac{\sigma^0\overline{\sigma}^0\overline{\alpha}^0 + \frac{1}{2}\Psi_1^0}{r^3} + \mathcal{O}(r^{-4}),\ \gamma = \gamma^0 - \frac{\Psi_2^0}{2r^2} + \frac{2\bar{\eth}\Psi_1^0 + \alpha^0\Psi_1^0 - \bar{\alpha}^0\bar{\Psi}_1^0}{6r^3} + \mathcal{O}(r^{-4}),$$
$$\mu = \frac{\mu^0}{r} - \frac{\sigma^0\lambda^0 + \Psi_2^0}{r^2} + \frac{\sigma^0\bar{\sigma}^0\mu^0 + \frac{1}{2}\bar{\eth}\Psi_1^0}{r^3} + \mathcal{O}(r^{-4}),\ \nu = \nu^0 - \frac{\Psi_3^0}{r} + \frac{\bar{\eth}\Psi_2^0}{2r^2} + \mathcal{O}(r^{-3}),$$
$$\lambda = \frac{\lambda^0}{r} - \frac{\overline{\sigma}^0\mu^0}{r^2} + \frac{\sigma^0\bar{\sigma}^0\lambda^0 + \frac{1}{2}\bar{\sigma}^0\Psi_2^0}{r^3} + \mathcal{O}(r^{-4}),$$

$$\begin{aligned}
X^\zeta &= \overline{X^{\bar{\zeta}}} = \frac{\bar{P}\Psi_1^0}{6r^3} + \mathcal{O}(r^{-4}), \quad \omega = \frac{\bar{\eth}\sigma^0}{r} - \frac{\sigma^0\bar{\eth}\overline{\sigma}^0 + \frac{1}{2}\Psi_1^0}{r^2} + \mathcal{O}(r^{-3}),\\
U &= -r(\gamma^0 + \overline{\gamma}^0) + \mu^0 - \frac{\Psi_2^0 + \overline{\Psi}_2^0}{2r} + \frac{\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0}{6r^2} + \mathcal{O}(r^{-3}),\\
L^\zeta &= \overline{L^{\bar{\zeta}}} = -\frac{\sigma^0\bar{P}}{r^2} + \mathcal{O}(r^{-4}), \quad L^{\bar{\zeta}} = \overline{L^\zeta} = \frac{P}{r} + \frac{\sigma^0\overline{\sigma}^0 P}{r^3} + \mathcal{O}(r^{-4}),
\end{aligned}$$

where

$$\begin{aligned}
&\alpha^0 = \frac{1}{2}\bar{P}\partial\ln P, \quad \gamma^0 = -\frac{1}{2}\partial_u \ln\bar{P}, \quad \nu^0 = \bar{\eth}(\gamma^0 + \overline{\gamma}^0),\\
&\mu^0 = -\frac{1}{2}P\bar{P}\partial\bar{\partial}\ln P\bar{P} = -\frac{1}{2}\bar{\eth}\eth\ln P\bar{P} = -\frac{R}{4}, \quad \lambda^0 = \dot{\overline{\sigma}}^0 + \overline{\sigma}^0(3\gamma^0 - \overline{\gamma}^0),\\
&\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0 + \bar{\sigma}^0\bar{\lambda}^0 - \sigma^0\lambda^0\\
&\Psi_3^0 = -\eth\lambda^0 + \bar{\eth}\mu^0,\\
&\Psi_4^0 = \bar{\eth}\nu^0 - (\partial_u + 4\gamma^0)\lambda^0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_u\Psi_0^0 + (\gamma^0 + 5\overline{\gamma}^0)\Psi_0^0 &= \eth\Psi_1^0 + 3\sigma^0\Psi_2^0,\\
\partial_u\Psi_1^0 + 2(\gamma^0 + 2\overline{\gamma}^0)\Psi_1^0 &= \eth\Psi_2^0 + 2\sigma^0\Psi_3^0,\\
\partial_u\Psi_2^0 + 3(\gamma^0 + \overline{\gamma}^0)\Psi_2^0 &= \eth\Psi_3^0 + \sigma^0\Psi_4^0,\\
\partial_u\Psi_3^0 + 2(2\gamma^0 + \overline{\gamma}^0)\Psi_3^0 &= \eth\Psi_4^0,\\
\partial_u\mu^0 &= -2(\gamma^0 + \overline{\gamma}^0)\mu^0 + \bar{\eth}\eth(\gamma^0 + \overline{\gamma}^0),\\
\partial_u\alpha^0 &= -2\gamma^0\alpha^0 - \bar{\eth}\overline{\gamma}^0,
\end{aligned}$$

$$\partial_u\Psi_0^1 + (2\gamma^0 + 6\overline{\gamma}^0)\Psi_0^1 = -\bar{\eth}(\eth\Psi_0^0 + 4\sigma^0\Psi_1^0),$$

$$\partial_u \Psi_0^2 + (3\gamma^0 + 7\overline{\gamma}^0)\Psi_0^2 = -\frac{1}{2}\bar{\eth}\eth\Psi_0^1 + 3\mu^0\Psi_0^1 + 5(\Psi_1^0\Psi_1^0 - \Psi_0^0\Psi_2^0 - \frac{1}{2}\Psi_0^0\overline{\Psi}_2^0)$$
$$+ 5\bar{\eth}\sigma^0\bar{\eth}\Psi_0^0 + 3\eth\overline{\sigma}^0\eth\Psi_0^0 + \frac{5}{2}\sigma^0\bar{\eth}^2\Psi_0^0 + \frac{5}{2}\eth^2\overline{\sigma}^0\Psi_0^0 + \frac{1}{2}\overline{\sigma}^0\eth^2\Psi_0^0 + \frac{9}{2}\sigma^0\overline{\sigma}^0\eth\Psi_1^0$$
$$+ 12\sigma^0\eth\overline{\sigma}^0\Psi_1^0 + 2\overline{\sigma}^0\eth\sigma^0\Psi_1^0 + \frac{15}{2}\overline{\sigma}^0(\sigma^0)^2\Psi_2^0 + \frac{5}{2}\sigma^0\lambda^0\Psi_0^0.$$

C.2 Parameters of residual gauge transformations

For computational purposes, it turns out to be more convenient to determine the parameters of residual gauge transformations by using the generating set given in (3.2.42) rather than the one in (3.2.44).

Asking that conditions (3.3.5) be preserved on-shell yields

- $0 = \delta_{\xi,\omega}\, e_1^u = -\partial_r \xi^u \Longrightarrow \xi^u = f(u,\zeta,\bar{\zeta})$.
- $0 = \delta_{\xi,\omega}\, e_2^u = -e_2^\alpha \partial_\alpha f + \omega^{12} \Longrightarrow \omega^{12} = \partial_u f + X^A\partial_A f$.
- $0 = \delta_{\xi,\omega}\, e_3^u = -e_3^\alpha \partial_\alpha f + \omega^{42} \Longrightarrow \omega^{24} = L^A\partial_A f$.
- $0 = \delta_{\xi,\omega}\, e_4^u = -e_4^\alpha \partial_\alpha f + \omega^{32} \Longrightarrow \omega^{23} = \bar{L}^A\partial_A f$.
- $0 = \delta_{\xi,\omega}\, e_1^r = -e_1^\alpha \partial_\alpha \xi^r + \omega^{2a} e_a^r \Longrightarrow \xi^r = -\partial_u f r + Z(u,\zeta,\bar{\zeta}) - \partial_A f \int_r^{+\infty} dr[\omega \bar{L}^A + \bar{\omega} L^A + X^A]$.
- $0 = \delta_{\xi,\omega}\, e_1^A = -e_1^\alpha \partial_\alpha \xi^A + \omega^{2a} e_a^A \Longrightarrow \xi^A = Y^A(u,\zeta,\bar{\zeta}) - \partial_B f \int_r^{+\infty} dr[L^A\bar{L}^B + \bar{L}^A L^B]$.
- $\delta_{\xi,\omega}\, \bar{\pi} = 0 \iff 0 = \delta_{\xi,\omega}\, \Gamma_{321} = l^\mu\partial_\mu \omega^{41} + \Gamma_{32a}\omega^{2a} \Longrightarrow \omega^{14} = \omega_0^{14}(u,\zeta,\bar{\zeta}) + \partial_A f \int_r^{+\infty} dr[\bar{\lambda}\bar{L}^A + \bar{\mu} L^A]$.
- $\delta_{\xi,\omega}\, \pi = 0 \iff 0 = \delta_{\xi,\omega}\, \Gamma_{421} = l^\mu\partial_\mu \omega^{31} + \Gamma_{42a}\omega^{2a} \Longrightarrow \omega^{13} = \omega_0^{13}(u,\zeta,\bar{\zeta}) + \partial_A f \int_r^{+\infty} dr[\lambda L^A + \mu\bar{L}^A]$.
- $\delta_{\xi,\omega}\, (\epsilon - \bar{\epsilon}) = 0 \iff 0 = \delta_{\xi,\omega}\, \Gamma_{431} = l^\mu\partial_\mu \omega^{43} + \Gamma_{43a}\omega^{2a} \Longrightarrow \omega^{34} = \omega_0^{34}(u,\zeta,\bar{\zeta}) - \partial_A f \int_r^{+\infty} dr[(\bar{\alpha} - \beta)\bar{L}^A + (\bar{\beta} - \alpha)L^A]$.
- $\epsilon + \bar{\epsilon} = 0 = \kappa = \bar{\kappa}$ is equivalent to $\Gamma_{211} = \Gamma_{311} = \Gamma_{411} = 0$, $\rho - \bar{\rho} = 0$ is equivalent to $\Gamma_{314} - \Gamma_{413} = 0$ while $\tau - \bar{\alpha} - \beta = 0$ is equivalent to $\Gamma_{213} - \Gamma_{312} = 0$. On-shell, i.e., in the absence of torsion, these conditions on spin coefficients hold as a consequence of the tetrad conditions imposed in (3.3.5). It follows that requiring these conditions to be preserved on-shell by gauge transformations does not give rise to new conditions on the parameters. This can also be checked by direct computation.

Asking that the fall-off conditions (3.3.6) be preserved on-shell yields

- $\delta_{\xi,\omega}\ e^A_2 = \mathcal{O}(r^{-1}) \Longrightarrow \partial_u Y^A = 0.$
- $\delta_{\xi,\omega}\ g_{\zeta\zeta} = \mathcal{O}(r^{-1}) \Longrightarrow \bar{\eth} Y^\zeta = 0 \iff Y^\zeta = Y(\zeta).$
- $\delta_{\xi,\omega}\ g_{\bar{\zeta}\bar{\zeta}} = \mathcal{O}(r^{-1}) \Longrightarrow \eth Y^{\bar{\zeta}} = 0 \iff Y^{\bar{\zeta}} = \bar{Y}(\bar{\zeta}).$
- $\delta_{\xi,\omega}\ \Gamma_{314} = \mathcal{O}(r^{-3}) \Longrightarrow Z = \frac{1}{2}\bar{\Delta} f.$
- $\delta_{\xi,\omega}\ \Gamma_{312} = \mathcal{O}(r^{-2}) \Longrightarrow \omega^{14}_0 = (\gamma^0 + \bar{\gamma}^0) P \bar{\partial} f - P \partial_u \bar{\partial} f.$
- $\delta_{\xi,\omega}\ \Gamma_{412} = \mathcal{O}(r^{-2}) \Longrightarrow \omega^{13}_0 = (\gamma^0 + \bar{\gamma}^0) \bar{P} \partial f - \bar{P} \partial_u \partial f.$
- $\delta_{\xi,\omega}\ \Psi_0 = \mathcal{O}(r^{-5})$ does not impose further constraints.

C.3 Action on solution space: original parametrization

Besides (3.3.18), if $s_o = (Y, \bar{Y}, f, \omega_0)$, one finds

$$\begin{aligned}
\delta_{s_o}\sigma^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + \partial_u f + 2\omega^{34}_0]\sigma^0 - \bar{\eth}^2 f,\\
\delta_{s_o}\Psi^0_0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f + 2\omega^{34}_0]\Psi^0_0 + 4\Psi^0_1 \bar{\eth} f,\\
\delta_{s_o}\Psi^0_1 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f + \omega^{34}_0]\Psi^0_1 + 3\Psi^0_2 \bar{\eth} f,\\
\delta_{s_o}\left(\frac{\Psi^0_2 + \bar{\Psi}^0_2}{2}\right) &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f]\left(\frac{\Psi^0_2 + \bar{\Psi}^0_2}{2}\right) + \Psi^0_3 \bar{\eth} f + \bar{\Psi}^0_3 \eth f.
\end{aligned} \tag{C.1}$$

When Ψ_0 can be expanded in powers of $1/r$, $\Psi_0 = \sum_{n=0}^{\infty} \frac{\Psi^n_0}{r^{n+5}}$, one also has

$$\begin{aligned}
\delta_{s_o}\Psi^1_0 = {}& [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 4\partial_u f + 2\omega^{34}_0]\Psi^1_0\\
&+ [-\frac{5}{2}\bar{\Delta} f - 5\bar{\eth} f \eth - \eth f \bar{\eth}]\Psi^0_0 - 4\sigma^0 \eth f \Psi^0_1,
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\delta_{s_o}\Psi^2_0 = {}& [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 5\partial_u f + 2\omega^{34}_0]\Psi^2_0 + [-3\bar{\Delta} f - 3\bar{\eth} f \eth - \eth f \bar{\eth}]\Psi^1_0\\
&+ [5\eth\sigma^0 \eth f + 15\bar{\eth}\bar{\sigma}^0 \bar{\eth} f + 5\sigma^0 \eth f \eth + 3\bar{\sigma}^0 \bar{\eth} f \bar{\eth}]\Psi^0_0 + 12\sigma^0\bar{\sigma}^0 \bar{\eth} f \Psi^0_1,
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
\delta_{s_o}\Psi^n_0 = {}& [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + (n+3)\partial_u f + 2\omega^{34}_0]\Psi^n_0\\
&+ (\text{inhomogeneous terms}).
\end{aligned} \tag{C.4}$$

For later purposes, we also give the variations of composite quantities in terms of free data,

$$\begin{aligned}
\delta_{s_o}\lambda^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 2\partial_u f - 2\omega_0^{34}]\lambda^0 - \partial_u \bar{\eth}^2 f + (\bar{\gamma}^0 - 3\gamma^0)\bar{\eth}^2 f,\\
\delta_{s_o}\Psi_2^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f]\Psi_2^0 + 2\Psi_3^0\eth f,\\
\delta_{s_o}\Psi_3^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f - \omega_0^{34}]\Psi_3^0 + \Psi_4^0\eth f,\\
\delta_{s_o}\Psi_4^0 &= [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f - 2\omega_0^{34}]\Psi_4^0.
\end{aligned} \tag{C.5}$$

C.4 Useful relations

Some useful relations for the computation of the current algebra are summarized here.

$$\begin{aligned}
&\partial_u f = \frac{1}{2}(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + f(\gamma^0 + \bar{\gamma}^0),\\
&\hat{f} = \frac{1}{2}f_1(\eth\mathcal{Y}_2 + \overline{\eth\mathcal{Y}}_2) + \mathcal{Y}_1\eth f_2 + \overline{\mathcal{Y}}_1\bar{\eth} f_2 - (1\leftrightarrow 2),\\
&\hat{\mathcal{Y}} = \mathcal{Y}_1\eth^2\mathcal{Y}_2 - \mathcal{Y}_2\eth^2\mathcal{Y}_1, \quad \hat{\overline{\mathcal{Y}}} = \overline{\mathcal{Y}}_1\bar{\eth}^2\overline{\mathcal{Y}}_2 - \overline{\mathcal{Y}}_2\bar{\eth}^2\overline{\mathcal{Y}}_1,\\
&\eth^2\hat{\mathcal{Y}} = \eth\mathcal{Y}_1\eth^2\mathcal{Y}_2 + \mathcal{Y}_1\eth^3\mathcal{Y}_2 - (1\leftrightarrow 2), \quad \eth\bar{\eth}\hat{\overline{\mathcal{Y}}} = \overline{\mathcal{Y}}_1\eth\bar{\eth}^2\overline{\mathcal{Y}}_2 - (1\leftrightarrow 2),\\
&\eth^3\hat{\mathcal{Y}} = 2\eth\mathcal{Y}_1\eth^3\mathcal{Y}_2 + \mathcal{Y}_1\eth^4\mathcal{Y}_2 - (1\leftrightarrow 2), \quad \eth^2\bar{\eth}\hat{\overline{\mathcal{Y}}} = \overline{\mathcal{Y}}_1\eth^2\bar{\eth}^2\overline{\mathcal{Y}}_2 - (1\leftrightarrow 2),\\
&\bar{\eth}\eth^3\mathcal{Y} = 2\mathcal{Y}\eth^2\mu^0 + 4\eth\mu^0\eth\mathcal{Y}, \quad \bar{\eth}^2\eth^2\mathcal{Y} = 2\bar{\eth}\eth\mu^0\mathcal{Y} + 2\bar{\eth}\mu^0\eth\mathcal{Y} + 4(\mu^0)^2\mathcal{Y},\\
&\eth\hat{f} = \frac{1}{2}f_1\eth(\eth\mathcal{Y}_2 + \overline{\eth\mathcal{Y}}_2) + \mathcal{Y}_1\eth^2 f_2 + \overline{\mathcal{Y}}_1\eth\bar{\eth} f_2 + \frac{1}{2}(\eth\mathcal{Y}_1 - \overline{\eth\mathcal{Y}}_1)\eth f_2 - (1\leftrightarrow 2),\\
&\eth\bar{\eth}\overline{\mathcal{Y}} = 2\mu^0\overline{\mathcal{Y}}, \quad \bar{\eth}\eth\mathcal{Y} = 2\mu^0\mathcal{Y}, \quad \partial_u\eth\mathcal{Y} = 2\bar{\nu}^0\mathcal{Y},\\
&\partial_u\eth f = \frac{1}{2}\eth(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + \eth f(\gamma^0 - \bar{\gamma}^0) + f\bar{\nu}^0,\\
&\partial_u\eth^2\mathcal{Y} = 2\eth\bar{\nu}^0\mathcal{Y} + 2\bar{\nu}^0\eth\mathcal{Y} - 2\bar{\gamma}^0\eth^2\mathcal{Y},\\
&\partial_u\eth\overline{\eth\mathcal{Y}} = 2\eth\nu^0\overline{\mathcal{Y}} - 2\bar{\gamma}^0\eth\overline{\eth\mathcal{Y}},\\
&\partial_u\eth^2 f = \frac{1}{2}\eth^2(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + \eth^2 f(\gamma^0 - 3\bar{\gamma}^0) + f\eth\bar{\nu}^0,\\
&\partial_u\eth\bar{\eth} f = \frac{1}{2}\eth\bar{\eth}(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) - \eth\bar{\eth} f(\gamma^0 + \bar{\gamma}^0) + \bar{\eth} f\bar{\nu}^0 + \eth f\nu^0 + f\eth\nu^0,\\
&\partial_u\eth\bar{\sigma}^0 = \eth\lambda^0 + \bar{\nu}^0\bar{\sigma}^0 - (\bar{\gamma}^0 + 3\gamma^0)\eth\bar{\sigma}^0,\\
&\partial_u\eth\mu^0 = \bar{\eth}\bar{\nu}^0 - 2\mu^0\bar{\nu}^0 - 2(\gamma^0 + 2\bar{\gamma}^0)\eth\mu^0,\\
&\bar{\eth}\eth\bar{\nu}^0 = \eth^2\nu^0 - 2\mu^0\bar{\nu}^0.
\end{aligned}$$

If one wants to compute the current algebra from the expressions derived in the standard Cartan formalism [181], one needs to transform the spin coefficients into a Lorentz connection with a space-time index in NU gauge. Using the notations of subsection 3.3.1, together with the gauge choice for the tetrads (3.3.5) (and thus also (3.3.12)), we have

$$
\begin{aligned}
&\Gamma_{12u} = -(\gamma+\bar{\gamma}) - \tau X^A \bar{L}_A - \bar{\tau} X^A L_A, && \Gamma_{12A} = \tau \bar{L}_A + \bar{\tau} L_A,\\
&\Gamma_{13u} = -\tau - \sigma X^A \bar{L}_A - \rho X^A L_A, && \Gamma_{13A} = \sigma \bar{L}_A + \rho L_A,\\
&\Gamma_{14u} = -\bar{\tau} - \bar{\sigma} X^A L_A - \rho X^A \bar{L}_A, && \Gamma_{14A} = \rho \bar{L}_A + \bar{\sigma} L_A,\\
&\Gamma_{23u} = \bar{\nu} + \bar{\lambda} X^A \bar{L}_A + \bar{\mu} X^A L_A, && \Gamma_{23A} = -\bar{\lambda} \bar{L}_A - \bar{\mu} L_A,\\
&\Gamma_{24u} = \nu + \mu X^A \bar{L}_A + \lambda X^A L_A, && \Gamma_{24A} = -\mu \bar{L}_A - \lambda L_A,\\
&\Gamma_{34u} = (\gamma-\bar{\gamma}) + (\beta-\bar{\alpha}) X^A \bar{L}_A + (\alpha-\bar{\beta}) X^A L_A, && \Gamma_{34A} = (\bar{\alpha}-\beta)\bar{L}_A + (\bar{\beta}-\alpha) L_A,\\
&\Gamma_{abr} = 0.
\end{aligned}
$$

Appendix D

Map from Bondi to Fefferman-Graham gauge

In this section, we find the explicit change of coordinates that maps a general vacuum asymptotically locally (A)dS$_4$ spacetime ($\Lambda \neq 0$) in Bondi gauge to Fefferman-Graham gauge [166]. This procedure will lead to the explicit map between the free functions defined in Bondi gauge $\{q_{AB}, \beta_0, U_0^A, \mathcal{E}_{AB}, M, N_A\}$ and the holographic functions defined in Fefferman-Graham gauge, namely the boundary metric $g_{ab}^{(0)}$ and the boundary stress-tensor encoded in $g_{ab}^{(3)}$.

We follow and further develop the procedure introduced in [167]. We first note that one can map the (A)dS$_4$ vacuum metric in retarded coordinates

$$ds^2 = \left(\frac{\Lambda r^2}{3} - 1\right) du^2 - 2dudr + r^2 \mathring{q}_{AB} dx^A dx^B, \tag{D.1}$$

to the global patch

$$ds^2 = -\Big(1 - \frac{\Lambda r^2}{3}\Big)dt^2 + \Big(1 - \frac{\Lambda r^2}{3}\Big)^{-1} dr^2 + r^2 \mathring{q}_{AB} dx^A dx^B \tag{D.2}$$

by using $u = t - r_\star$, where the tortoise coordinate is $r_\star \equiv \sqrt{-\frac{3}{\Lambda}} \left[\arctan\left(r\sqrt{-\frac{\Lambda}{3}}\right) - \frac{\pi}{2}\right]$ for $\Lambda < 0$ and $r_\star \equiv \sqrt{\frac{3}{\Lambda}} \left[\text{arcoth}\left(r\sqrt{\frac{\Lambda}{3}}\right)\right]$ for $\Lambda > 0$. The next step is to transform the radial coordinate r into the tortoise coordinate $r_\star$ which maps $r = \infty$ to $r_\star = 0$. The change of coordinates from $(t_\star, r_\star, x_\star^A)$ to Fefferman-Graham gauge (t, ρ, x^A) can then be performed perturbatively in series of ρ around $\rho = 0$, identified with $r_\star = 0$.

The general algorithm is then the following:

1. Starting from any asymptotically locally (A)dS$_4$ solution formulated in Bondi gauge (u, r, x^A), we perform the preliminary change to the tortoise coordinate

system,

$$
\begin{aligned}
&u \to t_\star - r_\star, \quad x^A \to x^A_\star, \\
&r \to \begin{cases} \sqrt{-\frac{3}{\Lambda}}\tan\left(r_\star\sqrt{-\frac{\Lambda}{3}} + \frac{\pi}{2}\right) & \text{if } \Lambda < 0 \\ \sqrt{\frac{3}{\Lambda}}\coth\left(r_\star\sqrt{\frac{\Lambda}{3}}\right) & \text{if } \Lambda > 0 \end{cases} = \frac{3}{\Lambda r_\star} + \frac{r_\star}{3} - \frac{\Lambda r_\star^3}{135} + \mathcal{O}(r_\star^5).
\end{aligned} \tag{D.3}
$$

2. We reach the Fefferman-Graham gauge at order $N \geqslant 0$ perturbatively,

$$
g_{\rho\rho} = -\frac{3}{\Lambda}\frac{1}{\rho^2}\Big(1 + \mathcal{O}(\rho^{N+1})\Big), \quad g_{\rho t} = \frac{1}{\rho^2}\mathcal{O}(\rho^{N+1}), \quad g_{\rho A} = \frac{1}{\rho^2}\mathcal{O}(\rho^{N+1}), \tag{D.4}
$$

thanks to a second change of coordinates,

$$
\begin{aligned}
r_\star &\to \sum_{n=1}^{N+1} R_n(t, x^A)\rho^n, \\
t_\star &\to t + \sum_{n=1}^{N+1} T_n(t, x^A)\rho^n, \\
x^A_\star &\to x^A + \sum_{n=1}^{N+1} X^A_n(t, x^B)\rho^n.
\end{aligned} \tag{D.5}
$$

To obtain all the free functions in γ_{ab}, we must proceed up to order $N = 3$. For each n, each gauge condition (D.4) can be solved separately and will algebraically determine R_n, T_n and X^A_n respectively. Only the function $R_1(t, x^A)$ remains unconstrained by these conditions, since it represents a Weyl transformation on the boundary metric that is allowed within Fefferman-Graham gauge. We fix this freedom by requiring the normalization $g^{(0)}_{AB} = q_{AB}$.

We use the following shorthand notations for subleading fields in Bondi gauge:

$$
\begin{aligned}
\frac{V}{r} &= \frac{\Lambda}{3}r^2 + r\, V_{(1)}(t, x^A) + V_{(0)}(t, x^A) + \frac{2M}{r} + \mathcal{O}(r^{-2}), \\
U^A &= U^A_0(t, x^B) + \frac{1}{r}U^A_{(1)}(t, x^B) + \frac{1}{r^2}U^A_{(2)}(t, x^B) + \frac{1}{r^3}U^A_{(3)}(t, x^B) + \mathcal{O}(r^{-4}), \\
\beta &= \beta_0(t, x^A) + \frac{1}{r^2}\beta_{(2)}(t, x^A) + \mathcal{O}(r^{-4}).
\end{aligned} \tag{D.6}
$$

whose explicit on-shell values can be read off in (6.2.44) and (6.2.41). That will state the equations in a more compact way. All the fields are now evaluated on (t, x^A) since the time coordinate on the boundary can be defined as t as well as u. We also define some recurrent structures appearing in the diffeomorphism as differential operators on boundary scalar fields $f(t,x^A)$:

$$\begin{aligned} P[f] &= \frac{1}{2}e^{-4\beta_0}(\partial_t f + U_0^A\partial_A f), \\ Q[f;g] &= P[f] - 2P[g]f, \\ B_A[f] &= \frac{1}{2}e^{-2\beta_0}(\partial_A - 2\partial_A\beta_0)f. \end{aligned} \tag{D.7}$$

$P^n[f]$ denotes n applications of P on f, for example $P^2[f] \equiv P[P[f]]$. Now we can write down the perturbative change of coordinate to Fefferman-Graham gauge:

$$\begin{aligned} R_1(t,x^A) &= -\frac{3}{\Lambda}, \\ R_2(t,x^A) &= \frac{9}{2\Lambda^2}e^{-2\beta_0}V_{(1)}, \\ R_3(t,x^A) &= \frac{3}{2\Lambda}\beta_{(2)} - \frac{3}{\Lambda^2}\Big(1+\frac{3}{4}e^{-2\beta_0}V_{(0)}\Big) + \frac{27}{2\Lambda^3}\Big(Q[V_{(1)};\beta_0] - \frac{3}{8}e^{-4\beta_0}V_{(1)}^2\Big), \\ R_4(t,x^A) &= \frac{3}{\Lambda^2}e^{-2\beta_0}\Big(M + 2e^{4\beta_0}P[\beta_{(2)}] - \frac{5}{2}V_{(1)}\beta_{(2)}\Big) \\ &\quad - \frac{9}{\Lambda^3}\Big\{Q[V_{(0)};\beta_0] + \frac{1}{4}e^{-4\beta_0}\Big[U^A_{(1)}\partial_A V_{(1)} - 2V_{(1)}U^A_{(1)}\partial_A\beta_0 - 3V_{(1)}(2e^{2\beta_0}+V_{(0)})\Big]\Big\} \\ &\quad + \frac{27}{\Lambda^4}e^{2\beta_0}\Big[P^2[V_{(1)}] - 2V_{(1)}\Big(P^2[\beta_0] + \frac{1}{2}e^{-4\beta_0}Q[V_{(1)};\beta_0] - \frac{3}{32}e^{-8\beta_0}V_{(1)}^2\Big) - 2P[\beta_0]P[V_{(1)}]\Big], \end{aligned}$$

$$\begin{aligned} T_1(t,x^A) &= (1-e^{-2\beta_0})R_1(t,x^A), \\ T_2(t,x^A) &= (1-e^{-2\beta_0})R_2(t,x^A) - \frac{18}{\Lambda^2}\Big(P[\beta_0] - \frac{1}{4}e^{-4\beta_0}V_{(1)}\Big), \\ T_3(t,x^A) &= (1-e^{-2\beta_0})R_3(t,x^A) - \frac{3}{\Lambda^2}e^{-2\beta_0}(1+e^{-2\beta_0}V_{(0)} - 2\partial^A\beta_0\partial_A\beta_0) \\ &\quad + \frac{9}{\Lambda^3}e^{-2\beta_0}\Big(Q[V_{(1)};\beta_0] - 4e^{4\beta_0}P^2[\beta_0] - \frac{1}{2}e^{-4\beta_0}V_{(1)}^2\Big), \\ T_4(t,x^A) &= (1-e^{-2\beta_0})R_4(t,x^A) \\ &\quad + \frac{9}{2\Lambda^2}\Big[e^{-4\beta_0}\Big(M - \beta_{(2)}V_{(1)} - \frac{1}{3}U^A_{(2)}\partial_A\beta_0\Big) - \frac{1}{2}(P[\beta_{(2)}] - 8\beta_{(2)}P[\beta_0])\Big] \\ &\quad - \frac{27}{\Lambda^3}\Big\{\frac{1}{8}e^{-2\beta_0}\Big(3Q[V_{(0)};\beta_0] - \frac{8}{3}P[\beta_0]V_{(0)} - 2e^{-4\beta_0}V_{(1)}V_{(0)}\Big) \\ &\qquad + \frac{1}{3}e^{-2\beta_0}\Big(P[U^A_{(1)}]\partial_A\beta_0 + \frac{3}{2}U^A_{(1)}\partial_A P[\beta_0]\Big) \\ &\qquad - \frac{1}{12}e^{-4\beta_0}\Big[U^A_{(1)}B_A[V_{(1)}] + 6V_{(1)} - 2(V_{(1)}\partial_A\beta_0 + 2\partial_B\beta_0\partial_A U^B_0)\partial^A\beta_0\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{81}{\Lambda^4}\Big\{-\frac{2}{3}e^{4\beta_0}\Big(P^3[\beta_0]+2P[\beta_0]P^2[\beta_0]\Big)+\frac{1}{4}\Big(P^2[V_{(1)}]-2V_{(1)}P^2[\beta_0]\Big)\\
&\qquad+\frac{1}{6}P[\beta_0]\Big[\Big(\frac{13}{4}e^{-4\beta_0}V_{(1)}-8P[\beta_0]\Big)V_{(1)}+P[V_{(1)}]\Big]\\
&\qquad-\frac{1}{16}e^{-4\beta_0}V_{(1)}\Big(5P[V_{(1)}]-e^{-4\beta_0}V_{(1)}^2\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
X_1^A(t,x^B) &= (T_1-R_1)U_0^A,\\
X_2^A(t,x^B) &= (T_2-R_2)U_0^A-\frac{3}{2\Lambda}e^{-2\beta_0}U_{(1)}^A+\frac{9}{\Lambda^2}P[U_0^A],\\
X_3^A(t,x^B) &= (T_3-R_3)U_0^A+\frac{1}{\Lambda}e^{-2\beta_0}U_{(2)}^A\\
&\quad-\frac{6}{\Lambda^2}\Big[Q[U_{(1)}^A;\beta_0]+\frac{1}{2}B^A[V_{(1)}]+\frac{1}{4}e^{-4\beta_0}(U_{(1)}^B\partial_BU_0^A-V_{(1)}U_{(1)}^A)\Big]\\
&\quad+\frac{18}{\Lambda^3}e^{2\beta_0}Q[P[U_0^A];\beta_0],\\
X_4^A(t,x^B) &= (T_4-R_4)U_0^A-\frac{3}{4\Lambda}e^{-2\beta_0}\Big[U_{(3)}^A+\frac{1}{2}e^{2\beta_0}(\partial^A\beta_{(2)}-8\beta_{(2)}\partial^A\beta_0)\Big]\\
&\quad+\frac{9}{2\Lambda^2}\Big[Q[U_{(2)}^A;\beta_0]-\frac{1}{2}e^{-4\beta_0}\Big(V_{(1)}U_{(2)}^A-\frac{1}{3}U_{(2)}^B\partial_BU_{(0)}^A\Big)-2\beta_{(2)}P[U_0^A]\\
&\qquad+\frac{1}{4}B^A[V_{(0)}]+\frac{1}{2}C^{AC}B_C[V_{(1)}]+\frac{1}{2}e^{-2\beta_0}U_{(1)}^CB_C[U_{(1)}^A]\Big]\\
&\quad-\frac{27}{\Lambda^3}\Big\{e^{2\beta_0}\Big(P[Q[U_{(1)}^A;\beta_0]]+P[B^A[V_{(1)}]]-\frac{1}{2}q^{AC}P[B_C[V_{(1)}]]\Big)\\
&\qquad-\frac{1}{2}e^{-2\beta_0}\Big[V_{(1)}P[U_{(1)}^A]-\frac{2}{3}\Big(P[U_{(1)}^C]+\frac{1}{2}B^C[V_{(1)}]-5P[\beta_0]U_{(1)}^C\Big)\partial_CU_0^A\\
&\qquad+\frac{1}{2}P[V_{(1)}]U_{(1)}^A-\frac{2}{3}(V_{(0)}-8e^{2\beta_0}\partial^B\beta_0\partial_B\beta_0)P[U_0^A]-U_{(1)}^BP[\partial_BU_0^A]\\
&\qquad+\frac{1}{2}(\partial^AU_0^C)B_C[V_{(1)}]\Big]+3P[\beta_0]V_{(1)}\partial^A\beta_0\\
&\qquad-e^{-4\beta_0}\Big[\frac{3}{32}\Big(\partial^A(V_{(1)}^2)-\frac{20}{3}\partial^A\beta_0V_{(1)}^2\Big)+\frac{1}{6}(V_{(1)}\partial^B\beta_0-\partial^C\beta_0\partial_CU_0^B)\partial_BU_0^A\Big]\Big\}.\\
&\quad+\frac{81}{\Lambda^4}\Big\{\frac{1}{3}e^{4\beta_0}P^3[U_0^A]+\Big[\frac{1}{4}e^{-4\beta_0}V_{(1)}^2-\frac{1}{3}Q[V_{(1)};\beta_0]-\frac{4}{3}e^{4\beta_0}(P^2[\beta_0]+P[\beta_0]^2)\Big]P[U_0^A]\Big\}.
\end{aligned}
$$

Several consistency checks can be performed at each stage of the computation. The boundary metric in Fefferman-Graham gauge must be equivalent to the pulled-back metric on the hypersurface $\{r\to\infty\}$ in Bondi gauge, up to the usual replacement $u\to t$:

$$
g_{ab}^{(0)}=\begin{bmatrix}\frac{\Lambda}{3}e^{4\beta_0}+U_0^CU_C^0 & -U_B\\ -U_A & q_{AB}\end{bmatrix}. \tag{D.8}
$$

At subleading orders, $g_{ab}^{(1)}$ and $g_{ab}^{(2)}$ must be algebraically determined by $g_{ab}^{(0)}$ and its first and second derivatives, which turns out to be the case. The constraint (6.2.45) forces $g^{(1)}_{ab} = 0$ while the annulation of $\mathcal{D}_{AB}(t, x^C)$ (6.2.46) results in

$$g^{(2)}_{ab} = \frac{3}{\Lambda}\Big[R^{(0)}_{ab} - \frac{1}{4}R_{(0)}g^{(0)}_{ab}\Big]. \tag{D.9}$$

We do not give the full general form of $g^{(3)}_{ab}$, but it can be proven that this tensor is traceless with respect to $g^{(0)}_{ab}$, and that the equations of motion in Bondi gauge are necessary and sufficient to show its conservation $D^{(0)}_a g^{ab}_{(3)} = 0$, as we argued in the main text.

After boundary gauge fixing $\beta_0 = 0$, $U^A_0 = 0$, the expressions of each coefficient in the diffeomorphism simplify drastically:

$$\begin{aligned}
R_1(t, x^A) &= -\frac{3}{\Lambda},\\
R_2(t, x^A) &= \frac{9}{2\Lambda^2}V_{(1)},\\
R_3(t, x^A) &= \frac{3}{2\Lambda}\beta_{(2)} - \frac{3}{\Lambda^2}\Big(1 + \frac{3}{4}V_{(0)}\Big) + \frac{27}{2\Lambda^3}\Big(\frac{1}{2}\partial_t V_{(1)} - \frac{3}{8}V^2_{(1)}\Big),\\
R_4(t, x^A) &= \frac{3}{\Lambda^2}\Big(M + \partial_t\beta_{(2)} - \frac{5}{2}V_{(1)}\beta_{(2)}\Big) - \frac{9}{\Lambda^3}\Big[\frac{1}{2}\partial_t V_{(0)} - \frac{3}{4}V_{(1)}(2 + V_{(0)})\Big]\\
&\quad + \frac{27}{\Lambda^4}\Big(\frac{1}{4}\partial_t^2[V_{(1)}] - \frac{1}{4}\partial_t V^2_{(1)} + \frac{6}{32}V^3_{(1)}\Big).
\end{aligned}$$

$$\begin{aligned}
T_1(t, x^A) &= 0,\\
T_2(t, x^A) &= \frac{9}{2\Lambda^2}V_{(1)},\\
T_3(t, x^A) &= -\frac{3}{\Lambda^2}(1 + V_{(0)}) + \frac{9}{2\Lambda^3}\Big(\partial_t V_{(1)} - V^2_{(1)}\Big),\\
T_4(t, x^A) &= \frac{9}{2\Lambda^2}\Big[M - \frac{1}{4}(\partial_t\beta_{(2)} + 4V_{(1)}\beta_{(2)})\Big] + \frac{9}{\Lambda^3}\Big[-\frac{9}{16}\partial_t V_{(0)} + \frac{3}{2}V_{(1)}(1 + \frac{1}{2}V_{(0)})\Big]\\
&\quad + \frac{81}{\Lambda^4}\Big(\frac{1}{16}\partial_t^2 V_{(1)} - \frac{5}{64}\partial_t V^2_{(1)} + \frac{1}{16}V^3_{(1)}\Big).
\end{aligned}$$

$$\begin{aligned}
X^A_1(t, x^B) &= X^A_2(t, x^B) = 0,\\
X^A_3(t, x^B) &= \frac{1}{\Lambda}U^A_{(2)} - \frac{3}{2\Lambda^2}\partial^A V_{(1)},\\
X^A_4(t, x^B) &= -\frac{3}{4\Lambda}\Big(U^A_{(3)} + \frac{1}{2}\partial^A\beta_{(2)}\Big) + \frac{9}{2\Lambda^2}\Big(\frac{1}{2}\partial_t U^A_{(2)} - \frac{1}{2}V_{(1)}U^A_{(2)} + \frac{1}{8}\partial^A V_{(0)}\Big)\\
&\quad - \frac{27}{16\Lambda^3}\, q^{AB}\Big(\partial_t\partial_B V_{(1)} + \frac{1}{2}V_{(1)}\partial_B V_{(1)}\Big).
\end{aligned}$$

Appendix E

Detailed computations in Fefferman-Graham gauge

E.1 Check of the conservation law

We explicitly check that the ansatz (6.3.23) satisfies (6.3.24). We start by computing the right-hand side of (6.3.24). The variations $\delta_\xi\sqrt{|g_{(0)}|}$ and $\delta_\xi T^{ab}$ are given by

$$\delta_\xi\sqrt{|g_{(0)}|} = \frac{1}{2}\sqrt{|g_{(0)}|}\, g^{ab}_{(0)}\delta_\xi g^{(0)}_{ab} = \sqrt{|g_{(0)}|}(D^{(0)}_a\xi^a_0 - 3\sigma_\xi), \tag{E.1.1}$$

$$\delta_\xi T^{ab} = \mathcal{L}_{\xi_0}T^{ab} + 5\sigma_\xi T^{ab}. \tag{E.1.2}$$

Recalling that T_{ab} obeys $T^a{}_a = 0$ and $D^{(0)}_a T^{ab} = 0$ on-shell, we get

$$\delta\theta^\rho_{\text{ren}}[g;\mathcal{L}_\xi g] = -\eta\delta\left(\sqrt{|g_{(0)}|}T^{ab}\right)D^{(0)}_a\xi^0_b - \eta\sqrt{|g_{(0)}|}T^{ab}\delta\left(D^{(0)}_a\xi^0_b\right), \tag{E.1.3}$$

$$-\delta_\xi\theta_{\text{ren}}[g;\delta g] = \frac{1}{2}\eta\sqrt{|g_{(0)}|}\left(D^{(0)}_c\xi^c_0 T^{ab} + \mathcal{L}_{\xi_0}T^{ab}\right)\delta g^{(0)}_{ab} + \eta\sqrt{|g_{(0)}|}T^{ab}\delta(D^{(0)}_a\xi^0_b). \tag{E.1.4}$$

The left-hand side reads as

$$\begin{aligned}\partial_a k^{\rho a}_\xi[g;\delta g] = &-\eta\delta\left(\sqrt{|g_{(0)}|}T^{ab}\right)D^{(0)}_a\xi^0_b - \eta\sqrt{|g_{(0)}|}T^{ab}\delta g^{(0)}_{bc}D^{(0)}_a\xi^c_0\\ &+\frac{1}{2}\eta\sqrt{|g_{(0)}|}D^{(0)}_a\xi^a_0 T^{bc}\delta g^{(0)}_{bc} + \frac{1}{2}\eta\sqrt{|g_{(0)}|}\xi^a_0 D^{(0)}_a T^{bc}\delta g^{(0)}_{bc}.\end{aligned} \tag{E.1.5}$$

Using $\mathcal{L}_{\xi_0}(T^{ab}) = \xi^c_0 D^{(0)}_c T^{ab} - 2T^{c(a}D^{(0)}_c\xi^{b)}_0$, we have

$$\begin{aligned}&\partial_a k^{\rho a}_\xi[g;\delta g] + \omega^\rho_{\text{ren}}[g;\delta g,\mathcal{L}_\xi g]\\ &\quad= \frac{1}{2}\eta\sqrt{|g_{(0)}|}\left(\mathcal{L}_{\xi_0}T^{ab}\delta g^{(0)}_{ab} + 2T^{ab}\delta g^{(0)}_{bc}D^{(0)}_a\xi^c_0 - \xi^c_0 D^{(0)}_c T^{ab}\delta g^{(0)}_{ab}\right) = 0,\end{aligned} \tag{E.1.6}$$

which finishes the verification.

E.2 Charge algebra

Here, we write the explicit computations leading to the result (6.3.38).

The computation is on-shell, so in particular $g_{(0)}^{ab}T_{ab}=0$ and $D_a^{(0)}T^{ab}=0$. Let us start by computing $\delta_\chi J_\xi^a[g]$. The computation is direct and takes benefit of (E.1.1) and (E.1.2):

$$\begin{aligned}\delta_\chi J_\xi^a[g] &= \eta\sqrt{|g_{(0)}|}D_d^{(0)}(\chi_0^d T^{ab})g_{bc}^{(0)}\xi_0^c - \eta\sqrt{|g_{(0)}|}T^{bd}D_d^{(0)}\chi_0^a g_{bc}^{(0)}\xi_0^c \\ &\quad + \eta\sqrt{|g_{(0)}|}T^{ab}(D_c^{(0)}\chi_b^0)\xi_0^c + J^a_{\delta_\chi\xi}[g].\end{aligned} \tag{E.2.1}$$

To obtain the second term is just a matter of replacement, so

$$\Xi_\chi^a[\delta_\xi g, g] = -\eta\sqrt{|g_{(0)}|}\chi_0^a T^{bc}D_b^{(0)}\xi_c^0 - J^a_{\delta_\xi\chi}[g]. \tag{E.2.2}$$

Summing both contributions and using the fact that T^{ab} is divergence-free, we get

$$\begin{aligned}&\delta_\chi J_\xi^a[g] + \Xi_\chi^a[\delta_\xi g, g] \\ &\quad = \eta\sqrt{|g_{(0)}|}\,T^a{}_b\left(\xi^c D_c^{(0)}\chi_0^b - \chi_0^c D_c^{(0)}\xi_0^b\right) + J^a_{\delta_\chi\xi} - J^a_{\delta_\xi\chi} - 2\eta\partial_b\left(\sqrt{|g_{(0)}|}\chi_0^{[a}T^{b]}{}_c\xi_0^c\right) \\ &\quad = \eta\sqrt{|g_{(0)}|}\,T^a{}_b[\xi,\chi]^b + J^a_{\delta_\chi\xi} - J^a_{\delta_\xi\chi} - 2\eta\partial_b\left(\sqrt{|g_{(0)}|}\chi_0^{[a}T^{b]}{}_c\xi_0^c\right) \\ &\quad = J^a_{[\xi,\chi]} + J^a_{\delta_\chi\xi} - J^a_{\delta_\xi\chi} - 2\eta\partial_b\left(\sqrt{|g_{(0)}|}\chi_0^{[a}T^{b]}{}_c\xi_0^c\right) \\ &\quad = J^a_{[\xi,\chi]_A} - 2\eta\partial_b\left(\sqrt{|g_{(0)}|}\chi_0^{[a}T^{b]}{}_c\xi_0^c\right).\end{aligned} \tag{E.2.3}$$

The last term exhibits the exterior derivative of a 2-form,

$$\boldsymbol{\ell} = \ell^{ab}(\mathrm{d}^2x)_{ab}, \quad \ell^{ab} = -2\eta\sqrt{|g_{(0)}|}\chi_0^{[a}T^{b]}{}_c\xi_0^c. \tag{E.2.4}$$

Therefore, we have shown that the charge algebra represents the vector algebra without additional 2-cocycle,

$$\{J_\xi[g], J_\chi[g]\}^a_\star = J^a_{[\xi,\chi]_A}[g] + \partial_b\ell^{ab}[g]. \tag{E.2.5}$$

Bibliography

[1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[2] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," *Proc. Roy. Soc. Lond.* **A270** (1962) 103–126.

[3] R. Sachs, "Asymptotic symmetries in gravitational theory," *Phys. Rev.* **128** (1962) 2851–2864.

[4] A. Ashtekar, "Geometry and Physics of Null Infinity," 1409.1800.

[5] G. Barnich and C. Troessaert, "Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited," *Phys. Rev. Lett.* **105** (2010) 111103, 0909.2617.

[6] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence," *JHEP* **05** (2010) 062, 1001.1541.

[7] G. Barnich and C. Troessaert, "Supertranslations call for superrotations," *PoS* **CNCFG2010** (2010) 010, 1102.4632. [Ann. U. Craiova Phys.21,S11(2011)].

[8] G. Compère, A. Fiorucci, and R. Ruzziconi, "Superboost transitions, refraction memory and super-Lorentz charge algebra," *JHEP* **11** (2018) 200, 1810.00377.

[9] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, "Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix," *JHEP* **08** (2014) 058, 1406.3312.

[10] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," 1703.05448.

[11] E. Himwich, Z. Mirzaiyan, and S. Pasterski, "A Note on the Subleading Soft Graviton," 1902.01840.

[12] M. Campiglia and A. Laddha, "Asymptotic symmetries and subleading soft graviton theorem," *Phys. Rev.* **D90** (2014), no. 12, 124028, 1408.2228.

[13] M. Campiglia and A. Laddha, "New symmetries for the Gravitational S-matrix," *JHEP* **04** (2015) 076, 1502.02318.

[14] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," *Commun. Math. Phys.* **104** (1986) 207–226.

[15] M. Henneaux and C. Teitelboim, "Asymptotically anti-De Sitter Spaces," *Commun. Math. Phys.* **98** (1985) 391–424.

[16] A. Ashtekar, J. Bicak, and B. G. Schmidt, "Asymptotic structure of symmetry reduced general relativity," *Phys. Rev.* **D55** (1997) 669–686, gr-qc/9608042.

[17] G. Barnich and G. Compere, "Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions," *Class. Quant. Grav.* **24** (2007) F15–F23, gr-qc/0610130.

[18] D. Grumiller and M. Riegler, "Most general AdS_3 boundary conditions," *JHEP* **10** (2016) 023, 1608.01308.

[19] A. Ashtekar, B. Bonga, and A. Kesavan, "Asymptotics with a positive cosmological constant: I. Basic framework," *Class. Quant. Grav.* **32** (2015), no. 2, 025004, 1409.3816.

[20] A. Strominger, "Asymptotic Symmetries of Yang-Mills Theory," *JHEP* **07** (2014) 151, 1308.0589.

[21] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, "New Symmetries of Massless QED," *JHEP* **10** (2014) 112, 1407.3789.

[22] H. Afshar, E. Esmaeili, and M. M. Sheikh-Jabbari, "Asymptotic Symmetries in p-Form Theories," *JHEP* **05** (2018) 042, 1801.07752.

[23] S. Detournay, P. M. Petropoulos, and C. Zwikel, "Asymptotic Symmetries of Three-Dimensional Black Strings," *JHEP* **06** (2019) 131, 1812.08764.

[24] G. Barnich and P.-H. Lambert, "Einstein-Yang-Mills theory: Asymptotic symmetries," *Phys. Rev.* **D88** (2013) 103006, 1310.2698.

[25] F. Capone and M. Taylor, "Cosmic branes and asymptotic structure," 1904.04265.

[26] H. Lu, P. Mao, and J.-B. Wu, "Asymptotic Structure of Einstein-Maxwell-Dilaton Theory and Its Five Dimensional Origin," 1909.00970.

[27] R. Ruzziconi, "Asymptotic Symmetries in the Gauge Fixing Approach and the BMS Group," *PoS* **Modave2019** (2020) 003, 1910.08367.

[28] G. 't Hooft, "Dimensional reduction in quantum gravity," *Conf. Proc.* **C930308** (1993) 284–296, gr-qc/9310026.

[29] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, hep-th/9409089.

[30] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, hep-th/9711200. [Adv. Theor. Math. Phys.2,231(1998)].

[31] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, hep-th/9802150.

[32] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, hep-th/9905111.

[33] S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse," *Phys. Rev.* **D14** (1976) 2460–2473.

[34] S. W. Hawking, M. J. Perry, and A. Strominger, "Soft Hair on Black Holes," *Phys. Rev. Lett.* **116** (2016), no. 23, 231301, 1601.00921.

[35] S. W. Hawking, M. J. Perry, and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," *JHEP* **05** (2017) 161, 1611.09175.

[36] S. Haco, S. W. Hawking, M. J. Perry, and A. Strominger, "Black Hole Entropy and Soft Hair," *JHEP* **12** (2018) 098, 1810.01847.

[37] S. Haco, M. J. Perry, and A. Strominger, "Kerr-Newman Black Hole Entropy and Soft Hair," 1902.02247.

[38] A. Strominger, "Black hole entropy from near horizon microstates," *JHEP* **02** (1998) 009, hep-th/9712251.

[39] M. Banados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, hep-th/9204099.

[40] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, "Geometry of the (2+1) black hole," *Phys. Rev.* **D48** (1993) 1506–1525, gr-qc/9302012. [Erratum: Phys. Rev.D88,069902(2013)].

[41] C. Dappiaggi, "BMS field theory and holography in asymptotically flat space-times," *JHEP* **11** (2004) 011, hep-th/0410026.

[42] A. Bagchi, "Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories," *Phys. Rev. Lett.* **105** (2010) 171601, 1006.3354.

[43] G. Barnich, "Entropy of three-dimensional asymptotically flat cosmological solutions," *JHEP* **10** (2012) 095, 1208.4371.

[44] M. Riegler, *How General Is Holography?* PhD thesis, Vienna, Tech. U., 2016. 1609.02733.

[45] R. Fareghbal and A. Naseh, "Aspects of Flat/CCFT Correspondence," *Class. Quant. Grav.* **32** (2015) 135013, 1408.6932.

[46] A. Bagchi, R. Basu, D. Grumiller, and M. Riegler, "Entanglement entropy in Galilean conformal field theories and flat holography," *Phys. Rev. Lett.* **114** (2015), no. 11, 111602, 1410.4089.

[47] R. Basu, S. Detournay, and M. Riegler, "Spectral Flow in 3D Flat Spacetimes," *JHEP* **12** (2017) 134, 1706.07438.

[48] A. Laddha, S. G. Prabhu, S. Raju, and P. Shrivastava, "The Holographic Nature of Null Infinity," 2002.02448.

[49] J. de Boer and S. N. Solodukhin, "A Holographic reduction of Minkowski space-time," *Nucl. Phys.* **B665** (2003) 545–593, hep-th/0303006.

[50] A. Ball, E. Himwich, S. A. Narayanan, S. Pasterski, and A. Strominger, "Uplifting AdS_3/CFT_2 to flat space holography," *JHEP* **08** (2019) 168, 1905.09809.

[51] S. Pasterski, S.-H. Shao, and A. Strominger, "Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere," *Phys. Rev.* **D96** (2017), no. 6, 065026, 1701.00049.

[52] D. Kapec, P. Mitra, A.-M. Raclariu, and A. Strominger, “2D Stress Tensor for 4D Gravity,” *Phys. Rev. Lett.* **119** (2017), no. 12, 121601, 1609.00282.

[53] C. Cheung, A. de la Fuente, and R. Sundrum, “4D scattering amplitudes and asymptotic symmetries from 2D CFT,” *JHEP* **01** (2017) 112, 1609.00732.

[54] R. K. Mishra and R. Sundrum, “Asymptotic Symmetries, Holography and Topological Hair,” *JHEP* **01** (2018) 014, 1706.09080.

[55] L. Donnay, A. Puhm, and A. Strominger, “Conformally Soft Photons and Gravitons,” *JHEP* **01** (2019) 184, 1810.05219.

[56] A. Puhm, “Conformally Soft Theorem in Gravity,” 1905.09799.

[57] L. Donnay, S. Pasterski, and A. Puhm, “Asymptotic Symmetries and Celestial CFT,” 2005.08990.

[58] C. Duval, G. W. Gibbons, and P. A. Horvathy, “Conformal Carroll groups and BMS symmetry,” *Class. Quant. Grav.* **31** (2014) 092001, 1402.5894.

[59] C. Duval, G. W. Gibbons, and P. A. Horvathy, “Conformal Carroll groups,” *J. Phys.* **A47** (2014), no. 33, 335204, 1403.4213.

[60] J. Figueroa-O’Farrill, R. Grassie, and S. Prohazka, “Geometry and BMS Lie algebras of spatially isotropic homogeneous spacetimes,” *JHEP* **08** (2019) 119, 1905.00034.

[61] V. E. Hubeny, S. Minwalla, and M. Rangamani, “The fluid/gravity correspondence,” in *Black holes in higher dimensions*, pp. 348–383. 2012. 1107.5780. [,817(2011)].

[62] M. Haack and A. Yarom, “Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT,” *JHEP* **10** (2008) 063, 0806.4602.

[63] L. Ciambelli, R. G. Leigh, C. Marteau, and P. M. Petropoulos, “Carroll Structures, Null Geometry and Conformal Isometries,” *Phys. Rev.* **D100** (2019), no. 4, 046010, 1905.02221.

[64] A. Campoleoni, L. Ciambelli, C. Marteau, P. M. Petropoulos, and K. Siampos, “Two-dimensional fluids and their holographic duals,” *Nucl. Phys.* **B946** (2019) 114692, 1812.04019.

[65] L. Ciambelli, A. C. Petkou, P. M. Petropoulos, and K. Siampos, “The Robinson-Trautman spacetime and its holographic fluid,” *PoS* **CORFU2016** (2017) 076, 1707.02995.

[66] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli, and K. Siampos, "Vorticity in holographic fluids," *PoS* **CORFU2011** (2011) 076, 1206.4351.

[67] F. E. Low, "Scattering of light of very low frequency by systems of spin 1/2," *Phys. Rev.* **96** (1954) 1428–1432.

[68] M. Gell-Mann and M. L. Goldberger, "Scattering of low-energy photons by particles of spin 1/2," *Phys. Rev.* **96** (1954) 1433–1438.

[69] D. R. Yennie, S. C. Frautschi, and H. Suura, "The infrared divergence phenomena and high-energy processes," *Annals Phys.* **13** (1961) 379–452.

[70] S. Weinberg, "Infrared photons and gravitons," *Phys. Rev.* **140** (1965) B516–B524.

[71] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations.* Cambridge University Press, 2005.

[72] F. E. Low, "Bremsstrahlung of very low-energy quanta in elementary particle collisions," *Phys. Rev.* **110** (1958) 974–977.

[73] T. H. Burnett and N. M. Kroll, "Extension of the low soft photon theorem," *Phys. Rev. Lett.* **20** (1968) 86.

[74] C. D. White, "Factorization Properties of Soft Graviton Amplitudes," *JHEP* **05** (2011) 060, 1103.2981.

[75] D. J. Gross and R. Jackiw, "Low-Energy Theorem for Graviton Scattering," *Phys. Rev.* **166** (1968) 1287–1292.

[76] R. Jackiw, "Low-Energy Theorems for Massless Bosons: Photons and Gravitons," *Phys. Rev.* **168** (1968) 1623–1633.

[77] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," 1404.4091.

[78] P. Di Vecchia, R. Marotta, and M. Mojaza, "Subsubleading soft theorems of gravitons and dilatons in the bosonic string," *JHEP* **06** (2016) 054, 1604.03355.

[79] M. Zlotnikov, "Sub-sub-leading soft-graviton theorem in arbitrary dimension," *JHEP* **10** (2014) 148, 1407.5936.

[80] Y. B. Zel'dovich and A. G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," *sovast* **18** (Aug, 1974) 17.

[81] V. B. Braginsky and L. P. Grishchuk, "Kinematic Resonance and Memory Effect in Free Mass Gravitational Antennas," *Sov. Phys. JETP* **62** (1985) 427–430. [Zh. Eksp. Teor. Fiz.89,744(1985)].

[82] V. B. Braginsky and K. S. Thorne, "Gravitational-wave bursts with memory and experimental prospects," *Nature* **327** (1987) 123–125.

[83] D. Christodoulou, "Nonlinear nature of gravitation and gravitational wave experiments," *Phys. Rev. Lett.* **67** (1991) 1486–1489.

[84] L. Blanchet and T. Damour, "Hereditary effects in gravitational radiation," *Phys. Rev.* **D46** (1992) 4304–4319.

[85] K. S. Thorne, "Gravitational-wave bursts with memory: The Christodoulou effect," *Phys. Rev.* **D45** (1992), no. 2, 520–524.

[86] M. Favata, "The gravitational-wave memory effect," *Class. Quant. Grav.* **27** (2010) 084036, 1003.3486.

[87] A. Tolish, L. Bieri, D. Garfinkle, and R. M. Wald, "Examination of a simple example of gravitational wave memory," *Phys. Rev.* **D90** (2014), no. 4, 044060, 1405.6396.

[88] J. Winicour, "Global aspects of radiation memory," *Class. Quant. Grav.* **31** (2014) 205003, 1407.0259.

[89] M. Pate, A.-M. Raclariu, and A. Strominger, "Color Memory: A Yang-Mills Analog of Gravitational Wave Memory," *Phys. Rev. Lett.* **119** (2017), no. 26, 261602, 1707.08016.

[90] L. Susskind, "Electromagnetic Memory," 1507.02584.

[91] S. Pasterski, "Asymptotic Symmetries and Electromagnetic Memory," *JHEP* **09** (2017) 154, 1505.00716.

[92] S. Pasterski, A. Strominger, and A. Zhiboedov, "New Gravitational Memories," *JHEP* **12** (2016) 053, 1502.06120.

[93] D. A. Nichols, "Center-of-mass angular momentum and memory effect in asymptotically flat spacetimes," *Phys. Rev.* **D98** (2018), no. 6, 064032, 1807.08767.

[94] E. E. Flanagan, A. M. Grant, A. I. Harte, and D. A. Nichols, "Persistent gravitational wave observables: general framework," *Phys. Rev.* **D99** (2019), no. 8, 084044, 1901.00021.

[95] J. Podolsky and R. Steinbauer, "Geodesics in space-times with expanding impulsive gravitational waves," *Phys. Rev.* **D67** (2003) 064013, gr-qc/0210007.

[96] J. Podolsky and R. Svarc, "Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant," *Phys. Rev.* **D81** (2010) 124035, 1005.4613.

[97] L. Donnay, G. Giribet, H. A. González, and A. Puhm, "Black hole memory effect," *Phys. Rev.* **D98** (2018), no. 12, 124016, 1809.07266.

[98] P. Mao and X. Wu, "More on gravitational memory," *JHEP* **05** (2019) 058, 1812.07168.

[99] A. Strominger, "On BMS Invariance of Gravitational Scattering," *JHEP* **07** (2014) 152, 1312.2229.

[100] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS supertranslations and Weinbergś soft graviton theorem," *JHEP* **05** (2015) 151, 1401.7026.

[101] A. Strominger and A. Zhiboedov, "Gravitational Memory, BMS Supertranslations and Soft Theorems," *JHEP* **01** (2016) 086, 1411.5745.

[102] G. Compère and J. Long, "Vacua of the gravitational field," *JHEP* **07** (2016) 137, 1601.04958.

[103] G. Compère and J. Long, "Classical static final state of collapse with supertranslation memory," *Class. Quant. Grav.* **33** (2016), no. 19, 195001, 1602.05197.

[104] G. Compère, "Bulk supertranslation memories: a concept reshaping the vacua and black holes of general relativity," *Int. J. Mod. Phys.* **D25** (2016), no. 12, 1644006, 1606.00377.

[105] F. Scholler, "Distinct Minkowski spaces from Bondi-Metzner-Sachs supertranslations," *Phys. Rev.* **D97** (2018), no. 4, 046009, 1711.02670.

[106] E. Adjei, W. Donnelly, V. Py, and A. J. Speranza, "Cosmic footballs from superrotations," 1910.05435.

[107] V. Lysov, S. Pasterski, and A. Strominger, "Lowś Subleading Soft Theorem as a Symmetry of QED," *Phys. Rev. Lett.* **113** (2014), no. 11, 111601, 1407.3814.

[108] P. Mao and J.-B. Wu, "Note on asymptotic symmetries and soft gluon theorems," *Phys. Rev.* **D96** (2017), no. 6, 065023, 1704.05740.

[109] M. Campiglia and A. Laddha, "Subleading soft photons and large gauge transformations," *JHEP* **11** (2016) 012, 1605.09677.

[110] E. Conde and P. Mao, "Remarks on asymptotic symmetries and the subleading soft photon theorem," *Phys. Rev.* **D95** (2017), no. 2, 021701, 1605.09731.

[111] R. Bousso and M. Porrati, "Soft Hair as a Soft Wig," *Class. Quant. Grav.* **34** (2017), no. 20, 204001, 1706.00436.

[112] M. Mirbabayi and M. Porrati, "Dressed Hard States and Black Hole Soft Hair," *Phys. Rev. Lett.* **117** (2016), no. 21, 211301, 1607.03120.

[113] G. Barnich and F. Brandt, "Covariant theory of asymptotic symmetries, conservation laws and central charges," *Nucl. Phys.* **B633** (2002) 3–82, hep-th/0111246.

[114] G. Barnich, "Boundary charges in gauge theories: Using Stokes theorem in the bulk," *Class. Quant. Grav.* **20** (2003) 3685–3698, hep-th/0301039.

[115] G. Barnich and G. Compere, "Surface charge algebra in gauge theories and thermodynamic integrability," *J. Math. Phys.* **49** (2008) 042901, 0708.2378.

[116] R. M. Wald, "Black hole entropy is the Noether charge," *Phys. Rev.* **D48** (1993), no. 8, R3427–R3431, gr-qc/9307038.

[117] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," *Phys. Rev.* **D50** (1994) 846–864, gr-qc/9403028.

[118] R. M. Wald and A. Zoupas, "A General definition of 'conserved quantities' in general relativity and other theories of gravity," *Phys. Rev.* **D61** (2000) 084027, gr-qc/9911095.

[119] R. L. Arnowitt, S. Deser, and C. W. Misner, "The Dynamics of general relativity," *Gen. Rel. Grav.* **40** (2008) 1997–2027, gr-qc/0405109.

[120] G. Barnich and C. Troessaert, "Comments on holographic current algebras and asymptotically flat four dimensional spacetimes at null infinity," *JHEP* **11** (2013) 003, 1309.0794.

[121] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* **12** (2011) 105, 1106.0213.

[122] J. Podolsky, C. Samann, R. Steinbauer, and R. Svarc, "The global uniqueness and C^1-regularity of geodesics in expanding impulsive gravitational waves," *Class. Quant. Grav.* **33** (2016), no. 19, 195010, 1602.05020.

[123] L. Bieri, D. Garfinkle, and S.-T. Yau, "Gravitational wave memory in de Sitter spacetime," *Phys. Rev.* **D94** (2016), no. 6, 064040, 1509.01296.

[124] Y.-Z. Chu, "Transverse traceless gravitational waves in a spatially flat FLRW universe: Causal structure from dimensional reduction," *Phys. Rev.* **D92** (2015), no. 12, 124038, 1504.06337.

[125] A. Tolish and R. M. Wald, "Cosmological memory effect," *Phys. Rev.* **D94** (2016), no. 4, 044009, 1606.04894.

[126] Y. Hamada, M.-S. Seo, and G. Shiu, "Memory in de Sitter space and Bondi-Metzner-Sachs-like supertranslations," *Phys. Rev.* **D96** (2017), no. 2, 023509, 1702.06928.

[127] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP* **05** (2003) 013, astro-ph/0210603.

[128] C. Cheung, A. L. Fitzpatrick, J. Kaplan, and L. Senatore, "On the consistency relation of the 3-point function in single field inflation," *JCAP* **0802** (2008) 021, 0709.0295.

[129] P. Creminelli and M. Zaldarriaga, "Single field consistency relation for the 3-point function," *JCAP* **0410** (2004) 006, astro-ph/0407059.

[130] B. Horn, L. Hui, and X. Xiao, "Soft-Pion Theorems for Large Scale Structure," *JCAP* **1409** (2014), no. 09, 044, 1406.0842.

[131] R. Penrose, "Asymptotic properties of fields and space-times," *Phys. Rev. Lett.* **10** (1963) 66–68.

[132] R. Penrose, "Conformal treatment of infinity," *Gen. Rel. Grav.* **43** (2011) 901–922. [,565(1964)].

[133] R. O. Hansen, E. T. Newman, R. Penrose, and K. P. Tod, "The Metric and Curvature Properties of H Space," *Proc. Roy. Soc. Lond.* **A363** (1978) 445–468.

[134] A. Ashtekar and M. Streubel, "Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity," *Proc. Roy. Soc. Lond.* **A376** (1981) 585–607.

[135] T. Dray and M. Streubel, "Angular momentum at null infinity," *Class. Quant. Grav.* **1** (1984), no. 1, 15–26.

[136] Y. Herfray, "Asymptotic Shear and the Intrinsic Geometry of Null-Infinity," 2001.01281.

[137] A. Ashtekar and R. O. Hansen, "A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity," *J. Math. Phys.* **19** (1978) 1542–1566.

[138] A. Ashtekar and J. D. Romano, "Spatial infinity as a boundary of space-time," *Class. Quant. Grav.* **9** (1992) 1069–1100.

[139] A. Ashtekar, B. Bonga, and A. Kesavan, "Asymptotics with a positive cosmological constant. II. Linear fields on de Sitter spacetime," *Phys. Rev.* **D92** (2015), no. 4, 044011, 1506.06152.

[140] A. Ashtekar, B. Bonga, and A. Kesavan, "Asymptotics with a positive cosmological constant: III. The quadrupole formula," *Phys. Rev.* **D92** (2015), no. 10, 104032, 1510.05593.

[141] A. Ashtekar and S. Bahrami, "Asymptotics with a positive cosmological constant. IV. The no-incoming radiation condition," *Phys. Rev.* **D100** (2019), no. 2, 024042, 1904.02822.

[142] V. Chandrasekaran, E. E. Flanagan, and K. Prabhu, "Symmetries and charges of general relativity at null boundaries," *JHEP* **11** (2018) 125, 1807.11499.

[143] J. D. Jackson, *Classical Electrodynamics*. Wiley, 1998.

[144] G. Compère and D. Marolf, "Setting the boundary free in AdS/CFT," *Class. Quant. Grav.* **25** (2008) 195014, 0805.1902.

[145] A. Strominger and A. Zhiboedov, "Superrotations and Black Hole Pair Creation," *Class. Quant. Grav.* **34** (2017), no. 6, 064002, 1610.00639.

[146] G. Barnich, "Centrally extended BMS4 Lie algebroid," *JHEP* **06** (2017) 007, 1703.08704.

[147] T. Regge and C. Teitelboim, "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity," *Annals Phys.* **88** (1974) 286.

[148] M. Henneaux and C. Troessaert, "BMS Group at Spatial Infinity: the Hamiltonian (ADM) approach," *JHEP* **03** (2018) 147, 1801.03718.

[149] M. Henneaux and C. Troessaert, "Asymptotic symmetries of electromagnetism at spatial infinity," *JHEP* **05** (2018) 137, 1803.10194.

[150] M. Henneaux and C. Troessaert, "The asymptotic structure of gravity at spatial infinity in four spacetime dimensions," 1904.04495.

[151] L. Blanchet, "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries," *Living Rev. Rel.* **17** (2014) 2, 1310.1528.

[152] A. A. Starobinsky, "Isotropization of arbitrary cosmological expansion given an effective cosmological constant," *JETP Lett.* **37** (1983) 66–69.

[153] C. Fefferman and C. R. Graham, "Conformal invariants," in *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, no. S131 in Astérisque, pp. 95–116. Société mathématique de France, 1985.

[154] K. Skenderis, "Lecture notes on holographic renormalization," *Class. Quant. Grav.* **19** (2002) 5849–5876, hep-th/0209067.

[155] C. Fefferman and C. R. Graham, "The ambient metric," *arXiv e-prints* (Oct, 2007) arXiv:0710.0919, 0710.0919.

[156] I. Papadimitriou, "Holographic renormalization as a canonical transformation," *JHEP* **11** (2010) 014, 1007.4592.

[157] E. T. Newman and T. W. J. Unti, "Behavior of Asymptotically Flat Empty Spaces," *J. Math. Phys.* **3** (1962), no. 5, 891.

[158] G. Barnich and P.-H. Lambert, "A Note on the Newman-Unti group and the BMS charge algebra in terms of Newman-Penrose coefficients," *Adv. Math. Phys.* **2012** (2012) 197385, 1102.0589. [J. Phys. Conf. Ser.410,012142(2013)].

[159] L. Donnay, G. Giribet, H. A. Gonzalez, and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," *Phys. Rev. Lett.* **116** (2016), no. 9, 091101, 1511.08687.

[160] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Extended Symmetries at the Black Hole Horizon," *JHEP* **09** (2016) 100, 1607.05703.

[161] L. Donnay and C. Marteau, "Carrollian Physics at the Black Hole Horizon," *Class. Quant. Grav.* **36** (2019), no. 16, 165002, 1903.09654.

[162] L. Donnay and G. Giribet, "Cosmological horizons, Noether charges and entropy," *Class. Quant. Grav.* **36** (2019), no. 16, 165005, 1903.09271.

[163] D. Grumiller, A. Pérez, M. M. Sheikh-Jabbari, R. Troncoso, and C. Zwikel, "Spacetime structure near generic horizons and soft hair," 1908.09833.

[164] E. E. Flanagan, K. Prabhu, and I. Shehzad, "Extensions of the asymptotic symmetry algebra of general relativity," 1910.04557.

[165] D. Marolf, W. Kelly, and S. Fischetti, "Conserved Charges in Asymptotically (Locally) AdS Spacetimes," in *Springer Handbook of Spacetime*, A. Ashtekar and V. Petkov, eds., pp. 381–407. 2014. 1211.6347.

[166] G. Compère, A. Fiorucci, and R. Ruzziconi, "The Λ-BMS$_4$ group of dS$_4$ and new boundary conditions for AdS$_4$," 1905.00971.

[167] A. Poole, K. Skenderis, and M. Taylor, "(A)dS$_\mathbf{4}$ in Bondi gauge," *Class. Quant. Grav.* **36** (2019), no. 9, 095005, 1812.05369.

[168] P. Mao, "Asymptotics with a cosmological constant: The solution space," *Phys. Rev.* **D99** (2019), no. 10, 104024, 1901.04010.

[169] E. E. Flanagan and D. A. Nichols, "Conserved charges of the extended Bondi-Metzner-Sachs algebra," *Phys. Rev.* **D95** (2017), no. 4, 044002, 1510.03386.

[170] G. Barnich, "A Note on gauge systems from the point of view of Lie algebroids," *AIP Conf. Proc.* **1307** (2010), no. 1, 7–18, 1010.0899.

[171] G. Barnich and F. Del Monte, "Introduction to Classical Gauge Field Theory and to Batalin-Vilkovisky Quantization," 1810.00442.

[172] M. Crainic and R. L. Fernandes, "Lectures on Integrability of Lie Brackets," 0611259.

[173] G. Barnich, A. Gomberoff, and H. A. Gonzalez, "Three-dimensional Bondi-Metzner-Sachs invariant two-dimensional field theories as the flat limit of Liouville theory," *Phys. Rev.* **D87** (2013), no. 12, 124032, 1210.0731.

[174] G. Barnich and B. Oblak, "Holographic positive energy theorems in three-dimensional gravity," *Class. Quant. Grav.* **31** (2014) 152001, 1403.3835.

[175] G. Barnich and B. Oblak, "Notes on the BMS group in three dimensions: II. Coadjoint representation," *JHEP* **03** (2015) 033, 1502.00010.

[176] G. Barnich, H. A. Gonzalez, and P. Salgado-Rebolledo, "Geometric actions for three-dimensional gravity," *Class. Quant. Grav.* **35** (2018), no. 1, 014003, 1707.08887.

[177] B. Oblak, "Berry Phases on Virasoro Orbits," *JHEP* **10** (2017) 114, 1703.06142.

[178] G. Barnich and C. Troessaert, "Finite BMS transformations," *JHEP* **03** (2016) 167, 1601.04090.

[179] G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in the antifield formalism. 1. General theorems," *Commun. Math. Phys.* **174** (1995) 57–92, hep-th/9405109.

[180] G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in Einstein Yang-Mills theory," *Nucl. Phys.* **B455** (1995) 357–408, hep-th/9505173.

[181] G. Barnich, P. Mao, and R. Ruzziconi, "Conserved currents in the Cartan formulation of general relativity," in *About Various Kinds of Interactions: Workshop in honour of Professor Philippe Spindel Mons, Belgium, June 4-5, 2015*. 2016. 1611.01777.

[182] G. Barnich, P. Mao, and R. Ruzziconi, "BMS current algebra in the context of the Newman-Penrose formalism," 1910.14588.

[183] E. Frodden and D. Hidalgo, "Surface Charges Toolkit for Gravity," 1911.07264.

[184] R. Oliveri and S. Speziale, "Boundary effects in General Relativity with tetrad variables," 1912.01016.

[185] G. Barnich, P. Mao, and R. Ruzziconi, "Conserved currents in the Palatini formulation of general relativity," 2004.15002.

[186] G. Compère, "Advanced Lectures on General Relativity," *Lect. Notes Phys.* **952** (2019) 150.

[187] H. Godazgar, M. Godazgar, and C. N. Pope, "Subleading BMS charges and fake news near null infinity," *JHEP* **01** (2019) 143, 1809.09076.

[188] E. T. Newman and R. Penrose, "New conservation laws for zero rest-mass fields in asymptotically flat space-time," *Proc. Roy. Soc. Lond.* **A305** (1968) 175–204.

[189] G. Compère, R. Oliveri, and A. Seraj, "Gravitational multipole moments from Noether charges," *JHEP* **05** (2018) 054, 1711.08806.

[190] M. Campiglia and A. Laddha, "Sub-subleading soft gravitons: New symmetries of quantum gravity?," *Phys. Lett. B* **764** (2017) 218–221, 1605.09094.

[191] G. Compère, "Infinite towers of supertranslation and superrotation memories," *Phys. Rev. Lett.* **123** (2019), no. 2, 021101, 1904.00280.

[192] C. Troessaert, "Hamiltonian surface charges using external sources," *J. Math. Phys.* **57** (2016), no. 5, 053507, 1509.09094.

[193] G. Compère, A. Fiorucci, and R. Ruzziconi, "The Λ-BMS$_4$ Charge Algebra," 2004.10769.

[194] D. Harlow and J.-Q. Wu, "Covariant phase space with boundaries," 1906.08616.

[195] T. Azeyanagi, G. Compere, N. Ogawa, Y. Tachikawa, and S. Terashima, "Higher-Derivative Corrections to the Asymptotic Virasoro Symmetry of 4d Extremal Black Holes," *Prog. Theor. Phys.* **122** (2009) 355–384, 0903.4176.

[196] M. Henneaux and C. Teitelboim, *Quantization of gauge systems.* 1992.

[197] E. Newman and R. Penrose, "An Approach to gravitational radiation by a method of spin coefficients," *J. Math. Phys.* **3** (1962) 566–578.

[198] R. P. Geroch, A. Held, and R. Penrose, "A space-time calculus based on pairs of null directions," *J. Math. Phys.* **14** (1973) 874–881.

[199] R. M. Wald, *General Relativity.* Chicago Univ. Pr., Chicago, USA, 1984.

[200] R. Penrose and W. Rindler, *Spinors and Space-Time.* Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 2011.

[201] A. R. Exton, E. T. Newman, and R. Penrose, "Conserved quantities in the Einstein-Maxwell theory," *J. Math. Phys.* **10** (1969) 1566–1570.

[202] I. Robinson and A. Trautman, "Spherical Gravitational Waves," *Phys. Rev. Lett.* **4** (1960) 431–432.

[203] I. Robinson and A. Trautman, "Some spherical gravitational waves in general relativity," *Proc. Roy. Soc. Lond.* **A265** (1962) 463–473.

[204] M. Campiglia and J. Peraza, "Generalized BMS charge algebra," 2002.06691.

[205] J. Distler, R. Flauger, and B. Horn, "Double-soft graviton amplitudes and the extended BMS charge algebra," 1808.09965.

[206] R. Penrose, "The geometry of impulsive gravitational waves," in *General relativity: Papers in honour of J.L. Synge*, L. O'Raifeartaigh, ed., pp. 101–115. 1972.

[207] Y. Nutku and R. Penrose, "On Impulsive Gravitational Waves," *Twistor Newsletter* **34** (1992) 9.

[208] J. Podolsky and J. B. Griffiths, "Expanding impulsive gravitational waves," *Class. Quant. Grav.* **16** (1999) 2937–2946, gr-qc/9907022.

[209] J. B. Griffiths, J. Podolsky, and P. Docherty, "An Interpretation of Robinson-Trautman type N solutions," *Class. Quant. Grav.* **19** (2002) 4649–4662, gr-qc/0208022.

[210] J. B. Griffiths and P. Docherty, "A Disintegrating cosmic string," *Class. Quant. Grav.* **19** (2002) L109–L112, gr-qc/0204085.

[211] B. Zeldovich and A. G. Polnarev., "Radiation of gravitational waves by a cluster of superdense stars," *Ya. Sov.Astron.Lett.* **18** (1974) 17.

[212] L. Blanchet and T. Damour, "Tail Transported Temporal Correlations in the Dynamics of a Gravitating System," *Phys. Rev.* **D37** (1988) 1410.

[213] J. Frauendiener, "Note on the memory effect," *Classical and Quantum Gravity* **9** (1992), no. 6, 1639.

[214] S. Pasterski, "Implications of Superrotations," *Phys. Rept.* **829** (2019) 1–35, 1905.10052.

[215] G. Compère, R. Oliveri, and A. Seraj, "The Poincaré and BMS flux-balance laws with application to binary systems," 1912.03164.

[216] L. Ciambelli, C. Marteau, M. Petropoulos, and R. Ruzziconi, "Gauges in three-dimensional gravity and holographic fluids," *To be published.*

[217] L. Ciambelli, C. Marteau, M. Petropoulos, and R. Ruzziconi, "Fefferman-Graham and Bondi gauges in the fluid/gravity correspondence," *To be published*.

[218] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," *Commun. Math. Phys.* **208** (1999) 413–428, hep-th/9902121.

[219] K. Skenderis, "Asymptotically Anti-de Sitter space-times and their stress energy tensor," *Int. J. Mod. Phys.* **A16** (2001) 740–749, hep-th/0010138. [,394(2000)].

[220] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly," *JHEP* **07** (1998) 023, hep-th/9806087.

[221] G. Barnich, A. Gomberoff, and H. A. Gonzalez, "The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes," *Phys. Rev.* **D86** (2012) 024020, 1204.3288.

[222] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," *Commun. Math. Phys.* **217** (2001) 595–622, hep-th/0002230.

[223] J. Winicour, "Logarithmic asymptotic flatness," *Foundations of Physics* **15** (May, 1985) 605–616.

[224] P. T. Chrusciel, M. A. H. MacCallum, and D. B. Singleton, "Gravitational waves in general relativity: 14. Bondi expansions and the polyhomogeneity of Scri," gr-qc/9305021.

[225] J. A. Valiente Kroon, "Logarithmic Newman-Penrose constants for arbitrary polyhomogeneous space-times," *Class. Quant. Grav.* **16** (1999) 1653–1665, gr-qc/9812004.

[226] X. He and Z. Cao, "New Bondi-type outgoing boundary condition for the Einstein equations with cosmological constant," *Int. J. Mod. Phys.* **D24** (2015), no. 10, 1550081.

[227] V.-L. Saw, "Bondi mass with a cosmological constant," *Phys. Rev.* **D97** (2018), no. 8, 084017, 1711.01808.

[228] X. He, J. Jing, and Z. Cao, "Relationship between Bondi-Sachs quantities and source of gravitational radiation in asymptotically de Sitter spacetime," *Int. J. Mod. Phys.* **D27** (2017), no. 04, 1850046, 1803.05564.

[229] S. L. Lyakhovich and A. A. Sharapov, "Characteristic classes of gauge systems," *Nucl. Phys.* **B703** (2004) 419–453, hep-th/0407113.

[230] H. R. Safari and M. M. Sheikh-Jabbari, "BMS_4 algebra, its stability and deformations," *JHEP* **04** (2019) 068, 1902.03260.

[231] H. Adami, D. Grumiller, S. Sadeghian, M. M. Sheikh-Jabbari, and C. Zwikel, "T-Witts from the horizon," 2002.08346.

[232] A. Ishibashi and R. M. Wald, "Dynamics in nonglobally hyperbolic static space-times. 3. Anti-de Sitter space-time," *Class. Quant. Grav.* **21** (2004) 2981–3014, hep-th/0402184.

[233] C. Jana, R. Loganayagam, and M. Rangamani, "Open quantum systems and Schwinger-Keldysh holograms," 2004.02888.

[234] A. M. Awad and C. V. Johnson, "Holographic stress tensors for Kerr - AdS black holes," *Phys. Rev.* **D61** (2000) 084025, hep-th/9910040.

[235] S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, "Large rotating AdS black holes from fluid mechanics," *JHEP* **09** (2008) 054, 0708.1770.

[236] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi, and S. R. Wadia, "Forced Fluid Dynamics from Gravity," *JHEP* **02** (2009) 018, 0806.0006.

[237] D. Grumiller, W. Merbis, and M. Riegler, "Most general flat space boundary conditions in three-dimensional Einstein gravity," *Class. Quant. Grav.* **34** (2017), no. 18, 184001, 1704.07419.

[238] L. Ciambelli and R. G. Leigh, "Weyl Connections and their Role in Holography," 1905.04339.

[239] G. Compere, *Symmetries and conservation laws in Lagrangian gauge theories with applications to the mechanics of black holes and to gravity in three dimensions*. PhD thesis, Brussels U., 2007. 0708.3153.