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STOCHASTIC EVOLUTION OF RATIONALITY

ABSTRACT. Following up on previous results by Falmagne, this paper investigates possible mechanisms explaining how preference relations are created and how they evolve over time. We postulate a preference relation which is initially empty and becomes increasingly intricate under the influence of a random environment delivering discrete tokens of information concerning the alternatives. The framework is that of a class of real-time stochastic processes having interlinked Markov and Poisson components. Specifically, the occurence of the tokens is governed by a Poisson process, while the succession of preference relations is a Markov process. In an example case, the preference relations are the various possible semiorders on the set of alternatives. Asymptotic results are obtained in the form of the limit probabilities of any semiorder. The arguments extend to a much more general situation including interval orders, biorders and partial orders. The results provide (up to a small number of parameters) complete quantitative predictions for panel data of a standard type, in which the same sample of subjects has been asked to compare the alternatives a number of times.

KEY WORDS: Biorder, semiorder, Poisson process, Markov chain, random walk

INTRODUCTION

Presumably, a subject or consumer first confronted with a choice situation, be it the selection of a car or that of a political candidate to an elective office, initially knows little about the available alternatives. It is tempting to suppose that the mental structure underlying the manifest choices of a subject evolve over time, from some initially naive state where the subject is indifferent to all the alternatives into a sophisticated state where this mental structure can be represented by an intricate relation such as a linear order or a semiorder (in the sense of Luce, 1956). In a recent paper (Falmagne, 1993) hereafter referred to as the *source paper*, we proposed a stochastic theory describing such an evolution, focusing on the case of linear orders. The purpose of the present paper is to apply the same ideas to the case of other types of preference relations, such as semiorders, interval orders or partial orders. (The present paper is self contained, however.) The basic concepts at the core of this work are recalled informally below.

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We assume a naive state that can be represented by the empty relation. The transformations taking place over time result from the probabilistic occurence of quantum items of information, called 'tokens', which are delivered to the subject by the medium at random times. Tokens can arise as a result of a conversation with a neighbor, or features of a TV program presenting a debate on the comparative merits of the alternatives, or the viewing of a poster (to name but a few possibilities). Note that the occurences of these token events are not necessarily meant to be observable, or controllable by the experimenter: there are too many possible tokens, and their effect on an individual is not easily assessed. A telling analogy with statistical mechanics was given in the source paper. The status of the tokens resembles that of the particles whose combined effect is postulated, for instance, in the derivation of the Boltzmann distribution or the Bose–Einstein distribution. The existence of the particles can be ascertained, but these chance manifestations play no role in the computation of the results. In the source paper, these tokens were formalized as ordered pairs (x,y) of alternatives. In the most important case, the occurence of a token (x,y) (with $x \neq y$) signals that '*x is preferable to y*'. The occurence of a token may result in a modification of some edge of the graph of the preference relation. General axioms were given, casting the theory as a Markov process having as a state space the collection of all preference relations. In other words, the state of the Markov process is the current preference relation of the subject. The occurence of the tokens is governed by a Poisson process. A special case of the axioms was analyzed in the source paper which entails the existence of a unique ergodic set composed of all the linear orders on the set of alternatives. Asymptotic results were formulated and proved. In particular, an explicit formula for the asymptotic probabilities of all the linear orders was obtained. In other words, the axioms guarantee the asymptotic existence of a random utility model in the sense of Block and Marschak (1960). This theory provides explicit predictions–up to a small number of parameters–of data consisting of successive rankings of the alternatives in a set by the same subjects at any arbitrarily chosen times $t_1, \cdots, t_n$. Other types of sequential data (such as binary choices or approval voting) could also be predicted and were discussed in the source paper.

The present paper is devoted to a different class of preference relations, which includes as special cases the (strict) semiorders, the interval orders, the biorders and the partial orders. We take the semiorders as our leading example. The semiorders are interesting for two reasons. For one, the concept of an 'order with a threshold' has been a concern of economists and other behavioral scientists for a long time. In experimental psychology, the name of Fechner (1860 / 1966) comes to mind, with his many followers. But the references in the economics literature are also numerous (for surveys, see Fishburn, 1970, 1985; Roberts, 1970; and Suppes, Krantz, Luce and Tversky, 1989). The second reason is both conceptual and technical. A key concept of our approach is that the transformation of an initially naive state into an articulate relation results from the accumulated effects of token events over time. Two relevant technical features of semiorders are that the empty relation itself is a semiorder, and more importantly, that any semiorder can be transformed into another semiorder (on the same set) simply by either adding or removing a single pair (see Doignon and Falmagne, 1997 and Theorem 3 below), an operation which, in our theory, is induced by the occurence of a single token. It turns out that these two features of the semiorders hold for various types of relations, which also include the partial orders, the interval orders and the biorders (but not, obviously, the linear orders). This means that the results given here will be applicable to many other cases of relations. As in the source paper, the axioms given here will lead to specific predictions regarding the asymptotic probabilities of all the relations in any subclass deemed suitable for a particular empirical setup. This result is applicable to panel data of a standard type, in which the same sample of subjects have been asked to compare the alternatives a number of times. A discussion on the applicability of this theory to such data will be given, including detailed statistical considerations.

Two notable differences between this theory and that of the source paper must be pointed out. One concerns the tokens. In the theory described here, the feasible tokens are of two types: positive or negative. Accordingly these tokens are, represented by 'signed' ordered pairs of distinct elements. For convenience of notation, we always write xy for the (ordered) pair (x, y). A positive token is symbolized by a pair xy signalling that '*alternative x is preferable to alternative* *y*'. A negative token is denoted as $\widetilde{xy}$, which conveys the message that '*x is not preferable to y* '. In both cases, we suppose that $x \neq y$. The theory will specify how the occurence of a positive token may result in the possible addition of the corresponding pair to the preference relation, and how a negative token may determine the removal of such a pair. In the source paper, no distinction was made between a positive token xy and a negative token $\widetilde{yx}$. The introduction of the negative tokens is a generalization which, on the one hand, renders the theory more symmetrical, making some of the results easier to grasp, and on the other hand, opens the possibility of capturing some revealing aspects of the distribution of the tokens. For instance, a high density of the negative tokens could reflect a case of 'negative campaigning' or 'negative advertising.' According to the theory, this would result in a high proportion of the subjects being in the empty state or near it, i.e. uninterested or uncommitted.

The second difference from the source paper was implicit in the outline of the theory given above. We suppose here that the subject is, in a technical sense, rational from the start and remains rational through all the transformations induced by the occurences of the tokens. In the case of the semiorders, the initially empty state is a strict semiorder because all the defining conditions are vacuously satisfied. The occurences of the tokens will transform this empty semiorder into other semiorders, and may gradually become quite elaborate. At no time, however, will the state of the subject leave the set of all semiorders. A similar scheme would apply in all the other cases of the theory. By contrast, it was assumed in the source paper that the ergodic set (i.e. the set of all linear orders) was reachable only after some meandering in the set of all relations on the set of alternatives.

It is not the aim of this theory, in its present status, to provide a full theoretical account of all the phenomena that can arise when subjects repeatedly express preferences in real time. This paper deals with a special case in which the subjects are sampled from a relatively homogeneous population, and develop preferences over time according to the same stochastic mechanisms. We believe, however, that this special case is the potential cornerstone of a more comprehensive theory that would be able to account for all or most of the effects described in the literature (see e.g. Converse, 1964, 1970;

Feldman, 1989). A number of possible extensions of this theory are discussed in the paper, which should evoke how such a more general theory could be constructed.

Our second section recalls standard concepts on preference relations, and summarizes some results recently obtained by Doignon and Falmagne (1997) in preparation for the theory expounded here. The stochastic aspects of the theory are developed next, and a number of theoretical results are stated and proved. In passing, we propose a variety of extensions of the so-called Feigin and Cohen distribution (Feigin and Cohen, 1978). The applicability of the theory is then examined from a statistical viewpoint. The paper ends with a general discussion of this approach.

WELL GRADED FAMILIES OF PREFERENCE RELATIONS

DEFINITION 1. Let X and Y be two basic finite sets, with Y not necessarily distinct or disjoint from X. As indicated, we always write xy to denote a pair (x, y). A pair xy such that $x \neq y$ is called *disparate*. For any relation R from X to Y, that is $R \subseteq X \times Y$, we denote by $\bar{R} = (X \times Y) \setminus R$ the *complement* of R (w.r.t. $X \times Y$). More generally, the *complement* of a subset R of a basic set E is $\bar{R} = E \setminus R$. As is customary, we write $R^{-1} = \{xy | yRx\}$ for the converse of a relation R. The (*relative) product* of relations $R_1, R_2, \ldots, R_k$ is denoted as $R_1 R_2 \cdots R_k$. (Thus, $RS = \{xy | \exists z, xRz \wedge zSy\}$, etc.) We will always designate by I the *identity relation* on the set $X \cup Y$, that is $I = \{xx | x \in X \cup Y\}$.

Consider the following three axioms for a relation R from X to Y. For all $x, z \in X$ and $y, w \in Y$,

(B)	if xRy	and zRw,	then xRw or zRy;	(i.e. $R\bar{R}^{-1}R \subseteq R$)
(S)	if xRy	and yRz,	then wRz or xRw;	(i.e. $RR\bar{R}^{-1} \subseteq R$)
(I)	$\neg(xRx)$.			(i.e. $R \cap I = \emptyset$)

A compact formulation of each axiom in relative product notation is given in parentheses. With regard to Axiom (S), notice for further reference that $RR\bar{R}^{-1} \subseteq R$ is equivalent to $\bar{R}^{-1}RR \subseteq R$. Suppose first that $X = Y$. Condition (I) together with either Condition (B) or Condition (S) imply that R is transitive. The relation R is *an interval* *order* on X iff it satisfies Axioms (I) and (B). It is a *semiorder* iff it satisfies Axioms (I), (B) and (S). Both interval orders and semiorders are strict partial orders (i.e. they are irreflexive and transitive). The following generalization of interval orders is also of interest. A *biorder between* X and Y is any relation $R \subseteq X \times Y$ satisfying Condition (B), this time for X and Y possibly distinct or disjoint. Thus, an interval order is nothing but an irreflexive biorder between a set and itself.

We begin by focusing on the semiorders. The concept of a semiorder is often attributed to Luce (1956) and occupies a prominent place in ordinal measurement theory. The interest of semiorders lies in part in their numerical representation, which formalizes the idea of a 'preference with a threshold'. For example, if R is a semiorder on a finite set X, we can assert the existence of a function f on X satisfying, for all $x, y \in X$

$$xRy \Leftrightarrow f(x) > f(y) + 1$$

(Suppes and Zinnes, 1963; Suppes et al., 1989.) The relation $\bar{R} \cap \bar{R}^{-1}$ is called the *indifference* relation of R. Clearly, the indifference relation of a semiorder is reflexive and symmetric, but not necessarily transitive.

We say that x *covers* y (*for the semiorder R*) if $x(R \setminus RR)y$ (that is, if xRy and there is no z such that xRz and zRy). The set of covering pairs forms the *Hasse diagram* of R. Notice in passing that the empty relation on X is vacuously a semiorder, with the indifference relation $X \times X$.

A remarkable property of semiorders is the following: if R is a semiorder on X, we can always create another semiorder on the same set by adding or removing some pair. This is illustrated in Figure 1. At the center of the figure, we have a representation of the semiorder $R = \{yx, xw, yw, yz\}$ on the set $X = \{x, y, z, w\}$. The semiorder R at the center of Figure 1 is represented by its Hasse diagram and the corresponding indifference relation by the dotted lines. (The loops are omitted.) The same conventions apply to the representation of the two other semiorders in Figure 1, and will be used throughout.

Removing a pair from a semiorder does not always generate a semiorder, however. Three forbidden cases are pictured in Figure 2. (The irrelevant edges are omitted in this figure. We suppose that the

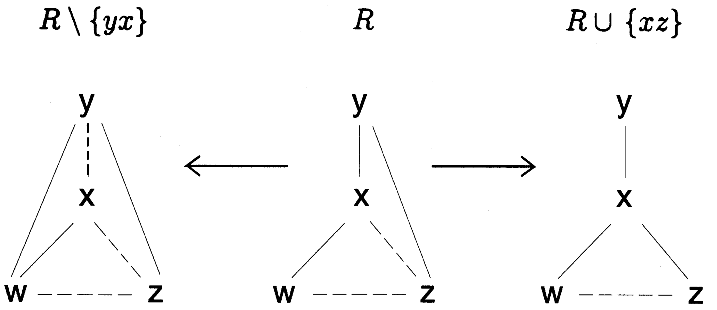

Figure 1. Two examples of semiorders obtained from the semiorder R at the center of the figure. The semiorder on the left is obtained by removing the pair yx. The semiorder on the right results from adding the pair xz. For each of the three semiorders, the solid lines represent the Hasse diagram and the dotted lines represent the corresponding indifference relation, omitting the loops.

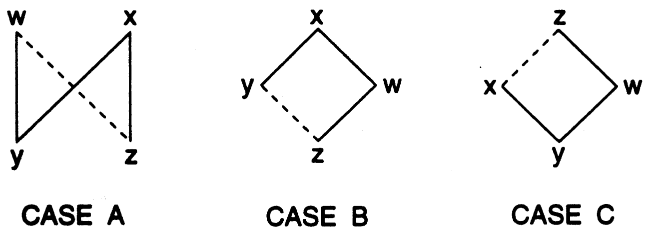

Figure 2. The three forbidden situations in which removing the pair xy from a semiorder would not yield a semiorder. Note that the representation is not complete: only the relevant edges of the Hasse diagram are indicated in each case.

three represented graphs are parts of the Hasse diagrams of three unspecified semiorders. Note that some vertices may coincide.) For example, removing xy from the semiorder of Case A to manufacture the relation $R' = R\setminus\{xy\}$ would yield the situation

$$wR'y,\ \ xR'z,\ \ \neg(wR'z),\ \ \neg(xR'y),$$

a contradiction of Condition (B) in the definition of a semiorder in Definition 1.

It is easy to see that removing the pair xy in Case B or in Case C would yield contradictions of Condition (S). By gathering those pairs xy whose removal from a semiorder R yield another semiorder, we define a new relation $R^{\mathcal{I}} \subseteq R$; formally

$$R^{\mathcal{I}} = R\setminus(R\bar{R}^{-1}R \cup RR\bar{R}^{-1} \cup \bar{R}^{-1}RR). \tag{1}$$

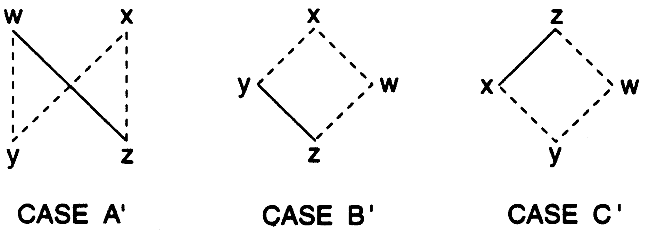

Figure 3. The three forbidden situations in which adding the pair xy to a semiorder would not yield a semiorder. The conventions are as in Figure 2.

The interpretation of the three sets removed by the union in the parenthesis of (1) should be clear from Figure 2 and the Axioms defining a semiorder. For example, the product $R\bar{R}^{-1}R$ corresponds to Axiom (B) and Case A of Figure 1; we cannot remove from R a pair xy belonging to $R\bar{R}^{-1}R$ since this would yield a relation $R' = R\backslash\{xy\}$ that would **not** satisfy $R'\bar{R}'^{-1}R' \subseteq R'$. The two other products removed from R correspond to Axiom (S) and Cases B and C of Figure 2. (Remember that $RR\bar{R}^{-1} \subseteq R$ and $\bar{R}^{-1}RR \subseteq R$ are two equivalent versions of Axiom (S).)

The situation concerning a possible addition of a disparate pair xy to a semiorder is symmetrical. Such an addition generates a semiorder except in three cases, which are illustrated by Figure 3. For example, adding xy to the semiorder in Case A' to manufacture the semiorder R' would yield a violation of Condition (B) (see Figure 3), namely

$$xR'y, \;\; wR'z, \;\; \neg(wR'y), \;\; \neg(xR'z).$$

Similarly adding xy in either Case B' or Case C' would create violations of Condition (S). Notice the symmetry between the three cases A, B and C of Figure 2 and the corresponding cases of Figure 3: any dotted line in Figure 2 is transformed into a solid line in Figure 3 and vice versa. All the cases in which the addition of a pair xy to a semiorder R would yield another semiorder are captured by the relation $R^{\mathcal{O}}$ defined below:

$$(2) \qquad R^{\mathcal{O}} = \bar{R}\backslash(I \cup \bar{R}R^{-1}\bar{R} \cup R^{-1}\bar{R}\bar{R} \cup \bar{R}\bar{R}R^{-1}).$$

Examining the union in right member of Equation (2) should lead the reader to parsing the exact relationships between each of the four

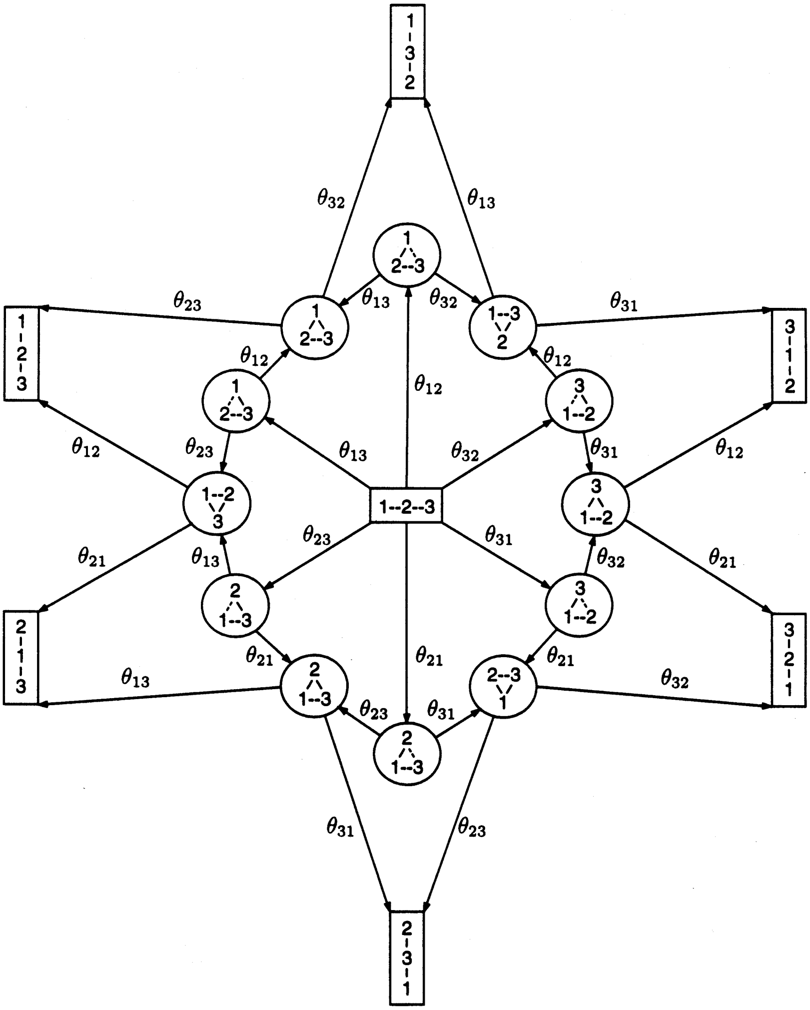

Figure 4. The 3-semiohedron for the set $\{1,2,3\}$. The directed edges and the θ_{ij} s refer to a random walk to be described later in this paper.

sets removed, Axioms (I), (B) and (S), and the three Case A', B' and C' of Figure 2.

To sum up, the situation for the semiorders is thus as follows. For any semiorder R, we can always manufacture another semiorder on the same set either by removing a pair from $R^{\mathcal{I}}$, or by adding to R a pair from $R^{\mathcal{O}}$. Notice that $R^{\mathcal{I}}$ may be empty (if $R = \emptyset$). Similarly $R^{\mathcal{O}}$ may be empty (when R is a linear order–regarded as a strict semiorder). But $R^{\mathcal{I}} \cup R^{\mathcal{O}}$ is never empty. Moreover, it can be shown (and will be stated formally later in this paper, cf. Theorem 3) that any two semiorders R and S can be linked by a sequence of semiorders $R = R_0, R_1, \ldots, R_k =$ S such that for $1 \leq i \leq k$, we have either $R_i = R_{i-1} \cup \{xy\}$ or $R_i = R_{i-1} \backslash \{xy\}$ for some pair xy. The number k is exactly the number of pairs by which R and S differ, namely $k = |(R \backslash S) \cup (S \backslash R)|$. Let $\mathcal{S}$ be the family of all the semiorders on a particular finite set X. The graph having the elements of $\mathcal{S}$ as vertices, and the pairs (R, S) such that either $S = R \cup \{xy\}$ or $S = R \backslash \{xy\}$ as edges, will be called a *semiohedron* or more precisely an m-semiohedron if X contains m elements. A picture of a 3-semiohedron for the set X = {1, 2, 3} is given in Figure 4. Note that there are 19 semiorders (in fact, partial orders) on a set of three elements.

The term 'semiohedron' extends in a natural way a terminology used for linear orders. In combinatorics, the graph of all permutations (or linear orders) on a finite set is sometimes called a *permutohedron*. The edges of this graph are the pairs of permutations (π, π') such that there is a transposition of adjacent values transforming π into π' (see e.g. Guilbaud and Rosenstiehl, 1963; Berge, 1968; Feldman Högaasen, 1969; Le Conte de Poly-Barbut, 1990; references can be found in Björner, 1984.)

To avoid repetitions, we postpone a formal statement of these general results for semiorders, since the situation that we just described actually holds for many types of relations which includes not only the semiorders but also the partial orders, the biorders and the interval orders. In the rest of this section, which sets the stage for the stochastic theory at the focus of this paper, we summarize some recent results of Doignon and Falmagne (1997).

DEFINITION 2. A standard distance d on the family of all subsets of a set E is obtained by defining $d(R, S) = |(R \backslash S) \cup (S \backslash R)|$ for any two subsets R and S. A collection $\mathcal{F}$ of subsets of E is *well graded* when, for any R and S in $\mathcal{F}$ at distance k, there always exist sets $R = F_0, F_1, \ldots, F_k = S$ in $\mathcal{F}$ such that $d(F_{i-1}, F_i)$ =1, for $i = 1, \ldots, k$. This definition applies here to specific families of relations regarded as sets of pairs. Well graded families of sets have also been investigated in the context of knowledge spaces, which are combinatoric structures playing a role in the design of efficient

TABLE I
Inner and outer fringes in the various families of relations.

Partial Orders	
(Inner Rim)	$R^{\mathcal{I}} = R \backslash RR$
(Outer Rim)	$R^{\mathcal{O}} = \bar{R} \backslash (I \cup \bar{R}R^{-1} \cup R^{-1}\bar{R})$
Biorders	
(Inner Rim)	$R^{\mathcal{I}} = R \backslash R\bar{R}^{-1}R$
(Outer Rim)	$R^{\mathcal{O}} = \bar{R} \backslash \bar{R}R^{-1}\bar{R}$
Interval Orders	
(Inner Rim)	$R^{\mathcal{I}} = R \backslash R\bar{R}^{-1}R$
(Outer Rim)	$R^{\mathcal{O}} = \bar{R} \backslash (I \cup \bar{R}R^{-1}\bar{R})$
Semiorders	
(Inner Rim)	$R^{\mathcal{I}} = R \backslash (R\bar{R}^{-1}R \cup RR\bar{R}^{-1} \cup \bar{R}^{-1}RR)$
(Outer Rim)	$R^{\mathcal{O}} = \bar{R} \backslash (I \cup \bar{R}R^{-1}\bar{R} \cup R^{-1}\bar{R}\bar{R} \cup \bar{R}\bar{R}R^{-1})$

algorithms for the assessment of knowledge (see e.g. Falmagne and Doignon, 1988).

Let $\mathcal{F}$ be a family of subsets of a finite set E, and let R be any set in $\mathcal{F}$. *The inner fringe* $R^{\mathcal{I}}$ of R (w.r.t. $\mathcal{F}$) consists of all elements e in R such that $R \backslash \{e\}$ is another set in $\mathcal{F}$. Similarly, the *outer fringe* $R^{\mathcal{O}}$ of R (w.r.t. $\mathcal{F}$) is formed by all elements e in R such that $R \cup \{e\}$ is another set in $\mathcal{F}$. By definition, we have thus for any $R, S \in \mathcal{F}$,

$$d(R,S) = 1 \Leftrightarrow (R^{\mathcal{O}} \cap S^{\mathcal{I}} \neq \emptyset \text{ or } R^{\mathcal{I}} \cap S^{\mathcal{O}} \neq \emptyset).$$

The results outlined above for the semiorders can be generalized as follows. The next Theorem is due to Doignon and Falmagne (1997). We omit the proof.

THEOREM 3. *If $\mathcal{F}$ is the family of all semiorders (resp. partial orders, interval orders) on a finite set, then $\mathcal{F}$ is well graded. Similarly, if $\mathcal{F}$ is the family of all biorders between two finite sets, then $\mathcal{F}$ is well graded. The inner and outer fringes of the relations in each of these four families of relations are specified in Table 1.*

BASIC CONCEPTS OF THE THEORY

In keeping with our introductory comments, we suppose that the subject is initially naive. In order words, the preference relation at time t = 0 is the empty set. We also assume that at some random times $t_1, t_2, \ldots, t_n, \ldots$ quantum tokens of information on the alternatives and their relationships are delivered by the environment. These tokens can be 'positive' or 'negative.' A positive token conveys a message such as *'x is preferable to y'*, while a negative token may carry the meaning that *'x is not preferable to y'*. These two types of tokens are denoted by xy and $\widetilde{xy}$, respectively (with $x \neq y$).

We suppose that these positive and negative tokens are delivered in random fashion by the environment through various means which are not monitored or controlled by the observer, such as TV commercials or programs, newspaper articles or adds, posters, conversations, etc. To represent these phenomena, we postulate the existence of a probabilistic mechanism with two components. One concerns the times of occurence of the tokens, which is assumed to be ruled by a renewal counting process (specifically, a homogeneous Poisson process, but this assumption is only critical for part of the predictions). The other concerns the nature of the occuring tokens, which is governed by a probability distribution on the set of all tokens, this distribution being characteristic of the environment. The occurence of a token can affect the current preference relation R—the state of the subject—only by adding a single disparate pair to R, or removing such a pair from it. A formal statement of the theory is given in the next definition and in the ensuing list of axioms. Comments follow the definition.

DEFINITION 4. Let X be the finite set of alternatives, and let $\mathcal{S}$ be a well graded family of (binary) relations on X, containing the empty relation. Examples are the semiorders, the partial orders and the interval orders. To avoid trivialities, we suppose that $|X|$ > 1. We denote the set of all tokens by

$$\mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_-$$

with

$$\mathcal{T}_+ = \{\tau | \tau = xy \text{ for } x, y \in X, x \neq y\},$$
$$\mathcal{T}_- = \{\tau | \tau = \widetilde{xy} \text{ for } x, y \in X, x \neq y\}.$$

A token is called *positive* or *negative* depending whether it belongs to $\mathcal{T}_+$ or $\mathcal{T}_-$.

The preference relations in $\mathcal{S}$ are called *states*. A state may be modified by the occurence of a token. The arrivals of the tokens at times $t_1, t_2, \ldots, t_n, \ldots$ are governed by a stochastic process. These transitions are specified by the operation $\Diamond : (R, \tau) \mapsto R\Diamond\tau$ mapping $\mathcal{S} \times \mathcal{T}$ into $\mathcal{S}$, defined by

$$(3) \qquad R \;\Diamond\; \tau = \begin{cases} R \cup \{xy\} & \text{if } \tau = xy \in R^{\mathcal{O}} \cap \mathcal{T}_+; \\ R \backslash \{xy\} & \text{if } \tau = \widetilde{xy} \in \mathcal{T}_- \text{and } xy \in R^{\mathcal{I}}; \\ R & \text{in all other cases.} \end{cases}$$

Thus, $R \;\Diamond\; \tau$ is the state resulting from the occurence of the token τ for an individual previously in state R. Since, with probability one, the initial state is the empty relation, the occurence of a token will always result in transforming any relation R in $\mathcal{S}$ into some relation R'in $\mathcal{S}$, which may be identical to R. In the case of the semiorder, for example, the token xy transforms the initial state $\emptyset$, which is vacuously a semiorder, into the semiorder $\{xy\}$.

An exemplary sequence of tokens occuring at times $t_1, \ldots, t_n, \ldots$ and their effects on the states, in the case of the semiorders, is pictured by Figure 5. Notice that the token wz presented at time t_2 is ignored, because wz is not in the outer fringe of the current state $\{xy\}$, regarded as a semiorder. However, if the set of states S had been the set of all partial orders on X, then the occurence of wz at time t_2 would have resulted in the partial order $\{xy, wz\}$: indeed wz **is** in the outer fringe of $\{xy\}$ regarded as a partial order.

We now turn to the probabilistic aspects of the theory. For terminology and basic concepts, see Parzen (1962) and Kemeny and Snell (1960). We assume that there exists a positive probability distribution

$$\theta : \mathcal{T} \to [0, 1] : \tau \mapsto \theta_\tau$$

on the collection $\mathcal{T}$ of tokens. Thus, $\theta_\tau > 0$ for any $\tau \in \mathcal{T}$, and $\sum_{\tau\in\mathcal{T}} \theta_\tau = 1$. The theory will be stated in terms of three collections of random variables. For any $t > 0$,

$\mathbf{S}_t$ specifies the state of the individual at time t,

$\mathbf{N}_t$ indicates the number of tokens arising in the half open interval of time $]0, t]$, and

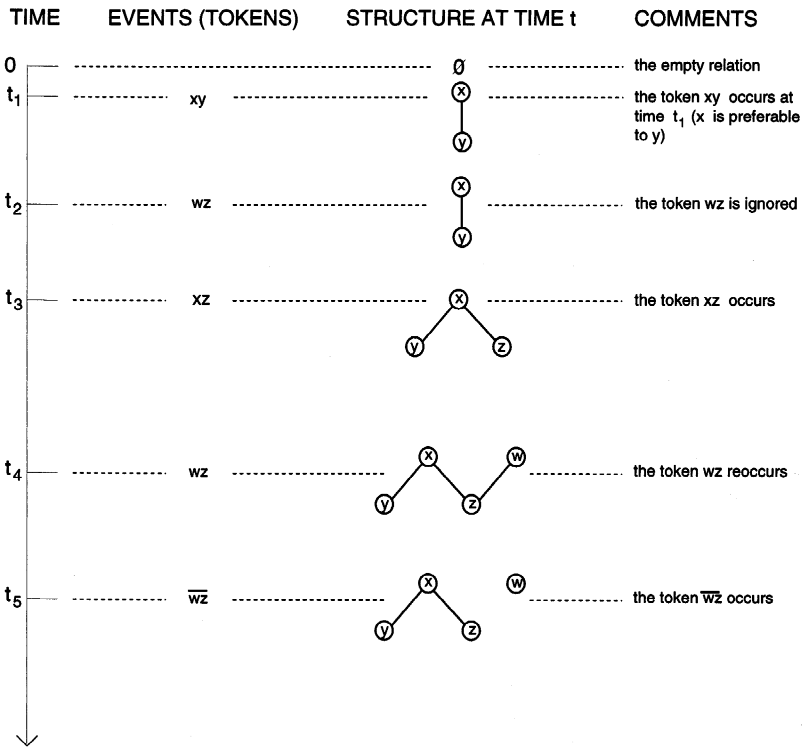

Figure 5. Exemplary sequence of tokens occuring at times $t_1, t_2, \ldots, t_5, \ldots$ The states are semiorders represented by their Hasse diagrams.

$\mathbf{T}_t$ means the last token presented before or at time t; we set $\mathbf{T}_t = 0$ if no tokens were presented, that is, if $\mathbf{N}_t = 0$.

We shall also use the random variable

$$\mathbf{N}_{t,t+\delta} = \mathbf{N}_{t+\delta} \Leftrightarrow \mathbf{N}_t$$

specifying the number of tokens arising in the half-open interval $]t, t + \delta]$. Thus, $\mathbf{S}_t$ takes its values in the set $\mathcal{S}$ of states; $\mathbf{N}_t$ and $\mathbf{N}_{t,t+\delta}$ are nonnegative integers, and $\mathbf{T}_t \in \mathcal{T} \cup \{0\}$. The random variable $\mathbf{N}_t$ will turn out to be the 'counting random variable' of a Poisson process regulating the number of Poisson events occuring in the interval $]0, t]$.

The three axioms [I], [T] and [L] below define a stochastic process $(\mathbf{N}_t, \mathbf{T}_t, \mathbf{S}_t)_{t>0}$ up to the parameters θ_τ's $(\tau \in \mathcal{T})$ and one parameter λ pertaining to the intensity of the Poisson process.

AXIOM 5. [**I**] (*Initial state*). Initially, the state of the individual is the empty relation. The subject remains in state $\varnothing$ until the realization of the first token. That is, for any $t > 0$

$$\mathbb{P}(\mathbf{S}_t = \varnothing | \mathbf{N}_{0,t} = 0) = 1.$$

The remaining axioms specify the process recursively. The notation $\mathcal{E}_t$ stands for any arbitrarily chosen history of the process before time $t > 0$, and $\mathcal{E}_0$ denotes the empty history.
[**T**] (*Occurence of the tokens*). The occurence of the tokens is governed by a homogeneous Poisson process of intensity λ. When a Poisson event is realized, the token τ occurs with probability θ_τ, regardless of past events. Thus, for any nonnegative integer k, any real numbers $t \geq 0$ and $\delta > 0$, and any history $\mathcal{E}_t$,

$$\text{(4)} \quad \mathbb{P}(\mathbf{N}_{t,t+\delta} = k) = \frac{(\lambda\delta)^k e^{-\lambda\delta}}{k!}$$

$$\text{(5)} \quad \mathbb{P}(\mathbf{T}_{t+\delta} = \tau | \mathbf{N}_{t,t+\delta} = 1, \mathcal{E}_t) = \mathbb{P}(\mathbf{T}_{t+\delta} = \tau | \mathbf{N}_{t,t+\delta} = 1) = \theta_\tau.$$

[**L**] (*Learning*). If R is the state at time t, and a single token τ occurs between times t and $t + \delta$, then the state at time $t + \delta$ will be $R \diamond \tau$ regardless of past events before time t, with the operation $\diamond$ defined as in Equation (3). Formally:

$$\begin{aligned}
&\mathbb{P}(\mathbf{S}_{t+\delta} = S | \mathbf{T}_{t+\delta} = \tau, \mathbf{N}_{t,t+\delta} = 1, \mathbf{S}_t = R, \mathcal{E}_t) \\
&\quad = \mathbb{P}(\mathbf{S}_{t+\delta} = S | \mathbf{T}_{t+\delta} = \tau, \mathbf{N}_{t,t+\delta} = 1, \mathbf{S}_t = R) \\
&\quad = \begin{cases} 1 & \text{if } S = R \diamond \tau, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The stochastic process defined by Axioms [I], [T], and [L] will be called the *stochastic rationality theory*. The special case of this theory in which the set of states $\mathcal{S}$ contains all the semiorders on the finite set X will be referred to as the *stochastic semiorder model*. A similar terminology will be adopted in the cases of the partial orders and the interval orders. It is easy to check that–in view of Theorem 3–essentially the same axioms would apply in a situation in which the set of states contains all the biorders between two finite sets. This case will be labelled the *stochastic biorder model*.

REMARKS 6. (a) Axiom [L], together with Equation (3) defining the operation $\Diamond$, forms the core of the theory, and formalize two simple ideas. One is that the individual is endowed with rationality in the guise of some well graded family of relations, of which the semiorders, the partial orders etc. are prime examples. The other is that the mental structure formalized by the states are rather rigid. They can be altered only minimally at any time. Specifically, the feasible minimal changes are represented either by the addition of a single disparate pair to the state, or by the removal of such a pair from the state.

(b) Some objections can be levelled against these axioms. In the case of a political election, for example, the Poisson process and the probability distribution on the set of tokens reflect the time course and the content of the political campaigning. It is unrealistic to suppose that the only difference between the voters is attributed to the chance occurence of the tokens, the distribution of which is supposed to be the same for all. (The voters read different newspapers, watch different TV programs, have different neighbors and co-workers.) An other objection concerns the homogeneity of the Poisson process. It would seem natural to suppose that the intensity of the campaigning changes in the course of the campaign, with perhaps a peak on the eve of the election. However serious, the objections bear only on superficial aspects of the theory. Relatively minor alterations of the axioms are easy to conceive and to implement, which would take care of these and some other shortcomings. We shall come back to these issues in a later section of this paper. In the mean time, the reader should keep in mind that some key asymptotic results do not specifically depend upon the assumption of a homogeneous Poisson process, and would also hold in the much more general setting of an arbitrary renewal process (see in particular Theorem 10).

(c) Finally, we mention that generalizations of this theory are easily conceivable, in which the elements of the basic wellgraded family $\mathcal{S}$ are not (necessarily) binary relations, but n-ary relations or even arbitrary sets. With an appropriately redefined set $\mathcal{T}$ of tokens, the developments would remain essentially the same. (Such a generalization is given in Falmagne, 1997.)

RESULTS

We begin by introducing a useful device.

DEFINITION 7. For a given realization of the Poisson process at times $t_1, \ldots, t_n, \ldots$, we partition the time axis into the segments

$$]0, t_1[, [t_1, t_2[, \ldots, [t_n, t_{n+1}[, \ldots \tag{6}$$

such that $\mathbf{N}_t = 0$ for $t < t_1$, $\mathbf{N}_t = 1$ for $t_1 \leq t < t_2$, and in general $\mathbf{N}_t = n$ for $t_n \leq t < t_{n+1}$. Fixing the sequence (t_n), and defining $\mathbf{S}_n^* = \mathbf{S}_{t_n}$, we obtain a discrete parameter process $(\mathbf{S}_n^*)$ with state space, $\mathcal{S}$. The process $(\mathbf{S}_n^*)$ will be called the *discrete companion* of $(\mathbf{S}_t)$. Even though this discrete parameter process is implicitly indexed by the particular sequence (6) of times of occurence of Poisson events, in some important sense it does not depend on it. In fact, the situation is that described in the Theorem below, which follows immediately from the definitions.

THEOREM 8. *The discrete companion $(\mathbf{S}_n^*)$ of $(\mathbf{S}_t)$ is a homogeneous Markov chain, with state space $\mathcal{S}$ and transition probabilities defined, for any distinct R, S in $\mathcal{S}$, with $t_1, t_2, \ldots, t_n, t_{n+1}, \ldots$, as in (6), by*

$$\begin{aligned} p^*_{R,S} &= \mathbb{P}(\mathbf{S}^*_{n+1} = S | \mathbf{S}^*_n = R) = \mathbb{P}(\mathbf{S}_{t_{n+1}} = S | \mathbf{S}_{t_n} = R) \\ &= \begin{cases} \theta_\tau & \textit{if } S = R \diamond \tau, \\ 0 & \textit{otherwise,} \end{cases} \end{aligned}$$

a result which does not depend upon the sequence (t_n) associated with a particular realization of the Poisson process.

Notice that

$$\begin{aligned} &\mathbb{P}(\mathbf{S}_{t+\delta} = S | \mathbf{S}_t = R) \\ &\quad = \sum_{k=0}^{\infty} \mathbb{P}(\mathbf{S}_{t+\delta} = S | \mathbf{N}_{t,t+\delta} = k, \mathbf{S}_t = R)\mathbb{P}(\mathbf{N}_{t,t+\delta} = k | \mathbf{S}_t = R) \\ &\quad = \sum_{k=0}^{\infty} \mathbb{P}(\mathbf{S}^*_{n+k} = S | \mathbf{S}^*_n = R)\mathbb{P}(\mathbf{N}_{t,t+\delta} = k). \end{aligned}$$

Writing

$$p^*_{R,S}(k) = \mathbb{P}(\mathbf{S}^*_{n+k} = S | \mathbf{S}^*_n = R) \tag{7}$$

for the k-step transition probability of the Markov chain $(\mathbf{S}_n^*)$, and using Equation (4), we obtain the following result:

THEOREM 9. *The stochastic process ($\mathbf{S}_t$) is a homogeneous Markov process, with transition probability function*

$$
\begin{aligned}
p_{R,S}(\delta) &= \mathbb{P}(\mathbf{S}_{t+\delta} = S|\mathbf{S}_t = R) \\
&= \sum_{k=0}^{\infty} p^*_{R,S}(k)\frac{(\lambda\delta)^k e^{-\lambda\delta}}{k!}. \qquad (8)
\end{aligned}
$$

(Thus, $p_{R,S}(\delta)$ is the probability of a transition from state R to state S in δ units of time.) Important aspects of the Markov process ($\mathbf{S}_t$) can be investigated via a study of the Markov chain ($\mathbf{S}^*_n$). In particular, we can use Equation (8) to predict the role of the passage of time on the evolution of the preference relations. It turns out that the asymptotic probabilities of the states in the Markov chain ($\mathbf{S}^*_n$) and in the Markov process ($\mathbf{S}_t$) exist and coincide. The next Theorem specifies these asymptotic probabilities. We use the notation for any relation $R \in \mathcal{S}$:

$$\hat{R} = \overline{R \cup I} = \bar{R} \setminus I. \qquad (9)$$

(Notice that $\hat{\hat{R}}$ = R.)

THEOREM 10. *The homogeneous Markov chain ($\mathbf{S}^*_n$) is irreducible and aperiodic. In this case, the asymptotic probabilities of the states exist and form the unique stationary distribution of the Markov chain ($\mathbf{S}^*_n$). We obtain for any $R \in S$:*

$$
\begin{aligned}
p_R &= \lim_{t\to\infty} \mathbb{P}(\mathbf{S}_t = R) = \lim_{n\to\infty} \mathbb{P}(\mathbf{S}^*_n = R) \\
&= \frac{\prod_{xy\in R} \theta_{xy} \cdot \prod_{zw\in\hat{R}} \theta_{\widetilde{zw}}}{\sum_{S\in\mathcal{S}} \prod_{st\in S} \theta_{st} \cdot \prod_{uv\in\hat{S}} \theta_{\widetilde{uv}}}. \qquad (10)
\end{aligned}
$$

We postpone the proof of this theorem for a moment. Considering the structure of $\mathcal{S}$ as a well graded family of states, it makes sense to describe the Markov chain $(\mathbf{S}^*_n)$ as a random walk on $\mathcal{S}$.

An example of such a random walk for $X = \{1, 2, 3\}$ was presented in Figure 4. In this situation, there are exactly five different types of semiorders, as can be seen from the figure. We write $\mathcal{S}_3$ for the collection of all 19 semiorders on X = $\{1, 2, 3\}$, and we denote the semiorders in $\mathcal{S}_3$ by $\succ, \succ'$ etc. Setting

$$K = \sum_{\succ'\in\mathcal{S}_3} \prod_{ij\in\succ} \theta_{ij} \prod_{lk\in\widehat{\succ'}} \theta_{\widetilde{lk}},$$

Equation (10) becomes for these five cases, with distinct $i, j, k \in \{1, 2, 3\}$ and with obvious notation for the asymptotic probabilities p_R:

$$\begin{aligned}
p_{[i \succ j \succ k]} &= \frac{1}{K} \theta_{ij} \theta_{\widetilde{ji}} \theta_{ik} \theta_{\widetilde{ki}} \theta_{jk} \theta_{\widetilde{kj}}, \\
p_{[i \succ j, i \succ k]} &= \frac{1}{K} \theta_{ij} \theta_{\widetilde{ji}} \theta_{ik} \theta_{\widetilde{ki}} \theta_{\widetilde{jk}} \theta_{\widetilde{kj}}, \\
p_{[j \succ k, i \succ k]} &= \frac{1}{K} \theta_{ik} \theta_{\widetilde{ki}} \theta_{jk} \theta_{\widetilde{kj}} \theta_{\widetilde{ij}} \theta_{\widetilde{ji}} \\
p_{[i \succ j]} &= \frac{1}{K} \theta_{ij} \theta_{\widetilde{ji}} \theta_{\widetilde{ik}} \theta_{\widetilde{ki}} \theta_{jk} \theta_{kj}, \\
p_{[\emptyset]} &= \frac{1}{K} \theta_{\widetilde{ij}} \theta_{\widetilde{ji}} \theta_{\widetilde{ik}} \theta_{\widetilde{ki}} \theta_{\widetilde{jk}} \theta_{\widetilde{kj}}. \qquad (11)
\end{aligned}$$

In the proof of Theorem 10, we use the following well-known result.

LEMMA 11. Let $(m_{R,S})_{R,S \in S}$ be the transition matrix of a regular Markov chain on a finite set $\mathcal{S}$, and let $\pi : R \mapsto \pi_R$ be a probability distribution on $\mathcal{S}$. Suppose that

$$\forall R, S \in \mathcal{S}, \quad \pi_R \cdot m_{R,S} = \pi_S \cdot m_{S,R}.$$

Then π is the unique stationary distribution of the Markov chain.

It suffices to show that $\pi_R = \sum_{S \in \mathcal{S}} \pi_S \cdot m_{S,R}$ for all $R \in \mathcal{S}$. We leave to the reader to check the algebra.

Proof (of Theorem 10) 12. The Markov chain ($\mathbf{S}_n^*$) is irreducible because: (i) $\mathcal{S}$ is well graded, and (ii) $\theta_\tau > 0$ for any token τ. Together, (i) and (ii) mean that any two states of the Markov chain communicate. To prove that the irreducible Markov chain ($\mathbf{S}_n^*$) is aperiodic, it suffices to show that $p^*_{R,R} > 0$ for some state R. Since $\mathcal{S}$ contains the empty relation and $|X| > 1$, there is a state R containing some disparate pair xy. (For example, we can take $R = \emptyset \diamond xy$). We necessarily have $R \diamond xy = R$, with $p^*_{R,R} \geq \theta_{xy} > 0$. We conclude that the Markov chain ($\mathbf{S}_n^*$) has a unique stationary distribution.

It remains to show that the stationary distribution is that given by Equation (10). We use Lemma 11, and prove that for any $R, S \in \mathcal{S}$, writing K for the denominator in (10),

$$\begin{aligned} (12) \qquad & \tfrac{1}{K}\left(\prod_{xy\in R}\theta_{xy}\cdot\prod_{zw\in\hat{R}}\theta_{\tilde{zw}}\right)p^*_{R,S} \\ & = \tfrac{1}{K}\left(\prod_{xy\in S}\theta_{xy}\cdot\prod_{zw\in\hat{S}}\theta_{\tilde{zw}}\right)p^*_{S,R}. \end{aligned}$$

In the Markov chain $(\mathbf{S}^*_n)$, the one-step transition probabilities between state R and state S are given by

$$\begin{array}{lll} p^*_{R,S}=\theta_{st}\wedge p^*_{S,R}=\theta_{\tilde{st}} & \text{if } S\backslash R=\{st\}, \text{for some } st\in X & \text{(Case 1)} \\ p^*_{R,S}=\theta_{\tilde{st}}\wedge p^*_{S,R}=\theta_{st} & \text{if } R\backslash S=\{st\}, \text{for some } st\in X, & \text{(Case 2)} \\ p^*_{R,S}=p^*_{R,S}=0 & \text{if } d(R,S)>1. & \text{(Case 3).} \end{array}$$

In Case 3, the two members of Equation (12) vanish, and it is easy to check that in each of Cases 1 and 2, the factors in the two members of (12) are identical. We leave the verification to the reader. ■

REMARK 13. (a) Note that the result of Theorem 10 would hold under much more general hypotheses on the process governing the delivery of the tokens. The assumption of homogeneity of the Poisson process plays no useful role in establishing the result. In fact, a much more general class of renewal counting processes can be assumed.

(b) A special case of the probability distribution $p : R \mapsto p_R$ defined by (10) on the well graded family $\mathcal{S}$ is conceptually close to a distribution on a set of linear orders proposed by Mallows (1957, see also Feigin and Cohen, 1978). Fix a particular relation $R_0 \in \mathcal{S}$ and define

$$(13) \qquad \theta_\tau = \begin{cases} \beta & \text{if } \exists \; xy=\tau\in\mathcal{T}_+\cap R_0; \\ \alpha & \text{if } \exists \; xy=\tau\in\mathcal{T}_+\cap \hat{R}_0; \\ \mu & \text{if } \exists \; \widetilde{xy}=\tau\in\mathcal{T}_-\text{with } xy\in R_0; \\ \xi & \text{if } \exists \; \widetilde{xy}=\tau\in\mathcal{T}_-\text{with } xy\in \hat{R}_0. \end{cases}$$

Replacing the θ's in Equation (10) by the values given by (13), we obtain

$$p_R\frac{\beta^{|R\cap R_0|}\cdot\alpha^{|R\cap\hat{R}_0|}\cdot\mu^{|\hat{R}\cap R_0|}\cdot\xi^{|\hat{R}\cap\hat{R}_0|}}{\sum_{S\in\mathcal{S}}\left(\beta^{|S\cap R_0|}\cdot\alpha^{|S\cap\hat{R}_0|}\cdot\mu^{|\hat{S}\cap R_0|}\cdot\xi^{\hat{S}\cap\hat{R}_0|}\right)}.$$

After setting $\alpha = \mu$ and $\beta = \xi$, this reduces to

$$p_R = \frac{\alpha^{d(R,R_0)} \cdot \beta^{d(R,\hat{R}_0)}}{\sum_{S\in\mathcal{S}} \alpha^{d(S,R_0)} \cdot \beta^{d(S,\hat{R}_0)}}. \tag{14}$$

Notice that with $m = |X|$, we have $d(R, S) + d(R, \hat{S}) = m(m \Leftrightarrow 1)$ for any two relations $R, S \in \mathcal{S}$. Dividing the numerator and the denominator of (14) by $\beta^{m(m-1)}$, we obtain

$$p_R = \frac{(\alpha/\beta)^{d(R,R_0)}}{\sum_{S\in\mathcal{S}} (\alpha/\beta^{d(S,R_0)})}, \tag{15}$$

which is similar to Feigin and Cohen's distribution for linear orders, but applies here to any well-graded family of relations. (The meaning of the exponent is different in the two cases though.) The two parameters of this distribution are thus the distinguished relation R_0 and the ratio $\alpha/\beta > 0$.

Using Theorem 10 and Theorem 9, an explicit expression can be obtained for the asymptotic joint probability of observing the states R and S at time t and time $t + \delta$, respectively.

THEOREM 14. *Writing, as before, $p^*_{R,S}(k)$ for the k-step transition probability between the states R and S in the random walk on $\mathcal{S}$, we have successively*

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}(\mathbf{S}_t = R, \mathbf{S}_{t+\delta} = S) &= \lim_{t\to\infty} [\mathbb{P}(\mathbf{S}_t) = R)\mathbb{P}(\mathbf{S}_{t+\delta} = S | \mathbf{S}_t = R)] \\
&= p_R \cdot p_{R,S(\delta)} \\
&= \frac{\prod_{xy\in R} \theta_{xy} \cdot \prod_{zw\in\hat{R}} \theta_{z\tilde{w}}}{\sum_{W\in\mathcal{S}} \prod_{st\in W} \theta_{st} \cdot \prod_{uv\in\hat{W}} \theta_{\tilde{u}\tilde{v}}} \sum_{k=0}^{\infty} p^*_{R,S}(k) \frac{(\lambda\delta)^k e^{-\lambda\delta}}{k!}
\end{aligned} \tag{16}$$

The proof is straightforward.

APPLICATION TO THE SEMIORDERS

We apply these results in a case in which $\mathcal{S}$ is the set of all semiorders on a finite set X. The Markov chain $(\mathbf{S}^*_n)$ is then a random walk on the semiohedron associated with X. In addition, we suppose that the empirical situation justifies the simplifying assumption

$$\theta_{xy} = \theta_{\widetilde{yx}} \tag{17}$$

for all disparate pairs xy. The example of the 3-semiohedron was displayed in Figure 4, for the set $X = \{1, 2, 3\}$. Only the centrifugal transitions are indicated in Figure 4 (the center being the empty semiorder). The opposite transition can be obtained by reversing the arrows in Figure 4, and replacing any transition probability θ_{ij} by θ_{ji}. As before, we denote a typical semiorder by $\succ$, and we write $\sim$ for the set of all disparate pairs which are also indifferent w. r. t. $\succ$. That is

$$x \sim y \Longleftrightarrow (\neg(x \succ y) \wedge \neg(y \succ x) \wedge x \neq y).$$

Similar conventions and notations apply to other semiorders appearing in the discussion or in the formulas. Those will be denoted $\succ'$, $\succ''$, etc. Specializing Theorems 10 and 14, we obtain:

THEOREM 15. *Suppose that $\mathcal{S}$ contains all the semiorders on a finite set X containing at least two elements, and that Equation (17) holds for any disparate pair xy. As an application of Theorem 10, we obtain for the asymptotic probability of any semiorder $\succ$:*

$$\begin{aligned} p_{\succ} &= \lim_{t\to\infty} \mathbb{P}(\mathbf{S}_t = \succ) \\ &= \frac{\prod_{xy\in\succ} \theta_{xy}^2 \prod_{zw\in\sim} \theta_{zw}}{\sum_{\succ''\in\mathcal{S}} \prod_{xy\in\succ''} \theta_{xy}^2 \prod_{zw\in\sim''} \theta_{zw}} \end{aligned} \tag{18}$$

Similarly, using Theorem 14, the asymptotic probability of a joint occurence of state $\succ$ at time t and state $\succ'$ at time t + δ is given by the formula:

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(\mathbf{S}_t &= \succ, \mathbf{S}_{t+\delta} = \succ') = \lim_{t\to\infty} [\mathbb{P}(\mathbf{S}_t = \succ)\mathbb{P}(\mathbf{S}_{t+\delta} = \succ' | \mathbf{S}_t = \succ)] \\ &= p_{\succ} p_{\succ,\succ'}(\delta) \\ &= \frac{\prod_{xy\in\succ} \theta_{xy}^2 \prod_{zw\in\sim} \theta_{zw}}{\sum_{\succ''\in\mathcal{S}} \prod_{xy\in\succ''} \theta_{xy}^2 \prod_{zw\in\sim''} \theta_{zw}} \sum_{k=0}^{\infty} p^*_{\succ,\succ'}(k) \frac{(\lambda\delta)^k e^{-\lambda\delta}}{k!} \end{aligned} \tag{19}$$

The proof is immediate. A conspicuous difference between the pair of Equations (10), (16) and the pair (18), (19), respectively, lies in the squares appearing in the denominators and the numerators of the latter two equations. These squares result from the simplifying assumption $\theta_{xy} = \theta_{\widetilde{yx}}$ in (17): each product $\theta_{xy}\theta_{\widetilde{xy}}$ in (10) has been replaced by θ_{xy}^2.

REMARKS ON STATISTICAL TESTING. In principle, this theory can be applied and tested using standard statistical techniques. An example is analyzed here for the semiorder model discussed above, in which a chi-square method is used, based on Equation (19). (We assume thus that $\theta_{xy} = \theta_{\widetilde{yx}}$ for every disparate pair xy.) Suppose that 2000 potential voters have been asked, on two occasions, to rate on a scale of 0 to 100 three major candidates in a presidential election. (There are many examples of data of this type.) For concreteness, suppose that the first survey was made on April 14th, and the second on June 30 of the same election year. Thus, the interval between the two surveys is 77 days. Each of the 2000 subjects has given two ratings of the three candidates. It is reasonable to only retain the ordinal information contained in these ratings. Accordingly we convert any rating r into a semiorder $\succ$ by the formula

$$x \succ y \Longrightarrow r(x) > r(y) + C$$

in which the constant C is to be determined empirically. This recoding results in assigning a pair $(\succ, \succ')$ of semiorders to each subject. Let $N(\succ, \succ')$ be the number of subjects being assigned the pair of semiorders $(\succ, \succ')$; we have thus: $\sum_{\succ,\succ' \in \mathcal{S}} N(\succ, \succ') = 2000$. We suppose that at the time of the first survey, all subjects were already fully exposed to the pros and cons regarding each of the three candidates. This means that asymptotic results can be used in the form of Equation (18).

A statistical test can be derived from the chi-square statistic

$$(20) \qquad \sum_{\succ,\succ' \in \mathcal{S}} \frac{[N(\succ, \succ') \Leftrightarrow 2000 \times [p_{\succ} \cdot p_{\succ,\succ'}(77)]]^2}{2000 \times [p_{\succ} \cdot p_{\succ,\succ'}(77)]}$$

with $p_{\succ}$ as in (18) and

$$(21) \qquad p_{\succ,\succ'}(77) = \sum_{k=0}^{\infty} p^*_{\succ,\succ'}(k) \frac{(\lambda 77)^k e^{-\lambda 77}}{k!}.$$

We use Equation (8) with δ = 77. Note that six parameters are used: there are six parameters θ_{xy} (these six having sum 1), plus the intensity parameter λ of the Poisson process. Since we have $19^2 \Leftrightarrow 1 = 360$ data points (independent response frequencies associated to the 19^2 pairs of semiorders), the chi-square statistic in (20) has $360 \Leftrightarrow 6 = 354$ degrees of freedom. The parameters θ_{xy}, and λ can be estimated for example by minimizing (20) using an optimizing routine such as PRAXIS (cf. Powel, 1964; Brent, 1074). In practice the series in (19) can be replaced by a finite sum, and the individual Poisson probabilities can be replaced by a standard approximation, such as

$$\frac{(77\lambda)^k e^{-77\lambda}}{k!} \approx (2\pi)^{-\frac{1}{2}} \int_{g_-(\lambda,k)}^{g_+(\lambda,k)} e^{-\frac{1}{2}u^2}\, \mathrm{d}u$$

with

$$g_+(\lambda, k) = (k \Leftrightarrow 77\lambda + \tfrac{1}{2})(77\lambda)^{\frac{1}{2}}$$

$$g_-(\lambda, k) = (k \Leftrightarrow 77\lambda \Leftrightarrow \tfrac{1}{2})(77\lambda)^{\frac{1}{2}}$$

(cf. Johnson and Kotz, 1969). Moreover, if q is a large enough positive integer, we get

$$p^*_{\succ,\succ'}(q) \approx p^*_{\succ'} = \frac{\prod_{xy\in\succ'} \theta_{xy} \prod_{zw\in\sim'} \theta_{zw}}{\sum_{\succ''\in\mathcal{S}} \prod_{st\in\succ''} \theta_{st} \prod_{uv\in\sim''} \theta_{uv}}$$

It follows that (21) can be rewritten in the form of the approximation

$$p_{\succ,\succ'}(77) \approx \sum_{k=0}^{q-1} p^*_{\succ,\succ'}(k)[\Phi(g_+(\lambda,k)) \Leftrightarrow \Phi(g_-(\lambda,k))] + p^*_{\succ'} \sum_{k=q}^{\infty} \frac{(\lambda 77)^k e^{-\lambda 77}}{k!}$$

in which Φ stands for the distribution function of a standard normal random variable, and the tail of the Poisson distribution in the last term can itself be replaced by an approximation such as that proposed by Peizer and Pratt (1968), for example.

In the application of such a test, the number of degrees of freedom may be smaller than 354, since some grouping of the cells may be necessary for the convergence of the chi-square statistic. Nevertheless, in view of the large number of subjects, the number of degrees of freedom will remain large, allowing the use of the normal approximation to the chi-square to evaluate the goodness of fit. It the fit is satisfactory, the estimated values of the parameters θ_{xy} and λ provide a quantitative assessment of the campaign. In principle, the parameters may also be used to predict the voters behavior at some later time.

It must be emphasized that the potential applications of the theory are by no means limited to the asymptotic predictions discussed in this section. Admittedly, an application in a non-asymptotic case would be computationally more difficult. In principle however, such an application is feasible.

VARIATIONS OF THE THEORY

A number of limitations of this theory were alluded to earlier, which we now address. In some situations, the assumptions of the theory should be altered, depending on some features of the alternatives, the subjects or the situation. A few examples are given below, which will evoke other possibilities. For concreteness, these examples are formulated in the framework of the semiorder model with the assumption $\theta_{xy} = \theta_{\widetilde{yx}}$ considered in the last section. Extensions to other cases should be obvious.

ASSESSING THE EFFECT OF A MAJOR EVENT. A variation of the application described in the preceding section arises in a slightly different situation. We still have three candidates campaigning for a political office. Suppose, however, that the researcher is especially interested in assessing in detail the effect of an important media event such as a debate between the candidates, and that the first rating was obtained just before the debate, and the second one week later. Equation (19) can still be used, but with some modifications. The θ_{xy}'s entering in the expression of $p_{\succ}$ in the equation must be regarded as reflecting the probabilities of the tokens before the debate. A different set of parameters θ'_{xy} must be postulated for the expression of the transition probabilities $p^*_{\succ,\succ'}(k)$, since these transitions may have been at least in part induced by the debate. In the same vein, the intensity parameter λ of the Poisson process should also reflect the effect of the debate. Moreover, we should set $\delta = 7$ (days). If a reasonable fit is obtained, a comparison between the estimated values of the θ and the θ' parameters may provide a quantitative assessment of the effect of the debate. The numbers of degrees of freedom in the chi-square statistic is of course decreased by five points, since five additional parameters have been used to predict the data.

Recalling Remark 13(a) we point out that this prediction does not depend on the assumption that the delivery of the tokens before the debate is governed by a homogeneous Poisson process. As indicated by Equation (19), a homogeneous Poisson process is explicitly used in the computation of the effect of the debate, in the form of the series in that equation. This assumption of homogeneity is not shocking, since it only concerns a short period of seven days. As for the other factor in that equation, representing the asymptotic probability of the semiorder $\succ$, a general class of renewal counting processes could be postulated which would yield the same expression.

FEATURES OF THE ALTERNATIVES. In some situations, the alternatives can be described by features which may play a role in the predictions. In the political example discussed above, it may be useful to know that candidate x is a woman in favor of 'free choice', since this may be relevant to the subjects' preferences. Several elaborations of the theory can be conceived to handle such cases, and two are sketched here. In both cases, we assume that the set of alternatives can be represented as a Cartesian product $X = X_1 \times X_2$. (Thus, (x_1, x_2) may represent a woman favoring 'free choice', while (y_1, y_2) represents a male candidate advocating the 'pro-life' option.)

We first consider the possibility that this Cartesian product structure is affecting the delivery of the tokens, which we typically write as (x_1x_2, y_1y_2). A concept of 'independence of the factors' could be formalized by a multiplicative decomposition of the token probabilities according to the formula:

$$(22) \qquad \theta_{x_1x_2,y_1y_2} = \theta^{(1)}_{x_1y_1}\theta^{(2)}_{x_2y_2}.$$

for any token (x_1x_2, y_1y_2). No change in the axioms of theory would be required, but the number of parameters would decrease substantially. A drawback of this formula however, is that the interpretation of the parameters $\theta^{(i)}_{x_iy_i} (i = 1, 2)$ in terms of the token events is not clear. A more drastic reformulation would arise from assuming that the collection of tokens contains 'single issue' tokens $xy \in X_i \times X_i$ for i = 1, 2). The effect of such tokens on the current relation would have to be formalized. In other words, a set of rules could be postulated that amount to redefining the operation $\lozenge$ used in the formulation of Axiom [L].

The Cartesian product structure of the tokens could also affect the nature of the preference relation itself, in the sense that the semiorder could reflect a combined effect of the factors. For example, we could investigate well graded families of 'conjoint' semiorders $\succ$, satisfying an additive representation

$$(x_1, x_2) \succ (y_1, y_2) \iff f_1(x_1) + f_2(x_2) > f_1(y_1) + f_2(y_2) + C,$$

where f_1 and f_2 are real valued functions. (See Luce, 1973, 1978, for some results regarding such a representation.) We shall not include such developments in this paper.

NON-HOMOGENEOUS POISSON PROCESS. The assumption that the time of delivery of the tokens is governed by a homogeneous Poisson process can easily be generalized to a much broader class of renewal counting processes. An example of how such a process can arise and how predictions can be made was mentioned, but a more systematic approach can be taken. In a typical election, the intensity of the flow of messages directed to the potential voters increases over time, with a peak on the eve before the election. A simple adaption of the theory would consist in replacing the homogeneity assumption of the Poisson process by another, in which the intensity parameter λ of the Poisson process increases over time, possibly exponentially. The asymptotic probabilities of the semiorders given in Equation (18) would not change, but some computation would be more complicate, such as that involved in the series of Equation (19).

INDIVIDUAL DIFFERENCES. The more serious limitation of the theory as it is stated here lies in the assumption that all subjects are embedded in the same medium and the only difference between them is to be attributed to the chance events associated with the delivery of the tokens. This is at best an approximation. There are two distinct criticisms here. One concerns the process governing the delivery of the tokens which, for a sufficiently diverse population, should not be assumed to be identical for all subjects. The other criticism would argue against the assumption that all subjects are alike in their reactions to the tokens. In this regard, note that the theory does not quite state that all subjects react to the tokens the same way since the effect of a token depends upon the current preference relation (the state of the subject). Nevertheless, one may wish to have some mechanisms in the theory that would explicitly model individual differences concerning both the exposure to the tokens, and their effects on the subjects.

Both of these criticisms are blunted in a situation in which the subjects belong to a reasonably homogeneous class, or can be partioned into a number of such classes by objective criteria. In the last case, the theory can be applied to each class separately. The estimated values of the parameters θ_τ and λ can be compared and analyzed, possibly revealing interesting differences between the classes.

When such a partition is not possible and the population of subjects is regarded as diverse, one could elaborate the theory beyond what has be done here, and axiomatize those individual differences. In principle, there are various ways to do this. The exposure to the tokens can be modelled by supposing that the intensity parameter λ of the Poisson process is actually a random variable with some specified distribution (e.g.gamma). This would complicate the application of the theory to some extent, but would only involve minor changes. For example, if the gamma distribution is assumed, two additional parameters are added, and Equation (19) becomes (for the semiorder model)

$$p_{\succ} p_{\succ,\succ'}(\delta) = \frac{\prod_{xy\in\succ}\theta_{xy}^2 \cdot \prod_{zw\in\sim}\theta_{zw}}{\sum_{\succ''\in\mathcal{S}}\prod_{xy\in\succ''}\theta_{xy}^2\cdot\prod_{zw\in\sim''}\theta_{zw}} \cdot \int_0^\infty \sum_{k=0}^{\infty} p^*_{\succ,\succ'}(k)\frac{(\lambda\delta)^k e^{-\lambda\delta}}{k!} g(\lambda)\,\mathrm{d}\lambda,$$

where g is the density of the intensity of the Poisson process.

As for the other criticism, one possible interpretation of the individual differences concerning the effect of the tokens would consist in assuming that the subjects come to the survey with different a priori conceptions concerning the alternatives. Formally, this leads to postulating the existence of some a priori (at time $t = 0$) distribution on the well graded family $\mathcal{S}$. From a mathematical viewpoint, this modification is a rather trivial one, since Axiom [I] also assumes such a distribution, but with a mass concentrated on the empty preference relation. Some reflection will convince the reader that, with this modification, the stochastic process has still the well graded family $\mathcal{S}$ as a unique ergodic set. (All the states communicate.) One could also suppose that a voter 'filters' the flow of tokens delivered by the environment according to some preconceptions. (The state of a voter insensitive to women's issues may be less likely to be affected by a positive token xy, in which x represents a woman and y a man.) The cases of voters totally impervious to the tokens and responding either idiosyncratically, or randomly, can be regarded as extreme forms of such filters. In general, the theory can be elaborated so as to give a formal status to such filters.

As suggested by the above examples, there are a various ways of incorporating individual differences within the theory, if the situation requires such developments. Needless to say, the elaborations of the theory evoked in this section were only meant as illustrations of some possibilities.

SUMMARY AND DISCUSSION

We have presented a theory purporting to explain how rationality could evolve from a naive state portrayed by the empty relation, to a sophisticated state represented by a semiorder or some other kind of order relation. This evolution was formalized by a stochastic process with three interlinked parts. One is a Poisson process governing the times $t_1, t_2, \ldots, t_n, \ldots$ of occurence of quantum events of information, called tokens, which are delivered by the medium. The second is a probability distribution on the collection of all possible tokens, which regulates the nature of the quantum event occuring at time t_n. Any token is formalized by some pair xy of distinct alternatives, bearing a positive or negative tag. The occurence of a positive token xy signals a quantum superiority of x over y, while the corresponding negative token $\widetilde{xy}$ indicates the absence of such a superiority. The last part of the stochastic process is a Markov process describing the changes of states occuring in the subject as a result of the occurence of particular tokens. The state space of the Maskov process is the set of all relations in a well graded family $\mathcal{S}$ of relations on the set X

of alternatives, assumed to contain the empty relation. Examples of well graded families include the partial orders, the semiorders, the biorders and the interval orders. The succession of states forms an irreducible Markov chain on $\mathcal{S}$. Asymptotic and sequential predictions were derived, which were applied in a special case involving the semiorders. The theory is capable of very strong predictions. For instance, on the basis of panel type data collected at time t_1 and t_2, an exact prediction can be made concerning the probability distribution on the set of preference relations at any later time t_3.

The axioms governing the transitions of the Markov process (which coincide with the changes of state of the subject), while apparently technical, were in fact inspired by simple and reasonable ideas. At first, when the preference relation is empty or lean, the subject's state is easily modified. Most tokens affect the state in a straightforward manner, by adding to or deleting from the current preference relation the pair corresponding to the token, thereby producing a new relation in the same well graded family and thus preserving rationality. These early tokens contribute to the subject's awareness of the alternatives, and influence the relative situations of these alternatives in the cognitive structure of the subject. Over time, however, an increasingly sophisticated state may be achieved which is endowed with a corresponding amount of rigidity. Few tokens are then capable of modifying the state. In fact, the current state can only be affected by tokens concerning pairs xy of alternatives which are in some sense contiguous with respect to the subject's current preference relation. Accordingly, the addition or the removal of such a pair would only involve a minimal change of the current relation. Reflecting on the basic mechanism of this theory, we see that it is partly exogenous, in that the tokens are delivered by the environment, and partly endogenous, in that the effect of the tokens depend upon the subject's state. This dual nature seems unavoidable.

An obvious alternative to the discrete theory described here would be a continuous analogue in which the effects on a subject of external events concerning the options would be formalized in terms of informative 'stuff' of variable magnitude. Such a direction can certainly be taken and may lead to workable predictions in the spirit of those obtained here. However, our experience with such dichotomies (quantity of 'stuff' vs volley of tokens) suggest that distinguishing between the two types of theorization on the basis of data may be elusive.

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