A Note on the Regularity of Center-Outward Distribution and Quantile Functions

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A note on the Regularity of Center-Outward Distribution and Quantile Functions

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Abstract

We provide sufficient conditions under which the center-outward distribution and quantile functions introduced in Chernozhukov et al. (2017) and Hallin (2017) are homeomorphisms, thereby extending a recent result by Figalli [17]. Our approach relies on Cafarelli's classical regularity theory for the solutions of the Monge-Ampère equation, but has to deal with difficulties related with the unboundedness at the origin of the density of the spherical uniform reference measure. Our conditions are satisfied by probabillities on Euclidean space with a general (bounded or unbounded) convex support which are not covered in [17]. We provide some additional results about center-outward distribution and quantile functions, including the fact that quantile sets exhibit some weak form of convexity.

Keywords: Optimal transportation; Monge-Ampère equation; multivariate ranks; quantile contours.

1 Introduction: center-outward distribution and quantile functions

Univariate distribution and quantiles functions, together with their empirical counterparts and the closely related concepts of ranks and order statistics, count among the most fundamental and useful tools in mathematical statistics. Ranks indeed are not just distribution-free: in models driven by noise with unspecified density, they generate the sub-σ-field of all distribution-free events (see [2]), which is also the largest sub-σ-field independent, irrespective of the underlying distribution, of the minimal sufficient σ-field generated by the order statistic; suitable rank-based procedures achieve optimality in several senses in nonparametric testing as well semiparametric efficiency (see, e.g. [20], [21], [22], [28]). A major limitation of the classical concepts of ranks and quantiles, however, is that, due to the absence of a canonical ordering of $\mathbb{R}^d$ for $d \geq 2$, they do not readily extend to the multivariate context.

The problem is not new, and numerous attempts have been made to fill that gap by defining multivariate versions of distribution and quantiles functions, with the ultimate goal of constructing suitable mutivariate versions of classical rank- and quantile-based inference procedures. The traditional definition of a multivariate distribution function is somewhat helpless in that respect, and does not produce any satisfactory concept of quantiles—let alone a satisfactory concept of ranks (see [18]). The componentwise approach, closely related with copula transforms, has been studied intensively (see [35]), but does not even enjoy distribution-freeness. Nor do the so-called *spatial ranks* ([33], [34]) inspired by the L_1 characterization of univariate quantiles. The whole theory of statistical depth (see [37], [38] for authoritative surveys), in a sense, is motivated by the same objective of providing a

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(data-driven) ordering of $\mathbb{R}^d$ and adequate concepts of multivariate ranks ([41]) and quantiles ([30]); here again, the resulting notions fail to be distribution-free. As for the *Mahalanobis ranks and signs* considered, e.g. in [25], [26] or [27], they do enjoy distribution-freeness and all the desired properties expected from ranks—under the restrictive assumption, however, of elliptical symmetry.

This shortcoming of all available solutions has motivated the introduction, in [8] and [23], of the measure transportation-based concepts of *Monge-Kantotovich depth*, *center-outward distribution* and *quantile functions, ranks,* and *signs*. These center-outward concepts, unlike all previous ones, are shown (see [23], [2]) to enjoy all the properties that make their univariate counterpart a fundamental and successful tool for statistical inference; we refer to [2] for more references and further discussion.

Let P be a Borel probability measure on the real line with finite second moment and continuous distribution function F and denote by $\mathrm{U}_{[0,1]}$ the uniform distribution on $(0,1)$: then F is a solution to *Monge's quadratic transportation problem*, that is,

$$\int_{\mathbb{R}} |x - F(x)|^2 d\mathrm{P}(x) = \min_{T\colon T\sharp \mathrm{P} = \mathrm{U}_{[0,1]}} \int_{\mathbb{R}} |x - T(x)|^2 d\mathrm{P}(x)$$

(see, e.g., [40]) where $T\sharp P$ denotes the *push forward of* P *by* T—namely, the distribution of $T(X)$ under $X \sim \mathrm{P}$ (T a measurable map from $\mathbb{R}$ to $(0,1)$). With generalization to higher dimension in mind, however, [8] and [23] rather consider $F_{\pm}(x) = 2F(x) - 1$, the so-called *center-outward distribution function* of P, satisfying the transportation problem

$$\int_{\mathbb{R}} |x - F_{\pm}(x)|^2 d\mathrm{P}(x) = \min_{T\colon T\sharp \mathrm{P} = \mathrm{U}_1} \int_{\mathbb{R}} |x - T(x)|^2 d\mathrm{P}(x),$$

where U_1 is the uniform distribution over $(-1,1)$, the one-dimensional unit ball $\mathbb{B}_1$. Clearly, $F_{\pm}$ and F carry the same information about P.

The latter definition, indeed, readily extends to arbitrary dimensions. Let P denote a Borel probability measure on $\mathbb{R}^d$ with finite second-order moments and Lebesgue density p. Measure transportation theory (see, e.g., Theorem 2.12 in [40]) tells us that there exists a P-a.s. unique map $\mathbf{F}_{\pm}$ such that

$$\int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{F}_{\pm}(\mathbf{x})|^2 d\mathrm{P}(\mathbf{x}) = \min_{\mathbf{T}\colon \mathbf{T}\sharp \mathrm{P} = \mathrm{U}_d} \int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{T}(\mathbf{x})|^2 d\mathrm{P}(\mathbf{x}) \tag{1.1}$$

where $|\mathbf{x}|$ stands for the Euclidean norm of $\mathbf{x}$ and U_d denotes the *uniform* distribution over the open d-dimensional unit ball $\mathbb{B}_d$. The center-outward distribution function is defined as a solution $\mathbf{F}_{\pm}$ of this optimal transportation problem.

By uniform over $\mathbb{B}_d$ we refer to *spherical* uniformity, that is, U_d here corresponds to the uniform choice of a direction on the unit sphere $\mathbb{S}_{d-1} := \bar{\mathbb{B}}_d - \mathbb{B}_d$ in $\mathbb{R}^d$ combined with an independent uniform choice in $(0,1)$ of a distance to the origin. A simple change of variable shows that U_d has density

$$u_d(\mathbf{x}) = \frac{1}{a_d |\mathbf{x}|^{d-1}} I\big[\mathbf{x} \in \mathbb{B}_d - \{\mathbf{0}\}\big] \tag{1.2}$$

where $a_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the area (the $(d-1)$-dimensional Hausdorff measure, see, e.g., [13]) of the sphere $\mathbb{S}_{d-1}$. Note the singularity at the origin since $\mathbf{x} \mapsto 1/|\mathbf{x}|^{d-1}$ is infinite at $\mathbf{x} = \mathbf{0}$; while we safely can neglect $\mathbf{0}$ itself, which has measure zero, by putting $u_d(\mathbf{0}) = 0$, u_d nevertheless remains unbounded in the vicinity of $\mathbf{0}$.

This definition of the center-outward distribution function as the solution of a quadratic transportation problem suffers from two major limitations. First, finite second-order moments are needed to make sense of the underlying optimization problem (1.1). Second, the distribution function $\mathbf{F}_{\pm}$ based on (1.1) is only defined P-a.s.; this means, for instance, that $\mathbf{F}_{\pm}$ is not well defined outside the support of P.

The first of these two limitations has been relaxed in [23] thanks to a celebrated theorem by McCann [31]. Under the assumption that P has finite second-order moments, Brenier in 1991 had shown that optimal transportation maps (hence, all versions of the P-a.s. unique solution $\mathbf{F}_{\pm}$ of

Monge's problem (1.1)) coincide P-a.s. with the Lebesgue-a.e. gradient $\nabla\varphi$ of a convex function[1] φ, which has the interpretation of a *potential*. More precisely, $\mathbf{F}_\pm$ a.s. is of the form $\nabla\varphi$ where φ *(i)* is lower semicontinous (lsc in the sequel), *(ii)* is convex, and *(iii)* is such that $\nabla\varphi\sharp\mathrm{P} = \mathrm{U}_d$. McCann [31] further showed that these last three conditions uniquely determine $\nabla\varphi$, even in the absence of second moment assumptions, while under finite second-order moments, $\nabla\varphi$ is a solution of Monge's problem (1.1). Thus, putting

$$\mathbf{F}_\pm(\mathbf{x}) := \nabla\varphi(\mathbf{x}) \quad \mathbf{x}\text{-a.e. in } \mathbb{R}^d, \tag{1.3}$$

the center-outward distribution function $\mathbf{F}_\pm$ is no longer characterized as the almost surely unique solution of an optimization problem (1.1) requiring finite moments of order two but as the unique a.e. gradient $\nabla\varphi$ of a convex function pushing P forward to U_d. We nevertheless conform to the common usage of improperly calling $\nabla\varphi$ the optimal transport pushing P forward to U_d.

While taking care of the moment assumption—existence of second-order moments indeed is an embarrassing assumption when distribution and quantile functions are to be defined—the second limitation still remains. The non-unicity of $\mathbf{F}_\pm := \nabla\varphi$, however, disappears if P is such that φ is everywhere differentiable. That this is indeed the case was shown by Figalli in 2018 [17] for P in the so-called class of distributions *with nonvanishing densities*[2]. For any P in that class of distributions, Figalli actually establishes that $\nabla\varphi(\mathbf{x})$ is a gradient for all $\mathbf{x}$ and, when restricted to

$$\mathbb{R}^d_{(\mathbf{0})} := \mathbb{R}^d \setminus \{\mathbf{x} : \nabla\varphi(\mathbf{x}) \neq \mathbf{0}\},$$

a homeomorphism between $\mathbb{R}^d_{(\mathbf{0})}$ and the punctured ball $\mathbb{B}_d\setminus\{\mathbf{0}\}$. The latter property is quite essential if sensible—namely, closed, continuous, connected, and nested—quantile regions and contours, based on an inverse[3] $\mathbf{Q}_\pm$ of $\mathbf{F}_\pm$, are to be defined: see [23] and [2].

The introduction by Hallin [23] of center-outward ranks and quantiles rapidly attracted the attention of the nonparametric community. It has triggered, among others, Faugeras and Rüschendorf ([14] and [15]) and de Valk and Segers [12]. Applications to the long-standing open problem of constructing distribution-free tests for the hypothesis of independence between vectors with unspecified densities have been proposed by Deb and Sen [11], Shi, Drton, and Han [39], and Ghosal and Sen [19]. Optimal center-outward R-estimators also have been derived (Hallin, La Vecchia, and Liu [24]) for VARMA models, while center-outward quantile-based methods for the measurement of multivariate risk are proposed in del Barrio, Beirlant, Buitendag, and Hallin [1].

The goal of this paper is to provide simple sufficient conditions for Figalli's results to hold beyond the assumption of nonvanishing densities; we more particularly consider distributions with (bounded or unbounded) convex supports. Beyond other theoretical considerations, these are the key properties required to prove a.s. convergence of the empirical center-outward distribution functions to their theoretical counterparts (see [2]). Hence, the results of the present paper also are extending the validity of the center-outward Glivenko-Cantelli theorem in that reference.

From a technical point of view, our main result is Theorem 2.5 below, which relies on the classical regularity theory for solutions of Monge-Ampère equations associated with the name of of Caffarelli (see [5, 6, 7]), as discussed in Section 2. The use of that theory to investigate the regularity of optimal transportation maps between two probabilities typically requires that both probabilities have densities that are bounded and bounded away from zero over their respective supports. Recently, under local versions of this condition, very general regularity results of this kind has been given in [10] and [19]. However, the spherical uniform reference measure U_d considered here, in dimension $d \geq 2$, yields unbounded densities at the origin, so that the results in [10] or [19] do not apply.[4] To

[1]The notation $\nabla\varphi$ here is used for the Lebesgue-a.e. gradient of φ, that is, $\nabla\varphi(\mathbf{x})$ is defined as the gradient at $\mathbf{x}$ of φ whenever φ is differentiable at $\mathbf{x}$—which, for a convex φ, holds Lebesgue-a.e. Note that, contrary to $\nabla\varphi$, which is a.e. unique, φ is not—unless we impose, without loss of generality (see, e.g., Lemma 2.1 in [3]), that $\varphi(\mathbf{0}) = 0$.

[2]Precisely, the distributions P with densities p and support $\mathcal{X} = \mathbb{R}^d$ satisfying Assumption A below.

[3]See Section 2.1 for a precise definition.

[4]Note that the choice of the spherical uniform reference is not a whimsical one. It preserves the independence between $\|\mathbf{F}_\pm\|$ and $\mathbf{F}_\pm/\|\mathbf{F}_\pm\|$ (extending the independence, for $d = 1$, between $|F_\pm|$ and $\mathrm{sign}(F_\pm)$) and produces simple and easily interpretable quantile contours with prescribed probability content (we refer to [23] for details).

our knowledge, the only reference dealing with this kind of unbounded density is [17] which, however, requires P to be supported on the whole space $\mathbb{R}^d$. Here we extend the result in [17] to cover the case of P with (bounded or unbounded) convex supports.

The sequel of this paper is organized as follows. Our main regularity result is established in Section 2, along with a succint account of the main elements of Cafarelli's theory and some auxiliary results. We conclude with Section 3, which presents some new results on center-outward distribution and quantile functions. These include an asymptotic invariance property extending a well-known feature of classical univariate distribution functions and the ability of quantile contours to capture the shape of the bounded support of a probability measure by convergencing (in Hausdorff distance) to the boundary of the support. Finally, we include a result on the geometry of quantile sets, showing that they turn out to exhibit a limiting form of "lighthouse convexity".

2 Regularity of center-outward distribution and quantile functions

2.1 Center-outward quantile functions

The Introduction was focused on the distribution functions $\mathbf{F}_\pm$. Exchanging the roles of P and U_d, we could have emphasized transportation from the unit ball to the support of P, leading to the definition of the center-outward quantile function $\mathbf{Q}_\pm$ with, *mutatis mutandis*, the same comments.

Let P denote a Borel probability measure over $\mathbb{R}^d$ with Lebesgue density p. While the center-outward distribution function is defined as the optimal (in the McCann sense) transport pushing P forward to U_d, the center-outward quantile map or quantile function $\mathbf{Q}_\pm$ of P is defined as the optimal transport pushing U_d forward to P. Namely,

$$\mathbf{Q}_\pm(\mathbf{u}) := \nabla\psi(\mathbf{u}) \quad \mathbf{u}\text{-a.e. in } \mathbb{B}_d \tag{2.1}$$

where $\nabla\psi$ is, in agreement with McCann's Theorem, the unique a.e. gradient of a convex function ψ with domain containing $\mathbb{B}_d$[5] such that $\nabla\psi\sharp\mathrm{U}_d = \mathrm{P}$. Again, imposing, without loss of generality,[6] that $\psi(\mathbf{0}) = 0$, the convex potential ψ is uniquely defined and a.e. differentiable over $\mathbb{B}_d$. We extend ψ to a lsc convex function on $\mathbb{R}^d$ with the standard procedure of setting $\psi(\mathbf{u}) := \liminf_{\mathbf{z}\to\mathbf{u},|\psi(\mathbf{z})|<1}\psi(\mathbf{z})$ if $|\mathbf{u}| = 1$ and $\psi(\mathbf{u}) := +\infty$ for $\mathbf{u} \notin \bar{\mathbb{B}}_d$ (see, e.g. (A.18) in [16]). With this extension, φ is the Legendre transform of ψ, that is,

$$\varphi(\mathbf{x}) = \psi^*(\mathbf{x}) := \sup_{\mathbf{u}\in\mathbb{B}_d}(\langle\mathbf{u},\mathbf{x}\rangle - \psi(\mathbf{u})), \quad \mathbf{x}\in\mathbb{R}^d. \tag{2.2}$$

We observe that the domain of φ is $\mathbb{R}^d$ and that φ, being the sup of a 1-Lipschitz function, is also 1-Lipschitz. In particular, for almost every $\mathbf{x}\in\mathbb{R}^d$, φ is differentiable with $|\nabla\varphi(\mathbf{x})| \le 1$ and, as a consequence (see, e.g., Corollary A.27 in [16]),

$$\partial\varphi(\mathbb{R}^d) \subseteq \bar{\mathbb{B}}_d; \tag{2.3}$$

here, and throughout this paper, $\bar{B}$ stands for the closure of a set B, $\partial\varphi(\mathbf{x})$ for the subdifferential[7] of the convex function φ at $\mathbf{x}$, and $\partial\varphi(A) := \bigcup_{\mathbf{x}\in A}\partial\varphi(\mathbf{x})$. Furthermore, Proposition 10 in [31] (see also Remark 16) shows that, since P has a density, $\nabla\psi(\nabla\varphi(\mathbf{x})) = \mathbf{x}$ for almost every $\mathbf{x}$ in the support of P and $\nabla\varphi(\nabla\psi(\mathbf{y})) = \mathbf{y}$ for almost every $\mathbf{y}\in\mathbb{B}_d$. In that sense, $\mathbf{Q}_\pm$ and $\mathbf{F}_\pm$ are the inverse of each other. In this way, we have defined $\mathbf{F}_\pm(\mathbf{x})$ for almost every $\mathbf{x}\in\mathbb{R}^d$ and $\mathbf{Q}_\pm(\mathbf{u})$ for almost every $\mathbf{u}\in\mathbb{B}_d$; the definitions coincide with those in [8] or [23] for $\mathbf{x}$ in the support of P.

[5]We adhere to the usual convention of considering that a function defined on $A\subset\mathbb{R}^d$ is convex if it can be extended to a convex function on $\mathbb{R}^d$ with values in $\mathbb{R}\cup\{\infty\}$; the domain of the convex function is then redefined as the set where it takes finite values.

[6]Indeed, two convex functions with a.e. equal gradients on an open convex set are equal up to an additive constant (see, e.g., Lemma 2.1 in [3]).

[7]Recall that the subdifferential of φ at $\mathbf{x}$ is the set of all $\mathbf{z}\in\mathbb{R}^d$ such that $\varphi(\mathbf{y})-\varphi(\mathbf{x}) \ge \langle\mathbf{z},\mathbf{y}-\mathbf{x}\rangle$ for all $\mathbf{y}$.

2.2 Some regularity results for Monge-Ampère equations

As announced in the Introduction, our approach to the regularity of the center-outward distribution and quantile functions is based on the classical regularity theory for Monge-Ampère equations. We refer to [16] for a comprehensive account of this theory, of which we present here a minimal account.

Given an open set $\mathcal{X} \subseteq \mathbb{R}^d$ and a (finite) convex function $\varphi : \mathcal{X} \to \mathbb{R}$, denoting by ℓ_d he Lebesgue measure on $\mathbb{R}^d$, the Monge-Ampère measure associated with φ is defined by

$$\mu_\varphi(E) := \ell_d\Big(\partial\varphi(E)\Big)$$

for every Borel set $E \subseteq \mathbb{R}^d$. It can be checked that μ_φ is indeed a locally finite Borel measure on $\mathcal{X}$. The crucial link between center-outward distribution functions and Monge-Ampère measures can be summarized as follows. Assume P is a probability on $\mathcal{X}$ with Lebesgue density p and let φ be a convex function from $\mathcal{X}$ to $\mathbb{R}$. Then, for every Borel set A,

$$\mathrm{Q}(A) := (\nabla\varphi\sharp\mathrm{P})(A) = \mathrm{P}\big(\partial\varphi^*(A)\big)$$

where φ^* is the Legendre transform of φ. We recall that convexity of φ implies that it is differentiable at almost every point in $\mathcal{X}$ (see, e.g., Theorem 25.4 in [36]) and, therefore,

$$(\nabla\varphi\sharp\mathrm{P})(A) = P(\{\mathbf{x} : \nabla\varphi(\mathbf{x}) \in A\}) = P(\{\mathbf{x} : \partial\varphi(\mathbf{x}) \subseteq A\}).$$

This and the fact that $\mathbf{y} \in \partial\varphi(\mathbf{x})$ if and only if $\mathbf{x} \in \partial\varphi^*(\mathbf{y})$ yield the last equality above. Hence, if Q has a density q, for every Borel set A,

$$\int_{\partial\varphi(A)} q(\mathbf{y})d\mathbf{y} = \int_A p(\mathbf{x})d\mathbf{x}$$

(see Lemma 4.6 in [40]); if, moreover, $\mathrm{Q} = \mathrm{U}_d$,

$$\int_{\partial\varphi(A)} u_d(\mathbf{y})d\mathbf{y} = \int_A p(\mathbf{x})d\mathbf{x}. \tag{2.4}$$

Observing that

$$\mu_\varphi(A) = \ell_d(\partial\varphi(A)) = \ell_d(\partial\varphi(A) \cap \mathbb{B}_d),$$

where the second equality follows from (2.3), we obtain from (2.4) that, for A such that $\ell_d(A) = 0$,

$$\mu_\varphi(A) \le a_d \int_{\partial\varphi(A)} u_d(\mathbf{y})d\mathbf{y} = a_d \int_A p(\mathbf{x})d\mathbf{x} = 0$$

with a_d as in (1.2). Thus, the Monge-Ampère measure μ_φ is Lebesgue-absolutely continuous. Since the density of the absolutely continuous part of the Monge-Ampère measure μ_φ is given by $\big(p(\mathbf{x})/u_d(\nabla\varphi(\mathbf{x}))\big)$ (see McCann [32] or Theorem 4.8 in [40]), we conclude that, for every Borel set $A \subseteq \mathbb{R}^d$,

$$\mu_\varphi(A) = \int_A \frac{p(\mathbf{x})}{u_d(\nabla\varphi(\mathbf{x}))}d\mathbf{x} = a_d \int_A p(\mathbf{x})|\nabla\varphi(\mathbf{x})|^{d-1}d\mathbf{x}. \tag{2.5}$$

Let us focus now on the Monge-Ampère measure μ_ψ associated (see Section 2.1) with $\mathbf{Q}_\pm$ and ψ (both defined over $\mathbb{B}_d$). Since $\nabla\psi$ pushes U_d forward to P, we have that $\nabla\psi(\mathbf{y}) \in \mathcal{X}$ $\mathbf{y}$-a.e. in $\mathbb{B}_d$. By continuity (see Theorem 25.5 in [36]), $\nabla\psi(\mathbf{y}) \in \bar{\mathcal{X}}$ for every point $\mathbf{y}$ of differentiability of ψ. Using again Corollary A.27 in [16], we conclude that $\partial\psi(\mathbb{B}_d)$ is included in the convex hull $\overline{\mathrm{conv}(\mathcal{X})}$ of $\mathcal{X}$. Hence, if $\mathcal{X}$ itself is convex, we obtain that

$$\partial\psi(\mathbb{B}_d) \subseteq \overline{\mathcal{X}}. \tag{2.6}$$

Analogous to (2.4), we have that

$$\int_{\partial\psi(B)} p(\mathbf{x})d\mathbf{x} = \int_B u_d(\mathbf{y})d\mathbf{y} \quad \text{for every Borel set } B \subseteq \mathbb{R}^d. \tag{2.7}$$

Now, denoting by $r\,\mathbb{B}_d$ the open ball with radius r centered at the origin, let us assume that the Borel set $B \subseteq r\,\mathbb{B}_d$, with $0 < r < 1$, has Lebesgue measure zero. Since $\bar{B} \subseteq r\,\mathbb{B}_d$ is compact, $\partial\psi(\bar{B})$ also is compact (see, e.g. Lemma A.22 in [16]). Hence, there exists $R > 0$ such that

$$\partial\psi(B) \subseteq \partial\psi(\bar{B}) \subseteq R\,\mathbb{B}_d.$$

The following assumption, which requires the density p of P to be bounded and bounded away from 0 on compact subsets of the support, is absolutely essential (the same assumption is also made by Figalli in [17]).

Assumption A. For every $R > 0$, there exist constants $0 < \lambda_R \leq \Lambda_R$ such that

$$\lambda_R \leq p(\mathbf{x}) \leq \Lambda_R \qquad \text{for all } \mathbf{x} \in \mathcal{X} \cap R\,\mathbb{B}_d. \tag{2.8}$$

Since $\mathcal{X}$ is convex (hence $\ell_d(\bar{\mathcal{X}} - \mathcal{X}) = 0$), Assumption A entails

$$\mu_\psi(B) \leq \frac{1}{\lambda_R}\int_{\partial\psi(B)} p(\mathbf{x})d\mathbf{x} = \frac{1}{\lambda_R}\int_B u_d(\mathbf{y})d\mathbf{y} = 0.$$

Assuming convexity of $\mathcal{X}$ and (2.8), we conclude that μ_ψ is absolutely continuous with respect to ℓ_d and, using Theorem 4.8 in [40] again, that, for every Borel set $B \subset \mathbb{B}_d$,

$$\mu_\psi(B) = \int_B \frac{u_d(\mathbf{y})}{p(\nabla\psi(\mathbf{y}))}d\mathbf{y} = \frac{1}{a_d}\int_B \frac{1}{p(\nabla\psi(\mathbf{y}))|\mathbf{y}|^{d-1}}d\mathbf{y}. \tag{2.9}$$

We summarize this discussion in the next proposition.

Proposition 2.1. *Let* P *be a probability measure with density p supported on the open set $\mathcal{X} \subseteq \mathbb{R}^d$. Denote by $\psi : \mathbb{B}_d \to \mathbb{R}$ the convex, lower semicontinuous function satisfying $\psi(\mathbf{0}) = 0$ and $\nabla\psi\sharp\mathrm{U}_d = \mathrm{P}$ and let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be defined as in* (2.2)*. Then,*

(i) μ_φ is absolutely continuous with respect to ℓ_d and, for every Borel $A \subseteq \mathbb{R}^d$,

$$\mu_\varphi(A) = a_d \int_A p(\mathbf{x})|\nabla\varphi(\mathbf{x})|^{d-1}d\mathbf{x};$$

(ii) if, moreover, $\mathcal{X}$ is convex and p satisfies Assumption A, then μ_ψ is absolutely continuous with respect to ℓ_d and, for every Borel set $B \subseteq \mathbb{B}_d$,

$$\mu_\psi(B) = \frac{1}{a_d}\int_B \frac{1}{p(\nabla\psi(\mathbf{y}))|\mathbf{y}|^{d-1}}d\mathbf{y}.$$

Next, let us show that, for well-behaved probability measures P (those with convex support and density p satisfying Assumption A), the center-outward distribution function $\mathbf{F}_\pm$ cannot map points in the interior of the support of P to extremal points of the unit ball.

Lemma 2.2. *Let* P *be a probability measure with density p supported on the convex open set $\mathcal{X} \subseteq \mathbb{R}^d$ and such that Assumption A holds. Then $(\partial\varphi)(\mathcal{X}) \cap \mathbb{S}_{d-1} = \emptyset$, where $\mathbb{S}_{d-1} = \bar{\mathbb{B}}_d \setminus \mathbb{B}_d$.*

Proof. Assume that there exists $x \in \mathcal{X}$ such that $|\mathbf{y}| = 1$ for some $\mathbf{y} \in \partial\varphi(\mathbf{x})$. Without loss of generality, we can assume $\mathbf{x} = \mathbf{0}$. Since $\mathcal{X}$ is open, there exists $\epsilon > 0$ such that $\epsilon\overline{\mathbb{B}}_d \subset \mathcal{X}$. For small $\theta > 0$, consider the sets

$$\mathcal{C}_{\epsilon,\vartheta} := \left\{\mathbf{x} \in \mathbb{R}^d : \left|\frac{\mathbf{x}}{|\mathbf{x}|} - \mathbf{y}\right| \leq \sin\vartheta, |\mathbf{x}| \leq \epsilon\right\}$$
$$\mathcal{D}_\vartheta := \{\mathbf{b} \in \mathbb{B}_d : \langle \mathbf{y} - \mathbf{b}, \mathbf{y}\rangle \leq 2\vartheta|\mathbf{y} - \mathbf{b}|\}.$$

Now, if $\mathbf{a} \in \mathcal{C}_{\epsilon,\vartheta}$ and $\mathbf{b} \in \partial\varphi(\mathbf{a})$, the monotonicity of $\partial\varphi$ implies that $\langle \mathbf{y} - \mathbf{b}, \mathbf{a}\rangle \leq 0$. Hence,

$$\langle \mathbf{y} - \mathbf{b}, \mathbf{y}\rangle = \langle \mathbf{y} - \mathbf{b}, \mathbf{y} - \frac{\mathbf{a}}{|\mathbf{a}|}\rangle + \langle \mathbf{y} - \mathbf{b}, \frac{\mathbf{a}}{|\mathbf{a}|}\rangle \leq |\mathbf{y} - \mathbf{b}|\sin(\vartheta) \leq |\mathbf{y} - \mathbf{b}|2\vartheta.$$

This shows that $\partial\varphi(\mathcal{C}_{\epsilon,\vartheta}) \subseteq \mathcal{D}_\vartheta$. But the density p, inside $\mathcal{C}_{\epsilon,\vartheta}$, is bounded from below by λ_ϵ and the density u_d is bounded from above by $2/a_d$ inside $\mathcal{D}_\vartheta$ for $\vartheta \ll 1$: then, in view of the transport equation (2.4), we have

$$\frac{2}{a_d}\ell_d\big(\mathcal{D}_\vartheta\big) \geq \int_{\mathcal{D}_\vartheta} u_d(\mathbf{b})d\mathbf{b} \geq \int_{\partial\varphi(\mathcal{C}_{\epsilon,\vartheta})} u_d(\mathbf{b})d\mathbf{b} = \int_{\mathcal{C}_{\epsilon,\vartheta}} p(\mathbf{x})d\mathbf{x} \geq \lambda_\epsilon\ell_d\big(\mathcal{C}_{\epsilon,\vartheta}\big).$$

This, however, cannot hold true since $\ell_d\big(\mathcal{C}_{\epsilon,\vartheta}\big) \approx \epsilon^d\vartheta^{d-1}$ and $\ell_d\big(\mathcal{D}_\vartheta\big) \approx \vartheta^{d+1}$ as $\theta \to 0$. The claim follows. $\square$

We now proceed to provide sufficient conditions under which the center-outward quantile function $\mathbf{Q}_\pm$ is continuous at every point in the open unit ball (except, possibly, at the origin). It is well known that differentiability of a lower semicontinuous convex function ψ (which entails continuity of its gradient) is equivalent to strict convexity of its convex conjugate (see Theorem 26.3 in [36]). As announced, the techniques we are using here are in the spirit of those developed by Caffarelli in [5], [6] or Figalli in [16], [17], which in turn largely rely on the fact that, under some control for the Monge-Ampère measure, the intersection between the graph and supporting hyperplanes of ψ either consists of a single point or has an extreme point (see Theorem 4.10 in [16]). A central result in Caffarelli's regularity theory (see Corollary 4.21 in [16]) is that a strictly convex function ψ on an open set Ω for which there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda\ell_d(A) \leq \mu_\psi(A) \leq \Lambda\ell_d(A) \tag{2.10}$$

for every Borel set $A \subseteq \Omega$ is automatically of class $\mathcal{C}^{1,\alpha}_{\text{loc}}$ for some $\alpha > 0$ that depends only on λ, Λ, and d (condition (2.10) in the sequel will be summarized, with a slight abuse of notation, as $\lambda d\mathbf{x} \leq \mu_\psi \leq \Lambda d\mathbf{x}$). The fact that U_d for $d \geq 2$ has an unbounded density adds some complication to the particular problem here, though. On the other hand, the density u_d is bounded away from 0, which allows to control the growth of the Monge-Ampère measure, as we show next.

Lemma 2.3. *If* P *satisfies the assumptions in Proposition 2.1(ii), denoting by* M *a compact subset of* $\mathbb{B}_d$*, there exist constants* α_M *and* A_M *such that, for every Borel set* $A \subseteq M$*,*

$$\alpha_M\ell_d(A) \leq \mu_\psi(A) \leq A_M(\ell_d(A))^{1/d}. \tag{2.11}$$

Proof. The compactness of M entails that of $\partial\psi(M)$; in particular, $\partial\psi(M) \subseteq R\,\mathbb{B}_d$ for some $R > 0$. Hence, using Proposition 2.1*(ii)* and taking $\lambda_R, \Lambda_R \in \mathbb{R}$ as in Assumption A, we obtain

$$\mu_\psi(A) = a_d\int_A \frac{1}{p(\nabla\psi(\mathbf{y}))|\mathbf{y}|^{d-1}}d\mathbf{y} \geq \frac{a_d}{\Lambda_R}\ell_d(A).$$

For the upper bound in (2.11), note that the ball $(\ell_d(A)/c_d)^{1/d}B_d$ (where $c_d = \pi^{d/2}/\Gamma(1 + d/2)$ denotes the volume of the d-dimensional unit ball) maximizes $\int_B |\mathbf{y}|^{1-d}d\mathbf{y}$ among all subsets of $\mathbb{B}_d$

with Lebesgue measure $\ell_d(A)$. On the other hand, by the co-area formula (see, e.g., Proposition 1, p. 118 in [13]),

$$\int_{r\mathbb{B}_d} |\mathbf{y}|^{1-d} d\mathbf{y} = \int_0^r \left[\int_{\partial s\mathbb{B}_d} |\mathbf{y}|^{1-d} d\mathcal{H}^{d-1}(\mathbf{y}) \right] ds = \int_0^r a_d \, ds = a_d \, r \tag{2.12}$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. Combining (2.12) with Proposition 2.1*(ii)*, we conclude that

$$\mu_\psi(A) \leq \frac{1}{\lambda_R \, a_d} \int_A |\mathbf{y}|^{1-d} d\mathbf{y} \leq \frac{1}{\lambda_R \, a_d^{1/d}} (\ell_d(A))^{1/d}. \qquad \square$$

Note that the lower bound in Lemma 2.3 remains valid for a compact subset M of $\bar{\mathbb{B}}_d$ provided that $\partial\psi(M)$ is bounded: indeed, that lower bound only requires the upper bound from Assumption A. A similar conclusion holds for the upper bound. Additionally, if the density p is uniformly bounded, the lower bound holds for any subset of $\bar{\mathbb{B}}_d$.

2.3 Main result

We are ready now for the main result of this note. Our proof follows the lines of [16], [17], and [10], and the related Proposition 3.3 in [19], but, unlike [17], we cover cases in which the support of the probabability P is is not the whole space $\mathbb{R}^d$. Similar to [10] and [19], we have to handle carefully the fact that $\mathcal{X}$ is not necessarily bounded and use a "minimal" extension of the quantile function potential, namely,

$$\tilde{\psi}(\mathbf{z}) := \sup_{\mathbf{b} \in \mathbb{B}_d, \; \mathbf{y} \in \partial\psi(\mathbf{b})} \{\langle \mathbf{y}, \mathbf{z} - \mathbf{b} \rangle + \psi(\mathbf{b})\}, \quad \mathbf{z} \in \mathbb{R}^d. \tag{2.13}$$

Obviously, $\tilde{\psi}$ is still a lower semicontinuous convex function and $\tilde{\psi}(\mathbf{z})$ coincides with $\psi(\mathbf{z})$ for $\mathbf{z} \in \mathbb{B}_d$. Since $\mathbf{Q}_\pm(\mathbf{z}) := \nabla\psi(\mathbf{z}) \in \mathcal{X}$ for every differentiability point $\mathbf{z}$ of ψ in $\mathbb{B}_d$, we see (using, once more, Corollary A.27 in [16]) that, provided that $\mathcal{X}$ is convex, $\partial\psi(\mathbb{B}_d) \subseteq \bar{\mathcal{X}}$. The "minimality" of the extension (2.13) refers to the fact that $\partial\tilde{\psi}(\mathbb{R}^d) \subset \bar{\mathcal{X}}$, as can be checked from a simple application of the Hahn-Banach separation theorem. Of course, the values of $\tilde{\psi}$ outside $\mathbb{B}_d$ are not relevant for the study of its differentiability inside $\mathbb{B}_d$, but the use of $\tilde{\psi}$ will be useful in the next proof. We note also that the discussion leading to Proposition 2.1 can be reproduced with $\tilde{\psi}$ substituted for ψ to conclude that $\mu_{\tilde{\psi}}$ is absolutely continuous with respect to the Lebesgue measure and that, for every Borel set $B \subseteq \mathbb{R}^d$,

$$\mu_{\tilde{\psi}}(B) = \int_{B \cap \mathcal{X}} \frac{u_d(\mathbf{y})}{p(\nabla\tilde{\psi}(\mathbf{y}))} d\mathbf{y}. \tag{2.14}$$

Finally, observe that $\mu_{\tilde{\psi}}$ in concentrated on $\mathbb{B}_d$, that is, if $B \subseteq \mathbb{R}^d \setminus \mathbb{B}_d$, then $\mu_{\tilde{\psi}}(B) = 0$, see Theorem 4.8 in [40] or [10] for further details.

The main result of this note follows from the following crucial lemma.

Lemma 2.4. *Under the assumptions of Theorem 2.5, $\tilde{\psi}$ is strictly convex on $\mathbb{B}_d$.*

Proof. To prove this, assume that the contrary holds true. Then, there exists $\mathbf{y} \in \mathbb{B}_d$ and $\mathbf{t} \in \partial\tilde{\psi}(\mathbf{y})$ such that, putting $l(\mathbf{z}) := \tilde{\psi}(\mathbf{y}) + \langle \mathbf{t}, \mathbf{z} - \mathbf{y} \rangle$, the convex set $\Sigma := \{\mathbf{z} : \tilde{\psi}(\mathbf{z}) = l(\mathbf{z})\}$ is not a singleton. By subtracting an affine function, we can assume $\tilde{\psi}(\mathbf{y}) = 0$ and $\tilde{\psi}(\mathbf{z}) \geq 0$ for all $\mathbf{z}$; then, $\Sigma = \{\mathbf{z} : \tilde{\psi}(\mathbf{z}) = 0\} = \{\mathbf{z} : \tilde{\psi}(\mathbf{z}) \leq 0\}$, which is closed since $\tilde{\psi}$ is lower semicontinuous. Also, by adding the convex function $w(\mathbf{z}) := \frac{1}{2}(|\mathbf{z}| - 1)_+^2$ (note that $\tilde{\psi} = \tilde{\psi} + w$ on $\bar{\mathbb{B}}_d$), we can assume that $\Sigma \subset \bar{\mathbb{B}}_d$. Being compact and convex, Σ equals the closed convex hull of its extreme points; as a consequence, it must have at least two exposed points (otherwise it would be empty or a singleton). Let $\bar{\mathbf{y}} \in \bar{\mathbb{B}}_d \setminus \{\mathbf{0}\}$ be one of them. If $\bar{\mathbf{y}} \in \mathbb{B}_d \setminus \{\mathbf{0}\}$, we consider a small ball $C_{\bar{\mathbf{y}}}$, say, around $\bar{\mathbf{y}}$, such that $\bar{C}_{\bar{\mathbf{y}}} \subset \mathbb{B}_d \setminus \{\mathbf{0}\}$. Then $\partial\tilde{\psi}(\bar{C}_{\bar{\mathbf{y}}})$ is a compact set, and hence $\partial\tilde{\psi}(\bar{C}_{\bar{\mathbf{y}}}) \subset R\,\mathbb{B}_d$ for some $R > 0$. By Proposition 2.1*(ii)*, we have constants $0 < \lambda_{C_{\bar{\mathbf{y}}}} \leq \Lambda_{C_{\bar{\mathbf{y}}}}$ such that the Monge-Ampère measure $\mu_{\tilde{\psi}}$ satisfies $\lambda_{C_{\bar{\mathbf{y}}}} d\mathbf{x} \leq \mu_{\tilde{\psi}} \leq \Lambda_{C_{\bar{\mathbf{y}}}} d\mathbf{x}$ in $C_{\bar{\mathbf{y}}}$. But the set Σ has an exposed point in $C_{\bar{\mathbf{y}}}$ and this contradicts

Theorem 4.10 in [16]. Consequently, we must assume that $\bar{\mathbf{y}} \in \partial\mathbb{B}_d$. Observe that $\tilde{\psi}(\bar{\mathbf{y}}) = 0$, hence $\bar{\mathbf{y}} \in \mathrm{dom}(\tilde{\psi})$. First consider the case where $\bar{\mathbf{y}} \notin \partial\big(\mathrm{dom}(\tilde{\psi})\big)$. Let $\mathbb{B}_r(\mathbf{x}) := \mathbf{x} + r\mathbb{B}_d$ and $\bar{\mathbb{B}}_r(\mathbf{x})$ denote, respectively the open and the closed ball of radius r centered at $\mathbf{x}$. Then, for $\eta > 0$ small enough, $\bar{\mathbb{B}}_\eta(\mathbf{y}) \subset \mathrm{dom}(\tilde{\psi})$; consequently, there exists some R_0 such that $\partial\tilde{\psi}\big(\bar{\mathbb{B}}_\eta(\bar{\mathbf{y}})\big) \subset R_0\,\mathbb{B}_d$. For η small enough, we further can ensure that $\bar{\mathbb{B}}_\eta(\bar{\mathbf{y}}) \subset 2\mathbb{B}_d$.

Without any loss of generality, let us assume that $\bar{\mathbf{y}} = \mathbf{e}_1$ where $\mathbf{e}_1$ stands for the first vector in the canonical basis of $\mathbb{R}^d$ (we can use a rotation otherwise):

$$\Sigma \subset \big\{\mathbf{z} = (z_1, \ldots, z_d)' \in \mathbb{R}^d :\ z_1 \leq 1\big\}, \quad \text{and} \quad \Sigma \cap \big\{\mathbf{z} = (z_1, \ldots, z_d)' \in \mathbb{R}^d :\ z_1 = 1\big\} = \{\mathbf{e}_1\}.$$

For $\sigma \in (0, 1)$ small enough, we have

$$\Sigma \cap \big\{\mathbf{z} \in \mathbb{R}^d :\ z_1 \geq 1 - \sigma\big\} \subset \mathbb{B}_d \cap \big\{\mathbf{z} \in \mathbb{R}^d :\ z_1 \geq 1 - \sigma\big\} \subset \mathbb{B}_\eta(\mathbf{e}_1).$$

For such σ, defining

$$\psi_\epsilon(\mathbf{z}) := \tilde{\psi}(\mathbf{z}) - \epsilon(z_1 - 1 + \sigma) \quad \text{and} \quad S_\epsilon := \{\mathbf{z} : \psi_\epsilon(\mathbf{z}) < 0\}, \tag{2.15}$$

observe that

$$S_\epsilon \longrightarrow \Sigma \cap \big\{\mathbf{z} \in \mathbb{R}^d :\ z_1 \geq 1 - \sigma\big\} \tag{2.16}$$

in the Hausdorff distance[8] d as $\epsilon \to 0$. Hence, for $\epsilon > 0$ small enough, the sets S_ϵ are bounded open convex subsets of the ball $\mathbb{B}_\eta(\mathbf{e}_1)$. By Lemma 2.3, there exists some $M > 0$ such that

$$\mu_{\psi_\epsilon}(A) = \mu_{\tilde{\psi}}(A) \leq M(\ell_d(A))^{1/d}$$

for every $A \subset S_\epsilon$ and ϵ small enough.

Next, fix $\mathbf{z}_0 \in \mathbb{B}_d \cap \Sigma$ and $\delta > 0$ such that $\bar{\mathbb{B}}_\delta(\mathbf{z}_0) \subset \mathbb{B}_d \cap \mathbb{B}_\eta(\mathbf{e}_1)$ and consider the normalizing map L_ϵ—namely, the affine transformation L_ϵ that normalizes[9] S_ϵ; denote by v_ϵ the normalized solution in $S^L_\epsilon := L_\epsilon(S_\epsilon)$ of $\mu_{v_\epsilon} = f \circ L_\epsilon^{-1}$ with the boundary condition $v_\epsilon = 0$ on ∂S^L_ϵ (v_ϵ is the convex map that has Monge-Ampère measure $d\mu_{v_\epsilon}(\mathbf{x}) = f \circ L_\epsilon^{-1}(\mathbf{x})d\mathbf{x}$ in S^L_ϵ and vanishes at the boundary of S^L_ϵ; its existence and uniqueness is guaranteed, for instance, by Proposition 4.2 in [16]). Since $\mathbb{B}_d \subset S^L_\epsilon$, we have that $L_\epsilon^{-1}(\mathbb{B}_d) \subset 2\mathbb{B}_d$ and, therefore, the map L^{-1} satisfies

$$|L_\epsilon(\mathbf{x}) - L_\epsilon(\mathbf{z})| \geq \frac{1}{2}|\mathbf{x} - \mathbf{z}| \quad \text{for all} \quad \mathbf{x}, \mathbf{z} \in \mathbb{R}^d.$$

This implies that

$$L_\epsilon(\mathbb{B}_d) \supset L_\epsilon\left(B_\delta\left(\mathbf{z}_0\right)\right) \supset B_{\delta/2}\left(L_\epsilon(\mathbf{z}_0)\right). \tag{2.17}$$

We consider the sets $S^L_{\epsilon,\delta} := \big\{\mathbf{z} \in S^L_\epsilon : \mathrm{d}_H\left(\mathbf{z}, \partial S^L_\epsilon\right) \geq \delta/4\big\}$. Now $L_\epsilon(\mathbf{z}_0) \in S^L_\epsilon$, a normalized set (it contains the unit ball and is contained in the ball of radius d, the dimension of the Euclidean space). This implies that there exists a constant $k_d > 0$, depending only on d such that (see Theorem 4.23 in [16] or Lemma 3 in [7])

$$\ell_d\big(S^L_{\epsilon,\delta} \cap \mathbb{B}_{\delta/2}(L_\epsilon(\mathbf{z}_0))\big) \geq k_d\delta^d.$$

In view of Lemma 2.3, the subsequent remark, and the fact that $\bar{\mathbb{B}}_\delta(\mathbf{z}_0) \subset \mathbb{B}_\eta(\mathbf{e}_1)$, we have that μ_{ψ_ϵ} is lower bounded over $\bar{\mathbb{B}}_\delta(\mathbf{z}_0)$, that is, there exists $\lambda > 0$ such that $\mu_{\psi_\epsilon}(A) \geq \lambda\ell_d(A)$ for every $A \subseteq \bar{\mathbb{B}}_\delta(\mathbf{z}_0)$. This and (2.17) thus imply that μ_{v_ϵ} is bounded from below on $\mathbb{B}_{\delta/2}(L_\epsilon(\mathbf{z}_0))$. It follows that, for some $\lambda > 0$,

$$\mu_{v_\epsilon}\left(S^L_{\epsilon,\delta}\right) \geq \lambda\ell_d\Big(S^L_{\epsilon,\delta} \cap \mathbb{B}_{\delta/2}\left(L_\epsilon(\mathbf{z}_0)\right)\Big) \geq C\delta^d.$$

[8]Recall that, for $A, B \subseteq \mathbb{R}^d$, $\mathrm{d}_H(A, B) := \max\big\{\sup_{\mathbf{a}\in A}\inf_{\mathbf{b}\in B}|\mathbf{a} - \mathbf{b}|, \sup_{\mathbf{b}\in B}\inf_{\mathbf{a}\in A}|\mathbf{a} - \mathbf{b}|\big\}$.

[9]A convex set $\Omega \subset \mathbb{R}^d$ is said to be normalized if $\mathbb{B}_d \subseteq \Omega \subseteq d\,\mathbb{B}_d$. For each open bounded convex set Ω there exists a unique invertible affine transformation L normalizing Ω (this is John's celebrated Lemma of convex analysis, see Lemma A.13 in [16]). We refer to L as the *normalizing map* and to $L(\Omega)$ as the *normalized version* of Ω.

This implies that, for c' small enough, no ball of radius $c'\delta/2$ can contain $\partial v_\epsilon\big(S^L_{\epsilon,\delta}\big)$. As a consequence, there exists $c > 0$ such that $\sup_{\mathbf{p}\in\partial v_\epsilon(S^L_{\epsilon,\delta})} |\mathbf{p}| \geq c\delta$. Using Corollary A.23 in [16], we conclude that

$$\big|\min_{S^L_\epsilon} v_\epsilon\big| \geq c''\,(\delta/2)^2$$

for some $c'' > 0$. On the other hand, using Lemma 2.11 again to upper bound μ_{ψ_ϵ}, we obtain

$$\mu_{v_\epsilon}\left(S^L_\epsilon\right) = \mu_{\psi_\epsilon}\left(S_\epsilon\right) \leq M(\ell_d(2\mathbb{B}_d))^{1/d}$$

and, by the Alexandrov maximum principle (e.g. Theorem 2.8. in [16]), this implies that

$$|v_\epsilon\left(L_\epsilon \mathbf{e}_1\right)| \leq C\left(\mathrm{d}_H(L_\epsilon \mathbf{e}_1, \partial S^L_\epsilon)\right)^{1/d}.$$

This means that the same arguments as in the proof of Theorem 4.10 in [16] yield

$$\lim_{\epsilon\to 0} \mathrm{d}_H\left(L_\epsilon \mathbf{e}_1, \partial(S^L_\epsilon)\right) = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \frac{v_\epsilon\left(L_\epsilon \mathbf{e}_1\right)}{\min_{S^L_\epsilon} v_\epsilon} = 1,$$

which is a contradiction.

Finally, consider the case where the exposed point of Σ belongs to $\partial\big(\mathrm{dom}(\tilde{\psi})\big)$; here again, it can be assumed that $\Sigma = \{\mathbf{z} : \tilde{\psi}(\mathbf{z}) = 0\}$ and that the exposed point of Σ is $\mathbf{e}_1$. We also can assume, without loss of generality, that $\mathrm{dom}(\tilde{\psi}) \subset \big\{\mathbf{z} \in \mathbb{R}^d : \ z_1 \leq 1\big\}$. Hence, $\{c\,\mathbf{e}_1 : c \geq 0\} \subset \partial\tilde{\psi}(\mathbf{e}_1)$. For small $\theta > 0$ we consider the sets

$$A_\theta := \mathbb{B}_d \cap \{\mathbf{z} = (z_1, \mathbf{z}') \in \mathbb{R}\times\mathbb{R}^{d-1} : -\theta|\mathbf{z}'| \leq z_1 - 1 \leq 0\}$$

and

$$C_\theta := \mathcal{X} \cap \{\mathbf{x} = (x_1, \mathbf{x}') \in \mathbb{R}\times\mathbb{R}^{d-1} : x_1 > 0, \, |\mathbf{x}'| \leq \theta x_1\}.$$

Let $\mathbf{x} \in C_\theta$ and $\mathbf{z} \in \partial\tilde{\psi}^*(\mathbf{x})$. Then $\mathbf{x} \in \partial\tilde{\psi}(\mathbf{z})$ and, thanks to the monotonicity of $\partial\tilde{\psi}$, we have that $\langle \mathbf{x} - t\,\mathbf{e}_1, \mathbf{z} - \mathbf{e}_1\rangle \geq 0$ for every $t \geq 0$, which entails $\langle \mathbf{x}, \mathbf{z} - \mathbf{e}_1\rangle \geq 0$ (take $t = 0$) and $\langle \mathbf{e}_1, \mathbf{z} - \mathbf{e}_1\rangle \geq 0$ (take $t \to \infty$). This means that $z_1 \leq 1$ and $x_1(z_1 - 1) + \langle \mathbf{x}', \mathbf{z}'\rangle \geq 0$, from which we deduce that $z_1 - 1 \geq \theta|\mathbf{z}'|$. Since, by Lemma 2.2, we have $\mathbf{z} \in \mathbb{B}_d$, it follows that $\partial\tilde{\psi}(A_\theta) \supset C_\theta$. Also, since both $\mathbf{0}$ and $\mathbf{e}_1$ belong to $\partial\tilde{\psi}(\mathbb{R})^d \subset \bar{\mathcal{X}}$, which is a convex set with nonempty interior, we can argue as in pp. 8-9 of [10], to conclude that $\ell_d(C_\theta \cap 2\mathbb{B}_d) \gtrsim \theta^{d-1}$ for $\theta > 0$ small enough. From the transport equation, we have that

$$\mathrm{U}_d(A_\theta) = \int_{\partial\tilde{\psi}(A_\theta)} p(\mathbf{x})d\mathbf{x} \geq \int_{C_\theta} p(\mathbf{x})d\mathbf{x} \gtrsim \ell_d(C_\theta \cap 2\mathbb{B}_d) \gtrsim \theta^{d-1}, \tag{2.18}$$

where we have used that p is lower bounded on bounded subsets of $\mathcal{X}$. However, for small θ, A_θ is well separated from $\mathbf{0}$ and, consequently, u_d is upper bounded on A_θ. This means that

$$\mathrm{U}_d(A_\theta) \lesssim \ell_d(A_\theta) \lesssim \theta^{d+1},$$

which contradicts (2.18). This completes the proof of the claim that $\tilde{\psi}$ (equivalently, ψ) is strictly convex in $\mathbb{B}_d$. □

We now can state and, based on Lemma 2.4, prove our main result, which extends Theorem 1.1 in Figalli [17] to the case of (bounded or unbounded) convexely supported distributions.

Theorem 2.5. *Let* P *be a probability measure with density* p *supported on the open convex set* $\mathcal{X} \subseteq \mathbb{R}^d$.

(i) If p *satisfies* (2.8)*, there exists a compact convex set* K *with Lebesgue measure* 0 *such that the center-outward quantile function* $\mathbf{Q}_\pm := \nabla\psi$ *and the center-outward distribution function* $\mathbf{F}_\pm := \nabla\psi^*$ *are homeomorphisms between* $\mathbb{B}_d \setminus \{\mathbf{0}\}$ *and* $\mathcal{X} \setminus K$*, inverses of each other.*

(ii) If, moreover, $p \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathcal{X})$ *for some* $k \in \mathbb{N}$ *and* $\alpha \in (0,1)$*, then* $\mathbf{Q}_\pm$ *and* $\mathbf{F}_pm$ *are diffeomorphisms of class* $\mathcal{C}^{k+1,\alpha}_{\mathrm{loc}}$ *between* $\mathbb{B}_d \setminus \{\mathbf{0}\}$ *and* $\mathcal{X} \setminus K$.

Proof. Assumption A implies, for any closed ball $B \subseteq \mathbb{B}_d \backslash \{\mathbf{0}\}$, the existence of constants $0 < \lambda_B \leq \Lambda_B$ such that $\lambda_B d\mathbf{x} \leq \mu_{\tilde{\psi}} \leq \Lambda_B d\mathbf{x}$. It follows from Caffarelli's regularity theory (see Corollary 4.21 in [16]) that ψ is locally of class $C^{1,\alpha}$. The constant $\alpha > 0$ depends on λ and Λ and, consequently, we cannot conclude that, for some $\alpha > 0$, $\psi \in C^{1,\alpha}_{\text{loc}}(\mathbb{B}_d \setminus \{\mathbf{0}\})$. However, ψ is continuously differentiable on $\mathbb{B}_d \setminus \{\mathbf{0}\}$ and, therefore, $\mathbf{Q}_\pm = \nabla\psi$ is a (single-valued) continuous function on $\mathbb{B}_d \setminus \{\mathbf{0}\}$. Furthermore, the strict convexity of ψ (Lemma 2.4) implies that $\mathbf{Q}_\pm$ is injective. By Brouwer's theorem on invariance of domain (see, e.g., Theorem 2B.3, p. 172 in [29]), $\mathbf{Q}_\pm(\mathbb{B}_d \setminus \{\mathbf{0}\})$ is open and $\mathbf{Q}_\pm$ is a homeomorphism between $\mathbb{B}_d \setminus \{\mathbf{0}\}$ and $\mathbf{Q}_\pm(\mathbb{B}_d \setminus \{\mathbf{0}\})$. But then, necessarily, $\mathbf{Q}_\pm(\mathbb{B}_d \setminus \{\mathbf{0}\}) = \mathcal{X} \setminus K$, which completes the proof of part *(i)* of the theorem.

Turning to part *(ii)*, assume that $p \in \mathcal{C}^{k,\alpha}_{\text{loc}}(X)$, fix $\mathbf{x} \in \mathbb{B}_d$, and consider a neighbourhood V of $\mathbf{x}$ such that its closure $\bar{V}$ is contained in $\mathbb{B}_d \setminus \{\mathbf{0}\}$. Now, ψ is strictly convex over V and there exist constants $0 < \lambda_V < \Lambda_V$ such that $\lambda_V d\mathbf{x} \leq \mu_{\tilde{\psi}} \leq \Lambda_V d\mathbf{x}$ on V. Note that $u_d \in \mathcal{C}^{k,\alpha}_{\text{loc}}(V)$ for every k and α. Hence, we can apply Remark 4.44 in [16] to conclude that $\nabla\psi \in \mathcal{C}^{k+1,\alpha}_{\text{loc}}(V)$. This completes the proof. □

To conclude this section, observe that the center-outward quantile function of a probability measure P satisfying the assumptions of Theorem 2.5 may fail to be continuous (singled-valued) at the origin. However, the center-outward distribution function is single-valued (and consequently continuous) at every point of the support of P), since the points in the set K which had to be removed to guarantee that $\mathbf{Q}_\pm$ is a homeomorphism are all mapped by $\mathbf{F}_\pm$ to the origin.

3 Some further properties of center-outward distribution and quantile functions.

To conclude this note, we present three results that more or less directly follow as consequences of Theorem 2.5. The first one is about the asymptotic invariance of center-outward distribution functions; the second one deals with the ability of center-outward quantile functions to capture the shape of a convex supporting set; the third one is a result on the shape of quantile contours, which turn out to satisfy a kind of relaxed version of convexity, connected to the so-called "lighthouse convexity" property (see, e.g., pp. 263-264 in [9]).

A classical univariate distribution function F trivially satisfies

$$\lim_{x\to-\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} F(x) = 1,$$

hence, in terms of the univariate center-outward distribution function $F_\pm := 2F - 1$,

$$\lim_{t\to\infty} F_\pm(tu) = u \qquad \text{for all } u \text{ such that } |u| = 1.$$

Let us show that this carries over to $\mathbf{F}_\pm$ in general dimension. Keeping the notation from the previous sections, we establish the following result.

Proposition 3.1. *Let the probability measure* P *have a density on* $\mathbb{R}^d$. *For any* $\mathbf{u}$ *on the unit sphere* $\mathbb{S}_{d-1}$, *any sequence* $(t_n)_{n\in\mathbb{N}}$ *of real numbers such that* $t_n \to \infty$, *and any* $\mathbf{y}_n \in \partial\varphi(t_n\mathbf{u})$,

$$\lim_{n\to\infty} \mathbf{y}_n = \mathbf{u}.$$

Proof. It follows from (2.3) that $\mathbf{y}_n \in \bar{\mathbb{B}}_d$. Hence, by compactness, there exists a subsequence along which $\mathbf{y}_n \to \mathbf{y}_\infty \in \bar{\mathbb{B}}_d$. On the other hand, monotonicity of the subdifferential implies that, for all $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \partial\varphi(\mathbf{x})$,

$$\langle \mathbf{y} - \mathbf{y}_n, \mathbf{x} - t_n\mathbf{u} \rangle \geq 0$$

or, equivalently, for all $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \partial\psi(\mathbf{x})$,

$$\langle \mathbf{y} - \mathbf{y}_\infty, \mathbf{x} \rangle + \langle \mathbf{y}_\infty - \mathbf{y}_n, \mathbf{x} \rangle \geq t_n \left(\langle \mathbf{y} - \mathbf{y}_\infty, \mathbf{u} \rangle + \langle \mathbf{y}_\infty - \mathbf{y}_n, \mathbf{u} \rangle \right).$$

Fixing $\epsilon > 0$ and $N = N(\epsilon)$ such that $|\mathbf{y}_n - \mathbf{y}_\infty| < \epsilon$ for all $n \geq N$, we obtain

$$\langle \mathbf{y} - \mathbf{y}_\infty, \mathbf{x}\rangle + \epsilon|\mathbf{x}| \geq t_n \left(\langle \mathbf{y} - \mathbf{y}_\infty, \mathbf{u}\rangle - \epsilon\right).$$

Hence, for n large enough, $\langle \mathbf{y} - \mathbf{y}_\infty, \mathbf{u}\rangle - \epsilon < 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \partial\varphi(\mathbf{x})$. Since $\epsilon > 0$ is arbitrary, we conclude that

$$\partial\varphi(\mathbb{R}^d) \subset S := \{\mathbf{y} : \ \langle \mathbf{y} - \mathbf{y}_\infty, \mathbf{u}\rangle \leq 0\},$$

which is a hyperplane. Now, the fact that $\nabla\varphi$ pushes P forward to U_d implies that $\partial\varphi(\mathbb{R}^d)$ contains almost every $\mathbf{x} \in \mathbb{B}_d$. Hence, $\mathbb{B}_d \subset S$, which only can happen if $\mathbf{y}_\infty = \mathbf{u}$. □

Under additional smoothness assumptions on P, the announced result for $\mathbf{F}_\pm$ follows as a corollary.

Corollary 3.2. *Let* P *satisfy the assumptions in Proposition 2.1(ii). Then, for any* $\mathbf{u}$ *on the unit sphere* $\mathbb{S}_{d-1}$ *and any sequence* $(t_n)_{n\in\mathbb{N}}$ *of real numbers such that* $t_n \to \infty$, *we have*

$$\lim_{n\to\infty} \mathbf{F}_\pm(t_n\mathbf{u}) = \mathbf{u}.$$

Next, we include the announced simple result showing that the outer quantile contours of a convexely supported P approach (in Hausdorff distance) the boundary of its support.

Lemma 3.3. *Let* P *be a probability measure on* $\mathbb{R}^d$ *with compact convex support* $\mathcal{X}$ *and a density* p *such that* $\lambda \leq p \leq \Lambda$ *for some* $0 < \lambda \leq \Lambda$. *Then, as* $R \to 1$, $\nabla\psi(R\,\mathbb{B}_d)$ *tends to* $\mathcal{X}$ *in Hausdorff distance:*

$$\lim_{R\to 1} \mathrm{d}_H\big(\nabla\psi(R\,\mathbb{B}_d), \mathcal{X}\big) = 0.$$

Proof. Since $\nabla\psi(R\,\mathbb{B}_d)$ is contained in $\mathcal{X}$, we only need to analyse one of the two members of the maximum defining the Hausdorff distance: indeed,

$$\begin{aligned}\mathrm{d}_H(\nabla\psi(R\,\mathbb{B}_d), \mathcal{X}) &= \max\{\sup_{\mathbf{a}\in\nabla\psi(R\,\mathbb{B}_d)} \inf_{\mathbf{x}\in\mathcal{X}} |\mathbf{a} - \mathbf{x}|, \sup_{\mathbf{x}\in\mathcal{X}} \inf_{\mathbf{a}\in\nabla\psi(R\,\mathbb{B}_d)} |\mathbf{a} - \mathbf{x}|\} \\ &= \sup_{\mathbf{y}\in\mathbb{B}_d} \inf_{\mathbf{b}\in R\,\mathbb{B}_d} |\nabla\psi(\mathbf{b}) - \nabla\psi(\mathbf{y})|.\end{aligned}$$

On the other hand, since $r\,\mathbb{B}_d \subset R\,\mathbb{B}_d \subset \mathcal{X}$, $\nabla\psi(r\,\mathbb{B}_d) \subset \nabla\psi(R\,\mathbb{B}_d) \subset \mathcal{X}$ for $r \leq R$, so that the mapping $R \mapsto \mathrm{d}_H(\nabla\psi(R\,\mathbb{B}_d), \mathcal{X})$ is a decreasing function. Suppose that $\mathrm{d}_H(\nabla\psi(R\,\mathbb{B}_d), \mathcal{X})$ does not tend to 0 when R tends to 1. Then, there exists $\epsilon > 0$ such that, for every R, $\mathrm{d}_H(\nabla\psi(R\,\mathbb{B}_d), \mathcal{X}) > \epsilon$; in particular, there exists $\mathbf{x}_R \in \mathcal{X}$ such that $|\mathbf{a}_R - \mathbf{x}_R| > \epsilon$ for all $\mathbf{a}_R \in \nabla\psi(R\,\mathbb{B}_d)$.

Now, for each $n \in \mathbb{N}$, consider the sequences $A_n := \nabla\psi\big((1 - 1/n)\mathbb{B}_d\big)$ and $\mathbf{y}_n := \mathbf{x}_{1-1/n} \in \mathcal{X}$. These sequences are such that

$$\inf_{\mathbf{a}\in A_m} \big|\mathbf{a} - \mathbf{y}_n\big| \geq \inf_{\mathbf{a}\in A_n} \big|\mathbf{a} - \mathbf{y}_n\big| > \epsilon \quad \text{ for all } m \leq n.$$

By compactness, the sequence $\mathbf{y}_n$ admits a convergent subsequence, with limit $\mathbf{y}_\infty$, say, with $\mathbf{y}_\infty \in \mathcal{X}$. This limit satisfies $\inf_{\mathbf{a}\in A_n} \big|\mathbf{a} - \mathbf{y}_\infty\big| > \epsilon$ for all $n \in \mathbb{N}$, which is not possible since $\mathcal{X} = \bigcup_{n\in\mathbb{N}} A_n$. □

Our final result concerns the shape of the quantile contours of smooth probability measures (those satisfying the assumptions of Theorem 2.5). As a consequence of Theorem 2.5, the sets $\mathbf{Q}_\pm(r\,\mathbb{B}_d)$ are bounded, with connected boundary. Beyond this type of topological properties, results on the geometry of the quantile regions are not available. Here we prove that they satisfy a weak form of convexity. Recall from [9] that a set $B \subseteq \mathbb{R}^d$ is ρ-lighthouse convex if, from every point $\mathbf{x}$ in the boundary of B, there exists an open cone with vertex $\mathbf{x}$ and opening angle $\rho > 0$ which is contained in $\mathbb{R}^d \setminus B$. The limiting version of this concept (obtained as $\rho \to 0$) is that for every point $\mathbf{x}$ in the boundary of B there exists a ray emanating from $\mathbf{x}$ that does not intersect B at any other point. This is precisely what can be proved for quantile sets.

Lemma 3.4. *Let* P *be a probability measure on* $\mathbb{R}^d$ *satisfying the assumptions of Theorem 2.5. Then, for all* $r \in (0,1)$ *and all* $\mathbf{y}$ *belonging to the boundary of* $\mathbf{Q}_\pm(r\,\mathbb{B}_d)$*, there exists a ray* T *emanating from* $\mathbf{y}$ *for which* $\overline{\mathbf{Q}_\pm(r\,\mathbb{B}_d)} \cap T = \{\mathbf{y}\}$*.*

Proof. Assume, on the contrary, that there exists $\mathbf{y}$ in the boundary of $\mathbf{Q}_\pm(r\,\mathbb{B}_d)$ such that for every ray $T = \{\mathbf{z} \in \mathbb{R}^d :\ \mathbf{z} = \mathbf{y} + t\mathbf{s},\ t \geq 0\}$, there exists in $\overline{\mathbf{Q}_\pm(r\,\mathbb{B}_d)} \cap T$ at least one point $\mathbf{z}$ distinct from $\mathbf{y}$. Note that, necessarily, that point can be chosen in the boundary of $\mathbf{Q}_\pm(r\,\mathbb{B}_d)$. Now, since $\mathbf{Q}_\pm$ is a homeomorphism, it maps boundaries into boundaries. Therefore, we can assume, up to a rotation, that $\mathbf{y} = \mathbf{Q}_\pm(r\mathbf{e}_1)$. Monotonicity of $\mathbf{Q}_\pm$ implies that

$$\langle \mathbf{u} - r\mathbf{e}_1, \mathbf{Q}_\pm(\mathbf{u}) - \mathbf{Q}_\pm(r\mathbf{e}_1)\rangle \geq 0 \qquad \text{for all } \mathbf{u} \in \mathbb{B}_d. \tag{3.1}$$

However, if $\mathbf{Q}_\pm(\mathbf{u}) \in T = \{\mathbf{z} \in \mathbb{R}^d :\ \mathbf{z} = \mathbf{y}+t\mathbf{e}_1,\ t \geq 0\}$, then $\mathbf{Q}_\pm(\mathbf{u}) = \mathbf{Q}_\pm(r\mathbf{e}_1)+t\mathbf{e}_1$ for some $t > 0$. Hence, by (3.1) $\langle \mathbf{u} - r\mathbf{e}_1, r\mathbf{e}_1\rangle \geq 0$. This implies that $\mathbf{u} \notin r\,\mathbb{B}_d$, thus contradicting the assumption that T has a common point with the boundary of $\mathbf{Q}_\pm(r\,\mathbb{B}_d)$ other than $\mathbf{y}$. □

References

[1] del Barrio, E., Beirlant, J., Buitendag, S. and Hallin, M. (2019). Center-outward quantiles and the measurement of multivariate risk. https://arxiv.org/abs/1912.04924

[2] del Barrio, E., Cuesta Albertos, J., Hallin, M., and Matrán, C. (2018). Smooth cyclically monotone interpolation and empirical center-outward distribution functions. Available at `https://ideas.repec.org/p/eca/wpaper/2013-271399.html`.

[3] del Barrio, E. and Loubes, J.M. (2019). Central limit theorems for empirical transportation cost in general dimension. *Ann. Probab.*, **47**, 926–951.

[4] Brenier, Y. (1991). Polar factorization and monotone rearrangement of vector-valued functions, *Comm. Pure Appl. Math.* **44**, 375–417.

[5] Caffarelli, L.A. (1990). A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity, *Ann. Math.*, **131**, 129–134.

[6] Caffarelli, L.A. (1991). Some regularity properties of solutions of Monge-Ampère equation. *Comm. Pure Appl. Math.*, **44**, 965–969.

[7] Caffarelli, L.A. (1992). The regularity of mappings with a convex potential. *J. Amer. Math. Soc.*, **5**, 99–104..

[8] Chernozhukov, V., Galichon, A., Hallin, M. and Henry, M. (2017). Monge-Kantorovich depth, quantiles, ranks and signs. *Ann. Statist.*, **45**, 223–256.

[9] Cholaquidis, A., Cuevas, A. and Fraiman, R. (2014). On Poincaré cone property. *Ann. Statist.*, **42**, 255–284.

[10] Cordero-Erausquin, D. and Figalli, A. (2019). Regularity of monotone transport maps between unbounded domains. https://arxiv.org/abs/1902.07621

[11] Deb, N. and Sen, B. (2019). Multivariate Rank-based Distribution-free Nonparametric Testing using Measure Transportation. https://arxiv.org/abs/1909.08733.

[12] de Valk, C. and Segers, J. (2018). Tails of optimal transport plans for regularly varying probability measures. https://arxiv.org/abs/1811.12061

[13] Evans, L.C. and Gariepy, R.F. (1992). *Measure Theory and Fine Properties of Functions.* CRC Press.

[14] Faugeras, O.P. and Rüschendorf, L. (2017). Markov morphisms: a combined copula and mass transportation approach to multivariate quantiles. *Mathematica Applicanda,* **45**, 21–63.

[15] Faugeras, O.P. and Rüschendorf, L. (2019). Functional, randomized and smoothed multivariate quantile regions. *TSE Working Paper* n. 19 - 1039.

[16] Figalli, A. *The Monge-Ampère Equation and Its Applications*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zurich, 2017.

[17] Figalli, A. (2018). On the continuity of center-outward distribution and quantile functions, *Nonlinear Analysis*, **177**, 413–421.

[18] Genest, Chr. and Rivest, L.-P. (2001). On the multivariate probability integral transformation. *Statistics & Probability Letters* **53**, 391–399.

[19] Ghosal, P. and Sen, B. (2019). Multivariate Ranks and Quantiles using Optimal Transportation and Applications to Goodness-of-fit Testing. https://arxiv.org/abs/1905.05340

[20] Hájek, J., Šidák, Z. and Sen, P. K. (1999). *Theory of Rank Tests, 2nd Ed,*, Wiley.

[21] Hallin, M. (1994). On the Pitman nonadmissibility of correlogram-based time series methods. *Journal of Time Series Analysis* **16**, 607–612.

[22] Hallin, M. and Tribel, O. (2000). The efficiency of some nonparametric competitors to correlogram-based methods. In F.T. Bruss and L. Le Cam, Eds, *Game Theory, Optimal Stopping, Probability, and Statistics, Papers in honor of T.S. Ferguson on the occasion of his 70th birthday*, I.M.S. Lecture Notes-Monograph Series, 249–262.

[23] Hallin, M. (2017). Distribution and quantile functions, ranks and signs in $\mathbb{R}^d$: a measure transportation approach. Available at `https://ideas.repec.org/p/eca/wpaper/2013-258262.html`.

[24] Hallin, M., La Vecchia, D. and Liu, H. (2019). Center-Outward R-Estimation for Semiparametric VARMA Models. https://arxiv.org/abs/1910.08442

[25] Hallin, M. and Paindaveine, D. (2002a). Optimal tests for multivariate location based on interdirections and pseudo-Mahalanobis ranks. *Ann. Statist.* **30**, 1103–1133.

[26] Hallin, M. and Paindaveine, D. (2002b). Optimal procedures based on interdirections and pseudo-Mahalanobis ranks for testing multivariate elliptic white noise against ARMA dependence. *Bernoulli* **8**, 787–815.

[27] Hallin, M., Paindaveine, D., and Verdebout, Th. (2010). Optimal rank-based testing for principal components, *Ann. Statist.* **38**, 3245–3299.

[28] Hallin, M. and Werker, B.J.M. (2003). Semiparametric efficiency, distribution-freeness and invariance. *Bernoulli*, **9**,137–165.

[29] Hatcher, A. (2002). *Algebraic Topology.* Cambridge University Press.

[30] He, X. and Wang, G. (1997). Convergence of depth contours for multivariate datasets, *Ann. Statist.* **25**, 495–504.

[31] McCann, R. J. (1995). Existence and uniqueness of monotone measure-preserving maps. *Duke Math. Journal*, **80**, 309–323.

[32] McCann, R.J. (1997). A convexity principle for interacting gases, *Adv. Math.*, **128**, 153–179.

[33] Oja, H. (1999). Affine invariant multivariate sign and rank tests and corresponding estimates: a review. *Scand. J. Statist.* **26**, 319–343.

[34] Oja, H. (2010). *Multivariate Nonparametric Methods with R: an approach based on spatial signs and ranks*. Springer, New York.

[35] Puri, M.L. and Sen, P.K. (1971). *Nonparametric Methods in Multivariate Analysis*. John Wiley & Sons, New York.

[36] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press.

[37] Serfling, R. (2002). Quantile functions for multivariate analysis: approaches and applications, *Stat. Neerl.* **56**, 214–232.

[38] Serfling, R. (2019). Perspectives on depth functions on general data spaces, with consideration of the halfspace, projection, spatial, "density", "local", and "contour" depths. Available at `http://www.utdallas.edu/~serfling/papers/Perspectives_May_2019.pdf`.

[39] Shi, H., Drton, M. and Han, F. (2019). Distribution-free consistent independence tests via Hallin's multivariate rank. https://arxiv.org/abs/1909.10024

[40] Villani, C. *Topics in Optimal Transportation*, Dover Publications, Graduate Studies in Mathematics, American Mathematical Society, USA, 2003.

[41] Zuo, Y. and He, X. (2006). On the limiting distributions of multivariate depth-based rank sum statistics and related tests, *Ann. Statist.* **34**, 2879–2896.