A Note on the Regularity of Center-Outward Distribution and Quantile Functions

Eustasio del Barrio  
IMUVA, Universidad de Valladolid, Spain

Alberto Gonzalez-Sanz  
IMUVA, Universidad de Valladolid, Spain

Marc Hallin  
SBS-EM, ECARES, and Département de Mathématique,  
Université libre de Bruxelles, Belgium

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A note on the Regularity of Center-Outward Distribution and Quantile Functions

Eustasio del Barrio\(^{(1)}\)*, Alberto González-Sanz\(^{(2)}\) and Marc Hallin\(^{(3)}\)

\(^{(1)(2)}\)IMUVA, Universidad de Valladolid, Spain
\(^{(3)}\)ECARES and Département de Mathématique, Université libre de Bruxelles, Belgium

\(^{(1)}\)tasio@eio.uva.es \(^{(2)}\)alberto.gonzalez.sanz.96@gmail.com \(^{(3)}\)mhallin@ulb.ac.be

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Abstract

We provide sufficient conditions under which the center-outward distribution and quantile functions introduced in Chernozhukov et al. (2017) and Hallin (2017) are homeomorphisms, thereby extending a recent result by Figalli [17]. Our approach relies on Caffarelli’s classical regularity theory for the solutions of the Monge-Ampère equation, but has to deal with difficulties related with the unboundedness at the origin of the density of the spherical uniform reference measure. Our conditions are satisfied by probabilities on Euclidean space with a general (bounded or unbounded) convex support which are not covered in [17]. We provide some additional results about center-outward distribution and quantile functions, including the fact that quantile sets exhibit some weak form of convexity.

Keywords: Optimal transportation; Monge-Ampère equation; multivariate ranks; quantile contours.

1 Introduction: center-outward distribution and quantile functions

Univariate distribution and quantiles functions, together with their empirical counterparts and the closely related concepts of ranks and order statistics, count among the most fundamental and useful tools in mathematical statistics. Ranks indeed are not just distribution-free: in models driven by noise with unspecified density, they generate the sub-$\sigma$-field of all distribution-free events (see [2]), which is also the largest sub-$\sigma$-field independent, irrespective of the underlying distribution, of the minimal sufficient $\sigma$-field generated by the order statistic; suitable rank-based procedures achieve optimality in several senses in nonparametric testing as well semiparametric efficiency (see, e.g. [20], [21], [22], [28]). A major limitation of the classical concepts of ranks and quantiles, however, is that, due to the absence of a canonical ordering of $\mathbb{R}^d$ for $d \geq 2$, they do not readily extend to the multivariate context.

The problem is not new, and numerous attempts have been made to fill that gap by defining multivariate versions of distribution and quantiles functions, with the ultimate goal of constructing suitable multivariate versions of classical rank- and quantile-based inference procedures. The traditional definition of a multivariate distribution function is somewhat helpless in that respect, and does not produce any satisfactory concept of quantiles—let alone a satisfactory concept of ranks (see [18]). The componentwise approach, closely related with copula transforms, has been studied intensively (see [35]), but does not even enjoy distribution-freeness. Nor do the so-called spatial ranks ([33], [34]) inspired by the $L_1$ characterization of univariate quantiles. The whole theory of statistical depth (see [37], [38] for authoritative surveys), in a sense, is motivated by the same objective of providing a

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(data-driven) ordering of $\mathbb{R}^d$ and adequate concepts of multivariate ranks ([41]) and quantiles ([30]); here again, the resulting notions fail to be distribution-free. As for the Mahalanobis ranks and signs considered, e.g. in [25], [26] or [27], they do enjoy distribution-freeness and all the desired properties expected from ranks—under the restrictive assumption, however, of elliptical symmetry.

This shortcoming of all available solutions has motivated the introduction, in [8] and [23], of the measure transportation-based concepts of Monge-Kantorovich depth, center-outward distribution and quantile functions, ranks, and signs. These center-outward concepts, unlike all previous ones, are shown (see [23], [2]) to enjoy all the properties that make their univariate counterpart a fundamental and successful tool for statistical inference; we refer to [2] for more references and further discussion.

Let $P$ be a Borel probability measure on the real line with finite second moment and continuous density $f$. The latter definition, indeed, readily extends to arbitrary dimensions. Let $P$ denote a Borel probability measure on the real line with finite second moment and continuous density $f$. Measure transportation theory (see, e.g., Theorem 2.12 in [40]) tells us that there exists a $P$-a.s. unique solution to this optimal transportation problem.

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The first of these two limitations has been relaxed in [23] thanks to a celebrated theorem by McCann [31]. Under the assumption that $P$ has finite second-order moments, Brenier in 1991 had shown that optimal transportation maps (hence, all versions of the $P$-a.s. unique solution $F_\pm$ of
Monge’s problem (1.1)) coincide P-a.s. with the Lebesgue-a.e. gradient \( \nabla \varphi \) of a convex function\(^1\) \( \varphi \), which has the interpretation of a potential. More precisely, \( \mathbf{F}_\pm \) a.s. is of the form \( \nabla \varphi \) where \( \varphi \) (i) is lower semicontinous (lsc in the sequel), (ii) is convex, and (iii) is such that \( \nabla \varphi \| \mathcal{P} = U_d \). McCann [31] further showed that these last three conditions uniquely determine \( \nabla \varphi \), even in the absence of second moment assumptions, while under finite second-order moments, \( \nabla \varphi \) is a solution of Monge’s problem (1.1). Thus, putting

\[
\mathbf{F}_\pm(x) := \nabla \varphi(x) \quad \text{x-a.e. in } \mathbb{R}^d,
\]

(1.3)

the center-outward distribution function \( \mathbf{F}_\pm \) is no longer characterized as the almost surely unique solution of an optimization problem (1.1) requiring finite moments of order two but as the unique a.e. gradient \( \nabla \varphi \) of a convex function pushing \( P \) forward to \( U_d \). We nevertheless conform to the common usage of improperly calling \( \nabla \varphi \) the optimal transport pushing \( P \) forward to \( U_d \).

While taking care of the moment assumption—existence of second-order moments indeed is an embarrassing assumption when distribution and quantile functions are to be defined—the second limitation still remains. The non-unicity of \( \mathbf{F}_\pm := \nabla \varphi \), however, disappears if \( P \) is such that \( \varphi \) is everywhere differentiable. That this is indeed the case was shown by Figalli in 2018 [17] for \( P \) in the so-called class of distributions with nonvanishing densities\(^2\). For any \( P \) in that class of distributions, Figalli actually establishes that \( \nabla \varphi(x) \) is a gradient for all \( x \) and, when restricted to

\[
\mathbb{R}^d_{(0)} := \mathbb{R}^d \setminus \{x : \nabla \varphi(x) \neq 0\},
\]

a homeomorphism between \( \mathbb{R}^d_{(0)} \) and the punctured ball \( B_d \setminus \{0\} \). The latter property is quite essential if sensible—namely, closed, continuous, connected, and nested—quantile regions and contours, based on an inverse\(^3\) \( Q_\pm \) of \( \mathbf{F}_\pm \), are to be defined: see [23] and [2].

The introduction by Hallin [23] of center-outward ranks and quantiles rapidly attracted the attention of the nonparametric community. It has triggered, among others, Faugeras and Rüschendorf ([14] and [15]) and de Valk and Segers [12]. Applications to the long-standing open problem of constructing distribution-free tests for the hypothesis of independence between vectors with unspecified densities have been proposed by Deb and Sen [11], Shi, Drton, and Han [39], and Ghosal and Sen [19]. Optimal center-outward \( R \)-estimators also have been derived (Hallin, La Vecchia, and Liu [24]) for VARMA models, while center-outward quantile-based methods for the measurement of multivariate risk are proposed in del Barrio, Beirlant, Buitendag, and Hallin [1].

The goal of this paper is to provide simple sufficient conditions for Figalli’s results to hold beyond the assumption of nonvanishing densities: we more particularly consider distributions with (bounded or unbounded) convex supports. Beyond other theoretical considerations, these are the key properties required to prove a.s. convergence of the empirical center-outward distribution functions to their theoretical counterparts (see [2]). Hence, the results of the present paper also are extending the validity of the center-outward Glivenko-Cantelli theorem in that reference.

From a technical point of view, our main result is Theorem 2.5 below, which relies on the classical regularity theory for solutions of Monge-Ampère equations associated with the name of of Caffarelli (see [5, 6, 7]), as discussed in Section 2. The use of that theory to investigate the regularity of optimal transportation maps between two probabilities typically requires that both probabilities have densities that are bounded and bounded away from zero over their respective supports. Recently, under local versions of this condition, very general regularity results of this kind has been given in [10] and [19]. However, the spherical uniform reference measure \( U_d \) considered here, in dimension \( d \geq 2 \), yields unbounded densities at the origin, so that the results in [10] or [19] do not apply.\(^4\) To

\(^{1}\)The notation \( \nabla \varphi \) here is used for the Lebesgue-a.e. gradient of \( \varphi \), that is, \( \nabla \varphi(x) \) is defined as the gradient at \( x \) of \( \varphi \) whenever \( \varphi \) is differentiable at \( x \)—which, for a convex \( \varphi \), holds Lebesgue-a.e. Note that, contrary to \( \nabla \varphi \), which is a.e. unique, \( \varphi \) is not—unless we impose, without loss of generality (see, e.g., Lemma 2.1 in [3]), that \( \varphi(0) = 0 \).

\(^{2}\)Precisely, the distributions \( P \) with densities \( p \) and support \( \mathcal{X} = \mathbb{R}^d \) satisfying Assumption A below.

\(^{3}\)See Section 2.1 for a precise definition.

\(^{4}\)Note that the choice of the spherical uniform reference is not a whimsical one. It preserves the independence between \( ||\mathbf{F}_\pm|| \) and \( \mathbf{F}_\pm/||\mathbf{F}_\pm|| \) (extending the independence, for \( d = 1 \), between \( |\mathbf{F}_\pm| \) and \( \text{sign}(\mathbf{F}_\pm) \)) and produces simple and easily interpretable quantile contours with prescribed probability content (we refer to [23] for details).
our knowledge, the only reference dealing with this kind of unbounded density is [17] which, however, requires \( P \) to be supported on the whole space \( \mathbb{R}^d \). Here we extend the result in [17] to cover the case of \( P \) with (bounded or unbounded) convex supports.

The sequel of this paper is organized as follows. Our main regularity result is established in Section 2, along with a succinct account of the main elements of Cafarelli’s theory and some auxiliary results. We conclude with Section 3, which presents some new results on center-outward distribution and quantile functions. These include an asymptotic invariance property extending a well-known feature of classical univariate distribution functions and the ability of quantile contours to capture the shape of the bounded support of a probability measure by converging (in Hausdorff distance) to the boundary of the support. Finally, we include a result on the geometry of quantile sets, showing that they turn out to exhibit a limiting form of “lighthouse convexity”.

2 Regularity of center-outward distribution and quantile functions

2.1 Center-outward quantile functions

The Introduction was focused on the distribution functions \( F_\pm \). Exchanging the roles of \( P \) and \( U_d \), we could have emphasized transportation from the unit ball to the support of \( P \), leading to the definition of the center-outward quantile function \( Q_\pm \) with, mutatis mutandis, the same comments.

Let \( P \) denote a Borel probability measure over \( \mathbb{R}^d \) with Lebesgue density \( p \). While the center-outward distribution function is defined as the optimal (in the McCann sense) transport pushing \( P \) forward to \( U_d \), the center-outward quantile map or quantile function \( Q_\pm \) of \( P \) is defined as the optimal transport pushing \( U_d \) forward to \( P \). Namely,

\[
Q_\pm(u) := \nabla \psi(u) \quad u \text{-a.e. in } \mathbb{B}_d
\]  

(2.1)

where \( \nabla \psi \) is, in agreement with McCann’s Theorem, the unique a.e. gradient of a convex function \( \psi \) with domain containing \( \mathbb{B}_d \) such that \( \nabla \psi|_{U_d} = P \). Again, imposing, without loss of generality,\(^6\) that \( \psi(0) = 0 \), the convex potential \( \psi \) is uniquely defined and a.e. differentiable over \( \mathbb{B}_d \). We extend \( \psi \) to a lsc convex function on \( \mathbb{R}^d \) with the standard procedure of setting \( \psi(u) := \liminf_{z \rightarrow u, |\psi(z)| < 1} \psi(z) \) if \( |u| = 1 \) and \( \psi(u) := +\infty \) for \( u \notin \mathbb{B}_d \) (see, e.g. (A.18) in [16]). With this extension, \( \varphi \) is the Legendre transform of \( \psi \), that is,

\[
\varphi(x) = \psi^*(x) := \sup_{u \in \mathbb{B}_d} \langle (u,x) - \psi(u) \rangle, \quad x \in \mathbb{R}^d.
\]  

(2.2)

We observe that the domain of \( \varphi \) is \( \mathbb{R}^d \) and that \( \varphi \), being the sup of a 1-Lipschitz function, is also 1 -Lipschitz. In particular, for almost every \( x \in \mathbb{R}^d \), \( \varphi \) is differentiable with \( |\nabla \varphi(x)| \leq 1 \) and, as a consequence (see, e.g., Corollary A.27 in [16]),

\[
\partial \varphi(\mathbb{R}^d) \subseteq \overline{\mathbb{B}_d};
\]  

(2.3)

here, and throughout this paper, \( \overline{B} \) stands for the closure of a set \( B \), \( \partial \varphi(x) \) for the subdifferential\(^7\) of the convex function \( \varphi \) at \( x \), and \( \partial \varphi(A) := \bigcup_{x \in A} \partial \varphi(x) \). Furthermore, Proposition 10 in [31] (see also Remark 16) shows that, since \( P \) has a density, \( \nabla \psi(\nabla \varphi(x)) = x \) for almost every \( x \) in the support of \( P \) and \( \nabla \varphi(\nabla \psi(y)) = y \) for almost every \( y \in \mathbb{B}_d \). In that sense, \( Q_\pm \) and \( F_\pm \) are the inverse of each other. In this way, we have defined \( F_\pm(x) \) for almost every \( x \in \mathbb{R}^d \) and \( Q_\pm(u) \) for almost every \( u \in \mathbb{B}_d \); the definitions coincide with those in [8] or [23] for \( x \) in the support of \( P \).

\(^6\)We adhere to the usual convention of considering that a function defined on \( A \subset \mathbb{R}^d \) is convex if it can be extended to a convex function on \( \mathbb{R}^d \) with values in \( \mathbb{R} \cup \{\infty\} \); the domain of the convex function is then redefined as the set where it takes finite values.

\(^7\)Indeed, two convex functions with a.e. equal gradients on an open convex set are equal up to an additive constant (see, e.g., Lemma 2.1 in [3]).

\(^7\)Recall that the subdifferential of \( \varphi \) at \( x \) is the set of all \( z \in \mathbb{R}^d \) such that \( \varphi(y) - \varphi(x) \geq \langle z, y - x \rangle \) for all \( y \).
2.2 Some regularity results for Monge-Ampère equations

As announced in the Introduction, our approach to the regularity of the center-outward distribution and quantile functions is based on the classical regularity theory for Monge-Ampère equations. We refer to [16] for a comprehensive account of this theory, of which we present here a minimal account.

Given an open set $X \subseteq \mathbb{R}^d$ and a (finite) convex function $\varphi : X \to \mathbb{R}$, denoting by $\ell_d$ he Lebesgue measure on $\mathbb{R}^d$, the Monge-Ampère measure associated with $\varphi$ is defined by

$$\mu_\varphi(E) := \ell_d(\partial \varphi(E))$$

for every Borel set $E \subseteq \mathbb{R}^d$. It can be checked that $\mu_\varphi$ is indeed a locally finite Borel measure on $X$. The crucial link between center-outward distribution functions and Monge-Ampère measures can be summarized as follows. Assume $P$ is a probability on $X$ with Lebesgue density $p$ and let $\varphi$ be a convex function from $X$ to $\mathbb{R}$. Then, for every Borel set $A$,

$$Q(A) := (\nabla \varphi^* P)(A) = P(\partial \varphi^*(A))$$

where $\varphi^*$ is the Legendre transform of $\varphi$. We recall that convexity of $\varphi$ implies that it is differentiable at almost every point in $X$ (see, e.g., Theorem 25.4 in [36]) and, therefore,

$$(\nabla \varphi^* P)(A) = P(\{x : \nabla \varphi(x) \in A\}) = P(\{x : \partial \varphi(x) \subseteq A\}).$$

This and the fact that $y \in \partial \varphi(x)$ if and only if $x \in \partial \varphi^*(y)$ yield the last equality above. Hence, if $Q$ has a density $q$, for every Borel set $A$,

$$\int_{\partial \varphi(A)} q(y)dy = \int_A p(x)dx$$

(see Lemma 4.6 in [40]); if, moreover, $Q = U_d$,

$$\int_{\partial \varphi(A)} u_d(y)dy = \int_A p(x)dx. \quad (2.4)$$

Observing that

$$\mu_\varphi(A) = \ell_d(\partial \varphi(A)) = \ell_d(\partial \varphi(A) \cap B_d),$$

where the second equality follows from (2.3), we obtain from (2.4) that, for $A$ such that $\ell_d(A) = 0$,

$$\mu_\varphi(A) \leq a_d \int_{\partial \varphi(A)} u_d(y)dy = a_d \int_A p(x)dx = 0$$

with $a_d$ as in (1.2). Thus, the Monge-Ampère measure $\mu_\varphi$ is Lebesgue-absolutely continuous. Since the density of the absolutely continuous part of the Monge-Ampère measure $\mu_\varphi$ is given by $(p(x)/u_d(\nabla \varphi(x)))$ (see McCann [32] or Theorem 4.8 in [40]), we conclude that, for every Borel set $A \subseteq \mathbb{R}^d$,

$$\mu_\varphi(A) = \int_A \frac{p(x)}{u_d(\nabla \varphi(x))} dx = a_d \int_A p(x)|\nabla \varphi(x)|^{d-1}dx. \quad (2.5)$$

Let us focus now on the Monge-Ampère measure $\mu_\psi$ associated (see Section 2.1) with $Q_\pm$ and $\psi$ (both defined over $B_d$). Since $\nabla \psi$ pushes $U_d$ forward to $P$, we have that $\nabla \psi(y) \in X$ $\psi$-a.e. in $B_d$. By continuity (see Theorem 25.5 in [36]), $\nabla \psi(y) \in X$ for every point $y$ of differentiability of $\psi$. Using again Corollary A.27 in [16], we conclude that $\partial \psi(B_d)$ is included in the convex hull $\text{conv}(X)$ of $X$. Hence, if $X$ itself is convex, we obtain that

$$\partial \psi(B_d) \subseteq X.$$

(2.6)
Analogous to (2.4), we have that
\[
\int_{\partial \psi(B)} p(x) \, dx = \int_B u_d(y) \, dy \quad \text{for every Borel set } B \subseteq \mathbb{R}^d.
\] (2.7)

Now, denoting by \( r \mathbb{B}_d \) the open ball with radius \( r \) centered at the origin, let us assume that the Borel set \( B \subseteq r \mathbb{B}_d \), with \( 0 < r < 1 \), has Lebesgue measure zero. Since \( \overline{B} \subseteq r \mathbb{B}_d \) is compact, \( \partial \psi(\overline{B}) \) also is compact (see, e.g. Lemma A.22 in [16]). Hence, there exists \( R > 0 \) such that
\[
\partial \psi(B) \subseteq \partial \psi(\overline{B}) \subseteq R \mathbb{B}_d.
\]

The following assumption, which requires the density \( p \) of \( P \) to be bounded and bounded away from 0 on compact subsets of the support, is absolutely essential (the same assumption is also made by Figalli in [17]).

**Assumption A.** For every \( R > 0 \), there exist constants \( 0 < \lambda_R \leq \Lambda_R \) such that
\[
\lambda_R \leq p(x) \leq \Lambda_R \quad \text{for all } x \in \mathcal{X} \cap R \mathbb{B}_d.
\] (2.8)

Since \( \mathcal{X} \) is convex (hence \( \ell_d(\overline{\mathcal{X}} - \mathcal{X}) = 0 \)), Assumption A entails
\[
\mu_\psi(B) \leq \frac{1}{\lambda_R} \int_{\partial \psi(B)} p(x) \, dx = \frac{1}{\lambda_R} \int_B u_d(y) \, dy = 0.
\]

Assuming convexity of \( \mathcal{X} \) and (2.8), we conclude that \( \mu_\psi \) is absolutely continuous with respect to \( \ell_d \) and, using Theorem 4.8 in [40] again, that, for every Borel set \( B \subseteq \mathbb{B}_d \),
\[
\mu_\psi(B) = \int_B \frac{u_d(y)}{p(\nabla \psi(y))} \, dy = \frac{1}{a_d} \int_B \frac{1}{p(\nabla \psi(y)) |y|^{d-1}} \, dy.
\] (2.9)

We summarize this discussion in the next proposition.

**Proposition 2.1.** Let \( P \) be a probability measure with density \( p \) supported on the open set \( \mathcal{X} \subseteq \mathbb{R}^d \). Denote by \( \psi : \mathbb{B}_d \rightarrow \mathbb{R} \) the convex, lower semicontinuous function satisfying \( \psi(0) = 0 \) and \( \nabla \psi_\sharp U_d = P \) and let \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \) be defined as in (2.2). Then,

(i) \( \mu_\varphi \) is absolutely continuous with respect to \( \ell_d \) and, for every Borel \( A \subseteq \mathbb{R}^d \),
\[
\mu_\varphi(A) = a_d \int_A p(x) |\nabla \varphi(x)|^{d-1} \, dx;
\]

(ii) if, moreover, \( \mathcal{X} \) is convex and \( p \) satisfies Assumption A, then \( \mu_\psi \) is absolutely continuous with respect to \( \ell_d \) and, for every Borel set \( B \subseteq \mathbb{B}_d \),
\[
\mu_\psi(B) = \frac{1}{a_d} \int_B \frac{1}{p(\nabla \psi(y)) |y|^{d-1}} \, dy.
\]

Next, let us show that, for well-behaved probability measures \( P \) (those with convex support and density \( p \) satisfying Assumption A), the center-outward distribution function \( F_\pm \) cannot map points in the interior of the support of \( P \) to extremal points of the unit ball.

**Lemma 2.2.** Let \( P \) be a probability measure with density \( p \) supported on the convex open set \( \mathcal{X} \subseteq \mathbb{R}^d \) and such that Assumption A holds. Then \( (\partial \varphi)(\mathcal{X}) \cap \mathbb{S}_{d-1} = \emptyset \), where \( \mathbb{S}_{d-1} = \overline{\mathbb{B}}_d \setminus \mathbb{B}_d \).
Proof. Assume that there exists \( x \in \mathcal{X} \) such that \( |y| = 1 \) for some \( y \in \partial \varphi(x) \). Without loss of
generality, we can assume \( x = 0 \). Since \( \mathcal{X} \) is open, there exists \( \epsilon > 0 \) such that \( \epsilon \mathbb{B}_d \subseteq \mathcal{X} \). For small \( \theta > 0 \), consider the sets
\[
C_{\epsilon, \theta} := \left\{ x \in \mathbb{R}^d : \left| \frac{x}{|x|} - y \right| \leq \sin \theta, |x| \leq \epsilon \right\}
\]
\[
D_\theta := \left\{ b \in \mathbb{B}_d : \langle y - b, y \rangle \leq 2\theta|y - b| \right\}.
\]
Now, if \( a \in C_{\epsilon, \theta} \) and \( b \in \partial \varphi(a) \), the monotonicity of \( \partial \varphi \) implies that \( \langle y - b, a \rangle \leq 0 \). Hence,
\[
\langle y - b, y \rangle = \langle y - b, y - \frac{a}{|a|} \rangle + \langle y - b, \frac{a}{|a|} \rangle \leq |y - b| \sin \theta \leq |y - b|2\theta.
\]
This shows that \( \partial \varphi(C_{\epsilon, \theta}) \subseteq D_\theta \). But the density \( p \), inside \( C_{\epsilon, \theta} \), is bounded from below by \( \lambda_\epsilon \) and the density \( u_d \) is bounded from above by \( 2/a_d \) inside \( D_\theta \) for \( \theta \ll 1 \): then, in view of the transport
equation (2.4), we have
\[
\frac{2}{a_d} \ell_d(D_\theta) \geq \int_{D_\theta} u_d(b)db \geq \int_{\partial \varphi(C_{\epsilon, \theta})} u_d(b)db = \int_{C_{\epsilon, \theta}} p(x)dx \geq \lambda_d \ell_d(C_{\epsilon, \theta}).
\]
This, however, cannot hold true since \( \ell_d(C_{\epsilon, \theta}) \approx \epsilon^d \vartheta^{d-1} \) and \( \ell_d(D_\theta) \approx \vartheta^{d+1} \) as \( \theta \to 0 \). The claim
follows.

We now proceed to provide sufficient conditions under which the center-outward quantile function \( Q_\pm \) is continuous at every point in the open unit ball (except, possibly, at the origin). It is well
known that differentiability of a lower semicontinuous convex function \( \psi \) (which entails continuity
of its gradient) is equivalent to strict convexity of its convex conjugate (see Theorem 26.3 in [36]).
As announced, the techniques we are using here are in the spirit of those developed by Caffarelli in
[5], [6] or Figalli in [16], [17], which in turn largely rely on the fact that, under some control for the
Monge-Ampère measure, the intersection between the graph and supporting hyperplanes of \( \psi \) either
consists of a single point or has an extreme point (see Theorem 4.10 in [16]). A central result in
Caffarelli’s regularity theory (see Corollary 4.21 in [16]) is that a strictly convex function \( \psi \) on an
open set \( \Omega \) for which there exist constants \( 0 < \lambda < \Lambda \) such that
\[
\lambda \ell_d(A) \leq \mu_\psi(A) \leq \Lambda \ell_d(A)
\]
for every Borel set \( A \subseteq \Omega \) is automatically of class \( \mathcal{C}^{\alpha, \lambda}_{\text{loc}} \) for some \( \alpha > 0 \) that depends only on
\( \lambda, \Lambda, \) and \( d \) (condition (2.10) in the sequel will be summarized, with a slight abuse of notation, as
\( \lambda dx \leq \mu_\psi \leq \Lambda dx \)). The fact that \( U_d \) for \( d \geq 2 \) has an unbounded density adds some complication to the particular problem here, though. On the other hand, the density \( u_d \) is bounded away from 0, which
allows to control the growth of the Monge-Ampère measure, as we show next.

Lemma 2.3. If \( P \) satisfies the assumptions in Proposition 2.1(ii), denoting by \( M \) a compact subset
of \( \mathbb{B}_d \), there exist constants \( \alpha_M \) and \( A_M \) such that, for every Borel set \( A \subseteq M \),
\[
\alpha_M \ell_d(A) \leq \mu_\psi(A) \leq A_M (\ell_d(A))^{1/d}.
\]

Proof. The compactness of \( M \) entails that of \( \partial \psi(M) \); in particular, \( \partial \psi(M) \subseteq R \mathbb{B}_d \) for some \( R > 0 \).
Hence, using Proposition 2.1(ii) and taking \( \lambda_R, \Lambda_R \in \mathbb{R} \) as in Assumption A, we obtain
\[
\mu_\psi(A) = a_d \int_A \frac{1}{p(\nabla \psi(y))|y|^{d-1}}dy \geq \left( \frac{a_d}{\Lambda_R} \right) \ell_d(A).
\]
For the upper bound in (2.11), note that the ball \( (\ell_d(A)/c_d)^{1/d} B_d \) (where \( c_d = \pi^{d/2}/\Gamma(1 + d/2) \)
denotes the volume of the \( d \)-dimensional unit ball) maximizes \( \int_B |y|^{1-d}dy \) among all subsets of \( \mathbb{B}_d \).
with Lebesgue measure $\ell_d(A)$. On the other hand, by the co-area formula (see, e.g., Proposition 1, p. 118 in [13]),
\begin{equation}
\int_{r\mathbb{B}_d} |y|^{1-d} dy = \int_0^r \left[ \int_{\partial s\mathbb{B}_d} |y|^{1-d} d\mathcal{H}^{d-1}(y) \right] ds = \int_0^r a_d ds = a_d r
\end{equation}
where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. Combining (2.12) with Proposition 2.1(ii), we conclude that
\begin{equation}
\mu_{\tilde{\psi}}(A) \leq \frac{1}{\lambda_R a_d} \int_A |y|^{1-d} dy \leq \frac{1}{\lambda_R a_d^{1/d}} (\ell_d(A))^{1/d}.
\end{equation}
\[\square\]

Note that the lower bound in Lemma 2.3 remains valid for a compact subset $M$ of $\mathbb{B}_d$ provided that $\partial \psi(M)$ is bounded: indeed, that lower bound only requires the upper bound from Assumption A. A similar conclusion holds for the upper bound. Additionally, if the density $p$ is uniformly bounded, the lower bound holds for any subset of $\mathbb{B}_d$.

### 2.3 Main result

We are ready now for the main result of this note. Our proof follows the lines of [16], [17], and [10], and the related Proposition 3.3 in [19], but, unlike [17], we cover cases in which the support of the probability $P$ is not the whole space $\mathbb{R}^d$. Similar to [10] and [19], we have to handle carefully the fact that $X$ is not necessarily bounded and use a “minimal” extension of the quantile function potential, namely,
\begin{equation}
\tilde{\psi}(z) := \sup_{b \in \mathbb{B}_d, \ y \in \partial \psi(b)} \{ \langle y, z - b \rangle + \psi(b) \}, \quad z \in \mathbb{R}^d.
\end{equation}

Obviously, $\tilde{\psi}$ is still a lower semicontinuous convex function and $\tilde{\psi}(z)$ coincides with $\psi(z)$ for $z \in \mathbb{B}_d$. Since $Q_{\pm}(z) := \nabla \psi(z) \in X$ for every differentiability point $z$ of $\psi$ in $\mathbb{B}_d$, we see (using, once more, Corollary A.27 in [16]) that, provided that $X$ is convex, $\partial \psi(\mathbb{B}_d) \subseteq X$. The “minimality” of the extension (2.13) refers to the fact that $\partial \tilde{\psi}(\mathbb{R}^d) \subseteq X$, as can be checked from a simple application of the Hahn-Banach separation theorem. Of course, the values of $\tilde{\psi}$ outside $\mathbb{B}_d$ are not relevant for the study of its differentiability inside $\mathbb{B}_d$, but the use of $\tilde{\psi}$ will be useful in the next proof. We note also that the discussion leading to Proposition 2.1 can be reproduced with $\tilde{\psi}$ substituted for $\psi$ to conclude that $\mu_{\tilde{\psi}}$ is absolutely continuous with respect to the Lebesgue measure and that, for every Borel set $B \subseteq \mathbb{R}^d$,
\begin{equation}
\mu_{\tilde{\psi}}(B) = \int_{B \cap X} \frac{u_d(y)}{p(\nabla \psi(y))} dy.
\end{equation}

Finally, observe that $\mu_{\tilde{\psi}}$ in concentrated on $\mathbb{B}_d$, that is, if $B \subseteq \mathbb{R}^d \setminus \mathbb{B}_d$, then $\mu_{\tilde{\psi}}(B) = 0$, see Theorem 4.8 in [40] or [10] for further details.

The main result of this note follows from the following crucial lemma.

**Lemma 2.4.** Under the assumptions of Theorem 2.5, $\tilde{\psi}$ is strictly convex on $\mathbb{B}_d$.

**Proof.** To prove this, assume that the contrary holds true. Then, there exists $y \in \mathbb{B}_d$ and $t \in \partial \tilde{\psi}(y)$ such that, putting $l(z) := \psi(y) + \langle t, z - y \rangle$, the convex set $\Sigma := \{ z : \psi(z) = l(z) \}$ is not a singleton. By subtracting an affine function, we can assume $\tilde{\psi}(y) = 0$ and $\tilde{\psi}(z) \geq 0$ for all $z$; then, $\Sigma = \{ z : \tilde{\psi}(z) = 0 \} = \{ z : \psi(z) \leq 0 \}$, which is closed since $\psi$ is lower semicontinuous. Also, by adding the convex function $w(z) := \frac{1}{2} |z|^2$ (note that $\tilde{\psi} = \tilde{\psi} + w$ on $\mathbb{B}_d$), we can assume that $\Sigma \subseteq \mathbb{B}_d$. Being compact and convex, $\Sigma$ equals the closed convex hull of its extreme points; as a consequence, it must have at least two exposed points (otherwise it would be empty or a singleton). Let $\bar{y} \in \mathbb{B}_d \setminus \{ 0 \}$ be one of them. If $\bar{y} \in \mathbb{B}_d \setminus \{ 0 \}$, we consider a small ball $C_\bar{y}$, say, around $\bar{y}$, such that $\bar{C}_\bar{y} \subseteq \mathbb{B}_d \setminus \{ 0 \}$. Then $\partial \tilde{\psi}(C_\bar{y})$ is a compact set, and hence $\partial \tilde{\psi}(C_\bar{y}) \subset R \mathbb{B}_d$ for some $R > 0$. By Proposition 2.1(ii), we have constants $0 < \lambda_{C_\bar{y}} \leq \lambda_{C_\bar{y}}$ such that the Monge-Ampère measure $\mu_{\tilde{\psi}}$ satisfies $\lambda_{C_\bar{y}} d\mu_{\tilde{\psi}} \leq \mu_{\tilde{\psi}} \leq \lambda_{C_\bar{y}} d\mu_{\tilde{\psi}}$ in $C_\bar{y}$. But the set $\Sigma$ has an exposed point in $C_\bar{y}$ and this contradicts
Theorem 4.10 in [16]. Consequently, we must assume that \( \tilde{\bar{y}} \in \partial \mathbb{B}_d \). Observe that \( \tilde{\psi}(\tilde{\bar{y}}) = 0 \), hence \( \tilde{\bar{y}} \in \text{dom}(\tilde{\psi}) \). First consider the case where \( \bar{y} \notin \text{dom}(\tilde{\psi}) \). Let \( \mathbb{B}_r(x) := \{x + r\mathbb{B}_d\} \) and \( \mathbb{B}_r(x) \) denote, respectively the open and the closed ball of radius \( r \) centered at \( x \). Then, for \( \eta > 0 \) small enough, \( \mathbb{B}_\eta(\bar{y}) \subset \text{dom}(\tilde{\psi}) \); consequently, there exists some \( R_0 \) such that \( \tilde{\psi}(\mathbb{B}_\eta(\bar{y})) \subset R_0 \mathbb{B}_d \). For \( \eta \) small enough, we further can ensure that \( \mathbb{B}_\eta(\bar{y}) \subset 2\mathbb{B}_d \).

Without any loss of generality, let us assume that \( \tilde{\bar{y}} = e_1 \) where \( e_1 \) stands for the first vector in the canonical basis of \( \mathbb{R}^d \) (we can use a rotation otherwise):

\[
\Sigma \subset \{z = (z_1, \ldots, z_d) \in \mathbb{R}^d : z_1 \leq 1\}, \quad \text{and} \quad \Sigma \cap \{z = (z_1, \ldots, z_d) \in \mathbb{R}^d : z_1 = 1\} = \{e_1\}.
\]

For \( \sigma \in (0, 1) \) small enough, we have

\[
\Sigma \cap \{z \in \mathbb{R}^d : z_1 \geq 1 - \sigma\} \subset \mathbb{B}_d \cap \{z \in \mathbb{R}^d : z_1 \geq 1 \} \subset \mathbb{B}_\eta(e_1).
\]

For such \( \sigma \), defining

\[
\psi_\epsilon(z) := \tilde{\psi}(z) - \epsilon(z_1 - 1 + \sigma) \quad \text{and} \quad S_\epsilon := \{z : \psi_\epsilon(z) < 0\}, \quad (2.15)
\]

observe that

\[
S_\epsilon \rightarrow \Sigma \cap \{z \in \mathbb{R}^d : z_1 \geq 1 - \sigma\} \quad (2.16)
\]

in the Hausdorff distance\(^8\) \( d \) as \( \epsilon \to 0 \). Hence, for \( \epsilon > 0 \) small enough, the sets \( S_\epsilon \) are bounded open convex subsets of the ball \( \mathbb{B}_\eta(e_1) \). By Lemma 2.3, there exists some \( M > 0 \) such that

\[
\mu_{\psi_\epsilon}(A) = \mu_{\psi_\epsilon}(A) \leq M(\ell_d(A))^{1/d}
\]

for every \( A \subset S_\epsilon \) and \( \epsilon \) small enough.

Next, fix \( z_0 \in \mathbb{B}_d \cap \Sigma \) and \( \delta > 0 \) such that \( \mathbb{B}_\delta(z_0) \subset \mathbb{B}_d \cap \mathbb{B}_\eta(e_1) \) and consider the normalizing map \( L_\epsilon \)—namely, the affine transformation \( L_\epsilon \) that normalizes\(^9\) \( S_\epsilon \): denote by \( \psi_\epsilon \) the normalized solution in \( S_{L_\epsilon}^L := L_\epsilon(S_\epsilon) \) of \( \mu_{\psi_\epsilon} = f \circ L_\epsilon^{-1} \) with the boundary condition \( \psi_\epsilon = 0 \) on \( \partial S_{L_\epsilon}^L \) (\( \psi_\epsilon \) is the convex map that has Monge-Ampère measure \( d\mu_{\psi_\epsilon}(x) = f \circ L_\epsilon^{-1}(x)dx \) in \( S_{L_\epsilon}^L \) and vanishes at the boundary of \( S_{L_\epsilon}^L \); its existence and uniqueness is guaranteed, for instance, by Proposition 4.2 in [16]). Since \( \mathbb{B}_d \subset S_{L_\epsilon}^L \), we have that \( L_\epsilon^{-1}(\mathbb{B}_d) \subset 2\mathbb{B}_d \) and, therefore, the map \( L_\epsilon^{-1} \) satisfies

\[
|L_\epsilon(x) - L_\epsilon(z)| \geq \frac{1}{2}|x - z| \quad \text{for all} \quad x, z \in \mathbb{R}^d.
\]

This implies that

\[
L_\epsilon(\mathbb{B}_d) \supset L_\epsilon(B_\delta(z_0)) \supset B_{\delta/2}(L_\epsilon(z_0)). \quad (2.17)
\]

We consider the sets \( S_{L_\epsilon}^{L,d} := \{z \in S_{L_\epsilon}^L : d_H(z, \partial S_{L_\epsilon}^L) \geq \delta/4\} \). Now \( L_\epsilon(z_0) \in S_{L_\epsilon}^L \), a normalized set (it contains the unit ball and is contained in the ball of radius \( d \), the dimension of the Euclidean space). This implies that there exists a constant \( k_d > 0 \), depending only on \( d \) such that (see Theorem 4.23 in [16] or Lemma 3 in [7])

\[
\ell_d(S_{L_\epsilon}^{L,d} \cap B_{\delta/2}(L_\epsilon(z_0))) \geq k_d \delta^d.
\]

In view of Lemma 2.3, the subsequent remark, and the fact that \( \mathbb{B}_\delta(z_0) \subset \mathbb{B}_\eta(e_1) \), we have that \( \mu_{\psi_\epsilon} \) is lower bounded over \( \mathbb{B}_\delta(z_0) \), that is, there exists \( \lambda > 0 \) such that \( \mu_{\psi_\epsilon}(A) \geq \lambda \ell_d(A) \) for every \( A \subset \mathbb{B}_\delta(z_0) \). This and (2.17) thus imply that \( \mu_{\psi_\epsilon} \) is bounded from below on \( \mathbb{B}_{\delta/2}(L_\epsilon(z_0)) \).

It follows that, for some \( \lambda > 0 \),

\[
\mu_{\psi_\epsilon}(S_{L_\epsilon}^{L,d}) \geq \lambda \ell_d(S_{L_\epsilon}^{L,d} \cap B_{\delta/2}(L_\epsilon(z_0))) \geq C \delta^d.
\]

\(^8\)Recall that, for \( A, B \subset \mathbb{R}^d \), \( d_H(A, B) := \max \{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\} \).

\(^9\)A convex set \( \Omega \subset \mathbb{R}^d \) is said to be normalized if \( \mathbb{B}_d \subset \Omega \subset d\mathbb{B}_d \). For each open bounded convex set \( \Omega \) there exists a unique invertible affine transformation \( L \) normalizing \( \Omega \) (this is John’s celebrated Lemma of convex analysis, see Lemma A.13 in [16]). We refer to \( L \) as the normalizing map and to \( L(\Omega) \) as the normalized version of \( \Omega \).
This implies that, for \( \epsilon' \) small enough, no ball of radius \( \epsilon' \delta/2 \) can contain \( \partial \nu_t(S^{L,s}_t) \). As a consequence, there exists \( c > 0 \) such that \( \sup_{x \in \partial \nu_t(S^{L,s}_t)} |\nu_t| \geq c \delta \). Using Corollary A.23 in [16], we conclude that

\[
| \min_{S^{L,s}_t} \nu_t | \geq c'' (\delta/2)^2
\]

for some \( c'' > 0 \). On the other hand, using Lemma 2.11 again to upper bound \( \mu_{\psi} \), we obtain

\[
\mu_{\psi}(S^{L,s}_t) = \mu_{\psi}(S_t) \leq M(\ell_d(2B_d))^{1/d}
\]

and, by the Alexandrov maximum principle (e.g. Theorem 2.8. in [16]), this implies that

\[
|v_t(L_t e_1)| \leq C \left( d_H(L_t e_1, \partial S^{L,s}_t) \right)^{1/d}.
\]

This means that the same arguments as in the proof of Theorem 4.10 in [16] yield

\[
\lim_{\epsilon \to 0} d_H(L_t e_1, \partial(S^{L,s}_t)) = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \min_{S^{L,s}_t} v_t = 1,
\]

which is a contradiction.

Finally, consider the case where the exposed point of \( \Sigma \) belongs to \( \partial(\text{dom}(\tilde{\psi})) \); here again, it can be assumed that \( \Sigma = \{ z : \tilde{\psi}(z) = 0 \} \) and that the exposed point of \( \Sigma \) is \( e_1 \). We also can assume, without loss of generality, that \( \text{dom}(\psi) \subset \{ z \in \mathbb{R}^d : z_1 \leq 1 \} \). Hence, \( \{ c e_1 : c \geq 0 \} \subset \partial \tilde{\psi}(e_1) \). For small \( \theta > 0 \) we consider the sets

\[
A_\theta := B_d \cap \{ z = (z_1, z') \in \mathbb{R} \times \mathbb{R}^{d-1} : \theta |z'| \leq z_1 - 1 \leq 0 \}
\]

and

\[
C_\theta := X \cap \{ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > 0, |x'| \leq \theta x_1 \}.
\]

Let \( x \in C_\theta \) and \( z \in \partial \tilde{\psi}^*(x) \). Then \( x \in \partial \tilde{\psi}^*(z) \) and, thanks to the monotonicity of \( \partial \tilde{\psi}^* \), we have that \( \langle x - t e_1, z - e_1 \rangle \geq 0 \) for every \( t \geq 0 \), which entails \( \langle x, z - e_1 \rangle \geq 0 \) (take \( t = 0 \)) and \( \langle e_1, z - e_1 \rangle \geq 0 \) (take \( t \to \infty \)). This means that \( z_1 \leq 1 \) and \( x_1 (z_1 - 1) + \langle x', z' \rangle \geq 0 \), from which we deduce that \( z_1 - 1 \geq \theta |z'| \). Since, by Lemma 2.2, we have \( z \in B_d \), it follows that \( \partial \tilde{\psi}^*(A_\theta) \supset C_\theta \). Also, since both \( 0 \) and \( e_1 \) belong to \( \partial \tilde{\psi}^*(\mathbb{R}^d) \subset X \) which is a convex set with nonempty interior, we can argue as in pp. 8-9 of [10], to conclude that \( \ell_d(C_\theta \cap 2B_d) \geq \theta^{d-1} \) for \( \theta > 0 \) small enough. From the transport equation, we have that

\[
U_d(A_\theta) = \int_{\partial \tilde{\psi}^*(A_\theta)} p(x)dx \geq \int_{C_\theta} p(x)dx \geq \ell_d(C_\theta \cap 2B_d) \geq \theta^{d-1},
\]

where we have used that \( p \) is lower bounded on bounded subsets of \( X \). However, for small \( \theta \), \( A_\theta \) is well separated from \( 0 \) and, consequently, \( u_d \) is upper bounded on \( A_\theta \). This means that

\[
U_d(A_\theta) \leq \ell_d(A_\theta) \leq \theta^{d+1},
\]

which contradicts (2.18). This completes the proof of the claim that \( \tilde{\psi} \) (equivalently, \( \psi \)) is strictly convex in \( B_d \).

We now can state and, based on Lemma 2.4, prove our main result, which extends Theorem 1.1 in Figalli [17] to the case of (bounded or unbounded) convexly supported distributions.

**Theorem 2.5.** Let \( P \) be a probability measure with density \( p \) supported on the open convex set \( X \subset \mathbb{R}^d \).

(i) If \( p \) satisfies (2.8), there exists a compact convex set \( K \) with Lebesgue measure 0 such that the center-outward quantile function \( Q_{\pm} := \nabla \psi \) and the center-outward distribution function \( F_{\pm} := \nabla \psi^* \) are homeomorphisms between \( B_d \setminus \{ 0 \} \) and \( X \setminus K \), inverses of each other.

(ii) If, moreover, \( p \in C^k_{\text{loc}}(X) \) for some \( k \in \mathbb{N} \) and \( \alpha \in (0, 1) \), then \( Q_{\pm} \) and \( F_{\pm} \) are homeomorphisms of class \( C^{k+1,\alpha}_{\text{loc}} \) between \( B_d \setminus \{ 0 \} \) and \( X \setminus K \).
Proposition 3.1. Let the probability measure P have a density on $\mathbb{R}^d$. For any $u$ on the unit sphere $S_{d-1}$, any sequence $(t_n)_{n \in \mathbb{N}}$ of real numbers such that $t_n \to \infty$, and any $y_n \in \partial \varphi(t_n u)$,

$$\lim_{n \to \infty} y_n = u.$$ 

Proof. It follows from (2.3) that $y_n \to y_{\infty}$ in $\mathbb{B}_d$. Hence, by compactness, there exists a subsequence along which $y_{n_k} \to y_{\infty} \in \mathbb{B}_d$. On the other hand, monotonicity of the subdifferential implies that, for all $x \in \mathbb{R}^d$ and $y \in \partial \varphi(x)$,

$$\langle y - y_{n_k}, x - t_n u \rangle \geq 0$$

or, equivalently, for all $x \in \mathbb{R}^d$ and $y \in \partial \varphi(x)$,

$$\langle y - y_{\infty}, x \rangle + \langle y_{\infty} - y_n, x \rangle \geq t_n \left( \langle y - y_{\infty}, u \rangle + \langle y_{\infty} - y_n, u \rangle \right).$$

3 Some further properties of center-outward distribution and quantile functions.

To conclude this note, we present three results that more or less directly follow as consequences of Theorem 2.5. The first one is about the asymptotic invariance of center-outward distribution functions; the second one deals with the ability of center-outward quantile functions to capture the shape of a convex supporting set; the third one is a result on the shape of quantile contours, which turn out to satisfy a kind of relaxed version of convexity, connected to the so-called "lighthouse convexity" property (see, e.g., pp. 263-264 in [9]).

A classical univariate distribution function $F$ trivially satisfies

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1,$$

hence, in terms of the univariate center-outward distribution function $F_{\pm} := 2F - 1$,

$$\lim_{t \to \infty} F_{\pm}(tu) = u \quad \text{for all } u \text{ such that } |u| = 1.$$

Let us show that this carries over to $F_{\pm}$ in general dimension. Keeping the notation from the previous sections, we establish the following result.

Proposition 3.1. Let the probability measure $P$ have a density on $\mathbb{R}^d$. For any $u$ on the unit sphere $S_{d-1}$, any sequence $(t_n)_{n \in \mathbb{N}}$ of real numbers such that $t_n \to \infty$, and any $y_n \in \partial \varphi(t_n u)$,

$$\lim_{n \to \infty} y_n = u.$$
Fixing $\epsilon > 0$ and $N = N(\epsilon)$ such that $|y_n - y_\infty| < \epsilon$ for all $n \geq N$, we obtain

$$\langle y - y_\infty, x \rangle + \epsilon|x| \geq t_n \left( \langle y - y_\infty, u \rangle - \epsilon \right).$$

Hence, for $n$ large enough, $\langle y - y_\infty, u \rangle - \epsilon < 0$ for all $x \in \mathbb{R}^d$ and $y \in \partial \varphi(x)$. Since $\epsilon > 0$ is arbitrary, we conclude that

$$\partial \varphi(\mathbb{R}^d) \subset S := \{y : \langle y - y_\infty, u \rangle \leq 0\},$$

which is a hyperplane. Now, the fact that $\nabla \varphi$ pushes $P$ forward to $U_d$ implies that $\partial \varphi(\mathbb{R}^d)$ contains almost every $x \in \mathbb{B}_d$. Hence, $\mathbb{B}_d \subset S$, which only can happen if $y_\infty = u$. \hfill \Box

Under additional smoothness assumptions on $P$, the announced result for $F_\pm$ follows as a corollary.

**Corollary 3.2.** Let $P$ satisfy the assumptions in Proposition 2.1(ii). Then, for any $u$ on the unit sphere $\mathbb{S}_{d-1}$ and any sequence $(t_n)_{n \in \mathbb{N}}$ of real numbers such that $t_n \to \infty$, we have

$$\lim_{n \to \infty} F_\pm(t_n u) = u.$$

Next, we include the announced simple result showing that the outer quantile contours of a convexly supported $P$ approach (in Hausdorff distance) the boundary of its support.

**Lemma 3.3.** Let $P$ be a probability measure on $\mathbb{R}^d$ with compact convex support $\mathcal{X}$ and a density $p$ such that $0 < \lambda \leq p \leq \Lambda$ for some $0 < \lambda \leq \Lambda$. Then, as $R \to 1$, $\nabla \psi(R \mathbb{B}_d)$ tends to $\mathcal{X}$ in Hausdorff distance:

$$\lim_{R \to 1} d_H(\nabla \psi(R \mathbb{B}_d), \mathcal{X}) = 0.$$

**Proof.** Since $\nabla \psi(R \mathbb{B}_d)$ is contained in $\mathcal{X}$, we only need to analyse one of the two members of the maximum defining the Hausdorff distance: indeed,

$$d_H(\nabla \psi(R \mathbb{B}_d), \mathcal{X}) = \max\left\{ \sup_{a \in \nabla \psi(R \mathbb{B}_d)} \inf_{x \in \mathcal{X}} |a - x|, \sup_{x \in \mathcal{X}} \inf_{a \in \nabla \psi(R \mathbb{B}_d)} |a - x| \right\}$$

$$= \sup_{y \in \mathbb{B}_d} \inf_{b \in R \mathbb{B}_d} |\nabla \psi(b) - \nabla \psi(y)|.$$

On the other hand, since $r \mathbb{B}_d \subset R \mathbb{B}_d \subset \mathcal{X}$, $\nabla \psi(r \mathbb{B}_d) \subset \nabla \psi(R \mathbb{B}_d) \subset \mathcal{X}$ for $r \leq R$, so that the mapping $R \mapsto d_H(\nabla \psi(R \mathbb{B}_d), \mathcal{X})$ is a decreasing function. Suppose that $d_H(\nabla \psi(R \mathbb{B}_d), \mathcal{X})$ does not tend to 0 when $R$ tends to 1. Then, there exists $\epsilon > 0$ such that, for every $R$, $d_H(\nabla \psi(R \mathbb{B}_d), \mathcal{X}) > \epsilon$; in particular, there exists $x_R \in \mathcal{X}$ such that $|a_R - x_R| > \epsilon$ for all $a_R \in \nabla \psi(R \mathbb{B}_d)$.

Now, for each $n \in \mathbb{N}$, consider the sequences $A_n := \nabla \psi((1 - 1/n)\mathbb{B}_d)$ and $y_n := x_{1 - 1/n} \in \mathcal{X}$. These sequences are such that

$$\inf_{a \in A_n} |a - y_n| \geq \inf_{a \in A_n} |a - y_n| > \epsilon$$

for all $m \leq n$.

By compactness, the sequence $y_n$ admits a convergent subsequence, with limit $y_\infty$, say, with $y_\infty \in \mathcal{X}$. This limit satisfies $\inf_{a \in A_n} |a - y_\infty| > \epsilon$ for all $n \in \mathbb{N}$, which is not possible since $\mathcal{X} = \bigcup_{n \in \mathbb{N}} A_n$. \hfill \Box

Our final result concerns the shape of the quantile contours of smooth probability measures (those satisfying the assumptions of Theorem 2.5). As a consequence of Theorem 2.5, the sets $Q_\pm(r \mathbb{B}_d)$ are bounded, with connected boundary. Beyond this type of topological properties, results on the geometry of the quantile regions are not available. Here we prove that they satisfy a weak form of convexity. Recall from [9] that a set $B \subset \mathbb{R}^d$ is $\rho$-lighthouse convex if, from every point $x$ in the boundary of $B$, there exists an open cone with vertex $x$ and opening angle $\rho > 0$ which is contained in $\mathbb{R}^d \setminus B$. The limiting version of this concept (obtained as $\rho \to 0$) is that for every point $x$ in the boundary of $B$ there exists a ray emanating from $x$ that does not intersect $B$ at any other point. This is precisely what can be proved for quantile sets.
Lemma 3.4. Let \( P \) be a probability measure on \( \mathbb{R}^d \) satisfying the assumptions of Theorem 2.5. Then, for all \( r \in (0, 1) \) and all \( y \) belonging to the boundary of \( Q_\pm(r \mathbb{B}_d) \), there exists a ray \( T \) emanating from \( y \) for which \( Q_\pm(r \mathbb{B}_d) \cap T = \{ y \} \).

Proof. Assume, on the contrary, that there exists \( y \) in the boundary of \( Q_\pm(r \mathbb{B}_d) \) such that for every ray \( T = \{ z \in \mathbb{R}^d : z = y + ts, \ t \geq 0 \} \), there exists in \( Q_\pm(r \mathbb{B}_d) \cap T \) at least one point \( z \) distinct from \( y \). Note that, necessarily, that point can be chosen in the boundary of \( Q_\pm(r \mathbb{B}_d) \). Now, since \( Q_\pm \) is a homeomorphism, it maps boundaries into boundaries. Therefore, we can assume, up to a rotation, that \( y = Q_\pm(re_1) \). Monotonicity of \( Q_\pm \) implies that

\[
\langle u - re_1, Q_\pm(u) - Q_\pm(re_1) \rangle \geq 0 \quad \text{for all } u \in \mathbb{B}_d. \tag{3.1}
\]

However, if \( Q_\pm(u) \in T = \{ z \in \mathbb{R}^d : z = y + te_1, \ t \geq 0 \} \), then \( Q_\pm(u) = Q_\pm(re_1) + te_1 \) for some \( t > 0 \). Hence, by (3.1) \( \langle u - re_1, re_1 \rangle \geq 0 \). This implies that \( u \notin r \mathbb{B}_d \), thus contradicting the assumption that \( T \) has a common point with the boundary of \( Q_\pm(r \mathbb{B}_d) \) other than \( y \). \( \square \)

References


