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Macdonald Polynomials and the Delta Conjecture

Ph.D. Thesis

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Introduction

Since their introduction in 1988, Macdonald polynomials (described in [39]) have played a central role in algebraic combinatorics. These polynomials are actually symmetric functions with coefficients in $\mathbb{Q}(q,t)$, and for appropriate specialisations of q and t they reduce to many well-known families of symmetric functions, such as Schur functions, Hall-Littlewood polynomials, Jack polynomials, and much more. Immediately after their introduction, a slightly modified version of the Macdonald polynomials has been conjectured to be Schur positive, i.e. to be a linear combination of Schur functions with coefficients in $\mathbb{N}[q,t]$.

Motivated by this conjecture, in the 90's Garsia and Haiman introduced the S_n-module of *diagonal harmonics*, i.e. the coinvariants of the diagonal action of S_n on polynomials in two sets of n variables, and they conjectured that its Frobenius characteristic was given by ∇e_n, where ∇ is the *nabla* operator on symmetric functions introduced in [3]. This conjecture was known as $(n+1)^{n-1}$ conjecture, the name coming from the dimension of the module. In 2001 Haiman proved the famous $n!$ *conjecture*, (now $n!$ theorem) in [33], and in 2002 he showed how this results implies the $(n+1)^{n-1}$ conjecture in [34]. Later the authors of [26] formulated the so called *shuffle conjecture*, i.e. they predicted a combinatorial formula for ∇e_n in terms of labelled Dyck paths, which refines the famous q,t-Catalan formulated by Haglund in [23] and then proved by Garsia and Haglund in [18]. Several years later in [28] Haglund, Morse and Zabrocki conjectured a *compositional* refinement of the shuffle conjecture, which specified also the points where the Dyck paths touched the main diagonal. Recently Carlsson and Mellit in [6] proved precisely this refinement, thanks to the introduction of what they called the *Dyck path algebra*.

In [29], Haglund, Remmel and Wilson conjectured a combinatorial formula for $\Delta'_{e_{n-k-1}}e_n$ in terms of decorated labelled Dyck paths, which they called *Delta conjecture*, after the so called Delta operators Δ_f (and their slightly modified version Δ'_f), defined for any symmetric function f, which have been introduced by Bergeron, Garsia, Haiman, and Tesler in [3]. In fact in the same article [29] the authors conjectured a combinatorial formula for the more general $\Delta_{h_m}\Delta'_{e_{n-k-1}}e_n$ in terms of decorated partially labelled Dyck paths, which we call *generalised Delta conjecture*.

These problems have attracted considerable attention since their formulation: a partial list of works about the Delta conjecture is [8, 16, 29, 30, 40, 41, 42, 45, 46]. One of the two results that we will present in this thesis is an important special case of the generalised Delta conjecture, the Schröder case, i.e. the case $\langle \cdot, e_{n-d}h_d\rangle$, which we proved in [11] by generalising some families of symmetric functions introduced in [8].

In [38] Loehr and Warrington conjectured a combinatorial formula for $\nabla\omega(p_n)$ in terms of labelled square paths (ending East), called *square conjecture*. The special case $\langle \cdot, e_n\rangle$ of this conjecture, known as *q,t-square*, has been proved earlier by Can and Loehr in [5]. Recently the full square conjecture has been proved by Sergel in [43] after the breakthrough of Carlsson and Mellit in [6].

The other main result that we will state in this thesis is the Schröder case of a new conjecture of ours, which extends the square conjecture of Loehr and Warrington. In fact, in [9], we conjectured a combinatorial formula for $\frac{[n-k]_t}{[n]_t}\Delta_{h_m}\Delta_{e_{n-k}}\omega(p_n)$ in terms of *decorated partially labelled square paths* that we call *generalised Delta square conjecture*. This reduces to the conjecture of Loehr and Warrington for $m = k = 0$. Moreover, it extends the generalised Delta conjecture in the sense that on decorated partially labelled Dyck paths it gives the same combinatorial statistics. Notice that our conjecture answers a question in [29]. In the same work, we also prove the Schröder case, i.e. the case $\langle \cdot, e_{n-d}h_d\rangle$; the proof, as we already mentioned, will be presented in this thesis.

Some of the results regarding the Delta conjectures, expression by which we mean all the formulations and variants provided so far, are also related to parallelogram polyominoes. These objects and their relevant statistics will also be presented in this thesis, as they provide interesting material to understand the combinatorics behind the Delta operators. The combinatorics of lattice paths and parallelogram polyominoes, and the proofs of the Schröder cases of the two aforementioned conjectures, will constitute the body of this thesis.

The statistics for parallelogram polyominoes have been first introduced in [1], then extended in [8] and [11] to decorated objects. These extensions happen to match several q,t-enumerators for some special cases of the Delta conjectures, and are especially useful in understanding some intermediate steps in the algebraic recursions we use in the proofs, which are otherwise lacking a combinatorial interpretation in terms of Dyck paths.

More results about these conjectures include the case $t = 0$ of the generalised Delta conjecture, which we proved in [10]; this case happens to coincide with the case $q = 0$ of both the generalised Delta conjecture and generalised Delta square conjecture, which extends the result for $m = 0$ proved in [16]. In [7] we also provided a non-compositional proof of the $\langle \cdot, e_{n-j-k}h_jh_k\rangle$ case of the shuffle conjecture, and a connection between the *newdinv* statistic introduced in [13] (further discussed in [35]) and a more natural statistic on partially labelled Dyck paths.

This thesis is organised in the following way. In the first chapter, we are going to introduce all the symmetric functions background needed to understand the setting, and then we prove an important algebraic identity that will be crucial later on. In the second chapter we deal with the combinatorics, giving all the relevant definitions of lattice paths, polyominoes, and their statistics; we also show some interesting bijections which we can use to switch from a bistatistic to another. In the third chapter we give the statement of the Delta conjectures, of which some are now theorems, and a brief explanation about how to interpret the scalar products combinatorially. In the fourth and the fifth chapter we state and prove the combinatorial and algebraic recursions respectively, show that they coincide, and deduce the desired identities for the the Schröder case of both the Delta and Delta square conjectures. In the sixth and last chapter we give an overview of the state of the art for this family of conjectures, as well as some open problems and ideas on how to approach them.

CHAPTER 1

Symmetric functions

The goal of this chapter is to introduce all the algebraic structures needed to state and prove the results. The background setting of this thesis is the algebra of symmetric functions.

1.1 Basics

Let $\mathbb{K}$ be a field, and let $\mathbb{K}[x_1, \ldots, x_n]$ be the polynomial algebra on n variables over $\mathbb{K}$ The symmetric group S_n acts naturally on this algebra by $\sigma \cdot x_i := x_{\sigma_i}$, where the action is defined on the generators $x_1, \ldots, x_n$ and extended as an algebra morphism. The fixed points of this action form a subalgebra known as *symmetric polynomials*. In order not to be limited by the number of variables, we introduce a bigger algebra that shares many relevant properties with the symmetric polynomials.

For $m > n$ we have a projection map $\rho_{mn} \colon \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}[x_1, \ldots, x_n]$ defined by $\rho_{mn}(x_i) = \delta_{i \le n} x_i$, i.e. it is the identity if $i \le n$, and 0 otherwise. These projections naturally restrict to the symmetric polynomials and allow us to define the projective limit algebra Λ.

Definition 1.1. We define the *symmetric functions* algebra as

$$\Lambda := \varprojlim_{n} \mathbb{K}[x_1, \ldots, x_n]^{S_n}$$

in the category of *graded rings*, meaning that it is the direct sum (over the degree) of the inverse limits of the homogeneous components in fixed degree.

We need to take the projective limit in the category of graded rings in order to only have formal power series of bounded degree. The symmetric functions algebra Λ is endowed with an extremely rich structure. In fact, it is graded, it has a non-degenerate scalar product, and it also has a Hopf algebra structure.

The grading is the natural one inherited from the polynomial algebras, and it is well defined as every element is a formal power series of bounded degree. We denote by $\Lambda^{(d)}$ the vector space of symmetric functions that are homogeneous of degree d. The Hopf algebra structure is more easily defined in terms of *plethystic notation*, which we will introduce later in this chapter. From now on we are going to assume $\mathbb{K}$ to be a field of characteristic 0.

There are three well-known families of symmetric functions that generate Λ as a $\mathbb{K}$-algebra, defined as follows.

Definition 1.2. For $n \in \mathbb{N}$, we define $e_0 = h_0 = p_0 = 1$, and for $n > 0$ we define

- $e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ to be the n-th *elementary* symmetric function,
- $h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ to be the n-th *complete homogeneous* symmetric function,
- $p_n = \sum_{i \geq 1} x_i{}^n$ to be the n-th *power* symmetric function.

In other words, e_n is the sum of all the squarefree monomials of degree n, h_n is the sum of all the monomials of degree n, and p_n is the sum of all the n-th powers of the variables. It turns out that all these three families generate Λ as an algebra, as per the following theorem.

Theorem 1.3 ([44, Theorem 7.4.4, Corollary 7.6.2, Corollary 7.7.2])**.**

$$\Lambda = \mathbb{K}[e_1, e_2, \dots] = \mathbb{K}[h_1, h_2, \dots] = \mathbb{K}[p_1, p_2, \dots].$$

Now we can look for bases of Λ as vector space.

Definition 1.4. A *partition* $\lambda \vdash n$ of $n \in \mathbb{N}$ is an element $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $\lambda_i \geq \lambda_j$ for $i \geq j$ (i.e. the sequence is weakly decreasing) and $\sum \lambda_i = n$.

Its *length* is the minimum index $\ell(\lambda)$ such that $\lambda_{\ell(\lambda)+1} = 0$.

Since all the three families $\{e_n \mid n \in \mathbb{N}\}$, $\{h_n \mid n \in \mathbb{N}\}$, $\{p_n \mid n \in \mathbb{N}\}$ generate Λ as an algebra, the monomials $e_\lambda \coloneqq e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$ for $\lambda \vdash n$ (and the analogously defined h_λ and p_λ) generate $\Lambda^{(n)}$ as a vector space. Thus we have three bases of Λ indexed by partitions.

Definition 1.5. For $\lambda \vdash n$, we define its Ferrers diagram to be the set of cells

$$F(\lambda) \coloneqq \{(i,j) \mid 1 \leq i \leq \lambda_i, 1 \leq j \leq \ell(\lambda)\}.$$

Definition 1.6. For any partition λ, we define its *transpose* λ' by $\lambda'_i \coloneqq \#\{j \mid \lambda_j \geq i\}$, i.e. the partition whose Ferrers diagram is the transpose of the one of λ.

It is convenient to also introduce the *multiplicity* of i in λ, defined as $m_i(\lambda) := \#\{j \mid \lambda_j = i\}$. It is immediate that $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$. Let $z_\lambda = \prod k^{m_k(\lambda)} m_k(\lambda)!$, which is the size of the conjugacy class of a cycle of type λ in $S_{|\lambda|}$. One can show the following.

Proposition 1.7 ([44, Proposition 7.7.6])**.** For $n \in \mathbb{N}$, we have

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda.$$

The space Λ has two other notable bases, also indexed by partitions. We need a few preliminary definitions.

Definition 1.8. A *weak composition* $\alpha \vDash n$ of $n \in \mathbb{N}$ is an element $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $\sum \alpha_i = n$. Its *underlying partition* $\lambda(\alpha) \vdash n$ is the partition of n obtained from α by rearranging the α_i's in decreasing order.

We can now introduce the first of the two bases.

Definition 1.9. For $\lambda \vdash n$, we define

$$m_\lambda = \sum_{\substack{\alpha \vDash n \\ \lambda(\alpha) = \lambda}} {x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots$$

to be the *monomial* symmetric function indexed by λ.

In other words, m_λ is the sum of all the monomials whose exponents are exactly the parts of λ. It is clear that these elements form a basis of $\Lambda^{(n)}$.

Partitions admit two natural partial orderings.

Definition 1.10. The *containment order* on partitions is defined as

$$\mu \subseteq \lambda \iff \mu_i \leq \lambda_i \text{ for all } i \geq 1.$$

Definition 1.11. The *dominance order* on partitions is defined as

$$\mu \leq \lambda \iff \sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i \text{ for all } k \geq 1.$$

Definition 1.12. Let $\lambda \vdash n$ and let $F = F(\lambda)$ be its Ferrers diagram.

A *semi-standard filling* of F is a function $f\colon F \to \mathbb{N} \setminus \{0\}$ that is weakly increasing along rows and strictly increasing along columns.

A *standard filling* of F is a bijective function $f\colon F \to [n]$ that is strictly increasing along rows and columns, i.e. a semi-standard filling whose entries form the set $[n]$.

A *standard* (resp. *semi-standard*) *Young tableau* is a Ferrers diagram together with a standard (resp. semi-standard) filling.

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1} 6 \\
\cline{1-2} 4 & 6 \\
\cline{1-4} 2 & 5 & 5 & 8 \\
\hline 1 & 3 & 4 & 7 & 7 \\
\hline
\end{array}$$

Figure 1.1: A semi-standard Young tableau of shape $\lambda = (5, 4, 2, 1)$.

The underlying partition of a Ferrers diagram is called *shape* of the diagram. We denote by $\mathsf{SYT}(\lambda)$ (resp. $\mathsf{SSYT}(\lambda)$) the set of standard (resp. semi-standard) Young tableaux of shape λ. It is now possible to introduce the second basis.

Definition 1.13. For $\lambda \vdash n$, we define

$$s_\lambda = \sum_{(F,f) \in \mathsf{SSYT}(\lambda)} \prod_{c \in F} x_{f(c)}$$

to be the *Schur* symmetric function indexed by λ.

It is not obvious a priori that s_λ is a symmetric function, but in fact the Schur functions form a basis of Λ. To show that they are symmetric functions, the traditional argument is to give a different definition and then show that the two are equivalent by using a known lemma by Lindström–Gessel–Viennot. More details can be found in [44, Section 7.15-7.16], we will only restate a theorem we need, which is [44, Theorem 7.16.1].

Theorem 1.14 (Jacobi-Trudi identity)**.** For $\lambda \vdash n$, we have

$$s_\lambda = \det\left((h_{\lambda_i + j - i})_{i,j}^{\ell(\lambda) \times \ell(\lambda)} \right).$$

The easiest way to show that the Schur functions are, in fact, a basis of Λ is the following combinatorial formula, which gives a unitriangular relation.

Proposition 1.15. For $\lambda \vdash n$, we have

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu}$ is the number of semi-standard Young tableaux of shape λ and content μ (i.e. the filling has μ_1 1's, μ_2 2's, and so on).

Furthermore, $K_{\lambda\lambda} = 1$ and $K_{\lambda\mu} = 0$ unless $\mu \leq \lambda$ in the dominance order.

Proof. The combinatorial identity is trivial once we know that the Schur functions are symmetric.

It is easy to see that $K_{\lambda\lambda} = 1$, as the only way to fill a diagram of shape λ with content λ is to have all the 1's in the first row, all the 2's in the second row, and so on.

Finally, if $\mu \not\leq \lambda$ in the dominance order, there exists k such that $\mu_1 + \cdots + \mu_k > \lambda_1 + \cdots + \lambda_k$; by pigeonhole in a diagram of shape λ and content μ there must be a number lesser or equal than k in a cell (i, j) with $j > k$, which is impossible because the columns have to be strictly increasing. Thus $K_{\lambda\mu} = 0$ unless $\mu \leq \lambda$ in the dominance order.

We can introduce a notable endomorphism of Λ.

Definition 1.16. We define an algebra morphism $\omega\colon \Lambda \to \Lambda$ on the generators e_n by $\omega(e_n) = h_n$.

Proposition 1.17. The morphism ω is an involution. Moreover, $\omega(p_n) = (-1)^{n-1} p_n$ and $\omega(s_\lambda) = s_{\lambda'}$.

Proof. First of all, let us write the generating functions for e_n, h_n, and p_n.

$$\begin{aligned}
E(t) &\coloneqq \sum_{n \in \mathbb{N}} e_n t^n = \prod_{i>0} (1 + x_i t) \\
H(t) &\coloneqq \sum_{n \in \mathbb{N}} h_n t^n = \prod_{i>0} \frac{1}{1 - x_i t} \\
P(t) &\coloneqq \sum_{n \in \mathbb{N}} p_n t^n = \sum_{i>0} \frac{1}{1 - x_i t}
\end{aligned}$$

We have the identities

$$\begin{aligned}
t\frac{d}{dt} H(t) &= \left(\sum_{i>0} \frac{x_i t}{1 - x_i t} \right) \left(\prod_{i>0} \frac{1}{1 - x_i t} \right) = (P(t) - 1) H(t) \\
t\frac{d}{dt} E(t) &= \left(\sum_{i>0} \frac{x_i t}{1 + x_i t} \right) \left(\prod_{i>0} (1 + x_i t) \right) = (1 - P(-t)) E(t).
\end{aligned}$$

Equating the coefficients, we have

$$nh_n = \sum_{i=1}^{n} p_i h_{n-i} \quad \text{and} \quad ne_n = \sum_{i=1}^{n} (-1)^{i-1} p_i e_{n-i}.$$

Applying ω to the second identity and equating the coefficients again, we get $\omega(p_n) = (-1)^{n-1} p_n$, as desired. This implies that ω is an involution.

Now, notice that $H(t)E(-t) = 1$. We have the identity

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = \delta_{n,0}$$

which is equivalent to saying that, given $n \in \mathbb{N}$, the matrices $(h_{i-j})_{i,j}^{n\times n}$ and $((-1)^{i-j} e_{i-j})_{i,j}^{n\times n}$ are inverses of each other. If λ is any partition such that $\ell(\lambda) + \ell(\lambda') \leq n$, looking at the appropriate minors one can show that

$$\det\left((h_{\lambda_i+j-i})_{i,j}^{\ell(\lambda)\times\ell(\lambda)} \right) = \det\left((e_{\lambda'_i+j-i})_{i,j}^{\ell(\lambda)\times\ell(\lambda)} \right).$$

Applying ω and recalling Theorem 1.14, we have $\omega(s_\lambda) = s_{\lambda'}$, as desired.

The Hall scalar product

We can now define a scalar product on Λ.

Definition 1.18. We define the *Hall scalar product* on Λ by declaring that the Schur functions form an orthonormal basis, i.e.

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

This scalar product has several nice properties. First of all, notice that ω maps an orthonormal basis to itself, and thus it is an isometry. We also have that $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$ and $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$.

It is also worth noticing that the Hall scalar product is homogeneous, i.e. $\Lambda^{(i)} \perp \Lambda^{(j)}$ for $i \neq j$. This means that, when taking a scalar product, we can only look at the parts that are homogeneous of the same degree.

Definition 1.19. For $f \in \Lambda$, we define the operator $f^\perp$ as the adjoint of the multiplication by f. Namely, for every $g, h \in \Lambda$, we have $\langle f^\perp g, h \rangle \coloneqq \langle g, fh \rangle$.

Using the Hall scalar product, we can derive the following.

Corollary 1.20. For $\mu \vdash n$, we have

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda.$$

Proof. From Proposition 1.15, we have $\langle h_\lambda, s_\mu \rangle = K_{\lambda\mu}$. The statement follows immediately.

Definition 1.21. We define the classical statistic on partitions $n\colon \{\mu \mid \mu \vdash m, m \in \mathbb{N}\} \to \mathbb{N}$ as

$$n(\mu) \coloneqq \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i.$$

If we identify the partition μ with its Ferrers diagram, i.e. with the collection of cells

$$\{(i,j) \mid 1 \leq i \leq \mu_i, 1 \leq j \leq \ell(\mu)\},$$

then for each cell $c \in \mu$ we refer to the *arm*, *leg*, *co-arm* and *co-leg* (denoted respectively as $a_\mu(c)$, $l_\mu(c)$, $a_\mu(c)'$, $l_\mu(c)'$) as the number of cells in μ that are strictly to the right, above, to the left, and below c in μ, respectively (see Figure 1.2).

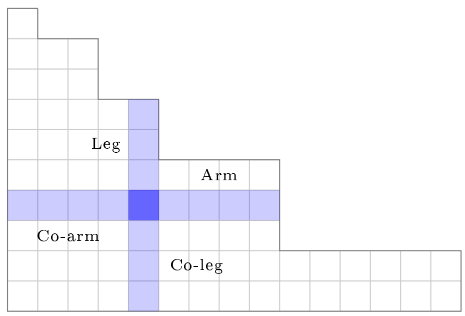


Figure 1.2: Statistics on a Ferrer diagram.

1.2 Plethystic notation

The algebra Λ is endowed with yet another operation, the composition (or *plethysm*). We need to introduce the *plethystic notation* first. See for example [37] for further details.

From now on, our base field will the field $\mathbb{K} \coloneqq \mathbb{Q}(q,t)$ of rational functions in two variables with rational coefficients. Let $\mathbb{Q}(q,t)((x_1, x_2, \dots))$ be the field of formal Laurent series in the indeterminates $x_1, x_2, \dots$ with coefficients in $\mathbb{Q}(q,t)$. Recall that $\Lambda = \mathbb{Q}(q,t)[p_1, p_2, \dots]$ as algebra. Let

$$f = \sum_\lambda f_\lambda(q,t) p_\lambda \in \Lambda$$

with $f_\lambda(q,t) \in \mathbb{Q}(q,t)$, and let $A(x_1, x_2, \dots; q, t) \in \mathbb{Q}(q,t)((x_1, x_2, \dots))$.

Definition 1.22. The *plethystic evaluation* of f in A is

$$f[A] \coloneqq \sum_\lambda f_\lambda(q,t) \prod_{i=1}^{\ell(\lambda)} A({x_1}^{\lambda_i}, {x_2}^{\lambda_i}, \dots; q^{\lambda_i}, t^{\lambda_i}) \in \mathbb{Q}(q,t)((x_1, x_2, \dots)).$$

Equivalently, $f[A]$ is the image of the $\mathbb{Q}(q,t)$-algebra homomorphism mapping p_n to the formal Laurent series obtained from A by rising every variable (including q,t) to the n-th power.

It is easy to check that $f[x_1+x_2+\dots] = f(x_1,x_2,\dots)$. More generally, if A has an expression as sum of monomials (possibly containing q,t, but all with coefficient 1), then $f[A]$ is the expression obtained by replacing the x_i's with such monomials. In this sense, we can interpret a sum of monomials as an alphabet, and a sum of expressions as concatenation of alphabets. We will write

$$X \coloneqq x_1 + x_2 + \dots$$

(and the same for Y, Z) as a shorthand for a sum of variables.

The plethystic evaluation has several nice properties.

- If $g \in \Lambda \subseteq \mathbb{Q}(q,t)((x_1,x_2,\dots))$, then $f[g] \in \Lambda$. This operation, called *plethysm*, is associative.
- If $f \in \Lambda^{(d)}$, then $f[uX] = u^d f[X]$ for any indeterminate u, and $f[-X] = (-1)^d \omega f[X]$. Notice that evaluating the indeterminates does not commute with the plethystic evaluation.
- The coproduct $\Delta(f[X]) = f[X+Y]$ and the antipodal map $S(f[X]) = f[-X]$ define a Hopf algebra structure on Λ.
- Let ϵ be the automorphism defined by $f[\epsilon X] \coloneqq \omega f[-X]$. It corresponds to the substitution $x_i \mapsto -x_i$ (which is not the same as $X \mapsto -X$).

Since the sum of two alphabets can be seen as the concatenation, we can easily derive the following summation formulae.

Proposition 1.23. For $n \in \mathbb{N}$, the following summation formulae hold.

$$e_n[X+Y] = \sum_{i=0}^{n} e_i[X] e_{n-i}[Y] \quad \text{and} \quad h_n[X+Y] = \sum_{i=0}^{n} h_i[X] h_{n-i}[Y].$$

A detailed proof of this statement can be found in [37]. Now recalling that, if $f \in \Lambda^{(n)}$, then $f[-X] = (-1)^n \omega f[X]$, we immediately get the following corollary.

Corollary 1.24. For $n \in \mathbb{N}$, the following subtraction formula holds.

$$e_n[X-Y] = \sum_{i=0}^{n} (-1)^{n-i} e_i[X] h_{n-i}[Y]$$

To deal with the products, we need the *Cauchy identity*.

Theorem 1.25 (Cauchy identity)**.** Let $\{u_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$, $\{v_\lambda \mid \lambda \vdash n, n \in \mathbb{N}\}$ be a pair of dual bases of Λ with respect to the Hall scalar product. Then for $n \in \mathbb{N}$,

$$h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X] v_\lambda[Y].$$

Proof. First of all, notice that the following are equivalent:

1. There exists a pair of dual bases $\{u_\lambda\}$, $\{v_\lambda\}$ such that $h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X] v_\lambda[Y]$;
2. For every pair of dual bases $\{u_\lambda\}$, $\{v_\lambda\}$ the identity $h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X] v_\lambda[Y]$ holds;
3. For every $f \in \Lambda$, $\langle h_n[XY], f[X] \rangle = f[Y]$.

It is immediate that (3.) $\implies$ (2.) $\implies$ (1.) $\implies$ (3.), hence the statements are all equivalent. To prove the statement is therefore sufficient to show (1.) for a pair of dual bases of our choice; we will do that for $\{p_\lambda\}$ and $\{\frac{p_\lambda}{z_\lambda}\}$.

In particular, we will show that

$$\sum_{n \in \mathbb{N}} h_n[XY] = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda[X] p_\lambda[Y]$$

and the statement will follow by isolating the part in degree n (because Λ is graded).

The first equality is trivial as $XY = \sum x_i y_j$ and the infinite product is precisely the generating function for the complete homogeneous symmetric functions. The second equality requires a little more work. We have

$$\begin{aligned}
\prod_{i,j} \frac{1}{1 - x_i y_j} &= \prod_{i,j} \exp(-\log(1 - x_i y_j)) \\
&= \prod_{i,j} \exp\left(\sum_k \frac{(x_i y_j)^k}{k} \right) = \exp\left(\sum_{i,j,k} \frac{(x_i y_j)^k}{k} \right) \\
&= \exp\left(\sum_k \frac{p_k[X] p_k[Y]}{k} \right) = \sum_n \frac{1}{n!} \left(\sum_k \frac{p_k[X] p_k[Y]}{k} \right)^n \\
&= \sum_n \frac{1}{n!} \sum_{\sum \alpha_k = n} \binom{n}{\alpha_1, \dots, \alpha_\ell} \prod_{k=0}^{\ell} \left(\frac{p_k[X] p_k[Y]}{k} \right)^{\alpha_k} \\
&= \sum_\lambda \frac{1}{z_\lambda} p_\lambda[X] p_\lambda[Y]
\end{aligned}$$

where in the last step we collect the compositions with the same parts sizes. This proves the theorem.

We introduce a translation operator.

Definition 1.26. We define the translation operator $\tau_z \colon \Lambda[[z]] \to \Lambda[[z]]$ as $\tau_z(f[X]) = f[X+z]$.

This operator can be computed using the following formula.

Proposition 1.27 ([20, Theorem 1.1])**.** For z any variable, we have

$$\tau_z = \sum_{r \in \mathbb{N}} z^r {h_r}^{\perp}.$$

1.3 q-notation

Before moving to the next family of symmetric functions, it is convenient to introduce the so called q-notation. In general, a q-analogue of an expression is a generalisation involving a parameter q that reduces to the original one for $q \to 1$.

Definition 1.28. For a natural number $n \in \mathbb{N}$, we define its q-analogue as

$$[n]_q \coloneqq \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Given this definition, one can define the q-factorial and the q-binomial as follows.

Definition 1.29. For $0 \leq k \leq n \in \mathbb{N}$, we define

$$[n]_q! \coloneqq \prod_{k=1}^{n} [k]_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q \coloneqq \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Many results about binomials naturally generalise to their q-analogues. For example, we have the following.

Proposition 1.30 (***q***-Pascal identities)**.** For $0 < k < n \in \mathbb{N}$, we have the two identities

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Proof. We have

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\
&= \frac{q^k [n-k]_q + [k]_q}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\
&= \frac{q^k [n-k]_q}{[k]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\
&= q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q
\end{aligned}$$

as desired. The second identity follows immediately by swapping k and $n-k$.

Using this identities, which generalise the standard Pascal identities for binomial coefficients, it is easy to show by induction that the q-binomials are actually polynomials with non-negative integer coefficients.

It is convenient to introduce another notation, namely the q-Pochhammer symbol.

Definition 1.31. For x any variable and $n \in \mathbb{N} \cup \{\infty\}$, we define the *$q$-Pochhammer symbol* as

$$(x;q)_n \coloneqq \prod_{k=0}^{n-1} (1 - xq^k) = (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}).$$

This notation is often handy. For example, we have the obvious identity $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$, or the following theorem due to Cauchy.

Theorem 1.32 (***q*-binomial theorem**)**.** For x any variable and $n \in \mathbb{N}$, we have

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

We also have the following plethystic expansions for the elementary and complete homogeneous symmetric functions.

Proposition 1.33. For $k, n \in \mathbb{N}$ we have

$$e_k[[n]_q] = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{and} \quad h_k[[n]_q] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q.$$

Furthermore we have the extension for $n = \infty$, namely

$$e_k\left[\frac{1}{1-q}\right] = q^{\binom{k}{2}} \frac{1}{(q;q)_k} \quad \text{and} \quad h_k\left[\frac{1}{1-q}\right] = \frac{1}{(q;q)_k}.$$

See [44, Theorem 7.21.2, Corollary 7.21.3] for a proof of these statements.

Before moving on, we need to extend the definitions of q-binomial, as we will need a slightly stronger version of Proposition 1.30 later.

Definition 1.34. For $n, k \in \mathbb{Z}$ we define the q-binomial as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \delta_{k\geq 0} \frac{(q^{n-k+1};q)_k}{(q;q)_k},$$

which agrees with Definition 1.29 for $0 \leq k \leq n$.

This modified definition is not necessarily symmetric in k and $n-k$ unless $0 \leq k \leq n$. Also notice that, since $(x;q)_0 = 1$ as the q-Pochhammer symbol yields an empty product, then

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \text{ for any } n \in \mathbb{Z}.$$

Proposition 1.35. The first q-Pascal identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

holds for any $n, k \in \mathbb{Z}$.

Proof. For $k < 0$ and any n the statement reduces to $0 = 0+0$; for $k = 0$ and any n the statement reduces to $1 = 1 + 0$; for $k > 0$ and $n = 0$ the statement reduces to

$$0 = \delta_{k,1} \cdot q \frac{1-q^{-1}}{1-q} + \delta_{k,1}$$

(which is true); finally for $k > 0$ and $n \neq 0$ we have

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{(q^{n-k+1};q)_k}{(q;q)_k} \\ &= \frac{q^k}{1-q^n} \frac{(q^{n-k+1};q)_k}{(q;q)_k} + \frac{1-q^k}{1-q^n} \frac{(q^{n-k+1};q)_k}{(q;q)_k} \\ &= q^k \frac{(q^{n-k};q)_k}{(q;q)_k} + \frac{(q^{n-k+1};q)_{k-1}}{(q;q)_{k-1}} \\ &= q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \end{aligned}$$

as desired.

1.4 Macdonald polynomials

The Macdonald polynomials form another basis of the symmetric functions. They have been introduced by Ian Macdonald in [39], and since then they played an important role in algebraic combinatorics.

Remark 1.36. Macdonald polynomials can be defined in abstract starting from any root system. We will only consider the one associated to the symmetric group, and we will actually use a modified version that fits our purposes better.

As all the bases of Λ that we already introduced, Macdonald polynomials are also indexed by partitions. Unlike the other bases, though, the definition of Macdonald polynomials relies on the fact that the base field is $\mathbb{Q}(q,t)$.

Definition 1.37 ([32, Proposition 2.6]). The (modified) Macdonald polynomials $\widetilde{H}_\mu[X;q,t]$ are defined by the triangularity and normalization axioms

$$\text{(T1)} \qquad \widetilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda \geq \mu} a_{\lambda\mu}(q,t) s_\lambda[X]$$

$$\text{(T2)} \qquad \widetilde{H}_\mu[X(1-t);q,t] = \sum_{\lambda \geq \mu'} b_{\lambda\mu}(q,t) s_\lambda[X]$$

$$\text{(N)} \qquad \langle \widetilde{H}_\mu[X;q,t], s_{(n)}[X] \rangle = 1$$

for suitable coefficients $a_{\lambda\mu}(q,t)$, $b_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$.

The modified Macdonald polynomials were actually first defined in terms of the original ones, but since these properties characterise them uniquely, we will use this result as definition to avoid introducing other families of symmetric functions that we are not going to use.

Let $M \coloneqq (1-q)(1-t)$. For every partition μ, we define the following constants:

$$\begin{aligned}
B_\mu &\coloneqq B_\mu(q,t) = \sum_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)}, \\
D_\mu &\coloneqq D_\mu(q,t) = MB_\mu(q,t) - 1, \\
T_\mu &\coloneqq T_\mu(q,t) = \prod_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)} = q^{n(\mu')} t^{n(\mu)} = e_{|\mu|}[B_\mu], \\
\Pi_\mu &\coloneqq \Pi_\mu(q,t) = \prod_{c\in\mu/(1,1)} (1 - q^{a'_\mu(c)} t^{l'_\mu(c)}), \\
w_\mu &\coloneqq w_\mu(q,t) = \prod_{c\in\mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}).
\end{aligned}$$

We need to introduce a new scalar product on Λ.

Definition 1.38. We define the *star scalar product* on Λ as

$$\langle f, g\rangle_* := \langle \omega f[MX], g\rangle.$$

It turns out that the Macdonald polynomials are orthogonal with respect to the star scalar product. More precisely,

$$\langle \widetilde{H}_\lambda, \widetilde{H}_\mu\rangle_* = w_\mu(q,t)\delta_{\lambda,\mu}.$$

Macdonals polynomials are needed to define some linear operators that will be crucial later.

Definition 1.39 ([2, 3.11])**.** We define the linear operator $\nabla\colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as

$$\nabla \widetilde{H}_\mu = T_\mu \widetilde{H}_\mu.$$

Definition 1.40. We define the linear operator $\mathbf{\Pi}\colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as

$$\mathbf{\Pi}\widetilde{H}_\mu = \Pi_\mu \widetilde{H}_\mu$$

where we conventionally set $\Pi_\varnothing := 1$.

The following result is extremely powerful, and it can be used to derive many of the identities in the remainder of this chapter.

Theorem 1.41 ([20, Theorem I.2])**.** For every $f \in \Lambda$, $\mu \vdash n$, we have

$$\langle f[X], \widetilde{H}_\mu[X+1]\rangle_* = \nabla^{-1}\tau_{-\epsilon} f[X]\big|_{X=D_\mu}.$$

We will need some classical identities involving the Macdonald polynomials. The first one is the Macdonald-Koorwinder reciprocity [39, VI (6.6)] (see also [25, Theorem 2.16])

Theorem 1.42 (Macdonald-Koorwinder reciprocity)**.** Let $\mu \vdash m$, $\nu \vdash n$. Then

$$\Pi_\nu \widetilde{H}_\mu[MB_\nu] = \Pi_\mu \widetilde{H}_\nu[MB_\mu].$$

The Cauchy identity holds for any scalar product and any pair of dual bases. In particular, using the star scalar product and the Macdonald polynomials (adequately normalised), we have the following.

Proposition 1.43. For $n \in \mathbb{N}$, we have

$$e_n \left[\frac{XY}{M} \right] = \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu[X] \widetilde{H}_\mu[Y]}{w_\mu}.$$

Proof. We can rewrite $\langle h_n[XY], f[X] \rangle = f[Y]$ as $\langle e_n \left[\frac{XY}{M} \right], f[X] \rangle_* = f[Y]$. Since $\{\widetilde{H}_\mu\}$ and $\{\frac{\widetilde{H}_\mu}{w_\mu}\}$ are dual with respect to the star scalar product, we can use the same argument as Theorem 1.25 and the statement follows immediately.

We need a small lemma.

Lemma 1.44. For $n \in \mathbb{N}$, we have

$$\widetilde{H}_{(n)}[(1-q)X] = (q;q)_n h_n[X].$$

Proof. By definition is immediate that (n) is maximal in the dominance order, i.e. $\lambda \vdash n, \lambda \geq (n) \implies \lambda = (n)$; now from Definition 1.37 (T1) we have $\widetilde{H}_{(n)}[(1-q)X] = a_{(n)(n)}(q,t) s_{(n)}[X]$ as the sum is composed of one term only.

Making the substitution $X \mapsto X/(1-q)$, we get

$$\widetilde{H}_{(n)}[X] = a_{(n)(n)}(q,t) s_{(n)} \left[\frac{X}{1-q} \right],$$

and now, since by Definition 1.37 (N) we have that $\langle \widetilde{H}_{(n)}[X], s_{(n)}[X] \rangle = 1$, we can take the scalar product with $s_{(n)}$ and get

$$a_{(n)(n)}(q,t) \left\langle s_{(n)}[X], s_{(n)} \left[\frac{X}{1-q} \right] \right\rangle = 1.$$

We have

$$\begin{aligned} a_{(n)(n)}(q,t)^{-1} &= \left\langle s_{(n)}[X], s_{(n)} \left[\frac{X}{1-q} \right] \right\rangle \\ (\text{as } s_{(n)} = h_n) &= \left\langle h_n[X], h_n \left[\frac{X}{1-q} \right] \right\rangle \\ (\text{by } 1.25) &= \left\langle h_n[X], \sum_{\lambda \vdash n} s_\lambda[X] s_\lambda \left[\frac{1}{1-q} \right] \right\rangle \\ (\text{by orthogonality}) &= h_n \left[\frac{1}{1-q} \right] \\ (\text{by } 1.33) &= \frac{1}{(q;q)_n} \end{aligned}$$

thus $a_{(n)(n)}(q,t) = (q;q)_n$ and the statement follows immediately.

As a corollary of Proposition 1.43, we get the following expansion.

Corollary 1.45. For $n, j \in \mathbb{N}$, we have

$$e_n[X[j]_q] = (1-q^j) \sum_{\mu \vdash n} h_j[(1-t)B_\mu] \frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu}.$$

Proof. We have

$$\begin{aligned}
e_n[X[j]_q] &= e_n\left[\frac{XM[j]_q}{M}\right] \\
\text{(by 1.43)} &= \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu[M[j]_q]\widetilde{H}_\mu[X]}{w_\mu} \\
(\text{as } B_{(j)} = [j]_q) &= \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu[MB_{(j)}]}{\Pi_\mu} \frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu} \\
\text{(by 1.42)} &= \sum_{\mu \vdash n} \frac{\widetilde{H}_{(j)}[MB_\mu]}{\Pi_{(j)}} \frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu} \\
\text{(by 1.44)} &= \sum_{\mu \vdash n} h_j[(1-t)B_\mu] \frac{(q;q)_j}{\Pi_{(j)}} \frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu} \\
(\text{as } \Pi_{(j)} = (q;q)_{j-1}) &= (1-q^j) \sum_{\mu \vdash n} h_j[(1-t)B_\mu] \frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu}
\end{aligned}$$

as desired.

Evaluating at $j = 1$, we get the following.

Corollary 1.46. For $n \in \mathbb{N}$, we have

$$e_n[X] = \sum_{\mu \vdash n} \frac{MB_\mu \Pi_\mu \widetilde{H}_\mu[X]}{w_\mu}.$$

We actually have a stronger result, but to prove it we need a lemma.

Lemma 1.47 ([39, Ex.2 p.362])**.** For every $\mu \vdash n$, we have

$$\langle \widetilde{H}_\mu[X], s_{(n-k,1^k)} \rangle = e_k[B_\mu - 1],$$

or equivalently

$$\langle \widetilde{H}_\mu[X], e_k h_{n-k} \rangle = e_k[B_\mu].$$

Now we can state another expansion.

Proposition 1.48. For $k \in Z$ and $n \in \mathbb{N}$, we have

$$h_k\left[\frac{X}{M}\right]e_{n-k}\left[\frac{X}{M}\right] = \sum_{\mu\vdash n}\frac{e_k[B_\mu]\widetilde{H}_\mu[X]}{w_\mu}.$$

Proof. The expression is equivalent to

$$\left\langle h_k\left[\frac{X}{M}\right]e_{n-k}\left[\frac{X}{M}\right], \widetilde{H}_\mu[X]\right\rangle_* = e_k[B_\mu].$$

By Definition 1.38 and Lemma 1.47 both terms are equal to $\langle \widetilde{H}_\mu, e_k h_{n-k}\rangle$. The statement follows.

Finally, we define the Pieri coefficients as follows.

Definition 1.49. For $k \in \mathbb{N}$ and $f \in \Lambda^{(k)}$, we define the Pieri coefficients $c_{\mu\nu}^{f^\perp}, d_{\mu\nu}^{f}$ by

$$f[X]^\perp \widetilde{H}_\mu[X] = \sum_{\nu\subset_k\mu} c_{\mu\nu}^{f^\perp}\widetilde{H}_\nu[X],$$
$$f[X]\widetilde{H}_\nu[X] = \sum_{\mu\supset_k\nu} d_{\mu\nu}^{f}\widetilde{H}_\mu[X].$$

where $\nu \subset_k \mu$ means that $\nu \subset \mu$ and $|\mu| - |\nu| = k$.

While the existence of the coefficients is a trivial because of linear algebra, the fact that the containments hold follows from [39, VI (6.7)] (see also [19]).

We can immediately derive that

$$w_\nu c_{\mu\nu}^{f^\perp} = \left\langle f^\perp \widetilde{H}_\mu[X], \widetilde{H}_\nu\right\rangle_* = \left\langle \widetilde{H}_\mu[X], \omega f\left[\frac{X}{M}\right]\widetilde{H}_\nu[X]\right\rangle_* = w_\mu d_{\mu\nu}^{\omega f[X/M]}$$

so these two families of coefficients determine each other. It is convenient to define $c_{\mu\nu}^{(k)}$, $d_{\mu\nu}^{(k)}$ by

$$h_k{}^\perp \widetilde{H}_\mu[X] = \sum_{\nu\subset_k\mu} c_{\mu\nu}^{(k)}\widetilde{H}_\nu[X] \quad \text{and} \quad e_k\left[\frac{X}{M}\right]\widetilde{H}_\nu[X] = \sum_{\mu\supset_k\nu} d_{\mu\nu}^{(k)}\widetilde{H}_\mu[X].$$

We have the following expansion.

Proposition 1.50. For any $\mu \vdash n$, we have

$$B_\mu = \sum_{\nu\subset_1\mu} c_{\mu\nu}^{(1)}.$$

Proof. We have

$$\begin{aligned}
B_\mu &= e_1[B_\mu] \\
\text{(by 1.47)} &= \langle \widetilde{H}_\mu, e_1 h_{n-1} \rangle \\
(e_1 = h_1) &= \langle \widetilde{H}_\mu, h_1 h_{n-1} \rangle \\
\text{(by 1.19)} &= \langle {h_1}^\perp \widetilde{H}_\mu, h_{n-1} \rangle \\
\text{(by 1.49)} &= \sum_{\nu \subset_1 \mu} c_{\mu\nu}^{(1)} \langle \widetilde{H}_\nu, h_{n-1} \rangle \\
\text{(by 1.47)} &= \sum_{\nu \subset_1 \mu} c_{\mu\nu}^{(1)}
\end{aligned}$$

as desired.

This concludes the section. More results involving Macdonald polynomials will be stated after the introduction of the Delta operators.

Delta operators

The study of the Delta operators began with the Nabla operator (see Definition 1.39), introduced by F. Bergeron and A. Garsia in [2], which shows a surprising amount of positivity properties.

A key role in the theory is played by ∇e_n. This symmetric function is Schur positive (i.e. its expansion in the Schur basis has coefficients in $\mathbb{N}[q,t]$), and in fact it does more: it is the Frobenius characteristic of the bigraded S_n module called *diagonal harmonics* (see [34]) and it q,t-counts parking functions with respect to two statistics (see [26], [6]). We will show more of this later.

The Delta operators generalise, in some sense, the Nabla operator. Many of the Delta operators show positivity properties as well, which lead to new conjectures about possible combinatorial interpretations for these symmetric functions.

Definition 1.51. For $f \in \Lambda$, we define the linear operators $\Delta_f, \Delta'_f \colon \Lambda \to \Lambda$ on the eigenbasis of Macdonald polynomials as

$$\Delta_f \widetilde{H}_\mu = f[B_\mu] \widetilde{H}_\mu, \qquad \Delta'_f \widetilde{H}_\mu = f[B_\mu - 1] \widetilde{H}_\mu.$$

Notice that, since $e_n[B_\mu] = T_\mu$ for $\mu \vdash n$, we have $\nabla = \bigoplus_n \Delta_{e_n}|_{\Lambda^{(n)}}$. So, while the Nabla operator is not strictly a Delta operator, it can be obtained by gluing Delta operators on the homogeneous subspaces. In this sense the Delta operators generalise the Nabla operator.

Proposition 1.52. We have that

$$\mathbf{\Pi} = \sum_{k \in \mathbb{N}} (-1)^k \Delta'_{e_k},$$

where, since $\Delta'_{e_k} f = 0$ for $f \in \Lambda^{(n)}$, $n \geq k$, the sum is locally finite (i.e. it has a finite number of non-zero addenda).

Proof. Since Macdonald polynomials form an eigenbasis for both $\mathbf{\Pi}$ and Δ_f for any $f \in \Lambda$, it is enough to check that the corresponding eigenvalues match. Let $\mu \vdash n$, let us identify it with its Ferrer's diagram, and for $c \in \mu$ let $p(c) = q^{a'_\mu(c)} t^{l'_\mu(c)}$. We have

$$\begin{aligned} \Pi_\mu &= \prod_{c \in \mu/(1,1)} (1 - q^{a'_\mu(c)} t^{l'_\mu(c)}) \\ \text{(expanding the product)} &= \sum_{k=0}^{n} (-1)^k \sum_{\substack{S \subseteq \mu/(1,1) \\ \#S = k}} \prod_{c \in S} q^{a'_\mu(c)} t^{l'_\mu(c)} \\ \text{(by definition of } B_\mu \text{ and plethysm)} &= \sum_{k=0}^{n} (-1)^k e_k[B_\mu - 1] \\ \text{(as } B_\mu - 1 \text{ has } n{-}1 \text{ terms)} &= \sum_{k \in \mathbb{N}} (-1)^k e_k[B_\mu - 1] \end{aligned}$$

so the corresponding eigenvalues are equal, as desired.

Definition 1.53. For $0 \leq k \leq n$, we define the symmetric function $E_{n,k}$ by the expansion

$$e_n\left[X \frac{1-z}{1-q}\right] = \sum_{k=0}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}.$$

Notice that setting $z = q^j$ we get

$$e_n\left[X \frac{1-q^j}{1-q}\right] = \sum_{k=0}^{n} \frac{(q^j;q)_k}{(q;q)_k} E_{n,k} = \sum_{k=0}^{n} \begin{bmatrix} k+j-1 \\ k \end{bmatrix}_q E_{n,k}$$

and in particular, for $j = 1$, we get

$$e_n = E_{n,0} + E_{n,1} + E_{n,2} + \cdots + E_{n,n},$$

so these symmetric functions split e_n, in some sense. Notice that $E_{n,0} = \delta_{n,0}$.

The following results will be useful later.

Lemma 1.54. For any symmetric function $f \in \Lambda$, we have

$$\langle \Delta_{e_k} f, h_n \rangle = \langle f, e_k h_{n-k} \rangle.$$

Proof. Since the Macdonald polynomials are a basis, and the identity is linear in f, it is enough to show it for $f = \widetilde{H}_\mu$, $\mu \vdash n$.

$$\begin{aligned}\langle \Delta_{e_k} \widetilde{H}_\mu[X], h_n\rangle &= \langle e_k[B_\mu]\widetilde{H}_\mu, h_n\rangle \\ \text{(by linearity)} &= e_k[B_\mu]\langle \widetilde{H}_\mu, h_n\rangle \\ \text{(by 1.47)} &= e_k[B_\mu] \\ \text{(by 1.47)} &= \langle \widetilde{H}_\mu, e_k h_{n-k}\rangle\end{aligned}$$

as desired.

1.5 A summation formula

The goal of this section is to prove the following theorem.

Theorem 1.55 ([8, Theorem 4.6])**.** For $m, n, s \in \mathbb{N}$, we have

$$\begin{aligned}&\sum_{\mu \vdash m+n} \frac{\widetilde{H}_\mu[X]}{w_\mu} h_s[(1-t)B_\mu] e_m[B_\mu] \\ &= \sum_{r=0}^{m} t^{m-r} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] e_{n-z}\left[\frac{X}{M}\right].\end{aligned}$$

We start with a lemma, which is the case $n = 0$.

Lemma 1.56 ([8, Theorem 4.8])**.** For $m, s \in \mathbb{N}$, we have

$$\sum_{\mu \vdash m} \frac{T_\mu \widetilde{H}_\mu[X]}{w_\mu} h_s[(1-t)B_\mu] = \sum_{r=0}^{m} t^{m-r} \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q h_r\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right].$$

Proof. By Definition 1.53 and by linearity of ∇, we have

$$\nabla e_n\left[X[s]_q\right] = \sum_{r=0}^{n} \begin{bmatrix} r+s-1 \\ s-1 \end{bmatrix}_q \nabla E_{n,r}.$$

By [25, Theorem 7.2] we have

$$\nabla E_{n,r} = t^{n-r}(1-q^r) \sum_{\nu \vdash n-r} \frac{T_\nu}{w_\nu} \sum_{\mu \supset_r \nu} \Pi_\mu \widetilde{H}_\mu d_{\mu\nu}^{h_r[X/(1-q)]},$$

which can easily be rewritten as [25, Equation (7.86)], namely

$$\nabla E_{n,r} = t^{n-r}(1-q^r)\mathbf{\Pi}\left(h_{n-r}\left[\frac{X}{M}\right] h_r\left[\frac{X}{1-q}\right]\right).$$

It follows that

$$\begin{aligned}\nabla e_m\left[X[s]_q\right] &= \sum_{r=0}^{m} \begin{bmatrix} r+s-1 \\ s-1 \end{bmatrix}_q t^{m-r}(1-q^r)\mathbf{\Pi}\left(h_{m-r}\left[\frac{X}{M}\right]h_r\left[\frac{X}{1-q}\right]\right) \\ &= (1-q^s)\sum_{r=0}^{m} \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q t^{m-r}\mathbf{\Pi}\left(h_{m-r}\left[\frac{X}{M}\right]h_r\left[\frac{X}{1-q}\right]\right).\end{aligned}$$

By Corollary 1.45 we have

$$\nabla e_m[X[s]_q] = \nabla(1-q^s)\sum_{\mu\vdash m} h_s[(1-t)B_\mu]\frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu}$$

which we can rewrite, using Definitions 1.39 and 1.40 as

$$\nabla e_m[X[s]_q] = (1-q^s)\sum_{\mu\vdash m} h_s[(1-t)B_\mu]\frac{T_\mu \mathbf{\Pi}\widetilde{H}_\mu[X]}{w_\mu}$$

and now, equating the two expressions and applying $\mathbf{\Pi}^{-1}$, we get

$$\sum_{\mu\vdash m}\frac{T_\mu \widetilde{H}_\mu[X]}{w_\mu}h_s[(1-t)B_\mu] = \sum_{r=0}^{m} t^{m-r}\begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q h_r\left[\frac{X}{1-q}\right]h_{m-r}\left[\frac{X}{M}\right]$$

as desired.

Now we need another lemma, which is due to J. Haglund. The ideas behind the proof of this lemma are extremely important, and following the steps carefully one can infer the general strategy we used to prove our summation formula, which is the core result behind the algebraic recursions that will appear later in the thesis.

Lemma 1.57. Let $f \in \Lambda$ such that $\nabla^{-1}\tau_{-\epsilon}f[X]\big|_{X=D_\mu} = h_s[(1-t)B_\mu]e_m[B_\mu]$. Then

$$\sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}h_s[(1-t)B_\mu]e_m[B_\mu] = \sum_{z=0}^{n} e_{n-z}\left[\frac{X}{M}\right](f[X])_z$$

where $(f[X])_z$ denotes the homogeneous component of $f[X]$ in degree z.

Proof. We know by Theorem 1.41 that

$$\langle f[X], \widetilde{H}_\mu[X+1]\rangle_* = \nabla^{-1}\tau_{-\epsilon}f[X]\big|_{X=D_\mu},$$

so we can rewrite the condition as

$$h_s[(1-t)B_\mu]e_m[B_\mu] = \langle f[X], \widetilde{H}_\mu[X+1]\rangle_*.$$

Now,

$$\begin{aligned}
\sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}h_s[(1-t)B_\mu]e_m[B_\mu] &= \sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}\langle f[X],\widetilde{H}_\mu[X+1]\rangle_* \\
&= \sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}\langle f[X],\tau_1\widetilde{H}_\mu[X]\rangle_* \\
&= \sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}\left\langle f[X],\sum_{r=0}^{n}h_r^\perp\widetilde{H}_\mu[X]\right\rangle_* \\
&= \sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}\left\langle \sum_{r=0}^{n}e_r\left[\frac{X}{M}\right]f[X],\widetilde{H}_\mu[X]\right\rangle_* \\
&= \sum_{\mu\vdash n}\frac{\widetilde{H}_\mu[X]}{w_\mu}\left\langle \sum_{r=0}^{n}e_r\left[\frac{X}{M}\right](f[X])_{n-r},\widetilde{H}_\mu[X]\right\rangle_* \\
&= \sum_{r=0}^{n}e_r\left[\frac{X}{M}\right](f[X])_{n-r}
\end{aligned}$$

as desired.

Now we need a couple elementary but technical lemmas about q-binomials.

Lemma 1.58 ([8, Lemma 4.11])**.** For $s,i\in\mathbb{N}$ we have

$$q^{i(i-1)}\begin{bmatrix}s\\i\end{bmatrix}_q=\sum_{k=0}^{i}(-1)^{i-k}q^{\binom{i-k}{2}}\begin{bmatrix}i-1\\i-k\end{bmatrix}_q\begin{bmatrix}k+s-1\\k\end{bmatrix}_q.$$

Proof. We have

$$\begin{aligned}
q^{i(i-1)}\begin{bmatrix}s\\i\end{bmatrix}_q &= q^{i(i-1)}h_i[[s-i+1]_q] \\
&= h_i[q^{i-1}[s-i+1]_q] \\
&= h_i\left[\frac{q^{i-1}-q^s}{1-q}\right] \\
\text{(by 1.23)} &= \sum_{k=0}^{i}h_{i-k}\left[-\frac{1-q^{i-1}}{1-q}\right]h_k\left[\frac{1-q^s}{1-q}\right] \\
&= \sum_{k=0}^{i}(-1)^{i-k}e_{i-k}[[i-1]_q]h_k[[s]_q] \\
\text{(by 1.33)} &= \sum_{k=0}^{i}(-1)^{i-k}q^{\binom{i-k}{2}}\begin{bmatrix}i-1\\i-k\end{bmatrix}_q\begin{bmatrix}k+s-1\\k\end{bmatrix}_q
\end{aligned}$$

as desired.

Lemma 1.59 ([8, Lemma 4.12])**.** For $r, s, z \in \mathbb{N}$, we have

$$q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q = \sum_{i=0}^{s} \sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i} \left[\frac{1}{1-q}\right] e_{i-z+j-r} \left[-\frac{1}{1-q}\right] e_{r-j} \left[\frac{1}{1-q}\right].$$

Proof of Lemma 1.59. We prove the identity for $z \in \mathbb{Z}$, $r + z \geq 0$, by induction on s and $r + z$. This clearly implies that it holds for $r, s, z \in \mathbb{N}$.

Since we require $r \geq 0$, we need the extra base case $r = 0$. In this case, the statement reduces to

$$q^{\binom{z}{2}} \begin{bmatrix} s-1 \\ s-z \end{bmatrix}_q = \sum_{i=0}^{s} q^{\binom{i}{2}} h_{s-i} \left[\frac{1}{1-q}\right] e_{i-z} \left[-\frac{1}{1-q}\right],$$

and we have

$$\begin{aligned}
q^{\binom{z}{2}} \begin{bmatrix} s-1 \\ s-z \end{bmatrix}_q &= q^{\binom{z}{2}} \prod_{i=1}^{s-z} \frac{(1-q^{z+i-1})}{(1-q^i)} \\
&= q^{\binom{z}{2}} h_{s-z} \left[\frac{1}{1-q}\right] \prod_{i=1}^{s-z} (1-q^{z+i-1}) \\
&= q^{\binom{z}{2}} h_{s-z} \left[\frac{1}{1-q}\right] \sum_{i=0}^{s-z} (-q^z)^i q^{\binom{i}{2}} \begin{bmatrix} s-z \\ i \end{bmatrix}_q \\
&= h_{s-z} \left[\frac{1}{1-q}\right] \sum_{i=0}^{s-z} (-1)^i q^{\binom{i+z}{2}} \begin{bmatrix} s-z \\ i \end{bmatrix}_q \\
(i \mapsto i-z) &= h_{s-z} \left[\frac{1}{1-q}\right] \sum_{i=0}^{s} (-1)^{i-z} q^{\binom{i}{2}} \begin{bmatrix} s-z \\ i-z \end{bmatrix}_q \\
&= h_{s-z} \left[\frac{1}{1-q}\right] \sum_{i=0}^{s} (-1)^{i-z} q^{\binom{i}{2}} \frac{[s-z]_q!}{[s-i]_q! [i-z]_q!} \\
&= \sum_{i=0}^{s} (-1)^{i-z} q^{\binom{i}{2}} \frac{(1-q)^{s-z}}{[s-i]_q! [i-z]_q!} \\
&= \sum_{i=0}^{s} q^{\binom{i}{2}} h_{s-z} \left[\frac{1}{1-q}\right] e_{i-z} \left[\frac{1}{1-q}\right]
\end{aligned}$$

as desired. For $s = 0$, the statement reduces to

$$\delta_{z,0} = \sum_{j=0}^{r} e_{-z+j-r} \left[-\frac{1}{1-q}\right] e_{r-j} \left[\frac{1}{1-q}\right],$$

and we have

$$
\begin{aligned}
\delta_{z,0} = e_{-z}[0] &= e_{-z}\left[\frac{1}{1-q} - \frac{1}{1-q}\right] \\
\text{(by 1.23)} &= \sum_{j=0}^{r} e_{-z-j}\left[-\frac{1}{1-q}\right] e_j\left[\frac{1}{1-q}\right] \\
(j \mapsto r-j) &= \sum_{j=0}^{r} e_{-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right]
\end{aligned}
$$

as desired. For $r+z=0$ the statement reduces to

$$
\delta_{r,0}\delta_{s,0} = \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i+j}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right]
$$

and we have

$$
\begin{aligned}
\delta_{r,0}\delta_{s,0} &= e_r\left[\frac{1}{1-q} - \frac{1}{1-q}\right]\left(h_s\left[\frac{1}{1-q}\right](1;q)_s\right) \\
\text{(by 1.23, 1.32)} &= \left(\sum_{j=0}^{r} e_j\left[\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right]\right)\left(h_s\left[\frac{1}{1-q}\right]\sum_{i=0}^{s}(-1)^i q^{\binom{i}{2}}\begin{bmatrix} s \\ i \end{bmatrix}_q\right) \\
&= \sum_{i=0}^{s}\sum_{j=0}^{r}(-1)^i q^{\binom{i}{2}} h_{s-i}\left[\frac{1}{1-q}\right] h_i\left[\frac{1}{1-q}\right] e_j\left[\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
&= \sum_{i=0}^{s}\sum_{j=0}^{r}(-1)^{i+j} q^{\binom{i}{2}} h_{s-i}\left[\frac{1}{1-q}\right] h_i\left[\frac{1}{1-q}\right] h_j\left[\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
&= \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i+j}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right]
\end{aligned}
$$

as desired. Finally, if $s>0$ and $r+z>0$ we have

$$
\begin{aligned}
& q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q \\
\text{(by 1.35)} &= q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \left(q^{s-z}\begin{bmatrix} r+s-2 \\ s-z \end{bmatrix}_q + \begin{bmatrix} r+s-2 \\ s-z-1 \end{bmatrix}_q\right) \\
\text{(by 1.35)} &= q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-2 \\ s-z-1 \end{bmatrix}_q + q^s q^{\binom{z-1}{2}}\left(q^z \begin{bmatrix} r+z-1 \\ z \end{bmatrix}_q + \begin{bmatrix} r+z-1 \\ z-1 \end{bmatrix}_q\right)\begin{bmatrix} r+s-2 \\ s-z \end{bmatrix}_q \\
&= q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+(s-1)-1 \\ (s-1)-z \end{bmatrix}_q + q^s q^{\binom{z}{2}} \begin{bmatrix} (r-1)+z \\ z \end{bmatrix}_q \begin{bmatrix} (r-1)+s-1 \\ s-z \end{bmatrix}_q \\
&\quad + q^s q^{\binom{z-1}{2}} \begin{bmatrix} r+(z-1) \\ z-1 \end{bmatrix}_q \begin{bmatrix} r+(s-1)-1 \\ (s-1)-(z-1) \end{bmatrix}_q,
\end{aligned}
$$

so now we can use the inductive hypothesis and get

$$
\begin{aligned}
\dots &= \sum_{i=0}^{s-1}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i-1}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
&\quad + q^s \sum_{i=0}^{s}\sum_{j=0}^{r-1} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r+1}\left[-\frac{1}{1-q}\right] e_{r-j-1}\left[\frac{1}{1-q}\right] \\
&\quad + q^s \sum_{i=0}^{s-1}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i-1}\left[\frac{1}{1-q}\right] e_{i-z+j-r+1}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right]
\end{aligned}
$$

and since as $r+z>0$ then either $r>0$ or $z>0$ we can shift the indices of the sums and get

$$
\begin{aligned}
\dots &= \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q (1-q^{s-i}) h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
(h \mapsto h-1) &\qquad + q^s \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j-1 \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
(i \mapsto i-1) &\qquad + q^s \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i-1}{2}} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
&= (1-q^{s-i}) \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
(\text{by } 1.35) &\qquad + q^{s-i} \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right] \\
&= \sum_{i=0}^{s}\sum_{j=0}^{r} q^{\binom{i}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q h_{s-i}\left[\frac{1}{1-q}\right] e_{i-z+j-r}\left[-\frac{1}{1-q}\right] e_{r-j}\left[\frac{1}{1-q}\right]
\end{aligned}
$$

as desired.

We need just two more results before being ready to prove our summation formula. The first one is due to A. Garsia, A. Hicks, and A. Stout. The second one is a consequence we proved.

Proposition 1.60 ([17, Proposition 2.6])**.** For $i,j \in \mathbb{N}$ we have

$$h_i\left[\frac{X}{1-q}\right] e_j\left[\frac{X}{M}\right] = \sum_{\mu \vdash i+j} \frac{\widetilde{H}_\mu[X]}{w_\mu} q^{-\binom{i}{2}} \sum_{k=0}^{i} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} i-1 \\ i-k \end{bmatrix}_q h_k[(1-t)B_\mu].$$

Proposition 1.61 ([8, Proposition 4.9])**.** For $i,j \in \mathbb{N}$, we have

$$\nabla\left(h_i\left[\frac{X}{1-q}\right] e_j\left[\frac{X}{M}\right]\right) = \sum_{r=0}^{j} t^{j-r} q^{\binom{i}{2}} \begin{bmatrix} i+r \\ i \end{bmatrix}_q h_{i+r}\left[\frac{X}{1-q}\right] h_{j-r}\left[\frac{X}{M}\right].$$

Proof. Using 1.60, we have

$$\begin{aligned}
\nabla\left(h_i\left[\frac{X}{1-q}\right]e_j\left[\frac{X}{M}\right]\right) &= \sum_{\mu\vdash i+j}\frac{T_\mu\widetilde{H}_\mu[X]}{w_\mu}q^{-\binom{i}{2}}\sum_{k=0}^{i}(-1)^{i-k}q^{\binom{i-k}{2}}\begin{bmatrix}i-1\\ i-k\end{bmatrix}_q h_k[(1-t)B_\mu] \\
(\text{by } 1.56) &= q^{-\binom{i}{2}}\sum_{k=0}^{i}(-1)^{i-k}q^{\binom{i-k}{2}}\begin{bmatrix}i-1\\ i-k\end{bmatrix}_q\sum_{r=0}^{i+j}t^{i+j-r}\begin{bmatrix}r+k-1\\ r-1\end{bmatrix}_q \\
&\quad\times h_r\left[\frac{X}{1-q}\right]h_{i+j-r}\left[\frac{X}{M}\right] \\
(\text{by } 1.58) &= \sum_{r=0}^{i+j}t^{i+j-r}q^{\binom{i}{2}}\begin{bmatrix}r\\ i\end{bmatrix}_q h_r\left[\frac{X}{1-q}\right]h_{i+j-r}\left[\frac{X}{M}\right] \\
(r\mapsto i+r) &= \sum_{r=0}^{j}t^{j-r}q^{\binom{i}{2}}\begin{bmatrix}i+r\\ i\end{bmatrix}_q h_{i+r}\left[\frac{X}{1-q}\right]h_{j-r}\left[\frac{X}{M}\right]
\end{aligned}$$

as desired.

Proof of Theorem 1.55. We want to find a symmetric function $f\in\Lambda$ such that

$$\nabla^{-1}\tau_{-\epsilon}f[X] = h_s\left[\frac{X+1}{1-q}\right]e_m\left[\frac{X+1}{M}\right]$$

as evaluating the expression at $X=D_\mu$ yields the hypothesis of Lemma 1.57.

$$\begin{aligned}
f[X] &= \tau_\epsilon\nabla h_s\left[\frac{X+1}{1-q}\right]e_m\left[\frac{X+1}{M}\right] \\
(\text{by } 1.23) &= \tau_\epsilon\nabla\sum_{i=0}^{s}\sum_{j=0}^{m}h_{s-i}\left[\frac{1}{1-q}\right]e_{m-j}\left[\frac{1}{M}\right]h_i\left[\frac{X}{1-q}\right]e_j\left[\frac{X}{M}\right] \\
(\text{by } 1.61) &= \tau_\epsilon\sum_{i=0}^{s}\sum_{j=0}^{m}h_{s-i}\left[\frac{1}{1-q}\right]e_{m-j}\left[\frac{1}{M}\right]\sum_{k=0}^{j}t^{j-k}q^{\binom{i}{2}}\begin{bmatrix}i+k\\ i\end{bmatrix}_q h_{i+k}\left[\frac{X}{1-q}\right]h_{j-k}\left[\frac{X}{M}\right] \\
(\text{by } 1.26) &= \sum_{i=0}^{s}\sum_{j=0}^{m}h_{s-i}\left[\frac{1}{1-q}\right]e_{m-j}\left[\frac{1}{M}\right]\sum_{k=0}^{j}t^{j-k}q^{\binom{i}{2}}\begin{bmatrix}i+k\\ i\end{bmatrix}_q h_{i+k}\left[\frac{X+\epsilon}{1-q}\right]h_{j-k}\left[\frac{X+\epsilon}{M}\right] \\
(\text{by } 1.23) &= \sum_{i=0}^{s}\sum_{j=0}^{m}h_{s-i}\left[\frac{1}{1-q}\right]e_{m-j}\left[\frac{1}{M}\right]\sum_{k=0}^{j}t^{j-k}q^{\binom{i}{2}}\begin{bmatrix}i+k\\ i\end{bmatrix}_q \\
&\quad\times\sum_{u=0}^{i+k}e_{i+k-u}\left[-\frac{1}{1-q}\right]h_u\left[\frac{X}{1-q}\right]\sum_{v=0}^{j-k}e_{j-k-v}\left[-\frac{1}{M}\right]h_v\left[\frac{X}{M}\right].
\end{aligned}$$

Isolating the part homogeneous in degree d, i.e. fixing $u+v=d$, we have

$$\begin{aligned}(f[X])_d = &\sum_{i=0}^{s}\sum_{j=0}^{m} h_{s-i}\left[\frac{1}{1-q}\right] e_{m-j}\left[\frac{1}{M}\right]\sum_{k=0}^{j} t^{j-k} q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q \\ &\times \sum_{u=0}^{i+k} e_{i+k-u}\left[-\frac{1}{1-q}\right] h_u\left[\frac{X}{1-q}\right] e_{j-k-d+u}\left[-\frac{1}{M}\right] h_{d-u}\left[\frac{X}{M}\right]\end{aligned}$$

Now, making the substitutions $d=m+z$ and $u=r+z$, we have

$$\begin{aligned}(f[X])_{m+z} &= \sum_{i=0}^{s}\sum_{j=0}^{m} h_{s-i}\left[\frac{1}{1-q}\right] e_{m-j}\left[\frac{1}{M}\right]\sum_{k=0}^{j} t^{j-k} q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q \\ &\quad\times \sum_{r+z=0}^{i+k} e_{i+k-r-z}\left[-\frac{1}{1-q}\right] e_{j-k-m+r}\left[-\frac{1}{M}\right] h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] \\
(r\le i+k-z) &= \sum_{i=0}^{s}\sum_{j=0}^{m}\sum_{k=0}^{j}\sum_{r=-z}^{m} h_{s-i}\left[\frac{1}{1-q}\right] q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q e_{i+k-r-z}\left[-\frac{1}{1-q}\right] \\ &\quad\times e_{m-j}\left[\frac{1}{M}\right] t^{j-k} e_{j-k-m+r}\left[-\frac{1}{M}\right] h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] \\
(j\mapsto m-j) &= \sum_{i=0}^{s}\sum_{j=0}^{m}\sum_{k=0}^{m-j}\sum_{r=-z}^{m} h_{s-i}\left[\frac{1}{1-q}\right] q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q e_{i+k-r-z}\left[-\frac{1}{1-q}\right] \\ &\quad\times e_{j}\left[\frac{1}{M}\right] t^{m-k-j} e_{r-k-j}\left[-\frac{1}{M}\right] h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] \\
&= \sum_{i=0}^{s}\sum_{k=0}^{m}\sum_{r=-z}^{m} t^{m-r} h_{s-i}\left[\frac{1}{1-q}\right] q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q e_{i+k-r-z}\left[-\frac{1}{1-q}\right] \\ &\quad\times \left(\sum_{j=0}^{r-k} e_j\left[\frac{1}{M}\right] t^{r-k-j} e_{r-k-j}\left[-\frac{1}{M}\right]\right) h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] \\
(\text{by } 1.23) &= \sum_{i=0}^{s}\sum_{k=0}^{m}\sum_{r=-z}^{m} t^{m-r} h_{s-i}\left[\frac{1}{1-q}\right] q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q e_{i+k-r-z}\left[-\frac{1}{1-q}\right] \\ &\quad\times e_{r-k}\left[\frac{1-t}{M}\right] h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] \\
(\text{as } r\ge k\ge 0) &= \sum_{r=0}^{m} t^{m-r} h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] \\ &\quad\times \sum_{i=0}^{s}\sum_{k=0}^{r} h_{s-i}\left[\frac{1}{1-q}\right] q^{\binom{i}{2}} \begin{bmatrix} i+k \\ i \end{bmatrix}_q e_{i+k-r-z}\left[-\frac{1}{1-q}\right] e_{r-k}\left[\frac{1}{1-q}\right] \\
(\text{by } 1.59) &= \sum_{r=0}^{m} t^{m-r} q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right].\end{aligned}$$

We showed that our function f satisfies

$$(f[X])_{m+z} = \sum_{r=0}^{m} t^{m-r} q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right].$$

Notice that the expression equals 0 for $z > s$. Now multiplying by $e_{n-z}[X/M]$ and summing over z we get

$$\begin{aligned} &\sum_{z=0}^{n} e_{n-z}\left[\frac{X}{M}\right] (f[X])_{m+z} \\ &\quad = \sum_{r=0}^{m} t^{m-r} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q h_{r+z}\left[\frac{X}{1-q}\right] h_{m-r}\left[\frac{X}{M}\right] e_{n-z}\left[\frac{X}{M}\right]. \end{aligned}$$

The statement now follows by Lemma 1.57.

CHAPTER 2

Combinatorial definitions

In this chapter we are going to introduce the combinatorial objects and their statistics that are relevant to the Delta conjectures.

2.1 Lattice paths

The various Delta conjectures can be stated in terms of certain sets lattice paths composed of North and East steps only, such as *Dyck paths* and *square paths*.

Definition 2.1. A *Dyck path* of size $n \in \mathbb{N}$ is a lattice path from $(0,0)$ to (n,n), composed of North and East steps only, that lies weakly above the diagonal $x=y$ (the *main diagonal*).

We denote by $\mathsf{D}(n)$ the set of Dyck paths of size n.

Dyck paths of size n are one of the many instances of objects counted by the *Catalan numbers*

$$C_n \coloneqq \frac{1}{n+1}\binom{2n}{n}.$$

Definition 2.2. An *area word* of length n in a well ordered alphabet A with successor function $S\colon A \to A$ is a sequence of letters $a_1, \dots, a_n$ such that for $1 \leq i \leq n-1$, we have $a_{i+1} \leq S(a_i)$.

Dyck paths of size n are in bijective correspondence with area words of length n in the alphabet $\mathbb{N}$ starting with 0, where the correspondence is given by defining a_i as the number of whole squares in the i-th row of the path that lie between the path and the main diagonal. Equivalently, a_i is the difference between the number of North steps and the number of East steps that occur *strictly before* the i-th North step. For example, the area word of the Dyck path in Figure 2.1 is $(0,1,2,1,2,0,1,1)$.

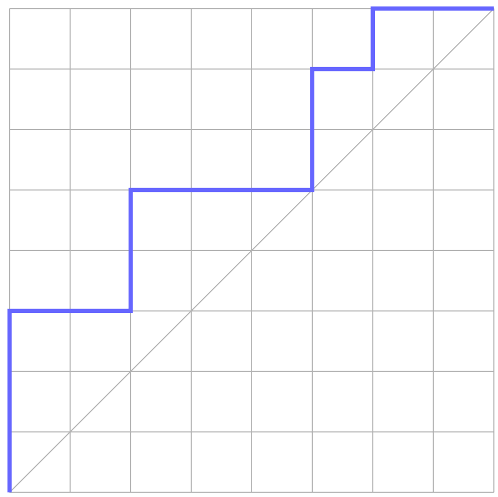


Figure 2.1: A Dyck path of size 8.

Definition 2.3. A *square path* of size $n \in \mathbb{N}$ is a lattice path from $(0, 0)$ to (n, n), composed of North and East steps only, ending with an East step.

We denote by $\mathsf{SQ}(n)$ the set of square paths of size n. These paths are rightfully called *square paths ending East* in the literature, but since we are not going to deal with *square paths ending North*, we omit the specification for brevity.

Definition 2.4. The *shift* of a square path is the maximum integer s such that the path intersects the diagonal $x = y + s$. If a square path has shift s, the diagonal $x = y + s$ is called *base diagonal*.

Dyck paths are precisely the subset of square paths with shift 0. Square paths of size n and shift s are in bijective correspondence with area words of length n in the alphabet $[-s] \cup \mathbb{N}$ such that the first letter is non-positive and the last letter is non-negative. The correspondence is the same as the one described for Dyck paths. Notice that we keep referring to the main diagonal (not the base diagonal), hence the letters of the area word can be negative (but not smaller than $-s$).

We now need to introduce *(partial) labellings* and *decorations*, which allow us to define more general objects. The following definitions are stated for square paths, but since Dyck paths are special cases of square paths, they are meant for both.

Definition 2.5. A *(partial) labelling* of a square path of size n is a sequence $l_1, \dots, l_n$ of (non-negative) positive integers such that $a_i < a_{i+1} \implies l_i < l_{i+1}$ (for a partial labelling, we also require $a_1 = 0 \implies l_1 \neq 0$ and $\{l_i \mid a_i = -s\} \neq \{0\}$). A *(partially) labelled* square path is a square path with a (partial) labelling.

More visually, a (partial) labelling of a square path is an assignment of (non-negative) positive integer labels to each North step of the path such that the labels are strictly increasing along columns (for a partial labelling, we also require that there is no 0 in the bottom-left corner, and that at least one of the steps starting from the base diagonal has a non-zero label).

We denote by $\mathsf{LD}(m,n)$ (resp. $\mathsf{LSQ}(m,n)$) the set of partially labelled Dyck (resp. square) paths of size $m+n$ with m 0 labels and n positive labels. We might omit the value of m if it is 0 (es. $\mathsf{LD}(n)$ or $\mathsf{LSQ}(n)$).

Partially labelled square paths have been introduced in [9], but the other definitions are prior (labelled Dyck paths in [26], labelled square paths in [38], partially labelled Dyck paths in [29]).

Definition 2.6. Let π be a (partially) labelled square path of size n. Its associated monomial is

$$x^\pi = \prod_{i=1}^{n} x_{l_i} \bigg|_{x_0=1}.$$

Notice that the evaluation at $x_0 = 1$ makes the 0 labels not contributing to the monomial (nor to the degree), which explains why the word *partially* is used for labellings where 0 is allowed. When dealing with partially labelled paths, it will be convenient to have two different parameters for the number of 0 labels and the number of positive ones.

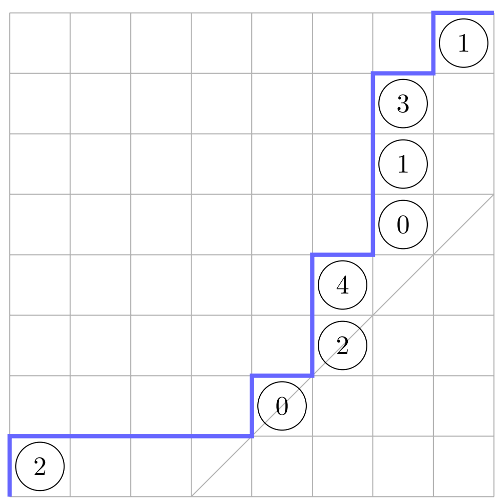


Figure 2.2: A partially labelled square path in $\mathsf{LSQ}(2,6)$ with shift 3 and monomial $x_1{}^2 x_2{}^2 x_3 x_4$.

The paths whose set of labels is exactly $[n]$, with n being their size, are in some sense a (finite) set of representatives for the whole (infinite) set of labelled paths, so it will be useful to give a special name to them. We do so in terms of *preference functions* and *parking functions*.

Definition 2.7. A *preference function* is a function $f\colon [n] \to [n]$. A *parking function* is a preference function such that $\#\{1 \leq j \leq n \mid f(j) \geq i\} \leq n+1-i$.

We denote by $\mathsf{PF}(n)$ (resp. $\mathsf{PR}(n)$) the set of parking (resp. preference) functions of size n.

Given a square path whose set of labels is exactly $[n]$, we can determine a preference function by defining $f(j) = i$ if the label j appears in the i-th column. It is easy to check that the correspondence is bijective, and that f is a parking function if and only if it comes from a Dyck path. From now on, we will identify preference functions and parking functions with the corresponding labelled paths.

Other than (partial) labellings, we need to extend our sets of objects by introducing *decorated rises*.

Definition 2.8. A *rise* of a square path is a North step preceded by another North step. A *decorated square path* is a square path together with a set of *decorated rises*.

We denote by $\mathsf{LD}(m,n)^{*k}$ (resp. $\mathsf{LSQ}(m,n)^{*k}$) the set of partially labelled Dyck (resp. square) paths of size $m+n$ with m 0 labels, n positive labels, and k decorated rises.

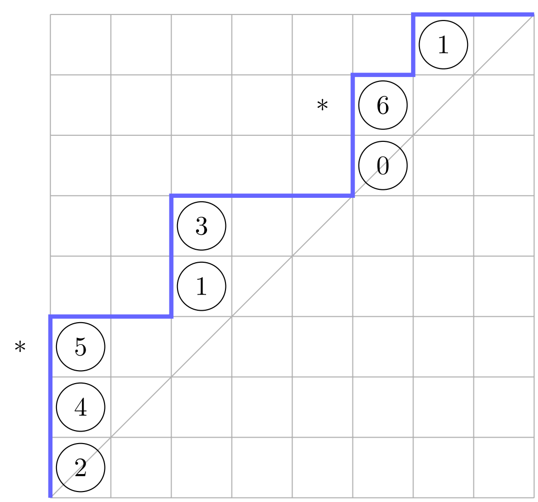


Figure 2.3: The Dyck path in Figure 2.1, with a partial labelling and two decorated rises.

Dyck paths and square paths can be both (partially) labelled and decorated. In fact, these objects will be relevant in the statement of the so called *generalised Delta conjecture* and *generalised Delta square conjecture*.

In order to get there, we need to introduce three statistics on these objects, which are extensions of the *area* (which is classical), the *dinv* (introduced by M. Haiman), and the *bounce* (introduced by J. Haglund).

Definition 2.9. The *area* of a (decorated) square path is the number

$$\sum_{i \notin R} (a_i + s),$$

where s is the shift of the path and R is the set of the indices of its decorated rises.

More visually, the area of a square path is the number of whole squares between the path and the base diagonal that do not lie in rows containing a decorated rise. The presence of a (partial) labelling does not influence the area in any way.

Definition 2.10. The *dinv* of a (partially) labelled square path of size n is the number of diagonal inversions of the path, where for $1 \leq i \leq j \leq n$ the pair (i, j) is a *diagonal inversion* if one of the following holds:

- $a_i = a_j$ and $l_i < l_j$ (*primary inversion*),
- $a_i = a_j + 1$ and $l_i > l_j$ (*secondary inversion*),
- $i = j$, $a_i < 0$, and $l_i > 0$ (*tertiary or bonus inversion*).

The presence of decorated rises does not influence the dinv in any way. Notice that a square path is a Dyck path if and only if it has no tertiary inversions.

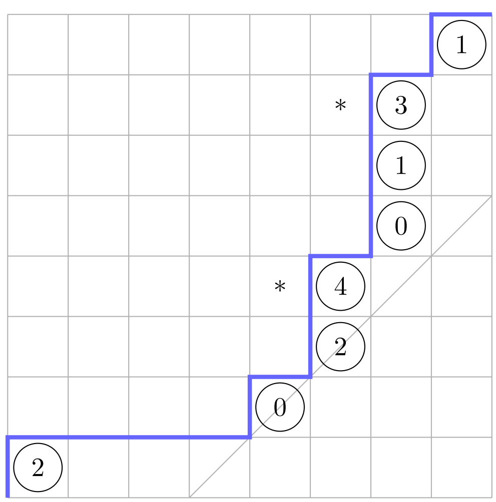


Figure 2.4: The square path in Figure 2.2 with 2 decorated rises. It has $\mathsf{area} = 9$, $\mathsf{dinv} = 6$ (2 primary, 1 secondary, 3 tertiary), and dinv reading word 241231.

Definition 2.11. The *dinv reading word* of a square path is the sequence of its non-zero labels read starting from the ones in the base diagonal going bottom to top, left to right; next the ones in the diagonal $x = y + s - 1$ bottom to top, left to right; then the ones in the diagonal $x = y + s - 2$ and so on.

This convention for the reading word is the inverse of the one that is commonly used in the literature.

If the path is not labelled, we define its dinv as the dinv of the path together with the labelling whose dinv reading word is $1, \dots, n$, i.e. the one such that all the inequalities appearing in the definition of diagonal inversion hold.

The third statistic it the *bounce*, introduced in [23], and its labelled extension *pmaj*. Unfortunately, a further generalisation to square paths is yet to be found, so we will only define it for Dyck paths.

Definition 2.12. The *bounce path* of a Dyck path is the lattice path from $(0, 0)$ to (n, n) computed in the following way: it starts in $(0, 0)$ and travels North until it encounters the beginning of an East step of the Dyck path, then it turns East until it hits the main diagonal, then it turns North again, and so on; thus it continues until it reaches (n, n).

We label the North steps of the bounce path starting from 0 and increasing the labels by 1 every time the path hits the main diagonal (so the steps in the first vertical segment of the path are labelled with 0, the ones in the next vertical segment are labelled with 1, and so on). We define the *bounce word* of the Dyck path to be the sequence of integers $b_1, \dots, b_n$, where b_i is the label attached to the i-th North step of the bounce path. See Figure 2.5 for an example.

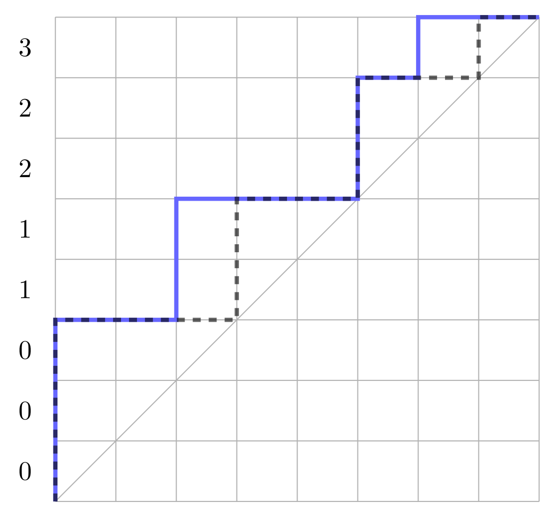


Figure 2.5: Construction of the bounce path (dashed) and the bounce word (left).

Definition 2.13. The *bounce* of a Dyck path is the sum of the values of the labels of its bounce word.

The bounce statistic actually has a generalisation for square paths, but it is rather complicated to describe, it does not generalise to labelled objects, and it is not very useful for our purposes. We will skip it.

Before introducing the pmaj, we need to recall what the major index of a word is.

Definition 2.14. Let $p_1, \dots, p_k$ be a sequence of integers. We define its *descent set*

$$\mathsf{Des}(p_1, \dots, p_k) \coloneqq \{1 \leq i \leq k-1 \mid p_i > p_{i+1}\}$$

and its *major index* $\mathsf{maj}(p_1, \dots, p_k)$ as the sum of the elements of the descent set.

Definition 2.15. The *pmaj* of a (partially) labelled Dyck path is the major index of its parking word, which is defined as follows.

Let C_1 be the set containing the labels appearing in the first column of D, and let $p_1 \coloneqq \max C_1$. At step i, let C_i be the multiset obtained from C_{i-1} by removing p_{i-1} and adding all the labels in the i-th column of the Dyck path; let

$$p_i \coloneqq \max \{x \in C_i \mid x \leq p_{i-1}\}$$

if this last multiset is non-empty, and $p_i \coloneqq \max C_i$ otherwise. The parking word is $p_1, \dots, p_n$.

As it happened for the dinv, the pmaj is also not influenced by the presence of decorated rises. It also has its own reading word (which is *not* the parking word).

Definition 2.16. The *pmaj reading word* of a Dyck path is the sequence of its non-zero labels read bottom to top.

It is not difficult to check that the bounce of a Dyck path agrees with the pmaj of the same path with the standard labelling $l_i = i$. As an example, the Dyck path in Figure 2.3 has parking word 54321061, $\mathsf{pmaj} = 2$, and pmaj reading word 2451361.

The Loehr-Remmel bijection

In [27], J. Haglund and N. Loehr describe a bijection $\zeta \colon \mathsf{D}(n) \to \mathsf{D}(n)$ mapping $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{bounce})$. In a subsequent paper [36], N. Loehr and J. Remmel extended the bijection to the set of parking functions of size n mapping $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{pmaj})$. We will describe the bijection passing through several intermediate steps.

Definition 2.17. Let $\sigma \in S_n$. A *run* of σ is a maximal decreasing sequence in the 1-line notation of σ.

For example, if $\sigma = 716429583$, its runs are 71, 642, 95, 83. We now define a function $w\colon S_n \to \mathbb{N}^n$ by saying that $w_i(\sigma)$ is the number of elements greater than σ_i in the run containing σ_i, plus the number of elements lesser than σ_i in the previous run (or plus 1 if σ_i is in the first run). In our case, $w(\sigma) = (1, 2, 1, 2, 3, 3, 3, 1, 1)$. Notice that $w_1 \equiv 1$.

Proposition 2.18. There is a bijection between $\mathsf{PF}(n)$ and the pairs (σ, u) with $\sigma \in S_n$ and $u \leq w(\sigma) \in \mathbb{N}^n$, where the inequality is componentwise.

We will actually show two such bijections, and the composition of the second one with the inverse of the first one will be the desired bijection of $\mathsf{PF}(n)$ with itself mapping $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{pmaj})$.

First bijection Let $f \in \mathsf{PF}(n)$, π its corresponding element in $\mathsf{LD}(n)$. We want to define a pair (σ, u) as in 2.18, such that $\mathsf{area}(\pi) = \sum_{i=1}^n u_i$ and $\mathsf{pmaj}(\pi) = \mathsf{maj}(\sigma_n \cdots \sigma_1)$. Let $\sigma := p(\pi)$ (the parking word of π), and $u_i := i - f(p_i(\pi))$.

Lemma 2.19. The map is well defined, i.e. $0 \leq u_i < w_i(\sigma)$.

Proof. Recall that $f(i)$ is the number of the column containing the label i. It is clear that $f(p_i(D)) \leq i$, because the label $p_i(D)$ must be in the first i columns by construction of the parking word. This implies that $u_i = i - f(p_i(D)) \geq 0$.

The number $i - f(p_i(\pi))$ is the delay between the moment in which $p_i(\pi)$ is scanned and the moment in which it appears in the parking word. In fact, since $f(p_i(\pi))$ is the column containing the label $p_i(\pi)$, then it is also the minimum among the indices j such that $p_i(\pi) \in C_j$; since it appears as i-th letter in the parking word, then i is the maximum among these indices. By construction of the parking word, for $f(p_i(\pi)) - 1 \leq j < i$ the label $p_j(\pi)$ must be either lesser than $p_i(\pi)$ and belonging to the previous run, or greater than $p_i(\pi)$ and belonging to the same run. The number of such labels at most $w_i(\sigma)$, hence $i - f(p_i(\pi)) < w_i(\sigma)$.

Since $f(i) = (\sigma^{-1})_i - u_{(\sigma^{-1})_i}$, the permutation σ and the sequence u completely determine f. This implies that the map is injective. To prove that it is actually a bijection, we have to show the following.

Lemma 2.20. For (σ, u) as in 2.18, $f\colon i \mapsto (\sigma^{-1})_i - u_{(\sigma^{-1})_i}$ is a parking function with parking word σ.

Proof. Recall that $f\colon [n] \to [n]$ is a parking function if and only if

$$\#\{1 \leq j \leq n \mid f(j) \geq i\} \leq n + 1 - i.$$

Since $u_i \geq 0$, then $f(\sigma_i) = i - u_i \leq i$, so we have $f(j) \geq i \implies j = \sigma_h$ for some $h \geq i$. There are $n + 1 - i$ such h's, thus f is a parking function.

Since the map is the inverse of the one we just defined, then the parking word of f must be σ.

Lemma 2.21. $\mathsf{area}(\pi) = \sum_{i=1}^{n} u_i$ and $\mathsf{pmaj}(\pi) = \mathsf{maj}(\sigma_n \cdots \sigma_1)$.

Proof. The equality $\mathsf{pmaj}(\pi) = \mathsf{maj}(\sigma_n \cdots \sigma_1)$ is trivial by construction.

We will now prove that $\sum_{i=1}^{n} u_i = \mathsf{area}(\pi)$. If $f(i)$ is the column containing the label i, then $n + 1 - f(i)$ is the number of whole squares in the row containing the label i between the path and the right edge of the square containing the path. The total number of squares below the path is also equal to the number of squares between the path and the main diagonal, which is $\mathsf{area}(\pi)$, plus the number of squares weakly below the main diagonal (including the ones containing it). It follows that

$$\sum_{i=1}^{n} (n + 1 - f(i)) = \mathsf{area}(\pi) + \binom{n+1}{2}$$

and so

$$\begin{aligned} \mathsf{area}(\pi) &= \sum_{i=1}^{n} (n + 1 - f(i)) - \binom{n+1}{2} = n(n+1) - \binom{n+1}{2} - \sum_{i=1}^{n} f(i) \\ &= \binom{n+1}{2} - \sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} (i - f(i)) = \sum_{i=1}^{n} u_i \end{aligned}$$

as desired.

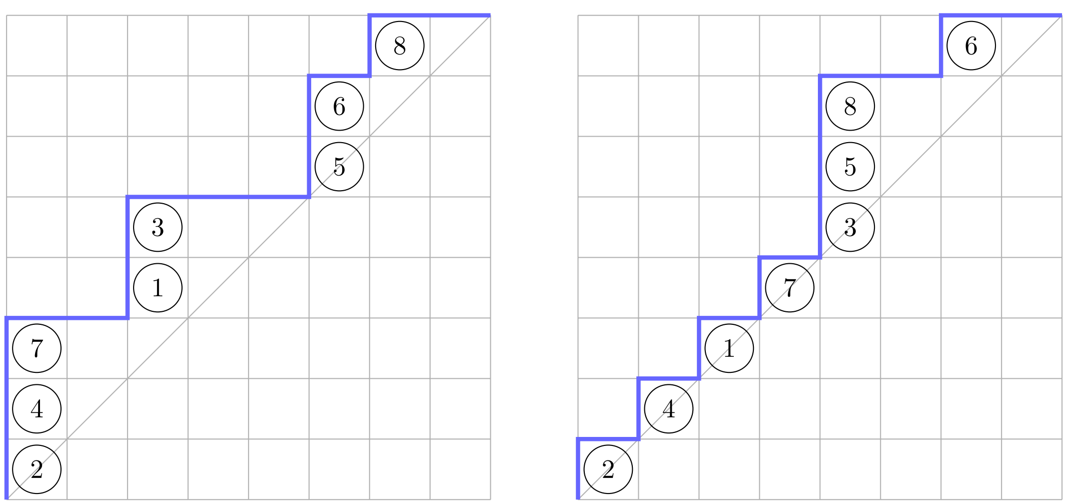


Figure 2.6: The Dyck path to the left is mapped to $\sigma = 74321658, u = 01032011$ by the first bijection, which is mapped to the path to the right by the inverse of the second one.

Second bijection Let $f \in \mathsf{PF}(n)$, π its corresponding element in $\mathsf{LD}(n)$. We want to define a pair (σ, u) as in 2.18, such that $\mathsf{dinv}(\pi) = \sum_{i=1}^{n} u_i$ and $\mathsf{area}(\pi) = \mathsf{maj}(\sigma_n \cdots \sigma_1)$. Let σ be the permutation whose i-th run is given by the labels in the diagonal $x + y = i - 1$ in π, in decreasing order. Let u_i be the number of inversions involving σ_i and any σ_j with $j \leq i$.

Lemma 2.22. The map is well defined, i.e. $0 \leq u_i < w_i(\sigma)$.

Proof. By construction $u_i \geq 0$. Furthermore, if the labels σ_j, σ_i form an inversion and $j \leq i$, then σ_j is counted by $w_i(\sigma)$. In fact, if σ_i and σ_j are in the same diagonal (i.e. the same run), by construction of σ we have that $j \leq i$ implies $\sigma_j > \sigma_i$, hence σ_j contributes to $w_i(\sigma)$ whether they form an inversion or not. For the same reason, if σ_i and σ_j are in two consecutive diagonals and $j \leq i$, then by construction of σ we must have σ_j in the lower diagonal (i.e. the previous run), so if they form an inversion then $\sigma_j < \sigma_i$, which means that σ_j contributes to $w_i(\sigma)$. It follows that $u_i \leq w_i(\sigma)$, but the inequality is strict because of we have to add 1 to $w_i(\sigma)$ if i is in the first run, and if it is not then before any label in the diagonal $x + y = i - 1$ there must be some label in the diagonal $x + y = i - 2$, and they cannot possibly form an inversion.

Lemma 2.23. The map is bijective.

Proof. Given (σ, u) as in 2.18, we can build a labelled Dyck path π recursively. More precisely, we can build a sequence of labelled Dyck paths of size k with labels in $\{\sigma_1, \ldots, \sigma_k\}$ such that the number of inversions involving σ_i and some σ_j with $j \leq i$ is exactly u_i.

For $k = 1$ the path is trivial. Supposing to have already built the path of size $k - 1$, then one has $w_k(\sigma)$ possibilities for the position of σ_k: since it must be in a fixed diagonal, it can be either one step North-East after any other label in the same diagonal (i.e. the number of elements in the same run that we already used, which are the ones greater than σ_k), or immediately on top any other label in the previous diagonal (i.e. the previous run), that must be strictly smaller by definition of labelled Dyck path (so, there are as many as the number of elements lesser than σ_k in the previous run), possibly moving one step towards North-East any label that were there before; if σ_k belongs to the first run, we have the extra option of putting it at the very beginning of the path.

This proves that the number of options we have is exactly $w_k(\sigma)$. It is easy to check that the contribution to the dinv of each of these options is different, and in particular it is 0 for the option that puts σ_k in the highest possible row, and increases by 1 every time we move down to the next possible spot (since we are adding exactly one inversion).

We can thus reconstruct π from (σ, u), hence the map is bijective.

Lemma 2.24. $\mathsf{dinv}(\pi) = \sum_{i=1}^{n} u_i$ and $\mathsf{area}(\pi) = \mathsf{maj}(\sigma_n \cdots \sigma_1)$.

Proof. The equality $\mathsf{area}(\pi) = \mathsf{maj}(\sigma_n \cdots \sigma_1)$ is trivial because the elements in the i-th run of σ contribute by $i - 1$ units each to $\mathsf{maj}(\sigma_n \cdots \sigma_1)$, and they correspond to the labels in the diagonal $x + y = i - 1$, which contribute by $i - 1$ units each to $\mathsf{area}(\pi)$.

The equality $\mathsf{dinv}(\pi) = \sum_{i=1}^{n} u_i$ is also trivial because both the left and the right hand side count the total number of diagonal inversions of π.

We can now conclude our proof.

Theorem 2.25. The composition of the second bijection with the inverse of the first bijection yields a map $\zeta \colon \mathsf{PF}(n) \to \mathsf{PF}(n)$ mapping $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{pmaj})$

Proof. It follows immediately by all the lemmas in this subsection.

2.2 Polyominoes

While most of the statements can be expressed in terms of lattice paths, in some cases polyominoes give more insight, for example when dealing with iterated recursions or complicated bijections.

Definition 2.26. A reduced (resp. standard) parallelogram polyomino of size $m \times n$ is a pair of lattice paths from $(0, 0)$ to (m, n), composed of North and East steps only, such that the first one, called *red path*, lies always weakly (resp. strictly) above the second one, called *green path*.

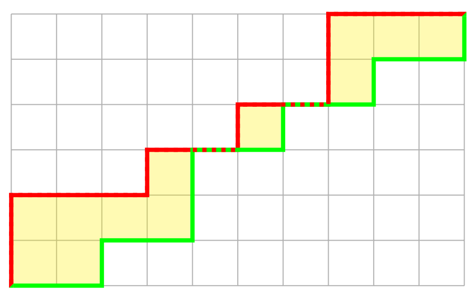


Figure 2.7: A 10×6 reduced polyomino.

We denote by $\mathsf{RP}(m, n)$ (resp. $\mathsf{PP}(m, n)$) the set of reduced (resp. standard) parallelogram polyominoes of size $m \times n$.

There is an obvious bijection between reduced parallelogram polyominoes of size $m \times n$ and standard parallelogram polyominoes of size $(n + 1) \times (m + 1)$, consisting of adding one North step at the beginning and one East step at the end of the red path, adding one East step at the beginning and one North step at the end of the green path, and then taking the symmetry with respect to the diagonal $x = y$ (thus swapping the red and the green path).

From now on, unless differently specified, the word *polyomino* has to be intended as *reduced parallelogram polyomino*. Details about the properties and the statistics on standard parallelogram polyominoes can be found in [1]. The bistatistics we are going to introduce are preserved (up to a normalisation) by either the bijection described above or its conjugate by another bijection we will describe later, hence the two descriptions are completely equivalent.

Reduced parallelogram polyominoes of size $m \times n$ are in bijective correspondence with area words $a_0, a_1, \ldots, a_{m+n}$ in the alphabet $\overline{\mathbb{N}} \coloneqq 0 < \bar{0} < 1 < \bar{1} < 2 < \bar{2} < \ldots$ starting with 0, with $m+1$ unbarred letters and n barred letters. The area word can be computed in two equivalent ways.

The first one consists of drawing a diagonal of slope -1 from the end of every horizontal green step, and attaching to that step the length of that diagonal (i.e. the number of squares it crosses). Then, one puts a dot in every square not crossed by any of those diagonals, and attaches to each vertical red step the number of dots in the corresponding row. Next, one bars the numbers attached to vertical red steps, and finally one reads those numbers following the diagonals of slope -1, reading the labels when encountering the end of its step and the red label before the green one. An artificial 0 is added at the beginning. See Figure 2.8 for an example.

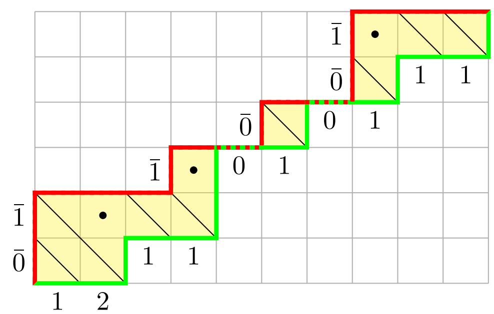


Figure 2.8: The area word for the polyomino in Figure 2.7 is $0\bar{0}1\bar{1}211\bar{1}0\bar{0}10\bar{0}1\bar{1}11$.

Equivalently, we can build a Dyck path of size $m+n+1$ from the polyomino in the following way. First we draw a North step; then, running over red and green steps alternatively, we draw a North step in our Dyck path if the corresponding step in the polyomino was either a red North step or a green East step, and we draw an East step in our Dyck path otherwise; finally we draw an East step. Now we take the area word of the Dyck path and replace the alphabet $\mathbb{N}$ with $\overline{\mathbb{N}}$. Notice that this gives a bijective correspondence between polyominoes with semi-perimeter $m+n$ and Dyck paths of size $m+n+1$.

It is not hard to check that those definitions are equivalent (see [1] and [8, Section 1.2] for detailed proofs and examples).

For $a \in \overline{\mathbb{N}}$ let $|a| = n \in \mathbb{N}$ if $a \in \{n, \bar{n}\}$ (i.e. its value disregarding the bar).

Definition 2.27. A *rise* of a polyomino is an index i such that $a_i > a_{i-1}$ in the alphabet $\overline{\mathbb{N}}$. A rise is *unbarred* if $|a_i| > |a_{i-1}|$ (i.e. if $a_i \in \mathbb{N}$) and it is *barred* if $|a_i| = |a_{i-1}|$ (i.e. if $a_i \in \overline{\mathbb{N}} \setminus \mathbb{N}$). A *rise-decorated polyomino* is a polyomino together with a set of *decorated unbarred rises*. A *doubly rise-decorated polyomino* is a polyomino together with a set of *decorated rises* (either barred or not).

Unbarred rises correspond in the picture to diagonals of slope -1 connecting the end point of a red North step with the end point of a green East step, with the value of the unbarred letter being the length of the diagonal. Barred rises correspond in the picture to to diagonals of slope -1 connecting the starting point of a red North step with the end point of a green East step, with the value of the unbarred letter being the length of the diagonal; this can be 0 (the two points may coincide) in the case of a $0\overline{0}$ rise.

Definition 2.28. The *area* of a (doubly rise-decorated) polyomino is the number

$$\sum_{i \notin R} |a_i|,$$

where R is the set of the indices of its decorated rises.

More visually, the area of a (doubly rise-decorated) polyomino is the number of squares between the red path and the green path that are neither crossed by a diagonal corresponding to a decorated unbarred rise, nor contain a dot in a row whose matching red North step is a decorated barred rise. For example, the area of the polyomino in Figure 2.7 (that has no decorated rises) is 12.

We denote by $\mathsf{RP}(m,n)^{*k,j}$ the set of doubly rise-decorated reduced polyominoes of size $m \times n$ with k decorated unbarred rises and j decorated barred rises.

Definition 2.29. The *dinv* of a polyomino of size $m \times n$ is the number of diagonal inversions of the polyomino, where for $0 \leq i < j \leq m+n$ the pair (i,j) is a *diagonal inversion* if $a_i = S(a_j) \in \overline{\mathbb{N}}$ and $S \colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ is the successor function.

For example, the dinv of the polyomino in Figure 2.7 is 24.

Definition 2.30. A *corner* of a polyomino is an index i such that neither $a_i = a_{i-1}$ nor $a_{i-2} < a_{i-1} < a_i$. A *green peak* of a polyomino is a corner such that a_i is unbarred, i.e the corresponding green step is an East step preceded by a North step. A *red valley* of a polyomino is a corner such that a_i is barred, i.e. the corresponding red step of the polyomino is a North step preceded by an East step. A *corner-decorated polyomino* is a polyomino together with a set of *decorated green peaks*. A *doubly corner-decorated polyomino* is a polyomino together with a set of *decorated corners* (either green peaks or red valleys).

We denote by $\mathsf{RP}(m,n)^{\circ k,j}$ the set of doubly rise-decorated reduced polyominoes of size $m \times n$ with k decorated green peaks and j decorated red valleys.

Definition 2.31. The *bounce path* of a polyomino path is the lattice path from $(0,0)$ to (m,n) computed in the following way: it starts in $(0,0)$ and travels East until it encounters the beginning of a green North step of the polyomino (which can happen immediately, after zero steps), then it turns North until it encounters the beginning of a red East step, then it turns East again, and so on; thus it continues until it reaches (m,n).

We label the steps of the bounce path starting from 0 and increasing the labels by taking the successor in $\overline{\mathbb{N}}$ every time the path hits changes directions (so the steps in the first horizontal segment of the bounce path - which is the only one that can be empty - are labelled with 0, the ones in the next vertical segment are labelled with $\bar{0}$, and so on). We define the *bounce word* of the polyomino to be the sequence of integers $b_0, b_1, \ldots, b_n$, where $b_0 \coloneqq 0$ and b_i is the label attached to the i-th step of the bounce path for $i > 0$. See Figure 2.9 for an example.

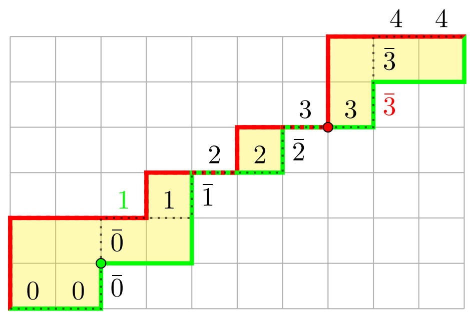


Figure 2.9: The polyomino in Figure 2.7 with a decorated green peak, a decorated red valley, and the bounce path (dotted) shown.

Definition 2.32. The *bounce* of a (doubly corner-decorated) polyomino is the number

$$\sum_{i \notin C} |b_i|,$$

where C is the set of the indices of its decorated corners.

For example, the bounce of the polyomino in Figure 2.9 is 25. The labels corresponding to decorated corners are highlighted and disregarded.

Decorated polyominoes and their statistics first appeared in [8] with one set of decorations, and in [11] with two.

Finally, we introduce a pmaj statistic for polyominoes. Here we switch to standard parallelogram polyominoes because the combinatorics of the labellings is more natural.

Definition 2.33. A *labelled parallelogram polyomino* is a parallelogram polyomino where the vertical steps of the first path are labelled with (not necessarily distinct) positive integers such that the labels appearing in each column are strictly increasing from bottom to top.

We denote by$\mathsf{LPP}(m, n)$ the set of labelled standard parallelogram polyominoes of size $m \times n$. For $\pi \in \mathsf{LPP}(m, n)$ we set $l_i(\pi)$ to be the label of the i-th vertical step. We define the associated monomial x^π as we did for the labelled square paths.

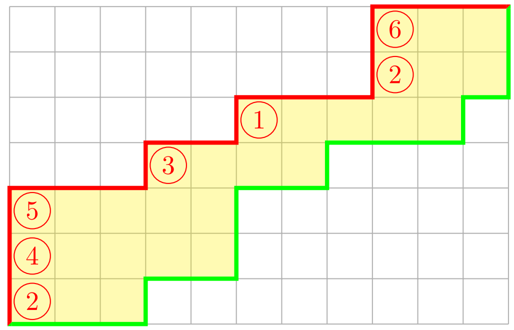


Figure 2.10: A 11×7 labelled standard parallelogram polyomino.

Definition 2.34. The *pmaj* of a labelled standard parallelogram polyomino is the major index of its parking word, which is defined as follows.

Let C_1 be the multiset containing the labels appearing in the first column of π, and let $p_1 \coloneqq \max C_1$. For $i > 1$, at step i, if the i-th step of the green path is a North step let $C_i = C_{i-1} \setminus \{p_{i-1}\}$; if the i-th step of the green path is an East step let C_i be the multiset obtained from C_{i-1} by replacing p_{i-1} with a 0, and then adding all the labels in the column of π containing the i-th green step (which we recall being an East step). Next, let $p_i \coloneqq \max \{a \in C_i \mid a \leq p_{i-1}\}$ if this set is non-empty, and $p_i \coloneqq \max C_i$ otherwise. The parking word is $p_1 \cdots p_{m+n-1}$.

For example, the parking word of the polyomino in Figure 2.10 is 54200003100000620 and hence its pmaj is 13. It is not hard to see that it the polyomino has the standard labelling $l_i = i$ then its pmaj agrees with the bounce of the corresponding reduced parallelogram polyomino.

The pmaj statistic for polyominoes has been introduced in [8].

The $\overline{\zeta}$ bijection for polyominoes

In [1], the authors describe a bijection between the set of $m \times n$ standard parallelogram polyominoes to the set of $n \times m$ standard parallelogram polyominoes, mapping $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{bounce})$. In [11] we slightly modified this bijection to suit reduced parallelogram polyominoes, and showed that it actually extends to doubly decorated objects.

Proposition 2.35. For $m, n, k, j \in \mathbb{N}$, there exists a bijection $\overline{\zeta} \colon \mathsf{RP}(m,n)^{*k,j} \to \mathsf{RP}(m,n)^{\circ k,j}$ mapping $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{bounce})$.

Proof. The map is essentially the same one described in [1, Section 4], adjusted to fit reduced polyominoes as in [8, Theorem 7.5]. We will actually describe its inverse.

Pick a reduced polyomino with some decorated red valleys and green peaks and draw its bounce path; then, project the labels of the bounce path on both the red and the green path. Let us

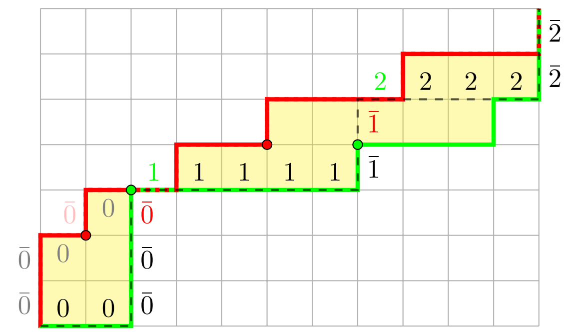


Figure 2.11: The first step needed to compute $\overline{\zeta}$, giving the partial word $0\bar{0}\bar{0}0\overset{*}{\bar{0}}0$. The final image will be the polyomino with area word $0\bar{0}\bar{0}0\overset{*}{\bar{0}}\overset{*}{1}\bar{1}\overset{*}{2}\bar{2}2211\overset{*}{\bar{1}}2\bar{2}110$

call *bounce point* a vertex of the bounce path in which it changes direction. Now, build the area word of the image as follows: start with a 0, then pick the first bounce point on the red path, and write down the 0's and the $\bar{0}$'s as they appear going upwards along the red path up to that point (in this case, the relative order will always be with the $\bar{0}$ first, and all the 1's next). Then, go to the first bounce point on the green path, and insert the 1's after the correct number of $\bar{0}$'s, in the same relative order in which they appear going upwards to the previous bounce point. If a letter is decorated, keep the decoration. Now, move to the second bounce point on the red path, and repeat. See Figure 2.11 for an example.

By construction the result will be the area word of a $m \times n$ reduced polyomino. It is also easy to see that the area is mapped to the dinv, since the squares of the starting reduced polyomino correspond to the inversions on the image.

Red valleys are mapped into barred rises, because when reading the red path bottom to top, one reads the horizontal step first, which corresponds to an unbarred letter, and the vertical step next, which correspond to the next barred letter. Moreover, the decoration is kept on a letter with the same value. The same argument applies to green peaks being mapped to unbarred rises. This implies that bounce is mapped to area.

Remark 2.36. Given a polyomino π, by construction we have that the area word of π is an anagram of the bounce word of $\overline{\zeta}(\pi)$. In particular, the number of 0's is preserved.

CHAPTER 3

Delta conjectures

One of the main reasons that motivates the study of the Delta operators is the surprising amount of conjectured positivity results related to them. While some have been proven over the years, most of them are still open.

3.1 The shuffle conjecture

The first one to have ever been introduced is known as *shuffle conjecture*, now a theorem by E. Carlsson and A. Mellit (see [6]).

Theorem 3.1 (Shuffle Theorem)**.** For $n \in \mathbb{N}$, we have

$$\nabla e_n = \sum_{\pi \in \mathsf{LD}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

In [36] the authors describe a bijection of $\mathsf{PF}(n)$ with itself mapping the bistatistic $(\mathsf{dinv}, \mathsf{area})$ to $(\mathsf{area}, \mathsf{pmaj})$. As a corollary, the following holds.

Corollary 3.2. For $n \in \mathbb{N}$, we have

$$\nabla e_n = \sum_{\pi \in \mathsf{LD}(n)} q^{\mathsf{area}(\pi)} t^{\mathsf{pmaj}(\pi)} x^\pi.$$

The shuffle conjecture is especially important because ∇e_n has another interpretation, as the bigraded Frobenius characteristic of the S_n module known as *diagonal harmonics*. This is one of the facts that first motivated the study of Macdonald polynomials, and it has been proved by M. Haiman in [34]. See also [33] for further details.

3.2 The Delta conjecture

The *Delta conjecture* is a generalisation of the shuffle conjecture, introduced by J. Haglund, J. Remmel, and A. Wilson in [29]. In the same paper, the authors suggest that an even more general conjecture should hold, which we call *generalised Delta conjecture*. It reads as follows.

Conjecture 3.3 (Generalised Delta conjecture)**.** For $m, n, k \in \mathbb{N}$, we have

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{\pi \in \mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

The Delta conjecture is simply the case $m = 0$ of the general case. Recalling that $\nabla|_{\Lambda^{(n)}} = \Delta'_{e_{n-1}}|_{\Lambda^{(n)}}$, it is clear that for $k = 0$ the Delta conjecture reduces to the shuffle conjecture.

In analogy with the shuffle conjecture, the generalised Delta conjecture also has a version in terms of $(\mathsf{area}, \mathsf{pmaj})$. Unfortunately the bijection by N. Loehr and J. Remmel does not generalise for $k > 0$ (but it does for $m > 0$), so in the general case it is not proved that the two conjectures are equivalent.

Conjecture 3.4. For $m, n, k \in \mathbb{N}$, we have

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{\pi \in \mathsf{LD}(m,n)^{*k}} q^{\mathsf{area}(\pi)} t^{\mathsf{pmaj}(\pi)} x^\pi.$$

We want to emphasise the $(\mathsf{area}, \mathsf{pmaj})$ version, despite it being less popular in the literature, because certain symmetric functions are easier to interpret in this case.

3.3 The square conjecture

The *square conjecture* was first suggested by N. Loehr and G. Warrington in [38], and it was then proved by E. Sergel in [43] using the shuffle theorem.

Theorem 3.5 (Square Theorem)**.** For $n \in \mathbb{N}$, we have

$$\nabla \omega(p_n) = \sum_{\pi \in \mathsf{LSQ}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

Unfortunately, adding zero labels and decorated rises to square paths in the trivial way and q,t-counting the resulting objects with respect to the bistatistic $(\mathsf{dinv}, \mathsf{area})$ gives a polynomial that does not match the expected symmetric function. This issue has been addressed by M. D'Adderio, A. Iraci, and A. Vanden Wyngaerd, who stated the *generalised Delta square conjecture* in [9].

Conjecture 3.6 (Generalised Delta square conjecture). For $m, n, k \in \mathbb{N}$, we have

$$\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{\pi \in \mathsf{LSQ}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

The square conjectures lack a $(\mathsf{area}, \mathsf{pmaj})$ version. The reason is that there is currently no extension of the pmaj statistic to square paths.

3.4 The polyominoes conjecture

For the sake of completeness, we also state the polyominoes conjecture, first introduced in [8].

Conjecture 3.7 (Polyominoes conjecture). For $m, n \in \mathbb{N}$, we have

$$\Delta_{h_m} e_{n+1} = \sum_{\pi \in \mathsf{LPP}(m+1,n+1)} q^{\mathsf{area}(\pi)} t^{\mathsf{pmaj}(\pi)} x^\pi$$

where the area of a standard parallelogram polyomino is the area of the corresponding reduced parallelogram polyomino (or equivalently, it is the number of squares between the two paths minus a normalisation factor $m + n + 1$).

Unlike the square conjectures, this conjecture lacks a $(\mathsf{dinv}, \mathsf{area})$ version instead. From some partial results that we have, it is plausible that the introduction of a pmaj statistic on partially labelled square paths will also enable the statement of a conjecture for $\Delta_{h_m} \omega(p_{n+1})$ in terms of pairs of paths from $(0, 0)$ to (m, n). This would allow for a generalisation of the polyominoes framework in the same way as square paths generalise the Dyck paths framework.

3.5 Shuffle theory

To prove any of these conjectures, it is enough to show that the scalar product of the corresponding symmetric function with any element of a given base agrees with the q, t-enumerator of the corresponding set. If we choose the basis of complete homogeneous symmetric functions $\{h_\lambda \mid \lambda \vdash n\}$, the Delta conjectures predict that the scalar product of the corresponding symmetric function with h_λ is the q, t-enumerator of the subset of the relevant paths whose elements have a reading word that is a $\varnothing, \lambda$-shuffle. The goal of this section is to give the relevant definitions and prove this result.

First of all, recall that

$$\sum_{\pi \in \mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} \quad \text{and} \quad \sum_{\pi \in \mathsf{LSQ}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi$$

are both symmetric functions, as it is immediate that they are positive sums of *LLT polynomials*. We also need to recall that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$, that is, the monomial and the complete homogeneous symmetric functions are dual bases. With that in mind, we can isolate the coefficient of m_λ by taking the scalar product with h_λ. But the coefficient of m_λ is the q,t-enumerator of the subset of the relevant lattice paths whose set of labels is composed of λ_1 1's, λ_2 2's, and so on (we are using the fact that the full series are symmetric functions). We can restate this result in term of shuffles.

Definition 3.8. Given two sequences $(a_1,\dots,a_m)$, $(b_1,\dots,b_n)$ two sequences of pairwise distinct elements, their *shuffle* $(a_1,\dots,a_m)\sqcup\!\sqcup(b_1,\dots,b_n)$ is the set of sequences $(c_1,\dots,c_{m+n})$ such that

- $\{c_k \mid 1\le k\le m+n\}=\{a_i,b_j \mid 1\le i\le m,\ 1\le j\le n\}$,
- $c_r=a_i,\ c_s=a_j,\ i<j \implies r<s$,
- $c_r=b_i,\ c_s=b_j,\ i<j \implies r<s$,

i.e. it is the set of sequences obtained by interlacing of the starting two sequences while preserving the relative order.

Definition 3.9. Given $\mu\vdash n-d$ and $\nu\vdash d$, a μ,ν-*shuffle* is a sequence of numbers from 1 to n in

$$\begin{aligned}(1,\dots,\mu_1)\sqcup\!\sqcup\cdots\sqcup\!\sqcup(n-\mu_{\ell(\mu)}+1,\dots,n-d)&\\ \sqcup\!\sqcup(n-d+\nu_1,\dots,n-d+1)\sqcup\!\sqcup\cdots\sqcup\!\sqcup(n,\dots,n-\nu_{\ell(\nu)}+1)&\end{aligned}$$

i.e. a shuffle of $\ell(\mu)$ increasing sequences of length $\mu_1,\dots,\mu_{\ell(\mu)}$, and $\ell(\nu)$ decreasing sequences of length $\nu_1,\dots,\nu_{\ell(\nu)}$, obtained by picking every time the smallest available positive integers.

We will write λ-shuffle as a short for $\varnothing,\lambda$-shuffle. As we did for the reading word, this is the inverse of the convention commonly used in the literature.

Notice that, for any $\lambda\vdash n$, the q,t-enumerator of the subset of any of the combinatorial sets defined in Chapter 2 whose objects have a set of labels composed of λ_1 1's, λ_2 2's, and so on, is the same as the q,t-enumerator, with respect to the bistatistic $(\mathsf{dinv},\mathsf{area})$, of the set of the objects of the same kind whose dinv reading word is a λ-shuffle. This is immediate, as replacing the 1's with the first decreasing sequence, the 2's with the second, and so on, preserves both the area (trivially) and the dinv (because the strict inequalities are preserved).

It follows that, to prove any of these conjectures, it is enough to show that the scalar product of the corresponding symmetric function with any h_λ yields the q,t-enumerator of the set of the objects of the right kind whose dinv reading word is a λ-shuffle. For this reason the first conjectured result was called *shuffle conjecture*.

From [25, Theorem 6.10] we have that if we take the scalar product with $e_\mu h_\nu$ instead, we get the q,t-enumerator, with respect to the bistatistic $(\mathsf{dinv},\mathsf{area})$, of the set of the objects whose dinv reading word is a μ,ν-shuffle. We omit the proof, as it would require introducing some extra background theory about quasisymmetric functions.

CHAPTER 4

Combinatorial recursions

In this chapter we are going to show the two combinatorial recursions that are needed to prove the so called *Schröder case* of the generalised Delta and Delta square conjectures, i.e. the $\langle \cdot, e_{n-j}h_j \rangle$ case. These are at the moment the most general results that don't require any specialization of the variables q, t.

4.1 Partially labelled Dyck paths

For further details on the content of this Section, see [11].

As it often happens when dealing with these recursions, it is convenient to split our set into smaller subsets and find a recursion for those. Define the subset

$$\mathsf{LD}(m, n\backslash s)^{*k,\circ j} \subseteq \mathsf{LD}(m,n)^{*k}$$

to consist of the paths $\pi \in \mathsf{LD}(m,n)^{*k}$ such that

$$\mathsf{dr}(\pi) \in (1, 2, \dots, n-j) \sqcup (n, n-1, \dots, n-j+1) \quad \text{and} \quad \#\{1 \leq i \leq m+n \mid a_i = 0 \wedge l_i \neq 0\} = s.$$

Let labels from 1 to $n-j$ be *small*, and the labels from $n-j+1$ to n be *big*. We set

$$\mathsf{LD}_{q,t}(m, n\backslash s)^{*k,\circ j} := \sum_{\pi \in \mathsf{LD}(m,n\backslash s)^{*k,\circ j}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}.$$

The generalised Delta conjecture predicts that

$$\sum_{s=1}^{n} \mathsf{LD}_{q,t}(m, n\backslash s)^{*k,\circ j} = \langle \Delta_{h_m} \Delta'_{e_{n-k-1}} e_n, e_{n-j}h_j \rangle$$

and our goal is to prove this result.

Theorem 4.1. For $0 \leq j,k,s \leq n$, $0 \leq m$, the polynomials $\mathsf{LD}_{q,t}(m,n\backslash s)^{*k,\circ j}$ satisfy the recursion

$$\begin{aligned}\mathsf{LD}_{q,t}(m,n\backslash s)^{*k,\circ j} &= t^{n-s-k}\sum_{r=0}^{m}\sum_{z=0}^{s} q^{\binom{z}{2}}\begin{bmatrix} s \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \\ &\times t^{m-r}\sum_{u=0}^{n-s-k}\sum_{h=0}^{r+z} q^{\binom{h}{2}}\begin{bmatrix} r+z \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u \end{bmatrix}_q \mathsf{LD}_{q,t}(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}\end{aligned}$$

with initial conditions

$$\mathsf{LD}_{q,t}(m,n\backslash n)^{*k,\circ j} = \delta_{k,0}\cdot q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q.$$

Proof. Let us start with the initial conditions. The set $\mathsf{LD}(m,n\backslash n)^{*k,\circ j}$ consists of the paths whose non-zero labels all lie on the main diagonal, namely $l_i \neq 0 \implies a_i = 0$. Therefore, it must also hold that $l_i = 0 \implies a_i = 0$, because the bottom-most label not on the main diagonal must be a rise, and hence it can't be a 0. It follows that all the a_i's must be 0, thus the area must be zero. Furthermore there can be no rises, which explains $\delta_{a,0}$. The primary dinv among small labels is counted by $q^{\binom{n-j}{2}}$. The primary dinv between small labels and big labels is taken into account by $\begin{bmatrix} n \\ j \end{bmatrix}_q$. Finally, the primary dinv between 0 labels and non-zero labels is counted by $\begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q$ because $l_1 \neq 0$.

For the recursive step, we first give an overview of the combinatorial interpretations of all the variables appearing in this formula. We say that a vertical step of a path is *at height* i if its corresponding letter in the area word equals i.

- z is the number of small labels on the main diagonal.
- $s-z$ is the number of big labels on the main diagonal.
- r is the number of zero labels on the main diagonal.
- h is the number of i's such that $a_i = 1$ and i is a decorated rise.
- u is the number of i's such that $a_i = 1$, i is not a decorated rise, and $l_i \neq 0$.

The strategy of this recursion is the following. Start from a path π in $\mathsf{LD}(m,n\backslash s)^{*k,\circ j}$. Remove all the 0's from the area word (with the corresponding labels), and remove the decorations on rises at height one (which are not rises any more). Then decrease all the remaining letters by 1. In this way we obtain a path in

$$\mathsf{LD}(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}.$$

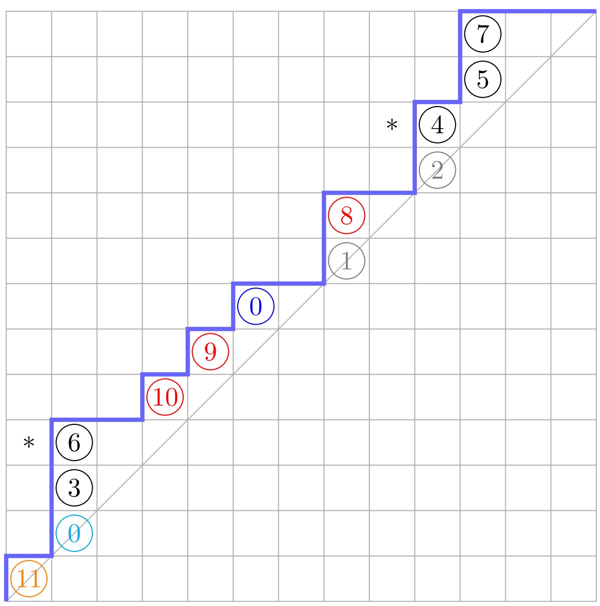
Let us look at what happens to the statistics of the path (see Figure 4.1).

The area goes down by the size (i.e. $m+n$), minus the number of zeros in the area word (i.e. $r+s$), minus the number of decorated rises (i.e. k), since these letters did not contribute to the area to begin with. This explains the term $t^{m+n-(r+s+k)}$.

The factor $q^{\binom{r}{2}}$ takes into account the primary dinv among 0's that have a small label. The factor $\begin{bmatrix} s \\ z \end{bmatrix}_q$ takes into account the primary dinv among 0's that have a small label, and 0's that have a big label. Indeed, each time a one of the former precedes one of the latter one unit of primary dinv is created. The factor $\begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q$ takes into account the primary dinv among 0's that have a zero labels and the other 0's, where we get $s-1$ because $l_1 \neq 0$.

The factor $q^{\binom{h}{2}}$ takes into account the secondary dinv between 1's that are decorated rises and 0's that are directly below a decorated rise. The factor $\begin{bmatrix} r+z \\ h \end{bmatrix}_q$ takes into account the secondary dinv between those 1's, and the remaining 0's that have either a zero or a small label. The factor $\begin{bmatrix} r+z+u-1 \\ u \end{bmatrix}_q$ takes into account the secondary between the remaining 1's and the 0's that have either a zero or a small label, where we get $s-1$ because the first 0 that has a non-big label must be before the first 1.

Summing over all the possible values of h, r, u, z, we obtain the stated recursion.



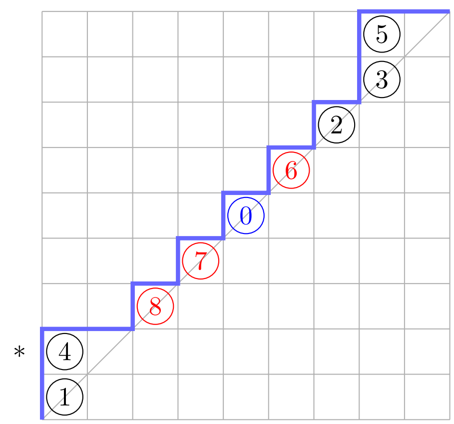


Figure 4.1: A Dyck path in $\mathsf{LD}(2, 11\backslash 3)^{*2,\circ 4}$ (left) and the resulting Dyck path in $\mathsf{LD}(1, 8\backslash 6)^{*1,\circ 3}$ after one step of the recursion (right).

The parameters for the recursive step of Theorem 4.1 shown in Figure 4.1 (left) are $n = 11$, $m = 2$ (blue/cyan), $j = 4$ (red/orange), $s = z + (s - z) = 2 + 1$ (grey/orange), $r = 1$ (cyan), $h = 1$ (label 4), $u = 5$ (labels $3, 5, 8, 9, 10$). In Figure 4.1 (right), the labels have been rescaled to make it a parking function.

This recursion is actually an iterated version of another one that is better explained in terms of polyominoes. Let

$$\mathsf{RP}(m\backslash r,n)^{*k,j}\subseteq \mathsf{RP}(m,n)^{*k,j}$$

be the subset of $m\times n$ reduced polyominoes with k unbarred and j barred decorated rises such that the area word has exactly r (unbarred) 0's, including the first one (hence $1\leq r\leq m+1$). We set

$$\mathsf{RP}_{q,t}(m\backslash r,n)^{*k,j}:=\sum_{\pi\in\mathsf{RP}(m\backslash r,n)^{*k,\circ j}} q^{\mathsf{dinv}(\pi)}t^{\mathsf{area}(\pi)}.$$

Theorem 4.2. For $0\leq j,k\leq m,n$, $1\leq r\leq m+1$, the polynomials $\mathsf{RP}_{q,t}(m\backslash r,n)^{*k,j}$ satisfy the recursion

$$\mathsf{RP}_{q,t}(m\backslash r,n)^{*k,j}=t^{m-r-k+1}\sum_{w=0}^{r}\sum_{s=0}^{n}q^{\binom{w}{2}}\begin{bmatrix}r\\w\end{bmatrix}_q\begin{bmatrix}r+s-w-1\\s-w\end{bmatrix}_q\mathsf{RP}_{q,t}(n{-}1\backslash s,\ m{-}r{+}1)^{*j-w,k}$$

with initial conditions

$$\mathsf{RP}_{q,t}(m\backslash m+1,n)^{*k,j}=\delta_{k,0}\cdot q^{\binom{j}{2}}\begin{bmatrix}m+1\\j\end{bmatrix}_q\begin{bmatrix}m+n-j\\m\end{bmatrix}_q.$$

Proof. Let us start with the initial conditions. The set $\mathsf{RP}_{q,t}(m\backslash m+1,n)^{*k,j}$ consists of the paths whose unbarred letters in the area word are all equal to 0. This implies that all the barred letters must be equal to $\bar{0}$, and also that there are no unbarred rises, hence the factor $\delta_{k,0}$. The dinv among the 0's and the decorated $\bar{0}$'s is counted by $q^{\binom{j}{2}}\begin{bmatrix}m+1\\j\end{bmatrix}_q$, while the dinv between the 0's and the non decorated $\bar{0}$'s is counted by $\begin{bmatrix}m+n-j\\m\end{bmatrix}_q$.

The recursive step consists of removing all the 0's, and going down by one step in the alphabet $0<\bar{0}<1<\dots$. The area drops by $m+1-k-r$ (the number of unbarred, non decorated letters, minus the number of 0's). The factor $q^{\binom{w}{2}}\begin{bmatrix}r\\w\end{bmatrix}_q$ takes care of the inversions formed by the the r 0's and the w decorated $\bar{0}$'s. The factor $\begin{bmatrix}r+s-w-1\\s-w\end{bmatrix}_q$ takes care of the inversions between the r 0's and the remaining $s-w$ $\bar{0}$'s, where s is the number of total $\bar{0}$'s.

Now, barred and unbarred letters switch roles, and the w decorations of rises of type $0\bar{0}$ disappear. The remaining letters form the area word of a polyomino in $\mathsf{RP}(n-1\setminus s,\ m-r+1)^{*j-w,k}$, and the statement follows.

Using the map $\bar{\zeta}$, thanks to Remark 2.36 we can translate this recursion into one for doubly corner-decorated polyominoes in terms of $(\mathsf{area},\mathsf{bounce})$. Let

$$\mathsf{RP}(m\backslash r,n)^{\circ k,j}\subseteq \mathsf{RP}(m,n)^{\circ k,j}$$

be the subset of $m\times n$ reduced polyominoes with k decorated green peaks and j decorated red valleys such that the bounce word has exactly r (unbarred) 0's, including the first one (hence $1\leq r\leq m+1$). We set

$$\mathsf{RP}_{q,t}(m\backslash r,n)^{\circ k,j}:=\sum_{\pi\in\mathsf{RP}(m\backslash r,n)^{\circ k,\circ j}} q^{\mathsf{area}(\pi)}t^{\mathsf{bounce}(\pi)}.$$

Theorem 4.3. For $0 \leq j, k \leq m, n$, $1 \leq r \leq m+1$, the polynomials $\mathsf{RP}_{q,t}(m \backslash r, n)^{\circ k,j}$ satisfy the recursion

$$\mathsf{RP}_{q,t}(m \backslash r, n)^{\circ k,j} = t^{m-r-k+1} \sum_{w=0}^{r} \sum_{s=0}^{n} q^{\binom{w}{2}} \begin{bmatrix} r \\ w \end{bmatrix}_q \begin{bmatrix} r+s-w-1 \\ s-w \end{bmatrix}_q \mathsf{RP}_{q,t}(n-1 \backslash s,\ m-r+1)^{\circ j-w,k}$$

with initial conditions

$$\mathsf{RP}_{q,t}(m \backslash m+1, n)^{\circ k,j} = \delta_{k,0} \cdot q^{\binom{j}{2}} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q \begin{bmatrix} m+n-j \\ m \end{bmatrix}_q .$$

Proof. This is a direct consequence of Theorem 4.2 and Proposition 2.35. We will, however, give a combinatorial interpretation.

Let us start with the initial conditions. The set $\mathsf{RP}_{q,t}(m \backslash m+1, n)^{\circ k,j}$ consists of the paths whose unbarred letters in the bounce word are all equal to 0. This implies that all the barred letters must be equal to $\bar{0}$, and also that the green path is a horizontal streak followed by a vertical streak, hence there are no green peaks, which explains the factor $\delta_{k,0}$. The area in the rows containing a decorated red valley is counted by $q^{\binom{j}{2}} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q$, while the area in the remaining rows is counted by $\begin{bmatrix} m+n-j \\ m \end{bmatrix}_q$.

The recursive step (see Figure 4.2) consists of removing the first horizontal streak of the bounce path and its first vertical step, i.e. taking the intersection with the rectangle going from $(r-1, 1)$ to (m, n) (orange in the picture). The factor $q^{\binom{w}{2}} \begin{bmatrix} r \\ w \end{bmatrix}_q$ takes care of the area outside the rectangle in the w rows containing a decorated red valley with horizontal coordinate from 0 to $r-1$. The factor $\begin{bmatrix} r+s-w-1 \\ s-w \end{bmatrix}_q$ takes care of the area outside the rectangle in the remaining $s-w$ rows, where s is the number of total $\bar{0}$'s in the bounce word. We then reflect along the line $x = y$. The bounce drops by $m-r-k+1$, because every unbarred letter decreases by one in the alphabet (so its value drops by 1), except the $r-1$ 0's, that are just removed, and the k letters corresponding to decorated green peaks, whose value actually decrease, but they are not counted while computing bounce and so they should be ignored. After one step of the recursion, the polyomino will be the one delimited by the orange rectangle, flipped along the line $x = y$.

Now, the green and the red path switch roles, and the w decorations in the first $r-1$ columns disappear. We are left with a polyomino in $\mathsf{RP}(n-1 \setminus s, m-r+1)^{\circ j-w,k}$, and the statement follows.

Iterating the recursion for $(\mathsf{dinv}, \mathsf{area})$ and making a suitable change of variables, we get the recursion in Theorem 4.1, hence the two families of polynomials are equal. We will show a bijective proof of this fact.

Theorem 4.4. For $0 \leq j, k, s \leq n$, $0 \leq m$, we have

$$\mathsf{LD}_{q,t}(m, n \backslash s)^{*k,\circ j} = \mathsf{RP}_{q,t}(n-1 \backslash s, m+n-j)^{*k,n-j}$$

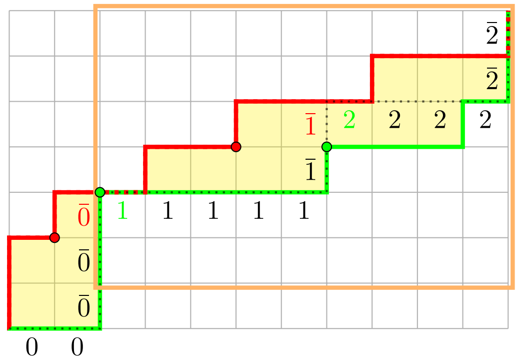


Figure 4.2: One step of the recursion for reduced polyominoes.

Proof. We prove this theorem by showing a bijection

$$\mathsf{LD}(m, n\backslash s)^{*k,\circ j} \to \mathsf{RP}(n-1\backslash s, m+n-j)^{*k,n-j}$$

that preserves the bistatistic $(\mathsf{dinv}, \mathsf{area})$.

Given the area word of such a decorated Dyck path, the first step of the bijection consists of putting bars on letters corresponding to 0 labels (of which there are m); notice that since 0 labels can only be assigned to valleys, and valleys are not rises, the word becomes an area word in the alphabet $\overline{\mathbb{N}}$ (i.e. we can't have jumps like $0\bar{1}$). The k decorations on the rises are kept as they are; notice that all these rises are unbarred, since all the barred letters are valleys and hence they can't be rises. Finally one adds a decorated barred letter after each unbarred letter who has a small label assigned (of which we have $n-j$). It is easy to check that in this way one obtains the area word of a doubly rise-decorated polyomino in the expected set. See Example 4.5 for an example.

This maps obviously preserves the area. The primary and the secondary dinv between 0 labels and positive ones is trivially preserved (the corresponding letters in the area word still form a diagonal inversion). The primary diagonal inversions among the small labels are replaced by the inversions between the decorated barred letters and the unbarred letters that are followed by a decorated barred one. The primary and secondary diagonal inversions between the small labels and the big labels are now replaced by inversions formed by decorated barred letters followed by unbarred letters that are not followed by a decorated barred letter (primary) or vice versa (secondary).

To build the inverse map, one proceeds as follows. Given the area word of such a polyomino, ignore the barred decorated letters. The remaining ones, disregarding bars, still form the area word of a Dyck path. This will be the actual path. If an unbarred letter is a decorated rise, then its image is still a rise, and we decorate it. We put zero labels on the steps corresponding to (non decorated) barred letters; those must be valleys, since there can't be a letter of strictly smaller value in the original area word of the polyomino, hence there is no restriction on their label (that can thus be 0). Next, we put a big label in all the rows, except those whose corresponding letter of the area word of a polyomino is an unbarred letter followed by a decorated rise. Notice that all the steps that are assigned a small label (i.e. a non-zero, non-big label) in the image must have this property (i.e. coming from an unbarred letter followed by a decorated rise).

It is clear that these two bijections are one the inverse of the other, and so the statement follows.

Example 4.5. Figure 4.3 shows a partially labelled decorated Dyck path in $\mathsf{LD}(2, 11\backslash 3)^{*2,\circ 7}$ and its image through the bijection. Its area word is $(0, 0, 1, \overset{*}{2}, 1, 1, 1, 0, 1, 0, \overset{*}{1}, 1, 1)$, where the asterisks denote decorated rises.

First we add bars to letters corresponding to 0 labels, getting $(0, \bar{0}, 1, \overset{*}{2}, 1, 1, \bar{1}, 0, 1, 0, \overset{*}{1}, 1, 1)$. Then we add a decorated barred letter after every letter corresponding to a small label, i.e. labels with value lesser or equal than $11 - 7 = 4$, getting $(0, \bar{0}, 1, \overset{*}{\bar{1}}, \overset{*}{2}, 1, 1, \bar{1}, 0, \overset{*}{\bar{0}}, 1, 0, \overset{*}{\bar{0}}, \overset{*}{1}, \overset{*}{\bar{1}}, 1, 1)$.

This is the area word of a polyomino in $\mathsf{RP}(10\backslash 3, 6)^{*2,4}$, as expected.

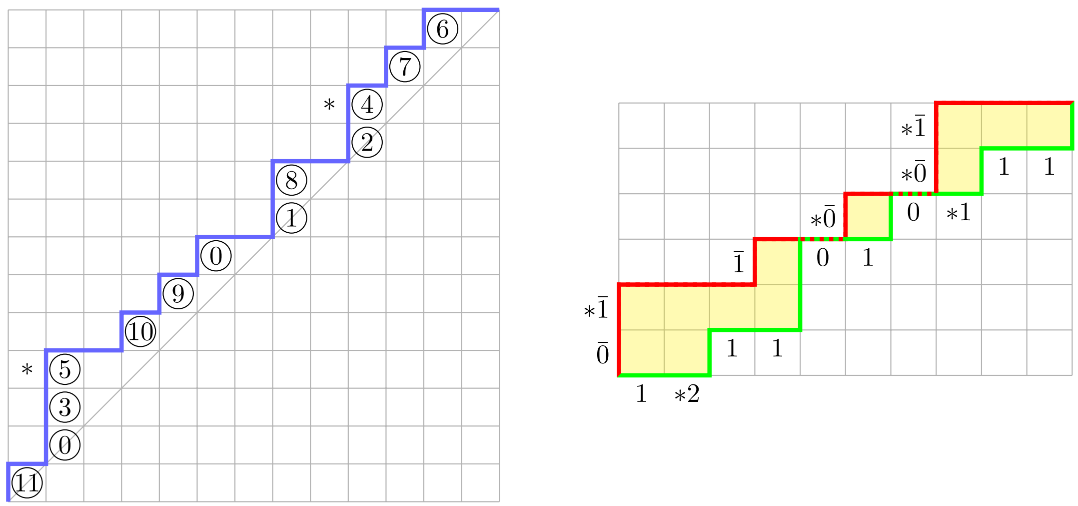


Figure 4.3: A Dyck path in $\mathsf{LD}(2, 11\backslash 3)^{*2,\circ 7}$ (left) and the polyomino that is its image through the bijection (right). The area word of the polyomino is $(0, \bar{0}, 1, \overset{*}{\bar{1}}, \overset{*}{2}, 1, 1, \bar{1}, 0, \overset{*}{\bar{0}}, 1, 0, \overset{*}{\bar{0}}, \overset{*}{1}, \overset{*}{\bar{1}}, 1, 1)$.

4.2 Partially labelled square paths

For further details on the content of this Section, see [9].

As we did for Dyck paths, it is convenient to split our set into smaller subsets and find a recursion for those. Define the subset

$$\mathsf{SQ}(m, n\backslash s)^{*k,\circ j} \subseteq \mathsf{LSQ}(m, n)^{*k}$$

to consist of the paths $\pi \in \mathsf{LSQ}(m, n)^{*k}$ such that

$$\mathsf{dr}(\pi) \in (1, 2, \dots, n-j) \sqcup (n, n-1, \dots, n-j+1) \quad \text{and} \quad \#\{1 \leq i \leq m+n \mid a_i = -a \wedge l_i \neq 0\} = s,$$

where a is the shift of the path. Let labels from 1 to $n - j$ be *small*, and the labels from $n - j + 1$ to n be *big*. We set

$$\mathsf{SQ}_{q,t}(m, n\backslash s)^{*k,\circ j} := \sum_{\pi \in \mathsf{SQ}(m,n\backslash s)^{*k,\circ j}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}.$$

Theorem 4.6. For $0 \leq j,k,s \leq n$, $0 \leq m$, the polynomials $\mathsf{SQ}_{q,t}(m,n\backslash s)^{*k,\circ j}$ satisfy the recursion

$$\mathsf{SQ}_{q,t}(m,n\backslash s)^{*k,\circ j} = \mathsf{LD}_{q,t}(m,n\backslash s)^{*k,\circ j} + q^s t^{n-k-s} \sum_{r=0}^{m} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q$$

$$\times t^{m-r} \sum_{u=0}^{n-k-s} \sum_{h=0}^{r+z} q^{\binom{h}{2}} \begin{bmatrix} u+h \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u+h-1 \end{bmatrix}_q \mathsf{SQ}_{q,t}(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}$$

with initial conditions

$$\mathsf{SQ}_{q,t}(m,n\backslash n)^{*k,\circ j} = \mathsf{LD}_{q,t}(m,n\backslash n)^{*k,\circ j} = \delta_{k,0} \cdot q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q.$$

Proof. The initial conditions are straightforward: if all the letters of the area word with a positive label are minima, since the condition of ending East implies that one of them must be on the main diagonal (i.e. the corresponding letter of the area word is 0), then all of them are on the main diagonal, hence the minimum of the area word is 0 and the path is actually a Dyck path.

For the recursive step, we first give an overview of the combinatorial interpretations of all the variables appearing in this formula. We say that a vertical step of a path is *at height i* if its corresponding letter in the area word equals $i-a$, where a is the shift of the path (i.e. the steps on the base diagonal are at height 0).

We have that

- z is the number of small labels on the base diagonal.
- $s-z$ is the number of big labels on the base diagonal.
- r is the number of zero labels on the base diagonal.
- h is the number of i's such that $a_i = 1-a$ and i is a decorated rise.
- u is the number of i's such that $a_i = 1-a$, i is not a decorated rise, and $l_i \neq 0$.

The strategy of this recursion is the following. Start from a path π in $\mathsf{SQ}(m,n\backslash s)^{*k,\circ j}$. If it is a Dyck path, it is counted by $\mathsf{LD}_{q,t}(m,n\backslash s)^{*k,\circ j}$. If it is not, remove all the minima from the area word (with the corresponding labels), and remove the decorations on rises at height one (which are not rises any more). In this way we obtain a path in

$$\mathsf{SQ}(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}.$$

Let us look at what happens to the statistics of the path.

The area goes down by the size (i.e. $m+n$), minus the number of zeros in the area word (i.e. $r+s$), minus the number of decorated rises (i.e. k), since these letters did not contribute to the area to begin with. This explains the term $t^{m+n-(r+s+k)}$.

The factor q^s takes into account the tertiary dinv that the minima generated (being them negative letters with a positive label). The factor $q^{\binom{r}{2}}$ takes into account the primary dinv among the minima that have a small label. The factor $\begin{bmatrix} r+z \\ z \end{bmatrix}_q$ takes into account the primary dinv among the minima that have a small label, and the minima that have a 0 label. Indeed, each time a one of the former precedes one of the latter one unit of primary dinv is created. The factor $\begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q$ takes into account the primary dinv among the minima that have a big label (which are $s-z$) and the other minima (which are $r+z$), where we get $r+z-1$ because the last minimum cannot have a big label (it must be followed by a rise).

The factor $q^{\binom{h}{2}}$ takes into account the secondary dinv betweensteps at height 1 that are decorated rises and steps at height 0 that are directly below a decorated rise. The factor $\begin{bmatrix} u+h \\ h \end{bmatrix}_q$ takes into account the secondary dinv among small labels at height 1, and 0 or small labels below a decorated rise. The factor $\begin{bmatrix} r+z+u-1 \\ u+h-1 \end{bmatrix}_q$ takes into account the secondary among all the non-zero labels at height 1 (of which we have $u+h$), and the 0 or small labels at height 0 that are not below a decorated rise (of which we have $r+z-h$), where we get $u+h-1$ because the last rise comes after all the minima (because the last letter of the area word is non-negative).

Summing over all the possible values of h, r, u, z, we obtain the stated recursion.

CHAPTER 5

Algebraic recursions

In this chapter we are going to show the two combinatorial recursions that are needed to prove the so called *Schröder case* of the generalised Delta and Delta square conjectures, i.e. the $\langle \cdot , e_{n-j}h_j\rangle$ case. We are going to show that they match the combinatorial ones, thus proving that the q,t-enumerators of the relevant agree with the symmetric functions as predicted by the Delta conjectures.

5.1 The family $F(m,n\backslash s)^{*k,\circ j}$

We now introduce the first of the families of symmetric functions that match our combinatorial q,t-enumerators. Let

$$F(m,n\backslash s)^{*k,\circ j} := t^{n-k-s}\langle \Delta_{h_{n-k-s}}\Delta_{e_k}e_{m+n-j}\left[X[s]_q\right], e_m h_{n-j}\rangle,$$

which is the family $F^{(j,k)}_{n,s;m}$ described in [11, Section 5]. We want to show the following.

Theorem 5.1. For $0 \leq j,k,s \leq n$, $0 \leq m$, we have

$$F(m,n\backslash s)^{*k,\circ j} = \mathsf{LD}_{q,t}(m,n\backslash s)^{*k,\circ j}.$$

We start with a lemma.

Lemma 5.2. For $0 \leq j,k,s \leq n$, $0 \leq m$, we have

$$\begin{aligned} F(m,n\backslash s)^{*k,\circ j} &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\Pi_\mu}{w_\mu}h_{n-k-s}[B_\mu]h_s[(1-t)B_\mu]e_k[B_\mu]e_m[B_\mu] \\ &= \sum_{\mu\vdash m+n-j}\left(\mathbf{\Pi}^{-1}\nabla E_{n-k,s}[X]\right)\Big|_{X=MB_\mu}\frac{\Pi_\mu}{w_\mu}e_k[B_\mu]e_m[B_\mu]. \end{aligned}$$

Proof. We have

$$\begin{aligned}
F(m,n\backslash s)^{*k,\circ j} &= t^{n-k-s}\langle \Delta_{h_{n-k-s}}\Delta_{e_k} e_{m+n-j}\left[X[s]_q\right], e_m h_{n-j}\rangle \\
\text{(by 1.45)} &= t^{n-k-s}(1-q^s)\left\langle \Delta_{h_{n-k-s}}\Delta_{e_k}\sum_{\mu\vdash m+n-j} h_s[(1-t)B_\mu]\frac{\Pi_\mu \widetilde{H}_\mu[X]}{w_\mu}, e_m h_{n-j}\right\rangle \\
\text{(by 1.51)} &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\Pi_\mu}{w_\mu} h_{n-k-s}[B_\mu]h_s[(1-t)B_\mu]e_k[B_\mu]\langle \widetilde{H}_\mu[X], e_m h_{n-j}\rangle \\
\text{(by 1.38)} &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\Pi_\mu}{w_\mu} h_{n-k-s}[B_\mu]h_s[(1-t)B_\mu]e_k[B_\mu] \\
&\quad \times \left\langle \widetilde{H}_\mu[X], h_m\left[\frac{X}{M}\right] e_{n-j}\left[\frac{X}{M}\right]\right\rangle_* \\
\text{(by 1.48)} &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\Pi_\mu}{w_\mu} h_{n-k-s}[B_\mu]h_s[(1-t)B_\mu]e_k[B_\mu]e_m[B_\mu]
\end{aligned}$$

which proves the first equality. Now,

$$\begin{aligned}
(*) &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\left(h_{n-k-s}\left[\frac{X}{M}\right]h_s\left[\frac{X}{1-q}\right]\right)\Big|_{X=MB_\mu}\frac{\Pi_\mu}{w_\mu}e_k[B_\mu]e_m[B_\mu] \\
\text{(by [25, (7.86)])} &= \sum_{\mu\vdash m+n-j}\left(\mathbf{\Pi}^{-1}\nabla E_{n-k,s}\right)\big|_{X=MB_\mu}\frac{\Pi_\mu}{w_\mu}e_k[B_\mu]e_m[B_\mu]
\end{aligned}$$

as desired.

Proof of Theorem 5.1. We prove the theorem by showing that $F(m,n\backslash s)^{*k,\circ j}$ satisfies the same recursion with the same initial conditions as $\mathsf{LD}_{q,t}(m,n\backslash s)^{*k,\circ j}$, stated in Theorem 4.1. In order to prove this, we show that it satisfies the recursion

$$\begin{aligned}
F(m,n\backslash s)^{*k,\circ j} = t^{n-k-s}\sum_{r=0}^{m}\sum_{z=0}^{s} q^{\binom{z}{2}}\begin{bmatrix} s \\ z\end{bmatrix}_q\begin{bmatrix} r+s-1 \\ r\end{bmatrix}_q \\
\times F(n-k-s, m+n-j\backslash r+z)^{*n-j-z,\circ m+n-j-k}
\end{aligned}$$

with initial conditions

$$F(m,n\backslash n)^{*k,\circ j} = \delta_{k,0}\cdot q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j\end{bmatrix}_q\begin{bmatrix} m+n-1 \\ m\end{bmatrix}_q.$$

This recursion is the one in Theorem 4.2 up to a change of variables. It is straightforward to check that iterating it we get the one in Theorem 4.1, which is exactly our statement.

Let us start with the initial conditions. We need to evaluate

$$\langle \Delta_{h_{-k}}\Delta_{e_k} e_{m+n-j}\left[X[n]_q\right], e_m h_{n-j}\rangle$$

which is clearly 0 if $k\neq 0$, as in that case either $\Delta_{h_{-k}}$ or Δ_{e_k} is the null operator. For $k=0$ we have

$$
\begin{aligned}
F(m,n\backslash n)^{*k,\circ j} &= \delta_{k,0}\langle e_{m+n-j}[X[n]_q], e_m h_{n-j}\rangle \\
\text{(by 1.38)} &= \delta_{k,0}\left\langle e_{m+n-j}[X[n]_q], h_m\left[\frac{X}{M}\right] e_{n-j}\left[\frac{X}{M}\right]\right\rangle_* \\
\text{(by 1.45)} &= \delta_{k,0}\left\langle \sum_{\mu\vdash m+n-j}\frac{\widetilde{H}_\mu[X]\widetilde{H}_\mu[M[n]_q]}{w_\mu}, h_m\left[\frac{X}{M}\right] e_{n-j}\left[\frac{X}{M}\right]\right\rangle_* \\
\text{(by 1.48)} &= \delta_{k,0}\sum_{\mu\vdash m+n-j}\frac{e_m[B_\mu]\widetilde{H}_\mu[M[n]_q]}{w_\mu} \\
\text{(by 1.48)} &= \delta_{k,0}e_{n-j}[[n]_q]h_m[[n]_q] = \delta_{k,0}q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j\end{bmatrix}_q\begin{bmatrix} m+n-1 \\ m\end{bmatrix}_q
\end{aligned}
$$

as desired. Now we have to deal with the recursive step. We have

$$
\begin{aligned}
F(m,n\backslash s)^{*k,\circ j} &= t^{n-k-s}\langle \Delta_{h_{n-k-s}}\Delta_{e_k}e_{m+n-j}\left[X[s]_q\right], e_m h_{n-j}\rangle \\
\text{(by 5.2)} &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\Pi_\mu}{w_\mu}h_{n-k-s}[B_\mu]h_s[(1-t)B_\mu]e_k[B_\mu]e_m[B_\mu] \\
\text{(by 1.48)} &= t^{n-k-s}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\Pi_\mu}{w_\mu}h_s[(1-t)B_\mu]e_m[B_\mu]\sum_{\nu\vdash n-s}e_{n-k-s}[B_\nu]\frac{\widetilde{H}_\nu[MB_\mu]}{w_\nu} \\
\text{(by 1.42)} &= t^{n-k-s}\sum_{\nu\vdash n-s}e_{n-k-s}[B_\nu]\frac{\Pi_\nu}{w_\nu}(1-q^s)\sum_{\mu\vdash m+n-j}\frac{\widetilde{H}_\mu[MB_\nu]}{w_\mu}h_s[(1-t)B_\mu]e_m[B_\mu] \\
\text{(by 1.55)} &= t^{n-k-s}\sum_{\nu\vdash n-s}e_{n-k-s}[B_\nu]\frac{\Pi_\nu}{w_\nu}(1-q^s)\sum_{r=0}^{m}t^{m-r}\sum_{z=0}^{s}q^{\binom{z}{2}}\begin{bmatrix} r+z \\ z\end{bmatrix}_q \\
&\quad\times\begin{bmatrix} r+s-1 \\ s-z\end{bmatrix}_q h_{r+z}[(1-t)B_\nu]h_{m-r}[B_\nu]e_{n-j-z}[B_\nu] \\
\text{(-)} &= t^{n-k-s}\sum_{r=0}^{m}\sum_{z=0}^{s}q^{\binom{z}{2}}\begin{bmatrix} s \\ z\end{bmatrix}_q\begin{bmatrix} r+s-1 \\ r\end{bmatrix}_q \\
&\quad\times t^{m-r}(1-q^{r+z})\sum_{\nu\vdash n-s}\frac{\Pi_\nu}{w_\nu}h_{m-r}[B_\nu]h_{r+z}[(1-t)B_\nu]e_{n-k-s}[B_\nu]e_{n-j-z}[B_\nu] \\
\text{(by 5.2)} &= t^{n-k-s}\sum_{r=0}^{m}\sum_{z=0}^{s}q^{\binom{z}{2}}\begin{bmatrix} s \\ z\end{bmatrix}_q\begin{bmatrix} r+s-1 \\ r\end{bmatrix}_q \\
&\quad\times F(n-k-s, m+n-j\backslash r+z)^{*n-j-z,\circ m+n-j-k}
\end{aligned}
$$

as desired. 🐥

We are left to prove that the sum over s of these polynomials yields $\langle \Delta_{h_m}\Delta'_{e_{n-k-1}}e_n, e_{n-j}h_j\rangle$, as predicted by the Delta conjecture. We need another lemma.

Lemma 5.3. For every $n \geq k \geq 0$ and $\lambda \vdash n$, we have

$$B_\lambda e_{n-k-1}[B_\lambda - 1] = \sum_{\mu \subset_k \lambda} c^{(k)}_{\lambda\mu} B_\mu T_\mu.$$

Proof. We prove the result by induction on $n-k$. If $n-k=0$ the statement holds trivially (we get $0=0$). Otherwise, we have

$$\begin{aligned}
B_\lambda e_{n-k-1}[B_\lambda - 1] &= B_\lambda e_{n-k-1}[B_\lambda] - B_\lambda e_{n-k-2}[B_\lambda - 1] \\
\text{(by 1.48)} &= B_\lambda \langle H_\lambda, e_{n-k-1}h_{k+1}\rangle - B_\lambda e_{n-k-2}[B_\lambda - 1] \\
\text{(by 1.19)} &= B_\lambda \langle h^\perp_{k+1} H_\lambda, e_{n-k-1}\rangle - B_\lambda e_{n-k-2}[B_\lambda - 1] \\
\text{(by 1.49)} &= B_\lambda \sum_{\nu \subset_{k+1} \lambda} c^{(k+1)}_{\lambda\nu} \langle H_\nu, e_{n-k-1}\rangle - B_\lambda e_{n-k-2}[B_\lambda - 1] \\
\text{(by 1.48)} &= B_\lambda \sum_{\nu \subset_{k+1} \lambda} T_\nu c^{(k+1)}_{\lambda\nu} - B_\lambda e_{n-k-2}[B_\lambda - 1] \\
\text{(by induction)} &= \sum_{\nu \subset_{k+1} \lambda} (B_\lambda - B_\nu) T_\nu c^{(k+1)}_{\lambda\nu} \\
\text{(by [4, Proposition 5])} &= \sum_{\nu \subset_{k+1} \lambda} (B_\lambda - B_\nu) T_\nu \frac{1}{B_\lambda - B_\nu} \sum_{\nu \subset_1 \mu \subset_k \lambda} c^{(k)}_{\lambda\mu} c^{(1)}_{\mu\nu} \frac{T_\mu}{T_\nu} \\
\text{(-)} &= \sum_{\nu \subset_{k+1} \lambda} \sum_{\nu \subset_1 \mu \subset_k \lambda} c^{(k)}_{\lambda\mu} c^{(1)}_{\mu\nu} T_\mu \\
\text{(-)} &= \sum_{\mu \subset_k \lambda} c^{(k)}_{\lambda\mu} \sum_{\nu \subset_1 \mu} c^{(1)}_{\mu\nu} T_\mu \\
\text{(by 1.50)} &= \sum_{\mu \subset_k \lambda} c^{(k)}_{\lambda\mu} B_\mu T_\mu
\end{aligned}$$

as desired.

Now we can finally show the last theorem.

Theorem 5.4. For $0 \leq j, k, s \leq n$, $0 \leq m$, we have

$$\sum_{s=0}^{n} F(m, n\backslash s)^{*k,\circ j} = \langle \Delta_{h_m}\Delta'_{e_{n-k-1}}e_n, e_{n-j}h_j\rangle.$$

Proof. We have

$$\begin{aligned}
\sum_{s=0}^{n} F(m,n\backslash s)^{*k,\circ j} &= \sum_{s=0}^{n} t^{n-k-s} \langle \Delta_{h_{n-k-s}} \Delta_{e_k} e_{m+n-j}[X[s]_q], e_m h_{n-j} \rangle \\
\text{(by 5.2)} &= \sum_{s=0}^{n} \sum_{\mu \vdash m+n-j} (\mathbf{\Pi}^{-1} \nabla E_{n-k,s}[X])\big|_{X=MB_\mu} \frac{\Pi_\mu}{w_\mu} e_k[B_\mu] e_m[B_\mu] \\
\text{(by 1.53)} &= \sum_{\mu \vdash m+n-j} (\mathbf{\Pi}^{-1} \nabla e_{n-k}[X])\big|_{X=MB_\mu} \frac{\Pi_\mu}{w_\mu} e_k[B_\mu] e_m[B_\mu] \\
\text{(by 1.46)} &= \sum_{\mu \vdash m+n-j} \left(\mathbf{\Pi}^{-1} \sum_{\nu \vdash n-k} \nabla \frac{MB_\nu \Pi_\nu \widetilde{H}_\nu[X]}{w_\nu} \right)\Bigg|_{X=MB_\mu} \frac{\Pi_\mu}{w_\mu} e_k[B_\mu] e_m[B_\mu] \\
\text{(by 1.51)} &= \sum_{\mu \vdash m+n-j} \sum_{\nu \vdash n-k} \frac{MB_\nu T_\nu}{w_\nu} \frac{\Pi_\mu e_m[B_\mu]}{w_\mu} e_k[B_\mu] \widetilde{H}_\nu[MB_\mu] \\
\text{(by 1.49)} &= \sum_{\mu \vdash m+n-j} \sum_{\nu \vdash n-k} \sum_{\lambda \supset_k \nu} d_{\lambda\nu}^{(k)} \frac{MB_\nu T_\nu}{w_\nu} \frac{\Pi_\mu e_m[B_\mu] \widetilde{H}_\lambda[MB_\mu]}{w_\mu} \\
\text{(by 1.49)} &= \sum_{\mu \vdash m+n-j} \sum_{\nu \vdash n-k} \sum_{\lambda \supset_k \nu} c_{\lambda\nu}^{(k)} \frac{MB_\nu T_\nu}{w_\lambda} \frac{\Pi_\mu e_m[B_\mu] \widetilde{H}_\lambda[MB_\mu]}{w_\mu} \\
\text{(by 1.42)} &= \sum_{\mu \vdash m+n-j} \sum_{\nu \vdash n-k} \sum_{\lambda \supset_k \nu} c_{\lambda\nu}^{(k)} \frac{MB_\nu T_\nu \Pi_\lambda}{w_\lambda} \frac{e_m[B_\mu] \widetilde{H}_\mu[MB_\lambda]}{w_\mu} \\
\text{(by 1.48)} &= \sum_{\lambda \vdash n} \sum_{\nu \subset_k \lambda} c_{\lambda\nu}^{(k)} \frac{MB_\nu T_\nu \Pi_\lambda}{w_\lambda} h_m[B_\lambda] e_{n-j}[B_\lambda] \\
\text{(by 5.3)} &= \sum_{\lambda \vdash n} \frac{MB_\lambda \Pi_\lambda}{w_\lambda} e_{n-k-1}[B_\lambda - 1] h_m[B_\lambda] e_{n-j}[B_\lambda] \\
\text{(by 1.48)} &= \sum_{\lambda \vdash n} \frac{MB_\lambda \Pi_\lambda}{w_\lambda} h_m[B_\lambda] e_{n-k-1}[B_\lambda - 1] \langle \widetilde{H}_\lambda, e_{n-j} h_j \rangle \\
\text{(by 1.51)} &= \langle \Delta_{h_m} \Delta'_{e_{n-k-1}} \sum_{\lambda \vdash n} \frac{MB_\lambda \Pi_\lambda \widetilde{H}_\lambda[X]}{w_\lambda}, e_{n-j} h_j \rangle \\
\text{(by 1.46)} &= \langle \Delta_{h_m} \Delta'_{e_{n-k-1}} e_n, e_{n-j} h_j \rangle
\end{aligned}$$

as desired.

5.2 The family $S(m, n\backslash s)^{*k,\circ j}$

The second family of symmetric functions that match our combinatorial q,t-enumerators is very similar to the first. Let

$$S(m,n\backslash s)^{*k,\circ j} := \frac{[n]_q}{[s]_q} F(m,n\backslash s)^{*k,\circ j} = \frac{[n]_q}{[s]_q} t^{n-k-s} \langle \Delta_{h_{n-k-s}} \Delta_{e_k} e_{m+n-j} \left[X[s]_q\right], e_m h_{n-j} \rangle,$$

which is the family $S_{n,s;m}^{(j,k)}$ described in [9, Section 4]. We want to show the following.

Theorem 5.5. For $0 \leq j, k, s \leq n$, $0 \leq m$, we have

$$S(m, n\backslash s)^{*k,\circ j} = \mathsf{SQ}_{q,t}(m, n\backslash s)^{*k,\circ j}.$$

Proof. We will show this using the recursion for $F(m, n\backslash s)^{*k,\circ j}$ and manipulating it slightly. The iterated version suits our needs better. Recall that $F(m, n\backslash s)^{*k,\circ j}$ satisfies the recursion

$$\begin{aligned} F(m, n\backslash s)^{*k,\circ j} &= t^{n-k-s} \sum_{r=0}^{m} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} s \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \\ &\times t^{m-r} \sum_{u=0}^{n-k-s} \sum_{h=0}^{r+z} q^{\binom{h}{2}} \begin{bmatrix} r+z \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u \end{bmatrix}_q F(m-r, n-s\backslash u+h)^{*k-h,\circ j-(s-z)} \end{aligned}$$

with initial conditions

$$F(m, n\backslash n)^{*k,\circ j} = \delta_{k,0} \cdot q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q.$$

If $s = n$ the two families of polynomials agree, hence the initial conditions are the same, as expected. If $s < n$, replacing F with S (i.e. multiplying both sides by $\frac{[n]_q}{[s]_q}$) we get

$$\begin{aligned} S(m, n\backslash s)^{*k,\circ j} &= \frac{[n]_q}{[s]_q} t^{n-k-s} \sum_{r=0}^{m} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} s \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \\ &\times t^{m-r} \sum_{u=0}^{n-k-s} \sum_{h=0}^{r+z} q^{\binom{h}{2}} \begin{bmatrix} r+z \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u \end{bmatrix}_q F(m-r, n-s\backslash u+h)^{*k-h,\circ j-(s-z)} \end{aligned}$$

$$\begin{aligned} &= \left(1 + q^s \frac{[n-s]_q}{[s]_q}\right) t^{n-k-s} \sum_{r=0}^{m} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} s \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \\ &\times t^{m-r} \sum_{u=0}^{n-k-s} \sum_{h=0}^{r+z} q^{\binom{h}{2}} \begin{bmatrix} r+z \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u \end{bmatrix}_q F(m-r, n-s\backslash u+h)^{*k-h,\circ j-(s-z)} \\ &= F(m, n\backslash s)^{*k,\circ j} + q^s \frac{[n-s]_q}{[s]_q} t^{n-k-s} \sum_{r=0}^{m} \sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} s \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ r \end{bmatrix}_q \\ &\times t^{m-r} \sum_{u=0}^{n-k-s} \sum_{h=0}^{r+z} q^{\binom{h}{2}} \begin{bmatrix} r+z \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u \end{bmatrix}_q \frac{[u+h]_q}{[n-s]_q} S(m-r, n-s\backslash u+h)^{*k-h,\circ j-(s-z)} \end{aligned}$$

$$= F(m,n\backslash s)^{*k,\circ j} + q^s t^{n-k-s} \sum_{r=0}^{m}\sum_{z=0}^{s} q^{\binom{z}{2}} \cdot t^{m-r} \sum_{u=0}^{n-k-s}\sum_{h=0}^{r+z} q^{\binom{h}{2}}$$
$$\times \frac{[s]_q!}{[z]_q![s-z]_q!}\frac{[r+s-1]_q!}{[r]_q![s-1]_q!}\frac{[r+z]_q!}{[h]_q![r+z-h]_q!}\frac{[r+z+u-1]_q!}{[u]_q![r+z-1]_q!}_q\frac{[u+h]_q}{[s]_q}\frac{[u+h-1]_q!}{[u+h-1]_q!}$$
$$\times S(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}$$

$$= F(m,n\backslash s)^{*k,\circ j} + q^s t^{n-k-s} \sum_{r=0}^{m}\sum_{z=0}^{s} q^{\binom{z}{2}} \cdot t^{m-r} \sum_{u=0}^{n-k-s}\sum_{h=0}^{r+z} q^{\binom{h}{2}}$$
$$\times \frac{[r+z]_q!}{[r]_q![z]_q!}\frac{[r+s-1]_q!}{[r+z-1]_q![s-z]_q!}\frac{[u+h]_q[u+h-1]_q!}{[h]_q![u]_q!}\frac{[r+z+u-1]_q!}{[u+h-1]_q![r+z-h]_q!}_q\frac{[s]_q!}{[s]_q[s-1]_q!}$$
$$\times S(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}$$

$$= F(m,n\backslash s)^{*k,\circ j} + q^s t^{n-k-s} \sum_{r=0}^{m}\sum_{z=0}^{s} q^{\binom{z}{2}} \begin{bmatrix} r+z \\ z \end{bmatrix}_q \begin{bmatrix} r+s-1 \\ s-z \end{bmatrix}_q$$
$$\times t^{m-r} \sum_{u=0}^{n-k-s}\sum_{h=0}^{r+z} q^{\binom{h}{2}} \begin{bmatrix} u+h \\ h \end{bmatrix}_q \begin{bmatrix} r+z+u-1 \\ u+h-1 \end{bmatrix}_q S(m-r,n-s\backslash u+h)^{*k-h,\circ j-(s-z)}$$

which is the same recursion of Theorem 4.6. The statement follows.

CHAPTER 6

State of the art

To conclude this thesis, we will give an overview on the available results on the Delta conjectures, the open problems, and possible directions for future research.

6.1 The q, t-Catalan

One of the first remarks made after the introduction of the nabla operator by Bergeron, Garsia, Haiman, and Tesler in [3], is that the scalar products $\langle \nabla e_n, e_n \rangle$, when evaluated at $q = t = 1$, yield the ubiquitous Catalan numbers. Since the possibly most iconic set counted by the Catalan numbers, the set of Dyck paths of size n, has a natural statistic given by the area, it was pretty soon clear that the identity

$$\langle \nabla e_n, e_n \rangle|_{t=1} = \sum_{\pi \in \mathsf{D}(n)} q^{\mathsf{area}(\pi)}$$

held. Moreover, this scalar product actually gives a polynomial which is symmetrical in q, t. Shortly after, people started chasing a second statistic $\mathsf{tstat} \colon \mathsf{D}(n) \to \mathbb{N}$, equidistributed with the area, such that

$$\langle \nabla e_n, e_n \rangle = \sum_{\pi \in \mathsf{D}(n)} q^{\mathsf{area}(\pi)} t^{\mathsf{tstat}(\pi)}.$$

In the meanwhile, Haiman proved the Schur positivity of the Macdonald polynomials [33] and ∇e_n [34], making the hunt even more interesting. After several years of failed attempts, almost simultaneously two statistics with that property have been found: the bounce, by Haglund, and the dinv, by Haiman. The conjectural identity

$$\langle \nabla e_n, e_n \rangle = \sum_{\pi \in \mathsf{D}(n)} q^{\mathsf{area}(\pi)} t^{\mathsf{bounce}(\pi)}$$

has been proved by Garsia and Haglund in [18], where they showed that bounce is indeed a valid choice for tstat; shortly afterwards, Haglund and Loehr in [27] found a bijection $\zeta \colon \mathsf{D}(n) \to \mathsf{D}(n)$

mapping $(\mathsf{area}, \mathsf{bounce})$ to $(\mathsf{dinv}, \mathsf{area})$, thus proving that

$$\langle \nabla e_n, e_n \rangle = \sum_{\pi \in \mathsf{D}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}$$

also held. Notice that, while we know by using the symmetric functions that $\langle \nabla e_n, e_n \rangle$ is symmetric in q, t, neither of those combinatorial results implies this symmetry, and in fact, up to today, we still lack a combinatorial interpretation of this apparently simple fact.

The so called q,t-Catalan is just the first chapter of the story. In [15], Egge, Haglund, Killpatrick, and Kremer extended the statistics area, bounce, and dinv to *Schröder paths*, i.e. Dyck paths in which diagonal steps are also allowed. In this thesis and in several other works in the literature, Schröder paths have been replaced by Dyck paths with decorated peaks, the identification being just replacing the diagonal steps with a North step followed by an East step, decorating the peak they form. They extended the bijection ζ to these Schröder paths, and conjectured the identity that, in our notation, states

$$\langle \nabla e_n, h_j e_{n-j} \rangle = \sum_{\pi \in \mathsf{D}(n)^{\circ j}} q^{\mathsf{area}(\pi)} t^{\mathsf{bounce}(\pi)} = \sum_{\pi \in \mathsf{D}(n)^{\circ j}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)}.$$

Shortly thereafter, Haglund proved the so called q,t-Schröder conjecture in [24].

6.2 The shuffle conjecture

In [26] we finally get the statement of the *shuffle conjecture* by Haiman, Haglund, Loehr, Remmel, and Ulyanov, in terms of labelled Dyck paths. They extended the definition of the dinv statistic to labelled objects, giving an interpretation of the full symmetric function ∇e_n. The original statement involved quasisymmetric functions and the scalar product with h_μ; without going into details, we restate it as

$$\nabla e_n = \sum_{\pi \in \mathsf{LD}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

In his work [24], Haglund actually proved not only the Schröder case $\langle \nabla e_n, h_j e_{n-j} \rangle$, but also the two-shuffle case $\langle \nabla e_n, h_j h_{n-j} \rangle$. In [36], Loehr and Remmel described the pmaj statistic, thus extending the bounce to labelled objects. They also generalised the previously known bijections to a map $\zeta \colon \mathsf{LD}(n) \to \mathsf{LD}(n)$ sending $(\mathsf{area}, \mathsf{pmaj})$ to $(\mathsf{dinv}, \mathsf{area})$. As a result, we got the pmaj version of the shuffle conjecture, equivalent to the dinv one, stating

$$\nabla e_n = \sum_{\pi \in \mathsf{LD}(n)} q^{\mathsf{area}(\pi)} t^{\mathsf{pmaj}(\pi)} x^\pi.$$

It is now 2007 when Loehr and Warrington, in [38], introduce the *square paths conjecture*. They removed the restriction of the path lying above the diagonal $x = y$, replacing it with the weaker condition of the path having to end with an East step. On this new set of objects, of which the Dyck paths form a subset, they extended the definitions of dinv and area and suggested a combinatorial interpretation of $\nabla \omega(p_n)$ in terms of labelled square paths, which in our notation reads

$$\nabla \omega(p_n) = \sum_{\pi \in \mathsf{LSQ}(n)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

The $\langle \nabla\omega(p_n), e_n\rangle$ case of this conjecture, known as q,t-square, has been previously proved by Can and Loehr in [5].

Several years passes before the next big step towards a proof of the shuffle conjecture. It is only in 2012 that Haglund, Morse, and Zabrocki introduce the *compositional shuffle conjecture* in [28], which later on turned out to be a key tool in the proof. In that paper, the authors introduce a family of operators on the symmetric functions, called C_α for $\alpha \vDash n$, which with an abuse of notation we identify with $C_\alpha(1)$. These operators have the notable properties that $\sum_{\alpha\vDash n} C_\alpha = e_n$ and that ∇C_α is Schur positive. A natural composition of n is associated to every Dyck path, namely the one given by the lengths of the segments between every pair of consecutive points in which the Dyck path touches the main diagonal. If we call $\mathsf{LD}(\alpha)$ the subset of Dyck paths whose *diagonal composition* is exactly α, then the compositional shuffle conjecture states that

$$\nabla C_\alpha = \sum_{\pi\in\mathsf{LD}(\alpha)} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

In [13], Duane, Garsia, and Zabrocki introduce a *newdinv* statistic, recursively defined, which they show that satisfy some two-shuffle compositional identity. Later on, in [35], Hicks and Kim gave a direct, but still algorithmic, definition of the newdinv statistic. Later on, thanks to this compositional refinement, Garsia, Xin, and Zabrocki managed to prove the two-shuffle $\langle \nabla C_\alpha, h_j h_{n-j}\rangle$ case of the compositional shuffle conjecture in [21], and then the case $\langle \nabla C_\alpha, h_j h_k e_{n-j-k}\rangle$ in [22], which was the first result that was not known in the non-compositional case. Their result remained the most general available one before the conjecture was fully proved.

In the meanwhile, in an apparently unrelated subject, Dukes and Le Borgne introduced in [14] a q,t-analogue of the Narayana numbers, that refine the Catalan numbers. Later on, Aval, D'Adderio, Dukes, Hicks, and Le Borgne, in [1], extended the story to parallelogram polyominoes. They defined three statistics area, bounce, and dinv on these objects, found a bijection sending $(\mathsf{area}, \mathsf{bounce})$ to $(\mathsf{dinv}, \mathsf{area})$, and gave a combinatorial interpretation of $\langle \Delta_{h_{m-1}} e_n, e_n\rangle$ in terms of $m \times n$ standard parallelogram polyominoes.

6.3 From the shuffle to the Delta

The year 2015 features not one, but two milestones in this field. The first one is the long awaited proof of the shuffle conjecture: Carlsson and Mellit, in [6] (see also [31]), introduce the *Dyck path algebra*, an algebra of operators which they use to prove the compositional refinement of the shuffle conjecture. The refinement is crucial, as without it there is no way to write the needed recursions for the symmetric functions side of the identity. The second milestone is the statement of the *generalised Delta conjecture* by Haglund, Remmel, and Wilson in [29], in terms of partially labelled decorated Dyck paths. In our notation, it reads

$$\Delta_{h_m}\Delta'_{e_{n-k-1}} e_n = \sum_{\pi\in\mathsf{LD}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

While the result by Carlsson and Mellit alone seemed to put an end to the story, if combined with the conjecture by Haglund, Remmel, and Wilson, it opened more questions than it closed. Can it be used to prove the square paths conjecture? Is there a compositional refinement of the Delta conjecture as well? What can we say about the Delta conjecture? Several people tried to answer these questions, and this is the setting we've been working in. The first question, regarding the square paths, was immediately answered by Sergel in [43], who showed that the shuffle theorem implies the square paths conjecture, thus proving it. A first result heading towards an answer to the second question, about a compositional Delta conjecture, has been shown by Zabrocki in [46], who proved a compositional refinement of the Schröder case of the Delta conjecture.

6.4 Our results

This is where our work starts. Together with D'Adderio and Vanden Wyngaerd, in [8] we proved the Schröder case $\langle \Delta'_{e_{n-k-1}} e_n, h_j e_{n-j} \rangle$, and the two-shuffle case $\langle \Delta'_{e_{n-k-1}} e_n, h_j h_{n-j} \rangle$ of the Delta conjecture. We also extended the statistics and the bijections in [1] to decorated polyominoes, proved the Schröder case $\langle \Delta_{h_{m-1}} e_n, h_k e_{n-k} \rangle$, and gave a bijection between the two-shuffle decorated Dyck paths and the Schröder rise-decorated polyominoes. We also introduced a pmaj statistic on labelled parallelogram polyominoes, stating the *polyominoes conjecture*

$$\Delta_{h_m} e_{n+1} = \sum_{\pi \in \mathsf{LPP}(m+1,n+1)} q^{\mathsf{area}(\pi)} t^{\mathsf{pmaj}(\pi)} x^\pi .$$

Finally, we showed that a special case of the Delta conjecture can be used to introduce new statistics on square paths that match a different q,t-square.

Later on, in [11] we proved the Schröder case $\langle \Delta_{h_m} \Delta'_{e_{n-k-1}} e_n, h_j e_{n-j} \rangle$ of the generalised Delta conjecture, and gave another interpretation in terms of doubly decorated parallelogram polyominoes, bijectively showing that the two are equivalent; in [9] we stated the generalised Delta square conjecture

$$\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \sum_{\pi \in \mathsf{LSQ}(m,n)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi$$

in terms of partially labelled decorated square paths, and proved the Schröder case

$$\frac{[n-k]_t}{[n]_t} \langle \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n), h_j e_{n-j} \rangle .$$

These results are the ones showed with full details in this thesis.

In the meanwhile, in [16] Garsia, Haglund, Remmel, and Yoo managed to prove the Delta conjecture at $q = 0$, and Romero in [42] did the same at $q = 1$. In [10], again together with D'Adderio and Vanden Wyngaerd, we proved the generalised Delta conjecture at $q = 0$ (or equivalently $t = 0$) and, as a corollary, the generalised Delta square conjecture at $q = 0$, since they happen to coincide.

Together with D'Adderio, in [7] we managed to give a proof of the $\langle \nabla e_n, h_j h_k e_{n-j-k} \rangle$ case of the shuffle conjecture that does not rely on the compositional refinement. In the same paper we showed, using a bijection, that the newdinv statistic in [13] coincides with the natural dinv statistics on a certain subset of decorated partially labelled Dyck paths.

In early 2019, Zabrocki conjectured in [47] a module, called super-diagonal coinvariants, that plays for the Delta conjecture the role that the module of the diagonal harmonics played for the shuffle conjecture, featuring a set of Grassmannian variables.

Finally, again together with D'Adderio and Vanden Wyngaerd, we introduced in [12] the Θ_f operators, which we used to state a compositional refinement of the Delta conjecture

$$\Theta_{e_k} \nabla C_\alpha = \sum_{\pi \in \mathsf{LD}(\alpha)^{*k}} q^{\mathsf{dinv}(\pi)} t^{\mathsf{area}(\pi)} x^\pi.$$

We also proved a touching refinement of the generalised shuffle conjecture (i.e. the case $k = 0$ of the generalised Delta conjecture) and showed that it implies the generalised square conjecture (i.e. the case $k = 0$ of the generalised Delta square conjecture). Furthermore, the Θ_{e_k} operators provide a conjectural formula for the Frobenius characteristic of super-diagonal coinvariants with two sets of Grassmanian variables, extending the one of Zabrocki in [47] for the case with one set of such variables.

6.5 Future directions

The game is far from being over. The obvious next step would be to find a way to generalise the tools provided by Carlsson and Mellit in order to prove the compositional Delta conjecture, and then show that it implies the Delta square conjecture. In the same framework, a dinv statistic for the generalised compositional versions of these two conjectures is still missing.

Even if the Delta conjectures were to be fully solved, there are still several other open questions. $\Delta_{h_{m-1}} \omega(p_n)$ is conjecturally Schur positive, and computational evidence suggests that it has an interpretation in terms of pairs of rectangular paths. This would generalise the polyominoes framework in the same way as square paths generalise Dyck paths, but the natural bistatistics we have do not match the symmetric functions unless we set $q = 1$ or $t = 1$. Notice that this is not a special case of the generalised Delta square conjecture, as it only holds for $k < n$.

For $\lambda \vdash m$ and $\alpha \vDash n$, computational evidence suggests that $\Theta_{e_\lambda} \nabla C_\alpha$ is Schur positive as well. While the part in low degree conjecturally agrees with the Frobenius characteristic of super-diagonal coinvariants with $\ell(\lambda)$ sets of Grassmanian variables (according to limited computational evidence), there is no interpretation for the full symmetric function. Moreover, there is no combinatorial interpretation whatsoever for $\ell(\lambda) > 2$, and even for $\ell(\lambda) = 2$ we only have conjectures for $q = 1$ or $t = 1$. The hope is that there exists a more general framework that explains all of these Schur positivity results, but it is yet to be found.

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