FAIR SOLUTIONS TO THE RANDOM ASSIGNMENT PROBLEM

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Abstract. We study the problem of assigning indivisible goods to individuals where each is to receive one good. To guarantee fairness in the absence of monetary compensation, we consider random assignments that individuals evaluate according to first order stochastic dominance (sd). In particular, we find that solutions which guarantee sd-no-envy (e.g. the Probabilistic Serial) are incompatible even with the weak sd-core from equal division. Solutions on the other hand that produce assignments in the strong sd-core from equal division (e.g. Hylland and Zeckhauser's Walrasian Equilibria from Equal Incomes) are incompatible with the strong sd-equal-division-lower-bound. As an alternative, we present a solution, based on Walrasian equilibria, that is sd-efficient, in the weak sd-core from equal division and satisfies the strong sd-equal-division-lower-bound.

Keywords: Probabilistic Serial; Walrasian Equilibrium; Sd-efficiency; Sd-envy-free; Sd-core from equal division; Sd-equal-division-lower-bound

JEL codes: C70, D63

1. Introduction

In many allocation problems, we have to assign indivisible objects to individuals where each is to receive at most one. Public housing associations assign apartments to residents, school districts assign seats to students and childcare cooperatives assign chores to its members.

If fairness is understood as equity, the indivisibility of assigned objects will often render any eventual allocation unfair. In order to guarantee fairness at least from an ex-ante perspective, many theorists as well as policy makers have considered lotteries. While the design of such lotteries has received a lot of attention in recent years, most of the work concentrates on their efficiency and incentive properties (i.e. what are the incentives for participants to reveal their true preferences) – see for example Erdil and Ergin [2008], Pathak and Sethuraman [2011], Abdulkadiroğlu et al. [2015], Kesten et al. [2017]. In this paper, we try to complement the literature by taking a closer look at the original motivation for applying a lottery and ask "when is a lottery fair?". For this, we draw on the rich literature on fair allocation (see Thomson [2011] for an overview) and adapt various equity criteria to random assignments.

Adherence to formal equity criteria may be particularly important when allocating publicly funded (or subsidised) private goods such as school seats, where no individual – or group of individuals – should be discriminated against. In the following, we focus

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on equity criteria that compare each individual's assignment to the assignments of others or to the average over all assignments. In addition we consider variants of the core from equal division, which can be seen as an equity criterion for groups. Perhaps surprisingly, we find that all equity criteria are compatible with Pareto-efficiency,[1] while (some) equity criteria for individuals are in conflict with (some) equity criteria for groups. To bridge this gap, we derive a new solution based on Walrasian Equilibria, an approach pioneered by Hylland and Zeckhauser [1979] that has enjoyed renewed attention in recent years – see for example Budish [2011], Miralles and Pycia [2014], He et al. [2018].

Since preferences over lotteries are often difficult to elicit, assignment mechanisms typically take individuals' preferences over sure objects as input. For example school choice mechanisms typically ask students to submit a ranking of schools that they would like to attend. To extend these preferences over sure objects to preferences over lotteries, we follow Bogomolnaia and Moulin [2001] and rely on first order stochastic dominance (sd). This extension can be seen as the most conservative possible extension, in the sense that an individual will sd-prefer one lottery over another only if she prefers it for any von Neumann Morgenstern utility function compatible with her preferences over sure objects.

The paper is organised as follows. In Section 2, we formally define the set of allocation problems under consideration. Section 3 lays out equity criteria and Section 4 describes which of these are satisfied by the most prominent existing solutions. Section 5 contains our main results, Section 6 discusses the new solution and Section 7 concludes.

2. Random Assignments

Let A be a set of objects and I be a set of individuals, both of size n.[2] Each individual $i \in I$ is to receive one object $a \in A$ and holds preferences over objects given by a weak order $\succsim_i$. Let $\succ_i$ and $\sim_i$ denote the associated strict preference and indifference relation, respectively.

A preference profile is denoted as $\succsim = (\succsim_i)_{i \in I}$. We restrict preference profiles to cases of objective indifference, i.e. an individual may only be indifferent between objects, if every other individual is indifferent as well:[3] $\forall\, a, b \in A, i, j \in I\colon\ a \sim_i b \iff a \sim_j b$.

We will refer to the tuple $(A, I, \succsim)$ as an *assignment problem (of size n)*. Let $p_{i,a}$ denote the probability that individual i is assigned object a. An *individual (random) assignment* is a probability distribution over A, i.e. a vector $p_i = (p_{i,a})_{a \in A}$ such that $\sum_{a \in A} p_{i,a} = 1$. The set of probability distributions over A is denoted $\Delta(A)$. A *random assignment*, $p = (p_i)_{i \in I}$, is a collection of individual assignments such that for all a, $\sum_{i \in I} p_{i,a} = 1$.[4] A solution maps assignment problems to (sets of) random assignments.

[1] This is in contrast to the class of (group) strategy-proof mechanisms, where equity and efficiency are conflicting objectives – see for example Nesterov [2017], Aziz and Kasajima [2017], Zhang [2017].

[2] The case of fewer objects than individuals can be accommodated by introducing null-objects.

[3] Objects that everyone is indifferent between may be interpreted as null-objects or as multiple copies of the same object, such as for example multiple seats at the same school.

[4] The Birkhoff–von Neumann-Theorem ensures that any random assignment can be represented as a convex combination of deterministic assignments where each individual receives one object.

In order to evaluate random assignments and solutions, we extend individuals' preferences over objects to preferences over individual assignments, using first order stochastic dominance (sd).[5] In words, an individual weakly prefers an individual assignment if it guarantees her a weakly higher chance of receiving her most preferred object(s) *and* a weakly higher chance of receiving the most or second most preferred object(s) *and* ... so on. Formally, define i's *weak upper contour set of* a as

$$U_i(a) = \{b \in A \,|\, b \succsim_i a\}$$

and write $p_i \succsim_i^{sd} \tilde{p}_i$ if

$$\forall a \in A: \sum_{b \in U_i(a)} p_{i,a} \geq \sum_{b \in U_i(a)} \tilde{p}_{i,a}.$$

If one of the inequalities is strict write $p_i \succ_i^{sd} \tilde{p}_i$. Note that stochastic dominance induces only a partial order over assignments.

At times, we will also evaluate individual assignments according to a vector of weights $w_i = (w_{i,a})_{a \in A} \in \mathbb{R}^n$. In some contexts – in particular where a social planner is able to elicit them – w_i may be interpreted as von Neumann–Morgenstern (vNM) utilities, associating an expected utility of $w_i \cdot p_i$ with each individual assignment p_i. The vector w_i is said to be *compatible with* $\succsim_i$ if

$$\forall a, b \in A: \quad w_{i,a} > w_{i,b} \iff a \succ_i b.$$

Analogously, a collection of weight vectors $w = (w_i)_{i \in I}$ is compatible with preference profile $\succsim$, if the same can be said for each component. The set of all such collections w is denoted $W(\succsim)$.

Under an alternative interpretation, the weights might constitute a value judgement on behalf of a social planer, who chooses between different random assignments. For example, a school board might find that moving to a different random assignment where in expectation some additional k students receive their first rather than their second choice school is preferable even as another l students receive only their third rather than their second most preferred school. Inevitably, such decisions have to be made and making them with respect to fixed weight vectors may increase transparency and accountability.

Finally, a random assignment p is *sd-efficient*[6] unless there exists another assignment $\tilde{p}$ such that

$$\forall i \in I: \quad \tilde{p}_i \succsim_i^{sd} p_i \qquad \text{and} \qquad \exists i \in I: \quad \tilde{p}_i \succ_i^{sd} p_i.$$

It is *weakly sd-efficient* unless there exists $\tilde{p}$ with $\tilde{p}_i \succ_i^{sd} p_i$, for all $i \in I$. A random assignment p is *efficient with respect to* w unless there exists another assignment $\tilde{p}$ such that

$$\forall i \in I: \quad w_i \cdot \tilde{p}_i \geq w_i \cdot p_i \qquad \text{and} \qquad \exists i \in I: \quad w_i \cdot \tilde{p}_i > w_i \cdot p_i.$$

[5]We abstract from consumption externalities, so preferences over random assignments only depend on the individual component.

[6]Bogomolnaia and Moulin [2001] introduced this concept as *ordinal efficiency*, to highlight the coarse informational underpinning of the preference relation $\succsim_i^{sd}$.

Observe that p is sd-efficient if there exists a collection of compatible weight vectors $w \in W(\succsim)$ such that p is efficient with respect to w[7] – absent possible improvements in expected utilities, there cannot be improvements w.r.t stochastic dominance.

3. Equity Criteria

3.1. **Individuals.** A minimal fairness requirement on random assignments demands equal treatment of equals: two individuals with identical preferences should receive the same amount of all objects that fall in the same indifference class.

Definition 1. Given an assignment problem $(A, I, \succsim)$, a random assignment p satisfies *equal treatment of equals* if for all i, j in I we have

$$\succsim_i = \succsim_j \quad \implies \left(\forall a \in A : \quad \sum_{b \sim_i a} p_{i,b} = \sum_{b \sim_i a} p_{j,b} \right).$$

Note that where preferences are strict, this reduces to $\succsim_i = \succsim_j \Rightarrow p_i = p_j$.

Equitable treatment of individuals who differ in their preferences is harder to conceptualize. If one refrains from interpersonal comparisons of utility (as we do here) envy-freeness is arguably the most prominent such criterion.[8] To check whether an allocation is envy-free, we need to compare individuals' assignments - each individual should then prefer her own over anyone else's assignment.

Definition 2. Given an assignment problem $(A, I, \succsim)$, a random assignment p is *sd-envy-free* if for all i, j in I we have $p_i \succsim_i^{sd} p_j$.[9]

Observe that sd-envy-freeness implies equal treatment of equals: if two individuals i, j share the same preferences, sd-envy-freeness implies $p_i \succsim_i^{sd} p_j \succsim_i^{sd} p_i$ which is only possible if both receive the same amount of all objects in the same indifference class.

Sd-envy-freeness is satisfied whenever p is envy-free with respect to *all* compatible weight vectors, i.e. if

$$\forall\, w \in W(\succsim),\, i, j \in I : \qquad w_i \cdot p_i \geq w_i \cdot p_j.$$

For a social planer who assumes that individuals evaluate random assignments in an expected utility framework, but who is informed only about their preferences over sure objects, sd-envy-freeness allows her to ensure envy-freeness with respect to individuals' expected utilities despite her limited information on the latter.

Another natural yardstick to measure individuals' assignments is equal division giving rise to the equal-division-lower-bound. In the context of random assignments, Nesterov [2017] shows that for strategy-proof mechanisms it conflicts with sd-efficiency and Heo [2014] invokes it to characterize a generalized version of Probabilistic Serial. To define it, denoted by $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$ the individual assignment that grants each object with probability $\frac{1}{n}$.

[7]The converse holds as well, as proven (non-constructively) by McLennan [2002] and (constructively) by Manea [2008].

[8]The criterion was introduced to economic theory by Tinbergen [1946] (p. 55 f.) who credits his professor, Dutch physicist Paul Ehrenfest, to have formulated the criterion in 1925 when they discussed the problem of interpersonal (non-)comparability. It was independently formulated in a dissertation by Foley [1967].

[9]Bogomolnaia and Moulin [2001] are the first to formulate this property in the context of random assignments and refer to it simply as 'envy-free'.

Definition 3. Given an assignment problem $(A, I, \succsim)$, a random assignment p satisfies

- the *strong sd-equal-division-lower-bound* if $\forall i \in I: \quad p_i \succsim_i^{sd} \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$.
- the *weak sd-equal-division-lower-bound* if $\nexists\, i \in I: \quad \left(\frac{\mathbf{1}}{\mathbf{n}}\right) \succ_i^{sd} p_i$.

The weak notion is satisfied if the equal division lower bound is met for some compatible weight vectors $w \in W(\succsim)$,[10] while the strong notion requires it to be met for all such w. A social planer who only knows individual preferences over objects but chooses a random assignment that meets the strong sd-equal-division-lower-bound ensures that each individual's expected utility is greater than under equal division.

Observe that any random assignment that is sd-envy-free also meets the strong sd-equal-division-lower-bound [Heo, 2014]: as each individual's assignment stochastically dominates all individual assignments, it also dominates the average $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$.

3.2. **Groups.** In addition, there are various equity criteria for groups of individuals, which ensure that no group receives less than their 'fair share' (see Thomson [2011]). Perhaps the most notable such criterion is the *core from equal division.*

Definition 4. Consider an assignment problem $(A, I, \succsim)$. A group of individuals $G \subset I$ objects to a random assignment $\tilde{p}$ if there is an alternative assignment p such that

- $\forall a \in A: \quad \sum_{i \in G} p_{i,a} = \frac{|G|}{n}$ and
- $\forall\, i \in G: \quad p_i \succ_i^{sd} \tilde{p}_i$.

If there is no such objection that can be raised against a random assignment, the assignment is said to be in the *weak sd-core (from equal division).*

Core assignments satisfy the weak sd-equal-division-lower-bound, as can be easily verified by restricting attention to cases $G = \{i\}$ in Definition 4. The core strengthens the weak sd-equal-division-lower-bound, in that groups of individuals whose assignments are barely above the bound (e.g. individuals who receive their least preferred object with probability $1/n - \varepsilon$ and with converse probability the second-least preferred) would object, provided there is some heterogeneity in preferences (e.g., in the case above, when they differ in their least preferred object).

For assignments that satisfy the strong sd-equal-division-lower-bound, being in the core ensures that the welfare improvements of trade from equal division are not too heavily concentrated within a particular subgroup:

Example 1. Consider $I = \{1, 2, 3, 4\}$ with preferences of the first individual given as $a \succ_1 b \succ_1 c \succ_1 d$. The other individuals mostly share these preferences, but with small deviations: $b \succ_2 a$, $c \succ_3 b$ and $d \succ_4 c$. If the first individual monopolizes trade – trading a for b with 2, b for c with 3 and c for d with 4 – we arrive at assignment p:

	a	b	c	d
p_1:	$1/2$	$1/4$	$1/4$	0
p_2:	0	$1/2$	$1/4$	$1/4$
p_3:	$1/4$	0	$1/2$	$1/4$
p_4:	$1/4$	$1/4$	0	$1/2$

[10] I.e. for all $i \in I$, $w_i \cdot p_i \geq w_i \cdot (\frac{\mathbf{1}}{\mathbf{n}})$.

While not in violation of the strong sd-equal-division-lower-bound, a less lopsided distribution of trades would make individuals $2, 3$ and 4 better off. In fact, faced with p, they would object as a group so that p is not in the weak sd-core.[11]

Finally, observe that any element of the weak sd-core is weakly sd-efficient as it could otherwise be improved upon by the grand coalition $G = I$. Still, the weak sd-core is comparatively large – strictly larger even, than the union over all w-cores[12] with weights $w \in W(\succsim)$ (see Basteck [2016], Example 4). To narrow it down and ensure sd-efficiency, we consider a prominent subset – the strong sd-core – that guards against objections by coalitions that would make their members weakly better off.

Definition 5. Consider an assignment problem $(A, I, \succsim)$. A group of individuals $G \subset I$ objects to a random assignment $\tilde{p}$ if there is an alternative assignment p such that

- $\forall a \in A: \quad \sum_{i \in G} p_{i,a} = \frac{|G|}{n}$ and
- $\forall i \in G: \quad p_i \succsim_i^{sd} \tilde{p}_i$ and $\exists j \in G: \quad p_j \succ_j^{sd} \tilde{p}_j$.

If there is no such objection that can be raised against a random assignment, the assignment is said to be in the *strong sd-core (from equal division)*.

The strong sd-core is a strict[13] subset of the weak sd-core. A fortiori, a random assignment in the strong sd-core will satisfy the weak sd-equal-division-lower-bound. However, it may violate equal treatment of equals and the strong sd-equal-division-lower-bound (see Appendix, Example 5).

Conversely, equal division necessarily satisfies the strong sd-equal-division-lower-bound and equal treatment of equals, but will typically not be an element of the strong sd-core,[14] as the latter implies sd-efficiency.

Figure 1 provides a summary of all equity concepts discussed thus far, including their logical relations. In the center column, there are two independent and comparatively weak equity criteria. The weak sd-equal-division-lower-bound in particular can be strengthened in different ways by either allowing for group comparisons (right hand side) or by replacing 'not strictly worse' by a stronger 'weakly better' (left hand side). Also note that sd-envy-freeness implies all other individual equity criteria.

The absence of any connecting arrow(s) between two properties marks their logical independence. To be explicit,

- (strong sd-equal-division-lower-bound + strong sd-core)
  $\not\Longrightarrow$ equal treatment of equals (see Appendix, Example 6).
- equal treatment of equals
  $\not\Longrightarrow$ weak sd-equal-division-lower-bound.[15]

[11] Proposition 1 revisits this example and shows that sd-envy-freeness forces us to be too generous to individual 1, once again leading to assignments outside of the weak sd-core.

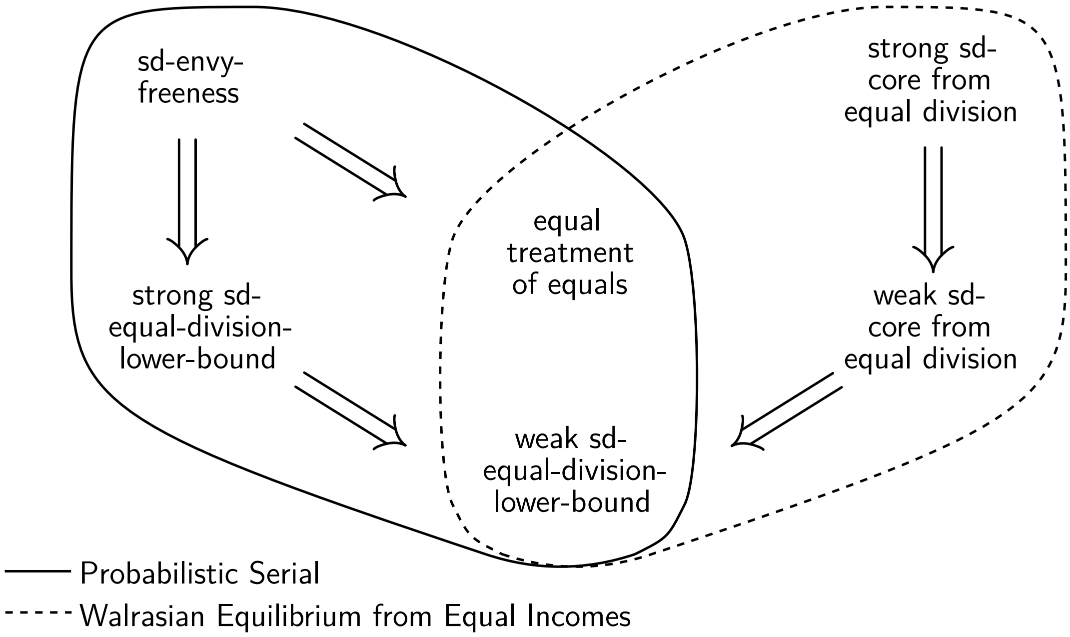
[12] A random assignment $\tilde{p}$ lies in the w-core, unless there exist $G \subset I$ and assignment p, such that for all $a \in A$ we have $\sum_G p_{i,a} = \frac{|G|}{n}$ and for each member i of G we have $w_i \cdot p_i > w_i \cdot \tilde{p}_i$.

[13] Consider $n = 3$, with $a \succ_{1,2} b \succ_{1,2} c$ and $b \succ_3 a \succ_3 c$. Then $p_{1,2} = (1/2, 1/6, 1/3)$ and $p_3 = (0, 2/3, 1/3)$ is in the weak but not in the strong core, as $\tilde{p}_1 = (2/3, 0, 1/3)$ and $\tilde{p}_3 = p_3$ is an objection by $G = \{1, 3\}$.

[14] Nor would it constitute an element of the weak sd-core – consider $n = 2$, $a \succ_1 b$ and $b \succ_2 a$.

[15] For example, consider the case $n = 3$ where $a \succ_{1,2} b \succ_{1,2} c$ and $b \succ_3 a \succ_3 c$, individuals 1 and 2 receive the same assignment $p_i = (p_{i,a}, p_{i,b}, p_{i,c}) = (1/2, 1/2, 0)$ and 3 receives object c.

FIGURE 1. Logical relations between equity criteria and 2 prominent solutions



- sd-envy-freeness

  $\not\Longrightarrow$ weak sd-core (follows from Proposition 1).
- (strong sd-core + equal treatment of equals)

  $\not\Longrightarrow$ strong sd-equal-division-lower-bound (follows from Proposition 2).

4. Prominent Solutions

So far, we have discussed efficiency and equity criteria for individual assignment problems. Let us extend these criteria to solutions that map individual assignment problems to sets of random assignments.

We say that a solution S satisfies criterion X (where X could stand for sd-efficiency, equal treatment of equals, sd-envy-freeness etc.) if for any assignment problem e in the domain of S, all random assignments in $S(e)$ satisfy X.

In the following, we consider three prominent solutions to see which efficiency and equity criteria they satisfy. As we will see, all three solutions can be interpreted as taking equal division as a starting point and differ only in the manner in which trades towards the efficiency frontier are conducted.

4.1. **Random Serial Dictatorship.** A common approach towards assignment problems, is Random Serial Dictatorship (RSD). It requires us to order our n individuals randomly (where all $n!$ orderings are equally likely); the first individual may then choose her most preferred object, while the next in line chooses the most preferred among the remaining objects.[16] So it continues, until the last in line receives the last

[16]If an individual is indifferent between multiple objects, this tie can be broken in any way – since under our assumption of objective indifference all others will similarly be indifferent between the same objects, her choice does not affect any individual that has to choose at a later stage.

object available. From an ex-ante perspective – that is before we have decided on a particular ordering of individuals – this procedure generates a random assignment. [17]

With respect to the equity criteria analysed in Section 3, let us first point out that a random assignment generated via RSD satisfies equal treatment of equals and meets the strong sd-equal-division-lower-bound: each individual has a chance of $\frac{k}{n}$ to be among the first k individuals to choose, in which case she is guaranteed one of her k-most preferred objects. However, for some preference profiles, RSD falls short of sd-envy-freeness [Bogomolnaia and Moulin, 2001].

The main weakness of RSD lies in the fact that it fails to ensure even weak sd-efficiency [Bogomolnaia and Moulin, 2001]:

Example 2. Consider the case $n = 4$ for the preferences profile $a \succ_{1,2} b \succ_{1,2} c \succ_{1,2} d$ and $b \succ_{3,4} a \succ_{3,4} d \succ_{3,4} c$. Here RSD produces the following random assignment

	a	b	c	d
p_1, p_2:	$5/12$	$1/12$	$5/12$	$1/12$
p_3, p_4:	$1/12$	$5/12$	$1/12$	$5/12$

which is Pareto inferior to $\tilde{p}_{1,2} = (1/2, 0, 1/2, 0)$, $\tilde{p}_{3,4} = (0, 1/2, 0, 1/2)$.

Moreover, such inefficient random assignments are not in the weak sd-core from equal division. This contrasts with an alternative description of RSD by Abdulkadiroğlu and Sönmez [1998], who characterize the solution on the domain of strict preferences as "core from random endowments". More precisely, they consider a random, uniformly distributed initial allocation of objects to individuals; given their endowments, individuals then trade towards the respective unique core allocation. From an ex-ante perspective, the convex combination of these core allocations coincides with the convex combination of allocations generated by fixed pecking orders.

4.2. **Probabilistic Serial.** One solution that overcomes RSD's lack of efficiency is the Probabilistic Serial (PS) mechanism, introduced by Bogomolnaia and Moulin [2001]. It generates random assignments via "simultaneous eating" where individuals accumulate probability shares, starting with their most preferred object until it is exhausted, before moving down to their second most preferred object and so on.[18] Not only does PS generate sd-efficient random assignments, it also ensures sd-envy-freeness [Bogomolnaia and Moulin, 2001]. However, as will follow from Proposition 1, these assignments can not in general lie in the weak sd-core from equal division.

Again, if we think of the core as the set of allocations that might be reached from an initial allocation through trade among individuals, this is in contrast with an alternative description of PS by Kesten [2009], who characterizes PS on the domain of strict preferences as "Top Trading Cycles from Equal Division". For each individual i, her initial assignment $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$ is managed by n "pseudo-agents" i_a, $a \in A$. Each i_a

[17] Typically, once we have found a random assignment p, we then need to construct a Birkhoff–von Neumann decomposition to implement p as a lottery over deterministic assignments. One of the practical advantages of RSD, is that the randomization occurs in the very first step where we choose an ordering of individuals. Given this order, the algorithm returns a deterministic assignment.

[18] Bogomolnaia and Moulin [2001] consider strict preferences. The mechanism can be easily generalized to objective indifferences, i.e. multiple copies of objects (e.g. Hashimoto et al. [2014]).

controls an initial probability share $\frac{1}{n}$ of object a and shares i's preferences over objects. In each round, pseudo-agent i_a will offer shares of a in exchange for an equal share of her most preferred among those objects still available in the market. Wherever there is a double coincidence of wants, probability shares are exchanged and withdrawn from the market. Over time i_a will have exchanged the whole of her initial share of object a, or she finds that object a is the most preferred among all remaining objects. In both cases, i_a exits the market. After at most n steps, this trading algorithm terminates and the sum of probability shares acquired by each i's pseudo-agents is found to coincide with the individual assignments p_i generated by simultaneous eating.

4.3. **Walrasian equilibrium from equal incomes.** A third prominent solution is offered by Hylland and Zeckhauser [1979], who adapt the familiar concept of a Walrasian equilibrium from equal incomes (WEEI) to assignment problems.[19] In contrast to our setting, individuals report vNM utilities w_i. Nevertheless, as a random assignment that is maximal with respect to expected utility will also be maximal with respect to stochastic dominance, we find that their solution not only satisfies sd-efficiency, but also many of the equity criteria formulated in Section 3.

Formally, define the set of price vectors as $Q = \{q = (q_a)_{a\in A} \in \mathbb{R}^n \mid \forall a \in A : q_a \geq 0\}$. Individuals purchase probability shares, maximizing their expected utility $w_i \cdot p_i$ subject to a constraining budget $B \in \mathbb{R}^+$ and the constraint $\sum_A p_{i,a} = 1$.

Fact 1. Hylland and Zeckhauser [1979]. *Consider an assignment problem $(A, I, \succsim)$. For any collection of compatible weights $w \in W(\succsim)$, their exists a Walrasian equilibrium from equal incomes, i.e. a tuple $(p, q, B) \in \Delta(A)^n \times Q \times \mathbb{R}^+$ such that both*

$$\forall\, i \in I, \tilde{p}_i \in \Delta(A): \qquad q \cdot p_i \leq B \ \textit{and} \ (w_i \cdot \tilde{p}_i > w_i \cdot p_i \Rightarrow q \cdot \tilde{p}_i > B)$$

$$\textit{(preference maximisation)},$$

$$\textit{and} \quad \forall\, a \in A: \qquad \sum_{i\in I} p_{i,a} = 1 \qquad \textit{(feasibility)}.$$

Not surprisingly, such a WEEI will be efficient and in the strong core with respect to w. Moreover, the associated random assignment will also be sd-efficient and an element of the strong sd-core from equal division: any trade (resp. objection) that would make everyone (resp. members of G) weakly better off with respect to first order stochastic dominance would also yield an increase in individuals expected utility. Similarly, preference maximisation and equal budgets guarantee envy-freeness with respect to w, i.e. for all i, j we have $w_i \cdot p_i \geq w_i \cdot p_j$.

One condition that is not automatically satisfied, is equal treatment of equals. If however, we chose $w_i = w_j$ whenever $\succsim_i = \succsim_j$, and constrain these individuals to consume the same probability shares (whenever they are indifferent and might choose different shares), this will guarantee equal treatment without violating preference maximisation or feasibility. Hence, using appropriately chosen weights w, there exists a (sub)solution of WEEIs that selects from the strong sd-core from equal division and satisfies equal treatment of equals.

[19] Hylland and Zeckhauser [1979] also allow for differences in income, justified for example by the seniority of individuals. In the spirit of our equity criteria identified in Section 3, we will concentrate on the case of equal incomes.

However, in contrast to both RSD and PS, Hylland and Zeckhauser's solution (and any sub-solution) will necessarily violate the strong sd-equal-division-lower-bound, at least for some preference profiles - see Section 5, Proposition 2.

Figure 1 relates the two sd-efficient solutions discussed so far to the equity criteria that they satisfy.

5. Main Results

In light of Figure 1, we may ask whether there exists a solution that is able to satisfy *all* of our equity criteria. Unfortunately, the answer is no.

Proposition 1. *For every $n \geq 4$ there exist assignment problems of size n, for which no random assignment simultaneously satisfies sd-envy-freeness and lies in the weak sd-core from equal division.*

The conflict between the two equity criteria arises in particular where, as it is common in many applications, preferences are (sufficiently) correlated. Intuitively then, objects may be seen as ordered from most to least valuable from a societal point of view. An individual whose preferences are very closely aligned with that ordering has 'expensive tastes' in that she, for example, most prefers the object that is also most valuable to society. To satisfy sd-envy-freeness, we would be forced to accommodate these expensive tastes to such a degree, that the remaining individuals would object and prefer to trade exclusively amongst themselves. The proof of Proposition 1 rests on an example that demonstrates such a situation.

Proof of Proposition 1. Consider an assignment problem $(A, I, \succsim)$ of size $n \geq 4$, label objects as a, b, c, d and $o_5, o_6, \ldots, o_n$, individuals as $1, 2, 3, ..., n$ and let their preferences over objects be given by the following rank-order lists:

$$\begin{array}{lllllllll}
1: & b, & a, & c, & d, & o_5, & o_6, & \ldots, & o_n, \\
2: & a, & c, & b, & d, & o_5, & o_6, & \ldots, & o_n, \\
3: & a, & b, & d, & c, & o_5, & o_6, & \ldots, & o_n, \\
j: & a, & b, & c, & d, & o_5, & o_6, & \ldots, & o_n, \quad \forall j = 4, 5 \ldots, n.
\end{array}$$

Intuitively, preferences of individuals $j \geq 4$ could be described as 'mainstream' and hence 'expensive' tastes while in the preferences of the first 3 individuals there are reversals in the ranking of objects a, b, c, d that create opportunities for welfare improving trade. We will consider an arbitrary sd-envy-free random assignment p and show that there exists a valid objection by $G = \{1, 2, 3\}$.

As p is assumed to be sd-envy-free, and all individuals agree on the ranking of alternatives $o_5, o_6, \ldots, o_n$, we know that $p_{i,o_k} = \frac{1}{n}$ for all $i \in I, k \geq 5$. As p also satisfies equal treatment of equals, we can express the individual assignment for $j \geq 4$ as

$$p_j = (p_{j,a}, p_{j,b}, p_{j,c}, p_{j,d}, ...p_{j,o_k}...) = (1/n + \alpha, 1/n - \alpha + \beta, 1/n - \beta + \gamma, 1/n - \gamma, ...1/n...)$$

with $\alpha, \beta, \gamma \geq 0$. Sd-envy-freeness then implies that p takes the form

	a	b	c	d	o_k
p_1:	$1/n-(n-1)\alpha$	$1/n+(n-1)\alpha+\beta$	$1/n-\beta+\gamma$	$1/n-\gamma$	$1/n$
p_2:	$1/n+\alpha$	$1/n-\alpha-(n-1)\beta$	$1/n+(n-1)\beta+\gamma$	$1/n-\gamma$	$1/n$
p_3:	$1/n+\alpha$	$1/n-\alpha+\beta$	$1/n-\beta-(n-1)\gamma$	$1/n+(n-1)\gamma$	$1/n$
p_j:	$1/n+\alpha$	$1/n-\alpha+\beta$	$1/n-\beta+\gamma$	$1/n-\gamma$	$1/n$

Individuals 2 and 3 agree with $j \geq 4$ on the most preferred object and hence receive it with probability $p_{2,a} = p_{3,a} = p_{j,a} = 1/n+\alpha$. Individual 1 receives object a with remaining probability $p_{1,a} = 1/n-(n-1)\alpha$. Similarly, individuals $i \neq 2$ agree on the weak upper contour set $U_i(b) = \{a,b\}$ and hence receive a or b with probability $p_{i,a}+p_{i,b} = 2/n+\beta$ - leaving individual 2 with the remaining probability $p_{2,b} = 1/n-\alpha-(n-1)\beta$. Finally, individuals $i \neq 3$ agree on the upper contour set $U_i(c) = \{a,b,c\}$ and hence receive a, b or c with probability $p_{i,a}+p_{i,b}+p_{i,c} = 3/n+\gamma$ - leaving individual 3 with the remaining probability $p_{3,c} = 1/n-\beta-(n-1)\gamma$. The entries $p_{i,d}$ then follow from the condition $\sum_{x\in A} p_{i,x} = 1$.

As all entries are non-negative, we find three additional constraints on α,β,γ:

$$\begin{array}{lll} \text{(I)} & \alpha \leq \frac{1}{n(n-1)} & (\Leftrightarrow p_{1,a} = 1/n-(n-1)\alpha \geq 0) \\ \text{(II)} & \beta \leq \frac{1}{n(n-1)} - \frac{1}{n-1}\alpha & (\Leftrightarrow p_{2,b} = 1/n-\alpha-(n-1)\beta \geq 0) \\ \text{(III)} & \gamma \leq \frac{1}{n(n-1)} - \frac{1}{n-1}\beta & (\Leftrightarrow p_{3,c} = 1/n-\beta-(n-1)\gamma \geq 0) \end{array}$$

We claim that the following random assignment $\tilde{p}$ constitutes a valid objection by group $G = \{1,2,3\}$, who can do better by trading exclusively amongst themselves:

	a	b	c	d	o_k
$\tilde{p}_1$:	0	$3/n-\alpha-\beta$	$\alpha+\beta+\gamma$	$1/n-\gamma$	$1/n$
$\tilde{p}_2$:	$1/n+\alpha$	0	$3/n-\alpha-\beta-\gamma$	$\beta+\gamma$	$1/n$
$\tilde{p}_3$:	$2/n-\alpha$	$\alpha+\beta$	0	$2/n-\beta$	$1/n$
$\tilde{p}_j$:	$1/n$	$1/n$	$1/n$	$1/n$	$1/n$

The random assignment is well defined, as all sums $\sum_{x\in A}\tilde{p}_{i,x} = 1 = \sum_{i\in I}\tilde{p}_{i,x}$ and all entries are non-negative, given that $\alpha,\beta,\gamma \leq 1/n$ - see (I)-(III). Moreover, G's resource constraint is met, as $\sum_{i\in G}\tilde{p}_{i,x} = \frac{3}{n}$, for all $x \in A$.

It remains to show that for all $i \in G$, $\tilde{p}_i \succ_i^{sd} p_i$. First, consider individual 1. Here we find that she receives her most preferred object with strictly greater probability

$$\tilde{p}_{1,b} - p_{1,b} = \frac{2}{n} - n\alpha - 2\beta > \frac{2}{n} - \frac{1}{n-1} - 2\frac{1}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0$$

where (I) and (II) are used in the inequality. Moreover, she also receives her first or second object with greater probability than before:

$$(\tilde{p}_{1,b}+\tilde{p}_{1,a}) - (p_{1,b}+p_{1,a}) = \frac{1}{n} - \alpha - 2\beta > \frac{1}{n} - \frac{3}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0.$$

As she receives her least preferred object d with the same probability as before ($\tilde{p}_{1,d} = p_{1,d} = \frac{1}{4} - \gamma$), we conclude that $\tilde{p}_1 \succ_1^{sd} p_1$.

Next, consider individual 2. She receives her most preferred object a with the same probability as before ($\tilde{p}_{2,a} = p_{2,a} = \frac{1}{n} + \alpha$) but receives her second most preferred object with higher probability:

$$\tilde{p}_{2,c} - p_{2,c} = \frac{2}{n} - \alpha - n\beta - 2\gamma \geq \frac{2}{n} - \alpha - \frac{1}{n-1} + \frac{n\alpha}{n-1} - 2\gamma \geq \frac{2}{n} - \frac{1}{n-1} - \frac{2}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0,$$

where we use (II) in the first and (III) in the second inequality. For the probability of receiving her least preferred object d, we find (using (III))

$$\tilde{p}_{2,d} - p_{2,d} = \beta - \frac{1}{n} < 0$$

so that in conclusion $\tilde{p}_2 \succ_2^{sd} p_2$. Finally, consider individual 3. Her most preferred object is a, which she now receives with strictly greater probability:

$$\tilde{p}_{3,a} - p_{3,a} = \frac{1}{n} - 2\alpha \geq \frac{1}{n} - \frac{2}{n(n-1)} = \frac{n-3}{n(n-1)} > 0.$$

As the probability of receiving one of her two most preferred objects remains unchanged ($\tilde{p}_{3,a} + \tilde{p}_{3,b} = p_{3,a} + p_{3,b} = \frac{2}{n} + \beta$) and as she now receives her least preferred object d with zero probability, she too strictly prefers $\tilde{p}$ over p, rendering $\tilde{p}$ a valid objection by group $G = \{1, 2, 3\}$. □

Proposition 1 reveals a weakness of the Probabilistic Serial and other sd-envy-free solutions in general, in that they allow individuals with 'expensive tastes' to seize too much of the gains from trade that arise out of preference heterogeneity.

This raises the question, whether there exist solutions that are able to satisfy all remaining equity criteria once we give up sd-envy-freeness. Again, the answer is no.

Proposition 2. *For every $n \geq 3$ there exist assignment problems of size n, for which no random assignment simultaneously satisfies the strong sd-equal-division-lower-bound and lies in the strong sd-core from equal division.*

Proof. Consider the assignment problem $(A, I, \succsim)$ where $I = \{1, 2, 3\}$ and preferences over A are given as $a \succ_{1,2} b \succ_{1,2} c$ and $b \succ_3 a \succ_3 c$.

Any random assignment p that satisfies the strong sd-equal-division-lower-bound will assign object c with probabilities $p_{i,c} \leq \frac{1}{3}$. But then, $p_{i,c} = \frac{1}{3}$ and p takes the form

	a	b	c
p_1:	$1/3 + \alpha$	$1/3 - \alpha$	$1/3$
p_2:	$1/3 + \beta$	$1/3 - \beta$	$1/3$
p_3:	$1/3 - \alpha - \beta$	$1/3 + \alpha + \beta$	$1/3$

with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq \frac{1}{3}$. For p to lie in the strong sd-core it has to be sd-efficient, i.e. $\alpha + \beta = \frac{1}{3}$. Either α or β will then be less than $\frac{1}{3}$ - assume w.l.o.g. that $\alpha < \frac{1}{3}$. But then, starting from equal division, individual 3 could exclusively trade with 1 and arrive at the following alternative random assignment $\tilde{p}$:

	a	b	c
$\tilde{p}_1$:	$1/3 + \alpha + \beta$	$1/3 - \alpha - \beta$	$1/3$
$\tilde{p}_2$:	$1/3$	$1/3$	$1/3$
$\tilde{p}_3$:	$1/3 - \alpha - \beta$	$1/3 + \alpha + \beta$	$1/3$

While this is a matter of indifference for individual 3, it is strictly preferred by 1. Thus, $\tilde{p}$ is a valid objection to p by the group of individuals $\{1,3\}$ which completes the proof for $n = 3$. The example extends immediately to the case $n > 3$ with preferences $a \succ_i b \succ_i c \succ_i o_4 \succ_i \cdots \succ_i o_n$ for $i \in \{1, 2..., n-1\}$ and $b \succ_n a \succ_n c \succ_n o_4 \succ_n \cdots \succ_n o_n$. □

Proposition 2 can be seen as a negative result with regard to extending the pseudo-market approach of Hylland and Zeckhauser to a situation where information on preferences is limited in that the true underlying vNM utilities are unknown. In their set-up, trade from equal division will both be in the strong core from equal division and satisfy the equal division lower bound. Yet if we account for the coarser information on preferences by modelling the result of such trade as an assignment in the strong sd-core from equal division, we will not be able to simultaneously satisfy the equal division lower bound for all possible underlying utility functions.

However, the next Proposition demonstrates that once we model trade as merely delivering a result in the weak sd-core, this goal can be achieved.

Proposition 3. *For any assignment problem $(A, I, \succsim)$, there exist random assignments that simultaneously satisfy equal treatment of equals, meet the strong sd-equal-division-lower-bound, are in the weak sd-core from equal division and sd-efficient.*

We will prove Proposition 3 by analysing a sequence of Walrasian equilibria with equal incomes. The limit of this sequence will then inherit many desirable properties, even if it is not itself a Walrasian equilibrium.

Our setting raises a number of problems for the existence of Walrasian equilibria. For one, individuals may be satiated and hence leave part of their budget unspent, leading to a violation of Walras' law.

Second, if we restrict individuals' consumption sets to random assignments that meet the sd-equal-division-lower-bound, an equal division endowment lies on the boundary of individuals' consumption sets. Then, depending on the price vector, there may be no assignment cheaper than the initial endowment. This violates the so called strong survival assumption, typically used to show that any quasi-equilibrium (whose existence may be established more easily) is in fact a Walrasian equilibrium.

Third, the preference relation given by first order stochastic dominance is not continuous. For example an individual with preference $a \succ_i b \succ_i c$ would (strictly) prefer $p_i = (p_{i,a}, p_{i,b}, p_{i,c}) = (1/3, 2/3, 0)$ over $(1/3, 1/3, 1/3)$ but not over $(1/3 + \varepsilon, 1/3 - \varepsilon, 1/3)$.

To overcome the third problem, we let individuals act as expected utility maximizers whose vNM utilities are compatible with their strict ordering of objects – a consumption bundle that is maximal with respect to these vNM utilities will then also be maximal with respect to first order stochastic dominance. To resolve the second problem, we relax the sd-equal-division-lower-bound by ε – letting ε go to zero will then yield a limit allocation satisfing all our desired criteria. To overcome the problem of satiated individuals, we allow for some 'slack' or a 'dividend' that increases the income of unsatiated individuals. In that, we follow Mas-Colell [1992]:

Let individuals' consumption sets $X_i \subset \mathbb{R}^n$ be closed, bounded and convex and let each individual's endowment y_i be in the interior of of X_i. Let the set of possible price vectors be given as $Q = \{q = (q_a)_{a\in A} \in \mathbb{R}^n | \|q\| = \sum |q_a| \leq 1\}$ and the state space be denoted as $Z = X_1 \times X_2 \times \cdots \times X_n \times Q$. Individuals' demand is guided by a (set-valued) preference map $P_i \colon X_i \rightrightarrows X_i$ and constrained by a budget $q \cdot y_i + \frac{1-\|q\|}{\|q\|}$ where the term $\frac{1-\|q\|}{\|q\|}$ may be used by the Walrasian auctioneer to increase budgets beyond the value of endowments.

If P_i is irreflexive (i.e. $x_i \notin P_i(x_i)$ for every $x_i \in X_i$), convex-valued (i.e. $P_i(x_i)$ is convex for every $x_i \in X_i$) and has an open graph (i.e. if $x_i \in P_i(x_i')$, the same holds for all $\tilde{v}_i$, $\tilde{w}_i$ in some small neighbourhood of v_i and w_i) we have the following.

Fact 2. Theorem 1 in Mas-Colell [1992].
There exists a Walrasian equilibrium with slack, i.e. a state $z = (x, q)$ such that both

$$\begin{aligned} &\forall\, i \in I: && q \cdot x_i \leq q \cdot y_i + \frac{1-\|q\|}{\|q\|} \ \text{ and } \left(\tilde{x}_i \in P_i(x_i) \implies q \cdot \tilde{x}_i > q \cdot y_i + \frac{1-\|q\|}{\|q\|}\right) \\ & && \text{(preference maximisation)}, \\ \text{and} \quad &\forall\, a \in A: && \sum_{i\in I} x_{i,a} = \sum_{i\in I} y_{i,a} \qquad \text{(feasibility)}. \end{aligned}$$

Proof of Proposition 3. Consider an assignment problem $(A, I, \succsim)$ and a compatible collection of weight vectors $w \in W(\succsim)$. Define individuals' consumption sets as

$$X_i^\varepsilon = \left\{ x_i = (x_{i,a})_{a\in A} \in \mathbb{R}^n | \sum_{a\in A} x_{i,a} \leq 1 + \varepsilon,\ \forall a \in A : x_{i,a} \geq 0 \ \text{ and } \sum_{b\in U_i(a)} x_{i,b} \geq \frac{|U_i(a)|}{n} - \varepsilon \right\}$$

and endow each individual with a share of $\frac{1}{n}$ of each object, i.e. $y_i = \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$. Note that consumption sets are closed, bounded and convex and that for $\varepsilon > 0$, endowments lie in the interior. Moreover, in the limit as ε goes to zero, individuals are restricted to consume bundles that can be interpreted as lotteries and that (weakly) stochastically dominate $\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$. Let the set of possible price vectors be given as

$$Q = \left\{ q = (q_a)_{a\in A} \in \mathbb{R}^n | \|q\| = \sum |q_a| \leq 1 \right\},$$

and the state space as $Z^\varepsilon = X_1^\varepsilon \times X_2^\varepsilon \times \cdots \times X_n^\varepsilon \times Q$. To provide individuals with continuous (strict) preferences, we define

$$P_i \colon X_i^\varepsilon \rightrightarrows X_i^\varepsilon : P_i(x_i) = \{\tilde{x}_i \in X_i^\varepsilon | w_i \cdot \tilde{x}_i > w_i \cdot x_i\}.$$

Intuitively one may say, that under P_i a consumption bundle is preferred over an other if it yields a higher expected utility with respect to vNM utilities w_i – except that bundles only approximate lotteries in that $1 - \varepsilon \leq \sum_A x_{i,a} \leq 1 + \varepsilon$. Clearly, P_i is irreflexive, convex-valued and has an open graph.

By Fact 2, for any ε, there exists a Walrasian equilibrium with slack. Moreover, if we assume that all individuals with the same ordinal preferences, $i \in G$, share the same weight vector w_i, there exists a Walrasian equilibrium with $x_i = x$ for all $i \in G$ –

for any other equilibrium, replacing individuals consumption bundles with

$$x_i = \frac{\sum_G x'_j}{|G|}$$

retains preference maximisation and feasibility yet attains equal treatment of equals.

Consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \searrow 0$ and a sequence of associated equilibria $e^k = (x^{\varepsilon_k}, q^{\varepsilon_k})$ satisfying equal treatment of equals. As the sequence of equilibria is bounded by $X_1^{\varepsilon_1} \times X_2^{\varepsilon_1} \times \cdots \times X_n^{\varepsilon_1} \times Q$, it has a convergent subsequence and we may assume w.l.o.g. that (e^k) is convergent itself. Denote its limit as $e^\star = (x^\star, q^\star)$. Then $x^\star$ satisfies equal treatment of equals and, by construction of our consumption sets, is a random assignment that satisfies the strong sd-equal-division-lower-bound.

Claim 1. *The random assignment $x^\star$ is in the weak sd-core from equal division.*

Proof of Claim: Towards a contradiction, assume there exists a group $G \subset I$ and another random assignment p such that $\sum_{i\in G} p_i = |G|\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$ and, for all $i \in G$, $p_i \succ_i^{sd} x_i^\star$. The latter implies

$$\forall i \in G, \varepsilon > 0: \qquad p_i \in X_i^\varepsilon,$$

and for some sufficiently large $\bar{k}$ we have

$$\forall i \in G, k > \bar{k}: \qquad w_i \cdot p_i > w_i \cdot x_i^{\varepsilon_k}.$$

Then by preference maximization we have

$$\forall i \in G, k > \bar{k}: \qquad q^{\varepsilon_k} \cdot p_i > q^{\varepsilon_k} \cdot \left(\frac{\mathbf{1}}{\mathbf{n}}\right) + \frac{1 - \|q^{\varepsilon_k}\|}{\|q^{\varepsilon_k}\|} \geq q^{\varepsilon_k} \cdot \left(\frac{\mathbf{1}}{\mathbf{n}}\right).$$

But this contradicts $\sum_G q^{\varepsilon_k} \cdot p_i = q^{\varepsilon_k} \sum_G p_i = q^{\varepsilon_k}|G|\left(\frac{\mathbf{1}}{\mathbf{n}}\right)$.

◇

Claim 2. *The random assignment $x^\star$ is sd-efficient.*

Proof of Claim: Towards a contradiction, assume there are individuals that can trade among themselves to be strictly better off.[20] That is, assume there exists another random assignment p and a group $G \subset I$ such that $\sum_G (p_i - x_i^\star) = 0$ and $p_i \succ_i^{sd} x_i^\star$ for all $i \in G$. As the trade $(p_i - x_i^\star)$ sd-improves individuals' assignments, we know that $w_i \cdot (p_i - x_i^\star) > 0$ for all $i \in G$. Moreover, as x^{ε_k} approaches x^*, we know that there exists an $\varepsilon \in (\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\forall\, i \in G, a \in A: \qquad x_{i,a}^\varepsilon \geq {}^1\!/\!_2 \cdot x_{i,a}^*.$$

Hence for a (scaled down) trade from x^ε, we find $x_{i,a}^\varepsilon + {}^1\!/\!_2(p_{i,a} - x_{i,a}^\star) \geq 0$. Since the trade moreover leaves the total sum of probability shares unchanged and constitutes an sd-improvement, this ensures that

$$\forall\, i \in G: \qquad x_i^\varepsilon + {}^1\!/\!_2(p_i - x_i^\star) \in X_i^\varepsilon.$$

Then, by preference maximization, we have

$$\forall\, i \in G: \qquad q^\varepsilon \cdot {}^1\!/\!_2(p_i - x_i^\star) > 0,$$

and in the aggregate $q^\varepsilon \cdot \sum_G (p_i - x_i^\star) > 0$. But this contradicts $\sum_G (p_i - x_i^\star) = 0$. ◇

[20]W.l.o.g, we can ignore individuals that are made only weakly better off, as they would exchange probability shares of objects that not only they, but everyone else is indifferent between.

This completes the proof. □

6. Discussion

While Proposition 3 establishes the existence of a solution satisfying a range of efficiency and equity criteria, its proof identifies a particular sub-solution – namely assignments x^* that arise in the limit of a sequence of Walrasian equilibria $e^k(x^{\varepsilon_k}, q^{\varepsilon_k})$ where along the sequence individuals evaluate bundles with respect to weights $w \in W(\succsim)$. In this section, we will describe further properties of the limit assignment x^*, illustrate it in an example, discuss how the weights w may be chosen in applications and how our approach protects against the negative effects of misspecified weights.

6.1. **Further properties of $e^*(x^*, q^*)$.** The first thing to note about the limit e^* is that it not necessarily constitutes a Walrasian equilibrium from equal incomes, as can be seen from the following example.

Example 3. Consider the assignment problem used in the proof of Proposition 2 – three individuals $I = \{1, 2, 3\}$, with preferences $a \succ_{1,2} b \succ_{1,2} c$ and $b \succ_3 a \succ_3 c$. Sd-efficiency, equal treatment of equals and the strong sd-equal-division-lower-bound together pin down the limit x^* as $x^*_{1,2} = (1/2, 1/6, 1/3)$, $x^*_3 = (0, 2/3, 1/3)$. For these individual assignments to be of equal value, requires prices $q^*_a = q^*_b$ – yet in a Walrasian equilibrium at these prices, individuals 1 and 2 would not demand b.

Nevertheless, $e^*(x^*, q^*)$ is a quasi-equilibrium with respect to individuals' consumption sets and the chosen weights $w \in W(\succsim)$ (which may be interpreted as vNM-utilities consistent with preferences over sure objects) – that is, there are no cheaper lotteries that stochastically dominate the uniform lottery and provide weakly higher expected utility. To see this, observe that if there were such lotteries for an individual i unsatiated at x^*_i, they would have been cheaper and in i's consumption sets along the sequence that led to x^*_i – violating preference maximization as preferred bundles had been affordable. Moreover at x^*, a satiated individual consumes probability shares of her most preferred object(s) that sum to one; if one of her most preferred objects was cheaper, she would have consumed more than one of that objects along the sequence where the consumption set was relaxed – violating feasibility.

To illustrate how a quasi-equilibrium can arise as the limit of Walrasian equilibria, consider again the preference profile of example 3.

Example 4. Assume normalized weights $w_{1,a}, w_{2,a}, w_{3,b} = 1$, $w_{1,c}, w_{2,c}, w_{3,c} = 0$ and $w_{1,b}, w_{2,b}, w_{3,a} \in (0, 1)$. As ε_k becomes sufficiently small, prices

$$q_a^{\varepsilon_k} = \frac{1 + 6\varepsilon_k}{2 + 6\varepsilon_k}, \qquad q_b^{\varepsilon_k} = \frac{1}{2 + 6\varepsilon_k}, \qquad q_c^{\varepsilon_k} = 0$$

support the bundles

$$x_1^{\varepsilon_k}, x_2^{\varepsilon_k} = (1/2, 1/6 - \varepsilon_k, 1/3 + 1/2\varepsilon_k), \qquad x_3^{\varepsilon_k} = (0, 2/3 + 2\varepsilon_k, 1/3 - \varepsilon_k)$$

as a Walrasian equilibrium from equal incomes. As ε_k goes to zero, these bundles converge to the random assignment x^* of example 3 above, which constitutes a quasi-equilibrium with respect to $q^* = (1/2, 1/2, 0)$.

Given that $e^*(x^*, q^*)$ is a quasi-equilibrium, the limit price vector q^* may be seen as an intuitive measure on the general desirability of objects, namely an ordering of objects from most to least expensive.[21] Relative to q^*, each individual's random assignment is of equal value – which may be seen as an additional, strong equity characteristic of the assignment, beyond the fact that it lies in the weak sd-core from equal division. If preferences are sufficiently correlated so that q^* yields a strict ordering of objects, an individual i whose preferences coincide with this ordering, will be assigned each object with equal probability, $x_i^* = \left(\frac{\mathbf{1}}{\mathbf{n}}\right)$, as any other assignment that stochastically dominates equal division would imply that the value of i's assignment exceeds the average value of others' assignments. In this sense, the limit assignment x^* ensures that 'expensive tastes' are not accommodated to an unwarranted degree.

Furthermore, an individual for whom an object is comparatively less valuable than for the rest of the society – in the sense that the individual ranks it lower than it is ranked according to q^* – will receive the object with zero probability, which can be seen intuitively as an expression of efficiency.

6.2. **Choosing $w \in W(\succsim)$.** On possible way of setting $w \in W(\succsim)$ is to estimate them as vNM-utilities using historical application data by means of a random utility model. For example in a school choice context, it seems plausible that applicants who live closer to a particular school or have siblings attending it, should receive a higher utility from that school. The size of these effects can be estimated using a random utility model, see e.g. Ashlagi and Shi [2015], Pathak and Shi [2017].[22]

Another approach is used in the current Israeli medical match [Bronfman et al., 2015a,b]. For that, applicants were asked to choose hypothetically between different exemplary lotteries that assigned them to different hospitals with various probabilities. Based on these survey answers, it was estimated that being assigned to the ith-most preferred option gives utility $(m - i + 1)^2$ where m denotes the number of hospitals. Using submitted rank order lists and these estimates, the matching algorithm uses Random Serial Dictatorship to find a first, intermediate random assignment before maximizing expected utility subject to the constraint that no applicant receives a lower expected utility than under the intermediate assignment.

A drawback of this approach is that if an applicants vNM-utilities do not coincide with the common estimates, the final random assignment may provide her with a lower expected utility than she would have received from Random Serial Dictatorship or even from the uniform lottery. The same problem arises of course for any other estimation procedure, say a random utility model, that is used in conjunction with an unconstrained Walrasian equilibrium approach, as in Hylland and Zeckhauser [1979]. Only the hard constraints that our approach puts on individuals consumption sets ensures that each individual indeed receives an assignment that yields (weakly) higher expected utility than the uniform lottery even when the estimated vNM-utilities are incorrect.

[21] In applications where assignment problems arise repeatedly, for example the assignment of school seats, such a measure may be helpful to guide supply.

[22] To use such additional information, one would have to redefine 'equal treatment of equals' – applicants who both submit the same rank order list *and* have the same additional characteristics should be treated equally.

7. Concluding Remarks

We end this paper with two remarks. First, our impossibility results Proposition 1 and 2 illuminate the difference between two of the most prominent efficient solutions to the random assignment problem, namely the Probabilistic Serial (which is sd-envy-free) and Hylland and Zeckhauser's WEEI (which selects from the strong sd-core) and show that no assignment mechanism may satisfy all equity criteria satisfied by either of the two. Such impossibility results may also be of practical importance where an assignment mechanism is challenged in court, for example by individuals who are unsatisfied with their eventual assignment. Here, claimants may argue against an assignment mechanism by identifying specific equity criteria that have been violated. To judge the validity of such arguments, we would have to know whether the identified equity criteria are at least feasible – if that is not the case, the violation cannot serve as an argument for rejecting the assignment mechanism or the specific assignment that it produced.

Proposition 3 offers a way to adapt Hylland and Zeckhauser's pseudo-market approach – which can be used to generate assignments in the core from equal division with respect to known vNM-utilites – to situations where the true underlying vNM utilities of individuals are unknown. Namely it shows, that it is possible to trade from equal division to arrive at assignments in the weak sd-core while simultaneously satisfying the equal-division-lower-bound for all possible underlying utility functions. Moreover, the proof can be straightforwardly adapted to account for general fractional endowments, as considered by Athanassoglou and Sethuraman [2011], to establish existence of allocations that are sd-efficient, lie in the sd-core from initial endowments and satisfy the strong sd-initial-endowment-lower-bound. For example in a school choice setting, a student's fractional initial endowments could be given by a convex combination between a deterministic assignment to the closest school and a uniform lottery over all schools (similar to Harless and Phan [2017]) or as a uniform lottery over a small number of close neighbourhood schools.

Appendix

Example 5. A random assignment in the strong sd-core from equal division, may neither satisfy the strong sd-equal-division-lower-bound, nor equal treatment of equals. Consider $n = 3$ and suppose $i \in 1, 2$ hold preference $a \succ_i b \succ_i c$ while individual 3 prefers object c. Then the random assignment given by $p_{1,a} = 1$, $p_{2,b} = 1$ and $p_{3,c} = 1$ lies in the strong sd-core – individual 2 cannot object on her own, as $\left(\frac{\mathbf{1}}{\mathbf{n}}\right) \not\succ_2^{sd} p_2$. Also, there is no objection involving either individuals 1 or 3 who both receive their most preferred object, and could not be made as well off by any coalition of size two. In a blocking coalition involving all three individuals, 1 and 3 would still have to receive their most preferred object, so that 2 could not be made better off.

Example 6. A random assignment in the strong sd-core from equal division, satisfying the strong sd-equal-division-lower-bound may violate equal treatment of equals. Suppose $i \in \{1, 2, 3\}$ hold preferences $a \succ_i b \succ_i c \succ_i d$ while 4 prefers object d. Then the following random assignment p satisfies the strong sd-equal-division-lower-bound:

	a	b	c	d
$p_{1,2}$:	$1/4$	$1/2$	$1/4$	0
p_3:	$1/2$	0	$1/2$	0
p_4:	0	0	0	1

To see that p lies in the strong sd-core from equal division, observe first that it is sd-efficient – 4 receives her most preferred object and $1, 2$ and 3 have identical preferences. Thus, the grand coalition will not object to p. Next, consider objections by groups of size $k < 4$. Individual 4 cannot support such an objection, as she would receive her most preferred object with probability $p_{4,d} \leq k/4 < 1$. Nor could the remaining individuals form a coalition, as someone would have to accept $p_{i,d} > 0$.

However, p does not satisfy equal treatment of equals, as $p_1 = p_2 \neq p_3$.

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