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# Quasi-exactly solvable polynomial extensions of the quantum harmonic oscillator

**Christiane Quesne**

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles,  
Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

E-mail: [cquesne@ulb.ac.be](mailto:cquesne@ulb.ac.be)

**Abstract.** The (analytic) sextic oscillator is often considered as the prototype of quasi-exactly solvable (QES) Schrödinger equations, i.e., those Schrödinger equations for which at some ad hoc couplings a finite number of eigenstates can be found explicitly by algebraic means, while the remaining ones remain unknown. Recently, a (non-analytic) QES symmetrized quartic oscillator was introduced and shown to complete the list of (analytic) QES anharmonic oscillators, which does not contain any quartic one. Here we prove that such a quartic oscillator is amenable to an  $\mathfrak{sl}(2, \mathbb{R})$  description. Furthermore, we generalize it to a symmetrized sextic oscillator. The latter is obtained by parting the real line into two subintervals  $\mathbb{R}^-$  and  $\mathbb{R}^+$  on which the corresponding Schrödinger equations are solved by using the functional Bethe ansatz method, and the resulting wavefunctions and their first derivatives are matched at  $x = 0$ . Two categories of QES potentials are obtained: the first one containing the well-known analytic sextic potentials as a subset, and the second one of novel potentials with no counterpart in such a class.

## 1. Introduction

Exact solutions of the Schrödinger equation are known to be very useful for approximating solutions of more realistic equations appearing in practical problems, because they may give a hint for suggesting a good starting point in perturbation theory or variational calculus. Such exact solutions essentially belong to one of two categories.

The first category consists of solutions of the so-called exactly solvable (ES) Schrödinger equations, for which all the eigenstates can be algebraically determined. Among them, one finds equations containing piece-wise constant point interactions, such as the rectangular potential hole [1], for which the solutions are obtained by matching the wavefunctions and their first derivatives at the non-analyticity points of the potential. Other ES Schrödinger equations are those containing an analytic potential that can be transformed into a second-order differential equation of hypergeometric type [2]. The eigenfunctions of such equations can be constructed in closed form by using the theory of special functions [3]. Still others can be obtained from the previous ones by using higher-order supersymmetric quantum mechanics or Darboux transformations and the corresponding potentials are rational extensions of the previous potentials [4, 5, 6, 7], while their wavefunctions can be expressed in terms of exceptional orthogonal polynomials [8, 9].

The second category of exact solutions comprises solutions of the so-called quasi-exactly solvable (QES) Schrödinger equations, for which a finite number of eigenstates can be algebraically determined at some ad hoc couplings, while the remaining ones remain unknown.



The simplest ones, which were discovered in the 1980s, are characterized by a hidden  $\mathfrak{sl}(2, \mathbb{R})$  algebraic structure [10, 11, 12, 13, 14] and are connected with polynomial solutions of the Heun equation [15]. More complicated ones, related to polynomial solutions of generalized Heun equations, can be solved by the recursion relation method [16] or by the functional Bethe ansatz method [17, 18]. In such a context, the latter has proven very effective [19, 20, 21, 22].

It was recently suggested [23, 24, 25, 26, 27] that the ES status should also be attributed to less common Schrödinger equations for which the real line of coordinates is splitted into subintervals wherein the potential admits different definitions, while being continuous on the whole line, and the wavefunctions remain piece-wise proportional to special functions, while being matched, as well as their first derivatives, at the subinterval limit points. For instance, the Schrödinger equation with the non-analytic potential  $V(x) = -g^2 \exp(-|x|)$  was solved by Sasaki and Znojil in terms of Bessel functions [24].

A similar extension to the QES case was also proposed by Znojil [28], who considered a symmetrized quartic oscillator and showed by using the recursion relation method that such an oscillator completes the list of (analytic) QES anharmonic oscillators, which contains sextic polynomial oscillators [10], but no quartic one.

In a recent work [29], we proved that Znojil's symmetrized quartic oscillator is amenable to an  $\mathfrak{sl}(2, \mathbb{R})$  description and we introduced a new QES symmetrized sextic oscillator, which we solved by the functional Bethe ansatz method.

It is the purpose of the present communication to review such a work.

## 2. Symmetrized quartic polynomial oscillator

The symmetrized quartic polynomial oscillator [28, 29] is described by the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x), \quad (1)$$

where the non-analytic potential

$$V(x) = x^4 - s|x|^3 + rx^2 - q|x| = \begin{cases} x^4 + sx^3 + rx^2 + qx & \text{if } x < 0, \\ x^4 - sx^3 + rx^2 - qx & \text{if } x > 0, \end{cases} \quad (2)$$

is defined separately on the two semi-infinite intervals  $x < 0$  and  $x > 0$  with  $q, r, s \in \mathbb{R}$ .

Physically acceptable wavefunctions should tend to zero for  $x \rightarrow \pm\infty$  and be continuous everywhere, as well as their first derivatives. Furthermore, as the potential is symmetric under parity, they must also be either symmetric or antisymmetric.

We may therefore solve the Schrödinger equation in one of the intervals, for instance  $x < 0$ , then prescribe that in the other interval  $x > 0$ ,  $\psi(x) = \epsilon\psi(-x)$  with  $\epsilon = +1$  or  $\epsilon = -1$  according to whether  $\psi$  is even or odd. On the resulting wavefunctions  $\psi^{(\epsilon)}(x)$  and their derivatives it then remains to impose continuity at  $x = 0$ , which amounts to setting either  $\psi^{(+)\prime}(0) = 0$  or  $\psi^{(-)}(0) = 0$ .

To get QES solutions of (1), it is appropriate to assume that

$$\psi_n^{(\epsilon)}(x) = e^{W(x)}\phi_n^{(\epsilon)}(x), \quad W(x) = -\frac{1}{3}|x|^3 + ax^2 - b|x|, \quad (3)$$

and

$$\phi_n^{(\epsilon)} = \begin{cases} \sum_{k=0}^n v_k x^k & \text{if } x < 0, \\ \epsilon \sum_{k=0}^n (-1)^k v_k x^k & \text{if } x > 0, \end{cases} \quad (4)$$

where  $a, b$  are two real constants and  $v_k$  ( $k = 0, 1, \dots, n$ ) are some expansion coefficients to be determined. Such functions tend to zero for  $x \rightarrow \pm\infty$ , as it should be, and they have the right behaviour at  $x = 0$  provided the additional condition

$$\begin{aligned} v_1 + bv_0 &= 0 & \text{if } \epsilon = +1, \\ v_0 &= 0 & \text{if } \epsilon = -1, \end{aligned} \quad (5)$$

is imposed.

On inserting potential (2) and wavefunctions (3) in (1), we obtain on the interval  $x < 0$  the differential equation

$$\left( -\frac{d^2}{dx^2} - 2(x^2 + 2ax + b)\frac{d}{dx} + (q - 4ab - 2)x - b^2 - 2a \right) \phi_n(x) = E_n \phi_n(x), \quad (6)$$

where higher-order terms have been eliminated by setting

$$s = 4a \quad \text{and} \quad r = 4a^2 + 2b. \quad (7)$$

This is a (triconfluent) Heun equation [15]. Hence, it can be solved by an  $\mathfrak{sl}(2, \mathbb{R})$  algebraic approach [14].

The first-order differential operators

$$J_n^+ = x^2 \frac{d}{dx} - nx, \quad J_n^0 = x \frac{d}{dx} - \frac{n}{2}, \quad J_n^- = \frac{d}{dx}, \quad (8)$$

satisfying the commutation relations

$$[J_n^0, J_n^\pm] = \pm J_n^\pm, \quad [J_n^+, J_n^-] = -2J_n^0, \quad (9)$$

provide a realization of  $\mathfrak{sl}(2, \mathbb{R})$  in the  $(n+1)$ -dimensional space of polynomials of order not higher than  $n$ ,  $\mathcal{P}_{n+1} = \langle 1, x, x^2, \dots, x^n \rangle$ . Their action in such a space is given by

$$J_n^+ x^k = (k-n)x^{k+1}, \quad J_n^0 x^k = \left(k - \frac{n}{2}\right) x^k, \quad J_n^- x^k = kx^{k-1}. \quad (10)$$

Provided we set

$$q = 4ab + 2n + 2, \quad (11)$$

the second-order differential operator  $h$  on the left-hand side of (6) can be written as a second-degree polynomial in the  $\mathfrak{sl}(2, \mathbb{R})$  generators (8),

$$h = -J_n^- J_n^- - 2J_n^+ - 4aJ_n^0 - 2bJ_n^- - 2(n+1)a - b^2. \quad (12)$$

On using relations (10), it is then straightforward to write the matrix of  $h$  in the space  $\mathcal{P}_{n+1}$ . The admissible energies  $E_n^{(\epsilon)}$  and wavefunctions  $\psi_n^{(\epsilon)}(x)$  can then be obtained by diagonalizing such a matrix and imposing constraint (5).

The simplest examples are provided by the  $n = 0$  and  $n = 1$  cases.

For  $n = 0$ , we get a  $1 \times 1$  matrix for  $h$ , so that we directly obtain  $E_0 = -2a - b^2$  and  $\phi_0(x) = 1$  on  $x < 0$ . Since  $v_0 = 1$  and  $v_1 = 0$ , it results from constraint (5) that the wavefunction on the whole real line cannot be odd, but it can be even provided the condition  $b = 0$  is satisfied. Hence the potential

$$V(x) = x^4 - 4a|x|^3 + 4a^2x^2 - 2|x| \quad (13)$$

has an energy eigenvalue  $E_0^{(+)}$  and associated wavefunction  $\psi_0^{(+)}(x)$ , given by

$$E_0^{(+)} = -2a, \quad \psi_0^{(+)}(x) = e^{-\frac{1}{3}|x|^3+ax^2}, \quad (14)$$

corresponding to a ground state.

For  $n = 1$ , the matrix for  $h$  is  $2 \times 2$ , so that its diagonalization leads to a second-degree equation for  $E_1$ . The solutions of the latter,  $E_{1\pm} = -4a - b^2 \pm 2\sqrt{a^2 - b}$ , are real provided  $a^2 \geq b$ , which can be satisfied if either  $b \leq 0$  or  $b > 0$  and simultaneously  $a \geq \sqrt{b}$  or  $a \leq -\sqrt{b}$ . The corresponding eigenfunctions on  $x < 0$  are  $\phi_{1\pm}(x) = x + a \pm \sqrt{a^2 - b}$ , showing that  $v_0 = a \pm \sqrt{a^2 - b}$  and  $v_1 = 1$ . On the whole line, we can now get an odd or even wavefunction according to the parameter choice.

The wavefunction is odd if  $a = \mp\sqrt{a^2 - b}$ , which imposes  $b = 0$ ,  $a < 0$  for  $\phi_{1+}$ , and  $b = 0$ ,  $a > 0$  for  $\phi_{1-}$ . We therefore obtain that the potential

$$V(x) = x^4 - 4a|x|^3 + 4a^2x^2 - 4|x| \quad (15)$$

has an energy eigenvalue  $E_1^{(-)}$  and associated wavefunction  $\psi_1^{(-)}(x)$ , given by

$$E_1^{(-)} = -6a, \quad \psi_1^{(-)}(x) = e^{-\frac{1}{3}|x|^3+ax^2}x, \quad (16)$$

corresponding to a first-excited state.

From (5), the wavefunction is even provided  $b \neq 0$  and  $a \pm \sqrt{a^2 - b} = -1/b$  for  $\phi_{1,\pm}$ . Examination of these conditions [29] leads to the conclusion that the potential

$$V(x) = x^4 + \frac{2(b^3 + 1)}{b}|x|^3 + \frac{b^6 + 4b^3 + 1}{b^2}x^2 + 2(b^3 - 1)|x|, \quad b \neq 0, \quad (17)$$

has an energy eigenvalue  $E_1^{(+)}$  and associated wavefunction  $\psi_1^{(+)}(x)$ , given by

$$E_1^{(+)} = \frac{2b^3 + 1}{b}, \quad \psi_1^{(+)}(x) = e^{-\frac{1}{3}|x|^3 - \frac{b^3+1}{2b}x^2 - b|x|} \left( |x| + \frac{1}{b} \right), \quad (18)$$

corresponding to a ground state for  $b > 0$  and to a second-excited state for  $b < 0$ .

A similar discussion can be carried out for higher  $n$  values and, in particular, for the  $n = 2$ ,  $\epsilon = +1$  case solved by Znojil [28] using the recursion relation method.

### 3. Symmetrized sextic polynomial oscillator

Instead of potential (2), let us consider now the symmetrized sextic polynomial oscillator potential

$$V(x) = x^6 - u|x|^5 + tx^4 - s|x|^3 + rx^2 - q|x| \quad (19)$$

in equation (1). Such a potential depends on five real constants  $q, r, s, t, u$  instead of three.

To get QES solutions of (1), let us assume that

$$\psi_n^{(\epsilon)}(x) = e^{W(x)}\phi_n^{(\epsilon)}(x), \quad W(x) = -\frac{1}{4}x^4 - a|x|^3 + bx^2 - c|x|, \quad (20)$$

with  $a, b, c \in \mathbb{R}$ , and  $\phi_n^{(\epsilon)}(x)$  given by (4), with the new constraint

$$\begin{aligned} v_1 + cv_0 &= 0 & \text{if } \epsilon = +1, \\ v_0 &= 0 & \text{if } \epsilon = -1. \end{aligned} \quad (21)$$

By setting

$$u = -6a, \quad t = 9a^2 - 4b, \quad s = 12ab - 2c, \quad (22)$$

to eliminate some higher-order terms, the Schrödinger equation (1) leads to the differential equation

$$\left( -\frac{d^2}{dx^2} + 2(x^3 - 3ax^2 - 2bx - c)\frac{d}{dx} + (r - 4b^2 - 6ac + 3)x^2 + (q - 4bc - 6a)x - c^2 - 2b \right) \phi_n(x) = E_n \phi_n(x), \quad (23)$$

on  $x < 0$ . This is a special case of a generalized Heun equation, whose polynomial solutions can be found by the functional Bethe ansatz method, as previously shown by Zhang [18]. Before reviewing this method, it is worth observing that a special case of (19) and (20), corresponding to  $u = s = q = a = c = 0$ , gives back the analytic sextic potential considered by Turbiner and Ushveridze [10]. In such a case, equation (23) can be transformed into a Heun-type equation by making the change of variable  $\xi = x^2$  and, consequently, can be dealt with by an  $\text{sl}(2, \mathbb{R})$  approach.

In the functional Bethe ansatz method, one considers a second-order differential equation

$$X(z)y''(z) + Y(z)y'(z) + Z(z)y(z) = 0 \quad (24)$$

with polynomial coefficients

$$X(z) = \sum_{l=0}^k a_l z^l, \quad Y(z) = \sum_{l=0}^{k-1} b_l z^l, \quad Z(z) = \sum_{l=0}^{k-2} c_l z^l, \quad (25)$$

such that  $a_l, b_l, c_l \in \mathbb{R}$ , and one looks for polynomial solutions of the type

$$y(z) = \prod_{i=1}^n (z - z_i) \quad (26)$$

with real and distinct roots  $z_i, i = 1, 2, \dots, n$ .

On inserting (25) in (24), the latter is transformed into

$$-c_0 = \left( \sum_{l=0}^k a_l z^l \right) \sum_{i=1}^n \frac{1}{z - z_i} \sum_{\substack{j=1 \\ j \neq i}}^n \frac{2}{z_i - z_j} + \left( \sum_{l=0}^{k-1} b_l z^l \right) \sum_{i=1}^n \frac{1}{z - z_i} + \sum_{l=1}^{k-2} c_l z^l, \quad (27)$$

where the left-hand side is a constant, while the right-hand one is a meromorphic function with simple poles at  $z = z_i$  and a singularity at  $z = \infty$ . Equation (27) will be satisfied provided the residues at the simple poles vanish and the coefficients of  $z, z^2, \dots, z^{k-2}$  vanish too. The first condition leads to the Bethe ansatz equations determining the roots,

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{2}{z_i - z_j} + \frac{\sum_{l=0}^{k-1} b_l z_i^l}{\sum_{l=0}^k a_l z_i^l} = 0, \quad i = 1, 2, \dots, n, \quad (28)$$

while the second one provides some relations between the coefficients  $a_l, b_l, c_l$  and the roots.

In the case of equation (23), we obtain the Bethe ansatz equations

$$\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{x_i - x_j} - x_i^3 + 3ax_i^2 + 2bx_i + c = 0, \quad i = 1, 2, \dots, n, \quad (29)$$

and the relations

$$\begin{aligned} r &= 4b^2 + 6ac - 2n - 3, & q &= -2 \sum_{i=1}^n x_i + 6a(n+1) + 4bc, \\ E &= 2 \sum_{i=1}^n x_i^2 - 6a \sum_{i=1}^n x_i - 2b(2n+1) - c^2. \end{aligned} \quad (30)$$

On the resulting  $\phi_n(x)$ , it then remains to impose constraint (21).

The  $n = 0$  case is easy to solve because then  $\phi_0(x) = 1$  on  $x < 0$  and there is no root. Relations (30) then become

$$r = 4b^2 + 6ac - 3, \quad q = 6a + 4bc, \quad E_0 = -2b - c^2. \quad (31)$$

Since  $v_0 = 1$ ,  $v_1 = 0$ , there is no odd wavefunction, but there is an even one provided  $c = 0$ . We then get the potential

$$V(x) = x^6 + 6a|x|^5 + (9a^2 - 4b)x^4 - 12ab|x|^3 + (4b^2 - 3)x^2 - 6a|x| \quad (32)$$

with an energy eigenvalue  $E_0^{(+)}$  and associated wavefunction  $\psi_0^{(+)}(x)$ , given by

$$E_0^{(+)} = -2b, \quad \psi_0^{(+)}(x) = e^{-\frac{1}{4}x^4 - a|x|^3 + bx^2}, \quad (33)$$

and corresponding to a ground state. For  $a = 0$ , we obtain, as a special case, the analytic sextic potential

$$V(x) = x^6 - 4bx^4 + (4b^2 - 3)x^2 \quad (34)$$

with

$$E_0^{(+)} = -2b, \quad \psi_0^{(+)}(x) = e^{-\frac{1}{4}x^4 + bx^2}, \quad (35)$$

as found by Turbiner and Ushveridze [10].

For  $n = 1$ , there is a single Bethe ansatz equation

$$x_1^3 - 3ax_1^2 - 2bx_1 - c = 0 \quad (36)$$

and relations (30) become

$$r = 4b^2 + 6ac - 5, \quad q = -2x_1 + 12a + 4bc, \quad E = 2x_1^2 - 6ax_1 - 6b - c^2. \quad (37)$$

Since  $v_0 = -x_1$  and  $v_1 = 1$ , we can get both an even and an odd wavefunction according to which parameters are chosen.

The wavefunction is odd if  $x_1 = 0$ , which imposes  $c = 0$ . The potential

$$V(x) = x^6 + 6a|x|^5 + (9a^2 - 4b)x^4 - 12ab|x|^3 + (4b^2 - 5)x^2 - 12a|x| \quad (38)$$

has therefore an energy eigenvalue  $E_1^{(-)}$  and associated wavefunction  $\psi_1^{(-)}(x)$ , given by

$$E_1^{(-)} = -6b, \quad \psi_1^{(-)}(x) = e^{-\frac{1}{4}x^4 - a|x|^3 + bx^2}, \quad (39)$$

and corresponding to a first-excited state. The special case resulting from  $a = 0$  is now the analytic sextic potential

$$V(x) = x^6 - 4bx^4 + (4b^2 - 5)x^2 \quad (40)$$

with

$$E_1^{(-)} = -6b, \quad \psi_1^{(-)}(x) = e^{-\frac{1}{4}x^4 + bx^2} x, \quad (41)$$

in accordance with Turbiner and Ushveridze results [10].

The even wavefunction is obtained for  $x_1 = 1/c$ , where  $c$  is a nonvanishing solution of the quartic equation  $c^4 + 2bc^2 + 3ac - 1 = 0$ . For such a  $c$  value, the potential

$$V(x) = x^6 + 6a|x|^5 + (9a^2 - 4b)x^4 - (12ab - 2c)|x|^3 + (4b^2 + 6ac - 5)x^2 - \left(12a + 4bc - \frac{2}{c}\right)|x| \quad (42)$$

has an eigenvalue  $E_1^{(+)}$  and an eigenfunction  $\psi_1^{(+)}(x)$ , given by

$$E_1^{(+)} = -6b - c^2 - \frac{6a}{c} + \frac{2}{c^2}, \quad \psi_1^{(+)}(x) = e^{-\frac{1}{4}x^4 - a|x|^3 + bx^2 - c|x|} \left(|x| + \frac{1}{c}\right), \quad (43)$$

describing a ground state for  $c > 0$  and a second-excited state for  $c < 0$ . Such a potential has no counterpart among the analytic sextic potentials of Ref. [10].

For  $n = 2$ , as detailed in Ref. [29], it is also possible to find both odd and even wavefunctions. An odd wavefunction  $\psi_2^{(-)}(x) \propto x(|x| + \frac{1}{c})$  is obtained if  $x_1 = 0$ ,  $x_2 = 1/c$ , and  $c \neq 0$  is a solution of a quartic equation  $2c^4 + 2bc^2 + 3ac - 1 = 0$ . The resulting potential has no counterpart in the set of analytic sextic potentials of Ref. [10]. On the other hand, an even wavefunction  $\psi_2^{(+)}(x) \propto (|x| + x_1)(|x| + x_2)$  corresponds to roots  $x_1, x_2 \neq 0, 1/c$  that are solutions of

$$x_1^3 - 3ax_1^2 - 2bx_1 - c = -(x_2^3 - 3ax_2^2 - 2bx_2 - c) = \frac{1}{x_1 - x_2}, \quad x_1 + x_2 = cx_1x_2. \quad (44)$$

This time, we get a counterpart in the set of analytic sextic potentials for  $a = c = x_1 + x_2 = 0$ . In this case, there indeed remains a single equation  $2x_1^4 - 4bx_1^2 - 1 = 0$  with the two solutions  $(x_1^2)_{\pm} = \frac{1}{2}(2b \pm \sqrt{4b^2 + 2})$ , giving rise to two eigenvalues and eigenfunctions

$$E_{2\pm}^{(+)} = -6b \pm 2\sqrt{4b^2 + 2}, \quad \psi_{2\pm}^{(+)}(x) = \exp\left(-\frac{1}{4}x^4 + bx^2\right) \left(x^2 - b \mp \frac{1}{2}\sqrt{4b^2 + 2}\right). \quad (45)$$

In Ref. [29], some examples of QES symmetrized sextic oscillators and associated wavefunctions have been displayed and compared with results for the analytic case whenever the latter exist.

#### 4. Results for QES symmetrized sextic polynomial oscillators

We have shown that the set of new QES symmetrized sextic polynomial oscillators can be separated into two categories.

The first one contains those symmetrized sextic oscillators whose known wavefunction is of ‘natural’ parity, i.e., for which  $\epsilon = (-1)^n$ . These include those associated with the cases  $(n, \epsilon) = (0, +1)$ ,  $(1, -1)$ , and  $(2, +1)$ , which we have explicitly considered. They comprise as special cases the QES analytic sextic potentials dealt with by Turbiner and Ushveridze [10], which constitute a subset.

The second category consists of those symmetrized sextic oscillators whose known wavefunction is of ‘unnatural’ parity, i.e., for which  $\epsilon = (-1)^{n+1}$ . They include, in particular,



those associated with the cases  $(n, \epsilon) = (0, +1)$  and  $(1, -1)$ , which have been explicitly studied. They do not comprise any QES analytic sextic potential, as found by Turbiner and Ushveridze [10].

In conclusion, considering symmetrized sextic oscillators gives rise to a novel type of QES potentials with no counterpart in the set of analytic sextic potentials.

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