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October 2018

ECARES working paper 2018-28

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www.ecares.org

A REPRESENTATION THEOREM FOR GENERAL REVEALED PREFERENCE

MIKHAIL FREER AND CÉSAR MARTINELLI

Abstract. We provide a representation theorem for revealed preference of an agent whose consumption set is contained in a general topological space (separable Hausdorff space). We use this result to construct a revealed preference test that applies to choice over infinite consumption streams and probability distribution spaces, among other cases of interest in economics. In particular, we construct a revealed preference test for best-responding behavior in strategic games.

Keywords: revealed preferences, representation theorem, preference extensions, equilibrium play

1. Introduction

We provide a necessary and sufficient revealed preference condition for an observed set of choices to be generated by maximization of a utility function. The condition applies to separable Hausdorff space of alternatives, thus requiring weaker topological restrictions than those imposed in the previous literature. In particular, we dispense with local compactness. This extension is important because there are at least two spaces frequently used in economic theory and modeling that do not have to satisfy local-compactness: (1) the space of infinite consumption streams (infinitely dimensional Hilbert space) and (2) the space of lotteries (measurable distribution functions) over a given set of alternatives; for instance mixed strategies in a strategic game. We illustrate our approach constructing a revealed preference test for best-responding behavior in strategic games.

Revealed preference theory, pioneered by the work of Samuelson (1938) and Afriat (1967), builds on the fact that, although we cannot observe complete preference relation profiles of players, we can observe their choices over some budget sets. Revealed preference theory allows data to speak for itself and therefore avoids problem of parametric misspecification of preferences. Recent developments in statistics (e.g. Chernozhukov et al., 2007; Aguiar and Kashaev, 2017) allow to use the revealed preference conditions as moment inequalities and therefore, allow for testing and identification of the models that accounts for measurement and decision making errors. Chambers and Echenique (2016) offer a general review of the revealed preference approach and its use for testing theories of individual behavior.

There is growing interest in developing a comprehensive approach to revealed preference that can be applied in a wide variety of contexts of interest. Recent research has proceeded along two lines: extending the scope of revealed preference tests for a large class of behavioral theories, and relaxing the standard assumptions about budget sets and commodity spaces. Along the first line, Demuynck (2009) provides a revealed preference test for the whole class of theories that can be represented by a "nice" function over preference relations, and Chambers et al. (2014) provides a sufficient condition for theories to be testable with a finite data set. Along the second line, Forges and Minelli (2009) generalizes the classical result from Afriat (1967) to the case of non-linear budgets, and Nishimura et al. (2017) provide a general revealed preference condition for locally compact Hausdorff space and compact budgets. This paper contributes to the second strand by further relaxing the assumptions on topological spaces.

The remainder of this paper is organized as follows. Section 2 contains basic definitions. Section 3 shows the main result and its application to construct the test of best-responding behavior in static games. Section 4 provides concluding remarks. All proofs omitted in the text are collected in an Appendix.

2. Preliminaries

Consider a second countable topological space (X, τ) representing the universal set of alternatives. A partial order $\leq$ is said to be **separable** if there is a countable set $Z \subseteq X$ such that for every $y < x$ there is $z \in Z$ such that $y \leq z \leq x$. Let $\leq$ be a separable partial order in the space (X, τ) with $<$ denoting the asymmetric part of the partial order. A utility function $u : X \to \mathbb{R}$ is said to be **monotonic** if $y \leq x$ implies $u(y) \leq u(x)$ and $y < x$ implies $u(y) < u(x)$.

Let $\mathcal{B}$ be a countable collection of compact subsets of X, representing possible budget sets. Denote by $C : \mathcal{B} \to 2^X$ a **choice correspondence** over $\mathcal{B}$ assigning to each $B \in \mathcal{B}$ a countable set $C(B) \subseteq B$. We can think of the elements of $C(B)$ as corresponding to different observed choices from the same budget. A **data set** is a tuple $(\mathcal{B}, C)$ that assigns choices to every budget from a collection. A data set $(\mathcal{B}, C)$ is

rationalizable if there is a monotonic utility function $u : X \to \mathbb{R}$ such that $u(y) \leq u(x)$ for every $y \in B_t$ and $x \in C(B_t)$ for every $B_t \in \mathcal{B}$.

Denote by

$$B^{\leq} = \{y \in X : \text{there is } x \in B \text{ such that } y \leq x\}$$

the downward closure of budget B, and by

$$B^{<} = \{y \in X : \text{there is } x \in B \text{ such that } y < x\}$$

the interior of the downward closure of B.

Definition 1. *A data set* $(\mathcal{B}, C)$ *satisfies the* **Generalized Axiom of Revealed Preferences (GARP)** *if for every sequence* $x_1, \ldots, x_n$ *such that* $x_t \in C(B_t)$ *for every* $t = 1, \ldots, n$,

$$x_t \in B_{t+1}^{\leq} \textit{ for every } t = 1, \ldots, n-1 \textit{ implies } x_n \notin B_1^{<}.$$

3. Results

3.1. A general revealed preference condition.

Theorem 1. *A data set* $(\mathcal{B}, C)$ *is rationalizable if and only if it satisfies GARP.*

The starting point for the proof of the theorem is an argument from the preference extension literature. We consider revealed preference as an incomplete preference relation, and show that there is a converging algorithm leading to a complete preference relation. Moreover, the algorithm guarantees that at every step (and therefore in the limit) the preference relation is transitive and separable, which guarantees the existence of a utility representation (see Debreu, 1954). While our definition of GARP incorporates monotonicity, the algorithm can incorporate other desiderata such as homotheticity and quasi-linearity (see Appendix for more details).

3.2. Revealed best responses.

Let $N = \{1, \ldots, n\}$ be a set of players, S_i be the set of pure strategies available to player $i \in N$, $S = \times_{i \in N} S_i$ be the set of strategy profiles, and $\phi_i : S \to \mathbb{R}$ be the monetary payoff function of player $i \in N$. Abusing terminology, we refer to the triple $G = \langle N, S, (\phi_i) \rangle$ as a game form, though note that we specify only the monetary payoffs of the players, not their actual utility payoffs. From now on we assume that players are concerned exclusively with their own monetary payoffs–the more money the better.

Let $\sigma_i \in \triangle(S_i)$ be a mixed strategy for player $i \in N$, and let $\sigma = \times_{i \in N} \sigma_i$ be a profile of mixed strategies. Each profile of strategies induces a profile of lotteries over monetary payoffs–a single lottery for every player. We denote by $\mathcal{L}_i$ the space of monetary lotteries for player i generated by profiles of mixed strategies. This is a space of probability distributions, which we equip with the topology of weak convergence. Note that the space as defined is a separable Hausdorff[1] space and therefore is second countable. Therefore, each mixed strategy profile σ indices a lottery over monetary payoffs for player i which can be described by its cumulative distribution function $F_\sigma : \mathbb{R} \to [0, 1]$ satisfying the usual properties i.e. F_σ is nondecreasing, right-continuous and satisfies $\lim_{x\to 0} F(x)_\sigma = 0$ and $\lim_{x\to+\infty} F(x)_\sigma = 1$. A lottery F_σ stochastically dominates a lottery $F_{\sigma'}$ (denoted by $F_\sigma \geq_{FSD} F_{\sigma'}$) if $F_\sigma(x) \leq F_{\sigma'}(x)$ for every $x \in \mathbb{R}$. First order stochastic dominance as just defined is an antisymmetric partial order and is separable in the space of probability distributions. Let $>_{FSD}$ denote the asymmetric part of $\geq_{FSD}$, and note that $F_\sigma >_{FSD} F_{\sigma'}$ if and only if $F_\sigma \geq_{FSD} F_{\sigma'}$ and $F_\sigma \neq F_{\sigma'}$. We can define a utility function $U_i : \mathcal{L}_i \to \mathbb{R}$ over the space of lotteries for each player i; we say that the utility function U_i is monotonic if $F_{\sigma_i,\sigma_{-i}} >_{FSD} F_{\sigma'_i,\sigma_{-i}}$ implies $U_i(F_{\sigma_i,\sigma_{-i}}) > U_i(F_{\sigma'_i,\sigma_{-i}})$.

A data set in this case is a finite set of games $\mathcal{G}$, and players' choices $C(G)$ for each $G \in \mathcal{G}$, where $C_i(G)$ denotes the mixed strategy chosen by player i in game G. Denote by S_i^t the set of pure strategies available to player i in the t^{th} observation, by σ^t the observed profile of mixed strategies in the t^{th} observation, and by F_σ^t the lottery over monetary payoffs for a given player generated by the mixed strategies in the t^{th} observation. Games may vary for different observations. We say that a data set is **rationalizable with best-responding behavior** if for each player i there is a monotonic utility function $U_i : \mathcal{L}_i \to \mathbb{R}$ such that $U_i(F^t_{\sigma_i^t,\sigma_{-i}^t}) \geq U_i(F^t_{\sigma'_i,\sigma_{-i}^t})$ for every $\sigma'_i \in \triangle(S_i^t)$ for every observation t. Note that we do not require expected utility maximization.

We consider two different assumptions about observability. First, suppose that we can observe perfectly the mixed strategy of the players. Assume, for instance, that each subject is asked to write down the exact mixed strategy (distribution) and this distribution would be used in order to generate the payoff. Hence, the input for testing best-responding behavior would be the exact mixed strategy. Corollary 1 relies on the first assumption. Second, assume that although subjects may have a full-support mixed strategy in mind, we only observe a finite number of choices (realizations of the mixed strategy). For instance, we only observe a finite number of repetitions of each game. In this case, the empirical distribution function would be a consistent estimate

[1]We claim this having in mind the topology of weak convergence, equivalent to the Levy-Prokhorov metrization of the space. Moreover, since the original space is separable, the space of Borel measures is also separable.

of the underlying mixed strategy at every strategy profile. Corollary 2 serves as an approximate condition for this case.

Corollary 1. *The data set $(\mathcal{G}, C)$ is rationalizable with best-responding behavior if and only if*

(i) *there is no $\tilde{\sigma}_i^t \in \triangle(S_i^t)$ such that $F^t_{\tilde{\sigma}_i^t,\sigma_{-i}^t} >_{FSD} F^t_{\sigma_i^t,\sigma_{-i}^t}$, and*

(ii) *there is no sequence $\tilde{\sigma}_i^{t_1},\dots,\tilde{\sigma}_i^{t_n}$ such that $F^{t_j}_{\tilde{\sigma}_i^{t_j},\sigma_{-i}^t} >_{FSD} F^{t_{j+1}}_{\sigma_i^{t_{j+1}},\sigma_{-i}^t}$ for every $j \in \{1,\dots,n-1\}$ and $F^{t_n}_{\tilde{\sigma}_i^{t_n},\sigma_{-i}^{t_n}} >_{FSD} F^{t_1}_{\sigma_i^{t_1},\sigma_{-i}^{t_1}}$.*

The first condition checks that none of the choices is stochastically dominated within the budget set. The second condition actually implements the GARP search over all possible cycles. Corollary 1 does not imply that game should be finite; it is applicable to continuous games as well. However, testing stochastic dominance in a game with a continuum of strategies may not be feasible in finite time. Moreover, due to finiteness of the data set, we may only observe finite support distributions. Below we provide conditions which can be used to test the consistency of data set with GARP given a finite number of observations.

For given F^t_σ, denote by $J(F^t_\sigma)$ the set of "jumps" of the empirical distribution function. Denote by $F^t_\sigma \gg_{FSD} F^s_{\sigma'}$ if $F^t_\sigma(x) \le F^s_{\sigma'}(x')$ for every $x \in J(F^t_\sigma)$ and $x' = \max\{y \in J(F^s_{\sigma'}) : \ y \le x\}$, with $F^t_\sigma(x) < F^s_{\sigma'}(x')$ for some $x \in J(F^t_\sigma)$ and $x' = \max\{y \in J(F^s_{\sigma'}) : \ y \le x\}$. Note that there are $\hat{F}^t_\sigma$ and $\hat{F}^s_{\sigma'}$ equal to F^t_σ and $F^s_{\sigma'}$ on $J(F^t_\sigma)$ and $J(F^s_{\sigma'})$ correspondingly such that $\hat{F}^t_\sigma >_{FSD} \hat{F}^s_{\sigma'}$ if and only if $F^t_\sigma \gg_{FSD} F^s_{\sigma'}$. If all games are finite, then F^t_σ, F^s_σ are step functions, so that $F^t_\sigma \gg_{FSD} F^s_{\sigma'}$ if and only if $F^t_\sigma >_{FSD} F^s_{\sigma'}$.

Corollary 2. *If*

(i) *there is no $\tilde{\sigma}_i^t$ such that $F^t_{\tilde{\sigma}_i^t,\sigma_{-i}^t} \gg_{FSD} F^t_{\sigma_i^t,\sigma_{-i}^t}$, and*

(ii) *there is no sequence $\tilde{\sigma}_i^{t_1},\dots,\tilde{\sigma}_i^{t_n}$ such that $F^{t_j}_{\tilde{\sigma}_i^{t_j},\sigma_{-i}^t} \gg_{FSD} F^{t_{j+1}}_{\sigma_i^{t_{j+1}},\sigma_{-i}^t}$ for every $j \in \{1,\dots,n-1\}$ and $F^{t_n}_{\tilde{\sigma}_n^t,\sigma_{-i}^t} \gg_{FSD} F^{t_1}_{\sigma_i^{t_1},\sigma_{-i}^t}$,*

then the data set $(\mathcal{G}, C)$ is rationalizable with best-responding behavior. Moreover, if all games are finite, then conditions are also necessary. In addition if the space of outcomes is compact, then utility is continuous.

Corollary 2 immediately follows from Corollary 1. The claim about continuity follows from the fact, that the space of probability distributions over compact separable space is compact and result from Nishimura et al. (2017) that guarantees the continuous extension in this case. Finally let us remark that Corollary 2 provides a computationally efficient way to test the conditions from Corollary 1 in the case of finite games.

4. Concluding Remarks

We provide revealed preference test for existence of utility that generates the observed choices in separable Hausdorff space and show that it can be applied to construct a revealed preference test for the best-responding behavior in static games. Moreover, additional spillover result is complementary to applicability of the test for the infinite data sets. Reny (2015) shows that in the case of linear budgets in the space of real vectors GARP is equivalent to the existence of utility function that generates the data even for the infinite data set. We extend this result further showing that GARP is equivalent to the existence of the utility function that generates the data even if the budgets are nonlinear.

Appendix: Proof of Theorem 1

Before we proceed with the proof let us introduce some additional notation. A set $R \subseteq X \times X$ is said to be a preference relation. We denote a set of all preference relations on X by $\mathcal{R}$. We denote the reverse relation $R^{-1} = \{(x, y) | (y, x) \in R\}$. We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \backslash I(R)$. We denote the incomparable part by $N(R) = X \times X \setminus (R \cup R^{-1})$. Note that

$$\geq \equiv \{(x, y) \in X^2 : y \leq x\} \quad \text{and} \quad > \equiv \{(x, y) \in X^2 : y < x\}$$

are preference relations.

Definition A.1. *A preference relation R satisfies:*

- ***completeness*** *if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \emptyset$).*
- ***transitivity*** *if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.*
- ***monotonicity*** *if $\geq \subseteq R$ and $> \subseteq P(R)$.*
- *Z**-separability** if there is a countable $Z \subseteq X$ such that $(x, y) \in P(R)$ implies that there is $z \in Z$ such that $(x, z), (z, y) \in R$.*

As we already mentioned, Theorem 1 is based in the preference extension literature.

Definition A.2. *A preference relation R' is an **extension** of R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

We say that R is **consistent** with R' if $P^{-1}(R) \cap R' = \emptyset$. Next we show that preference relation R' is an extension of $R \subseteq R'$ if and only if R is consistent with R'. Consistency is an operationalizable version of extension which will be useful in what follows.

Lemma A.1. *$R \preceq R'$ if and only if $R \subseteq R'$ and $P^{-1}(R) \cap R' = \emptyset$.*

Proof. (⇒) By definition, $R \subseteq R'$. Assume $P^{-1}(R) \cap R' \neq \emptyset$, then there is $(x, y) \in P^{-1}(R) \cap R'$. That is $(y, x) \in P(R)$ and $(x, y) \in R'$. At the same time $R \preceq R'$ implies that $(y, x) \in P(R')$, that is a contradiction.

(⇐) Assume that $R \subseteq R'$ but $R \npreceq R'$, that is $P(R) \nsubseteq P(R')$. Hence, there is $(x, y) \in P(R)$ and $(x, y) \notin P(R')$. At the same time $R \subseteq R'$ implies that $(x, y) \in R$. Therefore, $(y, x) \in R$, because $(x, y) \in I(R) = R \setminus P(R)$. Hence, $(y, x) \in P^{-1}(R) \cap R' \neq \emptyset$. □

Let $T : \mathcal{R} \to \mathcal{R}$ be the transitive closure, defined by $(x, y) \in T(R)$ if and only if there is a finite sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R$. The transitive closure is an example of a function over preference relations, which we develop in what follows.

Definition A.3. *For any given function $F : \mathcal{R} \to \mathcal{R}$, we let*

- $\mathcal{R}_F = \{R \in \mathcal{R} | R \preceq F(R)\}$,
- $\mathcal{R}_F^Z = \{R \in \mathcal{R}$ *and R is Z-separable* $|R \preceq F(R)\}$.

$\mathcal{R}_F$ and $\mathcal{R}_F^Z$ are different sets of preference relations that are extended by F.

Next, we define a set of properties of function over preference relations which allows to guarantee existence of complete fixed point extension of every consistent preference relation which can be represented by a utility function.

Definition A.4. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be*

- ***monotonic*** *if $R \subseteq R'$ implies $F(R) \subseteq F(R')$ for all $R, R' \in \mathcal{R}$,*
- ***closed*** *if $R \subseteq F(R)$ for all $R \in \mathcal{R}$,*
- ***idempotent*** *if $F(F(R)) = F(R)$ for all $R \in \mathcal{R}$,*
- ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*
- ***expansive*** *if for every $R = F(R)$ such that $N(R) \neq \emptyset$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \in \mathcal{R}_F$ and $P(R) = P(R \cup S)$,*
- ***transitivity-inducing*** *if every preference relation satisfying $R = F(R)$ is transitive,*

– ***separability-preserving*** *with respect to some countable set* Z*,*
if $P(F(P(R))) = P(R)$ *and* $R \in \mathcal{R}_F^Z$ *imply that* $F(R)$ *is* Z*-separable.*
– ***convergent*** *if* $F(R) = R$ *implies* $P(F(P(R))) = P(R)$*.*

The first four properties define an algebraic closure (see Demuynck, 2009). Further we refer to a function which satisfy all of the properties above as a **rational closure**. Using rational closures in the proofs allows us to simplify the reasoning as well as to allow for possible future extensions of the result to other theories. As we show below, the transitive closure is rational. Other examples are quasi-linear (Castillo and Freer, 2016) and homothetic (Demuynck, 2009) closures.

Lemma A.2. *T is a rational closure, and moreover $T(R) = R$ if and only if R is transitive relation.*

For the proof that T is an algebraic closure as well as for the proof that every fixed point of transitive closure is transitive see Demuynck (2009). It remains to be shown that T is expansive, separability-preserving and convergent.

Proof.

T **is expansive.**
Consider a relation $R = T(R)$ and assume that $N(R) \neq \emptyset$. Take any element $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y), (y, x)\}$. We claim that $R' \preceq T(R')$, which would prove that T is expansive. It is clear that $R' \subseteq T(R')$. Therefore, we only need to show that $P(R') \subseteq P(T(R'))$. Assume, on the contrary, that there are elements z and w for which $(z, w) \in P(R')$ and $(w, z) \in T(R')$, and note that $(x, y) \neq (z, w) \neq (y, x)$. From the definition of T, we know that there is some finite sequence $s_1, \ldots, s_n$ such that $s_1 = w$, $s_n = z$, and $(s_j, s_j + 1) \in R'$ for each $j = 1, \ldots, n - 1$. Let m be the minimal integer such that there is such sequence of length m, and let S be any such sequence of length m.

Given a sequence S as described above, there is some j such that either $(s_j, s_{j+1}) = (x, y)$ or $(s_j, s_{j+1}) = (y, x)$ for some $1 < j < m - 1$; otherwise $(w, z) \in T(R) = R$, contradicting $(z, w) \in P(R')$. Suppose without loss of generality that $(s_j, s_{j+1}) = (x, y)$ for some $1 < j < m - 1$; then there is no $k \neq j$ such that $(s_k, s_{k+1}) = (y, x)$ or $(s_k, s_{k+1}) = (x, y)$, otherwise S would not be the shortest sequence from w to z such that every consecutive pair is in R'. Since $(z, w) \in P(R')$, we have $(z, w) \in R'$. Now consider the finite sequence

$y, s_{j+2}, \ldots, s_{m-1}, z, w, s_1, \ldots, s_{j-1}, x$. Note that every pair of consecutive elements of the sequence is in R' and is different from (x, y) and (y, x), so every pair of consecutive elements of the sequence is in R. But then $(y, x) \in T(R) = R$, contradicting $(x, y) \in N(R)$.

T is convergent.
Since $T(R) = R$, we know that R is transitive (Demuynck, 2009). Note that $(x, y) \in T(P(R))$ if and only if there is a sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in P(R)$. This implies that $T(P(R)) = P(R)$, and hence $P(T(P(R))) = P(R)$.

T is separability-preserving.
Take Z such that R is Z-separable. Take $(x, y) \in P(T(R))$.Then there is a sequence $x = s_1, \ldots, s_n = y$ such that $(s'_j, s_{j'+1}) \in R$ for all $j' = 1, \ldots, n-1$ and an index j such that $(s_j, s_{j+1}) \in P(R)$. Z-separability of R implies that there is $z \in Z$ such that $(s_j, z), (z, s_{j+1}) \in R$. Moreover, by construction, $(x, z), (z, y) \in T(R)$. Hence, $T(R)$ is also separable. □

Finally we define a revealed preference relation. Given a data set $E = (\mathcal{B}, C)$, let $(x, y) \in R_E$ if there is $B \in \mathcal{B}$ such that $x \in C(B)$ and $y \in B$. In the remainder of the proof, we show that there is a complete, transitive and separable preference relation R^* such that $\geq \preceq R^*$ and $R_E \subseteq R^*$ if and only if the data set satisfies GARP.

We start by showing that rationalizability is sufficient for the existence of a monotonic utility function that is maximized by the observed choices.

Lemma A.3. *The data set $E = (\mathcal{B}, C)$ is rationalizable if and only if there is a complete, transitive and separable R^* such that $\geq \preceq R^*$ and $R_E \subseteq R^*$.*

Proof. Suppose there is such R^* as stated in the lemma. The existence of a utility function that represents R^* is immediately guaranteed by a classical theorem from Debreu (1954). Moreover, since $\geq \preceq R^*$, the utility function is monotonic. Finally, $R_E \subseteq R^*$ implies that $(x, y) \in R^*$ for every $x \in C(B)$ and $y \in B$. This in turn implies that $u(x) \geq u(y)$ for every $x \in C(B)$ and $y \in B$. The converse statement follows by construction of a preference relation using the utility function. □

Next, we show that $\geq \preceq T(R_E \cup \geq)$ is necessary and sufficient for the existence of a complete, transitive and separable relation R^*.

Proposition A.1. *The data set $E = (\mathcal{B}, C)$ is rationalizable if and only if $\geq \preceq T(R_E \cup \geq)$.*

Proof of necessity in Proposition A.1. We proceed by contradiction. Suppose $\geq \not\subseteq T(R_E \cup \geq)$; i.e. there is $(x, y) \in >^{-1} \cap T(R_E \cup \geq)$. Hence, there is $(y, x) \in >$ and $(x, y) \in T(R_E \cup \geq)$. Consider a sequence of minimal length $(s_j, s_{j+1}) \in R_E \cup \geq$ for $j \in \{1, \dots, n-1\}$, with $x = s_1$ and $y = s_n$, that is used to add (x, y) to $T(R_E \cup \geq)$. If the choices are generated by a monotonic utility function, then $(s_j, s_{j+1}) \in R_E \cup \geq$ implies that $u(s_j) \geq u(s_{j+1})$. Hence, we can conclude that $u(x) \geq u(y)$, by transitivity of $\geq$. At the same time $y > x$, hence monotonicity implies that $u(y) > u(x)$, a contradiction. $\square$

Proving sufficiency requires introducing some additional auxiliary results.

Lemma A.4. *If $F : \mathcal{R} \to \mathcal{R}$ is a closed, monotone and algebraic function, then for any countable Z and every chain*

$$R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$$

such that $R_\alpha \in \mathcal{R}_F^Z$ for all α, we have $\cup_{\alpha \geq 0} R_\alpha \in \mathcal{R}_F^Z$.

Proof. Let $\overline{R} = \cup_{\alpha \geq 0} R_\alpha$. If the chain is finite $\overline{R}$ is itself an element (the last element) of the chain, so that $\overline{R} \in \mathcal{R}_F$ is immediate. Thus, we only need to be concerned with infinite chains. We know that each element R_α of the chain is F-consistent (from Lemma A.1) and Z-separable, and we only need to show that $\overline{R}$ is consistent and separable.

For consistency of $\overline{R}$, assume that there is $(x, y) \in F(\overline{R})$ but $(y, x) \in P(\overline{R})$. By construction of $\overline{R}$ we know that $(y, x) \in R_a$ for some relation R_a (with finite index a), and therefore $(y, x) \in R_\alpha$ for $\alpha \geq a$. Since F is algebraic, there is some finite relation $R' \subseteq \overline{R}$ such that $(x, y) \in F(R')$. Moreover, since R' is finite, there is some R_b (with finite index b) in the chain such that $R' \subseteq R_b$. Since F is monotonic, $F(R') \subseteq F(R_b)$ and therefore $(x, y) \in F(R_b)$. By monotonicity again, $(x, y) \in F(R_\alpha)$ for $\alpha \geq b$. Hence, there is a finite $c = \max\{a, b\}$ such that R_c is not consistent, a contradiction.

For Z-separability of $\overline{R}$, suppose that $(x, y) \in P(\overline{R})$. By construction of $\overline{R}$ we know that $(x, y) \in R_d$ for some relation R_d (with finite index d), and $(y, x) \notin R_\alpha$ for any α. Hence $(x, y) \in P(R_d)$. From Z-separability of R_d, there is $z \in Z$ such that $(x, z) \in R_d$ and $(z, y) \in R_d$. But then $(x, z) \in \overline{R}$ and $(z, y) \in \overline{R}$. $\square$

Lemma A.4 shows that in the partially ordered space of separable and consistent extensions of a preference relation every chain has a maximal element. Therefore, one can apply Zorn's Lemma to show that there is a maximal element in the partially ordered space. Next, we use that and prove that this maximal element has to be complete, transitive and monotone. First we need to ensure that $T(R_E \cup \geq)$ is a separable preference relation.

Lemma A.5. *Let $E = (\mathcal{B}, C)$ be a data set. If $\geq$ is separable, then $T(R_E \cup \geq)$ is separable.*

Proof. Denote by $Z_{\geq}$ the countable set with regards to which $\geq$ is separable and let $Z_{R_E} = \cup_{B \in \mathcal{B}} C(B)$. Note that Z_{R_E} is countable, and therefore so is Z. Recall that $(x, y) \in T(R_E \cup \geq)$ if and only if there is a sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R_E \cup \geq$. Take $(x, y) \in P(T(R_E \cup \geq))$, then there is a $j \in \{1, \ldots, n-1\}$ such that $(s_j, s_{j+1}) \in P(R_E)$ or $(s_j, s_{j+1}) \in >$. In the first case $z = s_j$ has to be a chosen point and therefore an element of Z_{R_E}; in the second case there is an element $z \in Z_{\geq}$, such that $s_j \geq z \geq s_{j+1}$. Finally, by construction of the transitive closure we know that $(x, z) \in T(R_E)$ and $(z, y) \in T(R_E)$. Therefore, $T(R_E)$ is a separable preference relation. $\square$

Proof of sufficiency in Proposition A.1. We prove the result for rational closures in general; recall that T is a transitive closure. Let F be a rational closure and suppose $R \in \mathcal{R}^Z_F$. Let

$$\Omega = \{R' \in \mathcal{R}^Z_F : F(R) \preceq R' \text{ and } F(P(R')) = P(R')\}$$

be the set of extensions of $F(R)$ that are themselves Z-separable, can be extended by F, and are invariant towards applying F to their strict part. Clearly, $\preceq$ is a partial order (reflexive, antisymmetric and transitive binary relation) on Ω and we just showed that every chain has an upper bound. Hence, every preference relation in Ω extends R and by Zorn's lemma, there is maximal element of Ω, which we can denote by R^*.

We claim that R^* is complete. To see this, assume on the contrary that $N(R^*) \neq \emptyset$. If it is not a fixed point of F, then $F(R^*)$ is an extension of R^* which is Z-separable (since F is separability preserving) and $P(F(P(R^*))) = P(R^*)$ (since F is convergent). If R^* is a fixed point of F the existence of such extension is guaranteed by Lemma A.4.

We claim further that R^* is a fixed point of $F(R)$, i.e. $F(R^*) = R^*$. To see this, note that $R^* \subseteq F(R^*)$ follows from the fact that $R^* \preceq F(R^*)$. To get the reverse, assume that $(x, y) \in F(R^*)$ and $(x, y) \notin R^*$. From completeness of R^*, (y, x) must be an element of $P(R^*)$ which contradicts $R^* \preceq F(R^*)$. Therefore, $F(R^*) \subseteq R^*$.

We are left to show that there is a utility function that represents $R^* = F(R^*)$. We just showed that R^* is complete. Since F is a rational closure, R^* is transitive as well. As we already showed $R^* \in \Omega \subseteq \mathcal{R}^Z_F$, it follows that R^* is Z-separable. $\square$

To complete the proof of Theorem 1 we are only left to show that GARP is equivalent to the consistency condition from Proposition A.1.

Lemma A.6. *The data set $E = (\mathcal{B}, C)$ satisfies GARP if and only if $\geq \preceq T(R_E \cup \geq)$.*

Proof. **(⇒)** Assume that there is $(x, y) \in >^{-1} \cap T(R_E \cup \geq)$ so there is a violation of consistency by lemma A.1. Consider the shortest sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R_E \cup \geq$ which adds (x, y) to $T(R_E \cup \geq)$. For all $j = 1, \ldots, n-1$ either $s_t \geq s_{j+1}$ or there is some B_j such that $x_j \in C(B_j)$ and $s_t \in B_j$. Since $\geq$ is a partial order, it is transitive, so if $x_j \geq x_{j+1}$ for all $j \in 1, \ldots, n-1$ we get $x \geq y$ which contradicts $y > x$. Thus, there must be some subsequence $s'_1, \ldots, s'_m = y$ such that $s'_k \in C(B_k)$, $s'_{k+1} \in B_k^{\geq}$, and $y > s_1$. Since $y > s_1$ and $y \in B_{s_m}^{\geq}$, we get $s_1 \in B_{s_m}^{>}$, which contradicts GARP.

(⇐) Assume there is a sequence $s_1, \ldots, s_n$ such that $s_{j+1} \in B_j^{\leq}$ for $j = 1, \ldots, n-1$ and $s_1 \in B_n^{>}$, so there is a violation of GARP. By construction of $T(R_E \cup \geq)$ we know that $(x_1, y) \in T(R_E \cup \geq)$ for every $y \in B_n^{\geq}$. However, $x_1 \in B_n^{>}$ implies that there is $y \in B_n^{\geq}$ such that $(y, x_1) \in >$. Hence, $(x_1, y) \in >^{-1} \cap T(R_E \cup \geq)$, which contradicts consistency. □

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