String Dualities

and

Gaugings of Supergravity

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Statement of Originality

The current thesis, apart from the Chapter 2 and the introductory parts of each chapter which have been included for pedagogical reasons, is based on original materials of the following publications:

I) [HR17] M. Henneaux and A. Ranjbar, "Timelike duality, M'-theory and an exotic form of the Englert solution," JHEP **1708** (2017) 012 [arXiv: 1706.06948 [hep-th]].

II) [Hen+18] M. Henneaux, B. Julia, V. Lekeu and A. Ranjbar, "A note on gaugings in four spacetime dimensions and electric-magnetic duality," Class. Quant. Grav. **35** (2018) no.3, 037001 [arXiv: 1709.06014 [hep-th]].

III) [Bar+18] G. Barnich, N. Boulanger, M. Henneaux, B. Julia, V. Lekeu and A. Ranjbar, "Deformations of vector-scalar models," JHEP **1802** (2018) 064 [arXiv: 1712.08126 [hep-th]].

IV) [RZon] A. Ranjbar and J. Zanelli, "Parallelizable hyperbolic manifolds in three and seven dimensions," in preparation.

Besides above publications, the following are the papers published during the author's PhD studies but have not been included in the thesis:

V) J. P. Babaro, G. Giribet and A. Ranjbar, "Conformal field theories from deformations of theories with W_n symmetry," Phys. Rev. D **94** (2016) no.8, 086001 [arXiv:1605.01933 [hep-th]].

VI) G. Giribet and A. Ranjbar, "Screening Stringy Horizons," Eur. Phys. J. C **75** (2015) no.10, 490 [arXiv:1504.05044 [hep-th]].

Abstract

This thesis is devoted to various questions connected with duality. It is composed of two parts.

The first part discusses some aspects of timelike T-duality. We explore the possibility of compactification of supergravity theories with various signatures (low energy limit of M-theories which are dual under timelike T-dualities) on parallelizable internal seven dimensional (pseudo-)spheres. We show that, beside the standard theory, only one of the dual theories known as M'-theory can admit such a solution. The effective four dimensional theory is non-supersymmetric and due to the presence of torsion the symmetry of seven dimensional (pseudo-)sphere breaks down to $Spin(3, 4)$.

In the second part, in an attempt to have a systematic discussion of gaugings in supergravity, we show the isomorphism between the space of local deformations of the appropriate zero coupling limit of the embedding tensor Lagrangian and that of the second-order scalar-vector Lagrangian, describing the bosonic sector of supergravity ignoring gravity, in a chosen duality frame determined by embedding tensors. We analyze the BV-BRST deformation of a class of scalar-vector coupled Lagrangians, which contains supergravity Lagrangians as examples, and find a set of constraints that guarantee the consistency of the deformations of the Lagrangians. We show in principle that for a large class of theories considered in this thesis, the only deformations are those of the Yang-Mills type associated with a subgroup of the rigid symmetries.

Résumé

Cette thèse est consacrée à diverses questions de dualité. Elle est composée de deux parties.

La première partie discute quelques aspects de la T-dualité de genre temps. Nous explorons la possibilité de compactifier des théories de supergravité de différentes signatures (limites à basse énergie de la théorie M, duales selon les transformations de T-dualité de genre temps) sur des pseudo-sphères internes parallélisables de dimension sept. Nous montrons que, mise à part la théorie M standard, une seule des théories duales, dite théorie M', admet un telle réduction. La théorie effective en quatre dimensions n'est pas supersymétrique. De plus, la symétrie de la pseudo-sphère est réduite au groupe $Spin(3,4)$. Ceci est dû à la présence de "torsion aplatissante".

Dans la deuxième partie, nous entamons une discussion systématique des jaugeages en supergravité. Pour ce faire, nous démontrons d'abord l'isomorphisme entre l'espace des déformations locales du Lagrangien du formalisme du tenseur de plongement (dans la limite de couplage nul appropriée) avec celui du Lagrangien du second ordre usuel dans le repère de dualité spécifique déterminé par ce tenseur de plongement. Ce dernier décrit le secteur bosonique des théories de supergravité (en négligeant la métrique). Ensuite, nons analysons la déformation BV-BRST d'une classe très générale de Lagrangiens couplant des champs vectoriels et scalaires, qui contient les Lagrangiens de supergravité comme cas particuliers, et trouvons un ensemble de contraintes garantissant la cohérence de ces déformations. Nous démontrons que, pour une large classe de Lagrangiens considérés dans cette thèse, les seules déformations possibles sont celles de type Yang-Mills, associées à un sous-groupe du groupe de symétries rigides.

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Chapter 1

Introduction

This thesis is devoted to various questions connected with duality. It is composed of two parts. The first part explores some aspects of timelike T-duality in the context of comapctifications of the maximal supergravity in eleven-dimensions and its timelike T-duals. The second part studies the consistent deformations ("gaugings") of extended supergravities and sheds new light on the role played by the electric-magnetic duality in that context.

The question of why spacetime has Lorentzian signature $(-, +, +, ..., +)$ with one and only one time has been repeatedly raised by many authors in the literature (see [Bar01] and references therein). Duality provides a tool to address this question since the timelike T-dualities considered by Hull [Hul98a; Hul98b] change the spacetime signature. In the case of M-theory, one can generate through timelike dualities various "exotic" duals (M^*-theory and M'-theory as well as the reversed signature theories), which possess a different number of "times". Part I of the thesis studies a special class of solutions of the corresponding exotic supergravities. More preciesly, we investigate compactifications of these theories to four dimensions on internal seven-manifolds that are parallelizable.

The condition that allows a Riemannian geometry to admit an absolute parallelism is a classic question which was answered by Cartan and Schouten [CS26b; CS26a] in the case of Euclidean signature, concluding that the only possibilities for sphere topologies are S^3 and S^7. They also described curvature-free connections on Lie groups and as a result exhibited parallelisms on group manifolds. They then generalized it to give a local description of parallelism (in terms of curvature-free metric-compatible connections in the presence of torsion) on Riemannian manifolds which are products of flat manifolds, compact simple group manifolds, and 7-spheres. Moreover, the parallelism of the 3-sphere, which was

before understood as an isolated phenomenon, was provided as an example.

There is a close relation between parallelizability of spheres and division algebras. It is particularly interesting since it explains the reason behind the specific form [CS26b; CS26a] of the solutions that was found by Cartan and Schouten. It also allows one to go one step forward and to extend the result of Cartan and Schouten to pseudo-Riemannian manifolds [Wol72a; Wol72b]. This will lead to the result that there exist a parallelism on the pseudo-Riemannian manifolds which are of the form of the product of a flat manifold, irreducible symmetric spaces and the symmetric coset spaces $S^7 = SO(8)/SO(7)$, $S^{3,4} = SO(4,4)/SO(3,4)$ and $SO(8,\mathbb{C})/SO(7,\mathbb{C})$. Here "irreducible" means that the symmetric spaces are not locally product spaces. Equivalently, it means that the infinitesimal holonomy group acts irreducibly on the tangent space and therefore the holonomy group is given by the Berger classification of irreducible holonomy groups [Ber55; Sim62].

We are interested in the case of seven dimensions, which stands out also as a result of Adams theorem [Ada60] that relates S^0, S^1, S^3 and S^7 to normed division algebras. We will briefly discuss the three-dimensional case to set the stage for a more difficult case of seven dimensions. The generalization of Cartan-Schouten theorem to pseudo-Riemannian manifold will teach us that in three and seven dimensions (beside S^3 and S^7 already noted by Cartan and Schouten), respectively AdS$_3$ manifold and $S^{3,4} = SO(4,4)/SO(3,4)$ (or $H^{4,3} = SO(4,4)/SO(4,3)$) are also parallelizable.

Seven dimensions have been shown to play an important role in supergravity or M-theory, where one finds an effective theory of supergravity in four dimensions from compactification of the eleven-dimensional supergravity on a seven-manifold. If this seven-manifold is toroidal then the compactification will result in an ungauged theory in four-dimensional flat background [CJ78; CJ79]. The Freund-Rubin compactification [FR80] on S^7 will lead to a four-dimensional $SO(8)$ gauged supergravity in AdS$_4$ spacetime [WN82a]. The Freund-Rubin compactification is the only flux compactification of supergravity which has the maximum number of Killing spinors or, in other words, it preserves all supercharges. There is another type of flux compactification where one turns on other fluxes and therefore changes the internal geometry, which breaks part of the supersymmetry of the resulting theory. For example, G_2 compactification of eleven-dimensional supergravity [AW03; Duf02] preserves one Killing spinor, which leads to at most $\mathcal{N} = 1$ supergravity in four dimensions. The Englert solution [Eng82] is a specific solution of G_2 compactification where supersymmetry is completely broken. This solution is obtained by compactifying on parallelizable S^7.

Since S^7 plays an important role in this scenario, it might seem interesting to see if any role is played by $S^{3,4}$ for example in the compactification. At first sight, this looks impossible since we cannot compactify a theory in $10 + 1$ dimensions

on a manifold with more than one time direction. However, as we recalled above it turns out that there is a signature-changing duality in string theory [Hul98a; Hul98b]. This duality is called timelike T-duality as a result of which one finds an orbit of equivalent theories under this duality. The orbit involves three theories formulated in the same dimension but with different signatures. The theories in reversed signatures also belong to this orbit.

We will show that the compactification on parallelizable $S^{3,4}$ is possible if one works in the picture described by one of these dual theories, known as M'-theory [HR17]. This solution, similarly to the Englert solution, will result in a non-supersymmetric theory in AdS$_4$ background. This can be understood as a spontaneous compactification on $S^{3,4}$ with $G_{2,2}$ invariant fluxes. The compactification on parallelizable seven-manifolds for the other supergravity theories in eleven dimensions (the low energy limit of M- and M^*-theories) is not possible.

In Part II we focus on electric-magnetic duality of supergravity in four dimensions [Wit01]. Electric-magnetic duality is particularly interesting since the gauge group of a gauged supergravity in its second-order formulation is a subgroup of electric-magnetic duality symmetry group and moreover the available gaugings depend on the duality frame in which the theory is formulated[WN82b]. However, the electric-magnetic duality symmetry of a supergravity (or in general a scalar-vector coupled model) in the second-order formulation is not manifest off-shell since part of this symmetry acts non-locally (in space) on fields [DT76; Des82; BH10; GZ81]. As a result, the gauge group is normally a subgroup of an electric subgroup of electric-magnetic duality which acts locally on fields. On the other hand, the first-order formulation is manifestly duality invariant. The main purpose of Part II is to have a systematic discussion of the gaugings of a supergravity theory, or in other words to find the consistent deformations of an ungauged supergravity. We will show that even though the first-order formulation is manifestly duality invariant but due to the rigidity of the first-order formulation [Hen+18] it is not suited for the discussion of the deformations. Therefore, in order to discuss the consistent deformations of supergravity, it is inevitable to work in the second-order formulation.

The embedding tensor formalism introduced by [WST03; WST05; WST07; Sam08; Tri17], is a procedure of gauging a supergravity theory by promoting abelian vector fields to non-abelian ones, replacing the ordinary derivatives by covariant ones and adding extra interactions in order to restore the symmetry. In this formalism the embedding of the gauge group in the global symmetry group is described through constant parameters given by the so-called embedding tensor; hence the gauged Lagrangian depends on the embedding tensor which must satisfy a set of constraints. The embedding tensor formalism will provide a gauged Lagrangian for the supergravity theories where one can see that the possible deformations are those which require changing the gauge algebra. However, there

are in general other types of deformations where even though the gauge symmetry is deformed but the gauge algebra remains unchanged. It is then an interesting question to explore the possibility of gauging the symmetries corresponding to these type of deformations [BBH00]. In order to do that we use the BV-BRST deformation formalism [BV81; BV83a; BH93; Hen98] which provides precise tools to answer this question in a systematic manner.

We will first show that [Hen+18] the space of deformations of embedding tensor formalism is isomorphic to the space of deformation of a scalar-vector Lagrangian (describing the bosonic action of supergravity without spin-2 field) in a chosen duality frame determined by the embedding tensor. The infinitesimal local deformations are controlled by the cohomology of the BRST differential at ghost number zero in the space of local functionals. The BRST cohomology at ghost number one is related to the possible anomalies in the theory. In fact, we show that the BRST cohomology at all ghost numbers is equivalent for the embedding tensor Lagrangian and the one of scalar-vector coupled model. Then, performing a thorough analysis of BV-BRST deformation [Bar+18] of the scalar-vector Lagrangian, we will obtain all possible symmetry deformations controlled by the BRST cohomology at ghost number minus one. It includes the symmetries of both types; those which deform the gauge algebra and the types which do not. The consistent deformations then is implemented by a set of constraints (similarly to the embedding tensor formalism) and one finds that the only symmetries which satisfy these conditions are of the type already considered in the embedding tensor formalism. The other types of symmetries even though are part of the electric symmetry group but they cannot be gauged. We will demonstrate this explicitly in different examples including the one of $\mathcal{N} = 4$ supergravity in four dimensions.

The thesis is organized as follows:

In Chapter 2 we provide definitions regarding the notion of parallelism and the relation between parallelizability and division algebras. In Chapter 3 we use the division algebras to show the parallelizability on specific seven-dimensional topologies rediscovering the results of Cartan-Schouten and Wolf.

In Chapter 4, we start by reviewing the effect of time-like T-duality and the structure of dual M-theories. Then, we show all possible solutions of compact-ification of M-, M'- and M^*-theories on seven-dimensional (pseudo-)sphere or (pseudo-)hyperbolic manifolds in both cases of Freund-Rubin compactification, described by Hull [HK98], and the Englert compactification where the internal manifolds are the parallelizable ones of Chapter 3. We will show that the reduction of the $6 + 5$- dimensional supergravity, low energy limit of M'-theory, on the seven- dimensional parallelizable pseudo-sphere $S^{3,4}$, similarly to the reduction of standard supergravity on the parallelizable 7-sphere, will result in well defined theories in AdS_4 background. The difference with the reduction on a 'round' pseudo-sphere is that in the latter the dimensionally reduced four- dimensional

theories are maximally supersymmetric with $SO(8)$ gauge symmetry while the former solutions are non-supersymmetric with $Spin(3,4)$ gauge symmetry. The presence of fluxes due to the flattening torsions on a parallelizable manifold breaks the supersymmetry.

In Chapter 5, we review the electric-magnetic duality and explain the enhancement of the symmetry in the presence of scalars. In Chapter 6, we give a quick review of embedding tensor formalism and then we describe how the scalar-vector Lagrangian describing the maximal supergravity in four dimensions is captured as the undeformed limit of the embedding tensor gauged Lagrangian in a frame parameterized by the embedding tensor coefficients.

In Chapter 7, after explaining the BV-BRST deformation formalism we go on to prove the isomorphism between the space of local deformations of embedding tensor Lagrangian and the described scalar-vector coupled Lagrangian and therefore complete the analysis of previous chapter. In Section 7.5, we explicitly show the extra fields present in the embedding tensor formalism will disappear since whether they are auxiliary fields (like two-forms) or of pure gauge type.

In Chapter 8, we present a detailed computation of BRST cohomology in different ghost degrees analyzing the global symmetries, infinitesimal deformations and anomalies described respectively by $H^{-1}(s|d)$, $H^0(s|d)$ and $H^1(s|d)$. We provide then various examples to demonstrate the solutions to the obstruction equations for the consistent deformations and to show which symmetries can be gauged. We specifically show this for $\mathcal{N} = 4$ supergravity in four dimensions where one can in fact solve the constraints in practice. In principle, the result obtained in this chapter can be used for the gauging of supergravitis with $\mathcal{N} > 2$ in four dimensions.

Finally, we will give a summary of all results in Chapter 9. The future perspective of the works along the line of thoughts of this thesis is also discussed.

Four appendices were included at the end. Appendix A contains the list of all notations used in the thesis. Appendix B provides a quick review of Cayley-Dickson construction of division algebras and definitions of the algebra of octonions and split octonions relevant to Chapter 2. Appendix C includes the definitions of (pseudo-)sphere and (pseudo-)hyperbolic spaces and a presentation of two infinite families of parallelism (similarly to the ones of Cartan and Schouten) of the pseudo-sphere $S^{3,4}$. We also explained the relation between different groups $G_{2,2}$, $G^*_{2,2}$, $Spin(3,4)$ and $Spin^+(3,4)$ which are relevant in describing the parallelized $S^{3,4}$. The final appendix, Appendix D contains a proof of a critical relation being used in Chapter 8.

Part I

Parallelizable manifolds

Chapter 2

Parallelizability

2.1 Absolute Parallelism

In this chapter we provide some definitions and theorems necessary to understand the mathematics behind the next chapter.

We first start with explaining the meaning of absolute parallelism where we use the definition given by Wolf [Wol72a]. An **absolute parallelism** on a differentiable manifold $\mathcal{M}$, is an isomorphism map ϕ between tangent spaces at any two points $x, y \in \mathcal{M}$ which does not depend on additional choices. More precisely,

$$\phi = \{\phi_{yx}\}, \quad \phi_{yx} : T_x\mathcal{M} \to T_y\mathcal{M} \quad \forall x, y \in \mathcal{M} \tag{2.1}$$

which satisfies the consistency condition

$$\phi_{zy}\phi_{yx} = \phi_{zx}, \qquad \phi_{xx} = id. \quad \forall x, y, z \in \mathcal{M} \tag{2.2}$$

and the regularity condition that guarantees the smoothness of the isomorphism ϕ. Given ϕ, one says that the tangent vector $\xi_x \in T_x\mathcal{M}$ and the tangent vector $\xi_y \in T_y\mathcal{M}$ are parallel iff $\xi_y = \phi_{yx}\xi_x$.

On a simply connected differentiable manifold $\mathcal{M}$ one can show that there is a one-to-one correspondence between an absolute parallelism on $\mathcal{M}$, smooth trivialization of the frame bundle on $\mathcal{M}$ and the existence of flat connections (zero curvature) on the frame bundle, see [Wol72a] for the proof.

Cartan and Schouten [CS26b; CS26a] described curvature-free connections on Lie groups and as a result exhibited parallelisms on group manifolds. They then generalized it to provide a description of parallelism (in terms of curvature-free metric-compatible connections with torsion) on Riemannian manifolds which are products of flat manifolds, compact simple group manifolds and 7-spheres.

Later, Adams [Ada60] using a metric-free topological description of parallelism (i.e. existence of a global section on the frame bundle) showed that the only parallelizable spheres are S^0, S^1, S^3 and S^7. The 2-sphere is the simplest example where the parallelism fails to hold and one can still use the geometrical tools to see this. It is known that one cannot define a coordinate patch that covers the 2-sphere thoroughly. In fact, using the sterographic map it can be seen that one needs at least two different patches in order to cover the whole 2-sphere.

In the following, we shall focus on the relation between parallelizable spheres and normed division algebras. In the next section we define the normed division algebras and explain what will be necessary to use later on. We state a number of theorems without including the proofs since it would be a digression from our presentation.

2.2   Division Algebras

Consider an algebra A over the field F, that is A is a vector space equipped with a bilinear map $l : A \times A \to A$ and a nonzero unit element $e \in A$. Consider any two arbitrary elements a and b of the algebra A. In addition, we assume that a is a non-zero element of A. If linear equations $ax = b$ and $ya = b$ are uniquely solved for $x, y \in A$, then the algebra A is called a **division algebra** over the field F.

In other words, A is a division algebra if given $a, b \in A$ with $ab = 0$, then either $a = 0$ or $b = 0$. Equivalently, A is a division algebra if the operations of left and right multiplication by any nonzero element are invertible.

Whenever F is real, then A is a real division algebra. For example we will see later on in this section that real quaternion and octonion algebras are real division algebras.

A **normed** division algebra is a division algebra which is also a Banach algebra, i.e. it is a normed vector space with a multiplicative property $||ab|| = ||a|| \, ||b||$.

Associativity and Alternative Algebra

Given an algebra A, it need not be necessarily associative. One can define a trilinear map, called associator,

$$(x, y, z) = x(yz) - (xy)z \qquad x, y, z \in A, \tag{2.3}$$

which measures the non-associativity of the algebra. For an associative algebra, the associator vanishes. If the associator of an algebra is totally antisymmetric,

i.e. $(x, y, z) = -(x, z, y) = -(y, x, z) = -(z, y, x)$, then the algebra is called an **alternative** algebra. An alternative algebra is equivalently defined [Sch17; Sch66] as

$$x^2 y = x(xy), \tag{2.4}$$
$$yx^2 = (yx)x, \tag{2.5}$$

for any two elements x, y of the algebra. One very important feature of any alternative algebra is that even though the algebra itself is non-associative for elements $x, y, z \in A$, any subalgebra of an alternative algebra generated from any two elements $x, y \in A$ is associative [Sch17; Sch66]. For any alternative algebra, the following identities always hold for any three elements $x, y, z \in A$:

$$(xyx)z = x(y(xz)), \tag{2.6}$$
$$z(xyx) = ((zx)y)x, \tag{2.7}$$
$$(xy)(zx) = x(yz)x. \tag{2.8}$$

These relations are known as **Moufang identities** and those algebras that satisfy these identities are known as **Malcev algebras** [Sch17; Sch66].

Dimension of Normed Division Algebras

Proposition 2.1. *Let A be a non-associative algebra and x a fixed element of A. Then the left multiplication L_x and the right multiplication R_x are linear operators in the vector space A, i.e., $L_x, R_x : A \to A$, defined by*

$$L_x y = xy, \quad R_x y = yx, \tag{2.9}$$

for all $y \in A$.

Lemma 2.1. For any $x \in A$, we have $L_x L_{\bar{x}} = (x, x)I$ where I is the identity multiplication operator defined by $Ix = x$, $\bar{x}$ is the complex conjugate of x defined as $\bar{x} = 2(x, e)e - x$ and $(., .)$ is the scalar product.

Proof. For the proof of the lemma see Section 3.1 and Proposition 1 in Chapter one of [Oku95]. □

Theorem 2.2. (Bott-Milnor, Kervaire [Hur98; BM58; Ker58]) The dimension of a normed division algebra is restricted to $1, 2, 4$, or 8.

Proof. For the proof of theorem see Chapter three of [Oku95]. For completeness, a sketch of the proof is given in Appendix B. □

This theorem is particularly interesting since it relates the dimension of irreducible representations of Clifford algebra on $N - 1$ dimensional vector space to N the dimension of a normed division algebra, see Appendix B.

Quaternions and Octonions

The alternative property of an algebra is important. In fact, the only alternative division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ [Sch17; Sch66; Zor30]. This together with Theorem 2.2 will result in the following theorem:

Theorem 2.3. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only normed divison algebras.

Proof. The proof of this theorem requires identities for normed divison algebras which we do not repeat here. See Section 3.1 of [Oku95] for the complete proof. ☐

$\mathbb{R}$ and $\mathbb{C}$ are real and complex vector spaces respectively, $\mathbb{H}$ is the quaternion algebra and $\mathbb{O}$ is the octonion algebra. These division algebras have peculiar properties; $\mathbb{R}$ is an ordered, commutative and associative algebra, $\mathbb{C}$ is a commutative and associative algebra, $\mathbb{H}$ is an associative but non-commutative algebra and $\mathbb{O}$ is a non-commutative, non-associative algebra. However, they have one common feature that makes them distinct and that all these algebras are alternative. This is indeed the reason that the sequence of normed division algebras stops at octonions [Bae02].

In Appendix B, we give a short review of Cayley-Dickson construction of division algebras, in particular we provide the example of the construction of octonions.

As the final remark, we mention that it has been shown by Adams that for an n dimensional normed division algebra, S^{n-1} is parallelizable [Ada60], for an illustrative proof see [Bae02]. This is an important statement that we will use repeatedly in the next chapter.

Chapter 3

Lorentz Flatness

In the previous chapter, we pointed out that parallelism is in one-to-one correspondence with the existence of a flat connection. This can be interpreted as path-independence of parallel transport. It means that transporting a vector from the point $x \in \mathcal{M}$ to another point $z \in \mathcal{M}$ always leads to the same vector at z no matter what path is chosen. When there is no torsion then the condition that the parallel transport is path-independent is satisfied if the Riemann curvature – defined in (3.6) below as the curvature of the Levi-Civita connection – vanishes. Therefore, in the absence of torsion a manifold is parallelizable if its Riemann curvature vanishes. Flat (simply connected) space is the simplest example of a parallelizable manifold where the curvature vanishes and the Levi-Civita connection is the flat affine connection that provides a path-independent parallel transport.

However, in the presence of torsion, the parallel transport is not defined by the Levi-Civita connection and the vanishing of its curvature is not necessary for parallelism to hold. Torsion introduces new possibilities. This can be most easily discussed in local Lorentz frames where the metric takes the Minkowskian form. Parallel transport is captured by the differential equation

$$Du^a \equiv dx^\mu[\partial_\mu u^a + \omega^a{}_{b\mu}(x)u^b] = 0\,, \tag{3.1}$$

for a tangent vector $u^a(x)$ at x. This equation is a consequence of the fact that u^a transforms as a vector under the Lorentz group, i.e. $\delta u^a(x) = dx^\mu \omega^a{}_{b\mu}(x)u^b$.

This can be seen as $u^a(x + dx)$ obtained by parallel transport from $u^a(x)$ where Du^a is the covariant derivative of u^a. The 1-form $\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu$ is called the Lorentz connection because we consider local orthonormal frames, and the condition that the covariant derivative of the Lorentz metric vanishes identically $(D\eta^{ab} = 0)$ implies that $\omega_{ab} = -\omega_{ba}$. If the expression (3.1) is to have the same

meaning in any Lorentz frame, Du^a should transform under the Lorentz group in the same representation as u^a (see, e.g., [Sch80; GS89]).

The integrability condition for (3.1) or in other words the path-independence of the parallel transport is governed by

$$DDu^a \equiv R^a{}_b u^b = 0 \,, \tag{3.2}$$

where the curvature two-form $R^a{}_b$ is defined as[1]

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b \,. \tag{3.3}$$

If the condition (3.2) is to be fulfilled for an arbitrary u^a everywhere in an open region $\mathcal{U}$, the curvature $R^a{}_b$ must vanish identically in that region. Therefore a parallelizable manifold requires its Lorentz curvature $R^a{}_b$ to vanish. It is equivalent to the existence of a flat Lorentzian connection on the manifold.

3.1 Flatness of a (Pseudo-)Riemannian Manifold

For any $p, q \geq 0$, consider the manifold $R^{(p,q)}$ which is equipped with a symmetric non-degenerate bilinear form $b_{(p,q)}$ defined as

$$b_{p,q} = +\sum_{i=1}^{p} dx_i^2 - \sum_{j=1}^{q} dx_j^2 \,. \tag{3.4}$$

This bilinear form is invariant under the adjoint action of the $SO(p,q)$ group. Now consider an n dimensional $(p+q-1=n)$ manifold $\mathcal{M}$ defined by $SO(p,q)$ invariant equation

$$\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_j^2 = \pm 1. \tag{3.5}$$

The bilinear form on $R^{(p,q)}$ induces a generalized Lorentz invariant metric η_{ab} on the tangent space $T_x\mathcal{M}$ at any point $x \in \mathcal{M}$. The induced metric is diagonal $\eta_{ab} = diag(+,...,+;-,...,-)$ with the signature

- $(p-1, q)$ and $\mathcal{M} = \frac{SO(p,q)}{SO(p-1,q)}$ with a constant positive curvature,

- $(p, q-1)$ and $\mathcal{M} = \frac{SO(p,q)}{SO(p,q-1)}$ with a constant negative curvature.

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1 From now on wedge products ($\wedge$) of exterior forms will be implicitly understood.

$\mathcal{M}$ is also equipped with a generalized Lorentz connection one-form $\omega^a{}_b$ (with $\omega_{ab} = -\omega_{ba}$) and a local orthonormal basis $e^a = e^a_\mu dx^\mu$ that provides the metric, $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. The indices $a, b, ...$ are tangent space indices and run over $1, 2, ..., n$, while $\mu, \nu, ...$ are coordinate indices.

The geometry of a curved (pseudo-)Riemannian manifold with metric $g_{\mu\nu}$ is characterized by the Riemann curvature two-form,

$$\mathcal{R}^\alpha{}_\beta = d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \,, \tag{3.6}$$

where $\Gamma^\alpha{}_\beta = \Gamma^\alpha{}_{\beta\mu} dx^\mu$ is the Levi-Civita connection one-form. This Riemannian curvature is similar to –but not necessarily equivalent to– the curvature (3.3) defined for the Lorentz connection $\omega^a{}_b$. The difference stems from the fact that $\Gamma^\alpha{}_\beta$ is defined by the metric only while $\omega^a{}_b$ may involve a non-trivial torsion. The Levi-Civita connection is determined by the torsion-free part of the Lorentz connection, denoted by $\bar{\omega}^a{}_b$, and is defined by the condition

$$de^a + \bar{\omega}^a{}_b e^b \equiv 0 \implies \bar{\omega}^a{}_b = e^a_\nu [\delta^\nu_\mu d + \Gamma^\nu{}_\mu] E^\mu_b \,, \tag{3.7}$$

where E^μ_a is the inverse vielbein. The two connections $\omega^a{}_b$ and $\Gamma^\alpha{}_\beta$ define different notions of parallelism that give rise to different curvature forms, $R^a{}_b$ and $\mathcal{R}^\alpha{}_\beta$, respectively.

In Minkowski space, there is a torsion-free Lorentz connection $\bar{\omega}^a{}_b$. In this case, the connection is trivial, in the sense that the holonomy group of the connection is trivial in an obvious manner since the tangent bundle is naturally trivialized on Minkowski space. It is equivalent to say that the Riemann curvature vanishes. The question we examine here is what (pseudo-)Riemannian manifolds have flat connection. As we have already seen the Adams' theorem [Ada60] states that for Euclidean signature, the spheres S^1, S^3 and S^7 are parallelizable, that is, they admit globally defined orthonormal frames. The connection for the corresponding $SO(n)$ rotation groups have trivial holonomy group and the corresponding curvatures vanish. We would like to explore the extension of this result for other metric manifolds of arbitrary signature.

One way to address this question is to look for a necessary condition for flatness, which is equivalent to the statement

$$R^a{}_b u^b_{(\mu)} = 0, \quad \mu = 1, \cdots, n \tag{3.8}$$

where $\{u^b_{(\mu)}\}$ is a family of linearly independent vectors. In a smooth manifold, a weaker but necessary condition is

$$R^a{}_b \wedge e^b = 0, \tag{3.9}$$

which would be the case if the torsion two-form is covariantly constant,

$$T^a \equiv De^a = de^a + \omega^a{}_b e^b \implies DT^a = R^a{}_b e^b = 0. \tag{3.10}$$

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Hence, in a Lorentz-flat manifold, the torsion may not be zero but it must be covariantly constant, $DT^a = 0$. The torsion two-form can be expressed in a basis of local orthonormal frames as

$$T^a = \tau f^a{}_{bc} e^b e^c + \rho e^a \,, \tag{3.11}$$

where τ is a scalar zero-form, $f^a{}_{bc}$ is a zero-form antisymmetric in the lower indices, and ρ is a scalar one-form. Then, the necessary condition for Lorentz flatness (3.10) reads

$$(Df^a{}_{bc})\tau e^b e^c + 2\tau f^a{}_{bc} f^b{}_{df} e^c e^d e^f + (d\tau + \tau\rho) f^a{}_{bc} e^b e^c + d\rho\, e^a = 0\,, \tag{3.12}$$

A sufficient condition for solving this equation is to assume for the functions multiplying e^a, $e^a e^b$, and $e^a e^b e^c$ to be independent and therefore

$$d\rho = 0, \tag{3.13}$$

$$Df^a{}_{bc} + (d\log[\tau] + \rho) f^a{}_{bc} = 0, \tag{3.14}$$

$$f^a{}_{b[c} f^b{}_{df]} = 0. \tag{3.15}$$

By Poincaré's lemma, (3.13) means that in an open region $\rho = d\lambda$ and (3.14) becomes $D\tilde{f}^a{}_{bc} = 0$, where $\tilde{f}^a{}_{bc} \equiv \tau e^\lambda f^a{}_{bc}$. In other words, the coefficients $\{f^a{}_{bc}\}$ are, up to a conformal factor, covariantly constant. Moreover, since the connection is flat it can always be trivialized in an open region by a local Lorentz transformation and without loss of generality one can take $f^a{}_{bc} = \Omega(x)\bar{f}^a{}_{bc}$, where $d\bar{f}^a{}_{bc} = 0$. As we shall see in the next section, it is very easy to achieve these conditions in three dimensions, while the situation in seven dimensions is less straightforward but as it is discussed in Section (3.2) non-trivial solutions still exist.

3.1.1   Three Dimensional Example

As explained before, conditions (3.13) and (3.14) imply that $f^a{}_{bc} = \Omega(x)\tilde{f}^a{}_{bc}$, where $D\tilde{f}^a{}_{bc} = 0$, which is automatically satisfied if $\tilde{f}^a{}_{bc}$ are the structure constants of the Lorentz algebra. This means that $\tilde{f}$ is an invariant tensor of the Lorentz group, in which case (3.15) is the Jacobi identity. For $n = 3$ this means that modulo local rescalings, $f^a{}_{bc} \equiv \epsilon^a{}_{bc}$ where ϵ_{abc} is the Levi-Civita tensor defined with the convention[2] $\epsilon_{012} = +1$.

Consequently, in three-dimensions the torsion two-form,

$$T^a = \tau \epsilon^a{}_{bc} e^b e^c - d[\log\tau] e^a \,, \tag{3.16}$$

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2 The Lorentz indices of $a, b, c, ...$ are raised and lowered with η^{ab} and η_{ab}.

is covariantly constant by construction. By rescaling the frame as $\tilde{e}^a \equiv \lambda \tau e^a$ with λ an arbitrary constant, the torsion can be written (after dropping the tildes) as

$$T^a = \lambda \epsilon^a{}_{bc} e^b e^c. \tag{3.17}$$

The equation $De^a = \lambda \epsilon^a{}_{bc} e^b e^c$ is a condition for the metric structure of the manifold. But, what does this imply for the Riemannian tensor?

In order to address this question, let us separate the torsion-free part of the Lorentz connection $\bar{\omega}^a{}_b$ and the *contorsion* $\kappa^a{}_b$,

$$\omega^a{}_b = \bar{\omega}^a{}_b + \kappa^a{}_b, \tag{3.18}$$

where $\bar{\omega}^a{}_b$ is defined by the torsion-free condition,

$$\bar{D}e^a = de^a + \bar{\omega}^a{}_b e^b \equiv 0, \tag{3.19}$$

while the contorsion contains the information about the torsion, $T^a = \kappa^a{}_b e^b$. Equation (3.19) determines $\bar{\omega}$ as a function of the metric structure of the manifold, and corresponds to the projection onto the tangent space basis of the Levi-Civita connection. The Lorentzian curvature splits as

$$R^a{}_b = \bar{R}^a{}_b + \bar{D}\kappa^a{}_b + \kappa^a{}_c \kappa^c{}_b, \tag{3.20}$$

where $\bar{R}^a{}_b = d\bar{\omega}^a{}_b + \bar{\omega}^a{}_c \bar{\omega}^c{}_b$ is the projection onto the tangent space of the Riemann curvature tensor.

In three dimensions, (3.17) means that $\kappa^a{}_b = -\lambda \epsilon^a{}_{bc} e^c$, which in turn implies that $\bar{D}\kappa^a{}_b = 0$ and

$$\kappa^a{}_c \kappa^c{}_b = \lambda^2 \, det(\eta) \, \delta^a_f \, \eta_{bd} \, e^d e^f, \tag{3.21}$$

where we have used

$$\epsilon^a{}_{cd} \, \epsilon^c{}_{bf} = det(\eta) \left[\delta^a_f \, \eta_{bd} - \delta^a_b \, \eta_{df} \right]. \tag{3.22}$$

Then, (3.20) reads

$$R^a{}_b = \bar{R}^a{}_b - \lambda^2 \, det(\eta) \, \eta_{bd} \, e^a e^d. \tag{3.23}$$

Therefore, the Riemann curvature of a three-dimensional Lorentz-flat manifold is

$$\bar{R}^a{}_b = \lambda^2 \, det(\eta) \, \eta_{bd} \, e^a e^d. \tag{3.24}$$

This relation implies that for $\lambda \neq 0$ there are two distinct possibilities[3] for a three-dimensional manifold admitting a Lorentz-flat connection:

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3 There are two other options that one obtains from the solutions shown here just by a metric transformation which flips the sign of the metric and that too changes the sign of the curvature.

1. The metric η_{ab} is Riemannian, $ds^2 = \sum_{i=0}^{2} dx_i^2$. Then $det(\eta) = +1$ and the three-dimensional manifold is a constant positive curvature manifold, i.e. 3-sphere (of finite radius) S^3, in agreement with [Ada60].

2. The metric η_{ab} is pseudo-Riemannian, $ds^2 = -dx_0^2 + \sum_{i=1}^{2} dx_i^2$ then $det(\eta) = -1$ and the three-dimensional manifold has constant negative curvature, i.e. AdS$_3$. This is the case considered in [Alv+14].

There is also the degenerate solution $\lambda = 0$: flat Riemannian manifolds R^3 and pseudo-Riemannian flat manifolds $R^{2,1}$ and $R^{1,2}$.

3.1.2 A General Discussion

As we discussed before, an n dimensional manifold $\mathcal{M}$ is called parallelizable (i.e. it admits framing) if one can find n globally-defined linearly independent tangent vector fields on the manifold. This is always possible if there exists a flat connection in the frame bundle of the manifold. One can make sense of it as integrability of $Du^a = 0$ where u^a is a tangent vector defined at each point on the manifold.

Given a manifold with a connection in the fiber bundle $\omega^a{}_b$ and a local frame e^a at each point (vielbein)[4], the equations (3.3) and (3.10) can be viewed as structure equations for the connection [Che66; Mil58; KN65]

$$de^a = -\omega^a{}_b e^b + T^a, \quad d\omega^a{}_b = -\omega^a{}_c \omega^c{}_b + R^a{}_b. \tag{3.25}$$

Assuming there is a torsion-free connection, $\bar{\omega}^a{}_b$, then $de^a = -\bar{\omega}^a{}_b e^b$. From here, we can define the torsion 2-form T^a as

$$T^a = f^a{}_{bc} e^b e^c, \tag{3.26}$$

where $f^a{}_{bc} e^c$ in fact plays the role of the contorsion 1-form. Here the coefficient $f^a{}_{bc}$ is a zero-form Lorentz tensor whose main function is to map two vectors into a third one. In other words, it defines an antisymmetric bilinear map of the tangent space $T\mathcal{M}$ into itself,

$$T\mathcal{M} \otimes T\mathcal{M} \to T\mathcal{M}. \tag{3.27}$$

In (3.17) $f^a{}_{bc}$ was identified (up to a constant) with $\epsilon^a{}_{bc}$, which defines the vector cross product in three dimensions. The cross product is defined as a bilinear map which maps two elements of a vector space V to another element of V ($V \otimes V \to V$). In three dimensions, $f^a{}_{bc}$ is the structure constants of $\mathfrak{su}(2)$ Lie algebra and

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4 In fact, e^a is a solder form and defines a canonical isomorphism between the tangent space of $\mathcal{M}$ and the fibre E.

since $f^a{}_{bc} \equiv \epsilon^a{}_{bc}$ then $f^a{}_{bc}$ are invariant under $SO(3)$ Lorentz transformations. Now recalling the definition of the quaternion algebra, see Appendix (B), if we only consider the imaginary elements of quaternions, i.e. i, j, k with $i^2 = j^2 = k^2 = -1$, they form an $\mathfrak{su}(2)$ Lie algebra. Then one can define an isomorphism between the three dimensional vector space V and the imaginary quaternion vector space $\mathrm{Im}\,\mathbb{H}$.

A feature which is of great importance is the existence of the cross product in seven dimensions. In the same way that the three dimensional cross product is related to quaternions ($V \cong \mathrm{Im}\,\mathbb{H}$), the seven dimensional one is related to octonions ($V \cong \mathrm{Im}\,\mathbb{O}$). One defines this cross product as

$$v_a \times v_b = f^c{}_{ab} v_c, \tag{3.28}$$

where $v_a \in \mathrm{Im}\,\mathbb{O}$ and f_{abc} is a totally antisymmetric tensor, with[5]

$$f_{abc} = +1 \,, \text{for } abc = 123, 145, 176, 246, 257, 347, 365. \tag{3.29}$$

However, unlike the situation in three dimensions that the structure constant $f^a{}_{bc}$ was invariant under $SO(3)$ transformations, $f^a{}_{bc}$ is not invariant of $SO(7)$ transformations. The seven dimensional cross product defines a Moufang-Lie algebra (Malcev algebra) with $f^c{}_{ab}$ as corresponding structure constants [Nag93]

The exceptional Lie group G_2 is the automorphism group of the octonions [Hum12]. Its Lie algebra $\mathfrak{g}_2$ is $\mathfrak{der}(\mathbb{O})$, the derivations of the octonions [Sch66]. This can be considered as the definition of G_2 and its Lie algebra. The group G_2 is the isotropy group of $Spin(7)$[6] which fixes the identity and since it is the automorphism group of octonions, it thus preserves the space orthogonal to the identity. This seven dimensional orthogonal space is the space of imaginary octonions $\mathrm{Im}\,\mathbb{O}$ and there is an inclusion $G_2 \hookrightarrow SO(\mathrm{Im}\,\mathbb{O})$. Now, consider the adjoint representation of the algebra of imaginary octonions defined as $\mathrm{ad}_{\mathrm{Im}\,\mathbb{O}} = L_{\mathrm{Im}\,\mathbb{O}} - R_{\mathrm{Im}\,\mathbb{O}}$ where $L_{\mathrm{Im}\,\mathbb{O}}, R_{\mathrm{Im}\,\mathbb{O}}$ are spaces of linear transformations of $\mathbb{O}$ given by left and right multiplications by imaginary octonions. At the Lie algebra level, $\mathfrak{so}(\mathrm{Im}\,\mathbb{O})$ can then be written as

$$\mathfrak{so}(\mathrm{Im}\,\mathbb{O}) = \mathfrak{der}(\mathbb{O}) \oplus \mathrm{ad}_{\mathrm{Im}\,\mathbb{O}}, \tag{3.30}$$

where $\mathfrak{g}_2 \cong \mathfrak{der}(\mathbb{O})$ and $\mathrm{ad}_{\mathrm{Im}\,\mathbb{O}}$ is the annihilator of the identity element of $\mathbb{O}$. Therefore, G_2 has a 7-dimensional representation $\mathrm{Im}\,\mathbb{O}$.

There is another definition for the group G_2. The exceptional group G_2 which was introduced as the automorphism group of octonions can also be thought of

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5 Note that this is only one choice of octonion multiplication table and it is not unique.

6 $Spin(7)$ is the double cover of $SO(7)$ and can be built with the use of spinor representations of $\mathfrak{so}(7)$.

as a subgroup of $SO(7)$ which preserves a non-degenerate 3-form [Bry87; Agr08]

$$\varphi = dx^{123} + dx^{145} + dx^{176} + dx^{246} + dx^{257} + dx^{347} + dx^{365}, \qquad (3.31)$$

where $dx^{abc} = dx^a \wedge dx^b \wedge dx^c$. Also, G_2 is the group of all real linear transformations of $\text{Im}\,\mathbb{O}$ preserving the cross product.

3.2 Flat Connections in Seven Dimensions

3.2.1 Euclidean Signature

Let's consider a seven-dimensional simply-connected symmetric Riemannian manifold with the metric g and the connection ω. By imposing the flatness of ω, the corresponding curvature (3.3) vanishes. Then, it has been shown by Cartan-Schouten [CS26b; CS26a] and Wolf [Wol72a; Wol72b] that such a manifold is the 7-sphere given by the symmetric coset space $SO(8)/SO(7)$. The metric g is invariant under $SO(8)$ isometry group of S^7.

The group $SO(8)$ acts transitively on unit octonions $\mathbb{O}$ ($\mathbb{R}^8 \simeq \mathbb{O}$) with the isotropy group $SO(7) = \{h \in SO(8) : he_0 = e_0\}$ where $e_0 = 1$ is the identity element of $\mathbb{O}$. We can also write T^{AB} generators of $\mathfrak{so}(8)$ in terms of J^{ab} generators of the subalgebra $\mathfrak{so}(7)$ and J^a the remaining seven generators of $\mathfrak{so}(8)$,

$$T^{AB} = \{J^{ab}, J^{a8} = J^a\}, \quad A, B = 1, ..., 8 \quad \text{and} \quad a, b = 1, ..., 7. \qquad (3.32)$$

In the following we show that one can define another parallelism with respect to the metric g with the connection $\bar{\omega}$ in the presence of torsion. It turns out that the metric $\tilde{g}$ is $SO(8)$ invariant metric and the torsion, given by the structure constants of imaginary octonions, breaks the isometry group down to $Spin(7)$.

$Spin(7)$ acts on unit spinors in seven dimensions with G_2 a subgroup of which fixing a unit spinor.

We can define the flat $SO(8)$ connection as $\omega^A{}_B = \theta T^A{}_B$, with θ a closed 1-form, (or more generally $\omega^A{}_B = \theta^A{}_C T^C{}_B$ for any 1-form $\theta^A{}_C$). Given the decomposition of T^{AB} to J^{ab} and J^a, it is implied that the $Spin(7)$ connection $\bar{\omega}^a{}_b = \alpha^a{}_c J^c{}_b$ (with $\alpha^a{}_c$ a 1-form) cannot be a flat connection. However, one can construct a flat connection by adding a piece to $\bar{\omega}$, roughly speaking, in order to compensate for those terms in ω which are not in $\bar{\omega}$.

Precisely, it means the flat connection $\omega^a{}_b$ is written as

$$\omega^a{}_b = \alpha^a{}_c J^c{}_b + \beta^c (J_b)^a{}_c, \qquad (3.33)$$

where α^a_c and β^c are 1-forms, defined as $\alpha^a_c = \alpha^a{}_{c\mu} dx^\mu$ and $\beta^c = \beta e^c$. We can choose a basis where $(J_b)^a{}_c = f^a{}_{bc}$ and we get

$$\omega^a{}_b = \bar{\omega}^a{}_b + \beta f^a{}_{bc} e^c. \qquad (3.34)$$

Now, the connection ω is a flat connection and the corresponding curvature vanishes.

As we discussed, the coefficients f_{abc} define an invariant tensor under $G_2 \subset SO(7)$, which means that for any G_2 group element $g^a{}_b$, the structure coefficients[7] $f^a{}_{bc}$ transform as the components of an invariant tensor, i.e. $g: f \to f$, or

$$gf := g^a{}_d f^d{}_{eh}(g^{-1})^e{}_b (g^{-1})^h{}_c = f^a{}_{bc}. \tag{3.35}$$

Therefore one gets

$$g^a{}_d f^d{}_{eh}(g^{-1})^e{}_b (g^{-1})^h{}_c = f^a{}_{bc}. \tag{3.36}$$

Note, however, that f_{abc} does not define an invariant tensor under $SO(7)$, but only under its subgroup G_2. Let $g \in G_2$ be an element near the identity,

$$g^a{}_b = \delta^a_b + \theta^s (\mathbb{J}_s)^a{}_b, \tag{3.37}$$

where the generators $\mathbb{J}_s$ belong to the Lie algebra $\mathfrak{g}_2$ and satisfy

$$[\mathbb{J}_r, \mathbb{J}_s] = C^t_{rs} \mathbb{J}_t , \tag{3.38}$$

with C^t_{rs} the structure constants of $\mathfrak{g}_2$. The invariance of the cross product under the action of G_2 implies

$$(\mathbb{J}_s)^a{}_d f^d{}_{bc} - (\mathbb{J}_s)^d{}_b f^a{}_{dc} - (\mathbb{J}_s)^d{}_c f^a{}_{bd} = 0. \tag{3.39}$$

An important property of G_2 which was taken as a definition of G_2 in Section (3.1.2) is stated in the following theorem:

Theorem 3.1. (Bryant [Bry87]) The subgroup of $GL(\operatorname{Im} \mathbb{O})$ that leaves invariant the 3-form

$$\varphi = \frac{1}{2} f_{abc} e^a e^b e^c, \tag{3.40}$$

where e^a is a local basis of vector 1-forms in seven dimensions, is a compact, simply connected Lie group of dimension 14, isomorphic to $G_2 = \operatorname{Aut} \mathbb{O}$. Moreover, G_2 is also the holonomy group of squashed 7-sphere.

The S^7 manifold with the new structure, the metric g which is invariant under $SO(8)$ and the torsion given by $f^a{}_{bc}$ which is invariant under G_2, is defined by the coset $Spin(7)/G_2$ and is a squashed[8] S^7.

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7 We assume Euclidean signature so that $f^a{}_{bc}$ takes the same values as f_{abc}, but we still keep track of upper and lower indices to facilitate the transition to other signatures.

8 The $SO(8)$ invariant metric g induces a $Spin(7)$ invariant metric $\bar{g}$ on squashed S^7. Since the metric g is $SO(8)$ invariant, the described squashed sphere is known in physics literature as the (round) seven-sphere with torsion.

We can then compute the curvature corresponding to the connection $\bar{\omega}$. From (3.34), we obtain

$$R^a{}_b = \bar{R}^a{}_b + \beta \bar{D} f^a{}_{bc} e^c + \beta^2 f^a{}_{cd} f^c{}_{bg} e^d e^g. \tag{3.41}$$

Since ω is the flat connection, then $R^a{}_b = 0$. The next job is to compute $\bar{D} f^a{}_{bc}$.

We can consider different notions of parallel transport associated to G_2, $Spin(7)$, and torsion-free $Spin(7)$ connections[9], denoted by $\hat{\omega}$, ω, and $\bar{\omega}$ respectively. Each of these connections has a corresponding covariant derivative, which we denote by $\hat{D}$, D, and $\bar{D}$. Since $G_2 \subset SO(7)$, the connection $\hat{\omega}$ can also be viewed as a piece of the connection of $Spin(7)$. These connections are enjoying independent properties,

- $Spin(7)$ connection ω is torsionful, i.e. $\omega = \bar{\omega} + k$.

- $Spin(7)$ connection $\bar{\omega}$ is a torsion-free connection, i.e. $\bar{D} e^a = 0$.

- G_2 is the automorphism group of octonions, hence G_2 connection $\hat{\omega}$ satisfies $\hat{D} f^a{}_{bc} = 0$.

The G_2 connection $\hat{\omega}(x)$ can be expressed as

$$\hat{\omega}^a{}_b = \hat{\omega}^s(x) \left(\mathbb{J}_s\right)^a{}_b, \qquad s = 1, \cdots, 14 \tag{3.42}$$

where $\{\mathbb{J}_s\}$ are the 14 generators of G_2 in a 7×7 antisymmetric representation of the Lie algebra $\mathfrak{g}_2$ (for concrete examples of this 7×7 representation, see [GG73]).

We can define the torsion $Spin(7)$ connection $\bar{\omega}$ by splitting to G_2 connection and another 1-form which then has to be a multiple of $f^a{}_{bc} e^c$,

$$\bar{\omega}^a{}_b = \hat{\omega}^a{}_b + \mu f^a{}_{bc} e^c, \tag{3.43}$$

with μ a constant parameter. This is a consequence of (3.34) together with splitting torsionful $Spin(7)$ connection ω to a G_2 connection and an extra piece proportional to $f^a{}_{bc} e^c$, i.e. $\omega^a{}_b = \hat{\omega}^a{}_b + (\mu + \beta) f^a{}_{bc} e^c$.

This can be made more precise in the following way. Consider the generators of $\mathfrak{so}(7)$ algebra J^{ab} and the split $J^{ab} = \{\mathbb{J}_s, \mathbb{X}_c\}$ where $\mathbb{J}_s$ are generators of $\mathfrak{g}_2$ subalgebra and $\mathbb{X}_c$ are the remaining seven generators of $\mathfrak{so}(7)$. As we mentioned before $\mathbb{X}_c$ are the generators of the adjoint representation of the algebra of imaginary octonions. Then $Spin(7)$ connection is defined as

$$\bar{\omega}^a{}_b = \bar{\omega}^s(x) \left(\mathbb{J}_s\right)^a{}_b + \bar{\omega}^c(x) \left(\mathbb{X}_c\right)^a{}_b, \tag{3.44}$$

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9 Note that $Spin(7)$ is the universal cover of $SO(7)$ and hence simply connected and acts transitively on S^7. In the following, we will use $SO(7)$ instead of $Spin(7)$ wherever we refer to the group representation or $\mathfrak{so}(7)$ for the Lie algebra of $Spin(7)$.

with 1-form functions $\bar{\omega}^s(x)$ and $\bar{\omega}^c(x)$. Writing $\bar{\omega}^s(x) = \hat{\omega}^s(x) + \bar{\theta}^s(x)$ for $\bar{\theta}^s(x)$ a 1-form function, we have

$$\bar{\omega}^a{}_b = \hat{\omega}^s(x)\,(\mathbb{J}_s)^a{}_b + \bar{\theta}^s(x)\,(\mathbb{J}_s)^a{}_b + \bar{\omega}^c(x)\,(\mathbb{X}_c)^a{}_b, \tag{3.45}$$

which simplifies to (3.43) where we write $\mu f^a{}_{bc}\,e^c = \bar{\theta}^s(x)\,(\mathbb{J}_s)^a{}_b + \bar{\omega}^c(x)\,(\mathbb{X}_c)^a{}_b$. By evaluating the left hand side of this relation (i.e., $\mu f^a{}_{bc}\,e^c$), one can determine $\bar{\theta}^s(x)$ and $\bar{\omega}^c(x)$ in terms of μ and $e^c(x)$.

Now from (3.43), we can simply find

$$\bar{D}f^a{}_{bc}\,e^c = \hat{D}f^a{}_{bc}\,e^c + \mu(f^a{}_{cd}f^c{}_{bg} - f^c{}_{bd}f^a{}_{cg} - f^c{}_{gd}f^a{}_{bc})e^d e^g. \tag{3.46}$$

Plugging this back in (3.41), we get the $Spin(7)$ curvature

$$\bar{R}^a{}_b = \beta \left[\mu(f^c{}_{bd}f^a{}_{cg} + f^c{}_{gd}f^a{}_{bc}) - (\beta + \mu)f^a{}_{cd}f^c{}_{bg} \right] e^d e^g. \tag{3.47}$$

Direct computation using the explicit form of $f^a{}_{bc}$ given in (3.29) for the Euclidean signature metric shows that for $\mu = -\beta/3$ the right hand side is proportional to $e^a e_b$, and finally we obtain

$$\bar{R}^a{}_b = \beta^2 e^a e_b. \tag{3.48}$$

This is the curvature of $Spin(7)$ connection $\bar{\omega}$ which has the form of the curvature of a symmetric space.

From (3.48), we find that

$$\bar{R}^a{}_{bcd} = \beta^2(\delta^a_c \eta_{bd} - \delta^a_d \eta_{bc}), \tag{3.49}$$

which by comparing to the curvature of a seven dimensional symmetric manifold, one can determine β in terms of $\bar{R}$ the scalar curvature of the manifold or γ the radius of 7-sphere $\beta^2 = \frac{\bar{R}}{42} = \frac{1}{\gamma^2}$. Therefore, there are two flat connections written as [RZon]

$$\omega^a{}_b = \bar{\omega}^a{}_b \pm \frac{1}{\gamma}f^a{}_{bc}\,e^c. \tag{3.50}$$

The occurrence of the prefactor $\frac{1}{\gamma}$ is not a surprise. In fact, in the theory of interacting quantum fields in the presence of torsion, new currents must be considered due to symmetries corresponding to the torsion which results in the appearance of new interacting terms in the action. Specifically, there is an interaction $\bar{\psi}\Gamma_{mnp}\psi$ which couples to the current J^{mnp}. This interaction term is proportional to the torsion S_{mnp} with the coupling $\frac{1}{\gamma}$ [Bir+83; Cas+84]. In Chapter 4, we will see how the torsion S_{mnp} is related to $f^a{}_{bc}$.

As the final remark, we emphasize that the coset space $Spin(7)/G_2$ which describes the parallelized S^7 is a non-symmetric space [Eng82; Bir+83; DFN83]. It is shown explicitly in [GK96].

3.2.2   Pseudo-Riemannian Signature

The discussion of pseudo-Riemannian manifolds is the same as previous section. Let's consider a seven-dimensional simply-connected symmetric pseudo-Riemannian manifold with the metric g and the connection ω. By imposing the flatness of ω, the corresponding pseudo-Riemannian curvature vanishes. Then, as shown by Wolf [Wol72a; Wol72b], such a manifold is the pseudo-sphere $S^{3,4}$ given by the coset space $SO(4,4)/SO(3,4)$ or the pseudo-hyperbolic space $H^{4,3}$ given by the coset space $SO(4,4)/SO(4,3)$. The metric g is invariant under $SO(4,4)$ isometry group of $S^{3,4}$ or $S^{4,3}$. In fact, Wolf showed that these are the only parallelizable pseudo-sphere or pseudo-hyperbolic spaces, see the proof of theorem 8.13 of [Wol72b].

The group $SO(4,4)$ acts on split-octonions $\mathbb{O}_s$ with the isotropy group $SO(3,4)$ or $SO(4,3)$ depending on the signature of the induced metric on the space of imaginary split-octonion which is orthogonal to the identity element of $\mathbb{O}_s$. More precisely, we should consider $SO^+(4,4)$, $SO^+(4,3)$, and $SO^+(3,4)$ which are the connected components of the corresponding $SO(p,q)$ group. In fact, $SO^+(4,4)$ is the group which acts transitively on $S^{3,4}$ or $S^{4,3}$.

Similarly to previous section, we show that one can define another parallelism with respect to the metric g with the connection $\bar{\omega}$ in the presence of torsion. In this case, the metric g is $SO(4,4)$ invariant and the torsion, given by the structure constants of imaginary split-octonions, breaks the isometry group down to $Spin(3,4)$.

The complex Lie algebra $\mathfrak{g}_2$ has, besides the compact form, a split real form $\mathfrak{g}_{2,2}$ to which corresponds the Lie group split G_2, referred to as $G_{2,2}$ in the following. This group is the group of automorphism of split octonions $\mathbb{O}_s$, see [Agr08; BH14]. The split octonion vector space defines a product with a bilinear form of the signature $(4,4)$. This induces a metric of the signature $(3,4)$ (or $(4,3)$) on the tangent space which is spanned by the basis of imaginary split octonions $\mathrm{Im}\,\mathbb{O}_s$, see appendix B.3. This defines a cross product on $R^{3,4}$ (or $R^{4,3}$) with the coefficients of cross product f_{abc} being components of a totally antisymmetric tensor and taking the values $\pm 1, 0$ with $f_{abc} = +1$ for[10] $abc = 123, 154, 176, 264, 257, 374, 365$. Therefore, there is a 3-form,

$$\varkappa = dx^{123} - dx^{145} + dx^{176} - dx^{246} + dx^{257} - dx^{347} + dx^{365}, \qquad (3.51)$$

which is invariant under $G_{2,2} \subset SO(3,4)$.

There is a definition for the group $G_{2,2}$ along the same line as Theorem 3.1:

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10 The ordering written here is once again only a choice, consistent with our chosen convention for the octonionic multiplication table.

Theorem 3.2. The subgroup of $GL(\mathrm{Im}\,\mathbb{O}_s)$ that leaves invariant the 3-form

$$\varkappa = \frac{1}{2} f_{abc} e^a e^b e^c, \tag{3.52}$$

where e^a is a local basis of vector 1-forms in 7 dimensions, is a non compact Lie group of dimension 14, isomorphic to $G_{2,2} = \mathrm{Aut}\,\mathbb{O}_s$.

The $S^{3,4}$ manifold with the new structure, the metric g which is invariant under $Spin(3,4)$ and the torsion given by $f^a{}_{bc}$ which is invariant under $G_{2,2}$, is written as the coset $Spin(3,4)/G_{2,2}$ and can be called a squashed $S^{3,4}$. We can write the flat connection, in a chosen basis, similar to the previous section as

$$\omega^a{}_b = \bar{\omega}^a{}_b + \tilde{\beta} f^a{}_{bc} e^c, \tag{3.53}$$

with the corresponding curvature given by

$$R^a{}_b = \bar{R}^a{}_b + \tilde{\beta}\bar{D}f^a{}_{bc}e^c + \tilde{\beta}^2 f^a{}_{cd} f^c{}_{bg} e^d e^g. \tag{3.54}$$

Similar to the Euclidean discussion of previous section, we can take into account different notions of parallel transport associated to $G_{2,2}$ and $Spin(3,4)$[11], and torsion-free $Spin(3,4)$ denoted by $\hat{\omega}$ and ω, and $\bar{\omega}$ respectively. Each of these connections has a corresponding covariant derivative, which we denote by $\hat{D}$, D, and $\bar{D}$. Since $G_{2,2} \subset SO(3,4)$, the connection $\hat{\omega}$ can also be viewed[12] as a piece of the connection of $Spin(3,4)$. These connections are enjoying independent properties,

- $spin(3,40$ connection ω is torsionful, i.e. $\omega = \bar{\omega} + k$.

- $Spin(3,4)$ connection $\bar{\omega}$ is a torsion-free connection, $\bar{D}e^a = 0$.

- $G_{2,2}$ is the automorphism group of split-octonions, $\hat{D}f^a{}_{bc} = 0$.

The structure constants of split octonions are invariant with respect to the $G_{2,2}$ connection $\hat{\omega}$,

$$\hat{D}f^a{}_{bc} = \hat{\omega}^a{}_k f^k{}_{bc} - \hat{\omega}^k{}_b f^a{}_{kc} - \hat{\omega}^k{}_c f^a{}_{bk} = 0. \tag{3.55}$$

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11 Note that $Spin(3,4)$ is the universal cover of $SO(3,4)$ and it is not connected (it has two components). Its connected component $Spin^+(3,4)$ acts transitively on $S^{3,4}$. In the following, we will use $SO(3,4)$ instead of $Spin^+(3,4)$ whenever we refer to the group representation or $\mathfrak{so}(3,4)$ for the Lie algebra of $Spin^+(3,4)$, but keeping in mind we are always referring to connected component of each group.

12 We should emphasize an important point that one has to in fact consider the group $G_{2,2}^*$ which is a centerless group to be obtained from G_2 and is the stabilizer of $Spin^+(3,4)$. See Appendix C.3 for details.

The $G_{2,2}$ connection $\hat{\omega}(x)$ can be expressed as

$$\hat{\omega}^a_{\ b} = \hat{\omega}^s(x)(\hat{\mathbb{J}}_s)^a_{\ b}, \qquad s = 1, \cdots, 14 \tag{3.56}$$

where $\{\hat{\mathbb{J}}_s\}$ are the 14 generators of $G_{2,2}$ in a 7×7 antisymmetric representation of the Lie algebra $\mathfrak{g}_{2,2}$.

Then, one can write a similar equation as (3.43), $\bar{\omega}^a_{\ b} = \hat{\omega}^a_{\ b} + \tilde{\mu} f^a_{\ bc} e^c$, in order to find a relation between $\bar{D} f^a_{\ bc}$ and $\hat{D} f^a_{\ bc}$ which leads us to get the curvature of $Spin(3,4)$ connection

$$\bar{R}^a_{\ b} = \tilde{\beta} \left[\tilde{\mu}(f^c_{\ bd} f^a_{\ cg} + f^c_{\ gd} f^a_{\ bc}) - (\tilde{\beta} + \tilde{\mu}) f^a_{\ cd} f^c_{\ bg} \right] e^d e^g. \tag{3.57}$$

Direct computation using the explicit form of $f^a_{\ bc}$ given in (3.51) for the pseudo-Riemannian signature metric, and in a chosen basis, shows that if the tangent space is spanned by the imaginary split-octonion basis, the induced metric on the tangent space is a pseudo-Riemannian metric of signature $(3,4)$ and the curvature of the manifold is

$$\bar{R}^{ab} = \tilde{\beta}^2 det(\xi) e^a e^b. \tag{3.58}$$

Here $det(\xi)$ is the determinant of the metric ξ_{ab} on the quaternionic subspace of $\text{Im} \, \mathbb{O}_s$ spanned by the basis e^a, e^b and e^c where c is the index for which f_{abc} is non-zero [RZon].

We can summarize the result in seven dimensions as follows:

- If $T\mathcal{M} \cong \text{Im} \, \mathbb{O}$, then the induced metric η_{ab} on the tangent space is Euclidean. The manifold is a seven dimensional constant positive curvature symmetric manifold, that is $S^7 = \frac{SO(8)}{SO(7)}$ or constant positive curvature non-symmetric coset space $S^7 = Spin(7)/G_2$ in the presence of torsion.

- If $T\mathcal{M} \cong \text{Im} \, \mathbb{O}_s$, then the induced metric η_{ab} on the tangent space is a diagonal metric

  1. with the signature $(4,3)$, $ds^2 = -\sum_{i=1}^3 dx_i^2 + \sum_{i=4}^7 dx_i^2$, and consequently $det(\xi) = -1$. The manifold $\mathcal{M}$ is a constant negative curvature symmetric manifold $S^{4,3}$ written as the coset $S^{4,3} = \frac{SO(4,4)}{SO(4,3)}$ if there is no torsion or a constant negative curvature non-symmetric coset space $S^{4,3} = \frac{Spin(4,3)}{G_{2,2}}$ in the presence of torsion.

  2. with the signature $(3,4)$, $ds^2 = -\sum_{i=1}^4 dx_i^2 + \sum_{i=5}^7 dx_i^2$, and consequently $det(\xi) = +1$. The manifold $\mathcal{M}$ is a constant positive curvature symmetric manifold $S^{3,4}$ written as the coset $S^{3,4} = \frac{SO(4,4)}{SO(3,4)}$ if there is no torsion or a constant positive curvature non-symmetric coset space as $S^{3,4} = \frac{Spin(3,4)}{G_{2,2}}$ in the presence of torsion.

This is consistent with the discussion of parallelizability for both S^7 and $S^{3,4}$ in seminal papers by Cartan-Schouten [CS26b; CS26a] and Wolf [Wol72a; Wol72b].

Chapter 4

$S^{3,4}$ Reduction and Exotic Solutions

4.1 Duality and Unorthodox Spacetime Signatures

In general, dualities are nontrivial isomorphisms relating two models that are seemingly completely unrelated. Dualities have been a cornerstone in different branches of mathematics. As examples of dualities in mathematics we can mention the isomorphism between a finite dimensional vector space and its dual vector space, the relation which maps points to lines and lines to points in Desarguesian geometry and many other examples in different areas of mathematics. A very well known example of duality in physics is the electric-magnetic duality where the Maxwell equations are invariant under the transformation $(F^{\mu\nu}, \star F^{\mu\nu}) \rightarrow (\star F^{\mu\nu}, -F^{\mu\nu})$, where $\star$ is the Hodge dual operator. Since the early 1980's, with the discovery of Mirror symmetry in string theory, it has been understood that dualities can relate theories with different gauge groups, different spacetime dimensions, different amounts of supersymmetry, and even relate theories of gravity to gauge theories. Moreover, there are dualities suggesting that even the spacetime signature is a relative concept that depends on the description [Hul98a; Hul98b].

In any duality prescription between two dual models, each model has a moduli space of physical quantities and duality is the mean to relate these moduli spaces in a one-to-one correspondence. In a sense, observable quantities depend on a "frame of reference" and duality is a map between these frames. As an example in string theory S-duality is a relation between a theory in strong coupling regime

and another theory in weak coupling regime where the coupling constant bears the relative concept of the "frame of reference". The other example is the T-duality where a string theory with strings wrapped around a circle of radius R is proved to be equivalent to another string theory with strings wrapped around a circle of radius $1/R$. In both S-duality and T-duality the dual theory can be in a different scale of coupling or in different spacetime dimensions but always the signature of spacetime is Lorentzian. In [Hul98a; Hul98b], Hull extended this to include the relativeness of the notion of signature. T-duality maps type IIB superstring theory to type IIA while Heterotic theory is mapped to itself. M-theory in $10+1$ dimensions compactified on a 2-torus $T^{2,0}$ in the limit where the torus shrinks to zero size, gives a type IIB theory in $9+1$ dimensions, while compactifying the $10+1$ dimensional M-theory on a 3-torus $T^{3,0}$ in the zero size limit gives again M-theory. In all these cases, the compactification was done on a spacelike circle. In order to change the signature of the dual theory one needs to compactify the theory on a timelike circle. This can in principle produce ghosts or tachyons due to presence of closed timelike curves and brings about instabilities. Therefore a dual theory obtained by compactifying on a timelike circle can be pathological as a theory itself, however there are few reasons to consider them. First, it is true that while doing the dimensional reduction, truncating the dependence on the internal dimensions leaves a dimensionally reduced theory in which some of the fields have kinetic terms of the wrong sign, but in the full compactified theory such ghosts can be gauged away [HJ98]. This is similar to the example of dimensional reduction of (super) Yang-Mills theory from $D+d$ dimensions to D dimensions by reducing on a Lorentzian torus $T^{d-1,1}$ which gives Yang-Mills plus d scalars in Euclidean space, acted on by an $SO(d-1,1)$ R-symmetry. The scalar field coming from A_0 has a 'wrong' sign for the kinetic term but in the full compactified theory A_0 can be gauged away leaving the theory with no ghost.

Secondly, one notices that even though the dual theories themselves can be pathological but the theories obtained through timelike duality can be written on flat space independently without resorting to the timelike duality and avoiding possible pathologies with which will come. Moreover, since all these theories are related through the chain of dualities to an M-theory which is known to be free of the mentioned pathologies, the dual theories are as 'good' as (or as 'bad' as) the $10+1$ dimensional standard M-theory.

Given the possibility of timelike reduction, the M-theory in $10+1$ dimensions will be linked to a theory in $9+2$ dimensions, which will be referred to as M^*-theory, and a theory in $6+5$ dimensions, M'-theory. Timelike T-duality maps Heterotic string theory to itself but type IIA and type IIB theories are mapped to type IIB* and type IIA* theories respectively. The strong coupling limit of the IIA* theory is the M^*-theory in $9+2$ dimensions. Compactifying M-theory on $T^{2,1}$ on the zero size limit of 3-torus gives M^*-theory in $9+2$ dimensions.

Then compactifying M^*-theory on a Euclidean 3-torus $T^{3,0}$ in the limit that torus shrinks to zero size gives the M'-theory in $6+5$ dimensions. One can understand the appearance of extra three dimensions by looking at the strong coupling limit of type IIA and IIA* theories in the following way: in the case of M-theory, the type IIA theory is T-dual to M-thoery. Type IIA theory has D0-branes, which in the compactification limit on 3-torus, are wrapped around the compact 3-torus. At strong coupling limit, i.e. $g_s \to \infty$ where g_s is the string coupling, three spacelike 2-cycles of $T^{3,0}$ become light and signals opening of the three extra spacelike dimensions. In the case of M^*-theory, one observes that the type IIA* theory and M^*-theory are T-dual. The IIA* theory has E1-branes wrapped around one spacelike and two timelike 2-cycles of compact $T^{2,1}$ geometry which becomes light at strong coupling limit and can be considered as the whole tower of Kaluza-Klein modes from the decompactification of extra dimensions. It hints towards opening one spacelike and two timelike dimensions. The complete list of string theories and their strong coupling limit and the net of dualities which relate them can be seen in [Hul98a]. It has been reviewed more recently in [Dij+18], however the notation is different than what we use here.

The change of signature induced by duality transformations can be given a group theoretical interpretation in terms of the infinite-dimensional Kac-Moody algebras E_{11} or E_{10} that have been conjectured to be "hidden symmetries" of 11-dimensional supergravity or an appropriate extension [Jul80b; Jul82; Jul81; Nic92; Jul97; Wes01; DH01; DHN02]. Weyl reflections in the exceptional root may change the spacetime signature. In the orbit of the standard signature $(10, 1, +)$ of M-theory appear also the signatures $(9, 2, -)$ M^*-theory, $(6, 5, +)$ M'-theory, and the theories with reversed signatures $(1, 10, -)$, $(2, 9, +)$ and $(5, 6, -)$ [EH04; Keu04a; Keu04b; EHH05]. In $(s, t, \pm)$, s is the number of space directions, t the number of time directions, and $\pm$ refers to the sign of the kinetic term of the 3-form of eleven-dimensional supergravity where $-$ means the kinetic term has a 'wrong' sign.

For all these reasons, it is of interest to explore solutions with exotic spacetime signatures in the context of M-theory. This is the the path we follow in this chapter. The analogues of Freund-Rubin solutions [FR80] have been already studied in [HK98]. In this chapter, we derive the analogue of Englert solution [Eng82].

As we discussed in Chapter 3, the seven-sphere S^7 is one of the four spheres that are parallelizable. If one considers pseudo-Riemannian signatures, in seven dimensions the pseudo-sphere $S^{3,4}$, which is the space of constant curvature $SO(4, 4)/SO(3, 4)$ or space of unit split octonions[1], turns out to be paralleliz-

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1 The negative curvature spaces obtained by multiplying the metric by an overall minus sign, which are the pseudo-hyperbolic spaces $H^{0,7}$ and $H^{4,3}$ (see appendix A for conventions) also enjoy this property.

able. The 7-sphere plays an important role in eleven-dimensional supergravity [CJS78], either through Freund-Rubin compactification where it leads to the $SO(8)$ gauged supergravity [WN82a] or through Englert solution which breaks the $SO(8)$ symmetry down to $Spin(7)$ [ERS83] due to the presence of internal fluxes. We shall see that there is an analogue of the Englert solution in M'-theory, which takes the form $AdS_4 \times S^{3,4}$. Similar to the original proposal by Englert where the solution characterized by internal fluxes provided by a torsion that parallelizes S^7, this exotic form of the Englert solution is characterized by internal fluxes provided by a torsion that flatten a connection on the tangent bundle of $S^{3,4}$. These internal fluxes break the $SO(4,4)$ symmetry of the pseudo-sphere $S^{3,4} \equiv SO(4,4)/SO(3,4)$ down to[2] $Spin^+(3,4)$.

4.2 Action Principle and General Ansatz

Action

Our starting point is the bosonic action of eleven-dimensional supergravity with mixed signature [CJS78; Hul98a; Hul98b], written as

$$S = \frac{1}{2\kappa^2} \int dx^{s+t} \mathcal{L}, \tag{4.1}$$

where the Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \sqrt{|g^{s+t}|}(R - \frac{\eta}{24} F_{MNPQ} F^{MNPQ}$$
$$- \frac{2\sqrt{2}}{12^4} \frac{1}{\sqrt{|g^{s+t}|}} \varepsilon^{M_1 \ldots M_{11}} F_{M_1 \ldots M_4} F_{M_5 \ldots M_8} A_{M_9 \ldots M_{11}}). \tag{4.2}$$

Here, $s + t$-dimensions ($s + t = 11$) refers to s spacelike directions and t timelike directions and $F_{MNPQ} = 4\partial_{[M} A_{NPQ]}$. The sign of η is determined as[3]

$$\eta = +1 \quad \text{if} \quad (s - t) \bmod 8 = 1 \quad \text{i.e. in dimensions} \quad 10 + 1, 6 + 5, 2 + 9, \tag{4.3}$$
$$\eta = -1 \quad \text{if} \quad (s - t) \bmod 8 = 7 \quad \text{i.e. in dimensions} \quad 1 + 10, 5 + 6, 9 + 2. \tag{4.4}$$

Therefore, in dimensions $10 + 1$, $6 + 5$ and $2 + 9$ the kinetic term of the 3-from has the correct sign ($\eta = +1$) while in dimensions $9 + 2$, $1 + 10$ and $5 + 6$ the

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2 For more information on the split octonions and the relevant associated groups see Appendix B and see Appendix C for information on the pseudo-sphere $S^{3,4}$.

3 From the point of view of fermionic content of the theory, whenever $\eta = +1$ there exists a purely real representation of Clifford algebra and as a result spinors are Majorana. However, if $\eta = -1$ then there exists a purely imaginary representation of Clifford algebra and the spinors are pseudo-Majorana [KT83].

kinetic term of 3-form comes with the 'wrong' sign ($\eta = -1$). As mentioned in Section 4.1, M-theory corresponds to $\eta = 1$, $(s,t) = (10,1)$, whereas M^*-theory corresponds to $\eta = -1$, $(s,t) = (9,2)$ and M'-theory corresponds to $\eta = 1$, $(s,t) = (6,5)$.

The equations of motions can be obtained by varying the Lagrangian (4.2) with respect to the metric g_{MN} and the 3-from A_{MNP},

$$R_{MN} - \frac{1}{2}g_{MN}R = -\frac{\eta}{48}(g_{MN}F^2 - 8F_{MPQR}F_N{}^{PQR}), \qquad (4.5)$$

$$F^{MNPQ}{}_{;M} = \frac{1}{\eta}\frac{18\sqrt{2}}{12^4}\frac{1}{\sqrt{|g^{11}|}}\varepsilon^{M_1...M_8NPQ}F_{M_1...M_4}F_{M_5...M_8}. \qquad (4.6)$$

The equations of motion are invariant under the simultaneous transformations

$$g_{MN} \to -g_{MN}, \qquad A_{MNP} \to A_{MNP}, \qquad (4.7)$$

accompanied with the sign change $\eta \to -\eta$. We recall that with our conventions, $\epsilon^{01\cdots10}$ changes sign since in the transformation $g_{MN} \to -g_{MN}$, the parity of the number of time directions changes. Because of this invariance, we concentrate on M-theory, M^*-theory and M'-theory, knowing that the solutions for the reversed signature theories corresponding to the other theories in the same duality orbit as ordinary supergravity can be readily obtained using the invariance.

Product of Constant Curvature Spaces

We assume that the eleven-dimensional spacetime manifold (referred to as the "**ambient space**" in the sequel) splits as the product of a four-dimensional manifold ("**background spacetime**") and a seven-dimensional manifold ("**internal space**")[4]. The metric itself is a direct sum and splits into two parts, the metric on the internal space, and the one on the background spacetime. Capital Latin indices $M, N, ...$ run over $0, ..., 10$. Small Latin indices $m, n, ...$ are internal space indices and run over $4, ..., 10$. Greek indices $\mu, \nu, ...$ are background spacetime indices running over $0, 1, 2, 3$. Notice that here $|g^{11}|$ is the absolute value of the determinant of the metric of the total space. We use the absolute value everywhere when needed for the determinant of the ambient, internal and background metrics since, a priori, there is no restriction on the signature of the aforementioned metrics.

We also assume that spacetime and the internal space are spaces of constant curvature, i.e., either a pseudo-sphere or a pseudo-hyperbolic space, with curva-

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4 We stress that the "internal space" is thus in our terminology always a seven-dimensional manifold, independently of its curvature or signature.

tures respectively given by

$$R_{mnrs} = \frac{a}{6} \left(g_{mr} g_{ns} - g_{ms} g_{nr} \right), \tag{4.8}$$

$$R_{\mu\nu\rho\sigma} = \frac{b}{3} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right), \tag{4.9}$$

where a and b are constants, which are positive for pseudo-spheres and negative for pseudo-hyperbolic spaces[5]. This implies in particular

$$R_{mn} = a\, g_{mn}, \tag{4.10}$$

$$R_{\mu\nu} = b\, g_{\mu\nu}, \tag{4.11}$$

$$R_{m\mu} = 0. \tag{4.12}$$

Assuming that some of the internal or background fluxes are turned on, we can find different type of solutions for the bosonic system of supergravity given the above ansatz for the internal and the spacetime geometry.

4.3   Freund-Rubin Type Solutions

We first derive the analogues of Freund-Rubin solutions. These were already considered in [HK98].

In addition to conditions (4.10)-(4.12) on the metric, Freund-Rubin ansatz assumes that the only 4-form flux is in spacetime and reads explicitly as

$$F^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{|g^4|}} \frac{f}{\sqrt{4!}} \varepsilon^{\mu\nu\rho\sigma}, \tag{4.13}$$

$$F^{mPQR} = 0, \tag{4.14}$$

where f is a constant. There is no internal flux.

Now, from (4.5) and the ansatz, one finds that

$$R_{mn} = \left(-\frac{1}{3}\right) \frac{\eta}{24} f^2 (-1)^{T'} g_{mn}, \tag{4.15}$$

$$R_{\mu\nu} = \left(+\frac{2}{3}\right) \frac{\eta}{24} f^2 (-1)^{T'} g_{\mu\nu}, \tag{4.16}$$

where T' is the number of timelike directions in the four-dimensional background spacetime metric. We shall also introduce T as the number of timelike directions

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5 The conventions and notations used in this chapter are collected in Appendix A.

in the internal space, so that t (the number of timelike directions of the eleven-dimensional ambient space) is equal to $t = T + T'$. Comparing with (4.10) and (4.11), we find

$$a = -\frac{1}{3}\frac{\eta}{24}f^2(-1)^{T'}, \qquad b = -2a, \tag{4.17}$$

as in [HK98]. For example, the standard Freund-Rubin solution is obtained by setting $\eta = +1$, $T' = 1$.

The Freund-Rubin solutions constitute a one-parameter family of solutions. One can take f as the free parameter characterizing the solutions. The curvature of the background spacetime and of the internal space are then determined by f and are of opposite signs. Note that the dependence of both curvatures on f is always through f^2 in (4.17) and therefore they are invariant under $f \to -f$.

The sign of a is easily seen to be always equal to $(-1)^T$. For M-theory and M'-theory, $\eta = 1$ and $t = T + T'$ is an odd number and therefore one has $(-1)^{T'} = -(-1)^T$. This is also true for the reversed M^*-theory with signature $(2,9)$ and $\eta = 1$. For M^*-theory, one has $\eta = -1$ but t is now an even number resulting in $(-1)^{T'} = (-1)^T$. The same is true for the reversed signature M- and M'-theories. Therefore, the sign of a is always equal to $(-1)^T$. In all cases, the curvatures in the four-dimensional spacetime and in the internal manifold have opposite signs.

All Freund-Rubin solutions that can be obtained in this way are listed in Table 4.1. All these solution are maximally supersymmetric and their corresponding brane construction can be found in [HK98].

4.4   Englert Type Solutions

The Englert type solutions [Eng82] can be considered as spontaneous breakings of the Freund-Rubin solutions through non-vanishing expectation values for some of the internal components of the 4-form. In other words there is now an internal flux which breaks completely the supersymmetry.

The Englert construction corresponds to the compactification on a parallelized internal manifold. This occurs when the internal space of constant curvature is either the sphere $S^{7,0}$, corresponding to the original Englert solution [Eng82], or the pseudo-sphere $S^{3,4}$ (as well as the cases trivially obtained from these ones by an overall change of sign of the metric, corresponding to the pseudo-hyperbolic spaces $H^{0,7}$ and $H^{4,3}$). All the other cases ($S^{6,1}$, $S^{5,2}$, $H^{6,1}$ etc) are not parallelizable.

By matching the signatures, one easily sees that there are a priori four cases where the parallelizable seven-(pseudo-)spheres might appear for the theories in the time-like duality orbit of M-theory. These are

$t = T + T'$	T	solution

	$t = T + T'$	T	solution
M-Theory $(\eta = +1)$	1	0	$S^7 \times AdS_4$
		1	$AdS_7 \times S^4$
M^*-Theory $(\eta = -1)$	2	0	$S^7 \times H^{2,2} = S^7 \times \frac{SO(2,3)}{SO(2,2)}$
		1	$AdS_7 \times dS_4$
		2	$S^{5,2} \times H^4 = \frac{SO(6,2)}{SO(5,2)} \times \frac{SO(4,1)}{SO(4)}$
M'-Theory $(\eta = +1)$	5	1	$AdS_7 \times S^{0,4} = AdS_7 \times \frac{SO(1,4)}{SO(0,4)}$
		2	$S^{5,2} \times H^{1,3} = \frac{SO(6,2)}{SO(5,2)} \times \frac{SO(1,4)}{SO(1,3)}$
		3	$H^{4,3} \times S^{2,2} = \frac{SO(4,4)}{SO(4,3)} \times \frac{SO(3,2)}{SO(2,2)}$
		4	$S^{3,4} \times AdS_4 = \frac{SO(4,4)}{SO(3,4)} \times AdS_4$
		5	$H^{2,5} \times S^4 = \frac{SO(2,6)}{SO(2,5)} \times S^4$

Table 4.1: Freund-Rubin type solutions of different timelike T-dual M-theories. Here, $t = T + T'$ is the number of timelike direction of the corresponding M-theory, T is the number of time direction of the internal space, and the background geometry is always considered to be four dimensional. The notation for different pseudo-spheres $S^{p,q}$ and pseudo-hyperbolic spaces $H^{p,q}$ is explained in the appendix C.

- Case 1: M-theory with spacetime manifold $M^{3,1} \times S^{7,0}$,

- Case 2: M^*-theory with spacetime manifold $M^{2,2} \times S^{7,0}$,

- Case 3: M'-theory with spacetime manifold $M^{3,1} \times S^{3,4}$,

- Case 4: reversed M'-theory with spacetime manifold $M^{2,2} \times S^{3,4}$,

as well as the reversed signature solutions. Here, $M^{p,q}$ is a priori either $S^{p,q}$ or $H^{p,q}$, i.e., we leave the sign of the curvature of four-dimensional spacetime undetermined. It will be determined by solving the equations of motion. We shall show that only cases 1 and 3 are actually compatible with the equations of motion and furthermore, that $M^{3,1}$ is then the anti-de Sitter space $H^{3,1}$.

It may be worth mentioning that in all above cases, the product $\eta(-1)^{T'}$, where T' is the number of time directions in the four-dimensional background spacetime, is equal to -1,

$$\eta(-1)^{T'} = -1. \tag{4.18}$$

Indeed, one has $\eta = 1$ for cases 1 and 3 with T' odd, and $\eta = -1$ for cases 2 and 4 but T' is now even.

Furthermore, we point out that in all cases the curvature a of the internal space ($S^{7,0}$ or $S^{3,4}$) is positive, $a > 0$, and T the number of time directions in the internal space is even. For the reversed signature solution, the curvature a would be negative and T odd.

The Englert ansatz [Eng82] consists in imposing the conditions

$$R_{mnrs} = \frac{a}{6}\left(g_{mr}g_{ns} - g_{ms}g_{nr}\right), \tag{4.19}$$

$$R_{\mu\nu\rho\sigma} = \frac{b}{3}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right), \tag{4.20}$$

$$F^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{|g^4|}}\frac{f}{\sqrt{4!}}\varepsilon^{\mu\nu\rho\sigma}, \tag{4.21}$$

$$F^{m\mu QR} = 0. \tag{4.22}$$

which are the conditions (4.8), (4.9) and (4.13) with an extra condition (4.22) on the fluxes. The condition that the internal flux should vanish is, however, relaxed, i.e., one allows $F^{mnpq} \neq 0$. In fact, one postulates instead

$$F_{mnpq} = \lambda S_{[npq,m]} = \lambda S_{npq;m}. \tag{4.23}$$

where S_{npq} is one of the (metric compatible) torsions that parallelizes the internal space, i.e., such that when added to the Levi-Civita torsion-free connection, the resulting connection is flat (zero curvature)[6]. In (4.23), λ is a constant parameter.

As Cartan and Schouten showed [CS26b; CS26a] (see also [Eng82]), the torsions satisfy the following identities:

$$S^{tr}_{\ \ m}S_{trn} = a\, g_{mn}, \tag{4.24}$$

$$S_{tm}^{\ \ r}S_{rn}^{\ \ s}S_{sp}^{\ \ t} = \frac{1}{2}a\, S_{mnp}, \tag{4.25}$$

$$S_{npq;m} = S_{t[np}S^t_{\ q]m} = S_{[npq,m]}. \tag{4.26}$$

In order to prove these identities, recall the equation (3.20) with $\kappa^a_{\ b} = -e^a_m S^m_{\ n}E^n_b$ where $S^m_{\ n} = S^m_{\ np}dx^p$. Now, given that for a flat connection $R^a_{\ b} = 0$, if one rewrites (3.20) in the coordinates then the Riemann curvature is written as

$$\bar{R}_{mnpq} = S_{rmn}S^r_{\ pq} - S_{mnp;q}, \tag{4.27}$$

where $S_{mnp;q} = \bar{\nabla}_q S_{mnp}$. Using the symmetries of curvature tensor and Bianchi identities, with a bit of work, we can show (4.26). To prove the equations (4.24)

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6 The relation between the torsion coefficients $f^a_{\ bc}$ in the equation (3.26) and S_{mnp} in (4.23) is given by $f^a_{\ bc} = e^a_m S^m_{\ np}E^n_b E^p_c$.

and (4.25), we can substitute the left hand side of (4.27) with (4.8) and multiply both sides with g^{mp} and $S^{mp}{}_r$ to get (4.24) and (4.25) respectively. It is noted that the torsion tensor is totally antisymmetric, i.e., $S_{mnp} = S_{[mnp]}$.

The parallelisms are respectively related to the octonion and split octonion algebras. Because these algebras are non-associative, there are in each case two infinite classes, denoted $+$ and $-$, which are related to left and right multiplications. We refer to [Roo84] for useful information concerning the $S^{7,0}$ sphere, easily extendable to $S^{3,4}$. The corresponding torsions can be expressed in terms of the structure constants of the octonion or split-octonion algebras. The invariance group (at any given point) of these 3-forms structure constants is isomorphic to G_2 or $G_{2,2}^*$ in the Riemannian or pseudo-Riemannian case respectively (see Appendix C). Once again, the fact that there are only two possible internal manifolds $S^{7,0}$ or $S^{3,4}$, and that $S^{6,1}$, $S^{5,2}$ etc are excluded, is connected with the fact that there are only two real forms of the octonion algebra, and two real forms of the Lie algebra $\mathfrak{g}_2$.

Furthermore, one finds from (4.24)-(4.26) that the torsion 3-forms are eigenfunctions of the (pseudo-)Laplace operator in seven dimensions,

$$S_{npq;m}{}^{;m} = -\frac{2}{3} a\, S_{npq}. \tag{4.28}$$

One can also prove that the torsions are dual or anti-dual to their curvature [Eng82],

$$S^{mnp} = \pm\sqrt{\frac{6}{a}} \frac{1}{4!} \frac{1}{\sqrt{|g^7|}} \varepsilon^{mnpqrst} S_{[rst,q]}, \tag{4.29}$$

$$S_{mnp} = \pm\sqrt{\frac{6}{a}} \frac{1}{4!} \sqrt{|g^7|} \varepsilon_{mnpqrst} S^{[rst,q]}. \tag{4.30}$$

where the $\pm$ sign depends on the $\pm$-class to which the torsion belongs.

With the above assumptions, the equation (4.6) for F_{MNPQ} reduces to

$$F_{mnpq}{}^{;m} = \frac{(-1)^{T'}}{\eta} \frac{f}{2(12)^{3/2}} \sqrt{|g^7|}\, \varepsilon_{rstunpq} F^{rstu}, \tag{4.31}$$

and is therefore solved in view of (4.23) and (4.28) if we impose

$$a = -\frac{3}{4} \frac{\eta}{24} f^2 (-1)^{T'} = \frac{1}{32} f^2. \tag{4.32}$$

Given a, the two choices $f = \pm|f| = \pm\sqrt{32a}$ correspond to the $+$ or $-$ parallelism, respectively.

Now, the equation (4.5) is equivalent to

$$R_{MN} = -\frac{\eta}{48}\left(\frac{2}{3}g_{MN}F^2 - 8F_{MPQR}F_N{}^{PQR}\right), \qquad (4.33)$$

Using the form of the fields obtained so far, the (mn)-components of (4.33) read

$$a = \frac{4}{9}a + \frac{1}{24}\frac{10}{9}\eta\lambda^2 a^2, \qquad (4.34)$$

yielding

$$\lambda^2 = \eta\frac{12}{a}. \qquad (4.35)$$

Since $a > 0$, this equation has no (real) solution when $\eta = -1$, i.e., for cases 2 and 4 above. When $\eta = 1$, this equation determines λ in terms of a,

$$\lambda = \pm\sqrt{\frac{12}{a}}. \qquad (4.36)$$

The $(\mu\nu)$-components of (4.33) determine then the radius of curvature b of the four-dimensional background spacetime,

$$b = -\frac{5}{3}a, \qquad (4.37)$$

which is thus negative. The four-dimensional background spacetime is therefore anti-de Sitter space $AdS_4 \equiv H^{3,1}$.

Summary

We thus see that there are only two cases among the cases listed above that can actually be realized (and the cases obtained by reversing the signature). These are [HR17]

1. The Englert solution $H^{3,1} \times S^{7,0}$ in M-theory.

2. The "exotic" Englert solution $H^{3,1} \times S^{3,4}$ in M'-theory.

For all these solutions $\lambda^2 = \frac{12}{a}$ and $b = -\frac{5}{3}a$, in consistency with [Eng82].

The parallelizable seven-sphere or seven-pseudo-sphere with internal torsion (fluxes) turned on is therefore a possible solution only in M-theory and M'-theory. M^*-theory does not accommodate such solutions. Furthermore, the four-dimensional background spacetime is in both cases anti-de Sitter, with a single time direction. There is no solution with two time directions in the four-dimensional background spacetime.

4.4.1 Breaking of Symmetry

All Freund-Rubin solutions listed table (4.1) are maximally supersymmetric. However, the Englert solution of M-theory not only breaks supersymmetry but also breaks the $SO(8)$-symmetry of the seven-sphere down to $Spin(7)$ [ERS83]. This is what we explained in Section 3.2 that the due to presence of torsion not all 28 Killing vectors of $SO(8)$ isometry group of S^7 are preserved. As we discussed, in the presence of torsion the $SO(4,4)$ isometry group of $S^{3,4}$ breaks down to $Spin(3,4)$.

A similar situation occurs in M'-theory. The internal fluxes of the "exotic" Englert solution break the $SO(4,4)$ symmetry of the pseudo-sphere $S^{3,4}$. This is because the 4-form F_{mnpq} at any given point is not invariant under the full isotropy subgroup $SO(3,4)$. In fact, the 4-form field strength F_{mnpq} is only invariant under the subgroup $G^*_{2,2}$ of $SO(3,4)$. The reason is that the torsion S_{mnp} is determined by the structure constants of the algebra of split octonions which is invariant under $G^*_{2,2}$. The group $G^*_{2,2}$ is the non compact group with the trivial center corresponding to the split real form of the complex Lie algebra $\mathfrak{g}_2$ and is the automorphism group of split octonions. Now, the pseudo-sphere is also equal to $Spin^+(3,4)/G^*_{2,2}$, where the group $Spin^+(3,4)$ is the connected component of $Spin(3,4)$ (see [Kat98] and Appendix C). This implies that the symmetry group of the M'-theory "exotic" Englert solution is $Spin(3,4)$. This solution, similarly to the Englert solution of M-theory, breaks the supersymmetry.

Part II

BRST Cohomology

Chapter 5

Electric-Magnetic Duality

5.1 Electric-magnetic duality in Maxwell theory

It has long been known that the vacuum Maxwell equations

$$\nabla.\mathbf{E} = 0, \qquad \nabla.\mathbf{B} = 0, \tag{5.1}$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \qquad \nabla \times \mathbf{B} = \partial_t \mathbf{E}, \tag{5.2}$$

are invariant under the $SO(2)$ transformations

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \tag{5.3}$$

in the two dimensional plane $(\mathbf{E}, \mathbf{B})$ of electric and magnetic fields. From the covariant form of vacuum Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\mu \star F^{\mu\nu} = 0, \tag{5.4}$$

one can see that the $SO(2)$ transformations rotate the Maxwell field strength $F^{\mu\nu}$ and its dual $\star F^{\mu\nu}$ to each other,

$$F^{\mu\nu} = \cos\alpha\, F^{\mu\nu} - \sin\alpha\, \star F^{\mu\nu}, \tag{5.5}$$
$$\star F^{\mu\nu} = \sin\alpha\, F^{\mu\nu} + \cos\alpha\, \star F^{\mu\nu}, \tag{5.6}$$

hence they are called electric-magnetic duality transformations. Even though it is not the concern of our thesis but it is worth to mention that this duality can also be generalized to realize as the symmetry of Maxwell equations in the presence

of sources. In that case, one requires to turn on not only an electric source j^μ but also a magnetic source k^μ,

$$\partial_\mu F^{\mu\nu} = j^\nu, \qquad \partial_\mu \star F^{\mu\nu} = k^\nu, \tag{5.7}$$

such that these sources transform under duality symmetry as

$$j^\mu = \cos\alpha\, j^\mu - \sin\alpha\, k^\mu, \tag{5.8}$$
$$k^\mu = \sin\alpha\, j^\mu + \cos\alpha\, k^\mu. \tag{5.9}$$

However, one has to be careful since $\partial_\mu \star F^{\mu\nu} \neq 0$, one cannot solve the homogeneous wave equation in order to find the electromagnetic field A_μ everywhere. In other words, only if the magnetic source is spatially localized, then we are able to find A_μ where k^μ vanishes. In the rest of this thesis we focus on the free theory without sources.

Even though we found the electric magnetic duality as the symmetry of the equation of motion, it is important to notice that the electric-magnetic duality transformations are in fact an off-shell symmetry which leaves the action invariant. The action for a free Maxwell vector field in four dimensions

$$S = -\frac{1}{4} \int d^4x\, F^{\mu\nu} F_{\mu\nu}, \tag{5.10}$$

is invariant under the duality transformations, although not manifestly [DT76; Des82; BH10]. This means that in order to realize the invariance of this action under (infinitesimal) duality transformations, the electromagnetic field A_k needs to transform (infinitesimally) non-locally in space (but local in time) such that transformations of E^k and B^k on-shell coincide with the transformations (5.5) written infinitesimally.

However, if one considers the first-order Maxwell action, which puts electric and magnetic potentials on the same footing and that is called the "first-order action", one loses the manifest Lorentz invariance but instead the duality symmetry of the action can be realized manifestly. In order to do so, one goes to the Hamiltonian formulation by introducing the conjugate momenta π^i (conjugate to A_i) as independent variables in the Hamiltonian action. Then one solves Gauss's law, $\partial_i \pi^i = 0$, for new variables π^i introducing a second vector potential Z_i, $\pi^i = -\epsilon^{ijk}\partial_j Z_k$, and at the same time getting rid of A_0 component of the electromagnetic field which appears as Lagrange multiplier in the Hamiltonian action. Putting back the new fields in the Hamiltonian action one obtains

$$S[A_i^M] = \frac{1}{2} \int dx^0\, d^3x\, (\epsilon_{MN}\, \mathbf{B}^M.\dot{\mathbf{A}}^N - \delta_{MN}\, \mathbf{B}^M.\mathbf{B}^N), \qquad M, N = 1, 2 \tag{5.11}$$

where $\mathbf{A}^M = (\mathbf{A}, \mathbf{Z})$, $\mathbf{B}^M = \nabla \times \mathbf{A}^M$ and ϵ_{MN} and δ_{MN} are the Levi-Civita tensor and the Kronecker delta respectively. Now, it is obvious that this action is

invariant under the duality transformations $\mathbf{A} \to R_2\mathbf{A}$, with R_2 the $SO(2)$ transformations, since both ϵ_{MN} and δ_{MN} are $SO(2)$ invariant tensors, i.e. $R_2^T \epsilon R_2 = \epsilon$ and $R_2^T I R_2 = I$.

As we saw the first order action is manifestly duality invariant at the expense of losing manifest Lorentz invariance, while in the manifestly Lorentz invariant second order action the duality symmetry is not locally manifest in the sense that part of duality symmetry acts non-locally (in space) on the fields. Later, we will see that for example in supergravity there is a subgroup of the duality symmetry group that acts locally on fields in the second order action.

Now, for n_v free abelian Maxwell vector fields $\mathbf{A}^I$, $I = 1, ..., n_v$, the duality invariant first order action takes the form

$$S[A_i^M] = \frac{1}{2} \int dx^0 \, d^3x \, (\sigma_{MN} \, \mathbf{B}^M.\dot{\mathbf{A}}^N - \delta_{MN} \, \mathbf{B}^M.\mathbf{B}^N), \qquad M, N = 1, 2, ..., 2n_v \tag{5.12}$$

where now σ_{MN} is the antisymmetric canonical symplectic form and $\mathbf{A}^M = (\mathbf{A}^I, \mathbf{Z}^I)$. Therefore, in order to keep both the kinetic term and the Hamiltonian invariant under duality transformations, the duality group needs to be a subgroup of $GL(2n_v, \mathbb{R})$ of linear transformations of potentials A^M which preserve both the symplectic form and the scalar product. It turns out that this group is $U(n_v)$, the maximal compact subgroup of the group of symplectic transformations $Sp(2n_v, \mathbb{R})$ in $2n_v$ dimensions [FSZ77; GZ81; Wit01; BH11].

5.2 Non-compact duality group

The duality group of Maxwell theory of n_v free abelian vector field is the unitary group $U(n)$ which is compact. The non-compact duality symmetry group is possible only in the presence of non-minimally coupled scalar fields where the non-linear transformations of scalars play an important role. Let's consider a general case of n_v abelian electromagnetic fields coupled to n_s scalar fields,

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_V, \tag{5.13}$$

$$\mathcal{L}_S = -\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j, \tag{5.14}$$

$$\mathcal{L}_V = -\frac{1}{4} \mathcal{I}_{IJ}(\phi) F_{\mu\nu}^I F^{J\mu\nu} + \frac{1}{8} \mathcal{R}_{IJ}(\phi) \, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I F_{\rho\sigma}^J, \tag{5.15}$$

with $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ and the matrices $\mathcal{I}$ and $\mathcal{R}$ give the non-minimal couplings between the scalars and the abelian vectors.

In order to realize a group G as the symmetry of this action, scalar fields have to transform in a specific way both to compensate for the transformations of vector fields under the symmetry group and to keep the scalar Lagrangian

invariant under G. This requires that g_{ij} and matrices $\mathcal{I}$ and $\mathcal{R}$ transform in a specific form. This can be achieved if for example scalar fields parameterize the coset G/H where H is the maximal compact subgroup of G. This is the case in the ungauged supergravity theories. In fact, neglecting gravity, the Lagrangian (5.13) is the generic bosonic sector of ungauged supergravity. We discuss the symmetries of this system in detail later in Sections 6.6 and 8.3.

We want to write the first-order Lagrangian obtained from (5.13)-(5.15) following the discussion of previous section and to find the duality symmetry group of this system. In what comes next in the thesis we only consider theories in four spacetime dimensions unless otherwise explicitly stated.

We start by writing the canonical momenta conjugate to the A^I which are given by

$$\pi_I^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i^I} = \mathcal{I}_{IJ} F^{J\,i}{}_0 - \frac{1}{2} \mathcal{R}_{IJ}\, \varepsilon^{ijk} F_{jk}^J, \tag{5.16}$$

along with the constraint $\pi_I^0 = 0$. This relation can be inverted to get

$$\dot{A}^{Ii} = (\mathcal{I}^{-1})^{IJ}\, \pi_J^i + \partial^i A_0^I + \frac{1}{2}(\mathcal{I}^{-1}\mathcal{R})^I{}_J\, \varepsilon^{ijk} F_{jk}^J, \tag{5.17}$$

from which we can compute the first-order Hamiltonian action

$$S_H = \int d^4x \left(\pi_I^i \dot{A}_i^I - \mathcal{H} - A_0^I\, \mathcal{G}_I \right), \tag{5.18}$$

where

$$\mathcal{H} = \frac{1}{2}(\mathcal{I}^{-1})^{IJ}\pi_I^i \pi_{Ji} + \frac{1}{4}(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{IJ} F_{ij}^I F^{Jij} + \frac{1}{2}(\mathcal{I}^{-1}\mathcal{R})^I{}_J\, \varepsilon^{ijk}\pi_{Ii} F_{jk}^J \tag{5.19}$$

$$\mathcal{G}_I = -\partial_i \pi_I^i. \tag{5.20}$$

The time components A_0^I appear in the action as Lagrange multipliers for the constraints

$$\partial_i \pi_I^i = 0. \tag{5.21}$$

These constraints can be solved by introducing new (dual) potentials Z_{Ii} through the equation

$$\pi_I^i = -\varepsilon^{ijk}\partial_j Z_{Ik}, \tag{5.22}$$

which determines Z_I up to a gauge transformation $Z_{Ii} \to Z_{Ii} + \partial_i \tilde{\epsilon}_I$. Note that the introduction of these potentials is non-local but permitted in (flat) contractible space. Putting this back in the action gives

$$S = \frac{1}{2} \int d^4x \left(\Omega_{MN} \mathcal{B}^{Mi} \dot{\mathcal{A}}_i^N - \mathcal{M}_{MN}(\phi) \mathcal{B}_i^M \mathcal{B}^{Ni} \right), \tag{5.23}$$

where the doubled potentials are packed into a vector

$$(\mathcal{A}^M) = \begin{pmatrix} A^I \\ Z_I \end{pmatrix}, \quad M = 1, \ldots, 2n_v, \tag{5.24}$$

and their curls $\mathcal{B}^{Mi}$ are

$$\mathcal{B}^{Mi} = \varepsilon^{ijk} \partial_j \mathcal{A}_k^M. \tag{5.25}$$

The matrices Ω and $\mathcal{M}(\phi)$ are the $2n_v \times 2n_v$ matrices

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \tag{5.26}$$

each block being $n_v \times n_v$. The dual vector potentials Z_I are canonically conjugate to the magnetic fields B^I.

The kinetic term for the potentials in the first-order action (5.23) can be rewritten

$$\frac{1}{2} \int dt\, d^3x\, d^3y\, \sigma_{MN}^{ij}(\vec{x}, \vec{y})\, \mathcal{A}_i^M(\vec{x})\, \dot{\mathcal{A}}_j^N(\vec{y}), \tag{5.27}$$

where $\sigma_{MN}^{ij}(\vec{x}, \vec{y})$ defines a (pre-)symplectic structure in the infinite-dimensional space of the $\mathcal{A}_i^M(\vec{x})$ and is antisymmetric for the simultaneous exchange of $(M, i, \vec{x})$ with $(N, j, \vec{y})$,

$$\sigma_{MN}^{ij}(\vec{x}, \vec{y}) = -\sigma_{NM}^{ji}(\vec{y}, \vec{x}). \tag{5.28}$$

Explicitly, $\sigma_{MN}^{ij}(\vec{x}, \vec{y})$ is given by the product

$$\sigma_{MN}^{ij}(\vec{x}, \vec{y}) = -\Omega_{MN}\varepsilon^{ijk}\partial_k\delta(\vec{x} - \vec{y}) \tag{5.29}$$

The internal part Ω_{MN} is antisymmetric under the exchange of M with N, while the spatial part $\varepsilon^{ijk}\partial_k\delta(\vec{x} - \vec{y})$ is symmetric under the simultaneous exchange of $(i, \vec{x})$ with $(j, \vec{y})$. A necessary condition for a linear transformation in the internal space of the potentials to be a symmetry of the action is that it should leave the kinetic term invariant, which in turn implies that Ω_{MN} must remain invariant under these transformations, i.e., they must belong to the symplectic group $Sp(2n_v, \mathbb{R})$.

At this point, it is worth mentioning that the situation will be completely different if one considers a system of 2-forms interacting with scalars in six spacetime dimensions.

In that case, for 2-forms $\mathcal{A}_{ij}^M(\vec{x}) = -\mathcal{A}_{ji}^M(\vec{x})$ in six spacetime dimensions, the expression analogous to (5.27) is

$$\frac{1}{2} \int dt\, d^5x\, d^5y\, \sigma_{MN}^{ij\,pq}(\vec{x}, \vec{y})\, \mathcal{A}_{ij}^M(\vec{x})\, \dot{\mathcal{A}}_{pq}^N(\vec{y}), \tag{5.30}$$

where the (pre-)symplectic form $\sigma_{MN}^{ij\,pq}(\vec{x}, \vec{y})$ is given by the product of a matrix in internal space and a matrix involving the spatial indices (including $(\vec{x})$),

$$\sigma_{MN}^{ij\,pq}(\vec{x}, \vec{y}) = -\eta_{MN}\varepsilon^{ijpqk}\partial_k\delta(\vec{x} - \vec{y}). \tag{5.31}$$

As a (pre-)symplectic form, $\sigma_{MN}^{ij\,pq}(\vec{x}, \vec{y})$ must be antisymmetric for the simultaneous exchange of $(M, ij, \vec{x})$ with $(N, pq, \vec{y})$. But now, the spatial part $\varepsilon^{ijpqk}\partial_k\delta(\vec{x} - \vec{y})$ is *antisymmetric*,

$$\varepsilon^{ijpqk}\partial_k\delta(\vec{x} - \vec{y}) = -\varepsilon^{pqijk}\partial_k\delta(\vec{y} - \vec{x})$$

and therefore the internal part η_{MN} must be *symmetric*. And indeed, this is what one finds [HT88; BH99]. The signature of η_{MN} is (n_v, n_v) and its diagonalization amounts to splitting the 2-forms into chiral (self-dual) and anti-chiral (anti-self-dual) parts. A necessary condition for a linear transformation in the internal space of the potentials to be a symmetry of the action is again that it should leave the kinetic term invariant, but this implies now that η_{MN} must be invariant under these transformations, i.e., they must belong to the orthogonal group $O(n_v, n_v)$.

In general, for a p-form in $D = 2p + 2$ spacetime dimensions, the electric and magnetic fields are exterior forms of the same rank $p + 1$ and one may consider electric-magnetic duality transformations that mix them while leaving the theory invariant. The above analysis proceeds in exactly the same way in higher (even) dimensions. The (pre-)symplectic form $\sigma_{MN}^{i_1 i_2 \cdots i_p\, j_1 j_2 \cdots j_p}(\vec{x}, \vec{y})$ for the p-form potentials $\mathcal{A}_{i_1 i_2 \cdots i_p}^M(\vec{x})$ is equal to the product $-\rho_{MN}\varepsilon^{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p k}\partial_k\delta(\vec{x} - \vec{y})$. The spatial part $\varepsilon^{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p k}\partial_k\delta(\vec{x} - \vec{y})$ is symmetric in spacetime dimensions D equal to $4m$ (p odd) and antisymmetric in spacetime dimensions D equal to $4m + 2$ (p even). Accordingly the matrix ρ_{MN} in electric-magnetic internal space must be antisymmetric for $D = 4m$ ($p = 2m - 1$) and symmetric for $D = 4m + 2$ ($p = 2m$), implying that the relevant electric-magnetic group is a subgroup of $Sp(2n_v, \mathbb{R})$ or of $O(n_v, n_v)$ in $D = 4m$ and $D = 4m + 2$ dimensions respectively. These results were established in [Des+97; Des+98; Jul97; Jul80b] which were further developed in [Cre+98a; Bre+98; DHS98; Jul05]. As it was pointed out in [Jul97], they do not depend on the spacetime signature.

As we have just discussed in $4m$ spacetime dimensions scalars parameterize a coset $Sp(2n_v, \mathbb{R})/U(n_v)$. If there is a smaller set of scalars in the theory, then the scalars parametrize a coset G/H where now $G \subset Sp(2n_v, \mathbb{R})$ is the duality symmetry group of the action and H is its maximal compact subgroup. In the case of maximal $\mathcal{N} = 8$ supergrvaity in four dimensions, the duality group is $E_{7,7}$ which is a subgroup of $Sp(56, \mathbb{R})$ and the 70 scalars are parameterizing the coset $E_{7,7}/SU(8)$.

5.3 A toy model inspired by $\mathcal{N} = 4$ supergravity in four dimensions

Here we consider as an example a model that can be easily extended to capture the gravity multiplet of the bosonic sector of $\mathcal{N} = 4$ supergravity in four dimensions. Consider the Lagrangian (5.13) for a system constitutes of one vector field and two scalars (ϕ, χ) (in fact one scalar ϕ and one pseudo-scalar χ), i.e. $n_v = 1$ and $n_s = 2$, with the scalar and vector Lagrangians

$$\mathcal{L}_S = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu\chi\,\partial^\mu\chi, \tag{5.32}$$

$$\mathcal{L}_V = -\frac{1}{4}e^{-\phi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8}\chi\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{5.33}$$

The full Lagrangian is invariant under

I) constant shift of χ $(\chi \to \chi + c)$,

II) simultaneous transformations

$$A_\mu \to \lambda A_\mu, \tag{5.34}$$
$$\phi \to \phi + 2\log\lambda, \tag{5.35}$$
$$\chi \to \lambda^{-2}\chi. \tag{5.36}$$

The non-linear scalar transformations are those of the coset $SL(2,\mathbb{R})/SO(2)$ generated by the upper-triangular matrices, B^+, in the matrix representation of $SL(2,\mathbb{R})$,

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.37}$$

The scalar Lagrangian is also invariant under another transformation

$$\delta_-\phi = 2\chi\epsilon, \qquad \delta_-\chi = (e^{-2\phi} - \chi^2)\epsilon, \tag{5.38}$$

which acts locally on scalars and enlarges the group of transformations to full $SL(2,\mathbb{R})$. However, this transformation acts non-locally on vector fields in the second-order formalism. In order to see that $SL(2,\mathbb{R})$ is in fact the group of electric-magnetic duality of this model, one can consider the first-order form of the action. The first-order action is written as (5.23) with $M, N = 1, 2$ and

$$\Omega_{MN} = \sigma_{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{M}_{MN} = \begin{pmatrix} \chi^2 e^\phi + e^{-\phi} & -\chi e^\phi \\ -\chi e^\phi & e^\phi \end{pmatrix}. \tag{5.39}$$

Now, all $SL(2, \mathbb{R})$ transformations act linearly on vector fields as

$$\delta A^M = \varepsilon^\alpha (X_\alpha)^M{}_N A^N \tag{5.40}$$

and $\mathcal{M}$ transforms as

$$\delta \mathcal{M} = -(\varepsilon X)^T \mathcal{M} - \mathcal{M}(\varepsilon X), \tag{5.41}$$

where X_α are generators of $SL(2, \mathbb{R})$, (5.37) together with

$$X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.42}$$

Therefore, in the first-order formulation the group $SL(2, \mathbb{R})$ is manifest as the group of electric-magnetic duality of the system.

The $SO(6)$ (or $SU(4)$) invariant-formulation [CJ79; CSF77; CSF78; Cha81] of $\mathcal{N} = 4$ supergravity in four dimensions is obtained by considering $n_v = 6$ vector fields in the gravity multiplet of theory, once again enlarging the duality group to $SO(6) \times SL(2, \mathbb{R})$. We will come back to this point later when we discuss examples of gauging of scalar-vector models in Section 8.4.3.

Chapter 6

Equivalent description in terms of the Embedding tensor formalism

6.1 Gauged Supergravity

As we discussed before in Section 4.1 the M-theory, or the eleven dimensional supergravity as its low energy limit, is unique (up to T-duality invariant theories) with supersymmetry fixing its field content to that of the gravity multiplet [CJS78]. The toroidal compactification of the M-theories or superstring theories results in the ungauged supergravity theories in lower dimensions in flat backgrounds. The ungauged supergravities, however, do not appear to fit phenomenology at least at the classical perturbative level. Also, since the moduli space of vacua of scalars are not determined dynamically, the appearance of scalars in the interaction terms in the Lagrangian destroys the predictive power of the theory. For example, the maximal supergravity in four dimensions contains 70 scalars part of which can be interpreted as the moduli of the internal manifold T^7 and the other part comes from the 3-form field of the eleven dimensional supergravity during the compactification [CSF77; CJ78]. There is no scalar potential in the classical theory which could fix the values of these scalars on the vacuum. One attempt to overcome this issue is to gauge the ungauged theories by 28 vector fields of gravity supermultiplet through minimally coupling of the vector fields and other fields. In this way, one can obtain an $SO(8)$ gauged supergravity in four dimensions [WN82a]. This in turn introduces a scalar potential in the Lagrangian which fixes the problem of moduli degeneracy already

at the classical level.

Moreover, if the M-theory is the correct UV-completion of the effective quantum field theories in four dimensions then the standard model of fundamental interactions needs to be realized as a solution of the supergravity theory. In order to describe the standard model of particle physics one needs ultimately to gauge the ungauged supergravities obtaining a low energy effective action describing the known interactions between the gauge fields of standard model. This is an ongoing program which despite promising progress has not yet been successful. It should be however mentioned that the most interesting supergravity theories are in Anti-de Sitter background. As of today, a supergravity theory with a non-supersymmetric de Sitter (quasi Minkowski) stable vacuum, a residual gauge symmetry $SU(3) \times SU(2) \times U(1)$, and with the matter content of the standard model has not been discovered.

One can build different gauged supergravities by gauging different subgroups of global symmetry of the ungauged theories by introducing the minimal coupling with the vector fields. In many cases, the non-perturbative strong coupling limit of these gauged supergravities is described by superstring/M-theories. The compactification of the superstring/M-theories on some non-toroidal internal manifolds leads to finding the gauged supergravities in lower dimensions, however, it is not always possible to uplift an effective lower dimensional gauged supergravity theory to its UV complete superstring/M-theory.

In order to obtain gauged supergravities from compactification, one has to turn on internal form fluxes, namely non-vanishing vacuum expectation values of field strengths. These fluxes will appear in the Einstein equation through their energy-momentum tensors and may affect in a non-trivial way the geometry of the background, including the internal manifold. They change the structure and topology of the internal manifolds.

In the presence of fluxes one is then led to consider spontaneous compactifications of superstring/M-theory on more general backgrounds which have the form of a warped product $M_D \times M_{int}$ of a non-compact D-dimensional spacetime and an internal manifold which in general is no longer Ricci-flat, at least in the sense of Ricci flatness of Levi-Civita connection, see Chapters 3 and 4. The problem of finding backgrounds of this kind which preserve a minimal amount of supersymmetries requires considering more general geometries for internal manifold and hence leads to the so called G-structure manifolds. This class of internal manifolds comprises geometries characterized by background quantities, called geometric fluxes, which define topology-deformation of traditional Ricci-flat manifolds like tori or Calabi-Yau spaces.

In Chapter 3 we discussed the G-structure manifolds where we focused on manifolds with G_2-structure. In the M-theory compactification, the global existence of Killing spinors restricts the structure group allowing the internal manifold

with G_2-structure. In Chapter 4 we explained two ways of compactification in the presence of internal fluxes; Freund-Rubin compactification which preserves all supersymmetries and Englert spontaneous compactification which prompts a non-supersymmetric low energy effective theory. Also, we have seen that in the case of Englert(-like) compactification, the torsion on the -parallelizable- internal manifold S^7 ($S^{3,4}$) plays the role of geometric fluxes.

The first examples of gauged extended supergravities [FD77; FS78; FV76; Zac78; Zac79] exhibited some of the characteristic features of these theories such as fermion masses, spontaneous supersymmetry breaking and a scalar potential, or a cosmological constant when scalars were not present.

The gauged $\mathcal{N} = 8$ maximal supergravity with $SO(8)$ gauge group is obtained from Kaluza-Klein reduction of eleven dimensional supergravity on round S^7 where the SO(8) local symmetry is related to the isometry group of S^7 [WN84; WN87; WN13]. This was later generalized to maximal gauged supergravities with non-compact gauge group in [Hul84a; Hul84c; Hul84b].

As we pointed out before, at the classical level supergravities in lower dimensions are consistently defined independently of their string or M-theory origin and are characterized by the amount of supersymmetry which they preserve, their field content and the local internal symmetry group gauged by the vector fields (for the gauged supergravities). They are also invariant under $U(1)^{n_v}$ abelian gauge symmetry. At the classical level, gauged supergravities are obtained from ungauged ones, with the same amount of supersymmetry, field content and $U(1)^{n_v}$ abelian gauge symmetry, through the well defined gauging procedure. It involves promoting a suitable subgroup of the global symmetry group to local symmetry of the theory and modifying the action by the introduction of fermion mass terms and of a scalar potential in order to restore the same number of supersymmetries as the original ungauged theory. This uniquely defines the gauged supergravity, provided a set of consistency conditions to be satisfied by the gauge group. The gauging procedure is the only known way for introducing fermion mass terms or a scalar potential in an extended supergravity without explicitly breaking supersymmetry. In the M-theory compactification, these additional terms in the action as well as the gauge group are determined by the background fluxes and therefore by the structure of internal manifold. The background fluxes typically induce, in the lower dimensional effective theory, minimal couplings as well as mass terms and a scalar potential.

It is important to emphasize again that not all gaugings can be obtained from compactification of supergravity theories. There are known examples of gauged supergravities in lower dimensions for which it is not possible to find an uplift to the M-theory [STW05; DH03; Kac+03]. Even in those cases where the gauged supergravities can be obtained from the dimensional reduction, it is not always an easy task to find a consistent reduction ansatz. Therefore, it is tempting to seek a

formulation that can provide all possible gaugings in a systematic way regardless of whether it can or cannot be obtained through dimensional reduction.

The BV-BRST formulation [BV81; BV83b; BV83a; BV84; BV85], which is described in Chapter 7, will provide a powerful tool in order to deal with the question of finding all possible consistent gaugings of supergravity theories. This is the path we pursue in this thesis.

6.2   Electric-Magnetic duality and Gauging

As we have seen in Chapter 5 in the second-order Lagrangian formulation of supergravity, only the electric vector fields appear in the Lagrangian. The minimal coupling then, involving only the electric vector fields, manifestly breaks the electric-magnetic duality symmetry of the original ungauged theory.

The electric-magnetic duality is therefore particularly relevant in the context of gaugings of supergravities in their second-order formulation since the available gaugings depend on the Darboux frame in which the theory is formulated. The Darboux frame is also known as the "duality frame" or the "symplectic frame". The use of symplectic frame is referring to the fact that the real symplectic group preserves the symplectic tensor, written in Darboux form, on the $2n_v$ electric and magnetic field directions. We should distinguish the action of the full symplectic group acting on the frame from the "duality" symmetry subgroup G that acts also on the scalar fields and may be strictly smaller.

To be specific, the second-order Lagrangian, describing the maximal supergravity toroidally reduced to four spacetime dimensions [CJ79], is invariant under the non-compact rigid symmetry $E_{7,7}$ (a subgroup of the symplectic group $Sp(56, \mathbb{R})$) which contains standard electric-magnetic duality rotations among the electric and magnetic fields. The invariance of the equations of motion under duality symmetry group was already demonstrated in [CJ79]. However, as we explained in Section 5.2, the invariance of the action is more easily verified in the first-order formalism. This was explicitly achieved in [Hil10], following the crucial pioneering work of [DT76] further developed in [HT88; Des+97].

Now, while the full $E_{7,7}$ group is always present as an off-shell symmetry group of the Lagrangian, only a subgroup of which acts locally on all fields in the second-order formulation. The other transformations act non-locally (in space). The explicit non-local form of the duality transformations for standard electromagnetism may be found in [Des82; BH10].

The subgroup of symmetries that acts locally on the fields of the second-order formalism depends on the symplectic frame and is called the "electric symmetry subgroup" in that symplectic frame. A subgroup of this electric symmetry subgroup is the seed for the local deformations of the abelian gauge symmetry and

of the action of the ungauged theory usually considered in the literature[1]. For instance, the original gaugings of maximal supergravity were performed in [WN82b] starting from the Lagrangian formulation of [CJ79] with a specific choice of 28 "electric" vector potentials out of the 56 potentials that one may introduce to render dualities manifest. The electric subgroup is $SL(8, \mathbb{R})$ and the subgroup into which the abelian gauge symmetry is deformed is $SO(8)$. Changing the duality frame, as in [And+02; Hul03], opens new inequivalent possibilities, the most recent examples of which were proposed in [DI12; DIT12; DI13]. A systematic investigation of the space of inequivalent deformations of the Yang-Mills type has been undertaken in [DIM14; Inv16].

Due to the dependence of the available deformations on the duality frame, it is of interest to keep the duality frame unspecified in the gaugings in order to be able to contemplate simultaneously all possibilities. There are at least three different ways to do so. One of them is simply to try to deform directly the manifestly duality invariant action of [Hil10]. In that first-order formulation, all duality transformations are local and no choice of duality frame is needed. However, as we show in Section 6.4, generalizing the argument of [BH11] by introduction of scalar fields one finds that all standard non-abelian Yang-Mills deformations are obstructed for this Lagrangian, so that one needs to turn to a second-order Lagrangian in order to have access to non-abelian deformations including those of the Yang-Mills type. As the change of variables from the second-order formulation to the first-order one is non-local (in space), the concept of local deformations changes.

The transition from a Hamiltonian formulation to the usual second-order formulation involves a choice of symplectic frame. Explicitly, one needs to determine what are the q's and p's where a point on a symplectic manifold is given by (q, p). We always refer to those potentials which are kept, during the transition from first-order to second-order formulation, as q's and to those which are eliminated as p's. Since the electric and magnetic potentials of the first-order formulation play the respective roles of coordinates and momenta, this amounts to choosing which potentials are electric and which ones are magnetic. It is the reason why this step breaks manifest duality. One can nevertheless keep track of the symplectic transformation that relates the chosen symplectic frame to a fixed reference symplectic

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1 We should point out an abuse of the word "ungauged" in the jargon of gauged supergravity. The original ungauged theory is really already a gauge theory, since it has abelian vector fields, each with its own $U(1)$ gauge invariance. The terminology "gauging" might for that reason not be very precise, since one is actually not gauging an initial theory without gauge invariance, but deforming the abelian gauge transformations of an original theory that is already gauged. In the process, one may charge not only the vector fields but also the matter fields through minimal coupling. For that reason, it might be more appropriate to use the terminology "charging" or "deforming" though we adopt the terminology "gauging" commonly used in the literature.

frame. The symplectic matrix coefficients appear then as constant parameters that define the second-order Lagrangian. So, one is really dealing with a family of Lagrangians depending on constant parameters. In order to treat all symplectic frames on the same footing, one should study the gaugings in a uniform manner independently of the values of these constants – although the number and features of the solutions will depend on their actual values. There are moreover redundancies, because many choices of the symplectic matrix coefficients, i.e. of symplectic frame, lead to locally equivalent second-order formulations. We will discuss this approach in Section 6.5.

A third way to keep track of the symplectic frame is through the "embedding tensor" formalism [WST03; WST05; WST07; Sam08; Tri17], where both electric and magnetic vector potentials are introduced from the beginning without choosing a priori which one is electric and which one is magnetic. Furthermore, extra 2-forms are added in order to prevent doubling of the number of physical degrees of freedom. In addition to these fields, the Lagrangian contains also constants which form the components of the embedding tensor Θ and indicate how the subgroup to be gauged is embedded in the duality group G. These constants are not to be varied in the action so that, up to some equivalences, one has a family of Lagrangians depending on external constants and any choice destroys the manifest symmetry between electricity and magnetism. In Section 6.3 we review the important features of this formalism and the gauging procedure using the embedding tensors.

The last two formulations are classically equivalent since their corresponding second-order Lagrangians yield equivalent equations of motion [WST03; WST05; WST07; Sam08; Tri17]. One may, however, wonder whether the spaces of available local deformations are isomorphic. In Section 7.5 we prove that this is indeed the case and the space of local deformations of both pictures are isomorphic.

6.3   Embedding Tensor

In the embedding tensor formulation of the gauging construction [WST03; WST05; WST07; Sam08; Tri17], all the deformations of the original ungauged model are expressed in terms of a single embedding tensor Θ which is covariant with respect to the global symmetries of the theory and defines the embedding of the gauge algebra into the global symmetry algebra. The choice of gauge group inside the global symmetry group is constrained by consistency conditions which depend on the original ungauged model. They hint to the point that the dimension of gauge group cannot exceed the number n_v of electric vector fields. The consistency conditions appear in the form of a set of linear and quadratic G-covariant (purely group-theoretical) constraints on the components of embedding tensor Θ.

We shall review a general formulation of the gauging procedure in four di-

mensions which was developed in [WST05; WST07] and does not depend on the symplectic matrices E. The matrices E relate the chosen symplectic frame to a fixed reference symplectic frame, so that the kinetic terms are not written in terms of the electric vector fields.

We start by introducing a symplectic-invariant gauge connection of the form

$$\Omega_g = \Omega_{g\mu} dx^\mu, \qquad \Omega_{g\mu} \equiv g A_\mu^M X_M = g(A_\mu^I X_I + \tilde{A}_{I\mu} \tilde{X}^I) = g A_\mu^M \Theta_M{}^\alpha t_\alpha, \tag{6.1}$$

where $A^M = (A^I, \tilde{A}_I)$ are the symplectic vector fields and $X_M = (X_I, \tilde{X}^I)$ are the generators of the gauge group G_g. Here the index $I = 1, ..., n_v$ and one should notice that even though there are $2n_v$ generators X_M but there are at most n_v independent ones which generate the gauge group. The embedding tensors describe the embedding of the gauge group in the global symmetry group G as

$$X_M = \Theta_M{}^\alpha t_\alpha, \tag{6.2}$$

with t_α generators of the algebra of global symmetry transformation, therefore $\alpha = 1, ..., dim(G)$. One can, for the sake of simplicity, reabsorb the gauge coupling constant g into Θ.

There is a redundancy in the choice of the dynamical fields between A_μ^I and $\tilde{A}_{I\mu}$; half of them play the role of auxiliary fields and at most n_v linear combinations $A_\mu^{\check{I}}$ of the $2n_v$ vectors A_μ^I and $\tilde{A}_{I\mu}$ effectively enter the gauge connection

$$A_\mu^M X_M = A_\mu^{\check{I}} X_{\check{I}}, \tag{6.3}$$

where the index $\check{I} = 1, ..., n_v$ is referring to the n_v physical vector fields[2].

The consistency of gauging requires that the embedding tensor satisfies a set of linear and quadratic algebraic G-covariant constraints [Tri17]. The constraints are written as

$$X_{(MNP)} = 0, \tag{6.4}$$

$$\Omega^{MN} \Theta_M{}^\alpha \Theta_N{}^\beta = 0, \tag{6.5}$$

$$[X_M, X_N] = -X_{MN}{}^P X_P, \tag{6.6}$$

where $X_{MN}{}^P = \Theta_M{}^\alpha (t_\alpha)_N{}^P$ and $X_{MNP} = X_{MN}{}^Q \Omega_{QP}$.

The linear constraint (6.4) is also known as the representation constraint [Tri17]. The first quadratic constraint (6.5) guarantees that there exists a symplectic matrix E which rotates the embedding tensor $\Theta_M{}^\alpha$ to an electric frame in which the magnetic components $\Theta^{\check{I}\,\alpha}$ vanish.

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2 This could be easily seen bearing in mind that there is a symplectic matrix E which takes the frame to the electric one where $\tilde{X}^{\check{I}}$ vanishes.

The second quadratic constraint (6.6) is the condition that the gauge algebra close within the global symmetry algebra $\mathfrak{g}$ and implies that Θ is a singlet with respect to G_g. To see this one can rewrite (6.6) as

$$\Theta_M{}^\alpha \Theta_N{}^\beta f_{\alpha\beta}^\gamma + \Theta_M{}^\alpha (t_\alpha)_N{}^P \Theta_P{}^\gamma = 0, \tag{6.7}$$

where $f_{\alpha\beta}^\gamma$ are generators of the adjoint representation of the global symmetry group G.

The locality constraint (6.5) is not in general independent of other constraints. As it was proved in [Tri17], it will pose as an independent constraint only in theories featuring scalar isometries with no duality action, namely those theories in which the symplectic duality representation $\mathcal{R}_v$ of the isometry algebra $\mathfrak{g}$ is not faithful. This is the case of the quaternionic isometries in $\mathcal{N} = 2$ theories. In general, in theories in which all scalar fields sit in the same supermultiplets as the vector fields, as it is the case of supergravities with $\mathcal{N} > 2$ or $\mathcal{N} = 2$ with no hypermultiplets, the locality condition (6.5) is not independent but follows from the other constraints.

Following the known prescription of gauging a theory through minimal coupling, the covariant derivatives are then defined in terms of (6.1) and replaces ordinary derivatives everywhere in the action; therefore the gauge fields A^M are transformed under the gauge symmetry as

$$\delta A_\mu^M = D_\mu \zeta^M \equiv \partial_\mu \zeta^M + A_\mu^N X_{NP}{}^M \zeta^P, \tag{6.8}$$

where $X_{MP}{}^R \equiv \mathcal{R}_{v*}[X_M]_P{}^R$. Here, $\mathcal{R}_{v*}$ is referring to the symplectic matrix representation of the generators X_M acting on covariant vectors. The symplectic covariant non-Abelian field strengths $\mathcal{F}^M$ is then defined as

$$\mathcal{F}^M \equiv dA^M + \frac{1}{2} X_{NP}{}^M A^N \wedge A^P. \tag{6.9}$$

Although in the present formulation both vectors A^I and $\tilde{A}_I$ appear in the kinetic term but in a symplectic frame which is not the electric one, A^I fields are not well defined since the Bianchi identity (Gauss constraint) is not satisfied for their corresponding curvature. This is due to the non-vanishing components $\Theta^{I\alpha}$ of the embedding tensor in such a frame; these are nothing but magnetic charges. It can be indeed seen from

$$DF^M \equiv d\mathcal{F}^M + X_{NP}{}^M A^N \wedge \mathcal{F}^P = X_{(PQ)}{}^M A^P \wedge \left(dA^Q + \frac{1}{3} X_{RS}{}^Q A^R \wedge A^S \right),$$
$$\tag{6.10}$$

where in particular one finds $D\mathcal{F}^I \neq 0$ since $X_{(PQ)}{}^I = -\frac{1}{2}\Theta^{I\alpha}(t_\alpha)_P{}^N \Omega_{NQ}$ vanishes only in the electric frame where $\Theta^{I\alpha} = 0$. This can be also interpreted

as the failure to satisfy Jacobi identity for the structure constants $X_{MP}{}^R$ of gauge algebra in the $\mathcal{R}_{v*}$-representation [Tri17]

$$X_{[MP}{}^R X_{[NR]}{}^Q + X_{[PN}{}^R X_{[MR]}{}^Q + X_{[NM}{}^R X_{[PR]}{}^Q = -(X_{NM}{}^R X_{(PR)}{}^Q)_{[MNP]} \, . \tag{6.11}$$

This problem can be resolved by adding extra 2-form gauge fields $B_{\alpha\mu\nu}$ and a set of gauge symmetries to preserve the number of degrees of freedom. Then by combining the non-abelian vector field strengths $\mathcal{F}_{\mu\nu}^M$ with a set of massless antisymmetric tensor fields $B_{\alpha\,\mu\nu}$ one is able to construct gauge-covariant field strength $\mathcal{H}_{\mu\nu}^M$ as

$$\mathcal{H}_{\mu\nu}^M \equiv \mathcal{F}_{\mu\nu}^M + \mathcal{Y}^{M\,\alpha} B_{\alpha\,\mu\nu} \; : \; \begin{cases} \mathcal{H}^I = \mathcal{F}^I + \frac{1}{2}\,\Theta^{I\alpha}\,B_\alpha \, , \\[2mm] \tilde{\mathcal{H}}_I = \tilde{\mathcal{F}}_I - \frac{1}{2}\,\Theta_I{}^\alpha\,B_\alpha \, . \end{cases} \tag{6.12}$$

where

$$\mathcal{Y}^{M\,\alpha} \equiv \frac{1}{2}\,\Omega^{MN}\,\Theta_N{}^\alpha \, . \tag{6.13}$$

In order for the field strength $\mathcal{H}_{\mu\nu}^M$ to be gauge covariant, the 2-form field $B_{\alpha\mu\nu}$ should transform under the gauge symmetry such that it cancels the non-covariant terms in the variation of $\mathcal{F}_{\mu\nu}^M$. We have thus introduced, together with the new fields $\tilde{A}_{I\mu}$ and $B_{\alpha\,\mu\nu}$, extra gauge symmetries parametrized by $\tilde{\zeta}_I$ and $\Xi_{\alpha\,\mu}$. This ensures the correct number of propagating degrees of freedom. The fields are transformed under these symmetries as

$$\delta A^M = D\zeta^M - \mathcal{Y}^{M\alpha}\,\Xi_\alpha, \tag{6.14}$$

$$\delta B_\alpha = D\,\Xi_\alpha + t_{\alpha NP} A^N \wedge \mathcal{Y}^{M\alpha}\,\Xi_\alpha - t_{\alpha NP}(2\zeta^N \mathcal{H}^P - A^N \wedge D\zeta^N), \tag{6.15}$$

where the covariant derivatives are defined as

$$D\zeta^M = d\zeta^M + X_{NP}{}^M A^N \wedge \zeta^P, \tag{6.16}$$

$$D\,\Xi_\alpha = d\,\Xi_\alpha + \Theta_M{}^\beta f_{\beta\alpha}{}^\gamma A^M \wedge \Xi_\gamma. \tag{6.17}$$

Finally the vector-tensor part of the gauged bosonic Lagrangian invariant under the above symmetries is written as [WST03; WST05],

$$\mathcal{L}_{VT} = -\frac{1}{4}\,\mathcal{I}_{IJ}\,\mathcal{H}_{\mu\nu}^I \mathcal{H}^{J\mu\nu} + \frac{1}{8}\mathcal{R}_{IJ}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}^I \mathcal{H}_{\rho\sigma}^J + \mathcal{L}_{top,\,B} + \mathcal{L}_{GCS} \, , \tag{6.18}$$

where

$$\mathcal{L}_{top,\,B} = -\frac{1}{8}g\,\varepsilon^{\mu\nu\rho\sigma}\,\Theta^{I\alpha}\,B_{\alpha\mu\nu}\left(2\,\partial_\rho\tilde{A}_{I\sigma} + X_{MNI}\,A_\rho^M A_\sigma^N - \frac{1}{4}g\,\Theta_I{}^{\beta}B_{\beta\rho\sigma}\right),$$

(6.19)

$$\mathcal{L}_{GCS} = -\frac{1}{3}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MNI}\,A_\mu^M A_\nu^N\left(\partial_\rho A_\sigma^I + \frac{1}{4}gX_{PQ}{}^I A_\rho^P A_\sigma^Q\right)$$

$$-\frac{1}{6}g\,\varepsilon^{\mu\nu\rho\sigma}X_{MN}{}^I A_\mu^M A_\nu^N\left(\partial_\rho\tilde{A}_{I\sigma} + \frac{1}{4}g\,X_{PQI}A_\rho^P A_\sigma^Q\right),$$

(6.20)

where we restored the coupling g and wrote them down explicitly.

The $\mathcal{L}_{top,\,B}$ is the topological term which is added in order to guarantee the invariance under the Peccei-Quinn transformations. Notice that if the magnetic charges $\Theta^{I\alpha}$ vanish, which will happen in the electric frame, B_α disappears from the action since (6.19) vanishes as well as the B-dependent Stueckelberg term in $\mathcal{H}^I$.

One can add the Einstein-Hilbert and the scalar Lagrangians to this action to obtain the complete bosonic action of gauged supergravity in four dimensions and $\mathcal{N} > 2$. One can do the same analysis in the presence of fermions since the only constraint required by supersymmetry is the linear constraint (6.4). In fact, the consistency of gauging in the bosonic Lagrangian already imposes this condition and one just needs to covariantize the fermionic sector, add the mass shift term and introduce the scalar potential in order to restore the supersymmetry of the original ungauged theory.

After this short review of embedding tensor formalism, it might be good to look back at the second way of gauging, mentioned in Section 6.1, where one uses the action (5.13)-(5.15) with n_v vector fields. Then, one can formulate the theory in any arbitrary frame by means of a symplectic transformation which relates the arbitrary frame to the reference frame. But first, we want to show that the first-order formulation is rigid even in the presence of scalars.

6.4 Yang-Mills deformations of first-order action

One may think that it is more tractable to consider the gauging procedure in the first-order formulation since the duality symmetry is manifest in this formulation. However, it was shown in [BH11] that the first-order action (6.32) without scalar fields,

$$S[A_i^M] = \frac{1}{2}\int dt\,d^3x\left(\Omega_{MN}B^{Mi}\dot{A}_i^N - \delta_{MN}B_i^M B^{Ni}\right)$$

(6.21)

does not admit non-abelian deformations of the Yang-Mills type.

For the future reference, it is good to make the abelian gauge invariance of the action (6.21) manifest by introducing the temporal component A_0^M of

the potentials and rewrite the action (6.21) in terms of the abelian curvatures $F_{\mu\nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M$. One can replace $\dot{A}_i^N$ by F_{0i}^M in (6.21) because the magnetic fields are identically transverse and one finds

$$S[A_\mu^M] = \frac{1}{4} \int dt\, d^3x \left(\Omega_{MN} \epsilon^{ijk} F_{ij}^M F_{0k}^N - \delta_{MN} F_{ij}^M F^{Nij} \right). \tag{6.22}$$

Then, due to the result of [BH11], there is no subgroup of the duality group $U(n_v)$ that can be deformed into a non-abelian one. A much stronger result was actually derived earlier in [BC01] through the BRST formalism, namely, that the action (6.21) admits no local deformation that deforms the gauge algebra at all.

In fact, this obstruction as described in [BH11] does not depend on the scalar sector and obstructs Yang-Mills deformations even when scalar fields are present. The obstruction prevails due to the incompatibility of an adjoint action, required by the Yang-Mills construction, and the symplectic condition as required by the invariance of the scalar-independent kinetic term. It was explained in [BH11], that the obstruction persists to remain even in the presence of scalar fields. This is because even in the presence of scalars, the kinetic term of vector fields is independent of scalars. Following [Hen+18], we give a detailed proof of the rigidity of Yang-Mills deformation in the presence of scalar fields.

Let's consider the Yang-Mills deformation

$$\delta A_i^M = \partial_i \zeta^M + g C^M{}_{NP} A_i^N \zeta^P \tag{6.23}$$

of the original abelian gauge symmetry of (6.22),

$$\delta A_i^M = \partial_i \zeta^M \tag{6.24}$$

with gauge parameters ζ^M. Here, the $C^M{}_{NP}$ are the structure constants of the gauge group G_g of dimension $2n_v$ into which the original abelian gauge group is deformed, and g is the deformation parameter. Since the $C^M{}_{NP}$ are the structure constants of the gauge group, they satisfy $C^M{}_{NP} = -C^M{}_{PN}$.

Under the Yang-Mills deformation, the abelian curvatures are replaced by the non-abelian ones,

$$\mathcal{F}_{\mu\nu}^M = F_{\mu\nu}^M + g C^M{}_{NP} A_\mu^N A_\nu^P, \tag{6.25}$$

which transform in the adjoint representation as

$$\delta \mathcal{F}_{\mu\nu}^M = g C^M{}_{NP} \mathcal{F}_{\mu\nu}^N \zeta^P, \tag{6.26}$$

and the ordinary derivatives $\partial_\mu \phi^i$ of the scalar fields are replaced by the covariant derivatives $D_\mu \phi^i$. These contain linearly the undifferentiated vector potentials A_μ^N.

Therefore the deformed action is written as

$$S[\mathcal{F}, \phi] = \frac{1}{2} \int d^4x \, g_{ij}(\phi) D_\mu \phi^i D^\mu \phi^j$$
$$+ \frac{1}{4} \int d^4x \left(\Omega_{MN} \epsilon^{ijk} \mathcal{F}_{ij}^M \mathcal{F}_{0k}^N - \mathcal{M}_{MN}(\phi) \mathcal{F}_{ij}^M \mathcal{F}^{Nij} \right). \quad (6.27)$$

The important point is that the kinetic term for the vector potentials is the same as in the absence of scalar fields. Invariance of the kinetic term under the Yang-Mills gauge transformations yields therefore the same conditions as when there is no scalar ϕ^i. The kinetic term must be invariant by itself since the scalar Lagrangian is invariant and there can be no compensation with the energy density $\mathcal{M}_{MN}(\phi) \mathcal{F}_{ij}^M \mathcal{F}^{Nij}$ since it does not contain any time derivative.

As in the case with no scalars [BH11], the compatibility conditions are that the adjoint representation (6.26) of the gauge group G_g should preserve the symplectic form Ω_{MN} in internal electric-magnetic space, i.e. writing $C^M{}_{NP} \equiv (C_N)^M{}_P$ then the matrices C_N must be symplectic hence to satisfy $(C_N)^T \Omega C_N = \Omega$ or explicitly

$$C_{NPM} = C_{MPN} \quad (6.28)$$

with $C_{MNP} \equiv \Omega_{MQ} C^Q{}_{NP}$. The symmetry of C_{MNP} under the exchange of its first and last indices, and its antisymmetry in its last two indices, force it to vanish,

$$C_{MNP} = -C_{MPN} = -C_{NPM} = C_{NMP} = C_{PMN} = -C_{PNM} = -C_{MNP}. \quad (6.29)$$

Hence, the structure constants $C^Q{}_{NP}$ must also vanish and there can be no non-abelian deformation of the gauge symmetries. The Yang-Mills construction in which the potentials become non-abelian connections is unavailable in the manifestly duality invariant first-order formulation [BH11]. However, one should notice that the other types of deformations, charging the matter fields or adding functions of the gauge-invariant abelian curvatures, are not excluded by the argument since they preserve the abelian nature of the gauge group.

In order to surpass the obstruction to the non-abelian deformations, one must avoid the condition (6.28) expressing the invariance of the symplectic form. This is naturally attainable once one goes to the second-order formalism by choosing a Lagrangian submanifold, on which the pull-back of Ω_{MN} is by definition zero. This is explained in the next section.

6.5 Transition to the second-order formalism

While the first-order and second-order formalisms are equivalent in terms of symmetries in the sense that any symmetry in one formalism is also a symmetry of

the other, they are not equivalent regarding the concept of locality, more specifically locality in space. A local function in one formulation may not be local in the other and consequently the spaces of consistent local deformations are not the same in both formalisms. This subtle difference may open a window of possibilities for the non-abelian deformations in second-order formalism which were not accessible in the first-order formalism. Below, we will shed light on the origin of this practically important difference.

6.5.1 Choice of symplectic frame and locality restriction

Let's consider the first-order Lagrangian discussed in Section 5.2 which is written as

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_V, \tag{6.30}$$

$$\mathcal{L}_S = \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j, \tag{6.31}$$

$$\mathcal{L}_V = \frac{1}{2}\Omega_{MN}B^{Mi}\dot{A}_i^N - \frac{1}{2}\mathcal{M}_{MN}(\phi)B_i^M B^{Ni}, \tag{6.32}$$

with Ω and $\mathcal{M}$ given by (5.26).

Of special interest are the symmetries of $\mathcal{L}$ for which the vector fields transform linearly,

$$\delta A_i^M(\vec{x}) = \epsilon^\alpha (t_\alpha)^M_{\ N} A_i^N(\vec{x}) \tag{6.33}$$

accompanied by a transformation of the scalars that may be nonlinear,

$$\delta\phi^i = \epsilon^\alpha R_\alpha^{\ i}(\phi). \tag{6.34}$$

Here, the ϵ^α's are infinitesimal parameters. One might consider more general (nonlinear canonical) transformations of the vector fields, and the scalars might have their own independent symmetries, but the concern of the thesis is duality symmetries which are governed by the transformations of type (6.33) and (6.34).

For the first-order action to be invariant, the transformation (6.33) should not only be symplectic, but also such that the accompanying transformation (6.34) of the scalar field is given by a Killing vector of the scalar metric $g_{ij}(\phi)$ (so as to leave the scalar Lagrangian $\mathcal{L}_S$ invariant) which implies that

$$\epsilon^\alpha \left(\frac{\partial\mathcal{M}_{MN}}{\partial\phi^i} R_\alpha^{\ i} + \mathcal{M}_{MP} (t_\alpha)^P_{\ N} . + \mathcal{M}_{PN} (t_\alpha)^P_{\ M} \right) = 0 \tag{6.35}$$

It also guarantees to leave the energy density of the electromagnetic fields invariant. In finite form, the symmetries are $A \to \bar{A}$, $\phi \to \bar{\phi}$ where the symplectic transformation

$$\bar{A} = SA, \tag{6.36}$$

and the isometry $\phi \to \bar{\phi}$ of the scalar manifold are such that

$$\mathcal{M}(\bar{\phi}) = S^{-T} \mathcal{M}(\phi) S^{-1}. \qquad (6.37)$$

The algebra of transformations of the type (6.33) and (6.34) that leave the action invariant is called the duality algebra $\mathfrak{g}$, while the corresponding group is the duality group G. As it was indicated in Section 5.2, the duality algebra is a subalgebra of $\mathfrak{sp}(2n_v, \mathbb{R})$. This symplectic condition holds irrespective of the scalar sector and its couplings to the vectors, since it comes from the invariance of the vector kinetic term which assumes always the same form. Which transformations among those of $\mathfrak{sp}(2n_v, \mathbb{R})$ are actually symmetries depend, however, on the number of scalar fields, their internal manifold and their couplings to the vectors [FSZ77; GZ81; Wit01; CSF77; CSF78; CS77].

In order to go from the first-order formalism to the standard second-order formalism, one has to tell what are the "q"'s (to be kept) and the "p"'s (to be eliminated in favour of the velocities through the inverse Legendre transformation). We have actually made the choice to use Darboux coordinates in (5.26) and we call the corresponding frame a Darboux frame or symplectic frame.

Since we consider here only linear canonical transformations, a choice of canonical coordinates amounts to a choice of an element of the symplectic group $Sp(2n_v, \mathbb{R})$ relating that symplectic frame (i.e., symplectic basis) to a reference symplectic frame.

Once a choice of symplectic frame has been made, one goes from (6.32) to the second-order formalism by following the steps reverse to those that led to the first-order formalism:

- One keeps the first half of the vector potentials $A^M|_{M=1,\dots,n_v} = A^I$, the "electric ones" in the new frame ($I = 1, \cdots, n_v$).

- One replaces the second half of the vector potentials $A^M|_{M=n_v+1,\dots,2n_v} = Z_I$ (the "magnetic ones") by the momenta

$$\pi_I^i = -\varepsilon^{ijk}\partial_j Z_{Ik} \qquad (6.38)$$

  subject to the constraints

$$\partial_i \pi_I^i = 0 \qquad (6.39)$$

  which one enforces by introducing the Lagrange multipliers A_0^I. This is the non-local step (in space).

- One finally eliminates the momenta π_I^i, which can be viewed as auxiliary fields, through their equations of motion which are nothing else but the inverse Legendre transformation expressing $\dot{A}_i^I$ in terms of π_I^i.

One can encode the choice of symplectic frame through the symplectic transformation that relates the chosen symplectic frame to a reference symplectic frame,

$$A^M = E^M{}_N A^{'N}, \tag{6.40}$$

where E is a symplectic matrix in the fundamental representation of $Sp(2n_v, \mathbb{R})$, while A and $A^{'}$ are respectively the potentials in the reference and new duality frames. The symplectic property $E^T \Omega E = \Omega$ ensures that the kinetic term in (6.32) remains invariant. The first-order action therefore takes the same form, but with matrices $\mathcal{M}$ and $\mathcal{M}^{'}$ related by

$$\mathcal{M}^{'} = E^T \mathcal{M} E. \tag{6.41}$$

Using the fact that E and E^T are symplectic, a straightforward but not very illuminating computation shows that $\mathcal{M}^{'}$ determines matrices $\mathcal{I}^{'}$, $\mathcal{R}^{'}$ uniquely such that equation (5.26) with primes holds, i.e, there exist unique $\mathcal{I}^{'}$ and $\mathcal{R}^{'}$ such that

$$\mathcal{M}^{'} = \begin{pmatrix} \mathcal{I}^{'} + \mathcal{R}^{'}\mathcal{I}^{'-1}\mathcal{R}^{'} & -\mathcal{R}^{'}\mathcal{I}^{'-1} \\ -\mathcal{I}^{'-1}\mathcal{R}^{'} & \mathcal{I}^{'-1} \end{pmatrix}. \tag{6.42}$$

The matrices $\mathcal{I}^{'}$ and $\mathcal{R}^{'}$ depend on the scalar fields, but also on the symplectic matrix E. Performing the steps described above in reverse order, one then gets the second-order Lagrangian

$$\mathcal{L}^{'}_V = -\frac{1}{4}\mathcal{I}^{'}_{IJ}(\phi)F^{'I}_{\mu\nu}F^{'J\mu\nu} + \frac{1}{8}\mathcal{R}^{'}_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}F^{'I}_{\mu\nu}F^{'J}_{\rho\sigma}. \tag{6.43}$$

The new Lagrangian (6.43) depends on the parameters of the symplectic transformation E used in equation (6.41). So we really have a family of Lagrangians labelled by an $Sp(2n_v, \mathbb{R})$ element E. By construction, these Lagrangians differ from one another by a change of variables that is in general non-local in space.

There are of course redundancies in this description. The presence of these redundancies will be important in the study of deformations. There are in general two sets of redundancies:

1. The stability subgroup T of a Lagrangian subspace consists of the block lower triangular symplectic transformations.

2. The duality symmetries.

First, the different symplectic transformations can lead to the same final n_v dimensional Lagrangian subspace of q's upon elimination of the momenta. The choice of q's is equivalent to a choice of Lagrangian linear subspace, or linear

polarization, because q's form a complete set of commuting variables (in the Poisson bracket). The same is true for a choice of p's. The stability subgroup T of a Lagrangian subspace consists of the block lower triangular symplectic transformations yielding

- The canonical *point* transformation where the second-order Lagrangians differ by a redefinition of the q's,

- The canonical *phase* transformation where the second-order Lagrangians modified by a total derivative.

Second, duality symmetries correspond also to redundancies in the transition to the second-order formalism since they do not modify the first-order Lagrangian (when the scalars are transformed appropriately) and hence clearly lead to the same final second-order Lagrangian.

One can characterize the redundancies as follows. The symplectic transformations (6.40) with matrices $E \in Sp(2n_v, \mathbb{R})$ and gEt are equivalent. Here, t belongs to the stability subgroup T of the final Lagrangian subspace and g belongs to the duality group G. Indeed, the first-order Lagrangian is left invariant under the transformation $A \to g^{-1}A$ provided the scalars are transformed appropriately, and two sets of new symplectic coordinates $A^{'}$ and $tA^{'}$ yield equivalent second-order Lagrangians after elimination of the momenta. One could also consider what happens when the coordinates $A^{'}$ defined by (6.40) are taken as the reference frame. Then, the changes of frame $A^{'} = UA^{''}$ and $A^{'} = g^{'}Ut^{'}A^{''}$ are equivalent, where $g^{'} = E^{-1}gE$ is the matrix associated with the duality symmetry g in the $A^{'}$–frame and $t^{'}$ is an element of the stability subgroup of the Lagrangian subspace in the $A^{''}$–frame. This gives an equivalent description of the redundancies.

The relevant space is thus the quotient $G \backslash Sp(2n_v, \mathbb{R})/T$. Here we considered the invariance of action that means the invariances up to a total derivative which does not matter classically. However, if one just considers the invariance of the Lagrangian instead of the action then the stability subgroup reduces to $T = GL(n_v, \mathbb{R})$.

6.5.2 Electric group

The duality transformations $A \to \bar{A} = SA$, $\phi \to \bar{\phi}$, where the symplectic matrix A and the isometry $\phi \to \bar{\phi}$ of the scalar manifold are such that the condition (6.37) holds, generically mix the electric and magnetic potentials. Accordingly, when expressed in terms of the variables of the second-order formalism they will, in general, take a non-local form since the magnetic potentials become non-local

functions of the electric potentials[3] and their time derivatives in the second-order formalism [DT76; Des82; BH10].

Thus, an electric symmetry transformation is characterized by the property that the matrix S_e is lower-triangular, i.e., given that it must be symplectic

$$S_e = \begin{pmatrix} M & 0 \\ BM & M^{-T} \end{pmatrix}, \ M \in GL(n_v), \ B^T = B. \tag{6.44}$$

An electric symmetry is therefore a transformation $\bar{A}^I_\mu = M^I{}_J A^J_\mu$ of the electric potentials for which there is a symmetric matrix B and an isometry $\phi \to \bar{\phi}$ of the scalar manifold such that

$$\mathcal{M}(\bar{\phi}) = S_e^{-T} \mathcal{M}(\phi) S_e^{-1} \tag{6.45}$$

with S_e given by (6.44). The terminology "electric group" for the group of transformations characterized by (6.44) is standard. The transformations with $B \neq 0$ involve transformations of Peccei-Quinn type describing the axion shift symmetry [PQ77b; PQ77a].

The electric group G_e in a given frame depends of course on the chosen duality frame. Indeed, going to another duality frame with the symplectic matrix E as in (6.40) will replace the matrices S_e by their conjugates $S'_e = E^{-1} S_e E$, and these might not be lower-triangular. The condition that S'_e has the lower-triangular form (6.44) therefore depends on the choice of E.

We have repeatedly emphasized that a local expression in the Z_I's need not be local when expressed in terms of the π_I's and thus of the Lagrangian variables. This is because the magnetic potentials are not local functions of the conjugate momenta. But the π_I's are local functions of the Z_I's given by $\pi_I^i = -\varepsilon^{ijk}\partial_j Z_{Ik}$ and so it would seem that any local function of the π_I's, and thus of the Lagrangian variables, should also be a local function of the Z_I's. This is true as long as the relation $\pi_I^i = -\varepsilon^{ijk}\partial_j Z_{Ik}$ is not modified. In deformations of the second-order Lagrangians one allows, however, deformations that could modify this relation. For instance, after Yang-Mills deformations, the abelian constraints $\partial_i \pi_I^i = 0$ get replaced by non-abelian ones and their local solution gets replaced by a non-local expression involving both kinds of potentials.

---

3 Given a duality symmetry of the action $S[A^M_k, \phi^i]$, one gets the corresponding duality symmetry of the first-order action $S[A^I_k, \pi^k_I, A^I_0, \phi^i]$ by (i) expressing in the variation δA^I_k the magnetic potentials Z_{Ik} (if they occur) in terms of π^k_I, which is a non-local expression and determined up to a gauge transformation that can be absorbed in a gauge transformation of the electric variables; (ii) computing the variation $\delta\pi^k_I$ from $\pi^i_I = -\varepsilon^{ijk}\partial_j Z_{Ik}$; and (iii) determining the variation of the Lagrange multipliers A^I_0 so that the terms proportional to the constraints $\partial_i \pi^i_I$ cancel in $\delta S[A^I_k, \pi^k_I, A^I_0, \phi^i]$. One gets the symmetry of the second-order action by expressing the auxiliary fields π^i_I that are eliminated in terms of the retained variables.

The interest of the electric group G_e is that its transformations are local in the second-order formalism. Therefore, among the duality transformations, it is the electric ones that are candidates for (local) gaugings.

6.6 Deformation leading to the embedding tensor formalism

In Section 6.1 we explained that another way to cover arbitrary choices of symplectic frames is given by the embedding tensor formalism of [WST05; WST07; Sam08; Tri17]. As we have seen in Section 6.3 in this formalism in addition to the n_v electric vector fields A_μ^I, the Lagrangian also contains n_v magnetic vector fields $\tilde{A}_{I\mu}$ and $\dim(G)$ two-forms $B_{\alpha\mu\nu}$ with $\alpha = 1, \ldots, \dim(G)$, where G is the full duality group in which one can also include pure scalar symmetries. The Lagrangian (6.18) also depends on the embedding tensor $\Theta_M{}^\alpha = (\Theta_I{}^\alpha, \Theta^{I\alpha})$, which are constants in spacetime and not to be varied in the action. They contain information about the embedding of the gauge group in the duality group through the $\mathfrak{g}$-valued connection

$$\mathcal{T}_\mu^\alpha = A_\mu^M \Theta_M^\alpha \tag{6.46}$$

where $\mathfrak{g}$ is the Lie algebra of G and we wrote $A_\mu^M = (A_\mu^I, \tilde{A}_{I\mu})$. $\tilde{A}_{Ik}$ are the magnetic potentials called Z_{Ik} in Section 6.5 but for more harmonious notation used below, we use $\tilde{A}_{IK}$ in the following. The embedding tensor also contains information about the choice of symplectic frame; an important feature upon which our analysis in the following based. The consistency of gaugings in this formalism controls by three sets of constraints (6.4)-(6.6). The locality constraint (6.5) guarantees the existence of a symplectic matrix which takes a chosen symplectic frame to the electric one and hence implies that the gauged group must be a subgroup of an electric subgroup G_e of G, i.e. $dim(G_g) = dim(\Theta_{\check{I}}{}^\alpha) \leq n_v$. Here $\check{I} = 1, ..., n_v$ is again referring to the n_v physical vector fields.

The gauged Lagrangian (6.18)-(6.20), after restoring the coupling constant g, can be reorganized as

$$\begin{aligned}
\mathcal{L}_{VT}^{\Theta,\text{int.}}(A, \tilde{A}, B) = &-\frac{1}{4}\mathcal{I}_{IJ}(\phi)\mathcal{H}_{\mu\nu}^I \mathcal{H}^{J\mu\nu} + \frac{1}{8}\mathcal{R}_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}^I \mathcal{H}_{\rho\sigma}^J \\
&-\frac{g}{8}\varepsilon^{\mu\nu\rho\sigma}\Theta^{I\alpha}B_{\alpha\mu\nu}\left(\tilde{\mathcal{F}}_{I\rho\sigma} - \frac{g}{4}\Theta_I{}^\beta B_{\beta\rho\sigma}\right) \\
&+ g\mathcal{L}^{\text{extra}}(A, \tilde{A}, gB). \tag{6.47}
\end{aligned}$$

where

$$\mathcal{H}_{\mu\nu}^I = \mathcal{F}_{\mu\nu}^I + \frac{g}{2}\Theta^{I\alpha}B_{\alpha\mu\nu} \tag{6.48}$$

and $\mathcal{F}^I_{\mu\nu}$, $\tilde{\mathcal{F}}_{I\mu\nu}$ are the Yang-Mills curvatures, differing from the abelian ones by $O(g)$-terms. The term $\mathcal{L}^{\text{extra}}$ in (6.47) are terms necessary to secure gauge invariance and will vanish in the limit explained below; their explicit form can be extracted from (6.18)-(6.20). The complete form of supersymmetric action in the presence of matter couplings can be found in [WST05; Tri17], but we just focus on the bosonic action with no matter coupling. Note that the full duality group is in general not faithfully represented.

The gauged Lagrangian (6.47) is an interacting Lagrangian. To view it as a deformation of an ungauged Lagrangian with the same field content, so as to phrase the gauging problem as a deformation problem, we observe that the straightforward free limit $g \to 0$ of (6.47) is just the Lagrangian (5.15), without any additional fields and without the embedding tensor. However, one notices that all terms in (6.47) depend on the two-forms B_α only through the combination $\hat{B}_\alpha = gB_\alpha$. The $\hat{B}_\alpha$'s are necessary to shift away some field strengths even for the undeformed limit $g = 0$. If we redefine the two-forms appearing in the Lagrangian (6.47) as $\hat{B}_\alpha = gB_\alpha$ and then take the limit $g \to 0$, one gets

$$
\begin{aligned}
\mathcal{L}^\Theta = &-\frac{1}{4}\mathcal{I}_{IJ}(\phi)H^I_{\mu\nu}H^{J\mu\nu} + \frac{1}{8}\mathcal{R}_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}H^I_{\mu\nu}H^J_{\rho\sigma} \\
&-\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Theta^{I\alpha}\hat{B}_{\alpha\mu\nu}\left(\tilde{F}_{I\rho\sigma} - \frac{1}{4}\Theta_I{}^\beta\hat{B}_{\beta\rho\sigma}\right),
\end{aligned}
\tag{6.49}
$$

with

$$
H^I_{\mu\nu} = F^I_{\mu\nu} + \frac{1}{2}\Theta^{I\alpha}\hat{B}_{\alpha\mu\nu}
\tag{6.50}
$$

and $F^I_{\mu\nu}$, $\tilde{F}_{I\mu\nu}$ the abelian curvatures. This Lagrangian contains the extra fields and has the ability to cover generic symplectic frames through the embedding tensor components $\Theta^{I\alpha}$ and $\Theta_I{}^\alpha$. These tensors satisfy the so-called "locality" quadratic constraint (6.5),

$$
\Theta^{I[\alpha}\Theta_I{}^{\beta]} = 0.
\tag{6.51}
$$

In addition to (6.51), the embedding tensor fulfills other constraints (6.4) and (6.6), however, we stress that the locality constraint is the only condition relevant to our analysis. The other constraints have to appear as the solution of obstruction equations in the process of deformation. We will come back to this in Chapter 8 while we discuss BV-BRST deformation of such a model.

The Lagrangian (6.49) is invariant under the $2n_v$ gauge transformations

$$
\delta A^I_\mu = \partial_\mu \lambda^I - \frac{1}{2}\Theta^{I\alpha}\hat{\Xi}_{\alpha\mu},
\tag{6.52}
$$

$$
\delta \tilde{A}_{I\mu} = \partial_\mu \tilde{\lambda}_I + \frac{1}{2}\Theta_I{}^\alpha\hat{\Xi}_{\alpha\mu},
\tag{6.53}
$$

and the dim G generalized gauge transformations

$$\delta \hat{B}_{\alpha\mu\nu} = 2\partial_{[\mu}\hat{\tilde{\Xi}}_{\alpha\nu]}, \tag{6.54}$$

where we redefined the gauge parameter as $\hat{\tilde{\Xi}}_\alpha = g\Xi_\alpha$ with respect to the gauge parameter of the corresponding symmetry of (6.18). The parameters λ^I, $\tilde{\lambda}_I$ and $\hat{\tilde{\Xi}}_{\alpha\nu}$ are arbitrary functions. The λ's and $\tilde{\lambda}$'s correspond to standard $U(1)$ gauge symmetries of the associated one-form potentials. The $\hat{\tilde{\Xi}}$'s define ordinary abelian two-form gauge symmetries and also appear as shift transformations of the one-form potentials. This set of gauge symmetries is reducible since adding a gradient to $\hat{\tilde{\Xi}}$ and shifting simultaneously the λ^I and $\tilde{\lambda}_I$, i.e.

$$\hat{\tilde{\Xi}}_{\alpha\mu} \to \hat{\tilde{\Xi}}_{\alpha\mu} + \partial_\mu \xi_\alpha, \qquad \lambda^I \to \lambda^I + \frac{1}{2}\Theta^{I\alpha}\xi_\alpha, \qquad \tilde{\lambda}_I \to \tilde{\lambda}_I - \frac{1}{2}\Theta_I{}^\alpha \xi_\alpha, \quad (6.55)$$

leads to no modification of the gauge transformations.

By construction, the non-abelian Lagrangian of (6.18) can be viewed as a consistent local deformation of (6.49), since one can charge the fields in (6.49) in a smooth way – by adding $O(g)$ and $O(g^2)$ terms – to get the non-abelian Lagrangian. The abelian Lagrangian (6.49) is thus a sensible starting point for the deformation procedure. The space of consistent local deformations of (6.49) will necessarily include the non-abelian Lagrangian (6.18), given in [WST05]. If the embedding tensor components $\Theta^{I\alpha}$ and $\Theta_I{}^\alpha$ do not fulfill the extra constraints (6.4) and (6.6) of the embedding formalism, one will simply find that there is no Yang-Mills type deformation, as in [WST05], among the consistent deformations of (6.49).

A natural question then is whether the space of local deformations of (6.49) is isomorphic to the space of local deformations of the conventional Lagrangian (6.43) in an appropriate symplectic frame. We will show that it is indeed the case.

To that end, we will now perform a sequence of field redefinitions and show that the Lagrangian (6.49) with a given definite (arbitrary) choice of embedding tensor differs from the Lagrangian (6.43) in a related symplectic frame by the presence of algebraic auxiliary fields that can be eliminated without changing the local BRST cohomology. The other extra fields are pure gauge with gauge transformations that are pure shifts so that they do not appear in the Lagrangian and do not contribute to the local BRST cohomology either. We shall explain this later in Section 7.5 but for now we recall that the local BRST cohomology at ghost number zero is what controls the space of non-trivial deformations [BH93].

The steps performed below for the limit $g = 0$ follow closely the steps given in [WST05] to prove that the extra fields appearing in the embedding tensor formalism do not add new degrees of freedom with respect to conventional gaugings. As shown by the examples of the first-order and second-order formulations discussed

in the previous sections, however, equivalent actions might admit inequivalent spaces of local deformations. Another example is given in [Bra97b; Bra97a; Bra97c]. Therefore it seems important to relate the local BRST cohomologies; particularly we monitor how locality goes through each field redefinition. Furthermore, we do *not* fix the gauge at any stage in order to make it clear that the analysis is gauge-independent. This is also important since gauge fixing might have an impact on local cohomology.

The magnetic components $\Theta^{I\alpha}$ of the embedding tensor form a rectangular $n_v \times (\dim G)$ matrix. Let $r \leq n_v$ be its rank. Then, there exists an $n_v \times n_v$ invertible matrix Y and a $(\dim G) \times (\dim G)$ invertible matrix Z such that

$$\Theta'^{I\alpha} = Y^I{}_J \, \Theta^{J\beta} Z_\beta{}^\alpha \tag{6.56}$$

is of the form

$$(\Theta'^{I\alpha}) = \left(\begin{array}{c|c} \theta & 0 \\ \hline 0 & 0 \end{array} \right), \tag{6.57}$$

where θ is an $r \times r$ invertible matrix[4].

We also define the new electric components by

$$\Theta'_I{}^\alpha = (Y^{-1})^J{}_I \Theta_J{}^\beta Z_\beta{}^\alpha, \tag{6.58}$$

so that the new components still satisfy the locality constraint (6.51)

$$\Theta'^{I[\alpha} \Theta'_I{}^{\beta]} = 0. \tag{6.59}$$

Similarly to what has been done in [WST05], one splits the indices as $I = (\hat{I}, \hat{U})$ and $\alpha = (i, m)$, where $\hat{I}, i = 1, \ldots, r$, $\hat{U} = r+1, \ldots, n_v$ and $m = r+1, \ldots, \dim G$. Moreover, we define

$$\tilde{\theta}_{\hat{I}}{}^i = \Theta'_{\hat{I}}{}^i. \tag{6.60}$$

With this split, equations (6.57) and (6.59) become

$$\Theta'^{\hat{I}i} = \theta^{\hat{I}i} \quad \text{(invertible)}$$
$$\Theta'^{\hat{I}m} = \Theta'^{\hat{U}i} = \Theta'^{\hat{U}m} = \Theta'_{\hat{I}}{}^m = 0$$
$$\theta^{\hat{I}i} \tilde{\theta}_{\hat{I}}{}^j = \theta^{\hat{I}j} \tilde{\theta}_{\hat{I}}{}^i. \tag{6.61}$$

Now, we make the field redefinitions

$$\hat{B}_\alpha = Z_\alpha{}^\beta B'_\beta, \qquad A^I = (Y^{-1})^I{}_J A'^J, \qquad \tilde{A}_I = Y^J{}_I \, \tilde{A}'_J. \tag{6.62}$$

---

4 The matrices Y and Z can be constructed as a product of the matrices that implement the familiar elementary operations on the rows and columns of $\Theta^{I\alpha}$.

In the new primed-variables, the Lagrangian takes the same form (6.49) but with primed quantities everywhere,

$$\mathcal{L}^{\Theta} = -\frac{1}{4}\mathcal{I}'_{IJ}(\phi)H'^{I}_{\mu\nu}H'^{J\mu\nu} + \frac{1}{8}\mathcal{R}'_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}H'^{I}_{\mu\nu}H'^{J}_{\rho\sigma}$$
$$-\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Theta'^{I\alpha}B'_{\alpha\mu\nu}\left(\tilde{F}'_{I\rho\sigma} - \frac{1}{4}\Theta'_{I}{}^{\beta}B'_{\beta\rho\sigma}\right), \tag{6.63}$$

with

$$H'^{I}_{\mu\nu} = F'^{I}_{\mu\nu} + \frac{1}{2}\Theta'^{I\alpha}B'_{\alpha\mu\nu}. \tag{6.64}$$

The new matrices $\mathcal{I}'$ and $\mathcal{R}'$ are given by

$$\mathcal{I}' = Y^{-T}\mathcal{I}Y^{-1}, \qquad \mathcal{R}' = Y^{-T}\mathcal{R}Y^{-1}. \tag{6.65}$$

These field redefinitions are local, so any local function of the old set of variables is also a local function of the new set of variables.

Using equation (6.61), it can be seen that the magnetic vector fields $\tilde{A}'_{\hat{U}}$ and the two-forms B'_{m} do not appear in the Lagrangian. This means that their gauge symmetries are in fact pure shift symmetries. In particular, a complete description of the gauge symmetries of the two-forms B'_{m} is actually given by $\delta B'_{m} = \epsilon'_{m}$ rather than $\delta B'_{m} = d\Xi'_{m}$, implying that the above set of gauge transformations is not complete.

Let us also redefine the gauge parameters as $\Xi'_{\alpha} = (Z^{-1})_{\alpha}{}^{\beta}\hat{\Xi}_{\beta}$, $\lambda'^{I} = Y^{I}{}_{J}\lambda^{J}$ and $\tilde{\lambda}'_{I} = (Y^{-1})^{J}{}_{I}\tilde{\lambda}_{J}$. The gauge variations of $A'^{\hat{U}}$, $A'^{\hat{I}}$, $\tilde{A}'_{\hat{I}}$ and B'_{i} are then

$$\delta A'^{\hat{U}}_{\mu} = \partial_{\mu}\lambda'^{\hat{U}}, \tag{6.66}$$

$$\delta A'^{\hat{I}}_{\mu} = \partial_{\mu}\lambda'^{\hat{I}} - \frac{1}{2}\theta^{\hat{I}i}\Xi'_{i\mu}, \tag{6.67}$$

$$\delta \tilde{A}'_{\hat{I}\mu} = \partial_{\mu}\tilde{\lambda}'_{\hat{I}} + \frac{1}{2}\tilde{\theta}_{\hat{I}}{}^{i}\Xi'_{i\mu}, \tag{6.68}$$

$$\delta B'_{i\mu\nu} = 2\partial_{[\mu}\Xi'_{i\nu]}. \tag{6.69}$$

They suggest the further changes of variables

$$\bar{A}^{i}_{\mu} = \theta^{\hat{I}i}\tilde{A}'_{\hat{I}\mu} + \tilde{\theta}_{\hat{I}}{}^{i}A'^{\hat{I}}_{\mu} \tag{6.70}$$

$$\Delta_{i\mu\nu} = B'_{i\mu\nu} + 2(\theta^{-1})_{i\hat{I}}F'^{\hat{I}}_{\mu\nu}, \tag{6.71}$$

which we complete in the A-sector by taking other independent linear combinations of the $A'^{\hat{I}}$, $\tilde{A}'_{\hat{I}}$, that can be taken to be for instance $A'^{\hat{I}}_{\mu}$. The change of variables is again such that any local function of the old set of variables is also a

local function of the new set of variables. Using the constraint $\theta^{\hat{I}i}\tilde{\theta}_{\hat{I}}{}^{j} = \theta^{\hat{I}j}\tilde{\theta}_{\hat{I}}{}^{i}$, one finds that the variation of the new variables simplifies to

$$\delta\bar{A}_\mu^i = \partial_\mu\eta^i, \quad \delta\Delta_{i\mu\nu} = 0, \quad \delta A_\mu^{'\hat{I}} = \epsilon_\mu^{\hat{I}}, \tag{6.72}$$

with the gauge parameters η^i and $\epsilon_\mu^{\hat{I}}$ defined as

$$\eta^i = \theta^{\hat{I}i}\tilde{\lambda}'_{\hat{I}} + \tilde{\theta}_{\hat{I}}{}^{i}\lambda'^{\hat{I}}, \tag{6.73}$$

$$\epsilon_\mu^{\hat{I}} = \partial_\mu\lambda'^{\hat{I}} - \frac{1}{2}\theta^{\hat{I}i}\Xi'_{i\mu}. \tag{6.74}$$

We have used a different symbol $\Delta_{i\mu\nu}$ to emphasize that it does not transform anymore as a generalized gauge potential. Because $\theta^{\hat{I}i}$ is invertible, the gauge parameters η^i and $\epsilon_\mu^{\hat{I}}$ provide an equivalent description of the gauge symmetries as (6.67)-(6.69). However, they provide an irreducible set of gauge symmetries contrary to description given by $\lambda'^{\hat{I}}$, $\tilde{\lambda}'_{\hat{I}}$ and $\Xi'_{i\mu}$.

Written in those variables, the Lagrangian only depends on n_v vector fields $A'^{\hat{U}}$ and $\bar{A}^i$ and on r two-forms Δ_i. The variables $A_\mu^{'\hat{I}}$ drop out in agreement with the shift symmetry $\delta A_\mu^{'\hat{I}} = \epsilon_\mu^{\hat{I}}$.

The Lagrangian is explicitly

$$\mathcal{L} = -\frac{1}{4}\mathcal{I}'_{IJ}(\phi)\bar{H}_{\mu\nu}^I\bar{H}^{J\mu\nu} + \frac{1}{8}\mathcal{R}'_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}\,\bar{H}_{\mu\nu}^I\bar{H}_{\rho\sigma}^J$$
$$-\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Delta_{i\mu\nu}\left(\bar{F}_{\rho\sigma}^i - \frac{1}{4}\tilde{\theta}_{\hat{I}}{}^{i}\theta^{\hat{I}j}\Delta_{j\rho\sigma}\right), \tag{6.75}$$

where the $\bar{H}^I$ are

$$\bar{H}^{\hat{I}} = \frac{1}{2}\theta^{\hat{I}i}\Delta_i, \quad \bar{H}^{\hat{U}} = F'^{\hat{U}}. \tag{6.76}$$

The gauge variations of the fields are

$$\delta A_\mu^{'\hat{U}} = \partial_\mu\lambda'^{\hat{U}}, \quad \delta\bar{A}_\mu^i = \partial_\mu\eta^i, \quad \delta\Delta_{i\mu\nu} = 0. \tag{6.77}$$

The two-forms Δ_i are auxiliary fields and can be eliminated from the Lagrangian (6.75), yielding a Lagrangian of the form (6.43) in a definite symplectic frame. This is because the relevant quadratic form is invertible, see section 5.1 of [WST05] for details.

One can get this Lagrangian more directly as follows. After the auxiliary fields are eliminated, the variables that remain are the scalar fields and the n_v vector fields $A'^{\hat{U}}$ and $\bar{A}^i$. We thus see that the embedding tensor determines a symplectic frame. The matrix $E \in Sp(2n_v, \mathbb{R})$ defining the symplectic frame

can be viewed in this approach as the function $E(\Theta)$ of the embedding tensor obtained through the above successive steps that lead to the final Lagrangian where only half of the potentials remain. Of course, there are ambiguities in the derivation of E from the embedding tensor, since choices were involved at various stages in the construction. One gets the matrix E up to a transformation of the stability subgroup of the Lagrangian subspace of the n_v electric potentials $A'^{\hat{U}}$ and $\bar{A}^i$.

More explicitly, one gets the matrix $E(\Theta)$ from the above construction as follows: the change of variables (6.62) can be written as

$$\begin{pmatrix} A^I \\ \tilde{A}_I \end{pmatrix} = \begin{pmatrix} (Y^{-1})^I{}_J & 0 \\ 0 & Y^J{}_I \end{pmatrix} \begin{pmatrix} A'^J \\ \tilde{A}'_J \end{pmatrix} \tag{6.78}$$

and the full change (6.70), including the trivial redefinitions of the other vector fields, is

$$\begin{pmatrix} A'^{\hat{I}} \\ A'^{\hat{U}} \\ \tilde{A}'_{\hat{I}} \\ \tilde{A}'_{\hat{U}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \delta^{\hat{I}}_{\hat{J}} & 0 \\ 0 & \delta^{\hat{U}}_{\hat{V}} & 0 & 0 \\ \delta^{\hat{J}}_{\hat{I}} & 0 & -(\theta^{-1})_{i\hat{I}}\,\tilde{\theta}_{\hat{j}}{}^i & 0 \\ 0 & 0 & 0 & \delta^{\hat{V}}_{\hat{U}} \end{pmatrix} \begin{pmatrix} \bar{A}_{\hat{j}} \\ \bar{A}^{\hat{V}} \\ \bar{A}^{\hat{J}} \\ \bar{A}_{\hat{V}} \end{pmatrix}, \tag{6.79}$$

where we defined

$$\bar{A}_{\hat{I}} = (\theta^{-1})_{i\hat{I}} \bar{A}^i \tag{6.80}$$

with respect to (6.70) in order to have the same kind of indices. The matrix $E(\Theta)$ is therefore simply given by

$$E(\Theta) = E_1 E_2, \tag{6.81}$$

where E_1 and E_2 are the matrices appearing in equations (6.78) and (6.79) respectively. Using the property

$$(\theta^{-1})_{i[\hat{I}}\tilde{\theta}_{\hat{j}]}{}^i = 0, \tag{6.82}$$

which follows from the last of (6.61) upon contractions with θ^{-1}, one can show that $E(\Theta)$ defined in this way is indeed a symplectic matrix. Once $E(\Theta)$ is known, the final Lagrangian in this symplectic frame, i.e. the Lagrangian that follows from the elimination of Δ_i in (6.75), is then simply the Lagrangian (6.43) with $\mathcal{I}'$ and $\mathcal{R}'$ determined from (6.42) where $\mathcal{M}'$ is given by

$$\mathcal{M}' = E(\Theta)^T \mathcal{M} E(\Theta). \tag{6.83}$$

In general, the matrix E will have an upper triangular part, and hence, will not belong to the stability subgroup of the original electric frame.

Finally, we make a remark that, as we have seen, it is the rank of the magnetic part of the embedding tensor that controls how many original magnetic fields become "electric". In particular, if the rank is zero, all original electric fields remain electric, while if the rank is maximum, all original magnetic fields become electric.

We delay the detailed discussion of deformations in the language of BRST cohomology to the end of Chapter 7 after we introduce BRST cohomology and the tools necessary to develop the discussion.

Chapter 7

BV-BRST Formalism and Deformation

7.1 A quick look at BV-BRST formalism

The nature as we know, including the phenomena we deal with in our every day life and those which happens in microscopic scales, is described by the gauge theory of fundamental interactions between particles. It is common for the gauge theories to be formulated in a local and manifestly covariant manner as it may facilitate otherwise computational difficulties. However, the price to be paid is to introduce the extra (non-propagating) degrees of freedom to render the locality and the covariance of the theories manifest. The theories then, in the presence of non-dynamical local degrees of freedom, show invariance under the symmetries which account for the freedom of field transformations while leaving the theories unchanged; these symmetries are known as gauge symmetries.

The presence of gauge symmetries specially for the non-abelian theories causes issues once one wants to quantize the classical gauge theories. Typically, a gauge-fixing procedure is used to remove the non-dynamical degrees of freedom. Ghost fields are used to compensate for the effects of the gauge degrees of freedom so as the final theory yet remains unitary. In the theory of electrodynamics in the linear gauges, ghosts decouple while in non-abelian gauge theories, convenient gauges generically involve interacting ghosts. The Faddeev-Popov quantization procedure [FP67; DeW67a; DeW67b; DeW67c], particularly well-defined in the path integral formulation of field theories, was proposed in order to address this issue. In this procedure, the presence of ghost fields is essential to produce the correct measure for the functional integral by introducing quadratic terms in the ghost fields in the Lagrangian. The gauge-fixed action then is invariant under

a nilpotent global symmetry, the so-called BRST symmetry [BRS74; BRS75; BRS76; Tyu75], involving transformations of both fields and ghosts.

There are though systems that cannot be treated by the Faddeev-Popov quantization procedure of which are the theories involving higher order ghost vertices and the gauge theories with reducible gauge group. An important example is supergravity where the algebra does not necessarily close off-shell, i.e. the commutator of two off-shell gauge symmetries will be written by terms proportional to the equations of motion. Moreover, in the presence of higher order p-forms, as in the supergravity, the gauge algebra is usually reducible.

Since the presence of ghost fields are useful in the discussion of quantization and renormalization of gauge theories (most importantly of the non-abelian ones), it is of interest to have a formalism that not only includes the ghost fields from the beginning but also encompasses the remnant BRST symmetry of the theory. The Batalin-Vilkovisky (BV) formulation, originally proposed by Zinn-Justin in the context of renormalization of Yang-Mills theories [Zin75; Zin02] and developed by Batalin and Vilkovisky to more generalized models [BV81; BV83b; BV83a; BV84; BV85], does indeed provide the appropriate means. It involves anti-fields which source the BRST symmetries and a canonical symplectic structure, given by an odd non-degenerate symplectic form on the space of fields and anti-fields known as the anti-bracket map, which controls the dynamics of the theory.

The BV formalism has many advantages; it enjoys the manifest gauge invariance, allows for a perturbative expansion of the quantum theory and makes possible to study the quantum corrections to the symmetry structure of the theory. Furthermore, the formalism can treat the theories with an open algebra or reducible gauge symmetries by introducing the extra fields and antifields, the ghosts of ghosts and their corresponding anti-fields and so on.

In the rest of this chapter, we introduce in Section 7.2 the BRST cohomology and in Section 7.3 show its importance in the BV deformation procedure. Afterward, in Section 7.5 we go back to the question asked in Chapter 6 that if the spaces of local deformations of the undeformed Lagrangian (6.49) of embedding tensor formalism and the one of scalar-vector model (6.43) are equivalent. We use the BV-BRST deformation techniques of Section 7.3 to have a thorough analysis and to answer to this question.

7.2 BRST cohomology

The BRST differential s is a nilpotent differential defined on the space of all fields (including ghost fields) $\Phi^A = \left(A^I_{p\mu}, \psi^i, C^I, ... \right)$ and their anti-fields $\Phi^*_A = \left(A^*_{pI\mu}, \psi^{*i}, C^*_I, ... \right)$ where $A^I_{p\mu}$ and ψ^i are p-forms and the matter fields respectively and ellipses stand for the ghosts of ghosts and their corresponding anti-

fields. Therefore, one has

$$s^2 = 0. \tag{7.1}$$

Then, the BRST cocycles are those objects K that are BRST-closed, i.e. elements of the kernel of s,

$$sK = 0, \tag{7.2}$$

and the BRST coboundaries are those BRST-closed objects that are BRST-exact, i.e. elements of the image of s,

$$sK = 0, \qquad K = sB. \tag{7.3}$$

The BRST cohomolgy is formally defined as

$$H(s) = \frac{\text{Ker}(s)}{\text{Im}(s)}. \tag{7.4}$$

An element in $H(s)$ is an equivalence class of BRST-cocycles, where two BRST-cocycles are identified if they differ by a BRST-coboundary.

One can consider the BRST cohomology in the space of local functions where the cochain K are local functions of the fields Φ^A and a finite number of their derivatives (and depend on spacetime coordinates) or in the space of local functionals where the elements of K are the integral of forms built from the local functions.

In the standard field theoretic setting, one insists on spacetime locality which implies that the cohomology is computed in the space of local functionals in the fields and antifields. In turn, this can be shown to be equivalent to the cohomology of s in the space of local functions up to total derivatives or, in form notation, to the cohomology of s in top form degree n, up to the horizontal exterior derivative of an $(n-1)$-form. The horizontal exterior derivative is not the de Rham differential but is instead given by $d = dx^\mu \partial_\mu$, where $\partial_\mu = \frac{\partial}{\partial x^\mu} + \partial_\mu z^\Sigma \frac{\partial}{\partial z^\Sigma} + \dots$ is the total derivative. Here z^Σ is referring to fields and antifields collectively, $z^\Sigma = (\Phi^A, \Phi_A^*)$.

Therefore the cocycle and coboundary conditions are written as

$$sa + dm = 0 \qquad \text{cocycle condition,} \tag{7.5}$$
$$a = sb + dn \qquad \text{coboundary condition,} \tag{7.6}$$

where a, b are n-form local functions and m, n are some $(n-1)$-form local functions. In the following we use $H(s)$ for the BRST-cohomology in the sapce of local functions and $H(s|d)$ for the BRST-cohomology in the space of local functionals. One could consider a grading of these cohomologies with the ghost numbers, then

one defines $H^{g,n}(s|d)$, isomorphic to $H^g(s)$, as the cohomology of local functions of form degree n and ghost number g modulo total derivatives

$$sa^{g,n} + da^{g+1,n-1} = 0, \quad a^{g,n} \sim a^{g,n} + sb^{g-1,n} + db^{g,n-1}. \qquad (7.7)$$

We already mentioned the importance of BV-BRST formalism in the quantization of gauge systems at the quantum and classical level. In the language of BRST cohomology, important problems at the quantum level like the gauge anomalies and the counterterms in order to render the gauge theories renormalizable are equivalent to finding $H^{1,n}(s|d)$ and $H^{0,n}(s|d)$, see [HT92].

At the classical level, the BRST cohomology is a useful tool to explain the generalized conservation laws and to build all consistent interactions that one can add to a free theory while preserving the number of gauge symmetries of starting theory. The former is captured by the characteristic cohomology $H^{n+g}_{\mathrm{char}}(d)$ and the latter is described by $H^{0,n}(s_0|d)$ and $H^{1,n}(s_0|d)$ of the undeformed theory [BBH00]. We will explain this in more detail in the next section.

7.3 Batalin-Vilkovisky antifield formalism

In order to systematically construct consistent interactions in gauge theories, it is useful to reformulate the problem in the context of algebraic deformation theory [Ger63; NR66; Jul86b; Jul86a]. The appropriate framework is provided by the Batalin-Vilkovisky antifield formalism [BV81; BV83a; BH93; Hen98].

The structure of an irreducible gauge system, i.e., the Lagrangian $\mathcal{L}_0$ with field content φ^a, generating set of gauge symmetries[1] $\delta_\epsilon \varphi^a = R^a{}_\alpha[\varphi^b](\epsilon^\alpha)$ and their algebra, is captured by the Batalin-Vilkovisky (BV) master action S (see e.g. [HT92; GPS95] for reviews). The master action is a ghost number 0 functional

$$S = \int d^n x \, \mathcal{L} = \int d^n x \left[\mathcal{L}_0 + \varphi_a^* R^a{}_\alpha (C^\alpha) + \frac{1}{2} C_\alpha^* f^\alpha{}_{\beta\gamma} (C^\beta, C^\gamma) + \dots \right], \quad (7.8)$$

that satisfies what is called the master equation

$$\frac{1}{2}(S, S) = 0. \qquad (7.9)$$

In this equation, the BV antibracket is the odd graded Lie bracket defined by

$$(X, Y) = \int d^n x \left[\frac{\delta^R X}{\delta \Phi^A(x)} \frac{\delta^L Y}{\delta \Phi_A^*(x)} - \frac{\delta^R X}{\delta \Phi_A^*(x)} \frac{\delta^L Y}{\delta \Phi^A(x)} \right] \qquad (7.10)$$

---

1 We use the condensed De Witt notation.

on the extended space $\Phi^A = (\varphi^a, C^\alpha, \dots)$ of original fields and ghosts (and ghosts for ghosts in the case of reducible gauge theories) and their antifields Φ^*_A. The ghost number and the Grassmann number of each fields and their anti-fields are summarized in Table (7.1). The Lagrangian, gauge variations and structure

A	puregh(A)	antifld(A)	gh(A)	$\epsilon(A)$
φ^a	0	0	0	0
C^I	1	0	1	1
φ^*_a	0	1	-1	1
C^*_a	0	2	-2	0

Table 7.1: The list of pure ghost number (puregh), anti-field (anti-ghost) number (antifld), ghost number (gh) and Grassmann number (ϵ) of each of the fields and anti-fields. The ghost number is defined as $\text{gh}(A) = \text{puregh}(A) - \text{antifld}(A)$, and we have in general $\text{gh}(\Phi^*_A) = -\text{gh}(\Phi^A) - 1$.

functions of the gauge algebra are contained in the first, second and third term of the master action (7.8) respectively.

For the deformation problem, one assumes the existence of an undeformed theory described by $S^{(0)}$ satisfying the master equation $\frac{1}{2}(S^{(0)}, S^{(0)}) = 0$ and one analyzes the conditions coming from the requirement that, in a suitable expansion, the deformed theory

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots, \tag{7.11}$$

satisfies the master equation (7.9). This results in a chain of equations at different order in deformation parameters

$$(S^{(0)}, S^{(0)}) = 0, \tag{7.12}$$

$$(S^{(0)}, S^{(1)}) = 0, \tag{7.13}$$

$$(S^{(0)}, S^{(2)}) + \frac{1}{2}(S^{(1)}, S^{(1)}) = 0, \tag{7.14}$$

$$\vdots$$

This can be easily proved using the properties of anti-bracket map, see for example section 4 of [GPS95] for a review of all properties of anti-bracket map. The deformed Lagrangian, gauge symmetries and structure functions can then be read off from the deformed master action (7.11).

The first condition (7.13) on the infinitesimal deformation $S^{(1)}$ is

$$(S^{(0)}, S^{(1)}) = 0. \tag{7.15}$$

This equation admits solutions $S^{(1)} = (S^{(0)}, \Xi)$, for all Ξ of ghost number -1. Such deformations can be shown to be trivial in the sense that they can be absorbed by (anticanonical) field-antifield redefinitions. Moreover, trivial deformations in that sense are always of the form $S^{(1)} = (S^{(0)}, \Xi)$ for some local Ξ. It thus follows that equivalence classes of deformations up to trivial ones are classified by $H^0(s)$, the ghost number zero cohomology of the antifield dependent BRST differential $s = (S^{(0)}, \cdot)$ of the undeformed theory,

$$[S^{(1)}] \in H^0(s). \tag{7.16}$$

For our problem of determining the most general deformation, we start by computing $H^0(s)$ and couple its elements with independent parameters to the starting point action to obtain $S^{(0)} + S^{(1)}$. The parameters thus play the role of generalized coupling constants. In a second step, we determine the constraints on these coupling constants coming from the existence of a completion such that (7.9) holds. The expansion is then in terms of homogeneity in these generalized coupling constants and not, as often done, in homogeneity of fields (in which case $S^{(0)}$ corresponds to an action quadratic in the fields). In particular, this approach treats the different types of symmetries involved in the determination of $H^0(s)$ on the same footing.

The BRST differential is defined on the undifferentiated fields and antifields by $s\Phi^A = -\frac{\delta^R \mathcal{L}}{\delta \Phi_A^*}$, $s\Phi_A^* = \frac{\delta^R \mathcal{L}}{\delta \Phi^A}$. It is extended to the derivatives through $[s, \partial_\mu] = 0$ resulting in $\{s, d\} = 0$. This reformulation allows one to use systematic homological techniques (descent equations) for the computation of these classes (see e.g. [DTV85a]).

At second order, the condition (7.14) on the infinitesimal deformation $S^{(1)}$ is

$$\frac{1}{2}(S^{(1)}, S^{(1)}) + (S^{(0)}, S^{(2)}) = 0. \tag{7.17}$$

The antibracket gives rise to a well defined map in cohomology[2],

$$(\cdot, \cdot) : H^{g_1}(s|d) \otimes H^{g_2}(s|d) \longrightarrow H^{g_1+g_2+1}(s|d). \tag{7.18}$$

For cocycles C_i with $[C_i] \in H^{g_i}(s|d)$, it is explicitly given by

$$([C_1], [C_2]) = [(C_1, C_2)] \in H^{g_1+g_2+1}(s|d). \tag{7.19}$$

---

2 In the following, whenever we refer to the cohomology of local functional of a top form-degree, we drop the index n and write it as $H^g(s|d)$.

Condition (7.17) constrains the infinitesimal deformation $S^{(1)}$ to satisfy

$$\frac{1}{2}([S^{(1)}],[S^{(1)}]) = [0] \in H^1(s|d). \tag{7.20}$$

If this is the case, $S^{(2)}$ in (7.17) is defined up to a cocycle in ghost number 0. Higher order brackets and constraints can be analyzed in a similar way, see e.g. [Ret93; FW01].

Besides the group $H^0(s|d)$ that describes infinitesimal deformations, and $H^1(s|d)$ that controls the obstructions to extending these to finite deformations, one can furthermore show [BBH95b] that $H^g(s|d) \simeq H^{n+g}_{\mathrm{char}}(d)$ for $g \leq -1$. The characteristic cohomology groups are defined by forms ω in the original fields φ^a such that

$$d\omega^{n+g} \approx 0, \quad \omega^{n+g} \sim \omega^{n+g} + d\eta^{n+g-1} + t^{n+g}, \tag{7.21}$$

with $t^{n+g} \approx 0$ and where ≈ 0 denote terms that vanish on all solutions to the Euler-Lagrange equations of motion. In particular, these groups can be shown to vanish for $g \leq -3$ in irreducible gauge theories [BBH95b; BBH00]. The group $H^{-2}(s|d)$ describes equivalence classes of "global" reducibility parameters, i.e., particular local functions f^α such that $R^a{}_\alpha(f^\alpha) \approx 0$ where $f^\alpha \sim f^\alpha + t^\alpha$ with $t^\alpha \approx 0$. This terminology reflects the fact that this cohomology may be non trivial even for (locally) irreducible gauge systems, in other words in the absence of p-form gauge fields with higher p. This will become clear momentarily and is crucial throughout our discussion. These classes correspond to global symmetries of the master action rather than of the original action alone [BHW96; BHW98]. The associated characteristic cohomology $H^{n-2}_{\mathrm{char}}(d)$ captures non-trivial (flux) conservation laws. More generally in the case of free abelian p-form gauge symmetry it was shown in [Jul80a] that one can generalize the first Noether theorem ($p=0$) and deduce by a similar formula a class of $H^{n-p-1}_{\mathrm{char}}(d)$ generalizing the electric flux which corresponds to the case $p=1$, i.e., to ordinary gauge invariance. The groups $H^{-1-p}(s|d)$ appear for p-form gauge theories and vanish for $p \geq 2$ in the irreducible case [HKS97]. The group $H^{-1}(s|d)$ describes and generates the inequivalent global symmetries, with $H^{n-1}_{\mathrm{char}}(d)$ encoding the associated inequivalent Noether currents. We mention these groups here since they play an important role in the determination of $H^0(s|d)$ as it will be seen in Section 8.1.

When $g_1 = -1 = g_2$, $(\cdot,\cdot) : H^{-1} \otimes H^{-1} \to H^{-1}$; in this case the antibracket map encodes the Lie algebra structure of the inequivalent global symmetries [BH96]. More generally, it follows from $(\cdot,\cdot) : H^{-1} \otimes H^g \to H^g$ that, for any ghost number g, the BRST cohomology classes form a representation of the Lie algebra of inequivalent global symmetries.

From here on, for notational simplicity, we will drop the square brackets when computing the antibracket map, but keep in mind that it involves classes and not their representatives.

7.3.1 Depth of an element

Consider a BRST-cocycle $\omega^{g,k}$ at ghost number g and form degree k. Using (7.5) or (7.7), any cocycle $\omega^{g,k}$ of the local BRST cohomology is associated with a (s,d)-descent

$$s\omega_l^{g,k} + d\omega_l^{g+1,k-1} = 0, \quad s\omega_l^{g+1,k-1} + d\omega_l^{g+2,k-2} = 0, \ \ldots, \ s\omega_l^{g+l,k-l} = 0, \ (7.22)$$

that stops at some BRST cocycle $\omega_l^{g+l,k-l}$. The length l of the shortest non trivial descent is called the "depth" of $[\omega^{g,k}] \in H^{g,k}(s|d)$. The last element $\omega_l^{g+l,k-l}$ is then non trivial in $H^{g+l,k-l}(s)$. The usefulness of the depth in analyzing the BRST cohomology is particularly transparent in [DTV85a; DTV85b; DTV86].

Local BRST cohomology classes $[\omega^{g,k}] \in H^{g,k}(s|d)$ are thus characterized, besides **ghost number** g and **form degree** k, by the **depth** l. Here, we explain how the antibracket map behaves with respect to the depth of its elements.

Both the covariantizable and non-covariantizable currents [3] as elements of $H^{-1,n}(s|d)$ are distinguished by the property that for which the depth is 1. The associated infinitesimal deformations as elements of $H^{0,n}(s|d)$ are distinguished by the property that the depth is deeper than one. Therefore, the following will be relevant when studying the obstruction to infinitesimal deformations.

Proposition 7.1. *The depth of an image of the antibracket map is less than or equal to the depth of its most shallow argument.*

Proof. Consider $[\omega_{l_1}^{g_1,n}], [\omega_{l_2}^{g_2,n}] \in H^{*,n}(s|d)$, where we can assume without loss of generality that $l_1 \geqslant l_2$. For the antibracket, let us not choose the expression with Euler-Lagrange derivatives on the left and right that is graded antisymmetric without boundary terms, but rather the one that satisfies a graded Leibniz rule on the right

$$(\omega^{g,n}, \cdot)_{\text{alt}} = \partial_{(\nu)} \frac{\delta^R(- \star \omega^{g,n})}{\delta\phi^A} \frac{\partial^L \cdot}{\partial\partial_{(\nu)}\phi_A^*} - (\phi^A \leftrightarrow \phi_A^*), \quad (7.23)$$

and the following version of the graded Jacobi identity without boundary terms,

$$(\omega^{g_1,n}, (\omega^{g_2,n}, \cdot)_{\text{alt}})_{\text{alt}} = ((\omega^{g_1,n}, \omega^{g_2,n})_{\text{alt}}, \cdot)_{\text{alt}}$$
$$+ (-)^{(g_1+1)(g_2+1)}(\omega^{g_2,n}, (\omega^{g_1,n}, \cdot)_{\text{alt}})_{\text{alt}} \quad (7.24)$$

(see appendix B of [BH96] for details and a proof). Furthermore,

$$(\omega^{g,n}, d(\cdot))_{\text{alt}} = (-)^{g+1} d((\omega^{g,n}, \cdot)_{\text{alt}}), \quad (d\omega^{g+1,n-1}, \cdot)_{\text{alt}} = 0. \quad (7.25)$$

---

3 By "covariantizable", we mean that one can choose the ambiguities in the Noether currents so as to take them gauge invariant; otherwise the current is "non-covariantizable".

Let $S = \int(-\star \mathcal{L})$ be the BV master action. We have $s\cdot = (-\star \mathcal{L}, \cdot)_{\text{alt}}$. Using these properties, we get

$$s(\omega_{l_1}^{g_1,n}, \omega_{l_2}^{g_2,n})_{\text{alt}} + d((\omega_{l_1}^{g_1,n}, \omega_{l_2}^{g_2+1,n-1})_{\text{alt}}) = 0, \dots, s(\omega_{l_1}^{g_1,n}, \omega_{l_2}^{g_2+l_2,n-l_2})_{\text{alt}} = 0, \tag{7.26}$$

which proves the proposition. $\square$

By using $[(\omega^{-1,n}, \omega^{g,n})] \in H^{g,n}(s|d)$, it follows that :

(i) Characteristic cohomology in degree $n-1$ described by $H^{-1,n}(s|d)$ is a Lie algebra. It is the Lie algebra of non trivial global symmetries. It also describes the Dirac or Dickey bracket algebra of non trivial conserved currents (up to constants or more generally topological classes),

(ii) $H^{-2,n}(s|d)$ is a module thereof (module structure of flux charges - Gauss or ADM type surface charges - under global symmetries). The proposition gives rise for instance to the following refinements:

Corollary 7.1. Covariantizable characteristic cohomology in form degree $n-1$ forms an ideal in the Lie algebra of characteristic cohomology in form degree $n-1$. The module action of covariantizable characteristic cohomology of degree $n-1$ on characteristic cohomology in degree $n-2$ is trivial.

Similar results hold for the associated infinitesimal deformations.

7.4 Properties of BRST differential

The BRST differential s is defined as $s = \gamma + \delta +$ "more" which acts on the extended phase space of fields and antifields. The differentials γ and δ are horizontal differential and Koszul-Tate differential respectively. The horizontal differential γ has a non-trivial action when acting on fields, while it vanishes on anti-fields. The Koszul-Tate differential δ has a non-trivial action on anti-fields but vanishes when acting on fields. The "more" part in the definition of BRST differential is introduced in order to guarantee the nilpotency of s, i.e. $s^2 = 0$. When the gauge transformations form an abelian group or the gauge algebra closes off-shell, one simply has [Hen90]

$$s = \gamma + \delta, \qquad s^2 = 0, \tag{7.27}$$

with δ and γ the graded commuting nilpotent differentials satisfying

$$\gamma^2 = 0, \quad \delta^2 = 0, \quad \gamma\delta + \delta\gamma = 0. \tag{7.28}$$

Also, δ and γ have the properties

$$\text{puregh}(\delta) = 0, \qquad \text{antifld}(\delta) = -1, \tag{7.29}$$

$$\text{puregh}(\gamma) = 1, \qquad \text{antifld}(\gamma) = 0, \tag{7.30}$$

such that the BRST differential s defined to increase the ghost number by 1.

In order to have a covariant formalism, the space of all observable quantities, the so-called phase space, has to be covariant. In the absence of gauge invariances, given the space of all possible smooth fields, a subspace of which is called the covariant phase space and is defined as the space Σ of all smooth fields which are the solutions of equations of motion. When there is a gauge symmetry, then the covariant phase space is comprised of all functions in Σ which are also gauge-invariant. Since the gauge transformations are integrable on-shell, therefore they generate a congruence of gauge orbits on Σ along which the gauge-invariant quantities are conserved. The Koszul-Tate differential δ and the horizontal differential γ in fact take care of the first and the second steps in defining gauge-invariant observables respectively.

The first step is possible to achieve with the help of the Koszul-Tate resolution [HT92], which states as

$$
\begin{aligned}
H_g(\delta) &= 0, \qquad g \neq 0, \\
H_0(\delta) &= C^\infty(\Sigma).
\end{aligned}
\tag{7.31}
$$

Then since the only non-vanishing homology of δ is $H_0(\delta)$, one can easily implement the second step by defining a differential along the orbits of gauge groups acting on the space $C^\infty(\Sigma)$ of smooth functions given by $H_0(\delta)$. It is shown that in general one has [Hen95]

$$
\begin{aligned}
H^g(s) &= H^g(\gamma, H_0(\delta)) \qquad g \geq 0, \\
H^g(s) &= 0 \qquad\qquad g < 0.
\end{aligned}
\tag{7.32}
$$

We should emphasize that the cohomology of s, γ and δ in the space of local functional are in general different and for example $H^{g,p}(s|d) \neq 0$ for $g < 0$. This is an important feature that we will discuss in the next chapter. For more details and complete review of definitions, properties of different differentials and the homological and cohomological arguments regarding each, see [HT92; BBH00; BBH95b].

7.5 Deformation of the Embedding tensor formalism

We discussed in Chapter 6 that the systems described by the Lagrangian (6.49), of the embedding tensor formalism, and the Lagrangian (6.43) of ungauged scalar-vector model are governed by the same physics and how one can obtain one Lagrangian from the other one by a series of symplectic transformations. However, the space of deformations of these models can in general be different. As we

just discussed through this chapter, the BRST cohomology provides us a strong tool to obtain the consistent deformation of a theory. In this section, we show that *the space of deformations of* (6.49) *and* (6.43) *are equivalent.*

We recall that the space of equivalence classes of non-trivial, consistent deformations of a local action is isomorphic to the BRST cohomology $H^0(s|d)$ in field-antifield space [BV81] at ghost number zero [BH93]. The question, then, is to determine how the variables appearing in addition to the standard variables of the Lagrangian (6.43) could modify $H^0(s|d)$.

From the discussion of Section 6.6, these extra variables are of two types:

- either they are of pure gauge type or equivalently invariant under arbitrary shifts, and therefore they drop out from the Lagrangian;

- or they are auxiliary fields appearing quadratically and undifferentiated in the Lagrangian, consequently they can be eliminated algebraically through their own equations of motion.

We collectively denote by W^α the fields of pure gauge type. So the W^α's stand for the n_v vector potentials $\tilde{A}'_{\hat{U}}$, $A'^{\hat{I}}$ and the $(\dim G - r)$ two-forms B'_m. The auxiliary fields are the r two-forms Δ_i. For convenience, we absorb in (6.75) the linear term in the auxiliary fields by a redefinition $\Delta_i \to \Delta'_i = \Delta_i + b_i$ where the term b_i is Δ_i-independent. Once this is done, the dependence of the Lagrangian on the auxiliary fields simply reads $\frac{1}{2}\kappa^{ij}\Delta'_i\Delta'_j$ with an invertible quadratic form κ^{ij}.

The BRST differential acting in the sector of the first type of variables read

$$sW^\alpha = C^\alpha\,, \quad sC^\alpha = 0\,, \quad sC^*_\alpha = W^*_\alpha\,, \quad sW^*_\alpha = 0 \tag{7.33}$$

where C^α are the ghosts of the shift symmetry and W^*_α, C^*_α the corresponding antifields, so that (C^α, W^α) and (W^*_α, C^*_α) form "contractible pairs" [HT92].

Similarly, the BRST differential acting in the sector of the second type of variables read

$$s\Delta'_i = 0\,, \quad s\Delta'^{*i} = \kappa^{ij}\Delta'_j \tag{7.34}$$

where Δ'^{*i} are the antifields conjugate to Δ'_i, showing that the auxiliary fields and their antifields form also contractible pairs since κ^{ij} is non degenerate.

Now, the above BRST transformations involve no spacetime derivatives so that one can construct a "contracting homotopy" [HT92] in the sector of the extra variables (W^α, Δ'_i), their ghosts and their antifields that commutes with the derivative operator ∂_μ. The algebraic setting is in fact the same as for the variables of the "non-minimal sector" of [BV81]. This implies that the extra variables neither contribute to $H^g(s)$ nor to $H^g(s|d)$ [BBH95b].

To be more specific, let us focus on the contractible pair (C^α, W^α). The analysis proceeds in exactly the same way for the other pairs. One can write the BRST differential in that sector as

$$s = \sum_{\{\mu\}} \partial_{\mu_1 \cdots \mu_s} C^\alpha \frac{\partial}{\partial_{\mu_1 \cdots \mu_s} W^\alpha} \tag{7.35}$$

where the sum is over all derivatives of the fields. Define the "contracting homotopy" ρ as

$$\rho = \sum_{\{\mu\}} \partial_{\mu_1 \cdots \mu_s} W^\alpha \frac{\partial}{\partial_{\mu_1 \cdots \mu_s} C^\alpha}. \tag{7.36}$$

One has by construction

$$[\rho, \partial_\mu] = 0, \tag{7.37}$$

just as

$$[s, \partial_\mu] = 0. \tag{7.38}$$

This is equivalent to $\rho d + d\rho = 0$. Furthermore, the counting operator N defined by

$$N = s\rho + \rho s \tag{7.39}$$

is explicitly given by

$$N = \sum_{\{\mu\}} \partial_{\mu_1 \cdots \mu_s} C^\alpha \frac{\partial}{\partial_{\mu_1 \cdots \mu_s} C^\alpha} + \sum_{\{\mu\}} \partial_{\mu_1 \cdots \mu_s} W^\alpha \frac{\partial}{\partial_{\mu_1 \cdots \mu_s} W^\alpha} \tag{7.40}$$

and commutes with the BRST differential,

$$[N, s] = 0. \tag{7.41}$$

The operator N gives the homogeneity degree in W^α, C^α and their derivatives. So, for the polynomial a, we have $Na = ka$ with k a non negative integer if and only if a is of degree k in W^α, C^α and their derivatives (by Euler theorem for homogeneous functions).

Because N commutes with s, we can analyze the cocycle condition

$$sa + db = 0 \tag{7.42}$$

at definite polynomial degree, i.e., assume that $Na = ka$, $Nb = kb$ where k is a non-negative integer. Our goal is to prove that the solutions of (7.42) are trivial when $k \neq 0$, i.e., of the form $a = se + df$, so that one can find, in any cohomological class of $H(s|d)$, a representative that does not depend on W^α or C^α – that is, W^α and C^α "drop from the cohomology".

To that end, we start from $a = N\left(\frac{a}{k}\right)$ $(k \neq 0)$, which we rewrite as $a = s\rho\left(\frac{a}{k}\right) + \rho s\left(\frac{a}{k}\right)$ using the definition of N. The first term is equal to $s\left(\rho\left(\frac{a}{k}\right)\right)$ and hence is BRST-exact. The second term is equal to $\rho\left(-d\left(\frac{b}{k}\right)\right)$ (using (7.42)), which is the same as $d\left(\rho\left(\frac{b}{k}\right)\right)$ since ρ and d anticommute. So it is d-exact. We thus have shown that

$$a = se + df \tag{7.43}$$

with $e = \rho\left(\frac{a}{k}\right)$ and $f = \rho\left(\frac{b}{k}\right)$. This is what we wanted to prove.

We can thus conclude that the local deformations ($H^0(s|d)$) – and in fact also $H^g(s|d)$ and in particular the candidate anomalies ($H^1(s|d)$) – are the same whether or not one includes the extra fields (W^α, Δ'_i). This is what we wanted to prove:

BRST cohomologies computed from the Lagrangians (6.49) and (6.43) are isomorphic.

As a final remark, we point out [BH08] where a dual formulation of three dimensional non-linear Einstein gravity with similar features have been studied.

Chapter 8

BRST cohomology of scalar-vector coupled models

In Chapter 6 we proved that not only the Lagrangian of embedding tensor formalism (6.49) and the one of ungauged scalar-vector model of the form (6.43) (in a symplectic frame determined by the coefficients of embedding tensor) are equivalent[1] but also the space of deformations of both theories are in fact isomorphic. In Chapter 7, we discussed that the space of infinitesimal deformations of a theory is entirely determined by the ghost number zero cohomology of BRST differential. We explicitly showed that $H_0(s|d)$ for both theories are isomorphic.

In this chapter, following the discussion of Section 7.3, we seek for a systematic study of all consistent deformations of general scalar-vector models described by the gauge invariant actions of the form

$$S_0[A_\mu^I, \phi^i] = \int d^4x \, \mathcal{L}_0, \tag{8.1}$$

depending on n_s uncharged scalar fields ϕ^i and n_v abelian vector fields A_μ^I. In what comes next we consider that the Lagrangian has Poincaré invariance (which is guaranteed in the presence of gravity) even though it is not much used for the most part of our computations in this chapter. It is in practice possible to work out the result in the case of non-Poincaré invariant Lagrangians with a little bit of effort. We assume that the only gauge symmetries of (8.1) are the standard $U(1)$ gauge transformations for each vector field, so that the gauge algebra is abelian and given by n_v copies of $\mathfrak{u}(1)$. A generating set of gauge invariances can

---

1 We recall that these Lagrangians are the same only after the elimination of auxiliary fields in (6.75) which leads to a Lagrangian of the form (6.43) in a definite symplectic frame.

be taken to be

$$\delta A_\mu^I = \partial_\mu \epsilon^I, \quad \delta\phi^i = 0. \tag{8.2}$$

The Lagrangian takes the form

$$\mathcal{L}_0 = \mathcal{L}_S[\phi^i] + \mathcal{L}_V[A_\mu^I, \phi^i], \tag{8.3}$$

where $\mathcal{L}_V$ is a function that depends on the vector fields through the abelian curvatures $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ only, and which can also involve the scalar fields ϕ^i. Derivatives of these variables are in principle allowed in the general analysis carried out below, but actually do not occur in the explicit Lagrangians discussed in more detail. The scalar fields can occur non linearly, e.g. terms of the form $\mathcal{I}_{IJ}(\phi)F_{\mu\nu}^I F^{J\mu\nu}$ as in the Lagrangian (5.15). In addition, the scalar Lagrangian need not to be quadratic in general case. We will discuss this in more detail below in Section 8.1.1.

The gauge transformations (8.2) are sometimes called "free abelian gauge transformations" to emphasize that the scalar fields are uncharged and do not transform under them. This does not mean that the abelian vector fields themselves are free since non linear terms (non minimal couplings) are allowed in (8.3).

This class of models contains the vector-scalar sectors of ungauged extended supergravities, of which $\mathcal{N} = 4$ [Das77; CS77; CSF78] and $\mathcal{N} = 8$ [CJS78; CJ79] supergravities offer prime examples. These will be considered in detail in Sections 8.3 and 8.4. Born-Infeld type generalizations [GR95] are also covered together with first order manifestly duality invariant formulations [DT76; BH11; BH10], which fall into this class when reformulated with suitable additional scalar fields [BG08].

From previous chapter, we recall that consistent deformations of a gauge invariant action are deformations that preserve the number (but not necessarily the form or the algebra) of the gauge symmetries. In the supergravity context, these are called "gaugings", and the deformed theories are called gauged supergravities, even though the undeformed theories possess already a gauge freedom. We shall consider only local deformations, i.e., deformations of the Lagrangian by functions of the fields and their derivatives up to some finite (but unspecified) order.

Gaugings in extended supergravities have a long history that goes back to [FD77; FS78; FV76; Zac78; Zac79]. For maximal supergravity, the first gauging has been performed in [WN82b] in the Lagrangian formulation of [CJ79], which involves a specific choice of so-called duality frame. More recent gaugings involving a change of the duality frame have been constructed in [DIT12].

These works consider from the very beginning deformations in which the vector fields become Yang-Mills connections for a non-abelian deformation of the

original abelian gauge algebra. The corresponding couplings are induced through the replacement of the abelian curvatures by non-abelian ones and the ordinary derivatives by covariant ones, plus possible additional couplings necessary for consistency. Given the question that whether this embraces all possible consistent deformations, it was shown in [Hen+18], see Sections 6.6 and 7.5, that the space of consistent deformations in the embedding formalism is isomorphic to the space of consistent deformations for the action (8.1) written in the duality frame picked by the choice of embedding tensor. For that reason, one can investigate the question of gaugings by taking (8.1) as starting point of the deformation procedure, provided one allows the scalar field dependence in the vector piece of the Lagrangian to cover all possible choices of duality frame. By doing so, one does not miss any of the gaugings available in the embedding tensor formalism. This is what we are going to do in this chapter.

Since the BV-BRST formalism [BH93] is a systematic way to explore deformations of theories with a gauge freedom, we completely characterize the BRST cohomology for the theories defined by (8.1), i.e., we completely characterize, in four spacetime dimensions, the deformations of abelian vector fields coupled non-minimally to scalar chargeless fields with a possibly non polynomial dependence on the (undifferentiated) scalar fields.

In particular, we show that besides the obvious deformations that consist in adding gauge invariant terms to the Lagrangian without changing the gauge symmetries, the gaugings can be related to the global symmetries of the action (8.1). These gaugings modify the form of the gauge transformations.

The global symmetries can be classified into two different types:

(i) global symmetries with covariantizable Noether currents, where by "covariantizable", we mean that one can choose the ambiguities in the Noether currents so as to take them gauge invariant (V-type symmetries),

(ii) global symmetries with non-covariantizable Noether currents which in turn can be subdivided into two subtypes:

(a) global symmetries with non-covariantizable Noether currents that lead to a deformation that does not modify the gauge algebra (W-type symmetries),

(b) global symmetries with non-covariantizable Noether currents that lead to a deformation that does modify also the gauge algebra (U-type symmetries).

Only the type (i) global symmetries directly gives rise to an infinitesimal consistent deformation through minimal coupling of the corresponding current to the vector potentials. The gaugings associated with the other types of global symmetries need to satisfy additional constraints.

The Noether current corresponding to the global symmetries of type (a) contains non-gauge invariant Chern-Simons terms that cannot be removed by suitably adjusting trivial contributions. The global symmetries of type (b) are associated with ordinary free abelian gauge symmetries with co-dimension 2 conservation laws (see e.g. [Jul80a] for an early discussion). The divergence of a current of type (a) is itself gauge invariant, while the divergence of a current of type (b) is not. Yang-Mills gaugings are associated with currents of type (b) and are hence of U-type. Topological couplings [WHR87] are associated with non-covariantizable Noether currents of either type (a) or (b). Charging deformations (if available), in which the scalar fields become charged but the gauge transformations of the vector fields are not modified and remain therefore abelian, are of V- or W-type.

The BRST deformation procedure applies not only to the consistent first order deformations, but also to higher orders where one might encounter obstructions. That procedure provides a natural deformation-theoretic interpretation of quadratic constraints and higher order constraints in terms of what is called the antibracket map.

After establishing general theorems on the BRST cohomology valid without assuming a specific form of the Lagrangian or the rigid symmetries, including the above classification of the deformations and useful triangular properties of their algebra, we turn to various models that have been considered in the literature, for which we completely compute the deformations of U and W-types.

In Section 8.1, we compute the local BRST cohomology of the models described by the action (8.1). This is done by following the method of [BBH95b; BBH95c] where the BRST cohomology was computed for arbitrary compact – in fact reductive – gauge group. The difficulty in the computation comes from the free abelian factors by which we mean abelian factors of the gauge algebra such that all matter fields are uncharged, i.e., invariant under the associated gauge transformations. This is precisely the case relevant to the action (8.1), which needs thus special care. The method of [BBH95b; BBH95c] is based on an expansion according to the antifield number. It makes direct contact with symmetries and conservation laws through the lowest antifield number piece of the BRST differential, that is the Koszul-Tate differential, which involves the equations of motion [FH90; Hen91]. The Noether charges appear through the characteristic cohomology, given by the local cohomology of the Koszul-Tate differential [BBH95b].

We then discuss in Section 8.2 the structure of the antibracket map, which is relevant for the consistency of the deformation at second order and the possible appearance of obstructions, and provide information on the structure of the global symmetry algebra.

The method of [BBH95b; BBH95c] provides the general structure of the BRST cocycles in terms of conserved currents. In order to reach more complete results,

one must use additional information specific to each model. We therefore specify further the models in Section 8.3, where we concentrate on scalar-coupled second order Lagrangians that are quadratic in the vector fields and their derivatives. These specialized models still cover the scalar-vector sectors of extended supergravities. Explicit examples are treated in detail to illustrate the method in Section 8.4, where complete results for the local BRST cohomology, up to the determination of V-type symmetries, are worked out. In Section 8.5, we then illustrate our techniques in the case of the manifestly duality-symmetric first order action (5.23) of [BH10], in the formulation of [BG08], which is adapted to the direct use of the methods developed here.

8.1 Abelian vector-scalar models in 4 dimensions

8.1.1 Structure of the models

We now apply the formalism discussed in Section 7.3 to the scalar-vector models described by the action (8.1). We write $\mathcal{L}_0 = \mathcal{L}_S[\phi^i] + \mathcal{L}_V[A_\mu^I, \phi^i]$. In four spacetime dimensions, there is no Chern-Simons term in the Lagrangian, which can be assumed to be strictly gauge invariant and not just invariant up to a total derivative. Gauge invariant functions are functions that depend on $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$, ϕ^i and their derivatives, but not on $A_\mu^I, \partial_{(\nu} A_{\mu)}^I, \partial_{(\nu_1} \partial_{\nu_2} A_{\mu)}^I$, etc. Thus $\mathcal{L}_V[A_\mu^I, \phi^i]$ depends on the vector potentials A_μ^I only through $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ and their derivatives.

We define

$$\frac{\delta \mathcal{L}_V}{\delta F_{\mu\nu}^I} = \frac{1}{2} (\star G_I)^{\mu\nu} \tag{8.4}$$

where the $(\star G_I)^{\mu\nu}$ are also manifestly gauge invariant functions. The equations of motion for the vector fields can be written as

$$\frac{\delta \mathcal{L}_0}{\delta A_\mu^I} = \partial_\nu (\star G_I)^{\mu\nu} \tag{8.5}$$

and the Lagrangian can be taken to be

$$\mathcal{L}_0 = \mathcal{L}_S[\phi^i] + \mathcal{L}_V[A_\mu^I, \phi^i], \quad d^4x \, \mathcal{L}_V = \int_0^1 \frac{dt}{t} [G_I F^I][t A_\mu^I, \phi^i]. \tag{8.6}$$

The vector Lagrangian $\mathcal{L}_V$ has been written in the form of a homotopic integral [And92]. It simply counts the number of vector fields in the integrand in order to produce the correct prefactor in each term coming from G_I. This is a compact way to include all possible terms in G_I. As an example consider the

case where G_I is a series of homogeneous functions of order n in the field strength F^I, i.e. $G_I = \sum_{n=1}(G_n)_I$ with

$$(G_n)_{I\mu\nu} = (\alpha_n)_{II_1\dots I_n}(F^n)^{I_1\dots I_n}_{\mu\nu} + (\beta_n)_{II_1\dots I_n}((\star F)^n)^{I_1\dots I_n}_{\mu\nu}$$
$$+ (\zeta_n)_{II_1\dots I_n}(F^k \wedge (\star F)^{n-k})^{I_1\dots I_n}_{\mu\nu}, \tag{8.7}$$

where $(\alpha_n)_{II_1\dots I_n}$, $(\beta_n)_{II_1\dots I_n}$ and $(\zeta_n)_{II_1\dots I_n}$ are in general functions of scalar fields ϕ^i. The Lagrangian is then written as

$$d^4x\,\mathcal{L}_V = \sum_n \frac{1}{n+1}(G_n)_I(A^J_\mu,\phi^i)F^I(A^J_\mu,\phi^i). \tag{8.8}$$

The associated solution to the BV master equation is given by

$$S^{(0)} = S_0 + \int d^4x\, A^{*\mu}_I \partial_\mu C^I. \tag{8.9}$$

The ghost number, antifield number and pure ghost number of the various fields and antifields are given in the Table (8.1).

	gh	antifld	puregh
ϕ^i	0	0	0
A^I_μ	0	0	0
C^I	1	0	1
ϕ^*_i	-1	1	0
$A^*_{I\mu}$	-1	1	0
C^*_I	-2	2	0

Table 8.1: The list of pure ghost number (puregh), anti-field number (antifld) and ghost number (gh) of each of the fields and anti-fields. The ghost number is defined as $\mathrm{gh}(A) = \mathrm{puregh}(A) - \mathrm{antifld}(A)$, and we have in general $\mathrm{gh}(\Phi^*_A) = -\mathrm{gh}(\Phi^A) - 1$.

The BRST differential s splits according to antifield number as

$$s = \delta + \gamma \tag{8.10}$$

and the action of each differentials on all field and antifields are summarized in the Table (8.2).

	s	γ	δ
ϕ^i	0	0	0
A^I_μ	$\partial_\mu C^I$	$\partial_\mu C^I$	0
C^I	0	0	0
ϕ^*_i	$\frac{\delta \mathcal{L}_0}{\delta \phi^i}$	0	$\frac{\delta \mathcal{L}_0}{\delta \phi^i}$
$A^*_{I\mu}$	$\partial_\nu (\star G_I)^{\mu\nu}$	0	$\partial_\nu (\star G_I)^{\mu\nu}$
C^*_I	$-\partial_\mu A^{*\mu}$	0	$-\partial_\mu A^{*\mu}$

Table 8.2: The action of BRST differential s, the horizontal differential γ and the Koszul-Tate differential δ on each field and antifield.

The Koszul-Tate differential δ and the differential γ have antifield number -1 and 0 respectively and we have

$$\delta^2 = 0, \quad \delta\gamma + \gamma\delta = 0, \quad \gamma^2 = 0. \tag{8.11}$$

In terms of the Koszul-Tate differential, the cocycle condition for m in characteristic cohomology takes the form $dm + \delta n = 0$. This equation is the same as the (co)cycle condition for n in the local (co)homology of δ, which is indeed $\delta n + dm = 0$. Using this observation, and vanishing theorems for $H(d)$ and $H(\delta)$ in relevant degrees, one can establish isomorphisms between the characteristic cohomology and $H(\delta|d)$ [BBH95b]. For example, the characteristic cohomology $H^{n-2}_{\text{char}}(d)$ is given by the 2-forms $\mu^I G_I$, while $H^n_2(\delta|d)$ is given by the 4-forms $d^4x\,\mu^I C^*_I$. The isomorphism is realized through the (δ, d)-descent

$$\delta\, d^4x\, C^*_I + d \star A^*_I = 0, \quad \delta \star A^*_I + dG_I = 0, \tag{8.12}$$

where $A^*_I = dx^\mu A^*_{I\mu}$.

8.1.2 Consistent deformations

One can characterize the BRST cohomological classes with non trivial antifield dependence in terms of conserved currents and rigid symmetries for all values of the ghost number. For definiteness, we illustrate explicitly the procedure for $H^0(s|d)$ in maximum form degree, which defines the consistent local deformations. We consider next the case of general ghost number.

The main equation to be solved, see (7.5), for a is

$$sa + db = 0, \tag{8.13}$$

where a has form degree 4 and ghost number 0. To solve it, we expand the cocycle a according to the antifield number,

$$a = a_0 + a_1 + a_2. \tag{8.14}$$

Because a has total ghost number zero, each term a_n has antifield number n and pure ghost number (degree in the ghosts) n as well. As shown in [BBH95c], the expansion stops at most at antifield number 2. The term a_0 is the (first order) deformation of the Lagrangian. A non-vanishing a_1 corresponds to a deformation of the gauge variations, while a non-vanishing a_2 corresponds to a deformation of the gauge algebra. All three terms are related by the cocycle condition (8.13).

8.1.2.1 Solutions of U-type (a_2 non trivial)

The first case to consider is when a_2 is non-trivial. This defines "class I" solutions in the terminology of [BBH95c], which we call here "U-type" solutions to comply with the general terminology introduced below. One has from the general theorems of [BBH95b; BBH95c] on the invariant characteristic cohomology that

$$a_2 = d^4x\, C_I^* \Theta^I \tag{8.15}$$

with

$$\Theta^I = \frac{1}{2!} f^I{}_{J_1 J_2} C^{J_1} C^{J_2}. \tag{8.16}$$

Here $f^I{}_{J_1 J_2}$ are some constants, antisymmetric in J_1, J_2. The reason why the coefficient $d^4x\, C_I^*$ of the ghosts in a_2 is determined by the characteristic cohomology follows from the equation $\delta a_2 + \gamma a_1 + db_1 = 0$ that a_2 must fulfill in order for a to be a cocycle of $H(s|d)$. Given that a_2 has antifield number equal to 2, and the isomorphisms $H^{g,n}(s|d) \simeq H^n_{-g}(\delta|d) \simeq H^{n+g}_{\text{char}}(d|\delta)$ for negative g, it is the characteristic cohomology in form degree $n - 2 = 2$ that is relevant[2]. The emergence of the characteristic cohomology in the computation of $H(s|d)$ will be observed again for a_1 below, where it will be the conserved currents that appear. This central feature follows from the fact that the Koszul-Tate differential, which encapsulates the equations of motion, is an essential building block of the BRST differential. We must now find the lower terms $a_1 + a_0$ and relate them as expected to Noether currents that correspond to $H^4_1(\delta|d)$.

By the argument of section 8 of [BBH95c] suitably generalized in section 12, the term a_1 is then found to be

$$a_1 = \star A_I^* A^K \partial_K \Theta^I + m_1 \tag{8.17}$$

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2 The precise way to express the relation between the local cohomology of δ and the highest term of the equation obeyed by a is given in section 7 of [BBH95c]: a_2 must be a non trivial representative of invariant cohomology $H_{inv}(\delta|d)$, more precisely it must come from $H^4_{inv,2}(\delta|d)$ in ghost number zero. This relates the U-type deformations to the free abelian factors of the undeformed gauge group.

where $\gamma m_1 = 0$ and $\partial_K = \frac{\partial}{\partial C^K}$. The term m_1 (to be determined by the next equation) is linear in C^I and can be taken to be linear in the undifferentiated antifields A_I^* and ϕ_i^* since derivatives of these antifields, which can occur only linearly, can be redefined away through trivial terms. We thus write

$$m_1 = \hat{K} = \star A_I^* \hat{g}^I - \star \phi_i^* \hat{\Phi}^i \tag{8.18}$$

with

$$\hat{g}^I = dx^\mu g^I_{\mu K} C^K , \qquad \hat{\Phi}^i = \Phi^i_K C^K. \tag{8.19}$$

Here $g^I_{\mu K}$ and Φ^i_K are gauge invariant functions which are arbitrary at this stage but will be constrained by the requirement that a_0 exists.

We must now consider the equation $\delta a_1 + \gamma a_0 + db_0 = 0$ that determines a_0 up to a solution of $\gamma a_0' + db_0' = 0$. This equation is equivalent to

$$\left(\frac{\delta \mathcal{L}_0}{\delta A^I_\mu} \delta_K A^I_\mu + \frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_K \phi^i \right) C^K + \gamma \alpha_0 + \partial_\mu \beta^\mu_0 = 0, \tag{8.20}$$

where we have passed to dual notations ($a_0 = d^4 x \, \alpha_0$, $db_0 = d^4 x \, \partial_\mu \beta^\mu_0$) and where we have set

$$\delta_K A^I_\mu = A^J_\mu f^I{}_{JK} + g^I_{\mu K}, \qquad \delta_K \phi^i = \Phi^i_K. \tag{8.21}$$

Writing $\beta^\mu_0 = j^\mu_K C^K +$ "terms containing derivatives of the ghosts", we read from (8.20), by comparing the coefficients of the undifferentiated ghosts, that

$$\frac{\delta \mathcal{L}_0}{\delta A^I_\mu} \delta_K A^I_\mu + \frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_K \phi^i + \partial_\mu j^\mu_K = 0. \tag{8.22}$$

A necessary condition for a_0 (and thus a) to exist is therefore that $\delta_K A^I_\mu$ and $\delta_K \phi^i$ define symmetries.

To proceed further and determine a_0, we observe that the non-gauge invariant term $\frac{\delta \mathcal{L}_0}{\delta A^I_\mu} A^J_\mu f^I{}_{JK}$ in $\frac{\delta \mathcal{L}_0}{\delta A^I_\mu} \delta_K A^I_\mu$ can be written as $\partial_\mu \left(\star G_I^{\nu\mu} A^J_\nu f^I{}_{JK} \right)$ plus a gauge invariant term, so that $j^\mu_K - \star G_I^{\mu\nu} A^J_\nu f^I{}_{JK}$ has a gauge invariant divergence. Results on the invariant cohomology of d [BDK90; Dub+92] imply then that the non-gauge invariant part of such an object can only be a Chern-Simons form, i.e. $j^\mu_K - \star G_I^{\mu\nu} A^J_\nu f^I{}_{JK} = J^\mu_K + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} A^I_\nu F^J_{\rho\sigma} h_{I|JK}$, or

$$j^\mu_K = J^\mu_K + \star G_I^{\mu\nu} A^J_\nu f^I{}_{JK} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} A^I_\nu F^J_{\rho\sigma} h_{I|JK} \tag{8.23}$$

where J^μ_K is gauge invariant and where the symmetries of the constants $h_{I|JK}$ will be discussed in a moment. It is useful to point out that one can switch the indices I and J modulo a trivial term.

The equation (8.20) becomes $-(\partial_\mu j^\mu_K) C^K + \gamma\alpha_0 + \partial_\mu\beta^\mu_0 = 0$, i.e., $j^\mu_K(\gamma A^K_\mu) + \gamma\alpha_0 + \partial_\mu\beta'^\mu_0 = 0$. The first two terms in the current yield manifestly γ-exact terms,

$$J^\mu_K(\gamma A^K_\mu) = \gamma(J^\mu_K A^K_\mu), \quad \star G^{\mu\nu}_I A^J_\nu f^I{}_{JK}(\gamma A^K_\mu) = \frac{1}{2}\gamma(\star G^{\mu\nu}_I A^J_\nu f^I{}_{JK} A^K_\mu) \quad (8.24)$$

and so $h_{I|JK}$ must be such that the term $A^I F^J dC^K h_{I|JK}$ is by itself γ-exact modulo d. This is a problem that has been much studied in the literature through descent equations (see e.g. [DTV86]). It has been shown that $h_{I|JK}$ must be antisymmetric in J, K and should have vanishing totally antisymmetric part in order to be "liftable" to a_0 and non-trivial,

$$h_{I|JK} = h_{I|[JK]}, \quad h_{[I|JK]} = 0. \quad (8.25)$$

Putting things together, one finds for a_0

$$a_0 = A^I \partial_I \hat{J} + \frac{1}{2} G_I A^K A^L \partial_L \partial_K \Theta^I + \frac{1}{2} F^I A^K A^L \partial_L \partial_K \Theta'_I \quad (8.26)$$

where

$$\hat{J} = \star dx^\mu J_{\mu K} C^K, \quad \Theta'_I = \frac{1}{2} h_{I|J_1 J_2} C^{J_1} C^{J_2}. \quad (8.27)$$

A non-trivial U-solution modifies the gauge algebra. Deformations of the Yang-Mills type belong to this class. A U-solution is characterized by constants $f^I{}_{J_1 J_2}$ which are antisymmetric in J_1, J_2. These constants must be such that there exist gauge invariant functions $g^I_{\mu K}$ and Φ^i_K such that $\delta_K A^I_\mu$ and $\delta_K \phi^i$ define symmetries of the undeformed Lagrangian. Here $\delta_K A^I_\mu$ and $\delta_K \phi^i$ are given by (8.21). Furthermore, the h-term in the corresponding conserved current (if any) must fulfill (8.25). The deformation a_0 of the Lagrangian takes the Noether-like form.

Given the "head" a_2 of a U-type solution, characterized by a set of $f^I{}_{J_1 J_2}$'s, the lower terms a_1 and a_0, and in particular the h-piece, are not uniquely determined. One can always add solutions of W, V or I-types described below, which have the property that they have no a_2-piece. Hence one may require that the completion of the "head" a_2 of a U-type solution should be chosen to vanish when a_2 itself vanishes. But this leaves some freedom in the completion of a_2, since for instance any W-type solution multiplied by a component of $f^I{}_{J_1 J_2}$ will vanish when the $f^I{}_{J_1 J_2}$'s are set to zero. The situation has a triangular nature since two U-type solutions with the same a_2 differ by solutions of "lower" types, for which there might not be a canonical choice.

Note that further constraints on $f^I{}_{J_1 J_2}$ (notably the Jacobi identity) arise at second order in the deformation parameter.

8.1.2.2 Solutions of W and V-type (vanishing a_2 but a_1 non trivial)

These solutions are called "class II" solutions in [BBH95c]. Here we have

$$a = a_0 + a_1 \tag{8.28}$$

and a_1 can be taken to be gauge invariant, i.e., annihilated by γ [BBH95c]. We thus find

$$a_1 = \hat{K} = \star A_I^* \hat{g}^I - \star \phi_i^* \hat{\Phi}^i \tag{8.29}$$

with

$$\hat{g}^I = dx^\mu g_{\mu K}^I C^K , \quad \hat{\Phi}^i = \Phi_K^i C^K . \tag{8.30}$$

Here $g_{\mu K}^I$ and Φ_K^i are again gauge invariant functions, which we still denote by the same letters as above, although they are independent from the similar functions related to the constants $f^I{}_{J_1 J_2}$. We also set

$$\delta_K A_\mu^I = g_{\mu K}^I, \quad \delta_K \phi^i = \Phi_K^i. \tag{8.31}$$

The equation $\delta a_1 + \gamma a_0 + db_0 = 0$ implies then, as above,

$$\frac{\delta \mathcal{L}_0}{\delta A_\mu^I} \delta_K A_\mu^I + \frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_K \phi^i + \partial_\mu j_K^\mu = 0 . \tag{8.32}$$

A necessary condition for a_0 (and thus a) to exist is therefore that $\delta_K A_\mu^I$ and $\delta_K \phi^i$ given by (8.31) define symmetries. Equation (8.32) take the same form as Eq. (8.22), but there is an important difference: the divergence of the current j_K^μ is now gauge invariant, unlike in (8.22) that the divergence of current, due to the contribution coming from a_2, is not gauge invariant.

The current takes the form

$$j_K^\mu = J_K^\mu + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} A_\nu^I F_{\rho\sigma}^J h_{I|JK}, \tag{8.33}$$

(with $h_{I|JK}$ fulfilling the above symmetry properties) yielding

$$a_0 = A^I \partial_I \hat{J} + \frac{1}{2} F^I A^K A^L \partial_L \partial_K \Theta_I' \tag{8.34}$$

where still

$$\hat{J} = \star dx^\mu J_{\mu K} C^K , \quad \Theta_I' = \frac{1}{2} h_{I|J_1 J_2} C^{J_1} C^{J_2}. \tag{8.35}$$

We define W-type solutions to have $h_{I|JK} \neq 0$, while V-type solutions have $h_{I|JK} = 0$. Both these types deform the gauge transformations but not their algebra (to first order in the deformation). They are determined by rigid symmetries of the undeformed Lagrangian with gauge invariant variations (8.31). The

V-type have gauge invariant currents, while the currents of the W-type contain a non-gauge invariant piece.

Note that again, the solutions of W- and V-types are determined up to a solution of lower type with no a_1-"head", and that there might not be a canonical choice. In fact one may require similarly that W-type transformations become trivial when $h_{I|JK}$ tends to zero.

8.1.2.3 Solutions of I-type (vanishing a_2 and a_1)

In that case,

$$a = a_0 \tag{8.36}$$

with $\gamma a_0 + db_0 = 0$.

Since there is no Chern-Simons term in four dimensions, one can assume that $b_0 = 0$. The deformation b_0 is therefore a gauge invariant function, i.e., a function of the abelian curvatures $F_{\mu\nu}^I$, the scalar fields, and their derivatives. The I-type deformations neither deform the gauge transformations nor (a fortiori) the gauge algebra. Born-Infeld deformations belong to this type. They are called "class III" solutions in [BBH95c].

8.1.3   Local BRST cohomology at other ghost numbers

8.1.3.1 h-terms

The previous discussion can be repeated straightforwardly at all ghost numbers. The analysis proceeds as above. The tools necessary to handle the "h-term" in the non gauge invariant "currents" have been generalized to higher ghost numbers through familiar means and can be found in [DTV85b; DTV85a; DTV86].

The h-terms belong to the "small" or "universal" algebra involving only the 1-forms A^I, the 2-forms $F^I = dA^I$, the ghosts C^I and their exterior derivative. The product is the exterior product. One describes the h-term through a (γ, d)-descent equation and what is called the "bottom" of that descent, which is annihilated by γ and has form degree < 4 in four dimensions. The only possibilities in the free abelian case are the 2-forms

$$\frac{1}{m} h_{I|J_1 \cdots J_m} F^I C^{J_1} \cdots C^{J_m} \tag{8.37}$$

where

$$h_{I|J_1 \cdots J_m} = h_{I|[J_1 \cdots J_m]}. \tag{8.38}$$

One can assume $h_{[I|J_1 \cdots J_m]} = 0$ since the totally antisymmetric part gives a trivial bottom. The lift of this bottom goes two steps, up to the 4-form

$$h_{I|J_1 J_2 \cdots J_m} F^I F^{J_1} C^{J_2} \cdots C^{J_m} \tag{8.39}$$

producing along the way a 3-form

$$h_{I|J_1 J_2 \cdots J_m} F^I A^{J_1} C^{J_2} \cdots C^{J_m} \tag{8.40}$$

which has the property of not being gauge (BRST) invariant although its exterior derivative is invariant (modulo trivial terms).

8.1.3.2 Explicit description of cohomology

By applying the above method, one finds that the local BRST cohomology of the models of Section 8.1.1 can be described along exactly the same lines as given below. Note that the cohomology at negative ghost numbers reflect general properties of the characteristic cohomology that go beyond the mere models considered here [BBH95b].

(i) $H^g(s|d)$ is empty for $g \leqslant -3$.

(ii) $H^{-2}(s|d)$ is represented by the 4-forms

$$U^{-2} = \mu^I d^4 x \, C_I^*. \tag{8.41}$$

If $A_I^* = dx^\mu A_{I\mu}^*$, the associated descent equations are

$$s \, d^4 x \, C_I^* + d \star A_I^* = 0, \quad s \star A_I^* + dG_I = 0, \quad sG_I = 0. \tag{8.42}$$

Characteristic cohomology $H_{\text{char}}^{n-2}(d)$ is then represented by the 2-forms $\mu^I G_I$.

(iii) Several types of cohomology classes in ghost numbers $g \geqslant -1$, which we call U, V and W-type, can be described by constants $f^I{}_{JK_1...K_{g+1}}$ which are antisymmetric in the last $g+2$ indices,

$$f^I{}_{JK_1...K_{g+1}} = f^I{}_{[JK_1...K_{g+1}]}, \tag{8.43}$$

and constants $h_{I|JK_1...K_{g+1}}$ that are antisymmetric in the last $g+2$ indices but without any totally antisymmetric part[3],

$$h_{I|JK_1...K_{g+1}} = h_{I|[JK_1...K_{g+1}]}, \quad h_{[I|JK_1...K_{g+1}]} = 0, \tag{8.44}$$

together with gauge invariant functions $g^I_{\mu K_1...K_{g+1}}, \Phi^i_{K_1...K_{g+1}}$ that are antisymmetric in the last $g+1$ indices. They are constrained by the requirement that the transformations

$$\delta_{K_1...K_{g+1}} A_\mu^I = A_\mu^J f^I{}_{JK_1...K_{g+1}} + g^I_{\mu K_1...K_{g+1}}, \quad \delta_{K_1...K_{g+1}} \phi^i = \Phi^i_{K_1...K_{g+1}}, \tag{8.45}$$

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3 We write $h_{IJ} := h_{I|J}$ for $g = -1$.

define symmetries of the action in the sense that

$$\frac{\delta \mathcal{L}_0}{\delta A_\mu^I} \delta_{K_1\dots K_{g+1}} A_\mu^I + \frac{\delta \mathcal{L}_0}{\delta \phi^i} \delta_{K_1\dots K_{g+1}} \phi^i + \partial_\mu j_{K_1\dots K_{g+1}}^\mu = 0, \qquad (8.46)$$

with currents $j_{K_1\dots K_{g+1}}^\mu$ that are antisymmetric in the last $g+1$ indices. This can be made more precise by making the gauge (non-)invariance properties of these currents manifest. One finds

$$j_{K_1\dots K_{g+1}}^\mu = J_{K_1\dots K_{g+1}}^\mu + \star G_I^{\mu\nu} A_\nu^J f^I{}_{JK_1\dots K_{g+1}} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} A_\nu^I F_{\rho\sigma}^J h_{I|JK_1\dots K_{g+1}},$$
$$(8.47)$$

where $J_{K_1\dots K_{g+1}}^\mu$ is gauge invariant and antisymmetric in the lower $g+1$ indices. When taking into account that

$$G_I F^J = d(G_I A^J + \star A_I^* C^J) + s(\star A_I^* A^J + d^4x C_I^* C^J), \quad F^I F^J = d(A^I F^J), \tag{8.48}$$

and defining

$$\Theta^I = \frac{1}{(g+2)!} f^I{}_{J_1\dots J_{g+2}} C^{J_1\dots J_{g+2}},$$

$$\Theta'_I = \frac{1}{(g+2)!} h_{I|J_1\dots J_{g+2}} C^{J_1\dots J_{g+2}},$$

$$\hat{J} = \star dx^\mu J_{\mu K_1\dots K_{g+1}} \frac{1}{(g+1)!} C^{K_1\dots K_{g+1}},$$

$$\hat{K} = (\star A_I^* \hat{g}^I - \star \phi_i^* \hat{\Phi}^i), \tag{8.49}$$

$$\hat{g}^I = \frac{1}{(g+1)!} dx^\mu g_{\mu K_1\dots K_{g+1}}^I C^{K_1\dots K_{g+1}},$$

$$\hat{\Phi}^i = \frac{1}{(g+1)!} \Phi_{K_1\dots K_{g+1}}^i C^{K_1\dots K_{g+1}},$$

where $C^{K_1\dots K_g} = C^{K_1}\dots C^{K_g}$, the "global symmetry" condition (8.46) is equivalent to a (s,d)-obstruction equation,

$$G_I F^J \partial_J \Theta^I + F^I F^J \partial_J \Theta'_I + s(\hat{K} + A^I \partial_I \hat{J}) + d\hat{J} = 0, \qquad (8.50)$$

with $\partial_I = \frac{\partial}{\partial C^I}$. Note that the last two terms combine into

$$d[\star dx^\mu J_{\mu K_1\dots K_{g+1}}] \frac{1}{(g+1)!} C^{K_1}\dots C^{K_{g+1}},$$

so that this equation involves gauge invariant quantities only. It is this form that arises in a systematic analysis of the descent equations. One can now distinguish the three types of solutions.

a) U-type corresponds to solutions with non vanishing $f^I{}_{JK_1...K_{g+1}}$ and particular ${}^U h_{I|JK_1...K_{g+1}}$, ${}^U g^I_{\mu K_1...K_{g+1}}$, ${}^U \Phi^i_{K_1...K_{g+1}}$, ${}^U J_{\mu K_1...K_{g+1}}$ that vanish when the f's vanish (and that may be vanishing even when the f's do not). As we explained above, different choices of the particular completion ${}^U h_{I|JK_1...K_{g+1}}$, ${}^U g^I_{\mu K_1...K_{g+1}}$, ${}^U \Phi^i_{K_1...K_{g+1}}$, ${}^U J_{\mu K_1...K_{g+1}}$ of a_2 exist and there might not be a canonical one, but a completion exists if the U-type solution is indeed a solution. Similar ambiguity holds for the solutions of W and V-types described below. A U-type solution is trivial if and only if f's vanish. Denoting by $\hat{K}_U, \hat{J}_U, (\Theta'_U)_I$, the expressions as in (8.49) but involving the particular solutions, the associated BRST cohomology classes are represented by

$$U = (d^4x C^*_I + \star A^*_I A^K \partial_K + \frac{1}{2} G_I A^K A^L \partial_L \partial_K) \Theta^I$$
$$+ \hat{K}_U + \frac{1}{2} F^I A^K A^L \partial_L \partial_K (\Theta'_U)_I + A^I \partial_I \hat{J}_U, \quad (8.51)$$

with $sU + d(\star A^*_I \Theta^I + G_I A^J \partial_J \Theta^I + F^I A^J \partial_J (\Theta'_U)_I + \hat{J}_U) = 0$;

b) W-type corresponds to solutions with vanishing f's but non vanishing $h_{I|JK_1...K_{g+1}}$ and particular ${}^W g^I_{\mu K_1...K_{g+1}}$, ${}^W \Phi^i_{K_1...K_{g+1}}$, ${}^W J_{\mu K_1...K_{g+1}}$ that may be chosen to vanish when the h's vanish. Such solutions are trivial when the h's vanish. With the obvious notation, the associated BRST cohomology classes are represented by

$$W = \hat{K}_W + \frac{1}{2} F^I A^K A^L \partial_L \partial_K \Theta'_I + A^I \partial_I \hat{J}_W, \quad (8.52)$$

with $sW + d(F^I A^J \partial_J \Theta'_I + \hat{J}_W) = 0$;

c) V-type corresponds to solutions with vanishing f's and h's. They are represented by

$$V = \hat{K}_V + A^I \partial_I \hat{J}_V, \quad (8.53)$$

with $sV + d\hat{J}_V = 0$ and $s\hat{J}_V = 0$. V and its descent have depth 1.

(iv) Lastly, I-type cohomology classes exist in ghost numbers $g \geqslant 0$ and are described by

$$\hat{I} = d^4x \frac{1}{g!} I_{K_1...K_g} C^{K_1} \ldots C^{K_g} \quad (8.54)$$

with $s\hat{I} = 0$, i.e., gauge invariant $I_{K_1...K_g}$ that are completely antisymmetric in the K indices. Such classes are to be considered trivial if the $I_{K_1...K_s}$ vanish on-shell up to a total derivative. This can again be made more precise

by making the gauge (non-)invariance properties manifest: an element of I-type class is trivial if and only if

$$d^4x\, I_{K_1\ldots K_g} \approx dJ_{K_1\ldots K_g} + m^I{}_{JK_1\ldots K_g}G_I F^J + \frac{1}{2}F^I F^J m'_{IJK_1\ldots K_g}, \quad (8.55)$$

where $J_{K_1\ldots K_g}$ are gauge invariant 3 forms that are completely antisymmetric in the K indices, while $m^I{}_{JK_1\ldots K_g}, m'_{IJK_1\ldots K_g}$ are constants that are completely antisymmetric in the last $g+1$ indices. Note also that the on-shell vanishing terms in (8.55) need to be gauge invariant. When there are suitable restrictions on the space of gauge invariant functions (such as for instance x^μ independent, Lorentz invariant polynomials with power counting restrictions) one may sometimes construct an explicit basis of non-trivial gauge invariant 4 forms, in the sense that if $d^4xI \approx \rho^{\mathcal{A}}I_{\mathcal{A}} + d\omega^3$ and $\rho^{\mathcal{A}}I_{\mathcal{A}} \approx d\omega^3$, then $\rho^{\mathcal{A}} = 0$. The associated BRST cohomology classes are then parametrized by constants $\rho^{\mathcal{A}}{}_{K_1\ldots K_g}$.

At a given ghost number $g \geqslant -1$, the cohomology is the direct sum of elements of type U, W, V and also I when $g \geqslant 0$.

This completes our general discussion of the local BRST cohomology. Reference [BBH95c] also considered simple factors in addition to the abelian factors, as well as any spacetime dimension ≥ 3. One can extend the above results to cover these cases. The computation of the local BRST cohomology $H^{*,*}(s|d)$ for gauge models involving general reductive gauge algebras has been done in [BB18] by following the different route adopted in [BBH00], which did not consider free abelian factors in full generality. As requested by the analysis of the deformations of the action (8.1), reference [BB18] generalizes Theorem 11.1 of [BBH00] to arbitrary reductive Lie algebras that include also (free) abelian factors (and in any spacetime dimension ≥ 3).

8.1.3.3 Depth of solutions

The depth of the various BRST cocycles plays a key role in the analysis of the higher-order consistency condition. The U-type and W-type solutions have depth 2 because they involve $A_\mu j^\mu$ with a non-gauge invariant current. The V-type solutions have depth 1 because the Noether term $A_\mu j^\mu$ of them involves a gauge invariant current. Finally, I-type solutions clearly have depth 0.

8.2 Antibracket map and structure of symmetries

8.2.1 Antibracket map in cohomology

We now investigate the antibracket map $H^g \otimes H^{g'} \to H^{g+g'+1}$ for the different types of cohomology classes described above. It follows from the detailed discussion of the cohomology in Section 8.1.3 that the shortest non trivial length of descents, the "depth", of elements of type U, W, V, I is 2, 2, 1, 0. In particular, the antibracket map is sensitive to the depth of its arguments: as we have seen in Section 7.3.1 the depth of the map is less than or equal to the depth of its most shallow element.

The antibracket map involving $U^{-2} = \mu^I d^4 x \, C_I^*$ in H^{-2} is given by

$$(\cdot, U^{-2}) : H^g \to H^{g-1}, \quad \omega^{g,n} \mapsto \frac{\delta^R \omega^{g,n}}{\delta C^I} \mu^I. \tag{8.56}$$

More explicitly, it is trivial for $g = -2$. It is also trivial for $g = -1$ except for U-type where it is described by $f^I{}_J \mapsto f^I{}_J \mu^J$. For $g > 0$, it is described by $\rho^{\mathcal{A}}_{K_1 \ldots K_g} \mapsto \rho^{\mathcal{A}}_{K_1 \ldots K_g} \mu^{K_g}$ for I-type, $k^{v_1}{}_{K_1 \ldots K_{g+1}} \mapsto k^{v_1}{}_{K_1 \ldots K_{g+1}} \mu^{K_{g+1}}$ for V-type, $h_{IJK_1 \ldots K_{g+1}} \mapsto h_{IJK_1 \ldots K_{g+1}} \mu^{K_{g+1}}$ and $f^I{}_{JK_1 \ldots K_{g+1}} \mapsto f^I{}_{JK_1 \ldots K_{g+1}} \mu^{K_{g+1}}$ for U- and W-type.

The antibracket map for $g, g' \geqslant -1$ has the following triangular structure:

$(\cdot, \cdot)$	U	W	V	I
U	$U \oplus W \oplus V \oplus I$	$W \oplus V \oplus I$	$V \oplus I$	I
W	$W \oplus V \oplus I$	$W \oplus V \oplus I$	$V \oplus I$	I
V	$V \oplus I$	$V \oplus I$	$V \oplus I$	I
I	I	I	I	0

$$\tag{8.57}$$

Indeed, $(\hat{I}, \hat{I}') = 0$ because I-type cocycles can be chosen to be antifield independent. For all other brackets involving I-type cocycles, it follows from discussion of Section 7.3.1 that the result must have depth 0 and the only such classes are of I type. Alternatively, since all cocycles can be chosen to be at most linear in antifields, the result will be a cocycle that is antifield independent and only classes of I-type have trivial antifield dependence. It thus follows that I-type cohomology forms an abelian ideal.

Again given the discussion of Section 7.3.1, the depth of the antibracket map of V-type cohomology with V, W, U-type is less or equal to 1, so it must be of V- or I-type.

Finally, the remaining structure follows from the fact that only brackets of U-type cocycles with themselves may give rise to terms that involve C_I^*'s.

8.2.2 Structure of the global symmetry algebra

Let us now concentrate on brackets between two elements that have both ghost number -1, i.e., on the detailed structure of the Lie algebra of inequivalent global symmetries when taking into account their different types.

In this case, one may use the table above supplemented by the fact that $I^{-1} = 0$. Let then

$$U_u, \quad W_w, \quad V_v, \tag{8.58}$$

be bases of symmetries of U, W, V-type[4]. At ghost number $g = -1$, equations (8.51), (8.52), (8.53) give

$$V_v = K_v, \quad W_w = K_w, \quad U_u = (f_u)^I{}_J [d^4 x\, C_I^* C^J + \star A_I^* A^J] + K_u. \tag{8.59}$$

It follows from (8.57) that V-type symmetries and the direct sum of V and W-type symmetries form ideals in the Lie algebra of inequivalent global symmetries.

The symmetry algebra $\mathfrak{g}_U$ is defined as the quotient of all inequivalent global symmetries by the ideal of $V \oplus W$-type symmetries. In particular, if U-type symmetries form a sub-algebra, it is isomorphic to $\mathfrak{g}_U$.

First, V-type symmetries are parametrized by constants k^v, $V^{-1} = k^v V_v$. The gauge invariant symmetry transformation on the original fields then are

$$\delta_v A_\mu^I = -(V_v, A_\mu^I) = g_{v\mu}{}^I, \quad \delta_v \phi^i = -(V_v, \phi^i) = \Phi_v^i. \tag{8.60}$$

Furthermore, there exist constants $C^{v_3}{}_{v_1 v_2}$ such that

$$([V_{v_1}], [V_{v_2}]) = -C^{v_3}{}_{v_1 v_2}[V_{v_3}] \tag{8.61}$$

holds for the cohomology classes. We choose the minus sign because

$$(V_{v_1}, V_{v_2}) = -d^4 x (A_I^{*\mu}[\delta_{v_1}, \delta_{v_2}] A_\mu^I + \phi_i^*[\delta_{v_1}, \delta_{v_2}]\phi^i), \tag{8.62}$$

so that the $C^{v_3}{}_{v_1 v_2}$ are the structure constants of the commutator algebra of the V-type symmetries, $[\delta_{v_1}, \delta_{v_2}] = C^{v_3}{}_{v_1 v_2} \delta_{v_3}$. For the functions $g_{v\mu}{}^I$ and Φ_v^i, this gives

$$\begin{aligned}
\delta_{v_1} g_{v_2\mu}{}^I - \delta_{v_2} g_{v_1\mu}{}^I &= C^{v_3}{}_{v_1 v_2} g_{v_3\mu}{}^I + \text{(trivial)} \\
\delta_{v_1} \Phi_{v_2}^i - \delta_{v_2} \Phi_{v_1}^i &= C^{v_3}{}_{v_1 v_2} \Phi_{v_3}^i + \text{(trivial)}.
\end{aligned} \tag{8.63}$$

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4 These are bases in the cohomological sense, i.e., $\sum_u \lambda^u [U_u] = [0] \Rightarrow \lambda^u = 0$ (and similarly for W_w and V_v). In terms of the representatives, this becomes $\sum_u \lambda^u U_u = sa + db \Rightarrow \lambda^u = 0$.

The "trivial" terms on the right hand side take the form "(gauge transformation) + (antisymmetric combination of the equations of motion)" which is the usual ambiguity in the form of global symmetries, see e.g. section 6 of [BBH00]. They come from the fact that equation (8.61) holds for classes: for the representatives V_v themselves, (8.61) is $(V_{v_1}, V_{v_2}) = -C^{v_3}{}_{v_1 v_2} V_{v_3} + sa + db$. The trivial terms in (8.63) are then the symmetries generated by the extra term $sa + db$, which is zero in cohomology. The graded Jacobi identity for the antibracket map implies the ordinary Jacobi identity for these structure constants,

$$C^{v_1}{}_{v_2[v_3} C^{v_2}{}_{v_4 v_5]} = 0. \tag{8.64}$$

Next, W-type symmetries are parametrized by constants k^w, $W^{-1} = k^w W_w$ and encode the gauge invariant symmetry transformations

$$\delta_w A_\mu^I = -(W_w, A_\mu^I) = g_{w\mu}{}^I, \quad \delta_w \phi^i = -(W_w, \phi^i) = \Phi_w^i \tag{8.65}$$

with associated Noether 3 forms $j_W = k^w (h_w)_{IJ} F^{(I} A^{J)} + k^w J_{Ww}$. There then exist $C^{v_2}{}_{wv_1}$, $C^{w_3}{}_{w_1 w_2}$, $C^v{}_{w_1 w_2}$ such that

$$\begin{aligned}
([W_w], [V_v]) &= -C^{v_2}{}_{wv} [V_{v_2}], \\
([W_{w_1}], [W_{w_2}]) &= -C^{w_3}{}_{w_1 w_2} [W_{w_3}] - C^v{}_{w_1 w_2} [V_v],
\end{aligned} \tag{8.66}$$

with associated Jacobi identities that we do not spell out. For the functions $g_{w\mu}{}^I$ and Φ_w^i, this implies

$$\delta_w g_{v\mu}{}^I - \delta_v g_{w\mu}{}^I = C^{v_2}{}_{wv} g_{v_2\mu}{}^I, \quad \delta_w \Phi_v^i - \delta_v \Phi_w^i = C^{v_2}{}_{wv} \Phi_{v_2}^i, \tag{8.67}$$

$$\delta_{w_1} g_{w_2\mu}{}^I - \delta_{w_2} g_{w_1\mu}{}^I = C^{w_3}{}_{w_1 w_2} g_{w_3\mu}{}^I + C^v{}_{w_1 w_2} g_{v\mu}{}^I, \tag{8.68}$$

$$\delta_{w_1} \Phi_{w_2}^i - \delta_{w_2} \Phi_{w_1}^i = C^{w_3}{}_{w_1 w_2} \Phi_{w_3}^i + C^v{}_{w_1 w_2} \Phi_v^i, \tag{8.69}$$

up to trivial terms, see the discussion below (8.63).

Finally, U-type symmetries are parametrized by k^u, $U^{-1} = k^u U_u$ and encode the symmetry transformations

$$\begin{aligned}
\delta_u A_\mu^I &= -(U_u, A_\mu^I) = (f_u)^I{}_J A_\mu^J + g_{u\mu}{}^I, \quad \delta_u \phi^i = -(U_u, \phi^i) = \Phi_u^i, \\
\delta_u A_I^{*\mu} &= -(f_u)^K{}_I A_K^{*\mu} - \frac{\delta}{\delta A_\mu^I} (A_K^{*\nu} g_{u\nu}{}^K + \phi_i^* \Phi_u^i), \\
\delta_u \phi_i^* &= -\frac{\delta}{\delta \phi^i} (A_K^{*\nu} g_{u\nu}{}^K + \phi_j^* \Phi_u^j), \\
\delta_u C^I &= (f_u)^I{}_J C^J, \quad \delta_u C_I^* = -(f_u)^K{}_I C_K^*.
\end{aligned} \tag{8.70}$$

Again, there exist constants C with various types of indices such that

$$([U_u], [V_v]) = -C^{v_2}{}_{uv}[V_{v_2}], \tag{8.71}$$

$$([U_u], [W_w]) = -C^{w_2}{}_{uw}[W_{w_2}] - C^{v}{}_{uw}[V_v], \tag{8.72}$$

$$([U_{u_1}], [U_{u_2}]) = -C^{v}{}_{u_1 u_2}[V_v] - C^{w}{}_{u_1 u_2}[W_w] - C^{u_3}{}_{u_1 u_2}[U_{u_3}], \tag{8.73}$$

with associated Jacobi identities. Working out the term proportional to C_I^* in (U_{u_1}, U_{u_2}) gives the commutation relations for the $(f_u)^I{}_J$ matrices,

$$[f_{u_1}, f_{u_2}] = -C^{u_3}{}_{u_1 u_2} f_{u_3}. \tag{8.74}$$

In turn, this implies Jacobi identities for this type of structure constants alone:

$$C^{u_1}{}_{u_2[u_3} C^{u_2}{}_{u_4 u_5]} = 0. \tag{8.75}$$

The $C^{u_3}{}_{u_1 u_2}$ are the structure constants of $\mathfrak{g}_U$.

From equation (8.71), we get the identities

$$\delta_u g_{v\mu}{}^I - \delta_v g_{u\mu}{}^I - (f_u)^I{}_J g_{v\mu}{}^J = C^{v_2}{}_{uv} g_{v_2\mu}{}^I, \quad \delta_u \Phi_v^i - \delta_v \Phi_u^i = C^{v_2}{}_{uv} \Phi_{v_2}^i. \tag{8.76}$$

Equation (8.72) gives the same identities with the right-hand side replaced by the appropriate sum, i.e.

$$\delta_u g_{w\mu}{}^I - \delta_w g_{u\mu}{}^I - (f_u)^I{}_J g_{w\mu}{}^J = C^{w_2}{}_{uw} g_{w_2\mu}{}^I + C^{v}{}_{uw} g_{v\mu}{}^I,$$
$$\delta_u \Phi_w^i - \delta_w \Phi_u^i = C^{w_2}{}_{uw} \Phi_{w_2}^i + C^{v}{}_{uw} \Phi_v^i. \tag{8.77}$$

Considering the terms proportional to antifields $A_I^{*\mu}$ and ϕ_i^*, the equation (8.73) gives

$$\delta_{u_1} g_{u_2\mu}{}^I - (f_{u_1})^I{}_J g_{u_2\mu}{}^J - (u_1 \leftrightarrow u_2) = C^{u_3}{}_{u_1 u_2} g_{u_3\mu}{}^I$$
$$+ C^{w}{}_{u_1 u_2} g_{w\mu}{}^I + C^{v}{}_{u_1 u_2} g_{v\mu}{}^I,$$
$$\delta_{u_1} \Phi_{u_2}^i - \delta_{u_2} \Phi_{u_1}^i = C^{u_3}{}_{u_1 u_2} \Phi_{u_3}^i + C^{w}{}_{u_1 u_2} \Phi_w^i + C^{v}{}_{u_1 u_2} \Phi_v^i. \tag{8.78}$$

Equations (8.76)-(8.78) are again valid only up to trivial symmetries.

Let us now concentrate on identities containing the h_{IJ}, which appear in the currents of U and W-type. We first consider (U_u, W_w) projected to W-type. Here, we use the alternative form of anti-bracket defined in Section 7.3.1 since it makes the computation more easier. Then, we have

$$s(U_u, W_w)_{\text{alt}} = -d(U_u, (h_w)_{IJ} F^I A^J + J_w)_{\text{alt}}$$
$$= d\{(h_w)_{IJ}[(f_u)^I{}_K F^K A^J + F^I (f_u)^J{}_K A^K] + \text{invariant}\}. \tag{8.79}$$

When comparing this to s applied to the right hand side of (8.72), using $sW_w = -dj_{Ww}$, $sV_v = -dJ_{Vv}$ and the fact that W-type cohomology is characterized by the Chern-Simons term in its Noether current, we get

$$(h_w)_{IN}(f_u)^I{}_M + (h_w)_{MI}(f_u)^I{}_N = C^{w_2}{}_{uw}(h_{w_2})_{MN}. \tag{8.80}$$

This computation amounts to identifying the Chern-Simons term in the U-variation $\delta_u j_{Ww}$ of a current of W-type. The same computation applied to (W_{w_1}, W_{w_2}) shows that $C^{w_3}{}_{w_1 w_2}(h_{w_3})_{MN} = 0$, which implies

$$C^{w_3}{}_{w_1 w_2} = 0 \tag{8.81}$$

since the matrices h_w are linearly independent (otherwise, the W_w would not form a basis). In other words, the W-variation $\delta_{w_1} j_{W w_2}$ of a current of W-type is gauge invariant up to trivial terms, i.e., is of V-type.

In order to work out (U_{u_1}, U_{u_2}) projected to W-type, a slightly involved reasoning gives

$$\delta_u G_I + (f_u)^J{}_I G_J \approx -2(h_u)_{IJ}F^J + \lambda_u^w (h_w)_{IJ}F^J + d(\text{invariant}) \tag{8.82}$$

for some constants λ_u^w. This is proved in Appendix D in the case where G_I does not depend on derivatives of F^I (but can have otherwise arbitrary dependence of F^I). We were not able to find the analog of (8.82) in the higher derivative case.

Applying then $(U_{u_1}, \cdot)_{\text{alt}}$ to the chain of descent equations for U_{u_2} and adding the chain of descent equations for $C^{u_3}_{u_1 u_2} U_{u_3}$ yields

$$(h_{u_2})_{IN}(f_{u_1})^I{}_M + (h_{u_2})_{MI}(f_{u_1})^I{}_N - (h_{u_1})_{IN}(f_{u_2})^I{}_M - (h_{u_1})_{MI}(f_{u_2})^I{}_N$$
$$+ \frac{1}{2}\left[(h_w)_{IN}(f_{u_2})^I{}_M + (h_w)_{IM}(f_{u_2})^I{}_N\right]\lambda_{u_1}^w$$
$$= C^{u_3}{}_{u_1 u_2}(h_{u_3})_{MN} + C^w{}_{u_1 u_2}(h_w)_{MN}. \tag{8.83}$$

Again, this amounts to identifying the Chern-Simons terms in the U-variation $\delta_{u_1} j_{u_2}$ of a U-type current. Equation (8.82) is crucial for this computation since U-type currents contain G_I. Using (8.80), this becomes

$$(h_{u_2})_{IN}(f_{u_1})^I{}_M + (h_{u_2})_{MI}(f_{u_1})^I{}_N - (h_{u_1})_{IN}(f_{u_2})^I{}_M - (h_{u_1})_{MI}(f_{u_2})^I{}_N$$
$$= C^{u_3}{}_{u_1 u_2}(h_{u_3})_{MN} + \left[C^w{}_{u_1 u_2} - \frac{1}{2}C^w{}_{u_2 w_2}\lambda_{u_1}^{w_2}\right](h_w)_{MN}. \tag{8.84}$$

We see that the effect of the λ_u^w is to shift the structure constants of type $C^w{}_{u_1 u_2}$. The constants λ_u^w vanish for the explicit models considered below; it would be interesting to find an explicit example where this is not the case. As a last

comment, we note that antisymmetry of equation (8.84) in u_1 and u_2 imposes the constraint

$$C^w{}_{u_2 w_2} \lambda^{w_2}_{u_1} + C^w{}_{u_1 w_2} \lambda^{w_2}_{u_2} = 0 \tag{8.85}$$

on the constants λ^w_u.

8.2.3 Parametrization through symmetries

It follows from the discussion of the antibracket map involving H^{-2} after (8.56) that cohomologies of U, W, V-type in ghost numbers $g \geqslant 0$ can be parametrized by symmetries of the corresponding type with suitably constrained coefficients

$$k^u{}_{K_1 \ldots K_{g+1}}, \quad k^v{}_{K_1 \ldots K_{g+1}}, \quad k^w{}_{K_1 \ldots K_{g+1}}. \tag{8.86}$$

In this way, for $g = 0$, the problem of finding all infinitesimal gaugings can be reformulated as the question of which of these symmetries can be gauged.

In order to do this, it is useful to first rewrite the $h_{I|JK_1 \ldots K_{g+1}}$ appearing in the cohomology classes of U and W-types in the equivalent symmetric convention

$$X_{IJ,K_1 \ldots K_{g+1}} := h_{(I|J)K_1 \ldots K_{g+1}} \iff h_{I|JK_1 \ldots K_{g+1}} = \frac{2(g+2)}{g+3} X_{I[J,K_1 \ldots K_{g+1}]} \tag{8.87}$$

where (8.44) is now replaced by

$$X_{IJ,K_1 \ldots K_{g+1}} = X_{(IJ),[K_1 \ldots K_{g+1}]}, \quad X_{(IJ,K_1)K_2 \ldots K_{g+1}} = 0 . \tag{8.88}$$

Note that for $g = -1$, $h_{IJ} = X_{IJ}$.

For cohomology classes of U, W-type, we can write

$$f^I{}_{JK_1 \ldots K_{g+1}} = (f_u)^I{}_J \, k^u{}_{K_1 \ldots K_{g+1}}, \tag{8.89}$$

$$^U X_{IJ,K_1 \ldots K_{g+1}} = (h_u)_{IJ} \, k^u{}_{K_1 \ldots K_{g+1}}, \tag{8.90}$$

$$X_{IJ,K_1 \ldots K_{g+1}} = (h_w)_{IJ} \, k^w{}_{K_1 \ldots K_{g+1}}, \tag{8.91}$$

where $(f_u)^I{}_J$, $(h_u)_{IJ}$ and $(h_w)_{IJ}$ appear in the basis elements U_u and W_w. (One has similar parametrizations for the quantities $g^I_{\mu K_1 \ldots K_{g+1}}$, $\Phi^i_{K_1 \ldots K_{g+1}}$, $J_{\mu K_1 \ldots K_{g+1}}$ in the cohomology classes of the various types.) This guarantees that condition (8.46) (or (8.50)) is automatically satisfied.

However, the symmetry properties (8.43) and (8.88) imply the following linear constraints on the parameters:

$$(f_u)^I{}_{(J} \, k^u{}_{K_1) K_2 \ldots K_{g+1}} = 0, \tag{8.92}$$

$$(h_u)_{(IJ} \, k^u{}_{K_1) K_2 \ldots K_{g+1}} = 0, \tag{8.93}$$

$$(h_w)_{(IJ} \, k^w{}_{K_1) K_2 \ldots K_{g+1}} = 0. \tag{8.94}$$

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From the discussion of the cohomology, it also follows that V-type cohomology classes are entirely determined by V-type symmetries in terms of $k^v{}_{K_1 \dots K_{g+1}}$ without any additional constraints.

8.2.4   2nd order constraints on deformations and gauge algebra

The most general infinitesimal gauging is given by $S^{(1)} = \int (U^0 + W^0 + V^0 + I^0)$. We have

$$\frac{1}{2}(S^{(1)}, S^{(1)}) = \int \left(U^1 + W^1 + V^1 + I^1 \right). \tag{8.95}$$

The infinitesimal deformation $S^{(1)}$ can be extended to second order whenever the right hand side vanishes in cohomology, resulting in quadratic constraints on the constants $k^{u_1}{}_K$, $k^{w_1}{}_K$, $k^{v_1}{}_K$ and $\rho^{\mathcal{A}}$. Working all of them out explicitly requires computing all brackets between U^0, W^0, V^0 and I^0.

However, it follows from the previous section that the only contribution to U^1 comes from $\frac{1}{2}(U^0, U^0)$. The vanishing of the terms containing the antighosts C_I^* requires

$$f^I{}_{J[K_1} f^J{}_{K_2 K_3]} = 0, \tag{8.96}$$

i.e., the Jacobi identity for the $f^I{}_{JK}$. The associated n_v-dimensional Lie algebra is the gauge algebra and is denoted by $\mathfrak{g}_g$.

Using $f^I{}_{JK} = (f_{u_1})^I{}_J k^{u_1}{}_K$ and equation (8.74), the Jacobi identity reduces to the following quadratic constraint on $k^{u_1}{}_K$:

$$k^{u_1}{}_I k^{u_2}{}_J C^{u_3}{}_{u_1 u_2} - (f_{u_4})^K{}_I k^{u_4}{}_J k^{u_3}{}_K = 0. \tag{8.97}$$

Note that the antisymmetry in IJ of the second term is guaranteed by the linear constraint (8.92). The terms at antifield number 1 give the constraints

$$\delta_I g^K_J + f^K{}_{MJ} g^M_I - (I \leftrightarrow J) = f^L{}_{IJ} g^K_L \tag{8.98}$$

$$\delta_I \Phi^i_J - (I \leftrightarrow J) = f^L{}_{IJ} \Phi^i_L. \tag{8.99}$$

Expressed with k's, this gives

$$k^\Gamma{}_I k^\Delta{}_J C^\Sigma{}_{\Gamma\Delta} - (f_u)^K{}_I k^u{}_J k^\Sigma{}_K = 0, \tag{8.100}$$

where the capital Greek indices take all values u, w, v. This gives three constraints, according to the type of the free index Σ. When $\Sigma = u$, we get the constraint (8.97), because the only non-vanishing structure constants with an

upper u index are the $C^{u_3}{}_{u_1u_2}$. When $\Sigma = w$, the possible structure constants are $C^{w_3}{}_{w_1w_2}$, $C^{w_3}{}_{u_1w_2} = -C^{w_3}{}_{w_2u_1}$ and $C^{w_3}{}_{u_1u_2}$, giving the constraint

$$k^{w_1}{}_I k^{w_2}{}_J C^{w_3}{}_{w_1w_2} + 2k^{u_1}{}_{[I} k^{w_2}{}_{J]} C^{w_3}{}_{u_1w_2} + k^{u_1}{}_I k^{u_2}{}_J C^{w_3}{}_{u_1u_2}$$
$$- (f_{u_4})^K{}_I k^{u_4}{}_J k^{w_3}{}_K = 0. \qquad (8.101)$$

When the free index Σ is of type v, one gets a similar identity with all possible types of values in the lower indices of the structure constants,

$$k^{v_1}{}_I k^{v_2}{}_J C^{v_3}{}_{v_1v_2} + 2k^{w_1}{}_{[I} k^{v_2}{}_{J]} C^{v_3}{}_{w_1v_2} + k^{w_1}{}_I k^{w_2}{}_J C^{v_3}{}_{w_1w_2} + 2k^{u_1}{}_{[I} k^{v_2}{}_{J]} C^{v_3}{}_{u_1v_2}$$
$$+ 2k^{u_1}{}_{[I} k^{w_2}{}_{J]} C^{v_3}{}_{u_1w_2} + k^{u_1}{}_I k^{u_2}{}_J C^{v_3}{}_{u_1u_2} - (f_{u_4})^K{}_I k^{u_4}{}_J k^{v_3}{}_K = 0. \qquad (8.102)$$

8.3 Quadratic vector models

8.3.1 Description of the model

To go further, one needs to specialize the form of the Lagrangian, which has been assumed to be quite general so far. In this section, we focus on second order Lagrangians (5.13)-(5.15) arising in the context of supergravities that contain n_s scalar fields and depend quadratically on n_v abelian vector fields, non-minimally coupled to each other, in four space-time dimensions.

To recall, we consider explicitly $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_V$, where

$$\mathcal{L}_V = -\frac{1}{4}\mathcal{I}_{IJ}(\phi)F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{8}\mathcal{R}_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma} \qquad (8.103)$$

and the scalar Lagrangian is of the sigma model form

$$\mathcal{L}_S = -\frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - V(\phi) \qquad (8.104)$$

where g_{ij} is symmetric and invertible. Both g_{ij} and V depend only on undifferentiated scalar fields. As we pointed out in Section 5.2 neglecting gravity, this is the generic bosonic sector of ungauged supergravity. The symmetric matrices $\mathcal{I}$ and $\mathcal{R}$, with $\mathcal{I}$ invertible, depend only on undifferentiated scalar fields and encode the non-minimal couplings between the scalars and the abelian vectors. The Bianchi identities and equations of motion for the vector fields are given by

$$\partial_\mu(\star F^I)^{\mu\nu} = 0, \qquad \partial_\nu(\star G_I)^{\mu\nu} \approx 0. \qquad (8.105)$$

The Lagrangian (8.103) falls into the general class of models described previously, with the gauge invariant two-form $G_I = \mathcal{I}_{IJ}\star F^J + \mathcal{R}_{IJ}F^J$ and $d^4x\,\mathcal{L}_V = \frac{1}{2}G_IF^I$.

We also assume [Bar+18]

$$\mathcal{R}_{IJ}(0) = 0. \tag{8.106}$$

Note that a constant part in $\mathcal{R}_{IJ}$ can be put to zero without loss of generality since the associated term in the Lagrangian is a total derivative. In most cases, we also take $V = 0$ or assume (writing $\partial_i = \dfrac{\partial}{\partial \phi^i}$) that

$$(\partial_i V)(0) = 0. \tag{8.107}$$

8.3.2   Constraints on U, W-type symmetries

We assume here and in the examples below that there is no explicit x^μ-dependence in the space of local functions in order to constrain U and W-type symmetries. For simplicity, we also assume that the potential vanishes, $V = 0$. In Section 8.4, these constraints will allow us to determine all symmetries of U and W-type for specific models.

We need the scalar field equations, which are encoded in

$$s \star \phi_i^* + d(g_{ij} \star d\phi^j) = - \star \partial_i(\mathcal{L}_S + \mathcal{L}_V), \tag{8.108}$$

where $\partial_i = \dfrac{\partial}{\partial \phi^i}$. For $g = -1$, equation (8.50) becomes

$$G_I F^J f^I{}_J + F^I F^J h_{IJ} + dI^{n-1} - dG_I\, g^I - [d(g_{ij} \star d\phi^j) + \star \partial_i(\mathcal{L}_S + \mathcal{L}_V)]\Phi^i = 0. \tag{8.109}$$

When putting all derivatives of $F_{\mu\nu}^I, \phi^i$ to zero, one remains with

$$G_I F^J f^I{}_J + F^I F^J h_{IJ} - \star \partial_i \mathcal{L}_V\, \Phi^i|_{\mathrm{der}=0} = 0. \tag{8.110}$$

It is here that the assumption that there is no explicit x^μ dependence in the gauge invariant functions $g^{I\alpha}, \Phi^{i\alpha}$ is used. Using $-\partial_i \star \mathcal{L}_V = \frac{1}{2}\partial_i G_I\, F^I$, and the decomposition $\Phi^i|_{\mathrm{der}=0} = \Phi_0^i + \Phi_1^i + \dots$, where the Φ_n^i depend on undifferentiated scalar fields and are homogeneous of degree n in $F_{\mu\nu}^I$, the equation implies that

$$\frac{1}{2} M_{IJ}(\phi) \star F^I F^J + \frac{1}{2} N_{IJ}(\phi) F^I F^J = 0, \tag{8.111}$$

where

$$M_{IJ} = 2\mathcal{I}_{K(I} f^K{}_{J)} + \partial_i \mathcal{I}_{IJ} \Phi_0^i, \tag{8.112}$$

$$N_{IJ} = 2\mathcal{R}_{K(I} f^K{}_{J)} + 2h_{IJ} + \partial_i \mathcal{R}_{IJ} \Phi_0^i, \tag{8.113}$$

by using that $h_{IJ} = h_{JI}$ on account of (8.44). When taking an Euler-Lagrange derivative of (8.111) with respect to A_μ^I, one concludes that both terms have to vanish separately,

$$M_{IJ} = 0, \quad N_{IJ} = 0. \tag{8.114}$$

Setting $\phi^i = 0$ and using (8.106) then gives

$$f_{IJ}^{(\mathcal{I}(0))} + f_{JI}^{(\mathcal{I}(0))} = -(\partial_i \mathcal{I}_{IJ})(0)\Phi_0^i(0), \quad 2h_{IJ} = -(\partial_i \mathcal{R}_{IJ})(0)\Phi_0^i(0), \tag{8.115}$$

where the abelian index is lowered and raised with $\mathcal{I}_{IJ}(0)$ and its inverse. Note that completely skew-symmetric $f_{IJ}^{(\mathcal{I}(0))}$ solve the equations with $\Phi_0^i(0) = 0$, $h_{IJ} = 0$. More conditions are obtained by expanding equations (8.114) in terms of power series in ϕ^i.

In all examples considered below, the algebra $\mathfrak{g}_U$ and the W-type symmetries can be entirely determined from the analysis of this subsection.

8.3.3   Electric symmetry algebra

An important result of our general analysis is that the symmetries of the action that can lead to consistent gaugings may have a term that is not gauge invariant. This term is present only in the variation of the vector potential and is restricted to be linear in the undifferentiated vector potential, i.e., $\delta A_\mu^I = f^I{}_J A_\mu^J + g_\mu^I$, $\delta \phi^i = \Phi^i$. Here $f^I{}_J$ are constants, and g_μ^I and Φ^i are gauge invariant functions. The symbol δ represents the variation of the fields and is of course not the Koszul-Tate differential. No confusion should arise as the context is clear.

It is of interest to investigate a subalgebra of the gaugeable symmetries, obtained by restricting oneself from the outset to transformations of the gauge potentials that are linear and homogeneous in the undifferentiated potentials and to transformations of the scalars that depend on undifferentiated scalars alone,

$$\delta A_\mu^I = f^I{}_J A_\mu^J, \quad \delta \phi^i = \Phi^i(\phi). \tag{8.116}$$

This means that one takes $g_\mu^I = 0$ and that the functions Φ^i only depend on the undifferentiated scalar fields. These symmetries form a sub-algebra $\mathfrak{g}_e$ that includes the symmetries usually considered in the supergravity which is called the "electric symmetry algebra" (in the given duality frame), see Section 6.5.2. As we discussed there, the electric group is a subgroup of the duality group $G \subset Sp(2n_v, \mathbb{R})$ [GZ81] which is comprised of lower triangular symplectic transformations. Although our Lagrangians are not necessarily connected with supergravity, we shall nevertheless call the symmetries of the form (8.116) "electric symmetries" and the subalgebra $\mathfrak{g}_e$ the "electric algebra". It need not to be a subalgebra of $Sp(2n_v, \mathbb{R})$. It generically does not exhaust all symmetries and does not contain for example the conformal symmetries of free electromagnetism.

The transformations of the form (8.116) are symmetries of the action (8.103) + (8.104) if and only if the scalar variations leave the scalar action invariant separately, and $f^I{}_J, \Phi^i(\phi)$ satisfy

$$\frac{\partial \mathcal{I}}{\partial \phi^i} \Phi^i = -f^T \mathcal{I} - \mathcal{I}f, \tag{8.117}$$

$$\frac{\partial \mathcal{R}}{\partial \phi^i} \Phi^i = -f^T \mathcal{R} - \mathcal{R}f - 2h, \tag{8.118}$$

where the h are constant symmetric matrices. In particular, when the scalar Lagrangian is given by $\mathcal{L}_S = \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$, the first condition means that Φ^i must be a Killing vector of the metric g_{ij}. If U and W-type symmetries are of electric type, the electric symmetry algebra contains in addition only V-type symmetries of electric type, i.e., transformations among the undifferentiated scalars alone that leave invariant both the scalar action and the matrices $\mathcal{I}, \mathcal{R}$ (i.e., that satisfy $\delta S_S = 0$ and (8.117), (8.118) with 0's on the right hand sides). This will be the case in all examples below. In particular, the f's, and thus also the gauge algebra, will be the same for $\mathfrak{g}_U$ and $\mathfrak{g}_e$. The h matrix is determined by the transformation parameters Φ^i and the parity-odd term $\mathcal{R}$ of the action via (8.118).

We then suppose that we have a basis of symmetries of the action of this form,

$$\delta_\Gamma A_\mu^I = (f_\Gamma)^I{}_J A_\mu^J, \tag{8.119}$$

$$\delta_\Gamma \phi^i = \Phi_\Gamma^i(\phi). \tag{8.120}$$

When compared to the previous sections, the index Γ can take u, v or w values. Only the f_u matrices are non-zero. The h_Γ matrices are non-vanishing only for $\Gamma = u$ or w. When $h_\Gamma \neq 0$, the Lagrangian is only invariant up to a total derivative.

Closure of symmetries of this form then implies

$$[f_\Delta, f_\Gamma] = -C^\Sigma{}_{\Delta\Gamma} f_\Sigma, \tag{8.121}$$

$$f_\Gamma^T h_\Delta - f_\Delta^T h_\Gamma + h_\Delta f_\Gamma - h_\Gamma f_\Delta = -C^\Sigma{}_{\Delta\Gamma} h_\Sigma, \tag{8.122}$$

$$\frac{\partial \Phi_\Delta^i}{\partial \phi^j} \Phi_\Gamma^j - \frac{\partial \Phi_\Gamma^i}{\partial \phi^j} \Phi_\Delta^j = -C^\Sigma{}_{\Delta\Gamma} \Phi_\Sigma^i, \tag{8.123}$$

where one can obtain the equation (8.122) by the action of commutator $[\delta_\Gamma, \delta_\Delta]$ on $\mathcal{R}$. Decomposing the indices into U, W and V-type, this is consistent with the relations of Section 8.2.2 with $\lambda_u^w = 0$. Let us note that (8.118) expresses the surface term in the variation of the action.

8.3.4   Restricted first order deformations

We now limit ourselves to first order deformations of the master action with the condition that all infinitesimal gaugings come from symmetries that belong to the electric symmetry algebra above. In order to simplify formulas, we will no longer make the distinction between U-, W- and V-type which can easily be recovered.

According to Section 8.2.3, the deformations are parametrized through electric symmetries by a matrix k_I^Γ, with

$$f^I_{\ JK} = (f_\Gamma)^I_{\ J} k_K^\Gamma, \qquad (8.124)$$

$$\Phi^i_I(\phi) = \Phi^i_\Gamma(\phi) k_I^\Gamma, \qquad (8.125)$$

$$X_{IJ,K} = (h_\Gamma)_{IJ} k_K^\Gamma. \qquad (8.126)$$

The linear constraints (8.92) – (8.94) on the matrix k_K^Γ become

$$(f_\Gamma)^I_{\ J} k_K^\Gamma + (f_\Gamma)^I_{\ K} k_J^\Gamma = 0, \qquad (8.127)$$

$$h_{\Gamma\,(IJ} k_{K)}^\Gamma = 0. \qquad (8.128)$$

They guarantee that the first order deformation of the master action is given by

$$S^{(1)} = \int d^4x \, (a_2 + a_1 + a_0), \qquad (8.129)$$

where

$$a_2 = \frac{1}{2} C_I^* f^I_{\ JK} C^J C^K \qquad (8.130)$$

encodes the first order deformation of the gauge algebra and

$$a_1 = A_I^{*\mu} f^I_{\ JK} A_\mu^J C^K + \phi_i^* \Phi^i_K C^K \qquad (8.131)$$

encodes the first order deformation of the gauge symmetries. When taking (8.124) and (8.125) into account, this deformation of the gauge symmetries corresponds to gauging the underlying global symmetries by using local parameters $\eta^\Gamma(x) = k_I^\Gamma \epsilon^I(x)$. The deformation a_0 of the Lagrangian is given by the sum of three terms:

$$a_0^{(\mathrm{YM})} = \frac{1}{2}(\star G_I)^{\mu\nu} f^I_{\ JK} A_\mu^J A_\nu^K, \qquad (8.132)$$

$$a_0^{(\mathrm{CD})} = J_K^\mu A_\mu^K, \qquad (8.133)$$

$$a_0^{(\mathrm{CS})} = \frac{1}{3} X_{IJ,K} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I A_\rho^J A_\sigma^K. \qquad (8.134)$$

The terms $a^{(\mathrm{YM})}$ and $a^{(\mathrm{CD})}$ are exactly those necessary to complete the abelian field strengths and ordinary derivatives of the scalars into covariant quantities.

The term $a^{(CD)}$ is responsible for charging the matter fields. The Chern-Simons term $a_0^{(CS)}$ appears when $h_\Gamma \neq 0$: its role is to cancel the variation

$$\delta\mathcal{L} = -\frac{1}{4}\eta^\Gamma h_{\Gamma\,IJ}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^I F_{\rho\sigma}^J \tag{8.135}$$

that is no longer a total derivative when $\eta^\Gamma = k_I^\Gamma \epsilon^I(x)$ [WLV85; WHR87].

8.3.5 Complete restricted deformations

The second order deformation $S^{(2)}$ to the master action is then determined by the first order deformation through equation (7.17). As discussed in Section 8.2.4, the existence of $S^{(2)}$ imposes additional quadratic constraints on the matrix k_I^Γ,

$$k^\Gamma{}_I k^\Delta{}_J C^\Sigma{}_{\Gamma\Delta} - (f_\Gamma)^K{}_I k^\Gamma{}_J k^\Sigma{}_K = 0. \tag{8.136}$$

Explicit computation shows that $S^{(2)}$ can be chosen such that there is no further deformation of the gauge symmetries or of their algebra. The second order terms in the Lagrangian are exactly those necessary to complete abelian field strengths $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ and ordinary derivatives of the scalars to non-abelian field strengths and covariant derivatives,

$$\mathcal{F}_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + f^I{}_{JK} A_\mu^J A_\nu^K, \tag{8.137}$$

$$D_\mu \phi^i = \partial_\mu \phi^i - \Phi_I^i(\phi) A_\mu^I. \tag{8.138}$$

One also finds a non-abelian completion of the Chern-Simons term $a_0^{(CS)}$. Putting everything together, the Lagrangian after adding the second order deformation is [Bar+18]

$$\mathcal{L} = \mathcal{L}_S(\phi^i, D_\mu\phi^i) - \tfrac{1}{4}\,\mathcal{I}_{IJ}(\phi)\mathcal{F}_{\mu\nu}^I \mathcal{F}^{J\mu\nu} + \tfrac{1}{8}\,\mathcal{R}_{IJ}(\phi)\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^I \mathcal{F}_{\rho\sigma}^J$$
$$+ \tfrac{2}{3}\,X_{IJ,K}\,\varepsilon^{\mu\nu\rho\sigma}A_\mu^J A_\nu^K \left(\partial_\rho A_\sigma^I + \tfrac{3}{8}\,f^I{}_{LM} A_\rho^L A_\sigma^M\right). \tag{8.139}$$

The associated action can be checked to be invariant under the gauge transformations

$$\delta A_\mu^I = \partial_\mu \epsilon^I + f^I{}_{JK} A_\mu^J \epsilon^K, \tag{8.140}$$

$$\delta\phi^i = \epsilon^I \Phi_I^i(\phi). \tag{8.141}$$

This is equivalent to the fact that the deformation stops at second order, i.e, that $S = S^{(0)} + S^{(1)} + S^{(2)}$ gives a solution to the master equation $(S, S) = 0$.

Checking directly the invariance of this action under (8.140) – (8.141) without first parametrizing $f^I{}_{JK}$, $\Phi_I^i(\phi)$ and X_{IJK} through symmetries requires the use of the linear identities

$$f^I{}_{JK} = f^I{}_{[JK]}, \quad X_{(IJ,K)} = 0 \tag{8.142}$$

and of the quadratic ones

$$f^I{}_{J[K_1} f^J{}_{K_2 K_3]} = 0, \qquad (8.143)$$

$$f^K{}_{I[L} X_{M]J,K} + f^K{}_{J[L} X_{M]I,K} - \tfrac{1}{2} X_{IJ,K} f^K{}_{LM} = 0, \qquad (8.144)$$

$$\frac{\partial \Phi^i_I}{\partial \phi^j} \Phi^j_J - \frac{\partial \Phi^i_J}{\partial \phi^j} \Phi^j_I + f^K{}_{IJ} \Phi^i_K = 0. \qquad (8.145)$$

In terms of k_I^Γ, these three quadratic identities all come from the single quadratic constraint (8.136) once the algebra of global symmetries (8.121) – (8.123) is taken into account.

8.3.6 Remarks on $GL(n_v)$ transformations

Consider a linear field redefinition of the abelian vector potentials, $A^I_\mu = M^I{}_J A'^J_\mu$ with $M \in GL(n_v)$. Such a transformation gives rise to a trivial infinitesimal gauging which corresponds to the antifield independent part of the trivial ghost number 0 cocycle

$$S^{(1)}_{triv.} = (S^{(0)}, \Xi_s), \quad \Xi_s = f_s{}^I{}_J [d^4 x\, C^*_I C^J + \star A^*_I A^J], \qquad (8.146)$$

with $f_s \in \mathfrak{gl}(n_v, \mathbb{R})$.

Two remarks are in order.

The first concerns the relation to the algebra $\mathfrak{g}_U$ defined in Section 8.2.2. It can also be defined as the largest sub-algebra of $\mathfrak{gl}(n_v, \mathbb{R})$ that can be turned into symmetries of the theory by adding suitable gauge invariant transformations of the vector and scalar fields, or in other words, for which there exists a gauge invariant K_u of ghost number -1 such that $(S^{(0)}, \Xi_u + K_u) = 0$.

In particular, (8.89) and (8.92) for $g = 0$, as well as (8.97), can be summarized as follows: non-trivial U-type gaugings require the existence of a map (described by k_K^u) from the defining representation of the symmetry algebra $\mathfrak{g}_U \subset \mathfrak{gl}(n_v, \mathbb{R})$ into the adjoint representation of the n_v-dimensional gauge algebra $\mathfrak{g}_g$.

The second remark is about families of Lagrangians related by linear transformations of the vector potentials among themselves. It is sometimes useful not to work with fixed (canonical) values for various $GL(n_v)$ tensors that appear in the action. Instead, one considers the deformation problem for sets of Lagrangians parametrized by arbitrary $GL(n_v)$ tensors, for instance generic non-degenerate symmetric $\mathcal{I}_{MN}$ and symmetric $\mathcal{R}_{MN}$ that vanish at the origin of the scalar field space.

If the tensors of two such Lagrangians are related by a $GL(n_v)$ transformation, they should be considered as equivalent. Indeed, the local BRST cohomology for all members of such an equivalence class are isomorphic and related by the

above anti-canonical field redefinition. In particular, all members of the same equivalence class have isomorphic gaugings.

All general considerations and results on local BRST cohomology above apply in a unified way to all equivalence classes. When one explicitly solves the obstruction equation (8.50) (for instance at $g = -1$ in order to determine the symmetries), the results on local BRST cohomologies do depend on the various equivalence classes.

8.3.7 Comparison with the embedding tensor constraints

In the embedding tensor formalism [WST03; WST05; WST07; Sam08; Tri17], the possible gaugings are described by the embedding tensor $\Theta_M{}^\alpha = (\Theta_I{}^\alpha, \Theta^{I\alpha})$ with electric and magnetic components[5], which satisfies a number of linear and quadratic constraints (6.4)–(6.6). Recalling from Section 6.3, the index I runs from 1 to n_v, while α runs from 1 to the dimension of the group G of invariances of the equations of motion of the initial Lagrangian (8.103). More precisely, G is defined only as the group of transformations that act linearly on the field strengths F^I and their "magnetic duals" G_I, and whose action on the scalars contains no derivatives. This coincides with the group of symmetries of the first order Lagrangian discussed in Section 6.5.1 and [Hen+18], which are of the restricted form (8.116).

As explained in Section 6.3, one can always go to a duality frame in which the magnetic components of the embedding tensor vanishes, $\Theta^{I\alpha} = 0$, see Section 6.6 for an explicit construction of such a symplectic matrix. Moreover, only the components $\Theta_{\check{I}}{}^\Gamma$ survive, where Γ runs over the generators of the electric subgroup $G_e \subset G$ that act as local symmetries of the Lagrangian in that frame and as before $\check{I} = 1, ..., n_v$ is referring to the n_v physical vector fields. Then, the gauged Lagrangian (6.18)-(6.20) in the electric frame is exactly the Lagrangian (8.139), where the matrix k is identified with the remaining electric components of the embedding tensor, $k_{\check{I}}^\Gamma = \Theta_{\check{I}}{}^\Gamma$. The linear constraint (6.4) in the electric frame corresponds to (8.127) and (8.128). The quadratic constraint (6.6) on the embedding tensor in the electric frame corresponds to the constraint (8.136) on k [Bar+18]. As explained in Sections 8.2.3 and 8.2.4, the constraints can be refined using the split corresponding to the various (U, W, V) types of symmetries.

We showed in Chapter 6 that the embedding tensor formalism does not allow for more general deformations [Hen+18] than those of the Lagrangian (8.103) studied in this chapter. Indeed, we have seen in Section 7.5 that their BRST cohomologies are isomorphic even though the field content and gauge transformations are different. Conversely, as long as one restricts the attention to the

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5 We refer the reader to [CDV10], where a relation between the embedding tensor formalism and the BV-BRST antifield formalism has been considered for a different reason.

symmetries of (8.103) that are of the electric type (8.116), we showed that the embedding tensor formalism captures all consistent deformations that deform the gauge transformations of the fields.

8.4   Applications

8.4.1   Abelian gauge fields: U-type gauging

As a first example, let us consider the case where we have no scalars, $\mathcal{I}_{IJ} = \delta_{IJ}$, $\mathcal{R}_{IJ} = 0$. The Lagrangian is then simply

$$\mathcal{L} = -\frac{1}{4}\delta_{IJ}F^I_{\mu\nu}F^{J\mu\nu}. \tag{8.147}$$

From (8.115) and comparing with (8.117) and (8.118), it can be shown that U-type symmetries are of electric form and moreover there are no W-type symmetries. We have in this case $\mathfrak{g}_U = \mathfrak{g}_e = \mathfrak{so}(n_v)$.

The vector fields transform in the fundamental representation of $\mathfrak{so}(n_v)$. A basis of the Lie algebra $\mathfrak{so}(n_v)$ may be labeled by an antisymmetric pair of indices $[LM]$ that now plays the role of the index u,

$$\delta_{[LM]}A^I_\mu = (f_{[LM]})^I{}_J A^J_\mu, \qquad (f_{[LM]})^I{}_J = \frac{1}{2}(\delta^I_L \delta_{JM} - \delta^I_M \delta_{JL}). \tag{8.148}$$

Concerning the associated gaugings, the matrices $f_{[LM]IJ} = \delta_{II'}(f_{[LM]})^{I'}{}_J$ are antisymmetric in I, J; therefore, the structure constants of the gauge group

$$f^{(\delta)}_{IJK} = (f_{[LM]})_{IJ} k^{LM}_K \tag{8.149}$$

are automatically antisymmetric in their first two indices. Here the superscript (δ) means that f is contracted by the kronecker δ, more precisely $f^{(\delta)}_{IJK} = \delta_{IL}f^{(\delta)L}{}_{JK}$. The constraint (8.127) on k^{LM}_K ensures antisymmetry in the last two indices which in turn implies total antisymmetry and the Jacobi identity is obtained from (8.136). Moreover, any set of totally antisymmetric structure constants can be obtained in this way by taking $k^{LM}_K = f^{LM}{}_K$, as can be easily seen using the expression for $f_{[LM]}$ given above.

We thereby recover the result of [BHT94] stating that the most general deformation of the free Lagrangian (8.147) that is not of V or I-type is given by the Yang-Mills Lagrangian with a compact gauge group of dimension equal to the number of vector fields.

Remark: Note that Poincaré (conformal) symmetries (for $n = 4$) are of V-type if one allows for x^μ-dependent local functions. If such a dependence is allowed for U, W-type symmetries and gaugings as well, results can be very

different. For instance, as shown by equation (13.21) of [BBH00], if $n \neq 4$, there are additional U-type symmetries described by the cohomology class

$$U^{-1} = d^n x f_{(IJ)} \left[C^{*I} C^J + A^{*\mu I} A_\mu^J + \frac{2}{n-4} F_{\mu\nu}^I x^\mu A^{*\nu J} \right], \tag{8.150}$$

where indices $I, J, \dots$ are raised and lowered with the Kronecker delta. The associated Noether current can be obtained by working out the descent equation following (8.51), $sU^{-1} + d(f_{(IJ)}[\star A^{*I} C^J + \star F^I A^J + J_U{}^{IJ}]) = 0$, where

$$J_U{}^{IJ} = \frac{2}{n-4} (T_{\mu\nu})^{IJ} x^\nu \star dx^\mu, \quad (T^\mu{}_\nu)^{IJ} = F^{(I|\mu\rho} F_{\rho\nu}^{|J)} + \frac{1}{4} F^{I\alpha\beta} F_{\alpha\beta}^J \delta_\nu^\mu. \tag{8.151}$$

In other words, $\mathfrak{g}_U = \mathfrak{gl}(n_v)$. Note also that these U-type symmetries involve a non-vanishing ${}^U g_\mu^I$. It has furthermore been shown in section 13.2.2 of [BBH00] that there are associated U-type gaugings and cohomology classes in higher ghost numbers. In the present context, they are obtained as follows: the role of u for the additional symmetries is played by a symmetric pair of indices (LM),

$$\delta_{(LM)} A_\mu^I = (f_{(LM)})^I{}_J A_\mu^J, \quad (f_{(LM)})^I{}_J = \frac{1}{2} (\delta_L^I \delta_{JM} + \delta_M^I \delta_{JL}). \tag{8.152}$$

Once the linear constraints (8.92) on $k_{K_1 \dots K_{g+1}}^{(LM)}$ are fulfilled, the associated U-type gaugings and higher cohomology classes can be read off from equation (8.51) when taking (8.89).

After multiplying (8.150) by $n-4$, it represents for $n = 4$ the V-type symmetry associated with the dilatation of the conformal group. The associated cubic and higher order vertices for the full conformal group have been studied in detail in [Bra01].

8.4.2 Abelian gauge fields with uncoupled scalars: U, V-type gaugings

We now take the case

$$\mathcal{L} = \mathcal{L}_S(\phi^i, \partial_\mu \phi^i) - \frac{1}{4} \delta_{IJ} F_{\mu\nu}^I F^{J\mu\nu}, \tag{8.153}$$

where there is no interaction between the scalars and the vector fields. The $\mathfrak{g}_U$ algebra is again $\mathfrak{so}(n_v)$ and there are no W-type symmetries.

The electric symmetry algebra is the direct sum of $\mathfrak{so}(n_v)$ with the electric V-type symmetry algebra $\mathfrak{g}_s$ of the scalar Lagrangian. The matrices f_Γ split into two groups and are given by

$$(f_\alpha)^I{}_J = 0, \qquad (f_{[LM]})^I{}_J = \frac{1}{2} (\delta_L^I \delta_{JM} - \delta_M^I \delta_{JL}) \tag{8.154}$$

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where $\alpha = 1, \ldots, \dim \mathfrak{g}_s$ labels the G_s generators and the antisymmetric pair $[LM]$ labels the $SO(n_v)$ generators as before. The matrix k_I^Γ accordingly splits in two components k_I^{LM} and k_I^α. The constraints on k_I^{LM} again amount to the fact that the quantities $f^I{}_{JK} = (f_{[LM]})^I{}_J k_K^{LM}$ are the structure constants of a compact Lie group. The constraint on k_I^α tells us that the gauge variations

$$\delta \phi^i = \epsilon^I k_I^\alpha \Phi_\alpha^i(\phi) \tag{8.155}$$

close according to the structure constants $f^I{}_{JK}$. In the case where these variations are linear, $\Phi_\alpha^i(\phi) = (t_\alpha)^i{}_j \phi^j$, the constraint is that the matrices $T_I = -k_I^\alpha t_\alpha$ form a representation of the gauge group, $[T_I, T_J] = f^K{}_{IJ} T_K$.

8.4.3 Bosonic sector of $\mathcal{N} = 4$ supergravity

Neglecting gravity, the bosonic sector of $\mathcal{N} = 4$ supergravity is given by two scalar fields parametrizing the coset $SL(2, \mathbb{R})/SO(2)$ along with $n_v = 6$ vector fields [CS77; Das77; CSF78; CJ79]. We study three formulations of this model, where we determine the symmetry algebras $\mathfrak{g}_U$ and $\mathfrak{g}_e$ and the allowed gaugings. In all formulations, the scalar Lagrangian is determined by

$$\phi^i = (\phi, \chi), \quad g_{ij} = \text{diag}(1, e^{2\phi}), \quad V = 0. \tag{8.156}$$

They differ by the form of the matrices $\mathcal{I}$ and $\mathcal{R}$.

$SO(6)$ formulation

The vector Lagrangian (8.103) is determined by

$$\mathcal{I}_{IJ} = e^{-\phi} \delta_{IJ}, \quad \mathcal{R}_{IJ} = \chi \delta_{IJ}. \tag{8.157}$$

When $f^{(\delta)}_{IJ}$ is antisymmetric, the transformations $\delta A_\mu^I = f^I{}_J A_\mu^J$ define an $\mathfrak{so}(6)$ sub-algebra of U-type symmetries on their own. Note also that we can assume h_{IJ} to be symmetric. Equations (8.115) then imply that the traceless parts of $f^{(\delta)}_{IJ}, h_{IJ}$ have to vanish.

If $f^{(\delta)}_{IJ} = \delta_{IJ} \eta^0$, equations (8.115) are solved with $h_{IJ} = 0$, $\Phi_0^\phi(0) = 2\eta^0$, and $\Phi_0^\chi(0) = 0$. It then follows that (8.114) are solved with

$$\Phi_0^\phi = 2\eta^0, \quad \Phi_0^\chi = -2\eta^0 \chi. \tag{8.158}$$

Equation (8.109) is then also solved with $g^I = 0$ and $I^3 = 2\eta^0(e^{2\phi} \star d\chi \chi - \star d\phi)$. According to equation (8.51), the associated cohomology class is given by

$$\omega^{-1,4} = \eta^0 [d^4 x C_I^* C^I + \star A_I^* A^I + 2(\star \phi^* - \star \chi^* \chi)], \tag{8.159}$$

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with $s\omega^{-1,4} + d[\eta^0(\star A_I^* C^I + G_I A^I) + I^3] = 0$. This cohomology class encodes the symmetry

$$\delta A_\mu^I = \eta^0 A_\mu^I, \quad \delta\phi = 2\eta^0, \quad \delta\chi = -2\eta^0\chi, \tag{8.160}$$

with $\delta\mathcal{L}_0 = 0$. The associated Noether current is given by

$$j^\mu = [-(e^{-\phi}F_I^{\mu\lambda} - \frac{1}{2}\chi\epsilon^{\mu\lambda\rho\sigma}F_{I\rho\sigma})A_\lambda^I - 2\partial^\mu\phi + 2\chi e^{2\phi}\partial^\mu\chi]. \tag{8.161}$$

It cannot be made gauge invariant through allowed redefinitions.

For $f_{IJ} = 0$, $h_{IJ} = \eta^+\delta_{IJ}$, $\Phi_0^\chi = -2\eta^+, \Phi_0^\phi = 0$ is a solution to the full problem (8.109) since $\frac{1}{2}F^I F_I = s \star \chi^* + d(e^{2\phi} \star d\chi)$. This gives then the only class of W-type, which is also of restricted type. More explicitly, $W^{-1} = \star\chi^*$ with $s \star \chi^* + d(-\frac{1}{2}A^I F_I + e^{2\phi} \star d\chi) = 0$. The symmetry it describes is $\delta\chi = \eta^+$ with the associated Noether current given above that can not be made gauge invariant.

The algebra $\mathfrak{g}_U$ is therefore isomorphic to $\mathfrak{so}(6) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the sub-algebra of $\mathfrak{sl}(2,\mathbb{R})$ generated by diagonal traceless matrices. It is a sub-algebra of the electric symmetry algebra $\mathfrak{g}_e = \mathfrak{so}(6) \oplus \mathfrak{b}^+$, where $\mathfrak{b}^+$ corresponds to the sub-algebra of $\mathfrak{sl}(2,\mathbb{R})$ of upper triangular matrices, see Section 5.3. The electric algebra acts as

$$\delta\phi = 2\eta^0, \quad \delta\chi = -2\eta^0\chi + \eta^+, \quad \delta A_\mu^I = \eta^0 A_\mu^I + \eta^{LM}(f_{[LM]})^I{}_J A_\mu^J. \tag{8.162}$$

Accordingly $f_\Gamma = (f_0, f_+, f_{[LM]})$ where $f_{[LM]}$ are given in (8.148), while

$$(f_0)^I{}_J = \delta_J^I, \qquad (f_+)^I{}_J = 0. \tag{8.163}$$

The matrix $\mathcal{R}_{IJ} = \chi\delta_{IJ}$ transforms as

$$\delta\mathcal{R}_{IJ} = -2\eta^0\mathcal{R}_{IJ} + \eta^+\delta_{IJ}. \tag{8.164}$$

Therefore, contrary to the previous examples, the tensor $h_{\Gamma IJ}$ has a non-vanishing component $h_{+IJ} = -\frac{1}{2}\delta_{IJ}$.

The generalized structure constants $f^I{}_{JK_1\ldots K_{g+1}} = (f_\Gamma)^I{}_J k^\Gamma_{K_1\ldots K_{g+1}}$ are then

$$f^I{}_{JK_1\ldots K_{g+1}} = \delta_J^I k^0_{K_1\ldots K_{g+1}} + \frac{1}{2}(f_{[LM]})^I{}_J k^{LM}_{K_1\ldots K_{g+1}}. \tag{8.165}$$

The linear constraint (8.92) now implies that $k^0_{K_1\ldots K_{g+1}} = 0$ as can be seen by taking the first three indices equal and using the antisymmetry of the matrices $f_{[LM]}$, while $k^{(\delta)}_{LMK_1\ldots K_{g+1}}$ is restricted to be completely skew-symmetric in all indices. In the same way, the linear constraint (8.94) implies that $k^+_{K_1\ldots K_{g+1}} = 0$. Indeed, it reduces to

$$\delta_{(IJ}k^+_{K_1)\ldots K_{g+1}} = 0, \tag{8.166}$$

from which we deduce $k^+_{K_1...K_{g+1}} = 0$ by taking the first three indices equal.

It follows that there are no cohomology classes of W-type when $g \geqslant 0$ and that the only cohomology classes of U-type when $g \geqslant 0$ are given by

$$[d^4 x C^{*I} \partial_I + \star A^{*I} A^J \partial_J \partial_I + \frac{1}{2} G^I A^J A^K \partial_K \partial_J \partial_I] \Theta, \qquad (8.167)$$

with Θ a polynomial in C^I of ghost number $\geqslant 1$.

In particular, the symmetries of $\mathfrak{b}^+$ cannot be gauged and the gauge algebra is given by a compact sub-algebra of $\mathfrak{so}(6)$. The gauged Lagrangian is the original one, except that the abelian field strengths and ordinary derivatives are replaced by non-abelian field strength and covariant derivatives.

Dual $SO(6)$ formulation

We now have

$$\mathcal{I}_{IJ} = \frac{1}{e^{-\phi} + \chi^2 e^\phi} \delta_{IJ}, \quad \mathcal{R}_{IJ} = -\frac{\chi e^\phi}{e^{-\phi} + \chi^2 e^\phi} \delta_{IJ}, \qquad (8.168)$$

and the same analysis gives similar conclusions:

1. The cohomology classes (8.167) are again present since $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ are still proportional to δ_{IJ}.

2. There are no additional gaugings or cohomology classes in ghost number higher than 0 of U or W-type.

3. The only additional non-covariantizable characteristic cohomology comes from two additional solutions to (8.115).

The first of these additional solutions is of electric U-type and comes from $f^{(\delta)}_{IJ} = \tilde{\eta}^0 \delta_{IJ}$, $h_{IJ} = 0$, with $\Phi^\phi_0(0) = -2\tilde{\eta}^0$, $\Phi^\chi_0(0) = 0$. Equation (8.114) reduces to

$$2\tilde{\eta}^0 \mathcal{I}_{IJ} + \partial_i \mathcal{I}_{IJ} \Phi^i_0 = 0, \quad 2\tilde{\eta}^0 \mathcal{R}_{IJ} + 2h_{IJ} \delta_{IJ} + \partial_i \mathcal{R}_{IJ} \Phi^i_0 = 0, \qquad (8.169)$$

and is solved by

$$h_{IJ} = 0 \quad \Phi^\phi_0 = -2\tilde{\eta}^0, \quad \Phi^\chi_0 = 2\tilde{\eta}^0 \chi. \qquad (8.170)$$

This gives also a solution to the full problem since this transformation leaves the scalar field Lagrangian invariant. According to equation (8.51), the associated cohomology class is given by

$$\omega^{-1,4} = \tilde{\eta}^0 [d^4 x C^*_I C^I + \star A^*_I A^I - 2(\star \phi^* - \star \chi^* \chi)]. \qquad (8.171)$$

The second solution is of restricted W-type and comes from $f_{IJ}^{(\delta)} = 0$ while $h_{IJ} = \tilde{\eta}^+ \delta_{IJ}$ with $\Phi_0^\phi(0) = 0$, $\Phi_0^\chi(0) = 2\tilde{\eta}^+$. Equation (8.114) reduces to

$$\partial_i \mathcal{I}_{IJ} \Phi_0^i = 0, \quad 2\tilde{\eta}^+ \delta_{IJ} + \partial_i \mathcal{R}_{IJ} \Phi_0^i = 0, \tag{8.172}$$

and is solved by

$$\Phi_0^\chi = 2\tilde{\eta}^+ (e^{-2\phi} - \chi^2), \quad \Phi_0^\phi = 4\tilde{\eta}^+ \chi, \tag{8.173}$$

This is also a solution to the full problem since these transformations leave the scalar field Lagrangian invariant. The associated cohomology class is given by

$$2\tilde{\eta}^+ [\star \phi^* 2\chi + \star \chi^* (e^{-2\phi} - \chi^2)]. \tag{8.174}$$

In this case, we therefore have $\mathfrak{g}_U = \mathfrak{so}(6) \oplus \mathfrak{h} \subset \mathfrak{g}_e = \mathfrak{so}(6) \oplus \mathfrak{b}^-$, where $\mathfrak{b}^-$ is the sub-algebra of $\mathfrak{sl}(2, \mathbb{R})$ of lower triangular matrices.

Again, the symmetries of $\mathfrak{b}^+$ cannot be gauged and the gauge algebra is given by a compact sub-algebra of $\mathfrak{so}(6)$.

$SO(3) \times SO(3)$ formulation

The indices split as $I = (A, A')$, where $A, A' = 1, 2, 3$, and we have

$$\mathcal{I}_{IJ} = \mathrm{diag}(\mathcal{I}_{AB}, \mathcal{I}_{A'B'}), \quad \mathcal{R}_{IJ} = \mathrm{diag}(\mathcal{R}_{AB}, \mathcal{R}_{A'B'}),$$

$$\mathcal{I}_{AB} = e^{-\phi} \delta_{AB}, \quad \mathcal{I}_{A'B'} = \frac{1}{e^{-\phi} + \chi^2 e^\phi} \delta_{A'B'}, \tag{8.175}$$

$$\mathcal{R}_{AB} = \chi \delta_{AB}, \quad \mathcal{R}_{A'B'} = -\frac{\chi e^\phi}{e^{-\phi} + \chi^2 e^\phi} \delta_{A'B'}.$$

Spelling out equation (8.114) gives

$$\mathcal{I}_{IL} f^L{}_J + \mathcal{I}_{LJ} f^L{}_I + \partial_i \mathcal{I}_{IJ} \Phi_0^i = 0,$$
$$\mathcal{R}_{IL} f^L{}_J + \mathcal{R}_{JL} f^L{}_I + h_{IJ} + h_{JI} + \partial_i \mathcal{R}_{IJ} \Phi_0^i = 0. \tag{8.176}$$

Choosing $I = A$, $J = A'$, the first equation reduces to

$$e^{-\phi} f_{AA'}^{(\delta)} + \frac{1}{e^{-\phi} + \chi^2 e^\phi} f_{A'A}^{(\delta)} = 0. \tag{8.177}$$

Putting $\phi = 0 = \chi$ gives $f_{AA'}^{(\delta)}$ is symmetric under exchange of A, A' while taking the derivative with respect to ϕ and then putting $\phi = 0 = \chi$ gives $f_{AA'}^{(\delta)}$ is antisymmetric under exchange of A, A', thus $f_{AA'}^{(\delta)} = 0 = f_{A'A}^{(\delta)}$. When combined with the linear constraint (8.92), this implies that the $f^I{}_{JK_1 \ldots K_{g+1}}$'s have to

vanish unless all indices are of A, or of A', type respectively, which is thus a necessary condition to have non trivial U-type solutions.

When the f's vanish, the second part of (8.176) for $I = A$, $J = A'$ gives $h_{AA'} + h_{A'A} = 0$. When combined with the linear constraint (8.94), this implies that for non-trivial solutions of W-type associated to $X_{IJ,K_1...K_{g+1}}$ one again needs all indices to be either of A or of A' type.

The discussion then reduces to the one we had before in each of the sectors. For U-type solutions, this gives in a first stage the symmetries, gaugings and higher ghost cohomology classes associated with each of the $SO(3)$ rotations separately. There are again no additional solutions of U or W type when $g \geqslant 0$.

Only the remaining non-covariantizable symmetries, i.e., solutions of type U and W at $g = -1$ that correspond to $\mathfrak{b}^\pm$, remain to be discussed. For the U type solutions, one finds in the first sector that $f_{AB} = \eta^0 \delta_{AB}$ with (8.158) holding, while for the second sector $f_{A'B'} = \tilde{\eta}^0 \delta_{A'B'}$ with (8.170) holding. This gives a solution to the full problem if and only if $\tilde{\eta}^0 = -\eta^0$. Hence $\mathfrak{g}_U = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{h}$. On the other hand the solutions of W type for both sectors are solutions to the full problem if and only if $\eta^+ = \tilde{\eta}^+ = 0$ so that there is no surviving W-type symmetry. In particular $\mathfrak{g}_e = \mathfrak{g}_U$. The symmetry of $\mathfrak{h}$ cannot be gauged, and the gauge algebra is a compact sub-algebra of $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

This concludes the discussion with the expected results (see [DFR77; FS78; GZ83]).

8.5 First order manifestly duality-invariant actions

8.5.1 Non-minimal version with covariant gauge structure

We now investigate the first order formulation [BH10; Hen+18] of the models discussed previously. Those models are interesting because they contain more symmetries and therefore potentially more gaugings. In the original, minimal version, they are given by the action[6]

$$S = \int d^4x \left(\frac{1}{2} \Omega_{MN} B^{Mi} \dot{A}_i^N - \frac{1}{2} \mathcal{M}_{MN}(\phi) B_i^M B^{Ni} \right), \qquad (8.178)$$

where the potentials are packed into a vector

$$(A^M) = (A^I, Z_I), \quad M = 1, \ldots, 2n_v, \qquad (8.179)$$

and the magnetic fields are

$$B^{Mi} = \epsilon^{ijk} \partial_j A_k^M. \qquad (8.180)$$

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6 Note that in this section the convention for Levi-Civita is $\varepsilon^{01234\cdots} = +1$.

The matrices Ω and $\mathcal{M}(\phi)$ are the $2n_v \times 2n_v$ matrices

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \tag{8.181}$$

each block being $n_v \times n_v$. The matrix $\mathcal{N} = \Omega^{-1}\mathcal{M}$ is symplectic, $\mathcal{N}^T\Omega\mathcal{N} = \Omega$.

Local BRST cohomology and gaugings for this class of models with non-covariant gauge symmetries $\delta A_i^M = \partial_i \epsilon^M$ could then be discussed by generalizing the results of [BC01] in the presence of coupled scalars.

However, in order to be able to directly use the discussion of local BRST cohomology developed for the second order covariant Lagrangian in the case of the first order manifestly duality invariant formulation, we consider a modification of the non-minimal variant [BG08] with additional scalar potentials for the longitudinal parts of electric and magnetic fields. More precisely, we now take instead of (8.178) the action

$$S[A_\mu^M, D^M, \pi_M, \phi^i] = S_S[\phi] + S_{DP}, \tag{8.182}$$

with

$$\mathcal{L}_{DP} = \frac{1}{2}[\Omega_{MN}(\mathcal{B}^{Mi} + \partial^i D^M)(\partial_0 A_i^N - \partial_i A_0^N) - \mathcal{B}^{Mi}\mathcal{M}_{MN}(\phi)\mathcal{B}_i^N]$$
$$+ \pi_M \partial_0 D^M - \frac{1}{2}\pi_M(\mathcal{M}^{-1})^{MN}\pi_N - \mathcal{V}(\phi, D). \tag{8.183}$$

Here

$$\mathcal{B}^{Mi} = \epsilon^{ijk}\partial_j A_k^M + \partial^i D^M, \tag{8.184}$$

spatial indices $i, j, k, \ldots$ are raised and lowered with δ_{ij} and its inverse, with Ω_{MN} the symplectic matrix, $\mathcal{M}_{MN}$ symmetric and invertible and

$$(\partial_i \mathcal{V})(0,0) = 0 = (\partial_M \mathcal{V})(0,0), \quad \mathcal{V}(\phi, 0) = 0. \tag{8.185}$$

The modification with respect to [BG08] consists in the addition of the kinetic and potential terms for the longitudinal electric and magnetic potentials in the last line of (8.183). Defining

$$\mathcal{F}_{\mu\nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M + \star S_{\mu\nu}^M,$$
$$\star S_{0i}^M = \Delta^{-1}(\Omega^{-1})^{MN}\partial_0\partial_i\pi_N, \tag{8.186}$$
$$\star S_{ij}^M = \epsilon_{ijk}\partial^k D^M,$$

we have $\mathcal{B}^{Mi} = \frac{1}{2}\epsilon^{ijk}\mathcal{F}_{jk}$ and can write

$$S_{DP} = \frac{1}{4}\int d^4x[\Omega_{MN}\epsilon^{ijk}(\mathcal{F}_{jk}^M + \star S_{jk}^M)\mathcal{F}_{0i}^N - \mathcal{F}_{ij}^M\mathcal{M}_{MN}\mathcal{F}^{Nij}$$
$$- 2\pi_M(\mathcal{M}^{-1})^{MN}\pi_N - 4\mathcal{V}], \tag{8.187}$$

where a total derivative has been dropped.

The gauge invariances are then doubled but still of the same covariant form as in the second order Lagrangian case,

$$\delta A_\mu^M = \partial_\mu \epsilon^M, \quad \delta D^M = 0, \quad \delta \pi_M = 0, \quad \delta \phi^i = 0. \tag{8.188}$$

The equations of motion for the gauge and scalar potentials are determined by the vanishing of

$$\frac{\delta \mathcal{L}_{DP}}{\delta \pi_M} = -(\mathcal{M}^{-1})^{MN}(\pi_N - \mathcal{M}_{NL}\partial_0 D^L),$$

$$\frac{\delta \mathcal{L}_{DP}}{\delta D^M} = \Omega_{MN}(\Delta A_0^N - \partial_0 \partial^i A_i^N) + \partial^i(\mathcal{M}_{MN}\mathcal{B}_i^N) - \partial_0 \pi_M - \frac{\partial \mathcal{V}}{\partial D^M},$$

$$\frac{\delta \mathcal{L}_{DP}}{\delta A_0^M} = -\Omega_{MN}\Delta D^N = -\frac{1}{2}\Omega_{MN}\epsilon^{ijk}\partial_i \mathcal{F}_{jk}^N,$$

$$\frac{\delta \mathcal{L}_{DP}}{\delta A_i^M} = \Omega_{MN}\partial_0 \mathcal{B}^{Ni} - \epsilon^{ijk}\partial_j(\mathcal{M}_{MN}\mathcal{B}_k^N) = \frac{1}{2}\Omega_{MN}\epsilon^{ijk}\partial_0 \mathcal{F}_{jk}^N - \partial_j(\mathcal{M}_{MN}\mathcal{F}^{Nij}). \tag{8.189}$$

The first set of equations then allows one to eliminate the momenta π_M by their own equations of motion. When Δ is invertible, the second and third set of equations allow one to solve D^M and A_0^M by their own equations of motion in the action, which yields (8.178). It is in this sense that these variants of the double potential formalism are equivalent, but of course not locally so. The third and fourth set of equations can be written as

$$\frac{\delta \mathcal{L}_{DP}}{\delta A_\mu^M} = \partial_\nu \star G_M^{\mu\nu}, \tag{8.190}$$

when defining

$$\star G_M^{i0} = \frac{1}{2}\Omega_{MN}\epsilon^{ijk}\mathcal{F}_{jk}^N, \quad \star G_M^{ij} = -\mathcal{M}_{MN}\mathcal{F}^{Nij}. \tag{8.191}$$

This definition implies that the components of

$$G_M = \frac{1}{2}G_{Mjk}dx^j dx^k + G_{Mi0}dx^i dx^0 \tag{8.192}$$

are explicitly given by

$$G_{Mjk} = -\Omega_{MN}\mathcal{F}_{jk}^N, \quad G_{Mi0} = \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{MN}\mathcal{F}^{Njk}. \tag{8.193}$$

After elimination of the π_M, the action of the theory can then also be written as the integral of $\mathcal{L}_0 = \mathcal{L}_{ES} + \mathcal{L}_V$ with

$$\mathcal{L}_{ES} = \mathcal{L}_S - \frac{1}{2}\partial_\mu D^M \mathcal{M}_{MN}\partial^\mu D^N - \mathcal{V}, \quad d^4 x \mathcal{L}_V = \int_0^1 \frac{dt}{t}[G_M F^M][tA^M, D^M, \phi^i]. \tag{8.194}$$

so that the scalar sector has been enlarged to $\phi^m = (\phi^i, D^M)$ and the scalar metric and potential are now $(g_{ij}, \mathcal{M}_{MN})$, respectively $(V, \mathcal{V})$. It is thus a particular case of the actions of the form (8.6) studied in Section 8.1.

8.5.2 Local BRST cohomology

The master action is given by

$$S = \int d^4x \left[\mathcal{L}_0 + A_M^{*\mu} \partial_\mu C^M \right] \tag{8.195}$$

with an antifield and ghost sector that is doubled as compared to the second order covariant formulation.

We then can copy previous results:

(i) $H^g(s) = 0$ for $g \leq -3$.

(ii) $H^{-2}(s)$ is doubled: $U^{-2} = \mu^M d^4x C_M^*$ with descent equation

$$s\, d^4x\, C_M^* + d \star A_M^* = 0, \quad s \star A_M^* + dG_M = 0, \quad sG_M = 0. \tag{8.196}$$

Characteristic cohomology $H_W^{n-2}(d)$ is then represented by the 2-forms $\mu^M G_M$.

For $g \geq -1$, the discussion in terms of U, W, V types is the same as before with indices $I, J, K, \cdots \to M, N, O, \ldots$ on vector potentials, ghosts and their antifields, and $i, j, k \cdots \to m, n, o \ldots$ on scalar fields.

The obstruction equation for symmetries, equation (8.109), becomes

$$G_M F^N f^M{}_N + F^M F^N h_{MN} + dI^{n-1}$$
$$- (dG_M g^M + [d(g_{ij} \star d\phi^j) + \star\partial_i(\mathcal{L}_{ES} + \mathcal{L}_V)]\Phi^i + \frac{\delta\mathcal{L}_0}{\delta D^M}\,\Phi^M) = 0. \tag{8.197}$$

8.5.3 Constraints on W, U-type cohomology

When there is no explicit x^μ dependence and $V = 0 = \mathcal{V}$, putting all derivatives of $F_{\mu\nu}^M, \phi^i, D^M$ to zero, one remains with

$$G_M|_{\mathrm{der}=0} F^N f^M{}_N + F^M F^N h_{MN} - \star\partial_m \mathcal{L}_V \Phi^m|_{\mathrm{der}=0} = 0, \tag{8.198}$$

where $G_M|_{\mathrm{der}=0}$ amounts to replacing $\mathcal{F}_{\mu\nu}^M$ by $F_{\mu\nu}^M$ in (8.193).

Using $-\partial_m \star \mathcal{L}_V = \delta_m^i \frac{1}{2} \partial_i G_M|_{\mathrm{der}=0} F^M$, and the decomposition $\Phi^m|_{\mathrm{der}=0} = \Phi_0^m + \Phi_1^m + \ldots$, where the Φ_n^m depend on undifferentiated scalar fields and are homogeneous of degree n in $F_{\mu\nu}^M$, the equation implies

$$G_M|_{\mathrm{der}=0} F^N f^M{}_N + F^M F^N h_{MN} + \frac{1}{2} \partial_i G_M|_{\mathrm{der}=0} F^M \Phi_0^i = 0. \tag{8.199}$$

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When taking account that

$$G_M|_{\text{der}=0}F^N = d^4x \frac{1}{2}[\Omega_{OM}\epsilon^{ijk}F^O_{jk}F^N_{0i} - \mathcal{M}_{MO}F^O_{jk}F^{Njk}], \tag{8.200}$$

this gives an equation of the type

$$\frac{1}{4}d^4x[\mathcal{O}_{MN}(\phi)\epsilon^{ijk}F^M_{jk}F^N_{0i} - \mathcal{P}_{MN}(\phi)F^M_{jk}F^{Njk}] = 0, \tag{8.201}$$

where

$$\mathcal{P}_{MN} = 2\mathcal{M}_{O(M}f^O{}_{N)} + \partial_i\mathcal{M}_{MN}\Phi^i_0, \tag{8.202}$$

$$\mathcal{O}_{MN} = 2\Omega_{MO}f^O{}_N + 2h_{MN}, \tag{8.203}$$

and $h_{MN} = h_{NM}$ on account of (8.44). Note that there is one less term as compared to (8.113) since the kinetic term does not depend on the scalars and also that $\mathcal{O}_{MN}$ is not symmetric.

Now both terms have to vanish separately because they involve different field strengths,

$$\mathcal{P}_{MN} = 0, \quad \mathcal{O}_{MN} = 0. \tag{8.204}$$

Setting $\phi^i = 0 = D^M$ then gives

$$\begin{aligned}
f^{(\mathcal{M}(0))}_{MN} + f^{(\mathcal{M}(0))}_{NM} + (\partial_i\mathcal{M}_{MN})(0)\Phi^i_0(0) &= 0, \\
f^{(\Omega)}_{MN} &= -h_{MN},
\end{aligned} \tag{8.205}$$

with $f^{(\Omega)}_{MN} = \Omega_{MO}f^O{}_N$. Consider first symmetries of W-type, i.e., take the case when the f's vanish. The first equation is then satisfied with $\Phi^i_0(0) = 0$, while the second equation then requires h_{MN} to vanish. This implies:

There are neither W-type symmetries nor W-type cohomology in ghost numbers $g \geq 0$ for the first order model.

As a consequence, $h_{MN} = h_{uMN}$, and the second of equation (8.205) is equivalent to

$$f^{(\Omega)}_{u[MN]} = 0, \quad f^{(\Omega)}_{u(MN)} = -h_{uMN}. \tag{8.206}$$

It follows that:

The algebra $\mathfrak{g}_U$ is the largest sub-algebra of $\mathfrak{sp}(2n_v, \mathbb{R})$ that can be turned into symmetries of the full theory. All non-trivial U-type symmetries require a non-vanishing h_{uMN} and thus involve a Chern-Simons term in their Noether currents.

On its own, the first equation of (8.205) is solved for skew-symmetric $f^{(\mathcal{M}(0))}_{MN}$ with vanishing $\Phi^i(0)$. Symmetric $f^{(\mathcal{M}(0))}_{MN}$ needs a non trivial scalar symmetry.

For U-type cohomologies in higher ghost number $g \geq 0$, the $k^u_{O_1 \ldots O_{g+1}}$ tensor has to satisfy (8.92), which becomes

$$f^{(\Omega)}_{u\,M(N} k^u_{O_1)\ldots O_{g+1}} = 0. \tag{8.207}$$

The object $D_{MO_1NO_2\ldots O_{g+1}} = f^{(\Omega)}_{u\,MN} k^u_{O_1\ldots O_{g+1}}$ is then symmetric in the first and third indices because $f^{(\Omega)}_{u\,MN}$ is symmetric, and antisymmetric in the second and third indices on account of (8.207). It thus has to vanish,

$$D_{MO_1NO_2\ldots O_{g+1}} = D_{NO_1MO_2\ldots O_{g+1}} = -D_{NMO_1O_2\ldots O_{g+1}} = -D_{O_1MNO_2\ldots O_{g+1}}$$
$$= D_{O_1NMO_2\ldots O_{g+1}} = D_{MNO_1O_2\ldots O_{g+1}} = -D_{MO_1NO_2\ldots O_{g+1}}. \tag{8.208}$$

It follows that $k^u_{O_1\ldots O_{g+1}} = 0$:

There are no U-type cohomology classes in ghost number $g \geq 0$.

In particular, there are no U-type gaugings even though there are U-type symmetries. We thus recover the results on gaugings of [BH11] from the current perspective.

8.5.4 Remarks on $GL(2n_v)$ transformations

The two remarks on linear changes of variables from Section 8.3.6 also apply in the first order case. More precisely, the second remark can be rephrased as follows.

The general discussion of the structure of the BRST cohomology of the first order model in Sections 8.5.2 and 8.5.3 goes through unchanged for arbitrary skew-symmetric non-degenerate Ω_{MN} and symmetric non-degenerate $\mathcal{M}_{MN}$. The local BRST cohomology for sets of Ω_{MN}, $\mathcal{M}_{MN}$ related by $GL(2n_v, \mathbb{R})$ transformations will be isomorphic, whereas explicit results for the local BRST cohomology do depend on the equivalence classes. For instance for the symmetries, this is the case when explicitly solving the obstruction equation (8.197). As concerns Ω_{MN}, there is just one equivalence class since all such matrices are related to a canonical Ω_{MN}, say $\Omega_{MN} = \delta_{IJ}\epsilon_{ab}$, by a $GL(2n_v, \mathbb{R})$ transformation. Hence, one can restrict oneself to equivalence classes of Ω_{MN}, $\mathcal{M}_{MN}$ with canonical Ω_{MN}, and $\mathcal{M}_{MN}$'s related by $Sp(2n_v, \mathbb{R})$ changes of variables.

The first remark of Section 8.3.6 then boils down to the statement that the algebra $\mathfrak{g}_U$ is the largest sub-algebra of $\mathfrak{sp}(2n_v, \mathbb{R})$ that can be turned into symmetries of the full theory, in agreement with the discussion of the previous section. In addition we have recovered there the result that the gauge algebra remains abelian.

8.5.5 Application to the bosonic sector of $\mathcal{N} = 4$ supergravity

For definiteness, let us again concentrate on the bosonic sector of four dimensional supergravity, without gravity. As in Section 8.4.3, we use the standard second order formulation for the $SL(2,\mathbb{R})/SO(2)$ sigma model. Alternatively, one could use a first order formulation in terms of fields parametrizing $SL(2,\mathbb{R})$, with a first class constraint eliminating the field for the $SO(2)$ subgroup. It would provide a first order formulation for all fields and make all global symmetries manifest.

To this scalar action, we first couple one vector field, i.e. add the action associated to (8.183) where $\mathcal{V} = 0$, the indices M, N take two values a, b, $\Omega_{ab} = \epsilon_{ab}$, and $\mathcal{M}_{ab} = M_{ab}^{-1}$. The matrix M and its inverse are given by

$$M = \begin{pmatrix} e^\phi & \chi e^\phi \\ \chi e^\phi & \chi^2 e^\phi + e^{-\phi} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \chi^2 e^\phi + e^{-\phi} & -\chi e^\phi \\ -\chi e^\phi & e^\phi \end{pmatrix} \tag{8.209}$$

and are such that M transforms as $M \to g^T M g$ under an $SL(2,\mathbb{R})$ transformation. The model is invariant under $SL(2,\mathbb{R})$ if the other fields transform as $A^a \to (g^T A)^a$, $D^a \to (g^T D)^a$, $\pi_a \to (g^{-1}\pi)_a$ because $SL(2,\mathbb{R})$ transformations are symplectic, $g\epsilon g^T = \epsilon$.

For the U-type symmetries, equation (8.206) requires $f_{ab}^{(\epsilon)} = \epsilon_{ac} f^c{}_b$ to be symmetric, so there are at most 3 linearly independent solutions. According to the above discussion, all of these give rise to symmetries, which need h_{uab} and also Φ_u^i. The U-type symmetries constitute the $\mathfrak{sl}(2,\mathbb{R})$ electric symmetry algebra.

We now consider the coupling to six vector fields in the different formulations of Section 8.4.3. For the $SO(6)$ invariant model, $M = (I, a)$, $\Omega_{MN} = \delta_{IJ}\epsilon_{ab}$ and $\mathcal{M}_{MN} = \delta_{IJ} M^{-1}{}_{ab}$, while the dual formulation corresponds to $\mathcal{M}_{MN} = \delta_{IJ} M_{ab}$. Finally, in the $SO(3) \times SO(3)$ formulation $\mathcal{M}_{MN} = (M_{ab}^{-1}\delta_{AB}, M_{ab}\delta_{A'B'})$.

It then follows from (8.205) that both in the $SO(6)$ invariant formulation and in the dual formulation, the electric symmetry algebra is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(6)$, where the $\mathfrak{sl}(2,\mathbb{R})$ transformations on the vectors $A^{(I,a)}$ and on $D^{(I,a)}, \pi_{(J,b)}$ in the dual formulation corresponds to the infinitesimal version of the above transformations where $g^T \to g^{-1}$.

Finally, in the $SO(3) \times SO(3)$ formulation, the electric symmetry algebra is also $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(6)$. This is so because the $SL(2,\mathbb{R})$ element ϵ is such that $M = \epsilon^T M^{-1} \epsilon$.

Chapter 9

Conclusions and Prospects

In **Part** I of this thesis, we discussed some aspects of timelike duality in the context of exotic versions of maximal supergravity. We first described the extension of Adams' theorem, which states that the only spheres which admit global parallelism are S^0, S^1, S^3 and S^7, to the pseudo-Riemannian geometry. There we argued that in three-dimensions, the pseudo-heperbolic manifold $H^{2,1} = SO(2,2)/SO(2,1)$, commonly known as AdS$_3$, and the pseudo-sphere $S^{1,2} = SO(2,2)/SO(1,2)$ are parallelizable. These two manifolds are interchangeable under the metric parity $(g_{\mu\nu} \rightarrow -g_{\mu\nu})$ which changes the sign of the curvature. For that reason, sometimes in literature $S^{1,2}$ is called $-$AdS$_3$ or AAdS$_3$. In seven-dimensions, the manifolds $S^{3,4} = SO(4,4)/SO(3,4)$ and $H^{4,3} = SO(4,4)/SO(4,3)$ are parallelizable. Once again, we can find one from one another by metric reversal. We discussed as well the construction of parallelizable seven-(pseudo-)spheres in the presence of torsion (or similarly with special holonomy) which are the non-symmetric coset spaces $Spin^+(3,4)/G_{2,2}^*$ and $Spin^+(4,3)/G_{2,2}^*$.

We discussed the S^7 manifold as it plays an important role in the compactification of eleven-dimensional supergravity, where one can find an $SO(8)$ gauged maximal supergravity in AdS$_4$ background via Freund-Rubin compactification on (round) S^7 or a $Spin(7)$ gauged non-supersymmetric theory in AdS$_4$ background via Englert compactification on a parallellized squashed $S^7 = Spin(7)/G_2$. We discussed the effect of timelike T-duality which propounds, beside the standard M-theory, the presence of M' and M^*-theories which are dual to the standard theory. We showed that the compactification on parallelizable $S^{3,4}$ is possible if one works in the picture described by M' dual theory [HR17]. *This solution similarly to the Englert one will result in a non-supersymmetric theory in AdS$_4$ background.* Besides the known Englert solution, the compactification on par-

allelizable 7-manifolds for the other supergravity theories in eleven-dimensions (the low energy limit of standard M- and M^*-theories) is not possible, however one can see that there are plenty of solutions as a result of Freund-Rubin compactification of all dual M theories which results in four dimensional theories in non-AdS backgrounds.

In **Part** II, we investigated gaugings of models inspired by the vector sector of extended supergravities. We first explained the changes that were brought upon by considering scalar-vector couplings of particular non-minimal type. It enhances the electric-magnetic duality group to a subgroup of symplectic group $Sp(2n_v, \mathbb{R})$ for a theory with n_v vectors coupled to scalars. Then we discussed the interplay between the first-order and second-order Lagrangians where the electric-magnetic duality is manifest in the former while only part of electric-magnetic duality acts locally in the latter and is manifest. However, the first-order formulation is more rigid; we presented a proof that there is no Yang-Mills deformations of first-order Lagrangians in the presence of scalars [Hen+18]. A result that in fact is true regardless of the presence of scalars.

Given the existence of the embedding tensor formalism to construct a gauged supergravity by minimally coupling scalars to vectors and promoting the vector fields to non-abelian connections on the fibre bundle (and adding the appropriate interactions to render the Lagrangian invariant), we showed that not only the standard second-order action with the appropriate choice of the corresponding duality frame is equivalent with the undeformed limit of the embedding tensor Lagrangian but also that the space of local deformations of both theories are isomorphic. In fact, all local cohomology groups $H^g(s|d)$ are isomorphic in both formulations. In particular the potential anomalies (cohomology at ghost number one) are also the same.

We then extended our analysis to examine the possibility of those deformations that change the gauge symmetry while keeping the gauged algebra untouched. We have systematically analyzed gaugings of vector-scalar models through a standard deformation theoretic approach. In the case of gauge systems, this is most naturally done in the BV-BRST antifield formalism.

We have shown that different types of symmetries behave differently when one tries to gauge them. The method allows one to find all the infinitesimal gaugings and higher order cohomology classes once all symmetries are known.

We classified the symmetries into U, W and V-types, where the U-**type** corresponds to those deformations that deform the abelian gauge algebra, W-**type** symmetries contain the topological gaugings of [WHR87] and V-**type** symmetries contain gauge invariant couplings and admit gauge-invariant Noether currents. The Noether currents of U and W-type symmetries cannot be made gauge invariant.

For the models explicitly considered, we have found that the only possible

gaugings of U and W-types are the ones previously considered in the literature, namely Yang-Mills and topological couplings among the gauge fields, with minimal couplings of the scalars.

We have shown in Section 8.2 that given the graded structure of the antibracket map, the leading obstruction to extend first order deformations of U-type to second order, leading to the Jacobi identity for the structure constants, cannot be eliminated by adding V-terms. Furthermore, in some cases, for instance when one imposes Poincaré invariance as relevant to relativistic theories, the V-type symmetries can be shown to be absent [Tor95]. It turns out that the effect of coupling the models to Einstein gravity justifies this assumption [BBH95a] and simplifies the problem.

We have analyzed the problem in the second order Lagrangian and in the first order manifestly duality invariant formulation, both of which are non-locally related in space (but not in time). The results are very different: whereas the former formulation allows for standard gaugings, the latter formulation allows for more (generalized) symmetries of U-type, but none of those can be gauged.

Prospects

As the final remark, we list the future research directions inspired by results of the thesis:

1) The no-go theorem which states that there is no local deformation available in the first-order formulation is highly based on the locality present in the formulation. The inevitable result that in all examples presented in the thesis, one could not gauge any part of global symmetry but the U-type one corresponding to the Yang-Mills deformations, can also be related to the local framework of our BV-BRST analysis. In order to go past such no-go results, one should presumably try to work in a controlled way with deformations that are spatially non-local. This as well may open a window to discuss the gaugings in the presence of sources, most importantly in the presence of Dirac magnetic monopoles, since it requires going beyond locality, see the comment under the equation (5.9) in Chapter 5.

2) It is interesting to extend the analysis of gauging of the bosonic sector of supergravity to include the fermion sector as well. Even though, one may in general expect no new constraints but it is worth to see explicitly that the supersymmetric extension does not require any extra constraints. There has been works along this way [BJT18], where the deformation of a free theory in the presence of both bosons and fermions were considered. It is then intriguing to consider scalar coupling from the beginning as it enhances the electric-magnetic duality symmetry group and to analyze the deformations of such a theory.

3) In any dimensions, the positive level generators of the very extended

Kac–Moody algebra E_{11} with completely antisymmetric spacetime indices are associated with the form fields of the corresponding maximal supergravity. In [Ber+08], a correspondence between the gaugings of maximal and half-maximal supergravity and certain generators of E_{11} Kac-Moody algebra has been developed. It has been generalized in [Ric10a; Ric10b] to include the gauging of local scaling symmetry (Trombone symmetry) [Cre+98b] of maximal supergravity. Later in [LST11], it was shown how it can be realized in the context of embedding tensor formalism. Since we have established a relation between our result and the one of embedding tensor formalism, it would be interesting to consider how the above correspondence is realized in our framework.

4) In Chapter 4, we considered the solutions to supergravity theories in $6+5$ dimensions with a non-degenerate metric. It may be of interest to consider a case where the eleven-dimensional metric is not of maximal rank. See for example [Dra92] where similar question has been addressed in the context of standard supergravity.

5) We discussed in Chapter 4 that there are Freund-Rubin solutions for each timelike T-dual M-theories. Given the standard M-theory in $10+1$ dimensions, the corresponding Freund-Rubin solution is a supergravity theory in AdS_4 background spacetime. There is a superconformal field theory in three-dimensions known as ABJM theory; an $\mathcal{N}=6$ supersymmetric Chern-Simons-matter theory [Aha+08]. It was shown that the ABJM theory can be seen as the large N limit of a stack of N M_2-branes sitting at the orbifold singularity $\mathbb{Z}_k$ such that it can be realized as the CFT dual to maximal supergravity in the near horizon geometry $AdS_4 \times S^7/\mathbb{Z}_k$. Therefore one finds an example of AdS/CFT correspondence among the very few existing examples.

Given the existence of Freund-Rubin solution of M'- and M^*-theories on $AdS_4 \times S^{3,4}$ and $-AdS_4 \times -dS_7$ respectively, it would be interesting to build the corresponding three-dimensional superconformal field theories. This would then provide us with new examples of AdS/CFT correspondence.

Appendix A

Notation

The following notation, beside the definition of Levi-Civita tensor, is specifically used in the Part I. We generally follow the notation of [FV12], when we discuss supergravity.

- (s, t): Signature of a spacetime with

    - s: Number of space directions of the ambient space manifold

    - t: Number of time directions of the ambient space manifold with $t = T + T'$

- T: Number of time directions of the internal manifold with

- T': Number of time directions of spacetime "background" manifold

- η_{ab}: the metric on the tangent space $T\mathcal{M}$ of a manifold $\mathcal{M}$, where $a, b = 1, 2, ..., dim(T\mathcal{M})$.

- $g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{mn} \end{pmatrix}$: the metric on 11 dimensional ambient space, $M, N = 0, ..., 10$.

- g_{mn}: the metric on seven dimensional internal space, $m, n = 4, ..., 10$.

- $g_{\mu\nu}$: the metric on four dimensional spacetime, $\mu, \nu = 0, ..., 3$.

- $F_{MNPQ} = 4\partial_{[M}A_{NPQ]}$.

- $\varepsilon_{M_1...M_{11}}$: the eleven dimensional Levi-Civita antisymmetric covariant tensor density of rank eleven and weight -1, with $\varepsilon_{01234...} = +1$.

- $\varepsilon^{M_1 \ldots M_{11}}$: the eleven dimensional Levi-Civita antisymmetric contravariant tensor density of rank eleven and weight $+1$, with $\varepsilon^{01234\cdots} = (-1)^t$. This choice is such that

$$\frac{1}{\sqrt{|g^{11}|}} \varepsilon^{M_1 \ldots M_p} = \sqrt{|g^{11}|} g^{M_1 N_1} g^{M_2 N_2} \cdots g^{M_{11} N_{11}} \varepsilon_{N_1 \ldots N_p},$$

  where g^{11} is the determinant of g_{MN}.

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$$\varepsilon_{M_1 \ldots M_k M_{k+1} \ldots M_{k+p}} \varepsilon^{M_1 \ldots M_k N_{k+1} \ldots N_{k+p}} = (-1)^t \, k! \, \delta^{N_{k+1} \ldots N_{k+p}}_{M_{k+1} \ldots M_{k+p}}$$
$$= (-1)^t \, k! \, p! \, \delta^{N_{k+1}}_{[M_{k+1}} \delta^{N_{k+2}}_{M_{k+2}} \ldots \delta^{N_{k+p}}_{M_{k+p}]}.$$

- $SO(p,q)$: the group of all transformations which leaves invariant the bilinear form $\eta_{p,q} = \sum_{i=1}^p dx_i^2 - \sum_{j=1}^q dx_j^2$.

- The pseudo-sphere $S^{p-1,q} = SO(p,q)/SO(p-1,q)$ ($p \geq 1, q \geq 0$) is a manifold with induced metric of signature $(p-1,q)$ and positive curvature. In particular, $S^{p-1,0}$ is the standard $(p-1)$-sphere S^{p-1}.

- The pseudo-hyperbolic space $H^{p,q-1} = SO(p,q)/SO(p,q-1)$ ($p \geq 0, q \geq 1$) is a manifold with induced metric of signature $(p,q-1)$ and negative curvature. While one removes a spacelike direction from the ambient space to get $S^{p-1,q}$, one removes a timelike direction for $H^{p,q-1}$ (see Appendix C for an explicit example). In particular, $H^{p,0}$ is the negative curvature standard hyperbolic space H^p of dimension p.

The following notations are specifically used in Part II:

- $H^{g,n}(s|d)$: BRST cohomology at ghost number g in the space of local functional n-forms.

- $H^g(s|d)$: BRST cohomology at ghost number g in the space of local functional of a top form-degree.

- $H^{n+g}_{\mathrm{char}}(d)$: Characteristic cohomology

- $H^g(s|d) \simeq H^{n+g}_{\mathrm{char}}(d)$ for $g \leq -1$.

- The Section 8.5, has the opposite notation for Levi-Civita symbol compared to the rest of the thesis, i.e. $\varepsilon_{0123} = -1$.

Appendix B

Octonion and Split Octonion

In this appendix, we give a quick review of Cayley-Dickson construction of division algebras. We discuss the algebra of octonions and split octonions and we present the corresponding Fano planes.

B.1 The Cayley-Dickson Construction of Divison Algebras

The Cayley-Dickson construction of the normed division algebras is a powerful tool as it can nicely explain the reason that why each of the normed division algebras can be obtained as a subalgebra of a larger normed division algebra. It also gives a manifestation of why as we go up in the dimension of division algebras, the algebras lose realness condition, commutativity and associativity gradually.

Let's consider $a, b \in \mathbb{R}$, then the complex number $a + bi$ can be thought of as a pair (a, b). The addition and multiplication for any two complex numbers (a, b) and (c, d) is defined as

$$(a, b) + (c, d) = (a + c, b + d), \tag{B.1}$$
$$(a, b)(c, d) = (ac - db, ad + cb). \tag{B.2}$$

The complex conjugate of a complex number thus is given by

$$\overline{(a, b)} = (a, -b). \tag{B.3}$$

Having defined the complex numbers, we can define the quaternions in a similar way. A quaternion can be thought of as a pair of complex numbers. Addition is

again like (B.1), and the multiplication is defined for $a, b \in \mathbb{C}$ as

$$(a, b)(c, d) = (ac - \bar{d}b, ad + b\bar{c}). \tag{B.4}$$

The quaternion conjugate is given by

$$\overline{(a, b)} = (\bar{a}, -b). \tag{B.5}$$

If one repeats the same procedure once again, the algebra of octonions is obtained. This iterative method of building up normed division algebras is called the Cayley-Dickson construction. If one continues to build up algebras by this construction, all algebras larger than octonions will be normed but not division algebras [Bae02].

B.2 Octonions

Let a and b be two quaternions, $a = a_0 + a_1 i + a_2 j + a_3 k$, $b = b_0 + b_1 i + b_2 j + b_3 k$ ($a_i, b_i \in \mathbb{R}$, $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$ etc). As we explained in the previous section, the octonions are pairs of quaternions $a + b\ell$, where ℓ is a new element, for which one defines the multiplication as (B.4) or explicitly as

$$(a + b\ell)(c + d\ell) = ac + (\ell)^2 \bar{d}b + (da + b\bar{c})\ell, \tag{B.6}$$

where

$$\ell^2 = -1. \tag{B.7}$$

In the equation (B.6), the overbar denotes quaternionic conjugation defined in (B.5), $\bar{d} = d_0 - d_1 i - d_2 j - d_3 k$.

Considering the basis $\{e_0, e_1, ..., e_7\}$ the algebra of unit octonion is defined as

$$e_0 e_i = e_i e_0 = e_i, \tag{B.8}$$
$$e_i e_j = -\delta_{ij} e_0 + f_{ijk} e_k, \tag{B.9}$$

where f_{ijk} is a totally antisymmetric tensor and

$$f_{ijk} = +1 \quad \text{for} \quad (ijk) = \{(123), (145), (176), (246), (257), (347), (365)\}. \tag{B.10}$$

One should notice that the above definition of f_{ijk} is not unique but fixing the unit element $e_0 = 1$, then all different definitions are isomorphic.

There is a mnemonic representation of the multiplication of octonions which is called Fano plane and is the simplest example of the projective plane. The Fano plane for the octonions is depicted in Figure B.1 and one can easily find the other isomorphic tables using the symmetries of Fano plane.
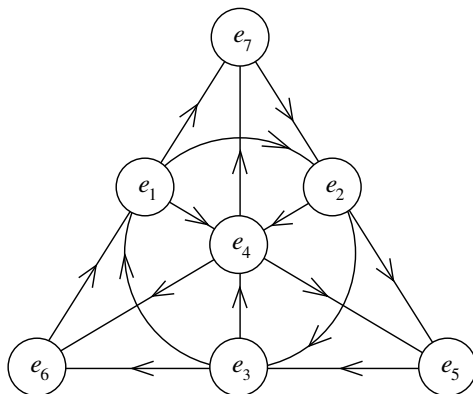
Figure B.1: The Fano plane for the octonions multiplication table.

B.3 Split Octonions

Let a and b be two quaternions, $a = a_0 + a_1 i + a_2 j + a_3 k$, $b = b_0 + b_1 i + b_2 j + b_3 k$ ($a_i, b_i \in \mathbb{R}$, $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$ etc). The split octonions are pairs of quaternions $a + b\ell$, where ℓ is a new element, for which one defines the product as

$$(a + b\ell)(c + d\ell) = ac + (\ell)^2 \bar{d} b + (da + b\bar{c})\ell, \tag{B.11}$$

where

$$\ell^2 = +1, \tag{B.12}$$

instead of -1 as for the standard octonions. In (B.11), the overbar denotes quaternionic conjugation, $\bar{d} = d_0 - d_1 i - d_2 j - d_3 k$.

Setting $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$, $e_4 = \ell$, $e_5 = i\ell$, $e_6 = j\ell$ and $e_7 = k\ell$, one gets the same products $e_i e_j$ as for the standard octonions, except when two ℓ's are involved, in which case one gets an additional minus sign. For instance $e_1 e_4 = e_5$ as for octonions, but $e_4 e_5 = -e_1$ while it is $+e_1$ for standard octonions. Similarly, $e_4^2 = e_5^2 = e_6^2 = e_7^2 = +1$ instead of -1.

One defines octonionic conjugation for a general split octonion

$$x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_4 + x^5 e_5 + x^6 e_6 + x^7 e_7, \quad x^i \in \mathbb{R} \tag{B.13}$$

as

$$\bar{x} = x^0 e_0 - x^1 e_1 - x^2 e_2 - x^3 e_3 - x^4 e_4 - x^5 e_5 - x^6 e_6 - x^7 e_7. \tag{B.14}$$

One has

$$\overline{xy} = \bar{y}\bar{x}, \tag{B.15}$$

and of course $\bar{\bar{x}} = x$. A scalar product can then be introduced as

$$(x, y) = \frac{1}{2} \left(x\bar{y} + y\bar{x} \right) = \frac{1}{2} \left(\bar{x}y + \bar{y}x \right). \tag{B.16}$$

Explicitly, one finds

$$\begin{aligned} (x, y) &= x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 - x^4 y^4 - x^5 y^5 - x^6 y^6 - x^7 y^7 \tag{B.17} \\ &= x^0 y^0 + \sum_{i=1}^{7} \eta_{ij} x^i y^j, \tag{B.18} \end{aligned}$$

where η_{ij} is the flat metric with signature $(3, 4)$.

An octonion is pure imaginary if $\bar{x} = -x$. The unit octonions e_i are pure imaginary. The pure imaginary condition is equivalent to $(e_0, x) = 0$.

One can rewrite the product of the unit octonions e_i as

$$e_i e_j = -\eta_{ij} e_0 + f^k{}_{ij} e_k, \tag{B.19}$$

where the structure constants $f^k{}_{ij}$ are such that the $f_{kij} \equiv \eta_{km} f^m{}_{ij}$ are completely antisymmetric in (i, j, k). The tensor f_{ijk} has value $+1$ for $(ijk) = (123)$, (154), (167), (264), (275), (374) and (356) (and cyclic permutations) and it vanishes otherwise. Similarly to octonions, there is a mnemonic picture of the multiplication table of split octonions. The corresponding Fano plane has been given in Figure B.2, where once again one can read off all possible multiplication tables using the symmetries of the Fano plane. The ordering of abc for the non-zero components of f_{abc} for the split octonions can be obtained from its octonionic counterpart and reversing the diagonal arrows in the Fano plane.

The squared norm $N(x) \equiv (x, x)$ of x reads

$$\begin{aligned} N(x) &= \left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 - \left(x^4\right)^2 - \left(x^5\right)^2 - \left(x^6\right)^2 - \left(x^7\right)^2 \\ &= \eta_{\lambda\mu} x^\lambda x^\mu, \quad (\lambda, \mu = 0, 1, \cdots, 7) \tag{B.20} \end{aligned}$$

where $\eta_{\lambda\mu}$ is the flat metric with signature $(4, 4)$. The split octonions form a composition algebra, i.e.,

$$N(xy) = N(x)N(y). \tag{B.21}$$

Just as the standard octonions, the split octonions do not form an associative algebra. However, in the same way as for the standard octonions, the associator $[x, y, z]$ of three split octonions x, y, z, defined through

$$(x, y, z) = (xy)z - x(yz), \tag{B.22}$$

is an alternating function of x, y, z, i.e.

$$(x, y, z) = (y, z, x) = (z, x, y) = -(y, x, z) = -(x, z, y) = -(z, y, x). \tag{B.23}$$

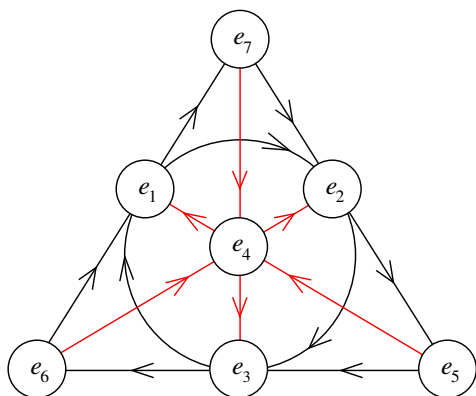More information on the split octonions can be found in [GG73; BH14].

Figure B.2: The Fano plane for split octonions multiplication table. This Fano plane can be obtained from its octonionic counterpart by reversing the diagonal arrows, shown in red, in the Fano plane.

B.4 Proof of Theorem 2.2

Using Lemma 2.1 and writing $y = y + \lambda x$ for any $\lambda \in F, x \in A$, we get

$$L_x L_{\bar{y}} + L_y L_{\bar{x}} = 2(x|y)I. \tag{B.24}$$

Now let $N = \mathrm{Dim}\, A$ be the dimension of the normed division algebra A. Let A_0 be a sub-vector space of A defined by

$$A_0 = \{x | (x, e) = 0, x \in A\}. \tag{B.25}$$

Any $x \in A$ can be uniquely rewritten as $x = \lambda e + y$ where $\lambda \in F$, $y \in A_0$, with $\lambda = (e, x)$ and $y = x - (e, x)e$. Then,

$$A = Fe \oplus A_0, \qquad \mathrm{Dim}\, A_0 = N - 1. \tag{B.26}$$

For $x \in A_0$, we have $\bar{x} = -x$ and therefore

$$L_x L_y + L_y L_x = -2(x, y)I, \tag{B.27}$$

for any $x, y \in A_0$. For the moment, let's assume the algebra A is defined over the field of complex numbers. One can find an N dimensional orthonormal basis $e_0(= e), e_1, e_2, ..., e_{N-1}$ of A satisfying the orthonormal condition [Oku95]

$$(e_\mu, e_\nu) = \delta_{\mu\nu}, \qquad \mu, \nu = 0, 1, ..., N - 1. \tag{B.28}$$

In particular, $e_1, ..., e_{N-1}$ realizes an orthonormal basis of A_0. Choosing $x = e_j$ and $y = e_k$ for $j, k \neq 0$ in (B.27), we obtain

$$L_j L_k + L_k L_j = -2\delta_{jk}, \qquad j, k = 1, ..., N-1. \tag{B.29}$$

which is a Clifford algebra in $N - 1$ dimensional vector space A_0. Any matrix representation of a Clifford algebra is reducible and the dimension of irreducible representation in $N - 1$ dimensional space is[Dix13]

$$d = 2^n, \qquad \begin{cases} n = \frac{N-1}{2} & (N-1)\,\text{even} \\ \\ n = \frac{N}{2} - 1 & (N-1)\,\text{odd} \end{cases} . \tag{B.30}$$

Suppose that the $N \times N$ matrix representation of L_js contains m irreducible components with its total matrix dimension of md. This must be equal to N, so that we have

$$N = \begin{cases} 2^{\frac{N-1}{2}} m & (N-1)\,\text{even} \\ 2^{\frac{N}{2}-1} m & (N-1)\,\text{odd} \end{cases} . \tag{B.31}$$

There are four solutions to this equation:

- $N = 1$, $m = 1$,

- $N = 2$, $m = 2$,

- $N = 4$, $m = 2$,

- $N = 8$, $m = 1$.

This proves the theorem. Even though we assumed that the algebra was defined on a complex field, the proof still holds for the real field [Oku95].

Appendix C

The Pseudo-Sphere $S^{3,4}$ and the Split Octonions

C.1 A Survey of $S^{3,4}$ as a Homogeneous Space

Let $\mathbb{R}^{4,4}$ be the 8-dimensional real vector space endowed with the flat metric of mixed signature $(4,4)$,

$$ds^2_{4,4} = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2 - \left(dy^1\right)^2 - \left(dy^2\right)^2 - \left(dy^3\right)^2 - \left(dy^4\right)^2.$$
(C.1)

We consider the hypersurfaces:

$$S^{3,4} : \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 - \left(y^1\right)^2 - \left(y^2\right)^2 - \left(y^3\right)^2 - \left(y^4\right)^2 = 1,$$
(C.2)

$$H^{4,3} : \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 - \left(y^1\right)^2 - \left(y^2\right)^2 - \left(y^3\right)^2 - \left(y^4\right)^2 = -1.$$
(C.3)

These are connected. Because the signature of the embedding space is symmetric for the exchange of the space and time directions, these hypersurfaces are clearly isomorphic. The "pseudo-sphere" $S^{3,4}$ has an induced metric with signature $(3,4)$ (three $+$ signs and four $-$ signs), while the "pseudo-hyperbolic space" $H^{4,3}$ has an induced metric with signature $(4,3)$. One goes from one to the other by an overall sign change of the metric.

The split group $O(4,4)$ and its subgroups $SO(4,4)$ and $SO^+(4,4)$, the connected component of $SO(4,4)$, act transitively on $S^{3,4}$ and $H^{4,3}$. Given a point on $S^{3,4}$, for example the north pole given by $(1,0,0,...,0)$, the subgroups of $O(4,4)$, $SO(4,4)$ and $SO^+(4,4)$ which leave the pointed fixed are $O(3,4)$, $SO(3,4)$ and

$SO^+(3, 4)$ respectively. This subgroups are called the stability subgroups. Similarly, the corresponding stability subgroups for a fixed point on $H^{4,3}$ are $O(4, 3)$, $SO(4, 3)$ and $SO^+(4, 3)$. The groups $O(3, 4)$ and $O(4, 3)$ (as well as $SO(3, 4)$ and $SO(4, 3)$, or $SO^+(3, 4)$ and $SO^+(4, 3)$) are isomorphic, but we prefer to adopt different notations in each case to keep track of the signature of the metric. Thus, the pseudosphere $S^{3,4}$ and pseudo-hyperbolic space $H^{4,3}$ are the homogeneous spaces:

$$S^{3,4} = \frac{O(4, 4)}{O(3, 4)} = \frac{SO(4, 4)}{SO(3, 4)} = \frac{SO^+(4, 4)}{SO^+(3, 4)}, \tag{C.4}$$

and

$$H^{4,3} = \frac{O(4, 4)}{O(4, 3)} = \frac{SO(4, 4)}{SO(4, 3)} = \frac{SO^+(4, 4)}{SO^+(4, 3)}. \tag{C.5}$$

The pseudo-sphere $S^{3,4}$ and pseudo-hyperbolic space $H^{4,3}$ are not only homogeneous spaces, they are in fact maximally symmetric and hence spaces of constant curvature,

$$R_{mnpq} = K \left(g_{mp} g_{nq} - g_{mq} g_{np} \right), \tag{C.6}$$

with $K = 1 > 0$ for the pseudo-sphere $S^{3,4}$ and $K = -1 < 0$ for the pseudo-hyperbolic space $H^{4,3}$, in agreement with the observation that under an overall change of sign of the metric, the Riemann curvature changes sign but the product $(g_{mp} g_{nq} - g_{mq} g_{np})$ does not, so that K changes sign.

C.2 Two Infinite Family of Parallelizations of the Pseudo-Sphere $S^{3,4}$

There exist two infinite families of parallelizations of the pseudo-sphere $S^{3,4}$. We first start by describing one of these parallelizations.

Consider the pseudo-sphere $S^{3,4}$ of unit split octonions, $x \in S^{3,4} \Leftrightarrow N(x) = 1$. The real number 1 is the "North pole". The tangent space at the North pole can be identified with the seven-dimensional vector space of imaginary split octonions. A Lorentz basis of this tangent space is given by the e_i's. Let $x \in S^{3,4}$. Right multiplication by x maps the North pole to x, and the tangent space at the North pole to the tangent space at x. Indeed, the vectors e_i at the North pole are mapped on $t_i^+ \equiv e_i x$. One has:

$$(t_i^+, x) = 0, \quad (t_i^+, t_j^+) = \eta_{ij}. \tag{C.7}$$

The first equality expresses that the seven imaginary split octonions t_i^+, viewed as vector in $\mathbb{R}^{4,4}$ are orthogonal to x and hence tangent to the pseudo-sphere $S^{3,4}$

at x. The second equality expresses that the t_i^+'s form a Lorentz basis of that tangent space.

We have thus defined at each point of the pseudo-sphere $S^{3,4}$ a Lorentz basis of the tangent space. This provides an absolute parallelism for $S^{3,4}$. The tangent vector at x parallel to the tangent vector e_i at 1 is the vector $e_i x$ obtained by right multiplication with x. The corresponding parallel transport preserves the metric of $S^{3,4}$ and its geodesics can be verified to coincide with those defined by the metric. The parallelization defined by $\{t_i^+\}$ is thus consistent with the metric.

Similarly, left multiplication also defines a parallelization of $S^{3,4}$ that maps the tangent basis $\{e_i\}$ at 1 on the tangent basis $\{t_i^- \equiv xe_i\}$ at x. The two parallelisms are inequivalent since the tangent vectors t_i^+ and t_i^- coincide only at the North pole and at the "South pole" -1.

Yet other parallelizations can be defined by using a reference point on $S^{3,4}$ different from unity. More precisely, let $\alpha \in S^{3,4}$. One goes from α to x by multiplying α by $\bar{\alpha}x$, $\alpha(\bar{\alpha}x) = x$. One defines the tangent vector ${}^{(\alpha)}t_i^+$ at x parallel to the tangent vector $e_i\alpha$ at α through right multiplication by the octonion $\bar{\alpha}x$ that connects α to x, ${}^{(\alpha)}t_i^+ = (e_i\alpha)(\bar{\alpha}x)$. The tangent vectors ${}^{(\alpha)}t_i^+$ and t_i^+ at x do not coincide because octonionic multiplication is not associative. A similiar construction yields the parallelism ${}^{(\alpha)}t_i^- = (x\bar{\alpha})(\alpha e_i)$.

Because the families of parallelisms given by the above construction are consistent with the metric, the corresponding torsion tensors obey the equations (4.24)-(4.26) given above [CS26b; CS26a; Wol72a; Wol72b].

The parallelisms of $S^{3,4}$ are related to the split octonions in the same way as the parallelisms of the seven-sphere are related to the standard octonions. For that reason, the reader can find more information on the parallelisms of $S^{3,4}$ in the literature on the parallelisms of the seven-sphere. A reference that we have found useful is [Roo84].

C.3 $Spin^+(3,4)$ and $G_{2,2}^*$

The complex Lie algebra $\mathfrak{g}_2$ possesses two real forms, the compact one and the split one. To the compact real form corresponds the unique compact group G_2. To the split real form correspond the simply connected non compact group $G_{2,2}$ with center $\mathbb{Z}_2$ and the quotient $G_{2,2}^* \equiv \frac{G_{2,2}}{\mathbb{Z}_2}$ ("adjoint real form") which has trivial center (and is not simply connected). The group $G_{2,2}^*$ is the automorphism group of the split octonions.

The group $Spin(7)$ is well-known to have a transitive action on the seven-sphere S^7, with isotropy group G_2. Similarly, the group $Spin^+(3,4)$ (connected component of $Spin(3,4)$) has a transitive action on the pseudo-sphere $S^{3,4}$ with

isotropy group $G_{2,2}^*$ [Kat98]. We can thus also identify $S^{3,4}$ with the homogeneous space $Spin^+(3,4)/G_{2,2}^*$,

$$S^{3,4} \simeq \frac{Spin^+(3,4)}{G_{2,2}^*} \tag{C.8}$$

Appendix D

Derivation of Equation (8.82)

In this appendix, we derive formula (8.82) for the variation $\delta_u G_I = -(U_u, G_I)$ of the two-form G_I under a U-type symmetry. This is done in two steps:

1. First, we show that

$$\delta_u G_I + (f_u)^J{}_I G_J \approx c_{IJ} F^J + d(\text{invariant}) \tag{D.1}$$

   for some constants c_{IJ}.

2. Then, we prove that the c_{IJ} take the form

$$c_{IJ} = -2(h_u)_{IJ} + \lambda_u^w (h_w)_{IJ}, \tag{D.2}$$

   where the constants $(h_u)_{IJ}$ and $(h_w)_{IJ}$ are those appearing in the currents associated with U_u and W_w respectively.

The proof is given in the case where the Lagrangian (or, equivalently, G_I) does not depend on the derivatives of $F^I_{\mu\nu}$.

A lemma

The proof of the above steps uses the following result on the W-type cohomology classes (with $g = -1$):

$$t_{IJ} F^I F^J \approx d(\text{invariant}) \;\Rightarrow\; t_{IJ} = \sum_w \lambda^w (h_w)_{IJ} \;\text{ for some } \lambda^w. \tag{D.3}$$

This is proven as follows: $t_{IJ}F^I F^J \approx d(\text{invariant})$ implies that

$$t_{IJ}F^I F^J + dI + \delta k = 0 \qquad (D.4)$$

for some gauge invariant I and some k of antifield number 1, where δ is here the Koszul-Tate differential. Now, it is proven in [BBH00] that k must be gauge invariant; hence, it can be written as

$$k = \hat{K} + dR, \quad \hat{K} = d^4 x [A_I^{*\mu} g_\mu^I + \phi_i^* \Phi^i] \qquad (D.5)$$

for some gauge invariant R, g_μ^I and Φ^i. Indeed, derivatives acting on the antifields contained in k are pushed to the term dR by integration by parts, leaving the form (D.5) where $\hat{K}$ contains only the undifferentiated antifields. Putting this back in (D.4) and using the fact that $\delta \hat{K} = s\hat{K}$ because $\hat{K}$ is gauge invariant, we get

$$s\hat{K} + d\left(t_{IJ}A^I F^J + J\right) = 0 \qquad (D.6)$$

for some gauge invariant $J = I - \delta R$. This shows that $\hat{K}$ is a W-type cohomology class: we can therefore expand $\hat{K}$ in the W_w basis as $\hat{K} = \sum \lambda^w W_w$. In particular, this implies that $t_{IJ} = \sum \lambda^w (h_w)_{IJ}$, which proves the lemma.

First step

We start from the chain of descent equations involving G_I,

$$s\, d^4 x\, C_I^* + d \star A_I^* = 0, \quad s \star A_I^* + dG_I = 0, \quad sG_I = 0. \qquad (D.7)$$

Applying $(U_u, \cdot)_{\text{alt}}$ to this chain, we get

$$s\left[d^4 x\, (f_u)^J{}_I C_J^* \right] + d\left[(f_u)^J{}_I \star A_J^* + \frac{\delta K_u}{\delta A^I} \right] = 0, \qquad (D.8)$$

$$s\left[(f_u)^J{}_I \star A_J^* + \frac{\delta K_u}{\delta A^I} \right] + d\left[-\delta_u G_I \right] = 0, \qquad (D.9)$$

$$s\left[-\delta_u G_I \right] = 0, \qquad (D.10)$$

which can be simplified to

$$d\left(\frac{\delta K_u}{\delta A^I} \right) = 0, \qquad (D.11)$$

$$s\left(\frac{\delta K_u}{\delta A^I} \right) + d\left(-\delta_u G_I - (f_u)^J{}_I G_J \right) = 0, \qquad (D.12)$$

$$s\left(-\delta_u G_I \right) = 0, \qquad (D.13)$$

using equations (D.7) again. Equation (D.11) implies that

$$\frac{\delta K_u}{\delta A^I} = d\eta^{-1,2} \tag{D.14}$$

for some $\eta^{-1,2}$ of ghost number -1 and form degree 2. Because the left-hand side is gauge invariant and $\eta^{-1,2}$ is of form degree two, $\eta^{-1,2}$ must also be gauge invariant. This follows from theorems on the invariant cohomology of d in form degree 2 [BDK90; Dub+92]. Equation (D.12) implies then

$$d\left(\delta_u G_I + (f_u)^J{}_I G_J + s\eta^{-1,2}\right) = 0, \tag{D.15}$$

i.e.

$$\delta_u G_I + (f_u)^J{}_I G_J + s\eta^{-1,2} = d\eta^{0,1} \tag{D.16}$$

for some $\eta^{0,1}$ of ghost number 0 and form degree 1. Again, the left-hand side of this equation is gauge invariant: results on the invariant cohomology of d in form degree 1 [BDK90; Dub+92] now imply that the non-gauge invariant part of $\eta^{0,1}$ can only be a linear combination of the one-forms A^I,

$$\eta^{0,1} = c_{IJ} A^J + \text{(gauge invariant)}. \tag{D.17}$$

Plugging this back in equation (D.16) and using the fact that $s\eta^{-1,2} \approx 0$ (since $\eta^{-1,2}$ is gauge invariant), we recover equation (D.1). This concludes the first step of the proof.

Second step

For the second step, we introduce

$$N = -\int d^4x \left(C_I^* C^I + A_I^{*\mu} A_\mu^I\right), \quad \hat{N} = (N, \cdot)_{\text{alt}}. \tag{D.18}$$

The operator $\hat{N}$ counts the number of A^I's and C^I's minus the number of A_I^*'s and C_I^*'s. Because it carries ghost number -1, it commutes with the exterior derivative, $\hat{N}d = d\hat{N}$. Applying this operator to the equation

$$sU_u + d\left[(f_u)^I{}_J(\star A_I^* C^J + G_I A^J) + (h_u)_{IJ} F^I A^J + J_u\right] = 0 \tag{D.19}$$

gives

$$\left(\int G_I F^I, U_u\right)_{\text{alt}} + d\left[(f_u)^I{}_J(\hat{N}+1)(G_I)A^J + 2(h_u)_{IJ} F^I A^J + \hat{N}(J_u)\right] \approx 0. \tag{D.20}$$

The second term is evident. The first term is

$$\hat{N}(sU_u) = (N, (S, U_u)_{\text{alt}})_{\text{alt}} = ((N, S), U_u)_{\text{alt}} + (S, (N, U_u)_{\text{alt}})_{\text{alt}} \qquad \text{(D.21)}$$

according to the graded Jacobi identity. The counting operator $\hat{N}$ kills the $A_I^{*\mu}\partial_\mu C^I$ term in the master action S, which implies

$$(N, S) = \int d^4x \, A_\mu^I \frac{\delta \mathcal{L}_V}{\delta A_\mu^I} = \int d^4x \, A_\mu^I \partial_\nu (\star G_I)^{\mu\nu} = \int G_I F^I. \qquad \text{(D.22)}$$

Similarly, $\hat{N}$ kills the first two terms of U_u given in (8.59), leaving $\hat{N}U_u = \hat{N}K_u$ which is gauge invariant. This implies $(S, (N, U_u)_{\text{alt}})_{\text{alt}} = s(\hat{N}U_u) \approx 0$. Therefore, we have indeed

$$\hat{N}(sU_u) \approx (\int G_I F^I, U_u)_{\text{alt}} \qquad \text{(D.23)}$$

which proves equation (D.20).

We now compute $(\int G_I F^I, U_u)_{\text{alt}}$ using the result of the first step. We have

$$\left(\int G_I F^I, U_u\right)_{\text{alt}} = \frac{\delta(G_K F^K)}{\delta A_\mu^I} \delta_u A_\mu^I + \frac{\delta(G_K F^K)}{\delta \phi^i} \delta_u \phi^i. \qquad \text{(D.24)}$$

This looks like the U-variation $\delta_u(G_I F^I)$, but it is not because there are Euler-Lagrange derivatives. For a top form ω, the general rule is [And92]

$$\delta_Q \omega = Q^a \frac{\delta \omega}{\delta z^a} + d\rho, \quad \rho = \partial_{(\nu)} \left[Q^a \frac{\delta}{\delta z_{(\nu)\rho}^a} \frac{\partial \omega}{\partial dx^\rho} \right]. \qquad \text{(D.25)}$$

In our case, this becomes

$$\delta_u(G_I F^I) = \frac{\delta(G_K F^K)}{\delta A_\mu^I} \delta_u A_\mu^I + \frac{\delta(G_K F^K)}{\delta \phi^i} \delta_u \phi^i + d\rho_A + d(\text{inv}), \qquad \text{(D.26)}$$

$$\rho_A = \partial_{(\nu)} \left((f_u)^I{}_J A_\mu^J \frac{\delta}{\delta A_{\mu,(\nu)\rho}^I} \frac{\partial(G_K F^K)}{\partial dx^\rho} \right). \qquad \text{(D.27)}$$

Using property (D.1) and putting together the terms of the form $d(\text{invariant})$, we get then from (D.20)

$$(c_{IJ} + 2(h_u)_{IJ})F^I F^J + d\left[(f_u)^I{}_J A^J(\hat{N}+1)(G_I) - \rho_A \right] + d(\text{inv}) \approx 0. \qquad \text{(D.28)}$$

Now, it is sufficient to prove that

$$d\left[(f_u)^I{}_J A^J(\hat{N}+1)(G_I) - \rho_A \right] \approx d(\text{inv}). \qquad \text{(D.29)}$$

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Indeed, this implies $(c_{IJ} + 2(h_u)_{IJ})F^I F^J \approx d(\text{inv})$, which in turn gives

$$c_{IJ} = -2(h_u)_{IJ} + \lambda_u^w (h_w)_{IJ} \tag{D.30}$$

for some constants λ_u^w using property (D.3) of the W-type cohomology classes.

Proof of (D.29)

We will actually prove the stronger equation

$$\rho_A = (f_u)^I{}_J A^J (\hat{N} + 1)(G_I) \tag{D.31}$$

in the case where G_I depends on F but not on its derivatives.

To do this, we can assume that G_I a homogeneous function of degree n in A^I, i.e. $\hat{N}(G_I) = nG_I$. If it is not, we can separate it into a sum of homogenous parts; the result then still holds because equation (D.31) is linear in G_I.

In components, equation (D.31) is

$$\frac{1}{2}\partial_{(\nu)}\left((f_u)^I{}_J A_\mu^J \frac{\delta}{\delta A_{\mu,(\nu)\rho}^I} G_{K\sigma\tau} F_{\lambda\gamma}^K \varepsilon^{\sigma\tau\lambda\gamma}\right) = (n+1)(f_u)^I{}_J A_\lambda^J G_{I\sigma\tau} \varepsilon^{\rho\lambda\sigma\tau}. \tag{D.32}$$

Under the homogeneity assumption $\hat{N}(G_I) = nG_I$, we have

$$G_{K\sigma\tau} F_{\lambda\gamma}^K \varepsilon^{\sigma\tau\lambda\gamma} = 4(n+1)\mathcal{L}_V. \tag{D.33}$$

Equation (D.31) now becomes

$$\frac{1}{2}\partial_{(\nu)}\left((f_u)^I{}_J A_\mu^J \frac{\delta \mathcal{L}_V}{\delta A_{\mu,(\nu)\rho}^I}\right) = \frac{1}{4}(f_u)^I{}_J A_\lambda^J G_{I\sigma\tau} \varepsilon^{\rho\lambda\sigma\tau}. \tag{D.34}$$

We now use the fact that G_I does not depend on derivatives of F, which implies that the higher order derivatives $\partial_{(\nu)}$ are not present and that the Euler-Lagrange derivatives are only partial derivatives. We then have

$$\frac{1}{2}\frac{\delta \mathcal{L}_V}{\delta A_{\mu,\rho}^I} = \frac{\delta \mathcal{L}_V}{\delta F_{\rho\mu}^I} = \frac{1}{4}\varepsilon^{\rho\mu\sigma\tau} G_{I\sigma\tau} \tag{D.35}$$

(see (8.4)), which proves (D.34) in this case.

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