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Partial actions in algebraic geometry

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Introduction

The coordinate algebras of algebraic groups provide classical examples of Hopf algebras and the interaction between Hopf algebra theory and algebraic geometry that arises from this construction has showed to be very fruitful for both worlds. One of the most fascinating examples of this interaction are the beautiful theorems of Deligne concerning Tannaka-Krein duality and the reconstruction of an algebraic group out of its category of representations, which correspond to the categories of comodules over the associated Hopf algebra (see [**23, 24**]). With the rise of quantum groups in the 80s of the 20th century, deformations of Hopf algebras associated to algebraic groups have inspired the field of non-commutative (algebraic) geometry, where non-commutative algebras play the role of non-commutative spaces and (non-commutative, non-cocommutative) Hopf algebras coacting on these algebras play the role of symmetry groups of these spaces, see [**39**].

The aim of the present thesis is to introduce a new type of symmetries in (non-commutative) algebraic geometry, that correspond to partial group actions.

It is well-known that (usual) actions of a (discrete) group G on a k-algebra A are in correspondence with semi-direct product structures, or smash product structures, on $A \otimes kG$. In order to describe certain algebras (such as Toeplitz algebras) as a generalized smash product, the notion of a partial group action was introduced about 25 years ago in the setting of C^*-algebras by Exel [**31**]. Roughly, a partial action of a group G on an object X associates to each element of G an isomorphism between two appropriate subobjects of X. In case these subobjects always coincide with the whole object X, the action is a usual (or as we will call them from now on: global) group action. Immediate examples of these partial actions can be obtained by restricting a (global) action to an arbitrary subobject of X. Since its introduction, this notion of partial group action, has been investigated from a purely algebraic point of view where one considers partial actions of a group on an algebra. Many interesting results have been obtained, such as the globalization of partial actions [1, 3], relations between partial actions and groupoids [2], a generalized Galois theory where the Galois groups act only partially on field extensions [29]. Closely related to partial actions is the notion of partial representations, which unify in some sense classical group representations and partial actions on algebras. It has been shown that these partial representations are not only interesting to study partial actions, but earn a place in classical representation theory. The study of partial representations, rather than classical ones, allows to probe deeper into the internal structure of the group. Indeed the associated "partial group algebra" is not only determined by the group itself, but also by its lattice of subgroups, which allowed to proof a partial version of the famous isomorphism problem for group algebras, see [27], [28]. A survey of the known results about partial actions and representations of groups can be found in [25, 26]

The above mentioned Galois theory for partial group actions was given an interpretation in terms of Galois corings in [19], motivated by the fact that such an interpretation exists for classical Galois theory [44]. Moreover, the formulation of Galois theory in terms of corings [17, 18, 30, 20] covers also many other (Galois) theories, such as the Galois theory of rings [22] and Hopf-Galois theory [40]. From the geometric point of view, the interest of Hopf-Galois extensions is that they describe (non-commutative) principle bundles [43]. This inspired Caenepeel and Janssen in [21] to bring partial actions from the setting of groups to the setting of Hopf algebras and to unify partial Galois theory with Hopf-Galois theory. This approach has shown to be very successful in the sense that many classical Hopf-algebraic results appear to have a partial counterpart [5, 7], and many results from partial group actions were given a Hopf-algebraic counter part [6]. For an overview of the results of partial actions in the setting of Hopf algebras we refer to the review [14]

However, in this initial approach, several aspects of the theory remained unclear. For example, the definition of Caenepeel and Janssen only allowed to describe partial (co)actions of Hopf algebras on other (co)algebras. It was not possible to define partial actions on vector spaces nor to define partial actions of algebras other than Hopf algebras. A next step was made in [9], where it was shown that, in analogy with classical actions of Hopf algebras, partial actions can be viewed as internal algebras in an appropriate monoidal category. However, in contrast to the classical case, the monoidal category in play is no longer the usual monoidal category of representations (or modules) of the Hopf algebra H, but rather the category of partial representations which coincides with the category of representations over a newly constructed Hopf algebroid H_{par}. Lately, it was shown in [10] how partial representations can be globalized and the partial representations of Sweedler's 4-dimensional Hopf algebra were completely classified.

A recent development in the theory of partial actions, is the approach of [42], where the initial theory of parial actions over C^*-algebras is merged with the Hopf-algebra setting, in the study of partial actions of C^*-quantum groups, which leads to a theory of partial action of multiplier Hopf algebras [13].

Despite the above described success of the theory of partial actions and coactions in the sense of Caenepeel and Janssen, it turns out that if one studies partial actions of Hopf algebras that arise from algebraic groups, the partial actions are not what one would expect. Indeed, it was observed in [15] that a partial coaction of a Hopf algebra $\mathcal{O}(G)$, which is the coordinate algebra of an algebraic group G, on an algebra $\mathcal{O}(X)$, which is the coordinate algebra of an algebraic space X, is always global unless X is a disjoint union of non-empty subspaces. The spirit of partial actions would however also ask for more involved examples, where the elements of the algebraic group G act as an isomorphism between arbitrary algebraic subspaces of X. Indeed, as we mentioned before examples of partial actions can be constructed by restricting global actions. If the algebraic group G acts (globally) on an algebraic variety X, we expect that the same group acts partially on arbitrary subvarieties of X. For a more concrete example, one could consider two circles in the real plane intersecting in two points. From the global point of view, such a configuration has only few symmetries. Nevertheless, each of the individual circles has a lot of symmetries. Partial actions allow to describe at once the (few) global symmetries of the pair of circles, and the (many) symmetries of the individual circles, as well as combinations of these.

To overcome this problem, we propose an alternative definition of partial (co)actions of Hopf algebras, that we call *geometric partial (co)actions* and that also allows us to bring partial action into the realm of non-commutative geometry as the algebraic structure to describe partial symmetries.

To arrive at this goal, we will first give a detailed study of partial actions of groups on sets, and provide a new approach to these. This approach is motivated by category theory, where partial morphisms have an interpretation as spans where one of the legs is a monomorphism. Given any category $\mathcal{C}$, one can build this way a bicategory of partial morphisms, which is a full subbicategory of the category of spans over $\mathcal{C}$. A partial action of a group G on on object X is then noting else than a lax functor from G into the endo-hom category of partial morphisms from X to X.

Based on this viewpoint, we generalize the notion of partial action of a group to partial (co)actions of (co)algebras in arbitrary categories with pullbacks (respectively pushouts). More precisely, given a coalgebra H in the monoidal category $\mathcal{C}$, a *partial comodule datum* for H is a quadruple $(X, X \bullet H, \pi, \rho)$, where $\pi : X \otimes H \to X \bullet H$ is an epimorphism and $\rho : X \to X \bullet H$ is a morhpism in $\mathcal{C}$. By considering 3 levels of strictness for the coassociativity condition on a given partial comodule datum, we consider then 3 versions of partial comodules: quasi, lax and geometric partial comodules. The name for the latter version is motivated by the fact that the above mentioned examples of partial actions of algebraic groups arise exactly as those 'geometric partial comodules'. The initial partial actions of groups coincide with geometric partial actions of groups, viewed as coalgebras in the opposite of the category of sets. In case of arbitrary (Hopf) algebras, this new notion covers the one of Caenepeel and Janssen, but allows to go beyond the notion of partial actions and partial representations as discussed above. Finally, our definition allows to consider partial (co)modules over arbitrary (co)algebras, where before it was only possible to consider such structures over Hopf algebras (with bijective antipode).

Although partial comodules are only a laxified version of classical comodules, they share surprisingly many properties with classical modules. In particular, we show that a version of the fundamental theorem for comodules is still valid for geometric partial comodules and the category of partial comodules is complete and cocomplete.

As we have mentioned before, one of the key features of Hopf algebras, is that their categories of (co)modules have a natural monoidal structure, inherited by the monoidal structure of the base category wherein the considered Hopf algebra is defined. At this point the theory of (geometric) partial modules becomes different from the global theory. Indeed, although the category of quasi partial comodules over a bialgebra can be shown to posses a monoidal structure, the more interesting category of geometric partial comodules has only a *oplax* monoidal structure [**36**]. By definition an oplax monoidal structure requires the existence of n-fold tensor products, along with suitable coherence conditions. Where the tensor product of global comodules over a Hopf k-algebra is given by the tensor product of the underlying vector spaces, the vector space tensor product of two geometric partial comodules is in general no longer a geometric partial comodule. Therefore, their tensor product is defined as the biggest geometric quotient of the underlying vector space product.

Using this oplax monoidal structure, one can give meaning to an algebra in the category of geometric partial comodules. We discuss these 'geometric partial comodule algebras' and initiate a Hopf-Galois theory for them, which allows us to consider 'partially principle bundles'.

The thesis is organized as follows. In Chapter 1 we recall several basic notions and theorems in category theory that will be used throughout the thesis. We also study the monoidal structures on categories and in particular, we present some new observations about *lax* monoidal categories. In Chapter 2 we recall the classical definition of partial group actions, then we give a categorical interpretation of them by means of *spans*. We introduce the concepts of lax and quasi partial actions. In Chapter 3 we recall the basics of the theory of Hopf algebras. We refer to Caenepeel and Janssen's definition of partial actions and coactions of Hopf algebras and we point out that their definition can not describe the partial actions on spaces that can *not* be decomposed as a disjoint union of subspaces, which, as mentioned above, is one of the motivations of this thesis. To overcome this problem, in Chapter 4 we introduce the concepts of *quasi*, *lax*, and *geometric* partial comodules over coalgebras. We study the properties of them and talk about the completeness and cocompleteness of the category of partial comodules. Furthermore in Chapter 5, we focus on partial comodules over Hopf algebras, we discuss the monoidal structures on the category of geometric partial comudules over Hopf algebras. Finally, we will mention the geometrically partial Hopf-Galois theory which can be studied using our new notions.

Most of the original results in this thesis are contained the paper [**33**].

CHAPTER 1

Categories and monoidal structures

In this chapter, we will recall the basic notions from category theory with a lot of examples. Of course, it is not our aim to reproduce the content of excellent text books such as [16] and [38] to which we refer the reader for further information, but we will focus on results that will be needed in the further parts of this thesis. We will recall the following results: The existence of limits is equivalent with the existence of products and equalizers; A cocomplete category that is well-copowered with a generator is complete. Then we focus on monoidal categories and present some new observations about lax monoidal categories.

1.1. Categories and limits

Definition 1.1.1. A *category* $\mathcal{C}$ consists of the following data:

- a class $\mathsf{Ob}(\mathcal{C})$ of objects whose elements is usually denoted by X, Y, Z... ;
- a class $\mathsf{Hom}(\mathcal{C})$ of morphisms whose elements is usually denoted by f, g, h... . Each morphism f has a source object X and a target object Y. We write $f : X \to Y$ and say "f is a morphism from X to Y". $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ denotes the class of all morphisms from X to Y;
- for any three objects X, Y and Z, there exists a binary operation called the composition of morphisms:

$$\circ : \mathsf{Hom}_{\mathcal{C}}(X, Y) \times \mathsf{Hom}_{\mathcal{C}}(Y, Z) \to \mathsf{Hom}_{\mathcal{C}}(X, Z),\ (f, g) \to (g \circ f)$$

  such that the following axioms hold:
- (associativity) for any objects X, Y, Z, U and morphisms $f \in \mathsf{Hom}_{\mathcal{C}}(X, Y), g \in \mathsf{Hom}_{\mathcal{C}}(Y, Z), h \in \mathsf{Hom}_{\mathcal{C}}(Z, U)$, we have:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- (identity) for every object X, there exists a morphism $id_X : X \to X$ called the identity morphism of X, such that for every morphism $f : X \to Y$ and every morphism $g : Z \to X$, we have:

$$f \circ id_X = f,\ id_X \circ g = g$$

Remarks 1.1.2. (1) $\mathsf{Hom}_{\mathcal{C}}(X, Y)$ is usually written as $\mathsf{Hom}(X, Y)$ for short and id_X is usually denoted by 1_X or X itself, if no confusion can be made.

(2) $\mathsf{Ob}(\mathcal{C})$ and $\mathsf{Hom}(\mathcal{C})$ are not necessarily sets. A category $\mathcal{C}$ is called a *small* category if both $\mathsf{Ob}(\mathcal{C})$ and $\mathsf{Hom}(\mathcal{C})$ are sets. A *locally small* category is a category such that for any two objects X and Y, $\mathsf{Hom}(X, Y)$ is a set.

Definition 1.1.3. Let $\mathcal{C}$ be a category and $f : C \to D$ a morphism in $\mathcal{C}$.

We call f a *monomorphism* if for any morphisms $g, h : X \to C$ with $f \circ g = f \circ h$, it follows that $g = h$.

We call f an *epimorphism* if for any morphisms $g, h : D \to Y$ with $g \circ f = h \circ f$, it follows that $g = h$.

We call f an *isomorphism* if there exists a morphism $f^{-1} : D \to C$ such that $f \circ f^{-1} = id_D, f^{-1} \circ f = id_C$.

Remarks 1.1.4. (1) If f is an isomorphism, then f^{-1} is unique.

(2) An isomorphism is a monomorphism and an epimorphism. Conversely, a morphism which is simultaneously a monomorphism and an epimorphism is not necessarily an isomorphism. For example, ring morphism $\mathbb{Z} \to \mathbb{Q}$.

Examples 1.1.5. (1) Category Set. Objects are sets and morphisms are functions between them.

(2) Category Grp. Objects are groups and morphisms are group morphisms between them. Similarly, we have the category of rings Ring.

(3) Let R be a commutative ring, then ${}_R\mathcal{M}$ denotes the category whose objects are (left) R-modules and whose morphisms are morphisms between R-modules. If $R = k$ is a field, we obtain the category of vector spaces Vect_k.

(4) Let R be a non-commutative ring, we can also consider the category of left R-modules ${}_R\mathcal{M}$, the category of right R-modules $\mathcal{M}_R$ and the category of R-bimodules ${}_R\mathcal{M}_R$.

(5) Alg_k is the category of k-algebras with k-algebras morphisms between them.

Here are some ways to construct new categories.

Examples 1.1.6. (1) Trivial category. It has only one element $\star$ as object and one morphism $id_\star$ as morphism.

(2) Discrete category. Let X be a set, then we can construct a category whose objects are elements in X and morphisms are identity morphisms of each element. This category is called a *discrete category.*

(3) Dual category. Let $\mathcal{C}$ be any category, then $\mathcal{C}^{op}$ is the *dual category* of $\mathcal{C}$ obtained by taking the same objects as in $\mathcal{C}$ and morphisms $\mathsf{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathsf{Hom}_{\mathcal{C}}(Y,X)$.

(4) Let M be a monoid, i.e. a set with an associative binary operation and an identity element. Then we can construct a category whose object is one element $\star$ and morphisms $\mathsf{Hom}(\star,\star) = M$.

As in category we have morphisms between two objects, let us introduce "morphisms" between two categories, which are called functors.

Definition 1.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A (covariant) *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- for every object $X \in \mathcal{C}$, we have an object $F(X) \in \mathcal{D}$;
- for every morphism $f : X \to Y$ in $\mathcal{C}$, we have a morphism $F(f) : F(X) \to F(D)$ in $\mathcal{D}$;

  such that the following conditions hold:
- for any $f \in \mathsf{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathsf{Hom}_{\mathcal{C}}(Y,Z)$, we have

$$F(g \circ f) = F(g) \circ F(f)$$

- for every object $X \in \mathcal{C}$, we have

$$F(id_X) = id_{F(X)}$$

Remarks 1.1.8. (1) A *contravariant* functor $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F^{op} : \mathcal{C}^{op} \to D$. Most of the functors we will encounter are covariant, therefore when we say functors, we will always mean covariant functors unless we specify explicitly.

(2) $F(X)$ and $F(f)$ are sometimes written as FX and Ff for short.

There exists an obvious way of composing two functors. If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor, where $(G \circ F)(X) = G(F(X))$ for every object $X \in \mathcal{C}$ and $(G \circ F)(f) = G(F(f))$ for every morphism $f : X \to Y$ in $\mathsf{Hom}(\mathcal{C})$.

Definition 1.1.9. Consider a functor $F : \mathcal{C} \to \mathcal{D}$ between locally small categories. For any two objects $X, Y \in \mathcal{C}$, we have a map:

$$F_{X,Y} : \mathsf{Hom}_{\mathcal{C}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}}(F(X), F(Y)),\ F_{X,Y}(f) = F(f)$$

The functor F is called:

- *faithful* if $F_{X,Y}$ is injective for any two objects $X, Y \in \mathcal{C}$;
- *full* if $F_{X,Y}$ is surjective for any two objects $X, Y \in \mathcal{C}$;
- *fully faithful* if $F_{X,Y}$ is bijective for any two objects $X, Y \in \mathcal{C}$;
- an *isomorphism* of categories if F is fully faithful and F induces a bijection on the classes of objects in $\mathcal{C}$ and $\mathcal{D}$.

For contravariant functors, these notions are defined similarly.

Examples 1.1.10. (1) The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, where $1_{\mathcal{C}}(C) = C$ for every object $C \in \mathcal{C}$ and $1_{\mathcal{C}}(f) = f$ for every morphism $f \in \mathsf{Hom}(\mathcal{C})$.

(2) The constant functor $c_D : \mathcal{C} \to \mathcal{D}$, assigns to every object of $\mathcal{C}$ the same fixed object $D \in \mathcal{D}$ and assigns to every morphism of $\mathcal{C}$ the identity morphism of D.

(3) Forgetful functor. A category $\mathcal{C}$ is called a *concrete* category if there exists a faithful functor $U : \mathcal{C} \to \mathsf{Set}$. U is often called a *forgetful functor*. Thus Grp, Ring, Vect_k, and Alg_k are all concrete categories, where the forgetful functor is to take the underlying sets as objects and the same morphisms as functions between sets.

Similarly, functor such as $U : \mathsf{Alg_k} \to \mathsf{Vect_k}$ which takes the underlying vector spaces as objects, is also called a forgetful functor. In other words, forgetful functors "forget" or "drop" some of the original structures.

(4) Hom functor. Let $\mathcal{C}$ be a locally small category, for any objects X and Y in $\mathcal{C}$, we can define two *Hom functors* (or *representable functors*) $\mathsf{Hom}(X, -)$ and $\mathsf{Hom}(-, Y)$ from $\mathcal{C}$ to Set as follows.

$\mathsf{Hom}(X, -)$ maps each object $C \in \mathcal{C}$ to the set of morphisms $\mathsf{Hom}(X, C)$ and each morphism $f : A \to B$ to the function $\mathsf{Hom}(X, f) : \mathsf{Hom}(X, A) \to \mathsf{Hom}(X, B)$, where $\mathsf{Hom}(X, f)(g) = f \circ g$.

$\mathsf{Hom}(-, Y)$ maps each object $C \in \mathcal{C}$ to the set of morphisms $\mathsf{Hom}(C, Y)$ and each morphism $f : A \to B$ to the function $\mathsf{Hom}(f, Y) : \mathsf{Hom}(B, Y) \to \mathsf{Hom}(A, Y)$, where $\mathsf{Hom}(f, Y)(g) = g \circ f$.

Note that $\mathsf{Hom}(X,-)$ is covariant and $\mathsf{Hom}(-,Y)$ is contravariant.

(5) Tensor product functor. The tensor product $-\otimes- : \mathsf{Vect}_\mathsf{k} \times \mathsf{Vect}_\mathsf{k} \to \mathsf{Vect}_\mathsf{k}$, associates two vector spaces V and W with their tensor product $V \otimes W$, which is covariant in both arguments.

Definitions 1.1.11. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\alpha : F \to G$, assigns to every object $C \in \mathcal{C}$ a morphism $\alpha_C : F(C) \to G(C)$ in $\mathcal{D}$, such that for every morphism $f : C \to C'$, the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ {\scriptstyle \alpha_C}\downarrow & & \downarrow{\scriptstyle \alpha_{C'}} \\ G(C) & \xrightarrow[G(f)]{} & G(C') \end{array}$$

We also say that morphism $\alpha_C : F(C) \to G(C)$ is *natural* in C.

If α_C is an isomorphism in $\mathcal{D}$ for every $C \in \mathcal{C}$, we say that $\alpha : F \to G$ is a *natural isomorphism*.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $H, K : \mathcal{D} \to \mathcal{E}$ be functors, $\alpha : F \to G$ and $\beta : H \to K$ be natural transformations. Then there is a natural transformation

$$\beta * \alpha : H \circ F \to K \circ G$$

where $(\beta * \alpha)_X = \beta_{G(X)} \circ H(\alpha_X) = K(\alpha_X) \circ \beta_{F(X)}$, for every object $X \in \mathcal{C}$. We call this natural transformation the Godement product of α and β.

Examples 1.1.12. (1) Let $F : \mathcal{C} \to \mathcal{D}$ be any functor. Then $1_F : F \to F$, defined by $(1_F)_X = 1_{F(X)} : F(X) \to F(X)$ is the identity natural transformation on F.

(2) Let X be a k-vector space. The canonical injection $\iota : X \to X^{**}$, $\iota(x)(f) = f(x)$, for all $x \in X$ and $f \in X^*$, induces a natural transformation $\iota : 1_{\mathsf{Vect}_\mathsf{k}} \to (-)^{**}$. If X is a finite dimensional vector space, then ι is a natural isomorphism.

(3) Let $\mathcal{C}$ be a locally small category and $f : X \to Y$ be a morphism in $\mathcal{C}$. Then there is a natural transformation between the Hom functors:

$$\mathsf{Hom}(f,-) : \mathsf{Hom}(Y,-) \to \mathsf{Hom}(X,-)$$

where for every object $C \in \mathcal{C}$,

$$\mathsf{Hom}(f, C) : \mathsf{Hom}(Y, C) \to \mathsf{Hom}(X, C),\ g \to g \circ f.$$

There is also a natural transformation between the Hom functors:

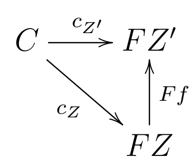
$$\mathsf{Hom}(-, f) : \mathsf{Hom}(-, X) \to \mathsf{Hom}(-, Y)$$

where for every object $C \in \mathcal{C}$,

$$\mathsf{Hom}(C, f) : \mathsf{Hom}(C, X) \to \mathsf{Hom}(C, Y),\ g \to f \circ g.$$

(4) Let A, B be rings, $M, N \in {}_B\mathcal{M}_A$ be bimodules and f be a $B - A$ bilinear map. Then $f \otimes_A - : M \otimes_A - \to N \otimes_A -$ is a natural transformation.

Definition 1.1.13. Let $F : \mathcal{Z} \to \mathcal{C}$ be a functor. A *cone* on F is a couple (C, c_Z), consisting of an object $C \in \mathcal{C}$ and a morphism $c_Z : C \to FZ$ for every object $Z \in \mathcal{Z}$, such that for any morphism $f : Z \to Z'$, the following diagram commutes:



A morphism between two cones (C, c_Z) and (D, d_Z) on F is a morphism $f : C \to D$ such that $d_Z \circ f = c_Z$ for every object $Z \in \mathcal{Z}$.

A *limit* of F is a cone (L, l_Z) such that for any other cone (C, c_Z), there exists a unique morphism $u : (C, c_Z) \to (L, l_Z)$ such that $c_Z = l_Z \circ u$, for every object $Z \in \mathcal{Z}$. If it exists, the limit of F is unique up to isomorphisms in $\mathcal{C}$, and we denote it by $\mathsf{lim}\, F = (L, l_Z)$.

Cocones and colimits are defined dually by inversing the arrow of the morphism c_Z.

Examples 1.1.14. (1) Let $\mathcal{Z}$ be a discrete category. For any functor $F : \mathcal{Z} \to \mathcal{C}$, the limit $\mathsf{lim}\, F$, if it exists, is called the *product* of all $F(Z) \in \mathcal{C}, Z \in \mathcal{Z}$, and the colimit $\mathsf{colim}\, F$, if it exists, is called the *coproduct* of all $F(Z) \in \mathcal{C}, Z \in \mathcal{Z}$.

(2) Let $\mathcal{Z}$ be a category of two objects X and Y with $\mathsf{Hom}(X, X) = id_X, \mathsf{Hom}(Y, Y) = id_Y, \mathsf{Hom}(X, Y) = \{f, g\}, \mathsf{Hom}(Y, X) = \varnothing$. For any functor $F : \mathcal{Z} \to \mathcal{C}$, the limit $\mathsf{lim}\, F$, if it exists, is called the *equalizer* of the pair $(F(f), F(g))$, and the colimit $\mathsf{colim}\, F$, if it exists, is called the *coequalizer* of the pair $(F(f), F(g))$.

(3) Let $\mathcal{Z}$ be a category with only three objects labeled by 0,1,2, apart from identity morphisms, there are only two morphisms among them, denoted by $a_1 : 1 \to 0$ and $a_2 : 2 \to 0$. Let F be a functor $F : \mathcal{Z} \to \mathcal{C}$ with $F(a_1) : A \to C$ and $F(a_2) : B \to C$. Then there is a bijective correspondence between cones on F and triples (M, m_A, m_B), where M is an object in $\mathcal{C}$, $m_A : M \to A$ and $m_B : M \to B$, such that $F(a_1) \circ m_A = F(a_2) \circ m_B$. If $\mathsf{lim}\, F$ exists, it is called the *pullback* of pairs $(F(a_1), F(a_2))$. We can define *pushout* similarly by taking the contravariant functor $G : \mathcal{Z} \to \mathcal{C}$ and $\mathsf{colim}\, G$ with $G(a_1) : C \to A$ and $G(a_2) : C \to B$.

Definition 1.1.15. A category $\mathcal{C}$ is said to be *(co)complete* when every functor $F : \mathcal{Z} \to \mathcal{C}$ has a (co)limit, where $\mathcal{Z}$ is a small category. A category is *bicomplete* if it's both complete and cocomplete.

We state a criterion without proof for (co)completeness.

Theorem 1.1.16. *A category is (co)complete if and only if it has all (small) (co)products and (co)equalizers.*

Remark 1.1.17. Since the equalizers may be constructed from pullbacks, a category is complete if and only if it has pullbacks and products. Similarly, a category is cocomplete if and only if it has pushouts and coproducts.

Examples 1.1.18. (1) The categories of sets, groups and rings are bicomplete.
(2) The categories of modules over algebras and comodules over coalgebras are bicomplete.
(3) The categories of finite sets, finite groups and finite vector spaces are neither complete nor cocomplete.

Definition 1.1.19. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $L : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{C}$ be two functors. We say that (L, R) is a pair of *adjoint* functors, or L is a left adjoint to R, or R is a right adjoint to L, if for any objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there is an isomorphism:

$$\theta_{C,D} : \mathsf{Hom}_{\mathcal{D}}(LC, D) = \mathsf{Hom}_{\mathcal{C}}(C, RD)$$

which is natural both arguments C and D.

If there is adjoint pair of functors between two categories $\mathcal{C}$ and $\mathcal{D}$, we say it is an *adjunction* and denote it by $(L, R) : C \rightleftarrows D$.

Theorem 1.1.20. *Consider functors $L : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{C}$. Then there is a bijective correspondence between natural isomorphisms $\theta_{C,D}$ which turns (L, R) into a pair of adjoint functors, and pairs of natural transformation (η, ϵ):*

$$\eta_C : C \to RLC \ ; \ \epsilon_D : LRD \to D$$

which makes the following diagrams commute (in which $*$ is the Godement product):

$$\begin{array}{ccc} RD & \xrightarrow{\eta_{RD}} & RLRD \\ & \searrow^{=} & \downarrow {\scriptstyle (1_R * \epsilon)_D} \\ & & RD \end{array}$$

$$\begin{array}{ccc} LC & \xrightarrow{(1_L * \eta)_C} & LRLC \\ & \searrow^{=} & \downarrow {\scriptstyle \epsilon_{LC}} \\ & & LC \end{array}$$

Remark 1.1.21. The natural transformation η and ϵ associated to an adjoint pair (L, R) are called respectively the *unit* and *counit* of the adjunction.

Examples 1.1.22. (1) A forgetful functor usually has a left adjoint given by a"free object functor". For example, the forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathsf{Grp}$ which assigns to every set the free group on this set. Similarly, let k be a field, the forgetful functor $U : \mathsf{Vect_k} \to \mathsf{Set}$ has a left adjoint functor which assigns to every set the free k-vector space over a set.

(2) Let $\varphi : R \to S$ be a ring morphism. This induces a functor $F_\varphi : \mathcal{M}_S \to \mathcal{M}_R$ from right S-module category to right R-module category, which gives right S-modules a right R-module structure by the formula:

$$m \cdot r = m \cdot \varphi(r), \ m \in \mathcal{M}_S, r \in R$$

This functor has a left adjoint, given by $- \otimes_R S : \mathcal{M}_R \to \mathcal{M}_S$.

(3) Let R, S be two rings and M a $R-S$-bimodule. Then the Hom functor $\mathsf{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$ has a left adjoint $- \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$. Note that the previous example is a special case of this one.

Definition 1.1.23. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. If there exists two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, such that we have natural isomorphisms $FG \cong 1_D$ and $GF \cong 1_F$, we say that F and G induce an *equivalence* of categories.

Theorem 1.1.24. *Given a functor $F : \mathcal{C} \to \mathcal{D}$, then F induces an equivalence of categories if and only if one of the conditions holds:*

(1) F is fully faithful and has a fully faithful left adjoint;
(2) F is fully faithful and has a fully faithful right adjoint;
(3) F has a left adjoint such that the unit and counit of the adjunction are natural isomorphism;
(4) F has a right adjoint such that the unit and counit of the adjunction are natural isomorphism;
(5) F is fully faithful and each object $D \in \mathcal{D}$ is isomorphic to an object of the form FC with $C \in \mathcal{C}$.

Examples 1.1.25. (1) Let R and S be two rings and M be a $R-S$-bimodule. Consider the adjoint functors $(- \otimes_R M, \mathsf{Hom}_S(M, -))$. These functors induces an equivalence of categories if and only if M is finitely generated and projective as a right R-module.

1.2. Monoidal structures

1.2.1. Monoidal categories and monoidal functors.

Definition 1.2.1. A *monoidal category* is given by the data $\mathcal{C} = (\mathcal{C}, \otimes, k, a, l, r)$ where

- $\mathcal{C}$ is a category;
- k is an object of $\mathcal{C}$;
- $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor;
- $a : \otimes \circ (\otimes \times id) \to \otimes \circ (id \times \otimes)$ is a natural isomorphism;
- $l : \otimes \circ (k \times id) \to id$ is a natural isomorphism;
- $r : \otimes \circ (id \times k) \to id$ is a natural isomorphism. Thus we have a family of isomorphisms:

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$$

$$l_M : k \otimes M \to M \ ; \ r_M : M \otimes k \to M$$

such that the following diagrams commute for all objects $M, N, P, Q \in \mathcal{C}$:

$$\begin{array}{ccccc}
((M\otimes N)\otimes P)\otimes Q & \xrightarrow{a_{M\otimes N,P,Q}} & (M\otimes N)\otimes(P\otimes Q) & \xrightarrow{a_{M,N,P\otimes Q}} & M\otimes(N\otimes(P\otimes Q)) \\
\downarrow{\scriptstyle a_{M,N,P\otimes Q}} & & & & \uparrow{\scriptstyle a_{M,N\otimes P,Q}} \\
(M\otimes(N\otimes P))\otimes Q & & \xrightarrow{\qquad a_{M,N\otimes P,Q} \qquad} & & M\otimes((N\otimes P)\otimes Q)
\end{array}$$

$$\begin{array}{ccc}
(M\otimes k)\otimes N & \xrightarrow{a_{M,k,N}} & M\otimes(k\otimes N) \\
{\scriptstyle r_M\otimes N}\searrow & & \swarrow{\scriptstyle M\otimes l_N} \\
& M\otimes N &
\end{array}$$

a is called the *associativity constraint*, l and r are called respectively the left and right *unit constraints* of $\mathcal{C}$. If a, l, r are identities, the monoidal category $\mathcal{C}$ is called *strict*.

Examples 1.2.2. (1) $((\mathsf{Set}), \times, \star)$ is a monoidal category, where $\star$ is a fixed set with only one element;
(2) Let R be a commutative ring, then $({}_R\mathcal{M}, \otimes, R)$ is a monoidal category.
(3) Let G be a monoid. Then $({}_{kG}\mathcal{M}, \otimes, k)$ is a monoidal category.

Definition 1.2.3. Let C and D be two monoidal categories. A *monoidal functor* from $\mathcal{C} \to \mathcal{D}$ is a triple (F, φ_0, φ) where

- F is a functor from $\mathcal{C} \to \mathcal{D}$;
- $\varphi_0 : k_D \to F(k_C)$ is a morphism in $\mathcal{D}$;
- $\varphi : \otimes \circ (F, F) \to F \circ \otimes$ is a natural transformation between functors from $\mathcal{C} \times \mathcal{C} \to D$, thus we have a family of morphisms:

$$\varphi_{M,N} : F(M) \otimes F(N) \to F(M \otimes N)$$

such that the following diagrams commute for all objects $M, N, P, Q \in \mathcal{C}$:

$$\begin{array}{ccc}
(F(M)\otimes F(N))\otimes F(P) & \xrightarrow{\alpha_{F(M),F(N),F(P)}} & F(M)\otimes(F(N)\otimes F(P)) \\
\downarrow{\scriptstyle \varphi_{M,N}\otimes F(P)} & & \downarrow{\scriptstyle F(M)\otimes\varphi_{P,Q}} \\
F(M\otimes N)\otimes F(P) & & F(M)\otimes F(N\otimes P) \\
\downarrow{\scriptstyle \varphi_{M\otimes N,P}} & & \downarrow{\scriptstyle \varphi_{M,N\otimes P}} \\
F((M\otimes N)\otimes P) & \xrightarrow{F(a_{M,N,P})} & F(M\otimes(N\otimes P))
\end{array}$$

$$\begin{array}{ccc} k_D \otimes F(M) & \xrightarrow{l_{F(M)}} & F(M) \\ {\scriptstyle \varphi_0 \otimes F(M)} \downarrow & & \uparrow {\scriptstyle F(l_M)} \\ F(k_C) \otimes F(M) & \xrightarrow{\varphi_{k_C, M}} & F(k_C \otimes M) \end{array}$$

$$\begin{array}{ccc} F(M) \otimes k_D & \xrightarrow{r_{F(M)}} & F(M) \\ {\scriptstyle F(M) \otimes \varphi_0} \downarrow & & \uparrow {\scriptstyle F(r_M)} \\ F(M) \otimes F(k_C) & \xrightarrow{\varphi_{M, k_C}} & F(M \otimes k_C) \end{array}$$

If φ_0 is an isomorphism and φ is a natural isomorphism, we say F is a *strong* monoidal functor. If φ_0 and $\varphi_{M,N}$ are identity morphisms for all $M, N \in C$, we say F is a *strict* monoidal functor.

Examples 1.2.4. (1) Let k be a field. Consider the functor $k- : \mathsf{Set} \to \mathsf{Vect_k}$, which assigns to a set $X \in \mathsf{Set}$ a k-vector space kX with base X. For every map $f : X \to Y$, the k-linear map kf is given by:

$$kf(\sum_{x \in X} a_x x) = \sum_{x \in X} a_x f(x)$$

$\varphi_0 : k \to k\star$ given by $\varphi_0(a) = a\star$ is an isomorphism.

$\varphi_{X,Y} : kX \otimes kY \to k(X \times Y)$ is given by $\varphi_{X,Y}(x \otimes y) = (x, y)$ for any $x \in X, y \in Y$.

Hence the functor $k-$ is a strong monoidal functor.

(2) Let G be a monoid. The forgetful functor $U : {}_{kG}\mathcal{M} \to {}_k\mathcal{M}$ is strong monoidal. Moreover, $\varphi_0 : k \to U(k) = k$ and $\varphi_{M,N} : U(M) \otimes U(N) = M \otimes N \to U(M \otimes N) = M \otimes N$ are identity maps for all $M, N \in {}_{kG}\mathcal{M}$, thus U is strictly monoidal.

(3) $\mathsf{Hom}(-, k) : \mathsf{Set}^{op} \to {}_k\mathcal{M}$ is a monoidal functor.

Definition 1.2.5. Let $\mathcal{C} = (\mathcal{C}, \otimes, k, a, l, r)$ be a monoidal category. If there exists a natural isomorphism

$$\gamma = (\gamma_{X,Y} : X \otimes Y \to Y \otimes X) : \otimes \to \otimes \circ \tau$$

for all $X, Y \in \mathcal{C}$, where τ is the flip functor, such that $l_X \circ \gamma_{X,k} = r_X$ for all $X \in \mathcal{C}$ and the following two hexagonal diagrams commute for all $X, Y, Z \in \mathcal{C}$:

$$\begin{array}{ccccccc}
 & & X \otimes (Y \otimes Z) & \xrightarrow{\gamma_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X & & \\
 & \nearrow^{a_{X,Y,Z}} & & & & \searrow^{a_{Y,Z,X}} & \\
(X \otimes Y) \otimes Z & & & & & & Y \otimes (Z \otimes X) \\
 & \searrow_{\gamma_{X,Y}\otimes 1_Z} & & & & \nearrow_{1_Y \otimes \gamma_{X,Z}} & \\
 & & (Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) & & \\
 & \nearrow^{a^{-1}_{X,Y,Z}} & & & & \searrow^{a^{-1}_{Z,X,Y}} & \\
X \otimes (Y \otimes Z) & & & & & & (Z \otimes X) \otimes Y \\
 & \searrow_{1_X \otimes \gamma_{Y,Z}} & & & & \nearrow_{\gamma_{X,Z} \otimes 1_Y} & \\
 & & X \otimes (Z \otimes Y) & \xrightarrow[a^{-1}_{X,Z,Y}]{} & (X \otimes Z) \otimes Y & &
\end{array}$$

Then we call γ a *braiding* on $\mathcal{C}$. A monoidal category with a braiding is called a *braided* monoidal category.

If in addition, the braiding γ satisfies $\gamma_{Y,X} \circ \gamma_{X,Y} = X \otimes Y$ for all $X, Y \in \mathcal{C}$, then we call it a *symmetry* and the category a *symmetric* monoidal category.

A monoidal functor $F = (F, \varphi_0, \varphi) : \mathcal{C} \to \mathcal{D}$ between two monoidal categories is called a *braided* monoidal functor if moreover, the following diagram commutes for all $X, Y \in \mathcal{C}$:

$$\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\gamma_{F(X),F(Y)}} & F(Y) \otimes F(X) \\
\downarrow{\scriptstyle \varphi_{X,Y}} & & \downarrow{\scriptstyle \varphi_{Y,X}} \\
F(X \otimes Y) & \xrightarrow{F(\gamma_{X,Y})} & F(Y \otimes X)
\end{array}$$

A braided monoidal functor between symmetric monoidal categories is called a *symmetric* monoidal functor.

Examples 1.2.6. (1) Set is a symmetric monoidal category, where the symmetry is given by the flip map $\gamma_{X,Y} : X \times Y \to Y \times X, \gamma_{X,Y}(x, y) = (y, x)$.

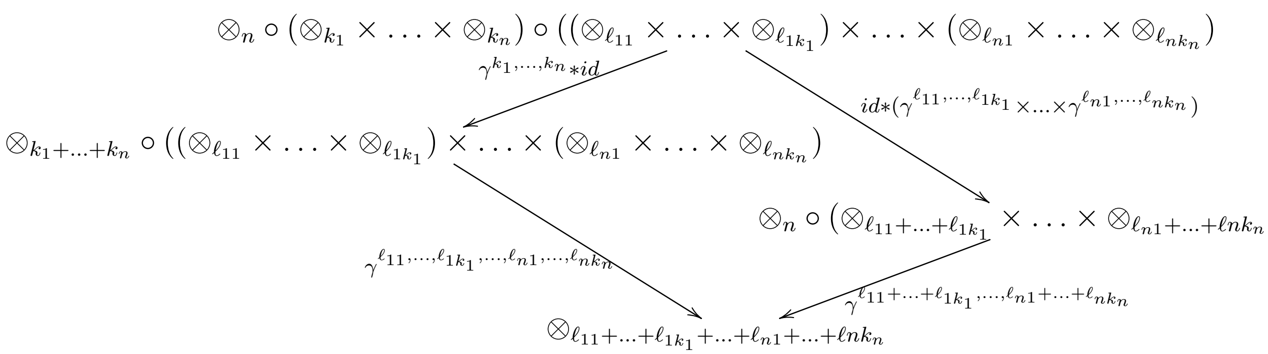
(2) $\mathcal{M}_k$ is a symmetric monoidal category, where the symmetry is given by $\gamma_{X,Y} : X \otimes Y \to Y \otimes X, \gamma_{X,Y}(x \otimes y) = y \otimes x$ (then linearly extend it).

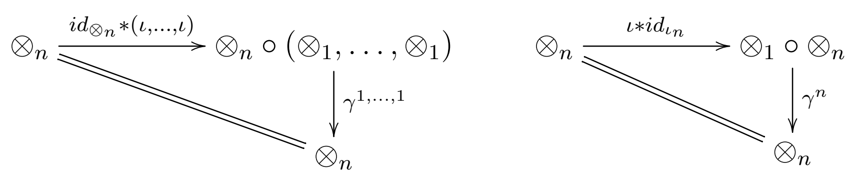
(3) In general, there is no braiding on the category of A-bimodules ${}_A\mathcal{M}_A$.

1.2.2. Lax monoidal categories. A category $\mathcal{C}$ is called *lax monoidal* [36] if

- for each $n \in \mathbb{N}$ there exists an n-fold tensor functor
$$\otimes_n : \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n} \to \mathcal{C};$$
- for each for each $(k_1, \ldots, k_n) \in \mathbb{N}^n$, there exists a natural transformation
$$\gamma^{k_1,\ldots,k_n} : \otimes_n \circ (\otimes_{k_1} \times \ldots \times \otimes_{k_n}) \to \otimes_{k_1+\ldots+k_n}$$
- there exists natural transformation
$$\iota : id_{\mathcal{C}} \to \otimes_1,$$

that satisfy the following associativity and unitality conditions.





Remark that the last two conditions imply in particular that the functor $\otimes_1$ is idempotent.

There is an obvious notion of *oplax monoidal categories*, where the direction of the natural transformations γ and ι is reversed. If the natural transformations γ and ι are invertible, then a lax monoidal category is just a monoidal category.

A *(lax) monoidal functor* between lax monoidal categories is a functor $F : \mathcal{C} \to \mathcal{D}$ that comes equipped with natural transformations

$$\zeta_n : \quad \otimes_n^{\mathcal{D}} F^n \to F \otimes_n^{\mathcal{C}} : \quad \mathcal{C}^n \to \mathcal{D}$$

for each $n \in \mathbb{N}$, satisfying the following compatibility conditions with γ and ι

$$\begin{array}{ccc}
\otimes_n^{\mathcal{D}}(\otimes_{k_1}^{\mathcal{D}} \times \ldots \times \otimes_{k_n}^{\mathcal{D}})F^{k_1+\ldots+k_n} & \xrightarrow{\gamma_{\mathcal{D}}^{k_1,\ldots,k_n} F^{k_1+\ldots+k_n}} & \otimes^{\mathcal{D}}_{k_1+\ldots+k_n} F^{k_1+\ldots+k_n} \\
\Big\downarrow {\scriptstyle \otimes_n(\zeta_{k_1}\times\ldots\times\zeta_{k_n})} & & \\
\otimes_n^{\mathcal{D}} F^n(\otimes_{k_1}^{\mathcal{C}} \times \ldots \times \otimes_{k_n}^{\mathcal{C}}) & & \Big\downarrow {\scriptstyle \zeta_{k_1+\ldots+k_n}} \\
\Big\downarrow {\scriptstyle \zeta_n(\otimes_{k_1}\times\ldots\times\otimes_{k_n})} & & \\
F \otimes_n^{\mathcal{C}} (\otimes_{k_1}^{\mathcal{C}} \times \ldots \times \otimes_{k_n}^{\mathcal{C}}) & \xrightarrow{F\gamma_{\mathcal{C}}^{k_1,\ldots,k_n}} & F\otimes^{\mathcal{C}}_{k_1+\ldots+k_n}
\end{array}$$

$$\begin{array}{ccc}
F & \xrightarrow{F\iota_{\mathcal{C}}} & F\otimes_1^{\mathcal{C}} \\
{\scriptstyle \iota_{\mathcal{D}}F} \searrow & & \nearrow {\scriptstyle \zeta_1} \\
& \otimes_1^{\mathcal{D}} F &
\end{array}$$

Similarly, a functor $G : \mathcal{C} \to \mathcal{D}$ is called *oplax monoidal* if there are natural transformations

$$\delta_n : \quad F\otimes_n^{\mathcal{C}} \to \otimes_n^{\mathcal{D}} F^n \quad \mathcal{C}^n \to \mathcal{D}.$$

satisfying appropriate compatibility conditions with γ and ι.

The following result might be well-known, but as we didn't found a reference we state it and give a sketch of the proof, which is quite elementary, but because of notational problems becomes quite technical. This result allows to construct many lax monoidal categories.

Theorem 1.2.7. *(i) Let $\mathcal{D}$ be a lax monoidal category and consider a pair (L, R) of adjoint functors*

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

then $\mathcal{C}$ is also a lax monoidal category such that R is a monoidal functor and L is an opmonoidal functor.

(ii) Let $\mathcal{D}$ be an oplax monoidal category and consider a pair (L, R) of adjoint functors

$$\mathcal{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{E}$$

then $\mathcal{E}$ is also an oplax monoidal category such that R is a monoidal functor and L is an opmonoidal functor.

Proof. We only give a sketch of the proof of part (i), the second follows by duality.

Denote the n-fold tensor products in $\mathcal{D}$ by $\otimes_n^{\mathcal{D}}$ and its associativity and unity constraints by $\gamma_{\mathcal{D}}$ and $\iota_{\mathcal{D}}$.

For any n-tuple $(c_1, \ldots, c_n)$ of objects in $\mathcal{C}$, define the n-fold tensor product in $\mathcal{C}$ as

$$\otimes_n^{\mathcal{C}}(c_1, \ldots, c_n) = R(\otimes_n^{\mathcal{D}}(Lc_1, \ldots, Lc_n)).$$

More precisely, the n-fold tensor product in $\mathcal{C}$ is defined as the following composition of functors

$$\otimes_n^{\mathcal{C}} = R \circ \otimes_n^{\mathcal{D}} \circ L^n : \mathcal{C}^n \to \mathcal{C}.$$

Let us denote by $\eta : id_{\mathcal{C}} \to RL$ and $\epsilon : LR \to id_{\mathcal{D}}$ the unit and counit of the adjunction (L, R). For any n-tuple $(c_1, \ldots, c_n)$ in $\mathcal{C}$, we can consider the morphism

$$\begin{aligned} \delta_n^{c_1,\ldots,c_n} = \epsilon_{\otimes_n(Lc_1,\ldots,Lc_n)} &: \\ L \otimes_n^{\mathcal{C}} (c_1, \ldots, c_n) &= LR \otimes_n^{\mathcal{D}} (Lc_1, \ldots, Lc_n) \to \otimes_n^{\mathcal{D}}(Lc_1, \ldots, Lc_n) \end{aligned} \tag{1.1}$$

which is natural in each of the entries c_i, defining in this way for each $n \in \mathbb{N}$ a natural transformation

$$\delta_n = \epsilon \otimes_n^{\mathcal{D}} L^n : L\otimes_n^{\mathcal{C}} = LR \otimes_n^{\mathcal{D}} L \to \otimes_n^{\mathcal{D}} L^n : \mathcal{C}^n \to \mathcal{D}.$$

Similarly, for each n-tuple $(d_1, \ldots, d_n)$ in $\mathcal{D}$ we put

$$\begin{aligned} \zeta_n^{d_1,\ldots,d_n} = R \otimes_n^{\mathcal{D}} (\epsilon_{d_1}, \ldots, \epsilon_{d_n}) &: \\ R \otimes_n^{\mathcal{D}} (LRd_1, \ldots LRd_n) &= \otimes_n^{\mathcal{C}}(Rd_1, \ldots Rd_n) \to R \otimes^{\mathcal{D}} (d_1, \ldots, d_n) \end{aligned} \tag{1.2}$$

which defines a natural transformation

$$\zeta_n = R \otimes_n^{\mathcal{D}} \epsilon^n : \otimes_n^{\mathcal{C}} R^n = R \otimes_n^{\mathcal{D}} (LR)^n \to R\otimes_n^{\mathcal{D}} : \mathcal{D}^n \to \mathcal{C}.$$

To define the associativity constraint of $\mathcal{C}$, first remark that

$$\begin{aligned} \otimes_n \circ (\otimes_{k_1} \times \ldots \times \otimes_{k_n}) &= (R \otimes_n^{\mathcal{D}} L^n) \circ ((R \otimes_{k_1}^{\mathcal{D}} L^{k_1}) \times \ldots \times (R \otimes_{k_n}^{\mathcal{D}} L^{k_n})) \\ &= R \otimes_n^{\mathcal{D}} (LR)^n (\otimes_{k_1}^{\mathcal{D}} \times \ldots \times \otimes_{k_n}^{\mathcal{D}}) L^{k_1+\ldots+k_n} \end{aligned}$$

We now define the associativity constraint $\gamma_{\mathcal{C}}$ of $\mathcal{C}$ as the following composition

$$\begin{array}{ccc} R \otimes_n^{\mathcal{D}} (LR)^n (\otimes_{k_1}^{\mathcal{D}} \times \ldots \times \otimes_{k_n}^{\mathcal{D}}) L^{k_1+\ldots+k_n} & \xrightarrow{\gamma_{\mathcal{C}}^{k_1,\ldots,k_n}} & R(\otimes_{k_1+\ldots+k_n}^{\mathcal{D}})) L^{k_1+\ldots+k_n} = \otimes_{k_1+\ldots+k_n}^{\mathcal{C}} \\ {\scriptstyle (\zeta_n (\otimes_{k_1}^{\mathcal{D}} \times \ldots \times \otimes_{k_n}^{\mathcal{D}}) L^{k_1+\ldots+k_n})} \searrow & & \nearrow {\scriptstyle (R\gamma_{\mathcal{D}}^{k_1,\ldots,k_n} L^{k_1+\ldots+k_n})} \\ & R \otimes_n^{\mathcal{D}} (\otimes_{k_1}^{\mathcal{D}} \times \ldots \times \otimes_{k_n}^{\mathcal{D}}) L^{k_1+\ldots+k_n} & \end{array}$$

The unitality constraint of $\mathcal{C}$ is defined as the composition

$$\iota_{\mathcal{C}} = (R\iota_{\mathcal{D}}L) \circ \eta : id_{\mathcal{C}} \to R \otimes_1^{\mathcal{D}} L = \otimes_1^{\mathcal{C}}.$$

The associativity conditions for the lax monoidal structure on $\mathcal{C}$ then follow directly from the naturality of ϵ and the associativity in $\mathcal{D}$. The unitality conditions in $\mathcal{C}$ follow from the unit-counit condition of the adjunction (L, R) and the unitality conditions in $\mathcal{D}$.

The monoidal structure on the functor R is given by (1.2), the op-monoidal structure on L is given by (1.1). □

The previous proposition can be applied in particular to a monoidal category category $\mathcal{D}$ and allows to produce in this way many natural examples of (op)lax monoidal categories.

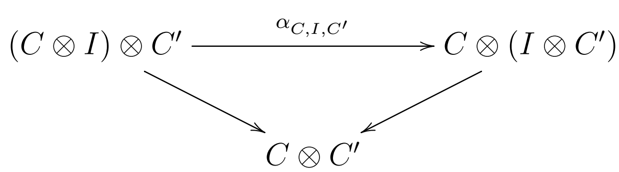
As an intermediate notion between lax monoidal categories and monoidal categories, one can consider a *monoidal category with lax unit*. This is a category $\mathcal{C}$ endowed with a monoidal tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, endowed with an associativity constraint

$$\alpha_{C,C',C''} : (C \otimes C') \otimes C'' \to C \otimes (C' \otimes C'')$$

which is a natural isomorphism that satisfies the usual pentagon condition. A *lax unit* I for such an associative tensor product is an object I in $\mathcal{C}$ such that for any $C \in \mathcal{C}$ there are natural transformations

$$\ell_C : I \otimes C \to C, \quad r : C \otimes I \to C$$

satisfying the usual compatibility constraints with α:



The following is now an easy observation.

Lemma 1.2.8. *If $(\mathcal{C}, \otimes, I)$ is a monoidal category with lax unit, then $\mathcal{C}$ is a lax monoidal category by defining*

- $\otimes_0 = I$, $\otimes_1 = id_{\mathcal{C}}$, $\otimes_2 = \otimes$;
- *for all* $n > 2$, $\otimes_n = \otimes \circ (id \times \otimes_{n-1})$;
- $\iota = id : id_{\mathcal{C}} \to \otimes_1$;
- *for all* $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$, $\gamma^{k_1,\ldots,k_n}$ *is canonically obtained from combinations of* α *and identities and are therefore invertible;*
- *for any* $(k_1, \ldots, k_n) \in \mathbb{N}^n$, *where* $k_{i_1} = \ldots = k_{i_m} = 0$ $(m < n)$, $\gamma^{k_1,\ldots,k_n}$ *is canonically obtained from combinations of* ℓ, r, α *and identities and are not invertible;*

Similarly, one introduces the notion of a monoidal category with an oplax unit, which gives rise to an oplax monoidal category.

CHAPTER 2

Partial actions of groups

In this chapter, we will first recall the classic definition of partial actions of groups. Then we will give a new result that characterizes the partial actions of groups in terms of category of spans.

2.1. The classical definition of partial group actions

Let G be a group and X a set. A *partial action datum* of G on X is a couple $(X_g, \alpha_g)_{g\in G}$, where

- $\{X_g\}_{g\in G}$, a family of subsets of X indexed by the group G;
- $\{\alpha_g : X_{g^{-1}} \to X\}_{g\in G}$ a family of maps indexed by the group G;

Recall from [**31**] that a *partial action* α of G on X is a partial action datum $(X_g, \alpha_g)_{g\in G}$ that satisfies the following axioms

(PA1) $X_e = X$ and $\alpha_e = id_X$, where e denotes the unit of G;
(PA2) $\alpha_g(X_{g^{-1}} \cap X_h) \subset X_g \cap X_{gh}$;
(PA3) $\alpha_h \circ \alpha_g = \alpha_{hg}$ on $X_{g^{-1}} \cap X_{(hg)^{-1}}$.

Remark that thanks to the second axiom (PA2), the third axiom (PA3) makes sense, since

$$\alpha_h \circ \alpha_g(X_{g^{-1}} \cap X_{(hg)^{-1}}) \subset \alpha_h(X_g \cap X_{h^{-1}}) \subset X_{hg} \cap X_h$$

and

$$\alpha_{hg}(X_{g^{-1}} \cap X_{(hg)^{-1}}) \subset X_h \cap X_{hg}.$$

Furthermore, combining (PA2) and (PA3), we find that

$$X_g \cap X_{gh} = \alpha_g \circ \alpha_{g^{-1}}(X_g \cap X_{gh}) \subset \alpha_g(X_{g^{-1}} \cap X_h)$$

and therefore, we can deduce the stronger axiom

(PA2') $\alpha_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}$.

If we take in particular $h = e$, then we find that $\alpha_g(X_{g^{-1}}) = X_g$. Moreover, since $\alpha_g \circ \alpha_{g^{-1}}(x) = x$ for all $x \in X_g$, we find that each map α_g induces a bijection $\alpha_g : X_{g^{-1}} \to X_g$. This last fact is often supposed as part of the definition of a partial action.

Many examples of partial actions have been observed in recent literature. It makes no sense to repeat them here, however, we will gave a few exemplary ones, which will make the transition to some of the new results in this paper more easy.

Examples 2.1.1.
(1) Consider a (global) action of the group G on a set Y, and let $X \subset Y$ be any (non-empty) subset of Y. Then G acts partially on X, by defining $X_g := \{x \in X \mid g^{-1} \cdot x \in X\}$ and defining $\alpha_g : X_{g^{-1}} \to X_g,\ \alpha_g(x) = g \cdot x$.
(2) As a particular case of the previous one, we consider the following geometric example. Let $G = (\mathbb{A}^2, +)$ be the group of 2-dimensional affine translations. This group acts strictly transitive on the affine plane $\mathbb{A}^2$. Consequently, this group acts partially on any subset of $\mathbb{A}^2$. In one of the next sections, we will discuss in more detail the case of the partial action of this group on two intersecting lines.
(3) Consider the additive group $\mathbb{Z}$. For any $z \in \mathbb{Z}$ with $z \geq 0$ we define its domain $X_{-z} = \mathbb{N}$ and its action $\alpha_z : \mathbb{Z} \to \mathbb{Z}, x \mapsto x + z$. On the other hand for each $z < 0$ we define its domain $X_{-z} = \{x \in \mathbb{Z} \mid x \geq -z\}$ and its action $\alpha_z : \mathbb{Z} \to \mathbb{Z}, x \mapsto x + z$. Then one easily verifies this defines a partial action which is obtained by restricting the action of $\mathbb{Z}$ on itself to $\mathbb{N}$.

2.2. A categorical interpretation of partial actions

2.2.1. Lax and quasi partial actions. As we explained, the axiom (PA2) in the definition of partial actions is designed to make sense of axiom (PA3) which expresses the associativity. However, this axiom can be weakened further.

Definition 2.2.1. Let G be a group, X a set and $\alpha = (X_g, \alpha_g)$ be a partial action datum. We say that α is a *lax* partial action of the following axioms hold

(LPA1) $X_e = X$ and $\alpha_e = id_X$, where e denotes the unit of G;
(LPA2) $X_{g^{-1}} \cap \alpha_g^{-1}(X_{h^{-1}}) \subset X_{(hg)^{-1}}$.
(LPA3) $\alpha_h \circ \alpha_g = \alpha_{hg}$ on $X_{g^{-1}} \cap \alpha_g^{-1}(X_{h^{-1}})$.

Axiom (LPA2) tells that if $x \in X_{g^{-1}}$ and $\alpha_g(x) \in X_{h^{-1}}$, then $x \in X_{(hg)^{-1}}$ and therefore axiom (LPA3) makes sense. As one can easily verify, any partial action is a lax partial action and the converse holds if and only if $\alpha_g(X_{g^{-1}}) \subset X_g$. The following example shows that lax partial actions are a proper generalization of partial actions.

Example 2.2.2. This example is a variation of Example 2.1.1 (3). Consider the additive group $\mathbb{Z}$. For any $z \in \mathbb{Z}$ with $z \geq 0$ we define its domain $X_{-z} = \mathbb{Z}$ and its action $\alpha_z : \mathbb{Z} \to \mathbb{Z}, x \mapsto x + z$. On the other hand for each $z < 0$ we define its domain $X_{-z} = \{x \in \mathbb{Z} \mid x \geq -z\}$ and its action $\alpha_z : X_{-z} \to \mathbb{Z}, x \mapsto x + z$. Then one can verify that this is indeed a lax partial action and moreover it is not a partial action, since $\rho_z : X_{-z} \to X_z = \mathbb{Z}$ is not a bijection for any $z < 0$.

For sake of completeness, we also state another weakening of the definition of partial action, which is, by our opinion, naturally the most general version of a partial action.

Definition 2.2.3. Let G be a group, X a set and $\alpha = (X_g, \alpha_g)$ be a partial action datum. We say that α is a *quasi* partial action of the following axioms hold

(QPA1) $X_e = X$ and $\alpha_e = id_X$, where e denotes the unit of G;
(QPA2) $\alpha_h \circ \alpha_g = \alpha_{hg}$ on $X_{g^{-1}} \cap \alpha_g^{-1}(X_{h^{-1}}) \cap X_{(gh)^{-1}}$.

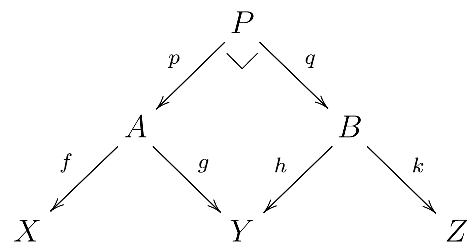
Remark that in this definition, we ask associativity to hold exactly there where both $\alpha_h \circ \alpha_g$ and α_{hg} make sense. The following construction shows that quasi partial actions properly generalize lax and usual partial actions.

Example 2.2.4. Let G be a group acting (globally) on a set X. For any $g \in G$ consider an arbitrary subset $X_g \subset X$ and let $\alpha_g : X_{g^{-1}} \to X$ be the restriction of the action of g on X. Then this defines a quasi partial action of G on X.

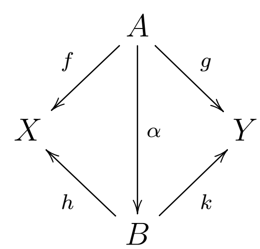
2.2.2. Partial actions and spans.

We will now reformulate the definition of a partial action, making no explicit reference to the elements of the set or the group, but stating everything internally in the category $\underline{\underline{\mathsf{Set}}}$ of sets. This way, the definition can be easily lifted to any (monoidal) category (with pullbacks). As we will show, quasi and lax partial actions arise naturally in this context.

Recall that in any category $\mathcal{C}$, a *span* from X to Y is a triple (A, f, g), where A is an object of $\mathcal{C}$ and $f : A \to X$ and $g : A \to Y$ are two morphisms of $\mathcal{C}$. If $\mathcal{C}$ has pullbacks and $(A, f, g), (B, h, k)$ are spans from X to Y and from Y to Z respectively, then one constructs a new span, called the *composition span*, from X to Z by the following pullback construction:



which we will denote as $(B, h, k) \bullet (A, f, g)$. Given two spans $(A, f, g), (B, h, k) : X \to Y$, a *morphism of spans* $\alpha : (A, f, g) \to (B, h, k)$ is a map $\alpha : A \to B$ such that the following diagram commutes



In this way, we obtain a bicategory $\mathsf{Span}\,(\mathcal{C})$, whose 0-cells are the objects of $\mathcal{C}$, 1-cells are spans and 2-cells are morphisms of spans. We can also consider the (usual) category $\mathsf{span}(C)$, whose objects are the objects of $\mathcal{C}$ and whose morphisms are isomorphism classes of spans.

In what follows, we will use the following variation on the usual category of spans.

Definition 2.2.5. By a *partial morphism* from X to Y in a category $\mathcal{C}$, we mean a morphism (A, f, g) in the category $\mathsf{Span}\,(\mathcal{C})$, with the additional property that $f : A \to X$ is a monomorphism. By $\mathsf{Par}(\mathcal{C})$ we denote the subbicategory of $\mathsf{Span}\,(\mathcal{C})$, with the 0-cells as $\mathsf{Span}\,(\mathcal{C})$ (and $\mathcal{C}$), whose 1-cells are given by partial morphisms in $\mathcal{C}$. By $\mathsf{par}(\mathcal{C})$ we denote the corresponding subcategory of $\mathsf{span}(\mathcal{C})$.

Remark that the above definition of $\mathsf{Par}(\mathcal{C})$ makes sense since the pullback of a monomorphism is a monomorphism. Moreover, if $\alpha, \beta : (A, f, g) \to (B, h, k)$ are 2-cells in $\mathsf{Par}(\mathcal{C})$ then $\alpha = \beta$ since $h \circ \alpha = f = h \circ \beta$ and h is a monomorphism. Hence $\mathsf{Par}(\mathcal{C})$ is locally a poset. In the particular case of $\mathsf{Par}(\underline{\underline{\mathsf{Set}}})$, there is a morphism of spans $\alpha : (A, f, g) \to (B, h, k)$ if and only if A is a subset of B and g is the restriction of k to A.

We will denote a partial morphism from X to Y by a dotted arrow $X \dashrightarrow Y$. When we consider a partial map as a triple (A, f, g), we will often omit to write explicitly the first map f, as it is an inclusion and supposed to be known if we know the object A, ie. we will write $(A, f, g) = (A, g) = g$.

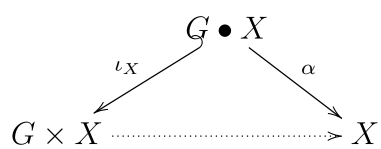
Lemma 2.2.6. *Let G be a group and X a set. Then there is a bijective correspondence between*

(1) partial action data of G on X;
(2) partial morphisms $G \times X \to X$;
(3) Maps $G \to \mathsf{Par}(X, X)$.

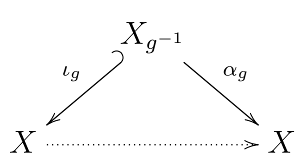
Proof. <u>(1) ⇔ (2)</u>. Let $(X_g, \alpha_g)_{g \in G}$ be a partial action datum of G on a set X. Then we can construct the set

$$G \bullet X = \{(g, x) \mid x \in X_{g^{-1}}\} \subset G \times X, \tag{2.1}$$

which is the set of all "compatible pairs" in $G \times X$. Clearly, the partial action then induces a well-defined map $\alpha : G \bullet X \to X, \alpha(g, x) = \alpha_g(x)$. Hence we obtain a partial morphism $G \times X \dashrightarrow X$,



Conversely, consider any partial morphism $\alpha = (G \bullet X, \iota, \alpha) : G \times X \to X$, where $G \bullet X$ is a subset of $G \times X$, $\iota : G \bullet X \to G \times X$ is the canonical inclusion and $\alpha : G \bullet X \to X$ is a map. Then for any $g \in G$, we can define $X_{g^{-1}} = \{x \in X \mid (g, x) \in G \bullet X\}$ and we recover formula (2.1).
<u>(1) ⇔ (3)</u>. Let $(X_g, \alpha_g)_{g \in G}$ be a partial action datum, then for any $g \in G$ we have that



where $\iota_g : X_g \to X$ is the canonical inclusion, is a partial endomorphism of X which defines a map $G \to \mathsf{Par}(X, X)$. Conversely, any map $G \to \mathsf{Par}(X, X)$ gives in the same way a family $(X_g, \alpha_g)_{g \in G}$, i.e. a partial action datum. □

The natural question that now arises is what are the conditions on a partial morphism $\alpha : G \times X \to X$ for the associated partial action datum to become an actual partial action. A first naive guess would be to impose the usual associativity and unitality conditions of an action expressed in the category $\mathsf{par}(\mathcal{C})$, or equivalently to impose that the map $G \to \mathsf{par}(X, X)$ is a morphism of monoids (where the the later is the endomorphism monoid of X in the (1-)category $\mathsf{par}(\underline{\underline{\mathsf{Set}}})$). However, as we will point out now, this leads to a global action.

Lemma 2.2.7. *Let G be a group with multiplication $m : G \times G \to G$, $m(g,h) = gh$ and the unit $e : \{*\} \to G, e(*) = e$. Consider a partial action datum $(X_g, \alpha_g)_{g \in G}$.*

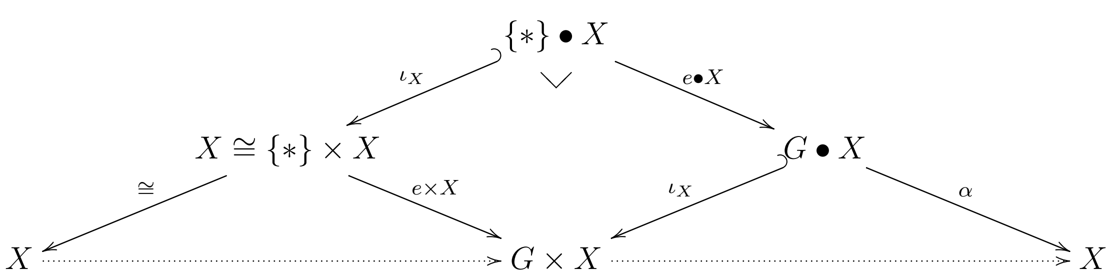
(1) the following assertions are equivalent
- *(i) The partial action datum satisfies axiom (PA1);*
- *(ii) The associated partial morphism $\alpha : G \times X \to X$ satisfies $\alpha \bullet (e \times X) \simeq X$ in $\mathsf{Par}(\underline{\underline{\mathsf{Set}}})$.*
- *(iii) The associated map $\alpha' : G \to \mathsf{Par}(X, X)$ preserves the unit.*

(2) The following assertions are equivalent
- *(i) The partial action datum defines a global action of G on X;*
- *(ii) The associated partial morphism $\alpha : G \times X \to X$ satisfies the following identities in $\mathsf{par}(\underline{\underline{\mathsf{Set}}})$ (i.e. isomorphism of spans)*

$$\begin{aligned} \alpha \bullet (e \times X) &\simeq X \\ \alpha \bullet (G \times \alpha) &\simeq \alpha \bullet (m \times X) \end{aligned}$$

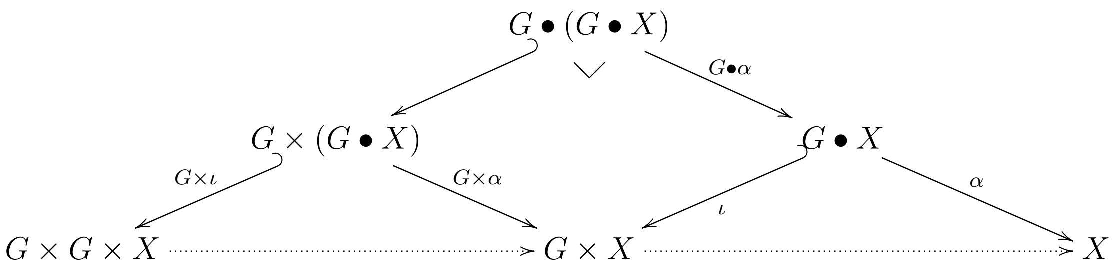
- *(iii) The associated map $\alpha' : G \to \mathsf{Par}(X, X)$ is a morphism of monoids.*

Proof. $\underline{(1)}$. Let us compute the composition of spans $\alpha \bullet (e \times X)$. This leads to the following pullback



where $\{*\} \bullet X = \{x \in X \mid x \in X_e\} \cong X_e$. Hence $\alpha \circ (e \times X)$ is the identity morphism on X in the category $\mathsf{Par}(\underline{\underline{\mathsf{Set}}})$, if and only if $X_e = X$ and $\alpha_e = id_X$, which is exactly axiom (PA1). Furthermore, it is clear

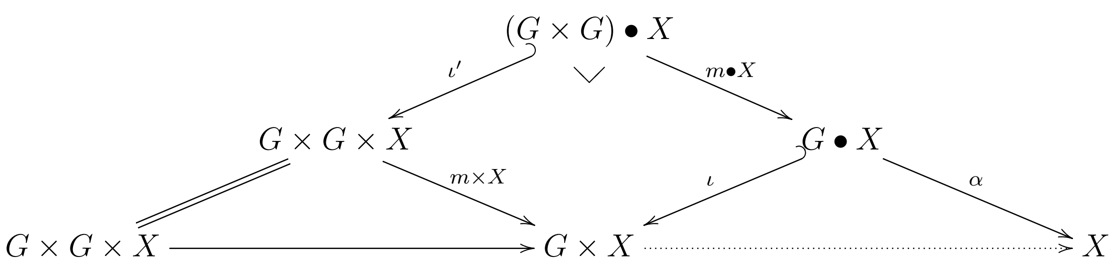
that this is equivalent to saying that $\alpha'(e) = (X_e, \iota_e, \alpha_e)$ is the span (X, id_X, id_X).
(2). By part (1), we only have to prove the equivalence of the associativity constraints. Let us compute the composition of spans $\alpha \bullet (G \times \alpha)$ in $\mathsf{Par}(\underline{\underline{\mathsf{Set}}})$, which is given by the following pullback.



Explicitly we find

$$G \bullet (G \bullet X) = \{(h, g, x) \in G \times G \times X \mid x \in X_{g^{-1}}, \alpha(g, x) \in X_{h^{-1}}\}.$$

Similarly, we can compute the composition $\alpha \bullet (m \times X)$ in $\mathsf{Par}(\underline{\underline{\mathsf{Set}}})$, which is again given by a pullback



where now

$$(G \times G) \bullet X = \{(h, g, x) \in G \times G \times X \mid x \in X_{(hg)^{-1}}\}.$$

We then find that $(g, g^{-1}, x) \in G \bullet (G \bullet X)$ if and only if $x \in X_g$ (and $\alpha(g^{-1}, x) \in X_{g^{-1}}$). On the other hand, $(g, g^{-1}, x) \in (G \times G) \bullet X$ if and only if $x \in X_e = X$. Hence, we obtain that the action is global if and only if $G \bullet (G \bullet X)$ and $(G \times G) \bullet X$ are isomorphic spans.

In the same way, if the action is global, then clearly α' is a morphism of monoids. Conversely, if α' is a morphism of monoids then we obtain in particular that $\alpha'(g^{-1}) \bullet \alpha'(g) = \alpha'(e) = (X, id_X, id_X)$. Since the underlying set of the span of $\alpha'(g^{-1}) \bullet \alpha'(g)$ is given by $\{x \in X_{g^{-1}} \mid \alpha(g, x) \in X_g\}$, we find that $\alpha'(g^{-1}) \bullet \alpha'(g) = \alpha'(e)$ implies that $X_e = X_g$ for all $g \in G$ and hence we have a global action. $\square$

As we have just observed, partial actions are not just actions in the category of partial morphisms. The monoid morphism $G \to \mathsf{par}(X, X)$ can also be viewed as a functor between 2 one-object categories. However, since the $\mathsf{Par}(\underline{\underline{\mathsf{Set}}})$ is a bicategory, the $\mathsf{Par}(X, X)$ becomes a (monoidal) category, or a one-object bicategory. Consequently, there is a natural laxified version of a partial action considering only a *lax* functor between G and $\mathsf{Par}(X, X)$. In the next proposition, we show that this coincides exactly with with the lax partial actions we introduced above.

Recall that a lax functor $F : \mathcal{B} \to \mathcal{B}'$ between 2 bicategories consists of

- a map from the 0-cells of $\mathcal{B}$ to the 0-cells of $\mathcal{B}'$,
- for any pair of 0-cells X, Y of $\mathcal{B}$, a functor $F_{X,Y} : \mathcal{B}(X, Y) \to \mathcal{B}'(X', Y')$
- for any 0-cell X in $\mathcal{B}$ a 2-cell $u_X : id_{FX} \to F(id_X)$'
- for any two 1-cells $a \in \mathcal{B}(X, Y)$ and $b \in \mathcal{B}(Y, Z)$ a 2-cell $\alpha_{a,b} : F(b) \bullet F(a) \to F(b \bullet a)$ (where $\bullet$ denotes the horizontal composition), which in natural in a and b;

satisfying the usual coherence axioms. If the category $\mathcal{B}'$ is locally a poset, then these coherence conditions follow automatically from the above information.

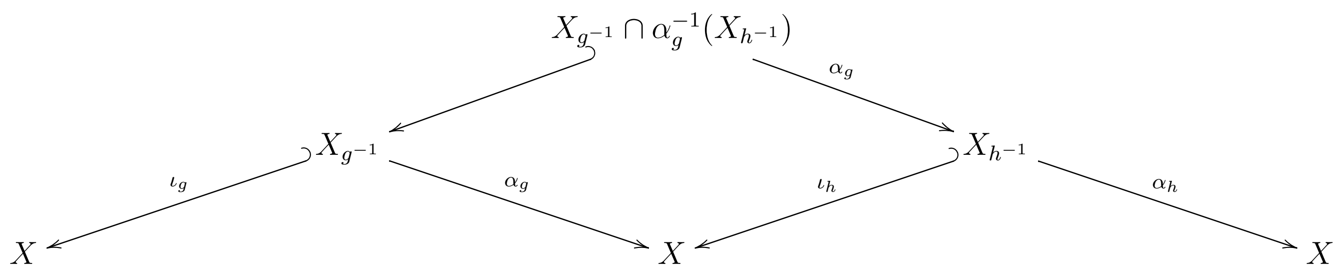
Proposition 2.2.8. *Let G be a group with multiplication $m : G \times G \to G$, $m(g, h) = gh$ and the unit $e : \{*\} \to G, e(*) = e$. Consider a partial action datum $(X_g, \alpha_g)_{g \in G}$. The following assertions are equivalent*

(i) The partial action datum defines a lax partial action of G on X;
(ii) For the associated partial morphism $\alpha : G \times X \to X$, there exist morphisms of spans $u : X \to \alpha \bullet (e \times X)$ and $\theta : \alpha \bullet (G \times \alpha) \to \alpha \bullet (m \times X)$;
(iii) The associated map $\alpha' : G \to \mathsf{Par}(X, X)$ is a lax functor where G is considered as a locally discrete 2-category with one 0-cell.

Proof. $\underline{(i) \Leftrightarrow (ii)}$. As we have shown in Lemma 2.2.7, $\alpha \bullet (e \times X)$ is given by the span $(X_e, \iota_e, \alpha_e) : X \to X$. The existence of a morphism of spans $u : X \to \alpha \bullet (e \times X)$, means that $X \subset X_e \subset X$, hence $X = X_e$, and $\alpha_e = id_X$.

Furthermore, we also know from Lemma 2.2.7 the explicit form of $\alpha \bullet (G \times \alpha)$ and $\alpha \bullet (m \times X)$. The existence of the morphism θ then means that $G \bullet (G \bullet X) \subset (G \times G) \bullet X$, which is exactly axiom (LPA2) and the restriction of the partial action α_{hg} to $X_{g^{-1}} \cap \alpha_g^{-1}(X_{h^{-1}})$ coincides with $\alpha_h \circ \alpha_g$, which is exactly axiom (LPA3).

$\underline{(i) \Rightarrow (iii)}$. Recall that the map $\alpha' : G \to \mathsf{Par}(X, X)$ is given by $\alpha'(g) = (X_{g^{-1}}, \iota_g, \alpha_g)$. Both G and $\mathsf{Par}(X, X)$ are considered as one-object bicategories and moreover G has only trivial 2-cells, $\mathsf{Par}(X, X)$ is a poset. Hence, $\alpha' : G \to \mathsf{Par}(X, X)$ induces a lax functor if and only if there exists morphism of spans $u' : (X, id_X, id_X) \to (X_e, \iota_e, \alpha_e)$ and $\theta' : \alpha'(h) \bullet \alpha'(g) \to \alpha'(hg)$. As in the first part of the proof, the existence of u' is equivalent to axiom (LPA1). Furthermore, remark that $\alpha'(h) \bullet \alpha'(g)$ is given by the span

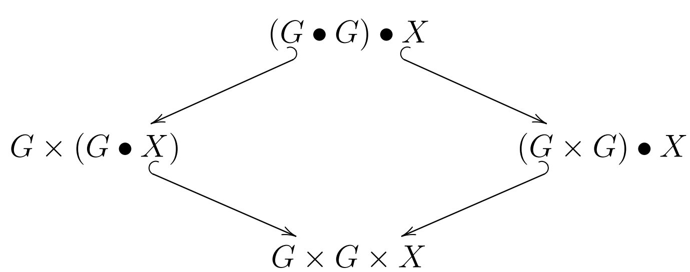


Hence the existence of θ' means that axioms (LPA2) and (LPA3) hold.

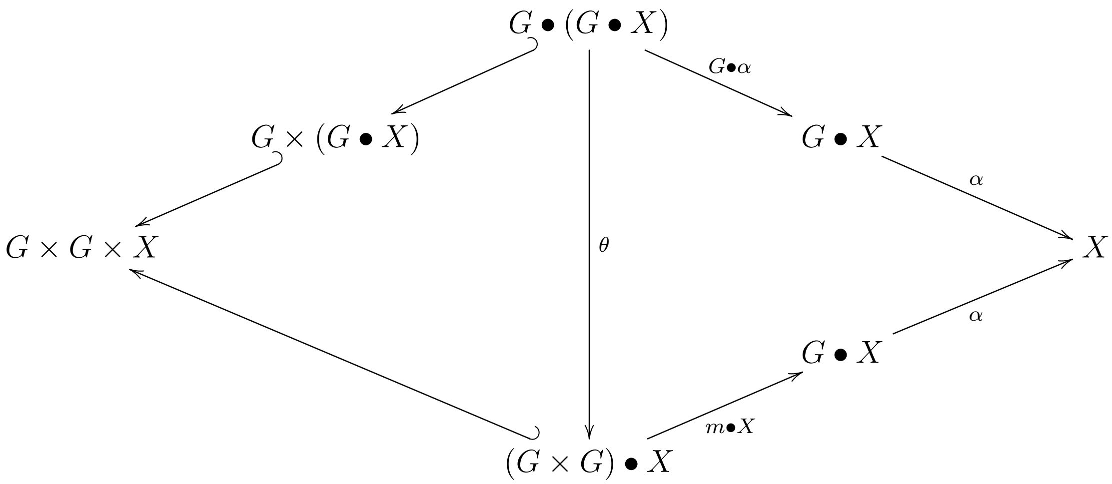
□

As we have pointed out before, partial actions are a special instance of lax partial actions. In the next result we provide equivalent conditions for a lax partial action to be partial.

Let us first make the following observation. Given a partial action datum, we can consider the pullback



which is nothing else than the intersection $G \times (G \bullet X) \cap (G \times G) \bullet X$. If the partial action datum defines a lax partial action, then the existence of the morphism of spans $\theta : \alpha \bullet (G \times \alpha) \to \alpha \bullet (m \times X)$ implies that the following diagram commutes



and therefore the image of θ lies in $(G \bullet G) \bullet X$, i.e. we can corestrict θ to a morphism $\bar{\theta} : G \bullet (G \bullet X) \to (G \bullet G) \bullet X$.

Proposition 2.2.9. *Let α be a lax partial action of the group G on the set X. Then the following statements are equivalent*

(i) α is a partial action;
(ii) $\bar{\theta} : G \bullet (G \bullet X) \to (G \bullet G) \bullet X$ is an isomorphism;
(iii) for each $g \in G$, we have that $\alpha_g : X_{g^{-1}} \to X_g$.

Proof. *(ii) ⇔ (i) ⇒ (iii).* By definition, partial actions and lax partial actions only differ in their second axiom. From the above discussion, we know that

$$\begin{aligned} (G \bullet G) \bullet X &= G \times (G \bullet X) \cap (G \times G) \bullet X \\ &= \{(h, g, x) \in G \times G \times X \mid x \in X_{g^{-1}} \cap X_{(gh)^{-1}}\} \end{aligned}$$

Therefore, $\bar{\theta}$ is an isomorphism we obtain that if

$$(h^{-1}g^{-1}, g, x) \in (G \bullet G) \bullet X, \text{ i.e. } \quad x \in X_{g^{-1}} \cap X_h$$

then also

$$((gh)^{-1}, g, x) \in G \bullet (G \bullet X), \text{ i.e. } \quad x \in X_{g^{-1}} \text{ and } gx \in X_{hg}$$

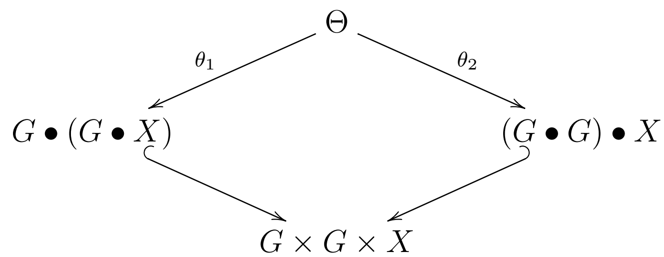
Hence we find that $\alpha_g(X_{g^{-1}} \cap X_h) \subset X_{gh}$. In particular, taking $h = e$, then we find that $\alpha_g(X_{g^{-1}}) \subset X_g$. Combining both, we recover exactly axiom (PA2).

$\underline{(iii) \Rightarrow (i)}$. For any $g \in G$ and $x \in X_g$ we find that $g^{-1}x \in X_{g^{-1}}$, we can apply g on $g^{-1}x$ and find that $x = g \cdot g^{-1}x$. So $x \in X_g$ if and only if $x = gy$ for some $y \in X_{g^{-1}}$.

Now take any $x \in X_{g^{-1}} \cap X_h$. Then by the above, we can write $x = hy$ for some $y \in X_{h^{-1}}$. Since we have that $y \in X_{h^{-1}}$ and $x = hy \in X_{g^{-1}}$, it follows by axiom (LPA2) that $y \in X_{(gh)^{-1}}$ and $gh \cdot y = g \cdot (hy) = gx$. In particular, we find that $gx = gh \cdot y \in X_{gh}$. Hence we obtain exactly axiom (PA2). □

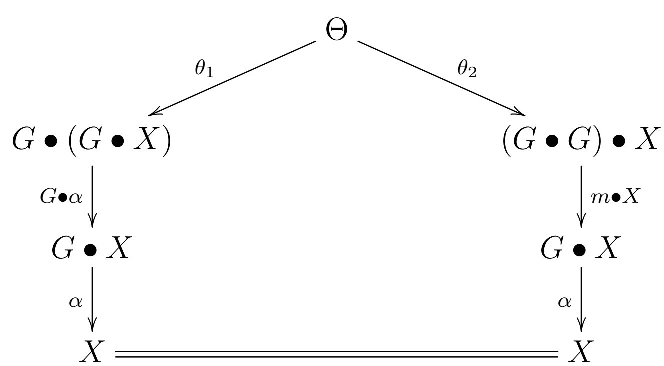
Finally, we also restate the definition of quasi partial action in terms of spans, the proof of which is clear.

Proposition 2.2.10. *A partial action datum (X_g, α_g) defines a quasi partial action of G on X if and only if the equivalent statements of Lemma 2.2.7 (1) hold and the associativity constraint $\alpha_h \circ \alpha_g = \alpha_{hg}$ holds on all elements of the following pullback*



Consequently, a quasi partial action is lax if and only if the span $(\Theta, \theta_1, \theta_2) : G \bullet (G \bullet X) \to (G \bullet G) \bullet X$ is induced by a morphism, and the quasi partial action is a partial action if and only if Θ is an isomorphism.

Proof. Let us just remark that the associativity on Θ means that the following diagram commutes



□

CHAPTER 3

Hopf algebras and their partial (co)actions

This chapter is the only one which does not contain any original results. However, as the notions recalled here are essential for the following chapters, we take the needed time to discuss them here. Firstly, we will recall the definition of algebras, coalgebras, modules and comodules with a lot of examples. We remark that all of these concepts can also be defined directly in monoidal categories. We will then recall the definition of bialgebras and Hopf algebras and their relations with monoidal categories. These results can for example be found in [**8**], [**35**]. Next, we will recall the notion of partial actions and coactions of Hopf algebras in the sense of Caenepeel-Janssen. And finely, we will recall the notion of partial representations.

3.1. Algebras and coalgebras

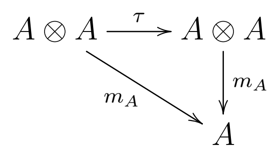
Let k be a field. Recall that an *algebra* is given by a triple (A, m_A, η) where A is a k-vector space, $m_A : A \otimes A \to A$ and $\eta : k \to A$ are linear maps such that the following diagrams commute:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m_A\otimes id} & A\otimes A \\ \downarrow{\scriptstyle id\otimes m_A} & & \downarrow{\scriptstyle m_A} \\ A\otimes A & \xrightarrow{m_A} & A \end{array}$$

$$\begin{array}{ccccc} k\otimes A & \xrightarrow{\eta\otimes id} & A\otimes A & \xleftarrow{id\otimes\eta} & A\otimes k \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle m_A} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

Denote $m_A(a \otimes a') = aa'$ and $\eta(1_k) = 1_A$. Then we have $(aa')a'' = a(a'a'')$ and $1_A a = a 1_A = a$, for all $a, a', a'' \in A$.

An algebra is said to be *commutative* if, in addition, the following diagram commutes:

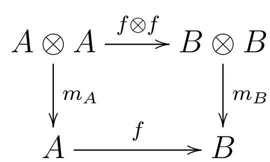


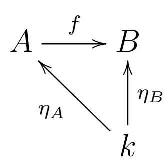
where τ is the flip map: $\tau(a \otimes a') = a' \otimes a$.

A morphism between two algebras (A, m_A, η_A) and (B, m_B, η_B) is a linear map $f : A \to B$ such that $m_B \circ (f \otimes f) = f \circ m_A$ and $f \circ \eta_A = \eta_B$, i.e.

$$f(aa') = f(a)f(a') \ , \ f(1_A) = 1_B$$

for all $a, a' \in A$, or equivalently, the following diagrams commute:

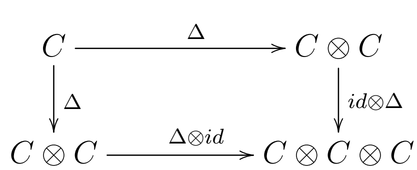


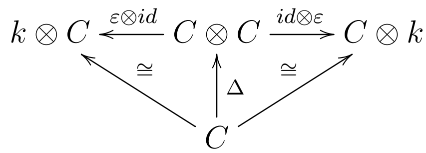


Example 3.1.1. Group algebra kG. kG is the free k-module generated by the elements $\{\sigma\}_{\sigma \in G}$ of a group G (or a monoid G). The multiplication in kG is defined on the base elements $\{\sigma\}_{\sigma \in G}$ by the multiplication of G and then linear extended to all the elements.

By reversing all the arrows of the above commutative diagrams, we get the definition of coalgebras.

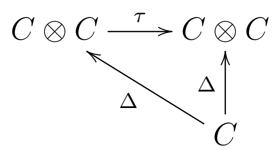
Definition 3.1.2. A *coalgebra* is a triple (C, Δ, ε) where C is a k-vector space, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are linear maps such that the following diagrams commute:





The map Δ is called the *comultiplication* and ε the *counit* of the coalgebra.

A coalgebra is said to be cocommutative if, in addition, the following diagram commutes:



To make equations easy to write, we now introduce the *Sweedler-Heyneman's notation*:

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

It can be simplified further by omitting the summation symbol:

$$\Delta(c) = c_{(1)} \otimes c_{(2)}$$

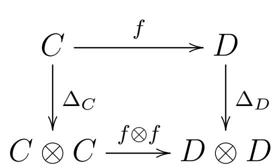
Note that the right hand side is not a monomial in general, but a finite sum of monomials. We will keep using this notation throughout the thesis.

Then we have $(c_{(1)(1)} \otimes c_{(1)(2)}) \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)(1)} \otimes c_{(2)(2)})$ and $\varepsilon(c_{(1)})c_{(2)} = c = c_{(1)}\varepsilon(c_{(2)})$.

A morphism between two coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ is a linear map $f : C \to D$ such that $(f \otimes f) \circ \Delta_C = \Delta_D \circ f$ and $\varepsilon_C = \varepsilon_B \circ f$, i.e.

$$f(c_{(1)} \otimes c_{(2)}) = f(c)_{(1)} \otimes f(c)_{(2)} \ , \ \varepsilon_D(f(c)) = \varepsilon_C(c)$$

for all $c \in C$, or equivalently, the following diagrams commute:



$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle \varepsilon_C} \searrow & \downarrow {\scriptstyle \varepsilon_D} \\ & & k \end{array}$$

It's easy to check that the composition of two morphisms of coalgebras is again a morphism of coalgebras. Here are some examples of coalgebras.

Examples 3.1.3. (1) The field k has a natural coalgebra structure with $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$. For any coalgebra (C, Δ, ε), the counit $\varepsilon : C \to k$ is a morphism of coalgebras.

(2) For any coalgebra (C, Δ, ε), define

$$\Delta^{op} = \tau \circ \Delta$$

Then $(C, \Delta^{op}, \varepsilon)$ is also a coalgebra called the opposite coalgebra of C and denote by C^{cop}.

(3) Let S be a set. X is the k-vector space generated by S. Then X has a coalgebra structure if we define $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$, for every $x \in X$. In particular, group algebra kG is also a coalgebra.

3.2. Bialgebras and Hopf algebras

Let H be a k-vector space equipped with an algebra structure (H, m_H, η) and coalgebra structure (H, Δ, ε). We give $H \otimes H$ an algebra structure by:

$$m_{H\otimes H}((h_1 \otimes h_2) \otimes (h'_1 \otimes h'_2)) = h_1 h'_1 \otimes h_2 h'_2 \ , \ 1_{H\otimes H} = 1_H \otimes 1_H$$

for all $h_1, h_2, h'_1, h'_2 \in H$. We give $H \otimes H$ a coalgebra structure by:

$$\Delta_{H\otimes H} = (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) \ , \ \varepsilon_{H\otimes H} = \varepsilon \otimes \varepsilon$$

that is

$$\Delta(h \otimes h') = h_{(1)} \otimes h_{(2)} \otimes h'_{(1)} \otimes h'_{(2)} \ , \ \varepsilon(h \otimes h') = \varepsilon(h) \otimes \varepsilon(h')$$

for all $h, h' \in H$.

With these structures, we have

Theorem 3.2.1. *The following statements are equivalent:*

(1) The maps m_H and η are morphisms of coalgebras;

(2) The maps Δ and ε are morphisms of algebras.

Proof. By definition, m_H is a morphism of coalgebras is equivalent to the following commutative diagrams

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\quad m_H \quad} & H \\ \Big\downarrow{\scriptstyle (id\otimes\tau\otimes id)(\Delta\otimes\Delta)} & & \Big\downarrow{\scriptstyle \Delta} \\ (H \otimes H) \otimes (H \otimes H) & \xrightarrow{\quad m_H\otimes m_H \quad} & H \otimes H \end{array}$$

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\varepsilon\otimes\varepsilon} & k \otimes k \\ \Big\downarrow{\scriptstyle m_H} & & \Big\downarrow{\scriptstyle id} \\ H & \xrightarrow{\quad \varepsilon \quad} & k \end{array}$$

where $\tau : A \otimes B \to B \otimes A, \tau(a \otimes b) = b \otimes a$ is the flip morphism.

η is a morphism of coalgebra is equivalent to the following commutative diagrams

$$\begin{array}{ccc} k & \xrightarrow{\quad \eta \quad} & H \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \Delta} \\ k \otimes k & \xrightarrow{\eta\otimes\eta} & H \otimes H \end{array}$$

$$\begin{array}{ccc} k & \xrightarrow{\ \eta\ } & H \\ & \searrow^{\cong} & \Big\downarrow{\scriptstyle \varepsilon} \\ & & k \end{array}$$

Similarly, by the definition of algebra morphisms Δ and ε, we write exactly these four commutative diagrams, which shows that they are equivalent. □

This leads to the following definition.

Definition 3.2.2. A *bialgebra* is a quintuple $(H, m_H, \eta, \Delta, \varepsilon)$ where (H, m_H, η) is an algebra and (H, Δ, ε) is a coalgebra such that they satisfying one of the conditions in the previous theorem. A morphism of bialgebras is a morphism of both the underlying algebra and coalgebra structures.

Examples 3.2.3. (1) We know that group algebra kG is both an algebra and a coalgebra. It's easy to check that kG is also a bialgebra. It is also true when G is a monoid.

(2) Let $M_n(k) = k[x_{11}, x_{12}, ..., x_{nn}]$ be the polynomial algebra of n^2 variables. For all $x_{ij}, 1 \leqslant i, j \leqslant n$ define

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \ ; \ \varepsilon(x_{ij}) = \delta_{ij}$$

These formulas make $M_n(k)$ into a bialgebra.

Given an algebra (A, m_A, η) and a coalgebra (C, Δ, ε) with linear maps f and g from C to A, we can define another map $f \star g$ from C to A as follows:

$$f \star g : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A$$

that is

$$(f \star g)(c) = f(c_{(1)})g(c_{(2)})$$

$\star$ is called the *convolution* of maps f and g.

Remark 3.2.4. $\mathsf{Hom}_k(C, A)$ is a monoid with convolution as product and $\eta_A \circ \varepsilon_C$ as unit. In particular, the dual vector space of a coalgebra is an algebra.

Definition 3.2.5. Let $(H, m_H, \eta, \Delta, \varepsilon)$ be a bialgebra. An endomorphism S of H is called an *antipode* of the bialgebra if

$$S \star id_H = id_H \star S = \eta \circ \varepsilon$$

or

$$S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \eta(\varepsilon(h))$$

for all $h \in H$.

A *Hopf algebra* is a bialgebra with an antipode. A morphism of Hopf algebras is a morphism of the underlying bialgebras.

Remark 3.2.6. A bialgebra does not need to have an antipode, once it does, it is unique. Indeed, if S and S' are both antipodes of H, then $S = S \star (\eta\varepsilon) = S \star (id_H \star S') = (S \star id_H) \star S' = (\eta\varepsilon) \star S' = S'$.

There are some important properties of antipodes that are very useful for calculation.

Theorem 3.2.7. *Let $(H, m_H, \eta, \Delta, \varepsilon)$ be a Hopf algebra. Then we have*

(1) $S(hk) = S(k)S(h)$;

(2) $\Delta(S(h)) = (S \otimes S)\tau\Delta(h)$, *for all* $h \in H$, *where* τ *is the flip morphism;*
(3) $S(1_H) = 1_H$*;*
(4) $\varepsilon(S(h)) = \varepsilon(h)$, *for all* $h \in H$.

Examples 3.2.8. (1) Group algebra kG becomes a Hopf algebra if we define $S(g) = g^{-1}$ for all $g \in G$.

(2) Hopf algebra coming from affine algebraic group. Let G be an affine algebraic group (i.e. an algebraic set with a group structure) over k, and let $\mathcal{O}(G)$ be the coordinate ring. If we define

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G), \Delta(f)(x, y) = f(xy)$$

for all $x, y \in G$, and

$$\varepsilon : \mathcal{O}(G) \to k, \varepsilon(f) = f(1_G)$$

and the antipode

$$S : \mathcal{O}(G) \to \mathcal{O}(G), Sf(x) = f(x^{-1})$$

for all $x \in G$. Then the coordinate ring $\mathcal{O}(G)$ is a Hopf algebra

Remark 3.2.9. Conversely, if $\mathcal{O}(G)$ is the coordinate ring of an affine algebraic set G and $\mathcal{O}(G)$ is a Hopf algebra, then G is an algebraic group. Moreover, the functor $G \to \mathcal{O}(G)$ defines a contravariant equivalence of categories: (affine algebraic groups over k, their morphisms) $\to$ (affine commutative semiprime Hopf k-algebra, Hopf algebra morphisms).

(3) $G = k$ is an algebraic group with usual addition as group operation. $\mathcal{O}(G) = k[X]$ and $X(a) = a$ for all $a \in k$. Then

$$\varepsilon(X) = X(0) = 0$$

and

$$\Delta(X) = X \otimes 1 + 1 \otimes X$$

and

$$S(X)(a) = X(-a) = -a$$

which implies $S(X) = -X$.

(4) $G = k\backslash\{0\}$ is an algebraic group with usual multiplication as group operation. $\mathcal{O}(G) \cong k[X, X^{-1}]$. Then

$$\varepsilon(X) = X(1) = 1$$

and

$$\Delta(X)(a \otimes b) = X(ab) = ab$$

which implies $\Delta(X) = X \otimes X$, and

$$S(X)(a) = X(a^{-1}) = a^{-1} = (X(a))^{-1} = X^{-1}(a)$$

which implies $S(X) = X^{-1}$.

3.3. Modules and comodules

Let (A, m_A, η) be an algebra. Recall that a left A-*module* is a couple (M, m_M), where M is a k-vector space and $m_M : A \otimes M \to M$ is a linear map such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{m_A \otimes id} & A \otimes M \\ \downarrow{\scriptstyle id \otimes m_M} & & \downarrow{\scriptstyle m_M} \\ A \otimes M & \xrightarrow{m_M} & M \end{array}$$

$$\begin{array}{ccc} k \otimes M & \xrightarrow{\eta \otimes id} & A \otimes M \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle m_M} \\ & & M \end{array}$$

Denote $m_M(a \otimes m) = a \cdot m$ or simply am. Then we have $a \cdot (b \cdot m) = ab \cdot m$ and $1_A \cdot m = m$.

A morphism between two A-modules (M, m_M) and (N, m_N) is a linear map $f : M \to N$ such that $m_N \circ (id \otimes f) = f \circ m_M$, or equivalently, $f(a \cdot m) = a \cdot f(m)$, for all $a \in A, m \in M$.

We can define right A-modules similarly.

Example 3.3.1. (1) Let A be an algebra and U, V be A-modules. Then $U \otimes V$ is a $A \otimes A$-module by:

$$(a_1 \otimes a_2)(u \otimes v) = a_1 u \otimes a_2 v$$

where $a_1, a_2 \in A$, $u \in U$ and $v \in V$. If, in addition, A is bialgebra $(A, m_A, \eta, \Delta, \varepsilon)$, the $A \otimes A$-module $U \otimes V$ has an A-module structure by:

$$a(u \otimes v) = \Delta(a)(u \otimes v) = \sum_{(a)} a_{(1)}u \otimes a_{(2)}v$$

.

(2) Representations of a group G over k are in one-to-one correspondence with kG-modules.

Dually, we have the definition of comodules over coalgebras.

Definition 3.3.2. Let (C, Δ, ε) be a coalgebra. A left C-*comodule* is a couple (N, Δ_N), where N is a k-vector space and $\Delta_N : N \to C \otimes N$ is a linear map such that the following diagrams commute:

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & C \otimes N \\ \downarrow{\scriptstyle \Delta_N} & & \downarrow{\scriptstyle id_C \otimes \Delta_N} \\ C \otimes N & \xrightarrow{\Delta \otimes id_N} & C \otimes C \otimes N \end{array}$$

$$\begin{array}{ccc} k \otimes N & \xleftarrow{\varepsilon \otimes id} & C \otimes N \\ & {\scriptstyle \cong} \nwarrow & \uparrow{\scriptstyle \Delta_N} \\ & & N \end{array}$$

Denote $\Delta_N(n) = n_{(-1)} \otimes n_{(0)}$. Then we have $(n_{(-1)(1)} \otimes n_{(-1)(2)}) \otimes n_{(0)} = n_{(-1)} \otimes (n_{(0)(-1)} \otimes n_{(0)(0)})$.

We will also say C *coacts* on N.

A morphism between two C-comodules (N, Δ_N) and (L, Δ_L) is a linear map $f : N \to L$ such that $(f \otimes id) \circ \Delta_N = \Delta_L \circ f$.

The composition of two morphisms of comodules is again a morphism of comodules.

We can define right C-comodules similarly.

Examples 3.3.3. (1) Let (C, Δ, ε) be any coalgebra, then (C, Δ) is a C-comodule.

(2) Let $(H, m_H, \eta, \Delta, \varepsilon)$ be a bialgebra and M, N both H-comodules. We can give a H-comodule structure on $M \otimes N$ by:

$$\Delta_{M \otimes N} = (m_H \otimes id_{M \otimes N})(id_H \otimes \tau \otimes id_N)(\Delta_M \otimes \Delta_N)$$

(3) Representations of an affine algebraic group G are in in one-to-one correspondence with $\mathcal{O}(G)$-comodules.

Theorem 3.3.4. *Let A be a k-algebra. Then there is a one-to-one correspondence between*

(1) A coalgebra structure on A such that A becomes a bialgebra;
(2) A monoidal structure on ${}_A\mathcal{M}$ such that the forgetful functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is strict monoidal.

Proof. (2)⇒(1):Assume we have a monoidal structure on ${}_A\mathcal{M}$ such that the forgetful functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is strict monoidal. This means in particular, after forgetting the A-module structure, the unit object of $\mathcal{M}$ is equal to k, and the tensor product in ${}_A\mathcal{M}$ is the same as in ${}_k\mathcal{M}$. Moreover, it follows from the commuting diagrams in the definition of monoidal functors that the associativity and unit constraints in ${}_A\mathcal{M}$ are the same as in ${}_k\mathcal{M}$.

$A \in {}_A\mathcal{M}$ via left multiplication, thus $A \otimes A \in_A \mathcal{M}$. Define a k-linear map

$$\Delta : A \to A \otimes A, \Delta(a) = a(1 \otimes 1)$$

for all $a \in A$, we denote $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Once Δ is known, we can define an A-action on $M \otimes N$, for all $M, N \in {}_A\mathcal{M}$ as in example3.3.1(1). Consider two A-linear maps:

$$f_m; A \to M, f_m(a) = am \; ; \; g_n : A \to N, g_n(a) = an$$

From the functoriality of tensor product, it follows that $f_m \otimes g_n$ is A-linear as a morphism in ${}_A\mathcal{M}$, thus:

$$\begin{aligned} a(m \otimes n) &= a((f_m \otimes g_n)(1 \otimes 1)) = (f_m \otimes g_n)(a(1 \otimes 1)) \\ &= (f_m \otimes g_n)(\Delta(a)) = (f_m \otimes g_n)(a_{(1)} \otimes a_{(2)}) \\ &= a_{(1)}m \otimes a_{(2)}n \end{aligned}$$

Using this equation, let us consider the associativity constraint $a_{A,A,A} : (A \otimes A) \otimes A \to A \otimes (A \otimes A)$, which is also a morphism in ${}_A\mathcal{M}$. Hence

$$a((1 \otimes 1) \otimes 1) = a_{(1)} \otimes a_{(2)}(1 \otimes 1) = a_{(1)} \otimes \Delta(a_{(2)})$$

On the other hand, it's also equal to

$$\begin{aligned} a(a_{A,A,A}((1 \otimes 1) \otimes 1)) &= a_{A,A,A}(a((1 \otimes 1) \otimes 1))) \\ &= a_{A,A,A}(a_{(1)}(1 \otimes 1) \otimes a_{(2)}) \\ &= a_{A,A,A}(\Delta(a_{(1)}) \otimes a_{(2)}) \end{aligned}$$

We conclude that $a_{(1)} \otimes \Delta(a_{(2)}) = \Delta(a_{(1)}) \otimes a_{(2)}$, which can also be expressed as

$$(A \otimes \Delta) \circ \Delta = (\Delta \otimes A) \circ \Delta$$

It's nothing but the first commutative diagram in the definition of coalgebras!

Next, we also know $k \in {}_A\mathcal{M}$, define a map:

$$\varepsilon : A \to k, \varepsilon(a) = a \cdot 1_k$$

Since the left unit map $l_A : k \otimes A \to A, l_A(x \otimes a) = xa$ is left A-linear, we have:

$$a = al_A(1_k \otimes 1_A) = l_A(a(1_k \otimes 1_A)) = l_A(\varepsilon(a_{(1)} \otimes a_{(2)})) = \varepsilon(a_{(1)})a_{(2)}$$

Similarly, from the left A-linearity of r_A, we have $a = a_{(1)}\varepsilon(a_{(2)})$. Thus

$$\varepsilon(a_{(1)})a_{(2)} = a_{(1)}\varepsilon(a_{(2)})$$

It is nothing but the second commutative diagram in the definition of coalgebras!

Moreover, we can compute:

$$\begin{aligned} \Delta(ab) &= (ab)(1 \otimes 1) = a(b(1 \otimes 1)) = a(b_{(1)} \otimes b_{(2)}) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \\ \Delta(1) &= 1(1 \otimes 1) = 1 \otimes 1 \\ \varepsilon(ab) &= (ab) \cdot 1_k = a \cdot (b \cdot 1_k) = a \cdot \varepsilon(b) = \varepsilon(a)\varepsilon(b) \\ \varepsilon(1) &= 1 \cdot 1_k = 1_k \end{aligned}$$

In other words, Δ and ε are algebra maps. Thus A becomes a bialgebra.

(1)⇒(2):Conversely, suppose A has a coalgebra structure (A, Δ, ε) which makes A into a bialgebra, we can define an A-module structure on $M \otimes N$ for all $M, N \in {}_A\mathcal{M}$ as following:

$$a(m \otimes n) = a_{(1)}m \otimes a_{(2)}n$$

k also has a A-module structure by the formula:

$$a \cdot x = \varepsilon(a)x;\ x \in k$$

It's straightforward to check ${}_A\mathcal{M}$ becomes a monoidal category with these structures such that the forgetful functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is strict monoidal.

□

For Hopf algebras, we have the following Tannaka reconstruction theorem, which we just state out the result.

Theorem 3.3.5. *There is a one-to-one correspondence between*

(1) Hopf algebra over k;

(2) Rigid monoidal category $\mathcal{C}$ together with a strict monoidal functor $U : \mathcal{C} \to \mathsf{Vect}$ that preserves duals.

Remarks 3.3.6. (1) We can also define algebras (or monoids) in monoidal categories. Given a monoidal category $(\mathcal{A}, \otimes, I)$, an algebra (or a monoid) in $\mathcal{A}$ is a triple (A, m_A, η), where $A \in \mathcal{A}$, $m_A : A \otimes A \to A$ and $\eta : I \to A$ are morphisms in $\mathcal{A}$ such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m_A \otimes id} & A \otimes A \\ \downarrow{\scriptstyle id \otimes m_A} & & \downarrow{\scriptstyle m_A} \\ A \otimes A & \xrightarrow{m_A} & A \end{array}$$

$$\begin{array}{ccccc} I \otimes A & \xrightarrow{\eta \otimes id} & A \otimes A & \xleftarrow{id \otimes \eta} & A \otimes I \\ & {\scriptstyle \cong} \searrow & \downarrow{\scriptstyle m_A} & \swarrow {\scriptstyle \cong} & \\ & & A & & \end{array}$$

We can define coalgebras (or comonoids) similarly. Furthermore, we can define modules and comudules in this way.

3.4. Partial actions and coactions of Hopf algebras

Caenepeel and Janssen defined partial actions and coactions of Hopf algebras on unital algebras in [21], which was motivated by examples of partial actions of groups on algebras.

Definition 3.4.1. A *partial action* of a Hopf algebra H on a unital algebra A is a linear map

$$\begin{array}{rrcl} \cdot : & H \otimes A & \to & A \\ & h \otimes a & \mapsto & h \cdot a \end{array}$$

satisfying the following identities for all $a, b \in A$ and $h, k \in H$:

(PHA1) $1_H \cdot a = a$;
(PHA2) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$;
(PHA3) $h \cdot (k \cdot a) = (h_{(1)} \cdot 1_A)(h_{(2)}k \cdot a)$.

In this case, the algebra A is called a *(left) partial H-module algebra.*

The partial action is called *symmetric* if in addition

(PHA4) $h \cdot (k \cdot a) = (h_{(1)}k \cdot a)(h_{(2)} \cdot 1_A)$.

A morphism between two partial H-module algebras A and B is an algebra map $f : A \to B$ such that for every $h \in H$ and $a \in A$ we have $h \cdot_B f(a) = f(h \cdot_A a)$.

One can define the right partial actions of H on A, or the right partial H-module algebras in a similar way.

Given a partial action of a Hopf algebra H on a unital algebra A, one can define an associative product on $A \otimes H$ by:

$$(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b) \otimes h_{(2)}k$$

for all $a, b \in A$ and $h, k \in H$. Then we can construct a new unital algebra called the *partial smash product* as $\underline{A\#H} = (A \otimes H)(1_A \otimes 1_H)$.

This algebra is generated by typical elements of the form

$$a\#h = a(h_{(1)} \cdot 1_A) \otimes h_{(2)}$$

One can then prove that $a\#h = a(h_{(1)} \cdot 1_A)\#h_{(2)}$ and $(a\#h)(b\#k) = a(h_{(1)} \cdot b)\#h_{(2)}k$.

Definition 3.4.2. A *partial coaction* of a Hopf algebra H on a unital algebra A is a linear map

$$\begin{array}{rccl} \bar{\rho}: & A & \to & A \otimes H \\ & a & \mapsto & \bar{\rho}(a) = a^{[0]} \otimes a^{[1]} \end{array}$$

satisfying the following identities for all $a, b \in A$:

(PHCA1) $\bar{\rho}(ab) = \bar{\rho}(a)\bar{\rho}(b)$;
(PHCA2) $(I \otimes \varepsilon)\bar{\rho}(a) = a$;
(PHCA3) $(\bar{\rho} \otimes I)\bar{\rho}(a) = [(I \otimes \Delta)\bar{\rho}(a)](\bar{\rho}(1_A) \otimes 1_H)$.

In this case, the algebra A is called a *(right) partial H-comodule algebra.*

The partial coaction is called *symmetric* if in addition

(PHCA4) $(\bar{\rho} \otimes I)\bar{\rho}(a) = (\bar{\rho}(1_A) \otimes 1_H)[(I \otimes \Delta)\bar{\rho}(a)]$.

A morphism between two partial H-comodule algebras A and B is an algebra map $f : A \to B$ such that $\bar{\rho}_B \circ f = (f \otimes I) \circ \bar{\rho}_A$

Of course, one can also define the left partial coactions of H on A, or the left partial H-comodule algebras in a similar way.

Denoting the partial coaction in Sweedler's notation

$$\bar{\rho}(a) = a^{[0]} \otimes a^{[1]}$$

we can rewrite the above axioms for partial coactions as:

(PHCA1) $(ab)^{[0]} \otimes (ab)^{[1]} = a^{[0]}b^{[0]} \otimes a^{[1]}b^{[1]}$;

(PHCA2) $a^{[0]}\varepsilon(a^{[1]}) = a$;

(PHCA3) $a^{[0][0]} \otimes a^{[0][1]} \otimes a^{[1]} = a^{[0]}1_A^{[0]} \otimes a_{(1)}^{[1]}1_A^{[1]} \otimes a_{(2)}^{[1]}$;

(PHCA4) $a^{[0][0]} \otimes a^{[0][1]} \otimes a^{[1]} = 1_A^{[0]}a^{[0]} \otimes 1_A^{[1]}a_{(1)}^{[1]} \otimes a_{(2)}^{[1]}$.

Let A be a k-algebra. An A-*coring* is a coalgebra object in the category of A-bimodules, i.e. it is a triple (C, Δ, ε) where C is an A-bimodule, comultiplication $\Delta : C \to C \otimes_A C$ and counit $\varepsilon : C \to A$ are A-bimodule maps that satisfying the usual coassociativity and counit conditions.

Given a (right) partial H-comodule algebra A, we can give the reduced tensor product $A\underline{\otimes}H = (A \otimes H)\bar{\rho}(1_A)$ an A-coring structure. The bimodule structure, comultiplication and counit are given by:

$$\begin{aligned} b \cdot (a1^{[0]} \otimes h1^{[1]}) \cdot b' &= bab'^{[0]} \otimes hb'^{[1]} \\ \tilde{\Delta}(a1^{[0]} \otimes h1^{[1]}) &= a1^{[0]} \otimes h_{(1)}1^{[1]} \otimes_A 1^{[0']} \otimes h_{(2)}1^{[1']} \\ \tilde{\varepsilon}(a1^{[0]} \otimes h1^{[1]}) &= a\varepsilon(h) \end{aligned}$$

Remarks 3.4.3. (1) It is easy to see that an H-module algebra is a partial H-module algebra. In fact, one can prove that a partial action is global if and only if for every $h \in H$ we have $h \cdot 1_A = \varepsilon(h)1_A$.

(2) As the partial coaction $\bar{\rho}$ is a morphism of non-unital algebras, $\bar{\rho}(1_A)$ is an idempotent in the algebra $A \otimes H$. For every $a \in A$, we have $\bar{\rho}(a) = \bar{\rho}(a)\bar{\rho}(1_A) = \bar{\rho}(1_A)\bar{\rho}(a)$. However, $\bar{\rho}(1_A)$ is only central in $\mathsf{Im}\,\bar{\rho}$, not in the whole of $A \otimes H$. The image of the coaction is contained in the unitary ideal $(A \otimes H)\bar{\rho}(1_A)$.

Examples 3.4.4. (1) We say that a group G acts partially on an algebra A if G acts partially on the underlying set of the algebra such that each A_g is an ideal of A and each α_g is multiplicative. If G acts partially on a set X with data $\{\alpha_g, X_g\}_{g \in G}$, then G acts partially on the algebra $A = \mathsf{Fun}(X, k)$ with data $A_g = \mathsf{Fun}(X_g, k)$ and $\beta_g(f)(x) = f(\alpha_{g^{-1}}(x))$, where $f \in A_{g^{-1}}$ and $x \in X_g$. In this example, the ideals A_g are unital algebras. Partial actions of the group algebra kG on any unital algebra A are one-to-one correspondence with partial actions of the group G on an algebra A in which A_g are unital ideals, that is, A_g has the form $1_g A$ where $1_g \in A$ is a central idempotent, and α_g are unital algebra isomorphisms for each $g \in G$.

(2) Let H be a Hopf algebra and A be an H-module algebra. If e is a central idempotent in A, then there exists a partial action of H on the ideal $B = eA$ by:

$$h \cdot b = e(h \cdot b)$$

where $h \in H, b \in B$ and the right hand side action is the global action of H on A.

(3) Let H be a Hopf algebra and A a right H-comodule algebra with coaction $\rho : A \to A \otimes H$. If $B \subset A$ is a unital ideal, then B is a right partial H-comodule algebra with coaction $\bar{\rho} : B \to B \otimes H, \bar{\rho}(b) = (1_B \otimes 1_H)\rho(b)$.

As we will now point out, Caenepeel and Janssen's definition of partial (co)actions can only describe the actions on spaces which are the disjoint union of subspaces. This observation is the main motivation for our work and was made in [**15**].

For our purpose, in the rest of this section, k will denote an algebraically closed field and $\mathbb{A}^n$ the n-dimensional affine space over k. By an affine algebraic set we mean a subset of points in $\mathbb{A}^n$ which are zeros of a finite set of polynomials $p_1, \dots, p_k$ in $k[x_1, \dots, x_n]$.

We will first list the definition and some results in [**15**].

Definition 3.4.5. Let G be an affine algebraic group and M an affine algebraic set. A partial action $(\{M_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on the underlying set M is said to be *algebraic* if

(1) for all $g \in G$, M_g and its complement $M'_g = M \setminus M_g$ are affine algebraic sets;
(2) for all $g \in G$, the maps $\alpha_g : M_{g^{-1}} \to M_g$ are polynomial;
(3) the set of "compatible couples" $G \bullet M := \{(g, x) \in G \times M \mid x \in M_{g^{-1}}\} \subset G \times M$ is an algebraic set and the map $\alpha : G \bullet M \to M, \alpha(g, x) = \alpha_g(x)$ is polynomial.

Proposition 3.4.6 ([**15**],4.12)**.** *Let G be an affine algebraic group and M be an affine algebraic set. Then each algebraic partial action of G on M defines a (symmetric) partial coaction of the commutative Hopf algebra $H = \mathcal{O}(G)$ on the commutative algebra $A = \mathcal{O}(M)$.*

Proposition 3.4.7 ([15],4.13)**.** *Let H be a commutative Hopf algebra and A be a commutative right partial H comodule algebra. Then there is a (symmetric) partial action of the affine algebraic group $G = \mathsf{Hom}_{\underline{Alg}}(H,k)$ on the affine algebraic set $M = \mathsf{Hom}_{\underline{Alg}}(A,k)$.*

Corollary 3.4.8 ([15],4.14)**.** *Let H be a commutative Hopf algebra and $G = \mathsf{Hom}_{\mathsf{Alg}}(H,k)$ the corresponding algebraic group, then there is an equivalence between the category of commutative right partial H-comodule algebras and the category of algebraic partial actions of the group G.*

Next, we prove a well-known lemma that will be used quite often.

Lemma 3.4.9. *Let R be a ring. Then there is a one-to-one correspondence between*

- *a decomposition $R = I_1 \oplus \cdots \oplus I_n$ as a direct sum of ideals;*
- *orthogonal idempotents $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1$.*

Proof. Suppose $R = I_1 \oplus \cdots \oplus I_n$, and write $1 = e_1 + \cdots + e_n$ where $e_i \in I_i, i = 1, \ldots, n$, then

$$e_i = e_i \cdot 1 = e_i(e_1 + \cdots + e_n) = e_i e_1 + \cdots + e_i e_n$$

note that $e_i e_j \in I_j$, and R is a direct sum of ideals, we have $e_i e_j = 0$ $(i \neq j)$, $e_i = e_i^2$ for $i = 1, \ldots, n$, i.e. these are orthogonal idempotents.

Conversely, for every idempotent e_i, define $I_i = Re_i$, which is an ideal of R. Then $R = I_1 + \cdots + I_n$ since $1 = e_1 + \cdots + e_n$. We can show that $I_i \cap \sum_{i \neq j=1}^{n} I_j = 0$. Indeed, if $r_i e_i = \sum_{i \neq j=1}^{n} r_j e_j$, then $r_i e_i^2 = \sum_{i \neq j=1}^{n} r_j e_i e_j = 0$ since $e_i e_j = 0$ $(j \neq i)$. Hence $R = I_1 \oplus \cdots \oplus I_n$.

Furthermore, the two constructions are mutually inverse. Given $R = I_1 \oplus \cdots \oplus I_n$ and write $1 = e_1 + \cdots + e_n$, we need to show $I_i = Re_i$, $i = 1, \ldots, n$. On one hand, $Re_i \subseteq I_i$. On the other hand, for every $x \in I_i$, write $x = x \cdot 1 = \sum_j x e_j$, where $x e_j \in Re_j \subseteq I_j$ and $x \in I_i$, the direct sum tell us $x e_j = 0 (j \neq i)$, hence $x = x e_i \in Re_i$ and $Re_i \supseteq I_i$. So $I_i = Re_i$, $i = 1, \ldots, n$.

Finally, given orthogonal idempotents $e_1, \cdots, e_n$ such that $e_1 + \cdots + e_n = 1$, define $I_i = Re_i$, clearly we can only write $1 = e_1 + \cdots + e_n$ in this way.

□

Now here comes the phenomenon that we really want to point out.

Lemma 3.4.10. *Let X be an affine algebraic set, $A = \mathcal{O}(X)$ be its coordinate ring. Then X is the disjoint union of two algebraic subsets $X_1 \bigsqcup X_2$ if and only if the unit of A can be decomposed as a sum of two orthogonal idempotents: $1 = e + f$. In this case $\mathcal{O}(X_1) = f\mathcal{O}(X)$ is an ideal of $\mathcal{O}(X)$ and f can be viewed as the characteristic function on X_1.*

Proof. Suppose X is an algebraic subset of $\mathbb{A}^n$ and $X = X_1 \bigsqcup X_2$. Then $\mathcal{O}(X)$, $\mathcal{O}(X_1)$ and $\mathcal{O}(X_2)$ are quotients of the polynomial algebra $k[x_1, x_2, \ldots, x_n]$. Denote the respective ideals by I, J, K, then $\mathbb{V}(J + K) = X_1 \cap X_2 = \varnothing$ and it follows from Hilbert's Nullstellensatz that $J + K = k[x_1, x_2, \ldots, x_n]$. So there exist $j \in J$ and $k \in K$ such that $j + k = 1$. Denote respectively the classes of j, k modulo JK by e and f, then e and f are orthogonal idempotents of $\mathcal{O}(X)$, indeed,

$$\begin{aligned} e^2 &= \bar{j}^2 = \bar{j}^2 + \overline{jk} = \bar{j}(\overline{j+k}) = \bar{j} = e \\ ef &= \bar{j}\bar{k} = \overline{jk} = 0 \end{aligned}$$

the same argument works for f. It follows from Lemma 3.4.9, $\mathcal{O}(X_1) = f\mathcal{O}(X)$.

Conversely, if $1 = e + f$ is a sum of two orthogonal idempotents, again from Lemma 3.4.9 we have $\mathcal{O}(X) = e\mathcal{O}(X) \oplus f\mathcal{O}(X)$, where $e\mathcal{O}(X)$ and $f\mathcal{O}(X)$ are ideals of $\mathcal{O}(X)$, and moreover $\frac{\mathcal{O}(X)}{e\mathcal{O}(X)} \cong f\mathcal{O}(X)$, $\frac{\mathcal{O}(X)}{f\mathcal{O}(X)} \cong e\mathcal{O}(X)$. Since $\mathcal{O}(X)$ is reduced, $e\mathcal{O}(X)$ and $f\mathcal{O}(X)$ are reduced and therefore radical ideals. Denote by X_1 and X_2 the corresponding algebraic subsets, we have $X = X_1 \bigsqcup X_2$.

□

Theorem 3.4.11. *Let G be an affine algebraic group, X be an algebraic set. If G acts partially on X, or equivalently, $\mathcal{O}(G)$ coacts partially on $\mathcal{O}(X)$ (in the sense of Caenepeel and Janssen), then $X = X_g \bigsqcup X'_g$ for all $g \in G$, where X_g and X'_g are certain algebraic subsets of X indexed by g.*

Proof. If $\mathcal{O}(G)$ coacts partially on $\mathcal{O}(X)$, that is, there exists a linear map

$$\bar{\rho} : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G)$$

Denote $\bar{\rho}(1_{\mathcal{O}(X)}) = 1^{[0]} \otimes 1^{[1]}$. For each $g \in G$, denote $1_g = 1^{[0]}1^{[1]}(g) \in \mathcal{O}(X)$, then 1_g is an idempotent, Indeed,

$$1_g 1_g = 1^{[0]}1^{[0']}1^{[1]}(g)1^{[1']}(g) = 1^{[0]}1^{[0']}1^{[1]}1^{[1']}(g) = 1^{[0]}1^{[1]}(g) = 1_g$$

Note that $1 - 1_g$ is also an idempotent, and $1 = 1_g + (1 - 1_g)$, from the above lemma we know that $X = X_g \bigsqcup X'_g$. □

Examples 3.4.12. Let M be the algebraic set in $\mathbb{R}^3$ which is the union of two horizontal circles of radius 1, one centered at $(0, 0, 0)$ and the other at (0,0,1). This is an affine algebraic set whose coordinate algebra is given by

$$A = k[x, y, z]/(x^2 + y^2 - 1, z^2 - z)$$

Let G be the affine algebraic group $\mathbb{S}^1 \rtimes \mathbb{Z}_2$. Geometrically, the group G can be thought as the union of two disjoint circles in $\mathbb{R}^3$: the circle G_1 whose elements are of the form $g = (x_1, x_2, 1)$ and the circle G_2 whose elements are of the form $(x_1, x_2, -1)$. The group operation is given by

$$(x_1, x_2, \lambda)(y_1, y_2, \mu) = (x_1 y_1 - \lambda x_2 y_2, y_1 x_2 + \lambda x_1 y_2, \lambda\mu)$$

where λ and μ are equal to $+1$ or -1. The coordinate algebra of G is given by

$$H = k[x_1, x_2, x_3]/(x_1^2 + x_2^2 - 1, x_3^2 - 1)$$

For $g \in G_1$, we have $M_g = M$, and the action is given by

$$\alpha_{(x_1,x_2,1)}(x, y, z) = (xx_1 - yx_2, xx_2 + yx_1, z) \qquad z = 0, 1$$

For $g \in G_2$, M_g is only the circle centered at $(0, 0, 0)$, and the action is given by

$$\alpha_{(x_1,x_2,-1)}(x, y, z) = (-xx_1 - yx_2, -xx_2 + yx_1, z) \qquad z = 0$$

If a space is not a disjoint union of subspaces, how to describe the partial actions on it? Here we just provide an example. We will look into more details of it in the following chapters.

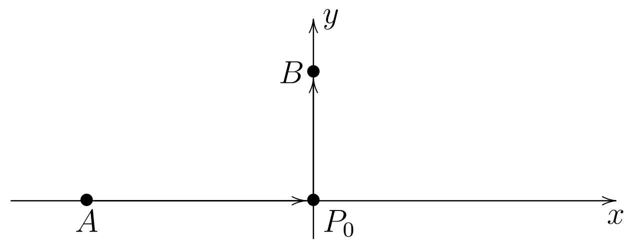
Examples 3.4.13. Let

- $\mathbb{A}^2$ be the usual plane with coordinates (x, y);
- X be the set $\{(x, y) \mid xy = 0\} \subset \mathbb{A}^2$;
- $G = \mathbb{A}^2$ be the usual additive group.

The (global) action of G on $\mathbb{A}^2$ is the usual translation, i.e.

$$(g_1, g_2) : (x, y) \mapsto (x + g_1, y + g_2).$$

Restricting this action on X, we get a partial action.



3.5. Partial representations and partial modules

The concept of partial representations is closely related to partial actions. Here we recall some definitions and important properties of partial representations from [9], which will be discussed further using our new notions in the next chapters.

Definition 3.5.1. Let H be a Hopf algebra and B be a unital k-algebra. A *partial representation* of H on B is a linear map $\pi : H \to B$ such that

(PR1) $\pi(1_H) = 1_B$;
(PR2) $\pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \pi(hk_{(1)})\pi(S(k_{(2)}))$;
(PR3) $\pi(h_{(1)})\pi(S(h_{(2)}))\pi(k) = \pi(h_{(1)})\pi(S(h_{(2)})k)$;
(PR4) $\pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \pi(hS(k_{(1)}))\pi(k_{(2)})$;
(PR5) $\pi(S(h_{(1)}))\pi(h_{(2)})\pi(k) = \pi(S(h_{(1)}))\pi(h_{(2)}k)$.

A morphism between two partial representations (B, π) and (B', π') of H is an algebra map $f : B \to B'$ such that $\pi' = f \circ \pi$.

Definition 3.5.2. Let H be a Hopf algebra and $T(H)$ be the tensor algebra of the vector space H. The *partial Hopf algebra H_{par}* is the quotient of $T(H)$ by the ideal I, where I is generated by elements of the form (for all $h, k \in H$)

(1) $1_H - 1_{T(H)}$;
(2) $h \otimes k_{(1)} \otimes S(k_{(2)}) - hk_{(1)} \otimes S(k_{(2)})$;
(3) $h_{(1)} \otimes S(h_{(2)}) \otimes k - h_{(1)} \otimes S(h_{(2)})k$;
(4) $h \otimes S(k_{(1)}) \otimes k_{(2)} - hS(k_{(1)}) \otimes k_{(2)}$;
(5) $S(h_{(1)}) \otimes h_{(2)} \otimes k - S(h_{(1)}) \otimes h_{(2)}k$.

Denoting the class of $h \in H$ by $[h]$, it's easy to see that the map

$$\begin{array}{rcl} [-]: H & \to & H_{par} \\ h & \mapsto & [h] \end{array}$$

satisfies the following conditions (for all $h, k \in H$)

(1) $[-]$ is a linear map;
(2) $[1_H] = 1_{H_{par}}$;
(3) $[h][k_{(1)}][S(k_{(2)})) = [hk_{(1)}][S(k_{(2)})]$;
(4) $[h_{(1)}][S(h_{(2)})][k] = [h_{(1)}][S(h_{(2)})k]$;
(5) $[h][S(k_{(1)})][k_{(2)}] = [hS(k_{(1)})][k_{(2)}]$;
(6) $[S(h_{(1)})][h_{(2)}][k] = [S(h_{(1)})][h_{(2)}k]$.

Thus the linear map $[-]$ is a partial representation of the Hopf algebra H on H_{par}.

The partial Hopf algebra H_{par} has the following universal property.

Theorem 3.5.3 ([9],4.2)**.** *Given a partial representation $\pi : H \to B$, there is a unique morphism of algebras $\hat{\pi} : H_{par} \to B$ such that $\pi = \hat{\pi} \circ [-]$. Conversely, given an algebra morphism $\phi : H_{par} \to B$, there is a unique partial representation $\pi_\phi : H \to B$ such that $\phi = \hat{\pi}_\phi$.*

In other words, the following functors establish an isomorphism between the category of partial representations ParRep_H *and the category of the co-slice category* H_{par}/Alg_k

$$\mathsf{ParRep}_H \underset{R}{\overset{L}{\rightleftarrows}} H_{par}/\mathsf{Alg}_k$$

where $L((B,\pi)) = (B,\hat{\pi})$ and $R((B,\phi)) = (B,\pi_\phi)$.

Definition 3.5.4. Let H be a Hopf algebra. A (left) *partial module* over H is a pair (M,π), where M is a k-vector space and $\pi : H \to \mathsf{End}_k(M)$ is a (left) partial representation of H.

A morphism between two partial modules (M,π) and (M',π') is a linear map $f : M \to M'$ such that $f \circ \pi(h) = \pi'(h) \circ f$ for all $h \in H$.

Using the classical Hom-Tensor relations, a k-vector space M is a partial H-module if and only if there exists a k-linear map $\bullet : H \otimes M \to M$ such that the following axioms hold for all $m \in M$ and $h, k \in H$

(PM1) $1_H \bullet m = m$;
(PM2) $h \bullet (k_{(1)}(S(k_{(2)}) \bullet m)) = (hk_{(1)})(S(k_{(2)}) \bullet m)$;
(PM3) $h_{(1)} \bullet (S(h_{(2)}) \bullet (k \bullet m)) = (h_{(1)})(S(h_{(2)})k \bullet m)$;
(PM4) $h \bullet (S(k_{(1)}) \bullet (k_{(2)} \bullet m)) = hS(k_{(1)}) \bullet (k_{(2)} \bullet m)$;
(PM5) $S(h_{(1)}) \bullet (h_{(2)} \bullet (k \bullet m)) = (S(h_{(1)})) \bullet (h_{(2)}k \bullet m)$.

Now we list some results from [9].

Theorem 3.5.5. *Let H be a Hopf algebra, then there exists a partial action of H on the subalgebra $A = \{[h_{(1)}][S(h_{(2)})] \mid h \in H\} \subseteq H_{par}$ such that $H_{par} \cong \underline{A\#H}$.*

Corollary 3.5.6. *Let H be a Hopf algebra, then there is an isomorphism between the category of partial H-modules ${}_H\mathcal{M}^{par}$ and the category of H_{par}-modules ${}_{H_{par}}\mathcal{M}$*

$$ {}_H\mathcal{M}^{par} \cong {}_{H_{par}}\mathcal{M} $$

These categories are both equivalent to the category of left $\underline{A\#H}$-modules.

Moreover, as H_{par} has a structure of a Hopf algebroid over A, ${}_H\mathcal{M}^{par}$ is a closed monoidal category that admitting a strict monoidal functor which preserves internal homs

$$ U : {}_H\mathcal{M}^{par} \to {}_A\mathcal{M}_A. $$

Lemma 3.5.7. *Let B be an algebra object in the monoidal category ${}_H\mathcal{M}^{par}$, then H acts partially on B with a symmetric action.*

Lemma 3.5.8. *Let B be a k-algebra on which H acts partially with a symmetric action, then B is an algebra object in the monoidal category $({}_H\mathcal{M}^{par}, \otimes_A, A)$.*

These lemmas lead immediately to the following theorem.

Theorem 3.5.9. *There is an isomorphism between the category of symmetric partial H-module algebras and the algebra objects in the category of partial H-modules, i.e.* $\mathsf{ParAct}_H \cong \mathsf{Alg}({}_{H_{par}}\mathcal{M})$.

CHAPTER 4

Geometric partial comodules over a coalgbra

This chapter is the heart of the thesis. We will introduce 3 kinds of partial comodules over arbitrary coalgebras in monoidal categories, and study their basic properties such as coassociativity and the completeness of the category of geometric partial comodules. These results will be published in [**33**].

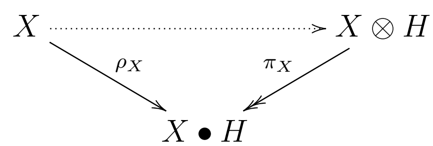
Let $\mathcal{C}$ be a braided monoidal category with pullbacks that are preserved by all endofunctors on $\mathcal{C}$ of the form $-\otimes X$ and $X\otimes -$. Then the observations from Chapter 2 allow us to define partial actions of a Hopf algebra in $\mathcal{C}$ on any object in $\mathcal{C}$ such that taking $\mathcal{C} = \underline{\underline{\mathsf{Set}}}$ we recover the classical definition of partial actions of groups on sets. Since we will rather be interested in examples inspired by algebraic geometry, hence in coactions rather than actions, we will take a dual point of view and consider from now on a braided monoidal category $\mathcal{C}$ with pushouts that are preserved by all endofunctors of the form $-\otimes X$ and $X\otimes -$, and Hopf algebras mentioned below are Hopf algebras in $\mathcal{C}$. Remark that in such a category, the tensor product of two epimorphisms is an epimorphism. Since pushouts are colimits, any braided closed monoidal category will serve as an example, in particular any category of modules over a commutative ring k. In what follows the latter will be our standard example, and we in fact mostly will restrict to the case where k is a field. Inspired by this example we will denote the unit of the monoidal category $\mathcal{C}$ by k.

4.1. Geometrically partial comodules

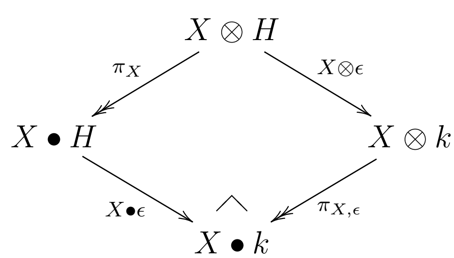
In [**9**], the notion of a "partial module" over a Hopf algebra H was introduced, by means of partial representations of Hopf algebras. In this section, we introduce alternative notions of partial (co)module over any (co)algebra. To prevent a clash of terminologies in case $C = H$, we will call our notions (in rising order of generality) *quasi*, *lax* and *geometric* partial (co)modules. We show that in the case of Hopf algebras, the partial modules of [**9**], and in particular, the partial actions of [**21**], appear as special cases of our quasi partial comodules. Examples arising from (usual) partial actions of (algebraic) groups on (algebraic) sets give rise to geometric partial comodules.

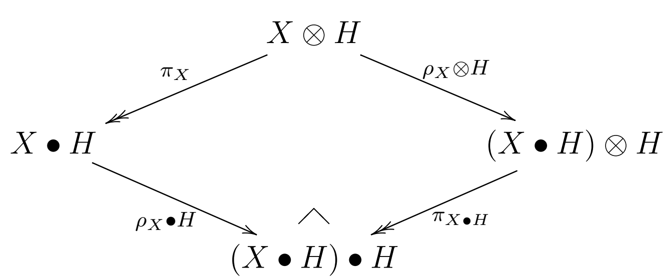
Definition 4.1.1. Let (H, Δ, ϵ) be a coalgebra in a monoidal category $\mathcal{C}$. A *partial comodule datum* is a quadruple $\underline{X} = (X, X \bullet H, \pi_X, \rho_X)$, where X and $X \bullet H$ are objects in $\mathcal{C}$, $\pi_X : X \otimes H \twoheadrightarrow X \bullet H$ is an epimorphism and $\rho_X : X \to X \bullet H$ is a morphism in $\mathcal{C}$.

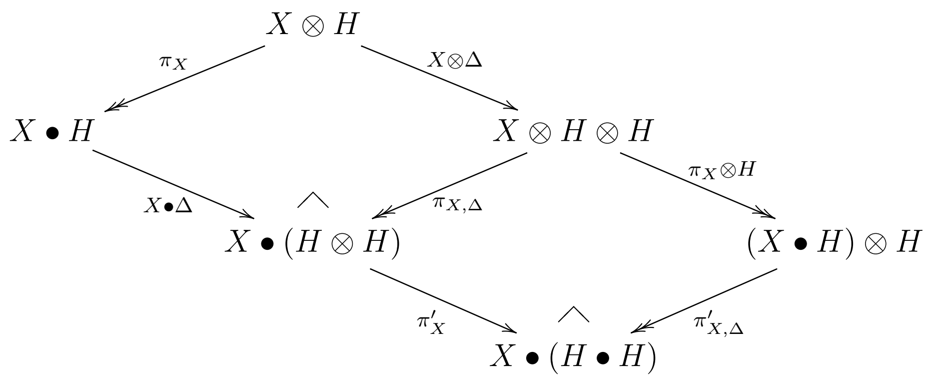
Remark that a partial comodule datum can be viewed as the following cospan in $\mathcal{C}$



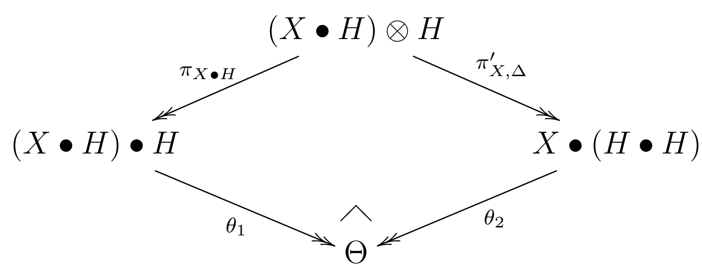
Suppose now that the category $\mathcal{C}$ has pushouts. Then any partial comodule datum induces canonically four pushouts, that we denote by $X \bullet k$, $(X \bullet H) \bullet H$, $X \bullet (H \otimes H)$ and $(X \bullet H) \bullet H$, and that are defined respectively by the following diagrams:







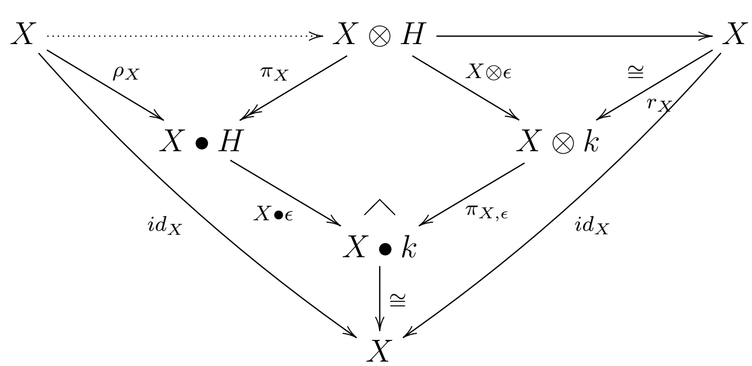
Finally, we consider a last pushout that we denote as Θ and that is given by the following diagram



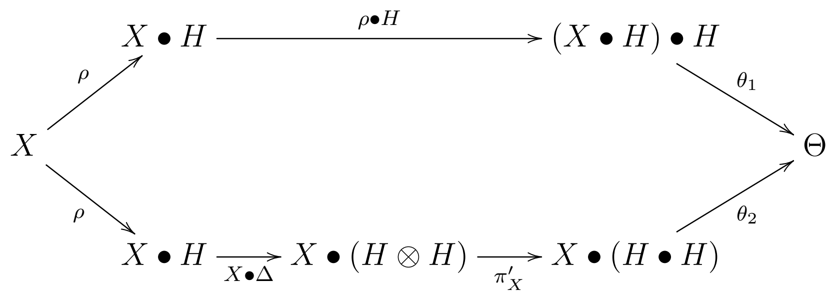
We are now ready to state the exact definitions of a partial comodule.

Definition 4.1.2. Let (H, Δ, ϵ) be a coalgebra in a monoidal category with pushouts $\mathcal{C}$. A *quasi partial comodule* is a partial comodule datum $(X, X \bullet H, \pi_X, \rho_X)$ that satisfies the following conditions

[QPC1] $(X \bullet \epsilon) \circ \rho_X = \pi_{X,\epsilon} \circ r_X : X \to X \bullet k$ are identical isomorphisms. I.e. the following diagram commutes



[QPC2] $\theta_1 \circ (\rho_X \bullet H) \circ \rho_X = \theta_2 \circ \pi'_X \circ (X \bullet \Delta) \circ \rho_X$, i.e. the following diagram commutes



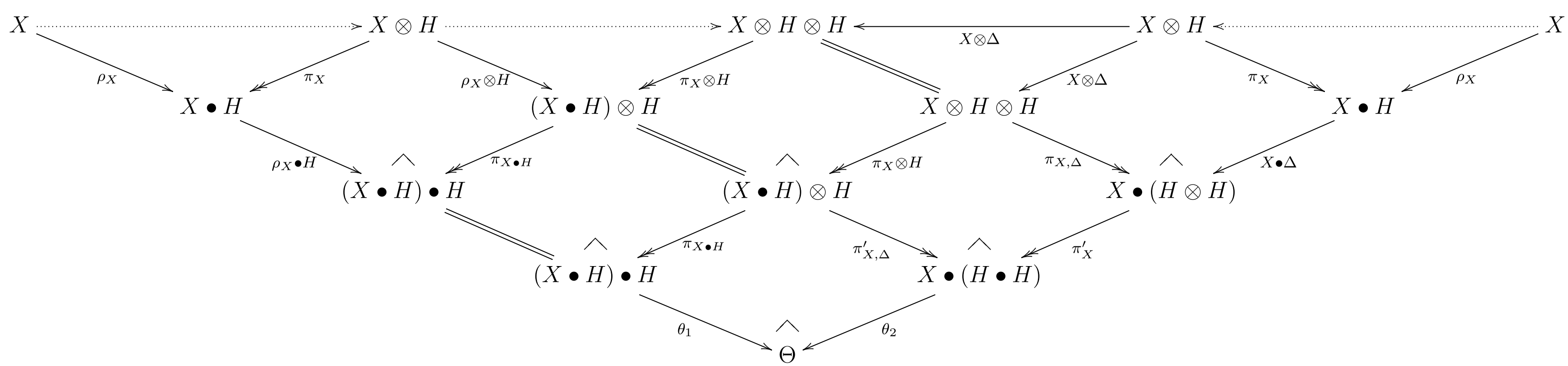
A quasi partial comodule will be called a *lax partial comodule* when the cospan $\Theta_X : X \bullet (H \bullet H) \dashrightarrow (X \bullet H) \bullet H$ is induced by a morphism θ (in $\mathcal{C}$). Furthermore a lax partial comodule is called a *geometric partial comodule* if θ is an isomorphism.

Remarks 4.1.3. (1) The property [QPC2] can be expressed in a larger diagram using the composition of spans, as one can see on the next page.

(2) Remark that by uniqueness of colimits, the pushout $\Theta_X = (\Theta, \theta_1, \theta_2)$ is unique up to isomorphism and hence is not part of the structure of a quasi partial comodule. Similarly, if Θ_X is induced by a morphism θ_X, then this morphism is uniquely determined by its property $\theta_X \circ \pi_{X \bullet H} = \pi'_{X,\Delta}$ since $\pi_{X \bullet H}$ is an epimorphism. Also, whenever there exists a morphism θ_X with this property, then $\Theta \cong (X \bullet H) \bullet H$.

(3) We will often denote a (quasi) partial comodule by (X, π_X, ρ_X) or just by X.

(4) Of course, one can state dual definitions of a *quasi*, *lax* and *geometric partial module* over an algebra. We leave the details to the reader, it suffices to apply the above definition to the opposite category $\mathcal{C}^{op}$.



X
$X \otimes H$
$X \otimes H \otimes H$
$X \otimes H$
X
$X \bullet H$
$(X \bullet H) \otimes H$
$X \otimes H \otimes H$
$X \bullet H$
$(X \bullet H) \bullet H$
$(X \bullet H) \otimes H$
$X \bullet (H \otimes H)$
$(X \bullet H) \bullet H$
$X \bullet (H \bullet H)$
Θ
$X \otimes \Delta$
ρ_X
π_X
$\rho_{X \otimes H}$
$\pi_{X \otimes H}$
$X \otimes \Delta$
π_X
ρ_X
$\rho_{X \bullet H}$
$\pi_{X \bullet H}$
$\pi_{X \otimes H}$
$\pi_{X,\Delta}$
$X \bullet \Delta$
$\pi_{X \bullet H}$
$\pi'_{X,\Delta}$
π'_X
θ_1
θ_2

(5) When working in the base category $\mathcal{C} = \mathcal{M}_k$ We will sometimes use the following Sweedler notation for quasi partial comodules. For any $x \otimes h \in X \otimes H$, we write $\pi_X(x \otimes h) = x \bullet h \in X \bullet H$. Remark that $X \bullet H$ is no longer a tensor product (see below for an interpretation of $X \bullet H$ as a monoidal product when H is a bialgebra). Hence $x \bullet h$ represents a certain class of tensors in $X \otimes H$ and by the surjectivity of π_X, any element of $X \bullet H$ can be represented in such a way, although non-uniquely. We then write $\rho(x) = x_{[0]} \bullet x_{[1]}$, which means that there exists an element $x_{[0]} \otimes x_{[1]} \in X \otimes H$ such that $\rho(x) = \pi(x_{[0]} \otimes x_{[1]})$. Again, the element $x_{[0]} \otimes x_{[1]} \in X \otimes H$ is not unique, so some care is needed in this notation. However, the class $x_{[0]} \bullet x_{[1]} \in X \bullet H$ is well-defined since ρ_X is a proper map. Axiom [PPC1] tells us then that, as for usual coactions, $x_{[0]}\epsilon(x_{[1]}) = x$ for all $x \in X$, and in particular this expression makes sense. We will treat axiom [PPC2] in a similar way by the expression

$$x_{[0][0]} \bullet x_{[0][1]} \bullet x_{[1]} = x_{[0]} \bullet x_{[1](1)} \bullet x_{[1](2)}$$

However, this expression now holds in the pushout Θ, and by definition, the left hand side in the above expression is the notation for $\theta_1 \circ (\rho_X \bullet H) \circ \rho_X(x)$ and the right hand side is $\theta_2 \circ \pi'_X \circ (X \bullet \Delta) \circ \rho_X(x)$, for the same $x \in X$.

A first class of examples is obtained from the results of the previous section by taking $\mathcal{C} = \underline{\underline{\mathsf{Set}}}^{op}$. Indeed, quasi, lax and (usual) partial actions of a group coincide in this way with quasi, lax and geometric partial (co)modules. Remark that in the above formation, these notions also allow to consider partial actions of arbitrary monoids rather than groups.

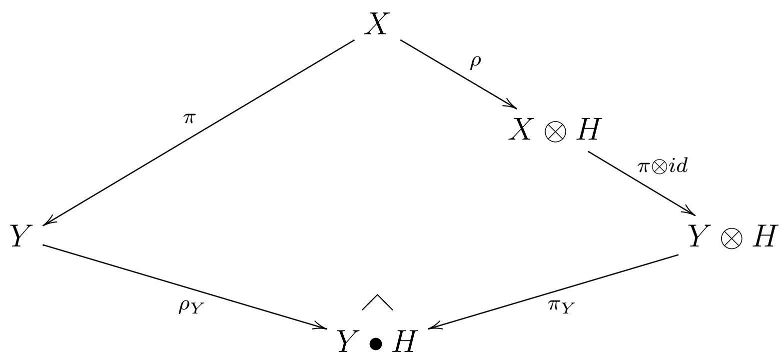
Before we give some more examples, let us first state the following (well-known) lemma that will be useful for our purposes.

Lemma 4.1.4. *Consider vector spaces U, V, W and linear maps $f : U \to V$, $g : U \to W$, where g is surjective. Then the pushout of the pair (f, g) is given by $P = V/f(\mathsf{Ker}\, g)$, where $\overline{g} : V \to P$ is the canonical surjection and $\overline{f} : W \to P$ is given by $\overline{f}(w) = f(u)$, where u is any element of U such that $g(u) = w \in W$.*

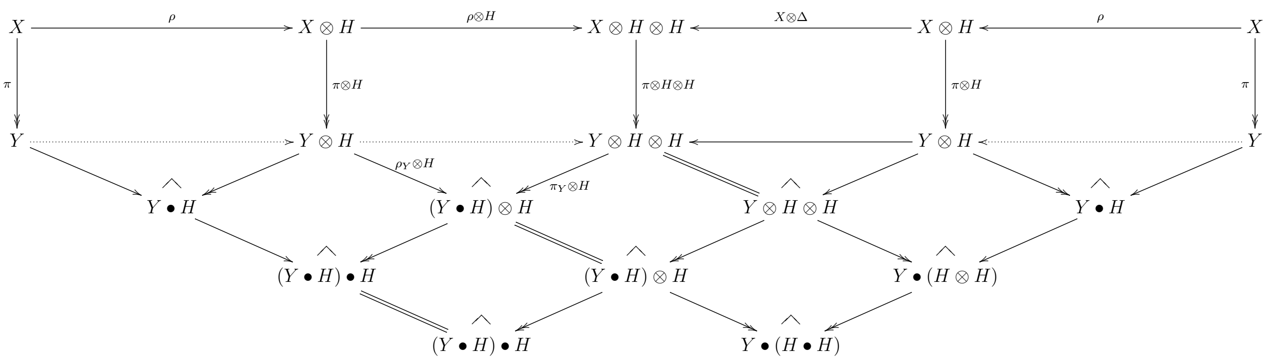
Let us now provide some examples.

Example 4.1.5 (Quotient of a global comodule)**.** Consider a global H-comodule X with coaction $\rho : X \to X \otimes H$ and any epimorphism

$\pi : X \to Y$ in $\mathcal{C}$. Then we can define a partial comodule datum over Y by taking the pushout of the pair $(\pi, (\pi \otimes H) \circ \rho)$



Considering the diagram



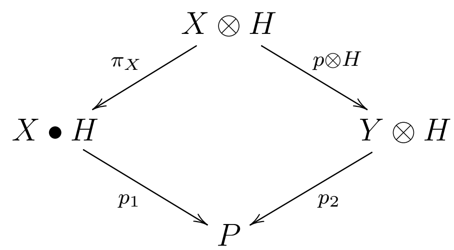
By composing pushouts in the diagram, we see that $(Y \bullet H) \bullet H$ is the pushout of the pair $(\pi, (\rho_Y \otimes H) \circ (\pi \otimes H) \circ \rho)$. Moreover, diagram chasing and the coassociativity of (X, ρ) tells us that

$$\begin{aligned}(\rho_Y \otimes H) \circ (\pi \otimes H) \circ \rho &= (\pi_Y \otimes H) \circ (\pi \otimes H \otimes H) \circ (\rho \otimes H) \circ \rho \\ &= (\pi_Y \otimes H) \circ (\pi \otimes H \otimes H) \circ (X \otimes \Delta) \circ \rho \\ &= (\rho_Y \otimes H) \circ (Y \otimes \Delta) \circ (\pi \otimes H) \circ \rho\end{aligned}$$

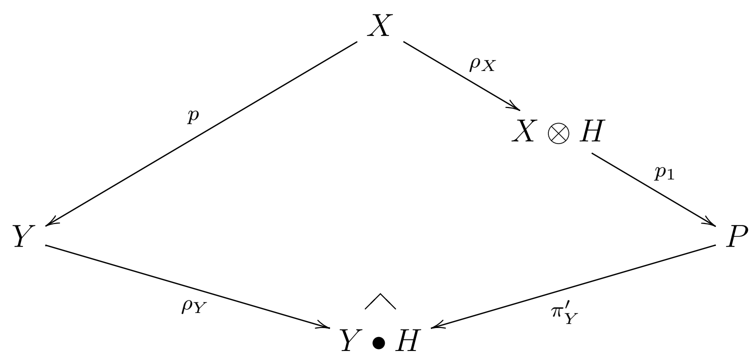
And hence $(Y \bullet H) \bullet H$ has to be isomorphic to $Y \bullet (H \bullet H)$, which is exactly the pushout of $(\pi, (\rho_Y \otimes H) \circ (Y \otimes \Delta) \circ (\pi \otimes H) \circ \rho)$. We can conclude that (Y, ρ_Y, π_Y) is a geometric partial comodule.

Performing this construction in $\mathcal{C} = \underline{\underline{\mathsf{Set}}}^{op}$, we recover Example 2.1.1 (1)

Example 4.1.6 (Quotient of a partial comodule)**.** The previous example can be generalized in the following way. Let $(X, X \bullet H, \pi_X, \rho_X)$ be a partial H-comodule datum, and $p : X \to Y$ an epimorphism. Then consider the pushout P of the pair $(\pi_X, p \otimes H)$:



Moreover, we can then define a partial comodule datum $(Y, Y \bullet H, \pi_Y, \rho_Y)$ by considering the following pushout



and taking $\pi_Y = \pi'_Y \circ p_2$. Similar to the previous example, one can show that Y is a quasi or geometric partial comodule if X is so.

Example 4.1.7 (Partial action in the affine plane)**.** Since the affine group $(\mathbb{A}^2, +)$ acts strictly transitive on the affine plane, the algebra $A = k[x, y]$ is a Galois object over the bialgebra $H = k[x, y]$. In particular, A is an H-comodule with coaction $\rho : k[x, y] \to k[x, y] \otimes k[x, y] \cong k[x, y, x', y']$, $\rho(f)(x, y, x', y') = f(x+x', y+y')$ where $f \in k[x, y]$. Considering the quotient $B = k[x, y]/(xy)$ we find by the previous example that B is a partial H-comodule with $B \bullet H = k[x, y, x', y']/\rho((xy))$. Remark that $\rho((xy))$ is not an ideal in $k[x, y, x', y']$, hence $B \bullet H$ is not an algebra quotient of $B \otimes H$. Furthermore, $\rho_B : B \to B \bullet H$ given by $\rho_B(\overline{f})(x, y, x', y') = \overline{f}(x + x', y + y')$ for all $\overline{f} \in B$.

Example 4.1.8 (A partial action on the quantum plane)**.** By a similar construction as in the previous example, we obtain a partial action on the quantum plane. Consider the tensor algebra $T(V)$ where V is a 2-dimensional vector space. Then this tensor algebra is known to be a Hopf algebra and it coacts on itself by the comultiplication. We can view $T(V)$ as the free algebra $k\langle x, y\rangle$ with two generators x, y

and the coaction is then given by the comultiplication $\Delta : k\langle x, y\rangle \to k\langle x, y\rangle \otimes k\langle x, y\rangle$, $\Delta(x) = x\otimes 1+1\otimes x, \Delta(y) = y\otimes 1+1\otimes y$. Now consider the quantum plane $k_q[x, y] = k\langle x, y\rangle / (xy - qyx)$. By Example 4.1.5, the quantum plane is a partial comodule over the tensor algebra.

Example 4.1.9. Consider a partial coaction in the sense of Caenepeel-Janssen [**21**]. This means that H is a Hopf algebra, A is an algebra and

$$\rho : A \to A \otimes H, \rho(a) = a_{[0]} \otimes a_{[1]}$$

is a linear map satisfying the following axioms:

(CJ1) $(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}$
(CJ2) $a_{[0][0]} \otimes a_{[0][1]} \otimes a_{[1]} = a_{[0]}1_{[0]} \otimes a_{[1](1)}1_{[1]} \otimes a_{[1](2)}$
(CJ3) $a_{[0]}\epsilon(a_{[1]}) = a$

Then we define $e = 1_{[0]} \otimes 1_{[1]} \in A \otimes H$, which is an idempotent, because of the first axiom. Then we get that

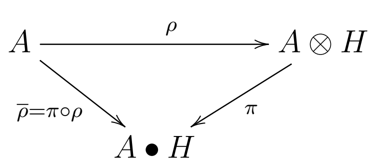
$$A \otimes H = (A \otimes H)e \oplus (A \otimes H)e'$$

where $e' = 1 - e$. If we put $A \bullet H = (A \otimes H)e$, then we have that the map

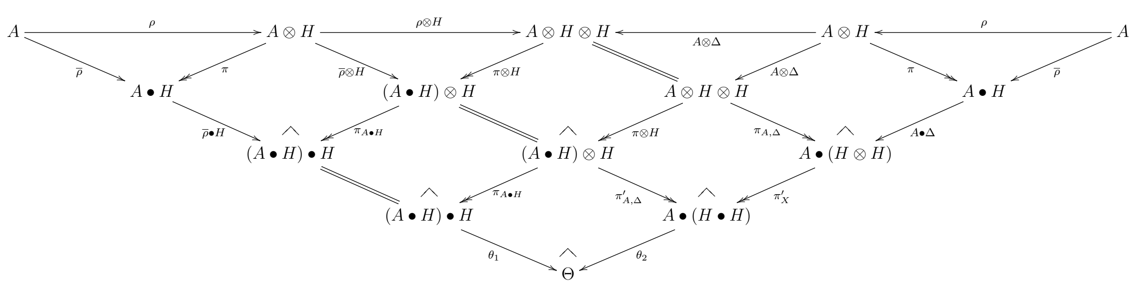
$$\pi : A \otimes H \to A \bullet H, a \otimes h \mapsto a1_{[0]} \otimes h1_{[1]}$$

is surjective with right inverse the inclusion map and kernel $(A\otimes H)e' = \{a \otimes h - a1_{[0]} \otimes h1_{[1]} \mid a \otimes h \in A \otimes H\} = N$ and $A \bullet H = (A \otimes H)/N$.

This allows us to define the partial action datum over A:



To check the coassociativity, we consider the diagram



Using Lemma 4.1.4, we find that $(A \bullet H) \bullet H \cong ((A \bullet H) \otimes H)/K$, $A \bullet (H \bullet H) = ((A \bullet H) \otimes H)/L$ and $\Theta = ((A \bullet H) \otimes H)/(K + L)$ where

$$K = \{a_{[0]} \otimes a_{[1]} \otimes h - a_{[0]} 1_{[0][0]} \otimes a_{[1]} 1_{[0][1]} \otimes h 1_{[1]} | a \otimes h \in A \otimes H\}$$

and

$$L = \{a 1_{[0]} \otimes h_{(1)} 1_{[1]} \otimes h_{(2)} - a 1_{[0]} 1_{[0']} \otimes h_{(1)} 1_{[1](1)} 1_{[1']} \otimes h_{(2)} 1_{[1](2)} | a \otimes h \in A \otimes H\}$$

Although in general K and L are not necessarily isomorphic subspaces of $(A \bullet H) \otimes H$, we see because of axiom (CJ2) that $(\pi \otimes H) \circ (\rho \otimes H) \circ \rho(a) = (\pi \otimes H) \circ (A \otimes \Delta) \circ \rho(a)$ in $(A \bullet H) \otimes H$. Hence the coassociativity holds in particular in the quotient Θ. We can conlude that a Caenepeel-Janssen partial action induces a quasi (and not geometric) partial comodule.

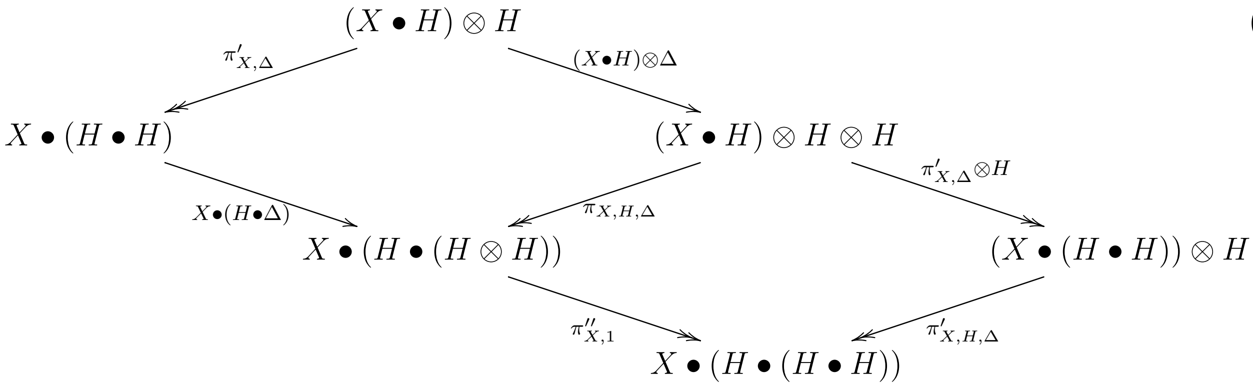
Example 4.1.10. In a similar way as the previous example, partial corepresentations [**11**] also lead to examples of quasi partial comodules.

Example 4.1.11. Let $(X, X \bullet H, \pi_X, \rho_X)$ be a quasi partial H-comodule. We know that $X \otimes H$ is a (global) right H-comodule with coaction $X \otimes \Delta$. By applying the result of Example 4.1.5, we find that the epimorphism $\pi_X : X \otimes H \to X \bullet H$ induces $X \bullet H$ with the structure of a geometric partial H-comodule under the partial coaction

$$X \bullet H \xrightarrow{\pi'_X \circ X \bullet \Delta} X \bullet (H \bullet H) \xleftarrow{\pi'_{X,\Delta}} (X \bullet H) \otimes H$$

which is geometric by Example 4.1.5. Therefore, we obtain that the following pushouts are isomorphic, where we denote $\overline{X \bullet \Delta} = \pi'_X \circ X \bullet \Delta$.

$$\begin{array}{ccc} & (X \bullet H) \otimes H & \\ \swarrow^{\pi'_{X,\Delta}} & & \searrow^{\overline{X \bullet \Delta} \otimes H} \\ X \bullet (H \bullet H) & & (X \bullet (H \bullet H)) \otimes H \\ \searrow_{\overline{X \bullet (\Delta \bullet H)}} & & \swarrow_{\pi'_{X,\Delta,H}} \\ & X \bullet ((H \bullet H)) \bullet H) & \end{array} \tag{4.1}$$



(4.2)

Denote $\overline{(X \bullet (H \bullet \Delta))} = \pi''_{X,1} \circ (X \bullet (H \bullet \Delta))$. Then we find that the following morphisms are identical up-to-isomorphism of their codomains.

$$\overline{(X \bullet (H \bullet \Delta))} \circ \overline{(X \bullet \Delta)} \simeq \overline{(X \bullet (\Delta \bullet H))} \circ \overline{(X \bullet \Delta)}$$

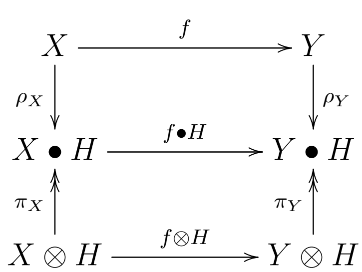
When X itself is a geometric partial comodule, one can use the isomorphism $\theta : X \bullet (H \bullet H) \cong (X \bullet H) \bullet H$ to rewrite the above pushouts as

$$\begin{aligned} (X \bullet (H \bullet H)) \bullet H &\cong X \bullet ((H \bullet H)) \bullet H) \\ \cong (X \bullet H) \bullet (H \bullet H) &\cong X \bullet (H \bullet (H \bullet H)) \end{aligned}$$

we will explain this in more detail in Section 4.3.

4.2. Partial comodule morphisms

Definition 4.2.1. If (X, π_X, ρ_X) and (Y, π_Y, ρ_Y) are two partial H-comodule data, then a *morphism* of partial H-comodule data is a couple $(f, f \bullet H)$ of morphisms in $\mathcal{C}$, where $f : X \to Y$ and $f \bullet H : X \bullet H \to Y \bullet H$ such that the following two squares commute



A morphism of a quasi, lax or geometric partial comodule is a morphism of the underlying partial comodule data. We denote the categories of quasi, lax and geometric partial H-comodules respectively by qPMod^H,

lPMod^H and gPMod^H. When we denote PMod^H, we mean any of the three partial comodule categories, without specifying which one.

If H is an algebra in $\mathcal{C}$, then H is a coalgebra in $\mathcal{C}^{op}$ and one defines the categories of partial modules as the opposite of the corresponding categories of partial comodules over the coalgebra H in $\mathcal{C}^{op}$

Remark 4.2.2. If $(f, f \bullet H)$ is a morphism of partial comodule data, then $f \bullet H$ is determined by f. Indeed, suppose that both $(f, f \bullet H), (g, g \bullet H) : X \to Y$ are morphisms of comodule data with $f = g$, then using the fact that π_X is an epimorphism, it follows that $f \bullet H = g \bullet H$. This justifies that from now on we will denote a partial morphism $(f, f \bullet H)$ just as f.

If moreover π_X is a regular epimorphism (that is, it is a coequalizer) in $\mathcal{C}$, then one can express the property of the existence of $f \bullet H$ more explicitly. We spell this out in the abelian case (where all epimorphisms are regular) in the next lemma.

Lemma 4.2.3. *Suppose that the category $\mathcal{C}$ is abelian. Let (X, π_X, ρ_X) and (Y, π_Y, ρ_Y) be partial H-comodule data in $\mathcal{C}$. If a morphism $f : X \to Y$ satisfies $(f \otimes H)(\mathsf{Ker}\, \pi_X) \subset \mathsf{Ker}\, \pi_Y$, then there exists a unique morphism $f \bullet H : X \bullet H \to Y \bullet H$ such that $\pi_Y \circ (f \otimes H) = (f \bullet H) \circ \pi_X$.*

Proof. The existence and uniqueness of $f \bullet H$ follows directly by universal property of $(X \bullet H, \pi_X) = \mathsf{coker}\,(\mathsf{Ker}\,(\pi_X))$ in the abelian category $\mathcal{C}$. □

Remark 4.2.4. In case $\mathcal{C} = \mathsf{Vect}$, one then finds that a map $f : X \to Y$ between two geometric partial modules is a morphism of partial comodules if and only if the following conditions hold:

(1) $f(x) \bullet h = 0$ if $x \bullet h = 0$;
(2) $f(x_{[0]}) \bullet x_{[1]} = f(x)_{[0]} \bullet f(x)_{[1]}$;

where we used the notation introduced in Remark 4.1.3, and where the second condition make sense thanks to the first one.

Lemma 4.2.5. *If $f : (X, \pi_X, \rho_X, \theta_X) \to (Y, \pi_Y, \rho_Y, \theta_Y)$ is a morphism of quasi partial H-comodules, then there exist unique morphisms $(f \bullet$*

$H) \bullet H$, $f \bullet (H \bullet H)$ and θ_f such that the following diagrams commute

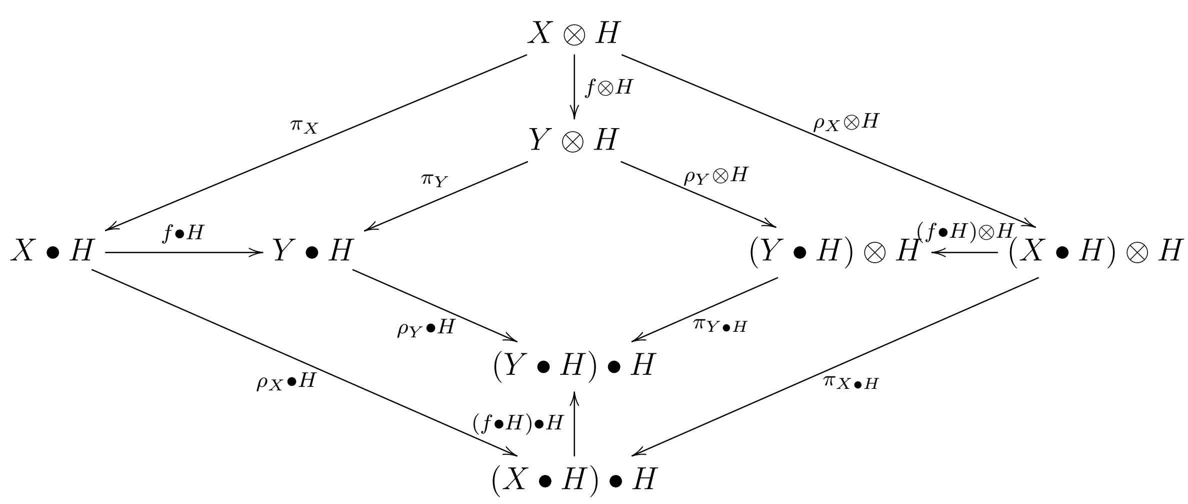
$$\begin{CD}
X \bullet H @>{\rho_X \bullet H}>> (X \bullet H) \bullet H @<{\pi_{X \bullet H}}<< (X \bullet H) \otimes H\\
@V{f \bullet H}VV @VV{(f \bullet H) \bullet H}V @VV{(f \bullet H) \otimes H}V\\
Y \bullet H @>{\rho_Y \bullet H}>> (Y \bullet H) \bullet H @<{\pi_{Y \bullet H}}<< (Y \bullet H) \otimes H
\end{CD}
\qquad
\begin{CD}
X \bullet H @>{\pi'_X \circ (X \bullet \Delta)}>> X \bullet (H \bullet H) @<{\pi_{X \bullet H}}<< (X \bullet H) \otimes H\\
@V{f \bullet H}VV @VV{f \bullet (H \bullet H)}V @VV{(f \bullet H) \otimes H}V\\
Y \bullet H @>{\pi'_Y \circ (Y \bullet \Delta)}>> Y \bullet (H \bullet H) @<{\pi_{Y \bullet H}}<< (Y \bullet H) \otimes H
\end{CD}$$

$$\begin{CD}
(X \bullet H) \bullet H @>{\theta_1^X}>> \Theta_X @<{\theta_2^X}<< X \bullet (H \bullet H)\\
@V{(f \bullet H) \bullet H}VV @VV{\theta_f}V @VV{f \bullet (H \otimes H)}V\\
(Y \bullet H) \bullet H @>{\theta_1^Y}>> \Theta_Y @<{\theta_2^Y}<< Y \bullet (H \otimes H)
\end{CD}$$

If moreover X and Y are lax, then the following diagram commutes as well.

$$\begin{CD}
(X \bullet H) \bullet H @>{(f \bullet H) \bullet H}>> (Y \bullet H) \bullet H\\
@V{\theta_X}VV @VV{\theta_Y}V\\
X \bullet (H \bullet H) @>{f \bullet (H \bullet H)}>> Y \bullet (H \bullet H)
\end{CD}$$

PROOF. This follows by the universal property of the considered pushouts. For example, $(f \bullet H) \bullet H : (X \bullet H) \bullet H \to (Y \bullet H) \bullet H$ is defined as the unique morphism that makes the following diagrams commute, where the inner and outer diamond are pushouts



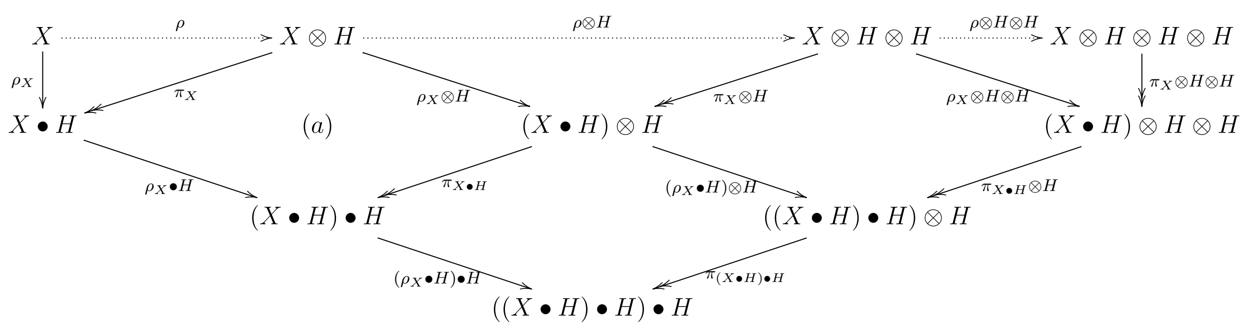
$\square$

4.3. Coassociativity

For a usual H-comodule (M, ρ), it is well-known that the coassociativity condition implies a generalized coassociativity condition saying that all morphisms from M to $M \otimes H^{\otimes n}$ that is constructed out of a combination of ρ, Δ and identity maps are identical. Our next aim is to prove a similar theorem for partial comodules. To this end, let consider the following compositions of partial mappings from X to $X \otimes H \otimes H \otimes H$. Let us first construct

$$\rho^1 = ((\rho \bullet H) \bullet H) \circ (\rho \bullet H) \circ \rho : X \to ((X \bullet H) \bullet H) \bullet H$$

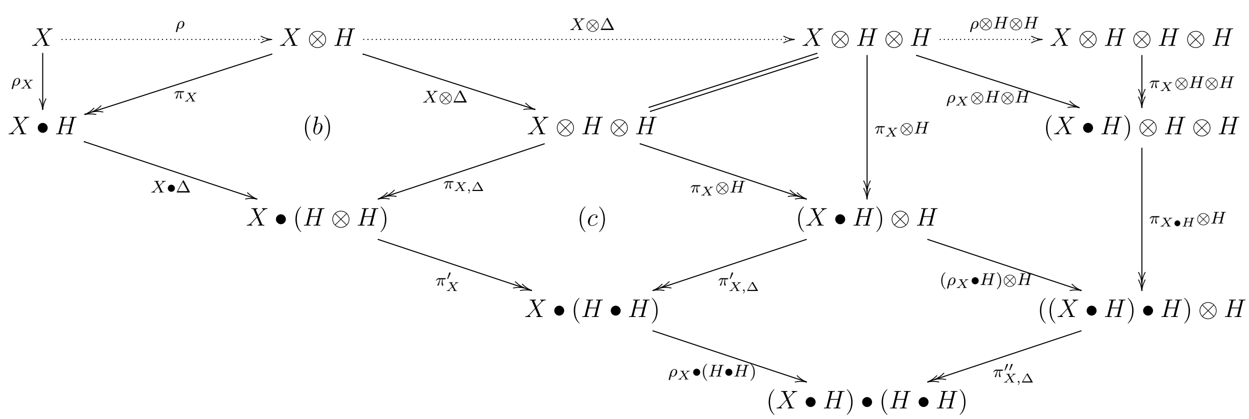
which is done in the following diagram, where all quadrangles are pushouts.



In the same way, we can construct

$$\rho^2 : ((\rho \bullet H) \bullet H) \circ \overline{(X \bullet \Delta)} \circ \rho : X \to (X \bullet H) \bullet (H \bullet H),$$

where we denote as before $\overline{X \bullet \Delta} = \pi'_X \circ (X \bullet \Delta)$ and which is defined by the following diagram.

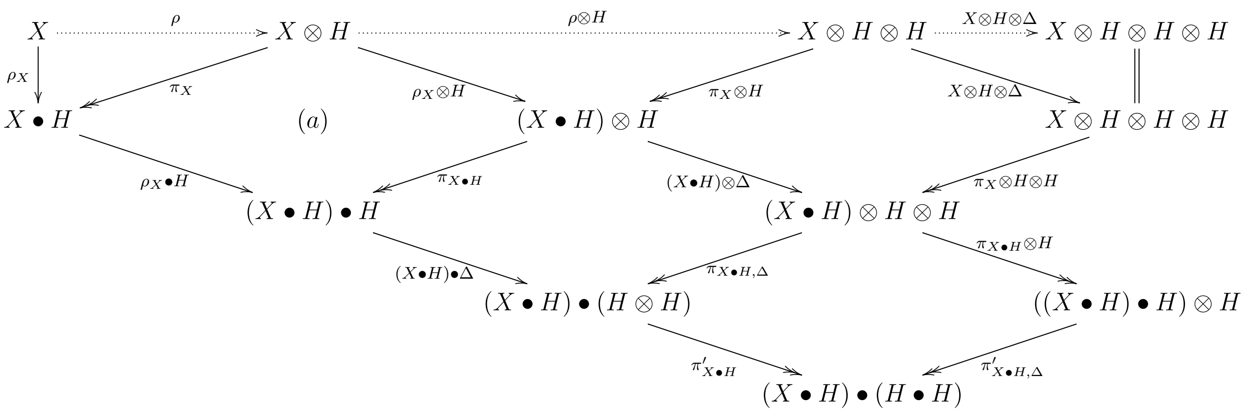


Since we know by the coassociativity on X that the pushouts $(X \bullet H) \bullet H$ given by the diagram (a) is isomorphic to the pushout $X \bullet (H \bullet H)$ which is the combinination of diagrams (b) and (c). Therefore it follows that the pushouts $((X \bullet H) \bullet H) \bullet H$ and $(X \bullet H) \bullet (H \bullet H)$ constructed above are isomorphic as well, in such a way that the constructed maps ρ^1 and ρ^2 from X into these pushouts are identical up to this isomorphism.

Next, we construct a morphism

$$\rho^3 : \overline{((X \bullet H) \bullet \Delta)} \circ (\rho \bullet H) \circ \rho : X \to (X \bullet H) \bullet (H \bullet H)$$

denoting $\overline{((X \bullet H) \bullet \Delta)} = \pi'_{X \bullet H} \circ ((X \bullet H) \bullet \Delta)$, as in the following diagram.



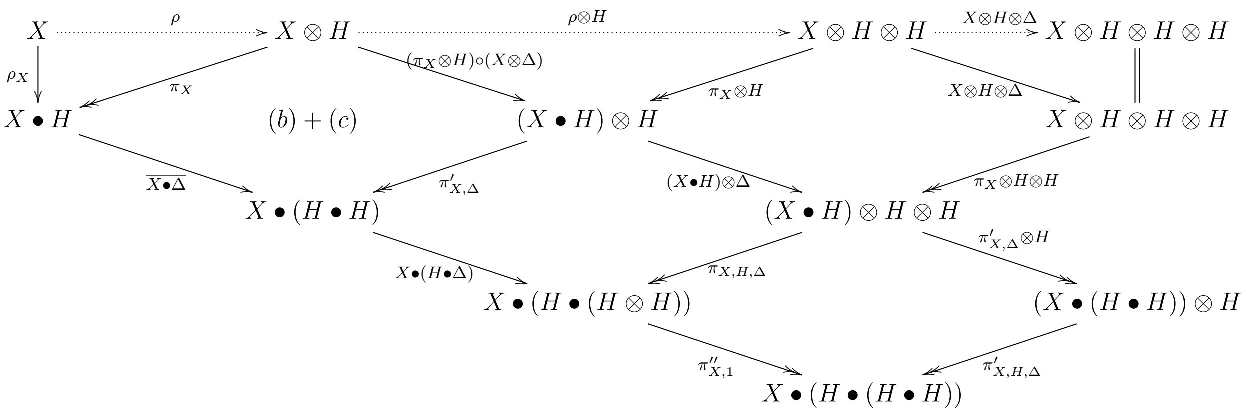
Let us first remark that the constructed pushout is the same as the one from the previous diagram. Indeed, we had constructed $(X \bullet H) \bullet (H \bullet H)$ as the pushout of π_X with

$$\begin{aligned} & ((\rho_X \bullet H) \otimes H) \circ (\pi_X \otimes H) \circ (X \otimes \Delta) \\ = \; & (\pi_{X \bullet H} \otimes H) \circ (\rho_X \otimes H \otimes H) \circ (X \otimes \Delta) \\ = \; & (\pi_{X \bullet H} \otimes H) \circ ((X \bullet H) \otimes \Delta) \circ (\rho_X \otimes H) \end{aligned}$$

It follows that the morphism ρ^3 is identical to ρ^2 (and to ρ^1).

Furthermore, one sees that the pushout (a) appears again in the last diagram, by a same argument as before, this can be replaced by the combination of the pushouts (b) and (c), since $\theta_X : X \bullet (H \bullet H) \to (X \bullet H) \bullet H$ is an isomorhpism. This leads us to the map

$$\rho^4 : \overline{((X \bullet H) \bullet \Delta)} \circ \overline{X \bullet \Delta} \circ \rho : X \to X \bullet (H \bullet (H \bullet H))$$

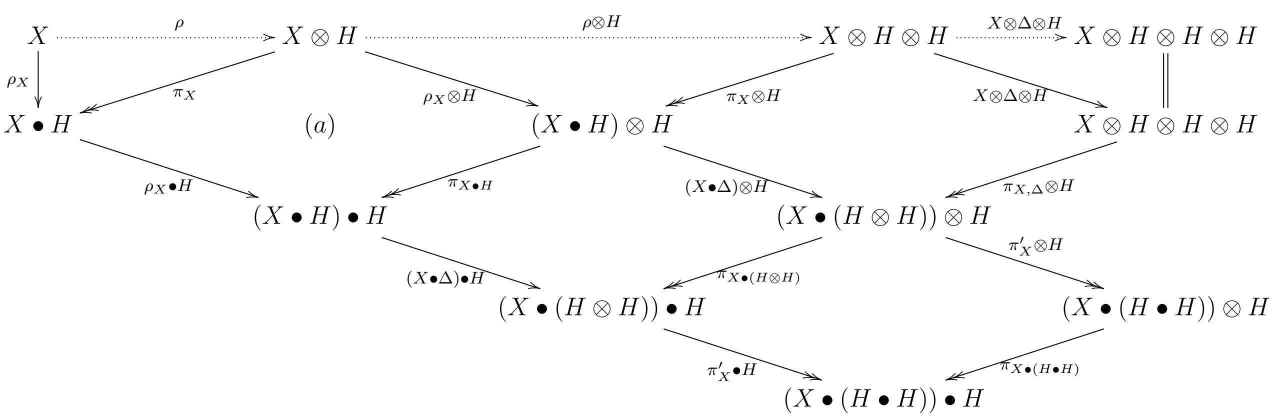


Remark that $(X \bullet H) \bullet (H \bullet H)$ is the pushout of the pair $(\pi_{X \bullet H}, (\pi_{X \bullet H} \otimes H) \circ ((X \bullet H) \otimes \Delta)$ and $X \bullet (H \bullet (H \bullet H))$ is the pushout of the pair $(\pi'_{X,\Delta}, (\pi'_{X,\Delta} \otimes H) \circ ((X \bullet H) \otimes \Delta)$. Since $\pi_{X \bullet H} = \theta_X \circ \pi'_{X,\Delta}$ and θ_X is an isomorphism, it follows that both pushouts are isomorphic and ϕ^3 and ϕ^4 are identical up to this isomorphism.

Let us now consider the morphism

$$\rho^5 : \overline{((X \bullet \Delta) \bullet H)} \circ (\rho \bullet H) \circ \rho : X \to (X \bullet (H \bullet H)) \bullet H$$

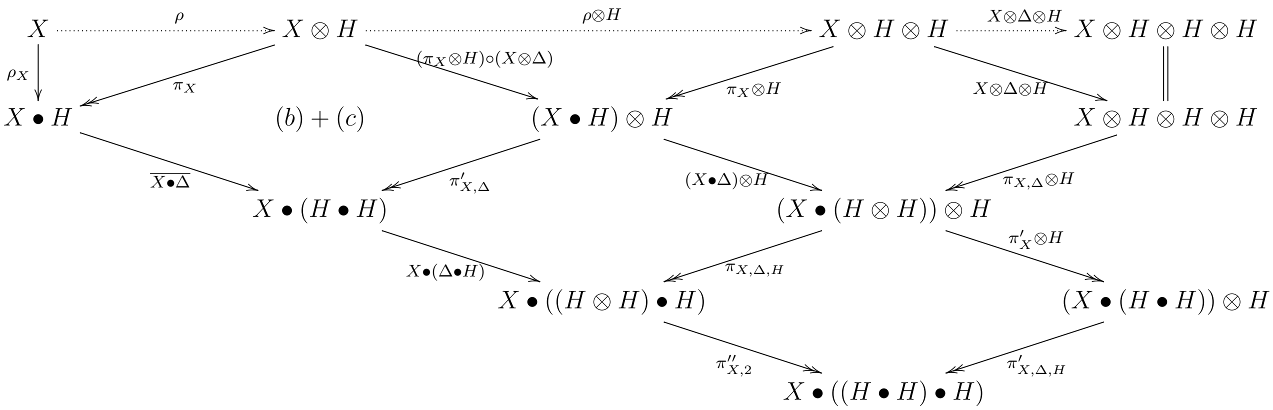
where $\overline{((X \bullet \Delta) \bullet H)} = (\pi'_X \bullet H) \circ ((X \bullet \Delta) \bullet H)$ and that is given by the following diagram



And again, by replacing the pullback (a) by the pullback (b)+(c), we obtain a map that is the same up-to-isomorphism the same as ρ^5:

$$\rho^6 : \overline{(X \bullet (\Delta \bullet H))} \circ \overline{X \bullet \Delta} \circ \rho : X \to X \bullet ((H \bullet H) \bullet H)$$

where $\overline{(X \bullet (\Delta \bullet H))} = \pi''_{X,2} \circ (X \bullet (\Delta \bullet H))$. This map ρ^6 is defined by the following diagram.



By Example 4.1.11, we know that $X \bullet ((H \bullet H)) \bullet H) \cong X \bullet (H \bullet (H \bullet H))$ and the maps ρ^4 and ρ^6 are identical up to this isomorphism.

Hence we have hereby proven that the all above constructed pushouts are isomorphic and the maps ρ^i $(i = 1, \dots, 6)$ are identical up to these isomorphisms. All this is summarized in the following result.

Theorem 4.3.1 (generalized coassociativity). *Let $(X, \pi_X, \rho_X, \theta_X)$ be a geometric partial comodule. Then the pushouts introduced above are all isomorphic*

$$\begin{aligned} X \bullet (H \bullet (H \bullet H)) &\cong (X \bullet H) \bullet (H \bullet H) \\ \cong ((X \bullet H) \bullet H) \bullet H &\cong (X \bullet (H \bullet H)) \bullet H \\ &\cong X \bullet ((H \bullet H) \bullet H) \end{aligned}$$

Moreover up to these isomorphisms, the following morphisms $X \to X \bullet H \bullet H \bullet H$ are identical

$$\begin{aligned} \overline{(X \bullet H \bullet \Delta)} \circ \overline{(X \bullet \Delta)} \circ \rho &\simeq (\rho \bullet H \bullet H) \circ \overline{(X \bullet \Delta)} \circ \rho \simeq \\ \overline{(X \bullet H \bullet \Delta)} \circ (\rho \bullet H) \circ \rho &\simeq (\rho \bullet H \bullet H) \circ (\rho \bullet H) \circ \rho \simeq \\ \overline{(X \bullet \Delta \bullet H)} \circ (\rho \bullet H) \circ \rho &\simeq \overline{(X \bullet \Delta \bullet H)} \circ \overline{(X \bullet \Delta)} \circ \rho \end{aligned}$$

Corollary 4.3.2. *If (X, π_X, ρ_X) is a geometrically partial H-comodule, then $(X \bullet H, (X \bullet H) \bullet H, \pi_{X \bullet H}, \rho_X \bullet H)$ is a geometrically partial H-comodule.*

Corollary 4.3.3. *All higher coassociativity conditions follow now by an induction argument from the previous two results.*

Remarks 4.3.4. (1) A lax version of the above results on generalized coassociativity can be proven in the same way. Indeed, analysing the reasoning at the start of this section, each of the isomorphisms between the pullbacks obtained in Theorem 4.3.1 follows from the isomorphism $\theta : X \bullet (H \bullet H) \to (X \bullet H) \bullet H$ at appropriate places. When θ_X is only assumed to be a morphism (not an isomorphism), then we also obtain only morphisms (and not isomorphisms) between the constructed pullbacks. The coassociativity will then hold up to composition with the induced morphisms onto $((X \bullet H) \bullet H) \bullet H$.

(2) As the isomorphisms between the respective pullbacks are constructed by applying the universal property of the pullback, one can moreover easily see, that these isomorphisms are compatible in a way that the following diagram commutes

$$\begin{array}{ccccc} X \bullet (H \bullet (H \bullet H)) & \longrightarrow & (X \bullet H) \bullet (H \bullet H) & \longrightarrow & ((X \bullet H) \bullet H) \bullet H \\ \downarrow & & & & \uparrow \\ X \bullet ((H \bullet H) \bullet H) & & \longrightarrow & & (X \bullet (H \bullet H)) \bullet H \end{array}$$

where all arrows are isomorphisms in the geometric case, and just morphisms in the lax case. The commutativity of this diagrams seems to suggest that there is an underlying (skew) monoidal structure with tensor product $- \bullet -$. In the next section, we will show that in case H is a bialgebra, there is at least a lax monoidal structure on the category of geometric partial modules, which coincides with the $\bullet$-product that we encountered so far.

4.4. Completeness and cocompleteness of the category of partial comodules

It is known that the category of comodules over a coalgebra over a field is complete and cocomplete, see e.g. [**37**], [**41**]. We will prove the same result for geometric partial comodules, and use for this the approach of [**4**], which relies on the fundamental theorem of comodules, for which we also provide a proof in the partial case.

For global comodules, the forgetful functor $U : \mathsf{Mod}^H \to \mathcal{C}$ allows a right adjoint given by the free functor $- \otimes H : \mathcal{C} \to \mathsf{Mod}^H$. Since every global comodule is also a partial module, the free functor $- \otimes H : \mathcal{C} \to \mathsf{PMod}^H$ still makes sense, however it no longer serves as a right adjoint for the forgetful functor $U : \mathsf{PMod}^H \to \mathcal{C}$, which is defined as $U(X, \rho_X, \pi_X, \theta_X) = X$ on objects and $U(f, f \bullet H) = f$ on morphisms. We now show that the forgetful functor still has a right adjoint.

Proposition 4.4.1. *Let V be any object in $\mathcal{C}$, then V can be endowed with a partial H-comodule structure putting $V \bullet H = V$, $\pi = V \otimes \epsilon_H$ and $\rho = id_V$. We call this the "trivial partial comodule structure" on V.*

Moreover a trivial partial comodule is always geometric and the functor $T : \mathcal{C} \to \mathsf{PMod}^H$ that assigns to each $\mathcal{C}$-object the trivial partial comodule structure, is fully faithful and a right adjoint for the forgetful functor $U : \mathsf{PMod}^H \to \mathcal{C}$.

Proof. It can be easily verified that $(V, V, V \otimes \epsilon_H, \mathbb{1}_V)$ is a geometric partial H-comodule with $(V \bullet H) \bullet H = V \bullet (H \bullet H) = V$.

Given a partial comodule $(X, X \bullet H, \pi_X, \rho_X)$, we find that $TU(X) = (X, X, X \otimes \epsilon_H, id_X)$ and we define the unit of the adjunction as $\eta_X = (id_X, X \bullet \epsilon_H) : X \to TU(X)$. For any object V in $\mathcal{C}$, we see that $UT(V) = V$. Then the unit-counit conditions become trivial. Since the counit is the identity, we obtain that T is fully faithful. □

Since the forgetful functor has a right adjoint, it preserves all colimits that exist in PMod^H. The main aim of this section is to show that colimits and limits indeed exist in PMod^H. Let us first show that thanks to the observation of the previous proposition, the category PMod^H is well-copowered.

Recall that a category is called *well-copowered* if and only if for any object X, there exist up-to-isomorphism only a set of epimorphisms $f : X \to Y$.

Corollary 4.4.2. *A morphism $f \in \mathsf{PMod}^H$ is an epimorphism if and only if $U(f) = f$ is an epimorphism in $\mathcal{C}$. Furthermore, the category PMod^H is well-copowered if $\mathcal{C}$ is so.*

Proof. Since the forgetful functor $U : \mathsf{PMod}^H \to \mathcal{C}$ has a right adjoint, U preserves epimorphisms. Conversely, if $f : X \to Y$ in PMod^H is such that $U(f)$ is an epimorphism, then f is an epimorphism as well. Indeed, suppose that we have $g, h : Y \to Z$ in PMod^H such that $g \circ f = h \circ f$. Then also $U(g) \circ U(f) = U(g) \circ U(f)$ in $\mathcal{C}$ and hence $U(g) = U(h)$. But in Remark 4.2.2, we remarked that for a morphism $g \in \mathsf{PMod}^H$, $g \bullet H$ is completely determined by g (or by $U(g)$ to be precise). Hence we find that $g = h$ in PMod^H.

Let $(X, X \bullet H, \rho, \pi)$ be a partial comodule datum. Since $\mathcal{C}$ is well-copowered, there exists up-to-isomorphism only a set of epimorphisms $f : X \to Y$ in $\mathcal{C}$. Moreover, for each Y, there exist again since $\mathcal{C}$ is well-copowered, only a set of epimorphisms $Y \otimes H \to Y \bullet H$, hence also only a set of partial comodule data over Y. We conclude that there will be only a set of epimorphisms $f : X \to Y$ in PMod^H and hence PMod^H is also well-copowered. □

Theorem 4.4.3. *Suppose that the endofunctor* $- \otimes H : \mathcal{C} \to \mathcal{C}$ *preserves colimits. Then the following statements hold.*

(i) If the category $\mathcal{C}$ *is* k*-linear then* PMod^H *is also* k*-linear and the forgetful functor is* k*-linear.*
(ii) If the category $\mathcal{C}$ *has all colimits of a shape* $\mathcal{Z}$*, then* PMod^H *also has colimits of shape* $\mathcal{Z}$*. Hence, if* $\mathcal{C}$ *is cocomplete then* PMod^H *is cocomplete and the forgetful functor* $U : \mathsf{PMod}^H \to \mathcal{C}$ *preserves colimits.*
(iii) If the category $\mathcal{C}$ *is additive, then* PMod^H *is also additive.*

Proof. (i) Let $X = (X, \rho_X, \pi_X, \theta_X)$ and $(Y, \rho_Y, \pi_Y, \theta_Y)$ be a two partial comodule data and $(f, f \bullet H), (g, g \bullet H) : X \to Y$ two morphisms. Let us verify that $(f + g, f \bullet H + g \bullet H)$ is again a morphism. Then we have

$$\begin{aligned} \rho_Y \circ (f+g) &= \rho_Y \circ f + \rho_Y \circ g \\ &= (f \bullet H) \circ \rho_X + (g \bullet H) \circ \rho_X \\ &= ((f \bullet H) + (g \bullet H)) \circ \rho_X \end{aligned}$$

And similarly, $((f \bullet H) + (g \bullet H)) \circ \pi_X = \pi_Y \circ ((f \otimes H) + (g \otimes H))$. Hence $(f + g, f \bullet H + g \bullet H)$ is indeed a morphism in PMod^H.

Similarly, for any $a \in k$, we define $a(f, f \bullet H) = (af, af \bullet H)$. One easily verifies that this is again a morphism, and using this addition and scalar multiplication, the Hom-sets in PMod^H are k-modules and composition is k-bilinear.

(ii) Let $\mathcal{Z}$ be any small category and $F : \mathcal{Z} \to \mathsf{PMod}^H$ a functor, where we denote for each $Z \in \mathcal{Z}$, $FZ = (FZ, FZ \bullet H, \rho_{FZ}, \pi_{FZ})$, i.e. we denote $UFZ = FZ$ for short. Consider the functor $UF : \mathcal{Z} \to \mathcal{C}$ and denote $(C, \gamma_Z) = \operatorname{colim} UF$, where $\gamma_Z : FZ \to C$ are such that $\gamma_Z = \gamma_{Z'} \circ Ff$ for any $f : Z \to Z'$ in $\mathcal{Z}$. Consider now the functor $UF^H : \mathcal{Z} \to \mathcal{C}$ given by $UF^H Z = FZ \otimes H$. Then by assumption we have that $\operatorname{colim} UF^H = (C \otimes H, \gamma_Z \otimes H)$. Finally consider the functor

$UF^\bullet : \mathcal{Z} \to \mathcal{C}$ given by $UF^\bullet Z = FZ \bullet H$ for all $Z \in \mathcal{Z}$, and denote $\mathsf{colim}\, UF^\bullet Z = (C \bullet H, \delta_Z)$ where $\delta_Z : FZ \bullet H \to C \bullet H$ are such that $\delta_Z = \delta_{Z'} \circ Ff \bullet H$. Let us verify that $(C \bullet H, \delta_Z \circ \rho_{FZ})$ is a cocone for UF. Indeed, for any morphism $f : Z \to Z'$ in $\mathcal{Z}$, Ff is a morphism in PMod^H and hence the following diagram commutes

$$\begin{array}{ccccc}
FZ & & \xrightarrow{Ff} & & FZ' \\
\downarrow{\scriptstyle \rho_{FZ}} & & & & \downarrow{\scriptstyle \rho_{FZ'}} \\
FZ \bullet H & & \xrightarrow{Ff \bullet H} & & FZ' \bullet H \\
& \searrow^{\delta_Z} & & \swarrow_{\delta_{Z'}} & \\
& & C \bullet H & &
\end{array}$$

By the universal property of the colimit $\mathsf{colim}\, UF$, we then obtain a unique morphism $\rho_C : C \to C \bullet H$ such that $\delta_Z \circ \rho_{FZ} = \rho_C \circ \gamma_Z$ for all $Z \in \mathcal{Z}$. In the same way, one shows that $(C \bullet H, \delta_Z \circ \pi_{FZ})$ is a cocone for UF^H, and hence there exists a morphism $\pi_C : C \otimes H \to C \bullet H$ such that $\delta_Z \circ \pi_{FZ} = \pi_C \circ \gamma_Z$ for all $Z \in \mathcal{Z}$. The situation is summarized in the next diagram.

$$\begin{array}{ccc}
FZ & \xrightarrow{\gamma_Z} & C \\
\downarrow{\scriptstyle \rho_{FZ}} & & \downarrow{\scriptstyle \rho_C} \\
FZ \bullet H & \xrightarrow{\delta_Z} & C \bullet H \\
\uparrow{\scriptstyle \pi_{FZ}} & & \uparrow{\scriptstyle \pi_C} \\
FZ \otimes H & \xrightarrow{\gamma_Z \otimes H} & C \otimes H
\end{array}$$

Let us show that $(C, C \bullet H, \rho_C, \pi_C)$ is a partial comodule datum, i.e. that $\pi_C : C \otimes H \to C \bullet H$ is an epimorphism in $\mathcal{C}$. To this end, consider $f, g : C \bullet H \to X$ in $\mathcal{C}$ such that $f \circ \pi_C = g \circ \pi_C$. Then for all $Z \in \mathcal{Z}$ we have that

$$\begin{aligned}
f \circ \pi_C \circ (\gamma_Z \otimes H) &= f \circ \delta_Z \circ \pi_{FZ} \\
= g \circ \pi_C \circ (\gamma_Z \otimes H) &= g \circ \delta_Z \circ \pi_{FZ}
\end{aligned}$$

Since each π_{FZ} is epi, we find $f \circ \delta_Z = g \circ \delta_Z$ for all Z and since the δ_Z are jointly epi, we obtain that $f = g$ and therefore π_C is indeed an epimorphism.

Furthermore, by the interchange law for colimits, it follows that the pushouts $C \bullet (H \bullet H)$, $(C \bullet H) \bullet H$ and Θ_C can be computed as the colimits of the respective functors $\mathcal{Z} \to \mathcal{C}$ that construct the pushouts

$Z \bullet H(\bullet H)$, $(Z \bullet H) \bullet H$ and Θ_Z. Hence, it follows that if all FZ are quasi, lax or geometric comodules, then Z will be such as well.
(iii). By part (i) we know already that PMod^H is pre-additive if $\mathcal{C}$ is so and by part (ii) we know that PMod^H has binary coproducts if $\mathcal{C}$ has so. It remains to prove that binary coproducts in PMod^H are also products. Consider two object $(X, X \bullet H, \rho_X, \pi_X)$ and $(Y, Y \bullet H, \rho_Y, \pi_X)$ in PMod^H and consider their coproduct which we know by part (ii) is of the form $(X \coprod Y, (X \bullet H) \coprod (Y \bullet H), \rho_X \coprod \rho_Y, \pi_X \coprod \pi_Y)$. Moreover, we know that $X \coprod Y$ $(X \bullet H) \coprod (Y \bullet H)$ are biproducts in $\mathcal{C}$. Hence we have the projections $p_X : X \coprod Y \to X$, $\pi_Y : X \coprod Y \to Y$, $p_{X \bullet H}$: $(X \bullet H) \coprod (Y \bullet H) \to X \bullet H$ and $p_{Y \bullet H} : (X \bullet H) \coprod (Y \bullet H) \to Y \bullet H$. Then by the properties of the coproduct in $\mathcal{C}$, we know that the following diagram commutes

$$
\begin{array}{ccc}
X \coprod Y & \longrightarrow & X \\
{\scriptstyle \rho_X \coprod \rho_Y} \downarrow & & \downarrow {\scriptstyle \rho_X} \\
(X \bullet H) \coprod (Y \bullet H) & \longrightarrow & X \bullet H \\
{\scriptstyle \pi_X \coprod \pi_Y} \uparrow & & \uparrow {\scriptstyle \pi_X} \\
(X \otimes H) \coprod (Y \otimes H) & \longrightarrow & X \otimes H
\end{array}
$$

Hence we find that $(p_X, p_{X \bullet H}) : X \coprod Y \to X$ is a morphism in PMod^H, and the same is true for $(p_Y, p_{Y \bullet H})$ and we obtain that $(X \coprod Y, (X \bullet H) \coprod (Y \bullet H), \rho_X \coprod \rho_Y, \pi_X \coprod \pi_Y)$ is indeed a biproduct in PMod^H. □

As we will show further in this section, there exist monomorphisms f in PMod^H such that $U(f)$ is not a monomorphism in $\mathcal{C}$. In particular, U does not have a left adjoint. Nevertheless, we have the following result.

Lemma 4.4.4. *Consider a morphism $f : X \to Y$ in PMod^H. If $Uf : UX \to UY$ is a monomorphism in $\mathcal{C}$, then f is also a monomorphism in PMod^H.*

Proof. Consider two morphisms $g, h : Z \to X$ in PMod^H such that $f \circ g = f \circ h$. Since Uf is a monomorphism, we obtain $Ug = Uh$. Then by Remark 4.2.2, we find that also $g \bullet H = h \bullet H$, i.e. $g = h$ in PMod. □

Definition 4.4.5. A *subcomodule* of a partial comodule $(X, X \bullet H, \rho_X, \pi_X)$ is a partial comodule datum $(Y, Y \bullet H, \rho_Y, \pi_Y)$, together with a morphism $f : Y \to X$ for which both f and $f \bullet H$ are monomorphisms in $\mathcal{C}$.

From now on, we restrict to our case of interest $\mathcal{C} = \mathsf{Vect}_k$ where k is a commutative ring.

Proposition 4.4.6. *Let $(X, X \bullet H, \rho_X, \pi_X)$ be a partial comodule datum and $j : Y \to X$ a subobject of X in Vect_k. Consider the epi-mono factorization of $\pi_X \circ (j \otimes H) : Y \otimes H \to X \bullet H$, which we denote as follows:*

$$Y \otimes H \xrightarrow{\ \pi_Y\ } Y \bullet H \xhookrightarrow{\ j \bullet H\ } X \bullet H$$

Then

(i) $\mathsf{Ker}\,\pi_Y \cong j(Y) \otimes H \cap \mathsf{Ker}\,\pi_X$*;*
(ii) Denote as usual by $Y \bullet (H \bullet H)$ the pushout of $(\pi_Y, (\pi_Y \otimes H) \circ Y \otimes \Delta)$. Then $Y \bullet (H \bullet H)$ is isomorphic to the image of the map $\pi'_{X,\Delta} \circ (j \bullet H) \otimes H$;

If moreover Y allows a partial comodule datum of the form $(Y, Y \bullet H, \rho_Y, \pi_Y)$ such that j is a morphism of partial comodule data, then

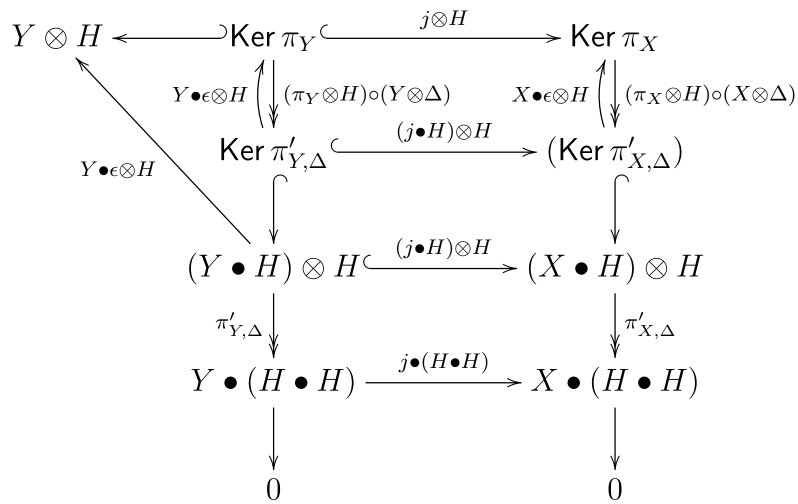
(iii) Y is a partial subcomodule of X.
(iv) $(Y \bullet H) \bullet H$ is isomorphic to the image of the map $\pi_{X \bullet H} \circ (j \bullet H) \otimes H$;
(v) if X is a lax (resp. geometric) partial comodule, then Y is as well a lax (resp. geometric) partial comodule.

Proof. (i). By construction we have the following commutative diagram

$$\begin{array}{ccc} Y \otimes H & \xhookrightarrow{\ j \otimes H\ } & X \otimes H \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Y \bullet H & \xhookrightarrow{\ j \bullet H\ } & X \bullet H \end{array}$$

Hence $y \otimes h \in \mathsf{Ker}\,\pi_Y$ iff $0 = (j \bullet H) \circ \pi_Y(y \otimes h) = \pi_X \circ (j \otimes H)(y \otimes h)$ iff $(j \otimes H)(y \otimes h) \in \mathsf{Ker}\,\pi_X$. I.e. $j(y) \otimes h \in \mathsf{Ker}\,\pi_X \cap j(Y) \otimes H$.
(ii). As in the case of partial comodules, we know by Lemma 4.1.4 that $Y \bullet (H \bullet H)$ can be computed as the quotient of $(Y \bullet H) \otimes H$ by the subspace $(\pi_Y \otimes H) \circ (Y \otimes \Delta)(\mathsf{Ker}\,\pi_Y)$. Furhtermore, the statement is true if and only if the canonical morphism $j \bullet (H \bullet H) : Y \bullet (H \bullet H) \to$

$X \bullet (H \bullet H)$ is injective. Consider the following diagram.



Consider any $y\bullet(h\bullet h') \in Y\bullet(H\bullet H)$, i.e. $y\bullet(h\bullet h') = \pi'_{Y,\Delta}((y\bullet h)\otimes h')$ for some $(y \bullet h) \otimes h' \in (Y \bullet H) \otimes H$ and $(y \bullet h) \otimes h' = \pi_Y(y \otimes h) \otimes h'$ with $y \otimes h \otimes h' \in Y \otimes H \otimes H$. Suppose that $j(y) \bullet (h \bullet h') = 0$, i.e. $(j(y) \bullet h) \otimes h' \in \operatorname{Ker} \pi'_{X,\Delta} = (\pi_X \otimes H) \circ (X \otimes \Delta)(\operatorname{Ker} \pi_X)$. Hence, $(j(y) \bullet h) \otimes h' = (x_i \bullet h_{i(1)}) \otimes h_{i(2)}$ for some $x_i \otimes h_i \in \operatorname{Ker} \pi_X$. Applying $(X \bullet \epsilon) \otimes H$ to the last identity, we obtain by part (i) that

$$x_i \otimes h_i = j(y) \otimes \epsilon(h)h' \in j(Y) \otimes H \cap \operatorname{Ker} \pi_X$$

Hence, $y \otimes \epsilon(h)h' \in \operatorname{Ker} \pi_Y$. Then we find

$$\begin{aligned} & (\pi_X \otimes H) \circ (X \otimes \Delta) \circ (j \otimes H)(y \otimes \epsilon(h)h') \\ = \; & (x_i \bullet h_{i(1)}) \otimes h_{i(2)} \\ = \; & (j(y) \bullet h) \otimes h' \\ = \; & ((j \bullet H) \otimes H) \circ (\pi_Y \otimes H) \circ (Y \otimes \Delta)(y \otimes \epsilon(h)h') \\ = \; & j(y) \bullet \epsilon(h)h'_{(1)} \otimes h'_{(2)} \end{aligned}$$

Since $(j\bullet H)\otimes H$ is injective, we have that $(y\bullet h)\otimes h' = y\bullet\epsilon(h)h'_{(1)}\otimes h'_{(2)}$ which is in $\operatorname{Ker} \pi'_{Y,\Delta}$ since $y\otimes\epsilon(h)h' \in \operatorname{Ker} \pi_Y$. Therefore $y\bullet(h\bullet h') = 0$ and $j \bullet (H \bullet H)$ is injective.

(iii). It is clear by construction that $(j, j \bullet H)$ is a morphism of partial comodule data and $j \bullet H$ is injective.

(iv). This is proven in the same way as in part (ii). We have to show that $(j\bullet H)\bullet H : (Y\bullet H)\bullet H \to (X\bullet H)\bullet H$ is injective. So suppose that $(y \bullet h) \bullet h' \in (Y \bullet H) \bullet H$ is such that $(j(y) \bullet h) \bullet h' = 0$ in $(X \bullet H) \bullet H$. Since $\pi_{Y\bullet H}$ is surjective, we find that $(y \bullet h) \bullet h' = \pi_{Y\bullet H}((y \bullet h) \otimes h')$

and $(j(y) \bullet h) \otimes h' \in \mathsf{Ker}\,\pi_{X\bullet H} = \rho_X(\mathsf{Ker}\,\pi_X)$. Hence, $(j(y) \bullet h) \otimes h' = (x_{i[0]} \bullet x_{i[1]}) \otimes h_i$ for some $x_i \otimes h_i \in \mathsf{Ker}\,\pi_X$. Applying $(X \bullet \epsilon) \otimes H$ to the last identity, we obtain by part (i) that

$$x_i \otimes h_i = j(y) \otimes \epsilon(h)h' \in j(Y) \otimes H \cap \mathsf{Ker}\,\pi_X \cong \mathsf{Ker}\,\pi_Y.$$

Hence $(j(y) \bullet h) \otimes h' = (x_{i[0]} \bullet x_{i[1]}) \otimes h_i \in (j \bullet H) \otimes H \circ \rho_Y \circ (\mathsf{Ker}\,\pi_Y) \cong \mathsf{Ker}\,\pi_{Y\bullet H}$, so $(y \bullet h) \bullet h' = 0$.
(v). Suppose that X is a lax partial module. Then by part (iii) and (iv) above, we can restrict and corestrict θ_X to obtain a morphism $\theta_Y : Y \bullet (H \bullet H) \to (Y \bullet H) \bullet H$. If moreover X is geometric, than we can also restrict and corestrict θ_X^{-1} to obtain an inverse θ_Y^{-1} of θ_Y and Y is again geometric. □

The following corollary describes a phenomenon that was also observed in [**10**] for the case of partial representations.

Corollary 4.4.7. *Any partial subcomodule of a global comodule is again global.*

Proof. By Proposition 4.4.6, we know that for partial subcomodule Y of partial comodule X that $\mathsf{Ker}\,\pi_Y \subset \mathsf{Ker}\,\pi_X$. Moreover, if X is global then $\mathsf{Ker}\,\pi_X = 0$ and therefore also $\mathsf{Ker}\,\pi_Y = 0$ so Y is global. □

We are now ready to prove the 'fundamental theorem for partial comodules'.

Theorem 4.4.8 (Fundamental theorem for partial comodules)**.** *Let $X = (X, X \bullet H, \rho_X, \pi_X)$ be a geometric partial comodule over the k-coalgebra H, and consider any $x \in X$. Then there exists a finite dimensional (geometric) partial subcomodule $Y \subset X$ such that $x \in Y$.*

Proof. Take $x \in X$ and write $\rho(x) = \sum_i y_i \bullet h_i = \pi(\sum y_i \otimes h_i)$, where h_i is a base of H. Denote Y the (finite dimensional) subspace of X generated by the y_i. We by coassociativity in the partial comodule X, we have the identity

$$\theta_X^{-1} \circ (\rho \bullet H)(\rho(x)) = \pi'_{X,\Delta} \circ (X \bullet \Delta)(\rho(x))$$

in $X \bullet (H \bullet H)$. But since $\rho(x) \in Y \bullet H$, by Proposition 4.4.6 we know that the above expressions are in fact in $Y \bullet (H \bullet H)$. Therefore there exists $\pi(y_{ij} \otimes h_{ij}) \otimes h_i$ in $\mathsf{Ker}\,\pi'_{Y,\Delta}$ such that

$$\rho(y_i) \otimes h_i = \sum \pi(y_k \otimes \sum a^k_{ji} h_j) \otimes h_i + \pi(y_{ij} \otimes h_{ij}) \otimes h_i$$

in $(Y \bullet H) \otimes H$ where we denoted $\Delta(h_i) = \sum a^i_{jk} h_j \otimes h_k$ for certain $a^i_{jk} \in k$. Since the h_i are linearly independent, we get $\rho(y_i) = \sum \pi(y_k \otimes \sum a^k_{ji} h_j) + \pi(y_{ij} \otimes h_{ij}) \in Y \bullet H$. Hence, it follows by Proposition 4.4.6 that Y is a geometric partial subcomodule of X. □

Corollary 4.4.9. *The category of geometric partial comodules has a generator.*

Proof. Let I be the set of isomorphism classes of finite dimensional geometric partial comodules over H. Since there exists clearly only a set of partial comodule structures over a given finite dimensional vector space, it follows that I is indeed a set. For any $i \in I$ choose one comodule M_i and denote by G the coproduct $\coprod_{i \in I} M_i$. Then the fundamental theorem implies there is a surjective morphism $G \to X$ for any geometric partial comodule X. Hence, G is a generator for gPMod^H. □

Corollary 4.4.10. *The category of geometric partial comodules is complete and cocomplete.*

Proof. This follows from the known fact that a cocomplete well-copowered category with a generator is complete. □

Remark 4.4.11. Although the category PMod^H is complete, its limits are not preserved by the forgetful functor U to Vect. More precisely, if L is a limit of a diagram D in PMod^H, then it is clear that $U(L)$ is a cone for the diagram $U(D)$ in Vect. Hence there is a morphism $u : U(L) \to L'$ in Vect where L' is the limit in Vect of $U(D)$. In general however, this morphism u is not a bijection. Rather, L can be understood as the biggest partial comodule inside L' that allows a cone on D. Remark however, that in order to be able to speak about the 'biggest' partial comodule inside L', we already use implicitly the existence of limits in PMod^H. This can be seen more explicitly by considering the kernel of a morphism $f : X \to Y$ in PMod^H which can be understood as the biggest partial subcomodule K of X such that $U(K)$ is contained in the vector space kernel of f. Thanks to the completeness and cocompleteness of PMod^H, we can construct from two partial subcomodules $v : V \to X$ and $w : W \to X$ the pushout of the pullback of v and w, which is then a partial subcomodule of X containing both V and W.

The following result will be important in the next section.

Corollary 4.4.12. *The forgetful functor* $\mathsf{gPMod}^H \to \mathsf{PCD}$ *(*PCD *denotes the category of partial comodule data), is fully faithful and has a left adjoint* B*.*

Proof. Let X be a partial comodule datum. Then BX is the biggest partial subcomodule of X which is geometric. As we explained in the previous remark this construction makes sense. It is easily verified that this provides a left adjoint to the forgetful functor. □

In the remaining part of this section, we will show that the category of Partial modules is not abelian. To this end, we will construct an example of a morphism f such that $\mathsf{Ker}\,\mathsf{coker}\, f$ and $\mathsf{coker}\,\mathsf{Ker}\, f$ are not isomorphic.

Consider a global comodule X and Y a linear subspace of X which is not a (global) subcomodule (recall that by Corollary 4.4.7 any subcomodule of global module is global). We can then construct the induced partial comodule X/Y as in Example 4.1.5 and consider the canonical projection $p : X \to X/Y$ which is a morphism of partial comodules. Then the vector space kernel of p is just Y. However as we assumed that Y was not a subcomodule of X, Y is also not a partial subcomodule of X and hence it can not be the kernel of p in PMod^H. Rather, this kernel is the biggest (global) subcomodule of X contained in Y. Suppose that Y was a one-dimensional subspace of X, then it follows that the kernel of p has to be 0. Then p is both a monomorphism (as morphisms with a zero kernel in additive categories are monomorphisms) and an epimorphism (as p is surjective and Corollary 4.4.2) but not an isomorphism. Hence PMod^H is not abelian. We also see as mentioned earlier that there exist monomorphisms that are not induced by monomorphisms.

CHAPTER 5

Partial comodules over a Hopf algebra and Hopf-Galois theory

In this chapter we study the notions of partial comodules and introduced in the previous chapter in case the coalgebra has moreover a bialgebra structure. Similar to the global case, we prove that the category of quasi partial comodules is monoidal, although with an oplax unit. The category of geometric partial comodules over a bialgebra is shown to be an oplax monoidal category. This is then used to study in the next section partial comodule algebras, partial relative Hopf modules and a partial Hopf Galois theory.

5.1. The lax monoidal category of geometric partial comodules over a bialgebra

The following result is essentially due to Johnstone [**34**], who formulated the proof in case of cartesian closed categories, but the argument easily generalizes to closed monoidal categories.

Let us recall first that a monoidal category is called *left closed monoidal* if for each object X in $\mathcal{C}$, the endofunctor $X \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint, that we denote by $[X, -]$ and that is called the internal hom. In other words, if $\mathcal{C}$ is right closed, then for any triple of objects X, Y, Z in $\mathcal{C}$ we have isomorphisms

$$\mathsf{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathsf{Hom}_{\mathcal{C}}(X, [Y, Z])$$

for any $f \in \mathsf{Hom}_{\mathcal{C}}(X \otimes Y, Z)$ we denote the corresponding element in $\mathsf{Hom}_{\mathcal{C}}(Y, [X, Z])$ by $\hat{f}$, and conversely for any $g \in \mathsf{Hom}_{\mathcal{C}}(Y, [X, Z])$, we have $\hat{g} \in \mathsf{Hom}_{\mathcal{C}}(X \otimes Y, Z)$ with $\hat{\hat{f}} = f$ and $\hat{\hat{g}} = g$. If one considers the evaluation and coevaluation maps

$$\mathsf{ev}^X_Y : X \otimes [X, Y] \to Y; \quad \mathsf{coev}^X_Y : Y \to [X, X \otimes Y],$$

then we can write

$$\begin{aligned} \hat{f} &= [X, f] \circ \mathsf{coev}^X_Y; \\ \hat{g} &= \mathsf{ev}^X_Z \circ (X \otimes g). \end{aligned}$$

Suppose that $\mathcal{C}$ is moreover right closed, and where the right internal hom denoted by $\{-,-\}$. Then we find for any three objects X, Y, Z in $\mathcal{C}$ that

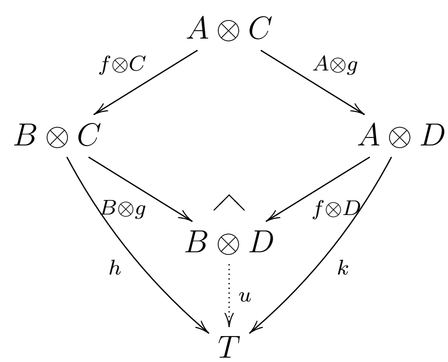
$$\mathsf{Hom}_{\mathcal{C}}(X,[Y,Z]) \cong \mathsf{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathsf{Hom}_{\mathcal{C}}(Y,\{X,Z\})$$

Hence

$$\mathsf{Hom}_{\mathcal{C}^{op}}([Y,Z],X) \cong \mathsf{Hom}_{\mathcal{C}}(Y,\{X,Z\})$$

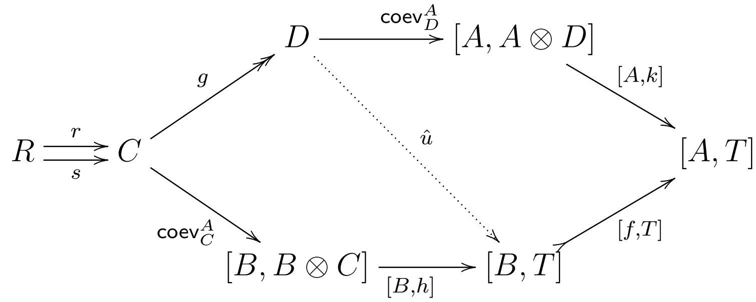
and the (contravariant) functor $[-,Z] : \mathcal{C} \to \mathcal{C}^{op}$ has a right adjoint $\{-,Z\}$, and therefore $[-,Z] : \mathcal{C} \to \mathcal{C}$ sends epimorphisms to monomorphisms.

Lemma 5.1.1. *Let $\mathcal{C}$ be a bi-closed monoidal category, $f : A \to B$ and epimorphism and $g : C \to D$ a regular epimorphism. Then the pushout of the pair $(f \otimes C, A \otimes g)$ is given by $(B \otimes D, B \otimes g, f \otimes D)$.*



Proof. Suppose that g is the coequalizer of the pair $r, s : R \to C$. Consider any object T and maps $h : B \otimes C \to T$, $k : A \otimes D \to T$ such that $\ell = h \circ (f \otimes C) = k \circ (A \otimes g) : A \otimes C \to T$. Using the left closure on $\mathcal{C}$, we find that ℓ corresponds uniquely to a morphism $\hat{\ell} \in \mathsf{Hom}(C,[A,T])$ and one checks that

$$\hat{\ell} = [A,k] \circ \mathsf{coev}_D^A \circ g = [f,T] \circ [B,h] \circ \mathsf{coev}_C^B.$$



Since g coequalizes the pair (r, s), it follows from the first equality that $\hat{\ell}$ also coequalizes the pair (r, s). Furthermore, since f is an epimorphism, $[f, T]$ is a monomorphism and we find that

$$[B, h] \circ \mathsf{coev}_C^B \circ r = [B, h] \circ \mathsf{coev}_C^B \circ s.$$

Therefore, the universal property of the coequalizer g leads to a unique morphism $\hat{u} : D \to [B, T]$ such that

$$\hat{u} \circ g = [B, h] \circ \mathsf{coev}_C^B : C \to [B, T].$$

Moreover, since g is an epimorphism, $\hat{u}$ also satisfies

$$[f, T] \circ \hat{u} = [A, k] \circ \mathsf{coev}_D^A$$

Consequently the induced morphism $\hat{\hat{u}} = u : B \otimes D \to T$ satisfies

$$u \circ (B \otimes g) = h, \quad u \circ (f \otimes D) = k$$

and is unique in this sense, which proves the universal property of the pushout $(B \otimes D, B \otimes g, f \otimes D)$. $\square$

Let $\mathcal{C}$ be a braided monoidal category with pushouts and consider a bialgebra H in $\mathcal{C}$. Let $(X, X \bullet H, \pi_X, \rho_X)$ and $(Y, Y \bullet H, \pi_Y, \rho_Y)$ be two partial comodule data over H. Then we can construct a new partial comodule datum $(X \otimes Y, (X \otimes Y) \bullet H, \pi_{X \otimes Y}, \rho_{X \otimes Y})$ in the following way. Consider the map $\mu_{X,Y} = (X \otimes Y \otimes \mu_H) \circ (X \otimes \sigma_{H,Y} \otimes H)$, where σ denotes the braiding of the category. Then define $(X \otimes Y) \bullet H$ and $\pi_{X \otimes Y}$ by the following pushout.

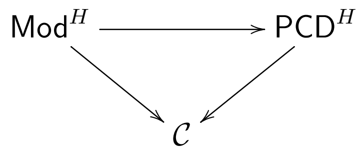
$$\begin{array}{ccccc} & & X \otimes H \otimes Y \otimes H & & \\ & \swarrow^{\pi_X \otimes \pi_Y} & & \searrow^{\mu_{X,Y}} & \\ (X \bullet H) \otimes (Y \bullet H) & & & & X \otimes Y \otimes H \\ & \searrow_{\overline{\mu}} & \wedge & \swarrow_{\pi_{X \otimes Y}} & \\ & & (X \otimes Y) \bullet H & & \end{array}$$

By taking $\rho_{X \otimes Y} = \overline{\mu} \circ (\rho_X \otimes \rho_Y)$, we obtain the desired partial comodule datum.

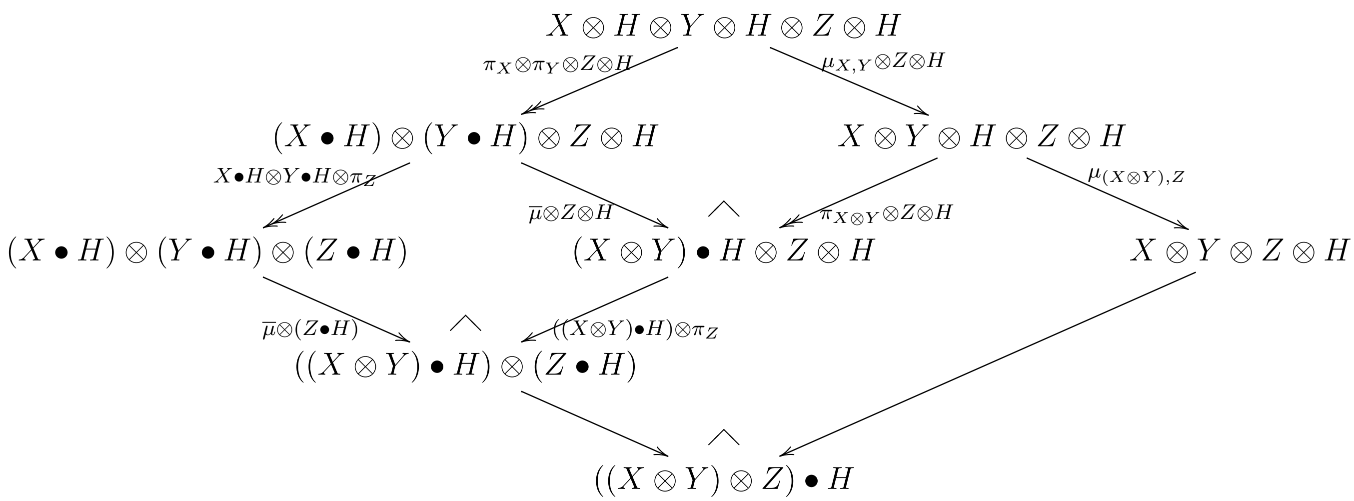
This construction leads to the following result.

Proposition 5.1.2. *Let $\mathcal{C}$ be a braided closed monoidal category where all epimorphisms are regular and let H be a bialgebra in $\mathcal{C}$.*

Then by the above defined tensor product, the category of partial comodule data over H is a monoidal category with an op-lax unit, such *that the following is a diagram of strict monoidal functors.*



Proof. Let us first verify the associativity of the defined tensor product for PCD. Consider 3 partial comodule data $\underline{X}, \underline{Y}, \underline{Z}$ and consider the following diagram.



The upper square is a pushout by definition of the tensor product and the fact that the functor $-\otimes Z \otimes H$ preserves pushouts since it has a right adjoint. The down square is a pushout by definition of the tensor product. The left square is a pushout by the Lemma 5.1.1. hence, by combining these pushouts we find that $((X \otimes Y) \otimes Z) \bullet H$ is the pushout of $(X \bullet H \otimes Y \bullet H \otimes \pi_Z) \circ (\pi_X \otimes \pi_Y \otimes Z) \simeq \pi_X \otimes \pi_Y \otimes \pi_Z$ along $\mu_{(X \otimes Y),Z} \circ (\mu_{X,Y} \otimes Z \otimes H)$. In the same way, $(X \otimes (Y \otimes Z)) \bullet H$ is shown to be the pushout of $\pi_X \otimes \pi_Y \otimes \pi_Z$ and $\mu_{X,Y \otimes Z} \circ (X \otimes H \otimes \mu_{Y,Z})$. One can easily verify that by the properties of the braiding in $\mathcal{C}$ and associativity of the multiplication of H, the maps $\mu_{(X \otimes Y),Z} \circ (\mu_{X,Y} \otimes Z \otimes H)$ and $\mu_{X,Y \otimes Z} \circ (X \otimes H \otimes \mu_{Y,Z})$ are the same up-to-isomorphism. Hence we find that $((X \otimes Y) \otimes Z) \bullet H \cong (X \otimes (Y \otimes Z)) \bullet H$, which induces the associativity constraint for the monoidal product in PCD.

Next, let us verify that the partial comodule datum $\underline{k} = (k, H, id_H, \eta)$ is an oplax unit for this monoidal product. Consider any partial comodule datum $\underline{X} = (X, X \bullet H, \pi_X, \rho_X)$ and construct the tensor product $\underline{X} \otimes \underline{k}$. We know that the underlying $\mathcal{C}$ object is just $X \otimes k \cong X$ via the isomorphism $r_X : X \to X \otimes k$. Futhermore, the object $(X \otimes k) \bullet H$ is constructed by the following pushout.

$$\begin{array}{ccccc} & & X \otimes H \otimes H & & \\ & \overset{\pi_X \otimes H}{\swarrow} & & \overset{X \otimes \mu}{\searrow} & \\ (X \bullet H) \otimes H & & & & X \otimes H \\ & \underset{\overline{\mu}}{\searrow} & \wedge & \underset{\pi_{X \otimes k}}{\swarrow} & \\ & & (X \otimes k) \bullet H & & \end{array}$$

Then consider the map $r_X \bullet H := \overline{\mu} \circ (X \bullet H) \otimes \eta : X \bullet H \to (X \otimes k) \bullet H$. One easily verifies that $r_X \bullet H \circ \pi_X = \pi_{X \otimes k}$ and therefore $(r_X, r_X \bullet H) : \underline{X} \to \underline{X} \otimes \underline{k}$ is a morphism of partial comodule data. □

Let us remark that the category of partial comodule data can not have a (strong) monoidal unit. Indeed, since the forgetful functor is strict monoidal, the underlying $\mathcal{C}$-object of the monoidal unit needs to be the monoidal unit k of $\mathcal{C}$. Hence the monoidal unit should be of the form (k, K, π, ρ), where K is a quotient of H. However, when computing the pushout

$$\begin{array}{ccccc} & & X \otimes H \otimes H & & \\ & \overset{\pi_X \otimes \pi}{\swarrow} & & \overset{X \otimes \mu}{\searrow} & \\ (X \bullet H) \otimes K & & & & X \otimes H \\ & \underset{\overline{\mu}}{\searrow} & \wedge & \underset{\pi_{X \otimes k}}{\swarrow} & \\ & & (X \otimes k) \bullet H & & \end{array}$$

In case $\mathcal{C} = \mathsf{Vect}$, we can compute this pushout explicitly via Lemma 4.1.4 and we see that $(X \otimes k) \bullet H$ is the quotient of $X \otimes H$ with respect to $(X \otimes \mu)(\mathsf{Ker}\, \pi_X \otimes H + X \otimes H \otimes \mathsf{Ker}\, \pi)$. However, this last set is strictly larger then $\mathsf{Ker}\, \pi_X$, so we can never get that $(X \otimes k) \bullet H \cong X \bullet H$. Nevertheless, the oplax monoidal unit from Proposition 5.1.2 becomes a strong unit for a suitable subcategory that we will define now.

Definition 5.1.3. We call a partial comodule datum X over a bialgebra H-*equivariant* if the kernel of the morphism $\pi_X : X \otimes H \to X \bullet H$ is an H sub-bimodule of $X \otimes H$ (hence $X \bullet H$ is an H-bimodule and π_X is H-bilinear). More explicitly, in case $\mathcal{C} = \mathsf{Vect}_k$, this means that if $x \otimes h \in \mathsf{Ker}\, \pi_X$, then also $x \otimes h'hh'' \in \mathsf{Ker}\, \pi_X$ for all $h', h'' \in H$.

Corollary 5.1.4. *The category of equivariant partial comodule data over a k-bialgebra H is monoidal.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & & X\otimes H\otimes H & & \\ & \swarrow^{\pi_X\otimes H} & & \searrow^{X\otimes\mu} & \\ (X\bullet H)\otimes H & & & & X\otimes H \\ & \searrow_{\overline{\sigma}} & \ulcorner & \swarrow_{\pi_X} & \\ & & X\bullet H & & \end{array}$$

Then we see that under the stated assumptions, $\mathsf{Ker}\,\pi_X$ is a right H-submodule of $X\otimes H$ and hence $(X\otimes\mu)\mathsf{Ker}\,(\pi_X\otimes H)=\mathsf{Ker}\,\pi_X$ and therefore the above diagram is a pushout, which implies that $X\otimes k\cong X$ in PCD. Similarly, using that $\mathsf{Ker}\,\pi_X$ is a left H-submodule of $X\otimes H$, we find that k is a left unit for the monoidal structure on PCD. □

The result from Proposition 5.1.2 leads to the natural question whether the full subcategories of the category of partial comodule data, consisting of quasi, lax and geometric partial comodules inherit a monoidal structure. To answer this question, let us compute the pushout $(X\otimes Y)\bullet(H\bullet H)$, which is given by the following composition of pushouts

$$\begin{array}{ccccccc} X\otimes H\otimes Y\otimes H & \xrightarrow{\mu_{X,Y}} & X\otimes Y\otimes H & \xrightarrow{X\otimes Y\otimes\Delta} & X\otimes Y\otimes H\otimes H & \xrightarrow{\pi_{X\otimes Y}\otimes H} & ((X\otimes Y)\bullet H)\otimes H \\ \downarrow{\scriptstyle \pi_X\otimes\pi_Y} & & \downarrow{\scriptstyle \pi_{X\otimes Y}} & & \downarrow{\scriptstyle \pi_{X\otimes Y,\Delta}} & & \downarrow{\scriptstyle \pi'_{X\otimes Y,\Delta}} \\ (X\bullet H)\otimes(Y\bullet H) & \xrightarrow[\overline{\mu}_{X,Y}]{} & \ulcorner\,(X\otimes Y)\bullet H & \xrightarrow[(X\otimes Y)\bullet\Delta]{} & \ulcorner\,(X\otimes Y)\bullet(H\otimes H) & \xrightarrow[\pi'_{X\otimes Y}]{} & \ulcorner\,(X\otimes Y)\bullet(H\bullet H) \end{array}$$

Furthermore, checks that the composition of the upper morphisms can be rewritten as

$$(\pi_{X\otimes Y}\otimes H)\circ(X\otimes Y\otimes\Delta)\circ\mu_{X,Y}=$$
$$(\overline{\mu}_{X,Y}\otimes H)\circ\mu_{X\bullet H,Y\bullet H}\circ(\pi_X\otimes H\otimes\pi_Y\otimes H)\circ(X\otimes\Delta\otimes Y\otimes\Delta)$$

In the same way, the pushout $((X\otimes Y)\bullet H)\bullet H$ is given by the following composition of pushouts

$$\begin{array}{ccccccc} X\otimes H\otimes Y\otimes H & \xrightarrow{\mu_{X,Y}} & X\otimes Y\otimes H & \xrightarrow{\rho_X\otimes\rho_Y\otimes H} & (X\bullet H)\otimes(Y\bullet H)\otimes H & \xrightarrow{\overline{\mu}_{X,Y}\otimes H} & ((X\otimes Y)\bullet H)\otimes H \\ \downarrow{\scriptstyle \pi_X\otimes\pi_Y} & & \downarrow{\scriptstyle \pi_{X\otimes Y}} & & & & \downarrow{\scriptstyle \pi_{(X\otimes Y)\bullet H}} \\ (X\bullet H)\otimes(Y\bullet H) & \xrightarrow[\overline{\mu}_{X,Y}]{} & (X\otimes Y)\bullet H & & \xrightarrow{\rho_{X\otimes Y}\bullet H} & & \ulcorner\,((X\otimes Y)\bullet H)\bullet H \end{array}$$

and again we can rewrite the composition of the upper morphisms

$$(\overline{\mu}_{X,Y} \otimes H) \circ (\rho_X \otimes \rho_Y \otimes H) \circ \mu_{X,Y} = \\ (\overline{\mu}_{X,Y} \otimes H) \circ \mu_{X \bullet H, Y \bullet H} \circ (\rho_X \otimes H \otimes \rho_Y \otimes H)$$

Hence, to study the relation between $(X \otimes Y) \bullet (H \bullet H)$ and $((X \otimes Y) \bullet H) \bullet H$, we have to compare the following pushouts

$$\begin{array}{ccccc} X \otimes H \otimes Y \otimes H & \xrightarrow{X \otimes \Delta \otimes Y \otimes \Delta} & X \otimes H \otimes H \otimes Y \otimes H \otimes H & \xrightarrow{\pi_X \otimes H \otimes \pi_Y \otimes H} & (X \bullet H) \otimes H \otimes (Y \bullet H) \otimes H \\ \downarrow{\scriptstyle \pi_X \otimes \pi_Y} & & & & \downarrow{\scriptstyle p_2} \\ (X \bullet H) \otimes (Y \bullet H) & & \xrightarrow{\quad p_1 \quad} & & P \end{array}$$

and

$$\begin{array}{ccc} X \otimes H \otimes Y \otimes H & \xrightarrow{\rho_X \otimes H \otimes \rho_Y \otimes H} & (X \bullet H) \otimes H \otimes (Y \bullet H) \otimes H \\ \downarrow{\scriptstyle \pi_X \otimes \pi_Y} & & \downarrow{\scriptstyle q_2} \\ (X \bullet H) \otimes (Y \bullet H) & \xrightarrow{\quad q_1 \quad} & Q \end{array}$$

Using Lemma 4.1.4, we find that P and Q are isomorphic to quotients of $(X \bullet H) \otimes H \otimes (Y \bullet H) \otimes H$ by the respective subspaces

$$(\pi_X \otimes H \otimes \pi_Y \otimes H) \circ (X \otimes \Delta \otimes Y \otimes \Delta)(\mathsf{Ker}\,(\pi_X \otimes \pi_Y)) = \\ (\pi_X \otimes H \otimes \pi_Y \otimes H) \circ (X \otimes \Delta \otimes Y \otimes \Delta)(\mathsf{Ker}\,\pi_X \otimes Y \otimes H + X \otimes H \otimes \mathsf{Ker}\,\pi_Y)$$

and

$$(\rho_X \otimes \rho_Y)(\mathsf{Ker}\,(\pi_X \otimes \pi_Y)) = (\rho_X \otimes \rho_Y)(\mathsf{Ker}\,\pi_X \otimes Y \otimes H + X \otimes H \otimes \mathsf{Ker}\,\pi_Y)$$

Since in general $(\pi_X \otimes H) \circ (X \otimes \Delta)(X \otimes H) \ncong (\rho_X \otimes H)(X \otimes H)$, we find that P and Q are non-isomorphic, and therefore also $(X \otimes Y) \bullet (H \bullet H)$ and $((X \otimes Y) \bullet H) \bullet H$ are non-isomorphic (even if X and Y are geometric partial comodules).

Hence, we can conclude that when X and Y are geometric (respectively lax) partial comodules, then $X \otimes Y$ is in general no longer a geometric (respectively lax) partial comodule. However, when X and Y are both quasi comodules, then $\theta_1 \circ (\rho_X \bullet H)$ and $\theta_2 \circ \overline{(X \bullet \Delta)}$ *do* have identical images when restricted to the image of ρ_X, we find that $X \otimes Y$ is still a quasi partial comodule. We can then conclude on the following.

Theorem 5.1.5. *The category of quasi partial comodules over a bialgebra is monoidal with an oplax monoidal unit, and the forgetful functor to vector spaces is strict monoidal.*

The category of equivariant quasi partial comodules over a bialgebra is a monoidal category.

Although the above introduced tensor product is not well-defined on the category geometric partial comodules, in case of working with a bialgebra over a field, we can combine Proposition 5.1.2 with Theorem 1.2.7 and obtain immediately the following result.

Theorem 5.1.6. *The category of geometric partial comodules over a bialgebra H over a field k is an op-lax monoidal category and the forgetful functor $U : \mathsf{gPMod}^H \to \mathsf{Vect}_k$ is monoidal.*

Remark 5.1.7. Let us describe the oplax tensor product of the category gPMod^H a bit more explicitly. Consider two geometric partial comodules M and N, and let $M \otimes N$ be the tensor product partial comodule datum (which we know is a quasi partial comodule). Then $M \bullet N$ is the geometric partial comodule that is uniquely defined by the following universal property. There exists a morphism $p : M \otimes N \to M \bullet N$ and for every other geometric partial comodule T with a morphism $M \otimes N \to T$, there exists a unique morphism $u : M \bullet N \to T$ such that $t = u \circ p$.

Since the zero module is a geometric partial comodule that is a minimal solution for the above problem, $M \bullet N$ will always exist. Given two $p : M \otimes N \to P$ and $q : M \otimes N \to Q$, Let R be the pullback of the pushout of p and q, i.e. firstly take the pushout of p, q and then take the pullback of the maps induced by this pushout (both exist since we proved that geometric partial comodules are bi-complete). Then there is a unique morphism $M \otimes N \to R$ compatible with both p and q. In this way, we can construct the "biggest" quotient $M \bullet N$ of $M \otimes N$ that is still a geometric partial comodule.

Remark that if one of the geometric partial comodules X and Y is global, then $X \bullet Y = X \otimes Y$.

5.2. Partial comodule algebras

In the partial case, it turns out that there are two kinds of 'comodule algebras': those which arise as partial comodules in the category of algebras, and those that arise as algebras in the (oplax monoidal) category of partial comodules. While these notions coincide in the global case, for partial coactions they differ, as a consequence of the fact that pushouts in the category of algebras are different from pushouts in the category of vector spaces.

5.2.1. Algebras in the category of partial comodules. Let $\mathcal{C}$ be an oplax monoidal category and denote $\otimes_0(\varnothing) = I$ and $\otimes_n(X_1, \ldots, X_n) = (X_1 \otimes \ldots \otimes X_n)$. An algebra $\mathcal{C}$, is an object A endowed with morphisms $m : (A \otimes A) \to A$ and $u : I \to A$ satisfying the following conditions

$$\begin{array}{ccccccc} & & ((A) \otimes (A \otimes A)) & \xrightarrow{\iota \otimes m} & (A \otimes A) & & \\ & \overset{\gamma}{\nearrow} & & & & \overset{m}{\searrow} & \\ (A \otimes A \otimes A) & & & & & & A \\ & \underset{\gamma}{\searrow} & & & & \underset{m}{\nearrow} & \\ & & ((A \otimes A) \otimes (A)) & \xrightarrow{m \otimes \iota} & (A \otimes A) & & \end{array}$$

$$\begin{array}{ccccccc} & & (A \otimes I) & \xrightarrow{(A \otimes u)} & (A \otimes A) & & \\ & \overset{\gamma}{\nearrow} & & & & \overset{m}{\searrow} & \\ (A) & & & \xrightarrow{\iota_A} & & & A \\ & \underset{\gamma}{\searrow} & & & & \underset{m}{\nearrow} & \\ & & (I \otimes A) & \xrightarrow[(u \otimes A)]{} & (A \otimes A) & & \end{array}$$

Then we obtain the following natural definitions.

Definition 5.2.1. Let H be a k-bialgebra. A *quasi (resp. geometric) partial H-comodule algebra* is a an algebra in the oplax monoidal category of quasi (resp. geometric) partial H-comodules.

Since the forgetful functors $\mathsf{gPMod}^H \to \mathsf{qPMod}^H \to \mathsf{Vect}_k$ are monoidal, each geometric partial comodule algebra is also a quasi partial comodule algebra and a quasi or geometric partial comodule algebra is also a k-algebra. More precisely, we have the following result.

Proposition 5.2.2. *Let $\mathcal{C}$ be a category satisfying the conditions of Proposition 5.1.2 and H a Hopf algebra in $\mathcal{C}$. If $(A, A \bullet H, \pi_A, \rho_A)$ be an algebra in the monoidal category with oplax unit qPMod^H, then A and $A \bullet H$ are algebras in $\mathcal{C}$ and the morphisms π_A and ρ_A are algebra morphisms.*

Proof. We already remarked that A is an algebra in $\mathcal{C}$, since the forgetful functor $\mathsf{qPMod}^H \to \mathcal{C}$ is monoidal. To see that $A \bullet H$ is an algebra consider the following, which expresses that the multiplication

$\mu : A\otimes A \to A$ is a morphism of partial comodules, and the construction of $(A \otimes A) \bullet H$ as pushout.

$$\begin{array}{ccccc}
 & & A\otimes A & \xrightarrow{\mu_A} & A \\
 & & \downarrow{\scriptstyle \rho_{A\otimes A}} & & \downarrow{\scriptstyle \rho_A} \\
(A\bullet H)\otimes(A\bullet H) & \xrightarrow{\overline{\mu}_{A,A}} & (A\otimes A)\bullet H & \xrightarrow{\mu_A\bullet H} & A\bullet H \\
\uparrow{\scriptstyle \pi_A\otimes\pi_A} & & \uparrow{\scriptstyle \pi_{A\otimes A}} & & \uparrow{\scriptstyle \pi_A} \\
A\otimes H\otimes A\otimes H & \xrightarrow{\mu_{A,A}} & A\otimes A\otimes H & \xrightarrow{\mu_A\otimes H} & A\otimes H
\end{array}$$

$$\rho_A\otimes\rho_A : A\otimes A \to (A\bullet H)\otimes(A\bullet H)$$

One can then easily verify that the morphism $(\mu_A \bullet H) \circ \overline{\mu}_{A,A}$ defines an associative multiplication on $A \bullet H$, and by construction π_A and ρ_A are then multiplicative. In a similar way, the unit morphism $u : k \to A$ is a morphism of partial comodules if the following diagram commutes

$$\begin{array}{ccc}
k & \xrightarrow{u} & A \\
\downarrow{\scriptstyle \eta_H} & & \downarrow{\scriptstyle \rho_A} \\
H & \xrightarrow{u\bullet H} & A\bullet H \\
\| & & \uparrow{\scriptstyle \pi_A} \\
H & \xrightarrow{u\otimes H} & A\otimes H
\end{array}$$

which means in Sweedler notation that

$$\rho_A(1_A) = 1_A \bullet 1_H \tag{5.1}$$

Then $\rho_A \circ u : k \to A \bullet H$ is a unit for the algebra $A \bullet H$ and the morphisms π_A and ρ_A are unital. □

Again, since the functor $\mathsf{gPMod}^H \to \mathsf{qPMod}^H$ is monoidal, it follows that for a geometric partial comodule algebra A, the vectorspace $A \bullet H$ is naturally a k-algebra and pi_A is a k-algebra morhpism. This implies that $\mathsf{Ker}\,\pi_A$ is a two-sided ideal in $A \otimes H$.

Remark 5.2.3. In contrast to what might think naively, the $\mathcal{C}$-objects $(A\bullet H)\bullet H$, $A\bullet(H\bullet H)$ or Θ_A do not posses a natural algebra structure in general. The main reason for this, is that these objects are defined as pushouts in $\mathcal{C}$ without any interaction with the multiplication μ_A. This is the main motivation to introduce a second type of comodule algebras in the next section.

Examples 5.2.4. Consider the example of geometric partial $k[x, y]$-comodule from Example 4.1.7, whose underlying object is $B = k[x, y]/(xy)$.

Then B has a natural algebra structure, however $B \bullet H = k[x, y, x', y']/\rho((xy))$ is not an algebra since $\rho((xy))$ is not an ideal. Hence, $(B, B \bullet H, \pi_B, \rho_B)$ is not a partial comodule algebra. However, consider $k[x, y, x', y']/(\rho(xy))$, where $(\rho(xy))$ is the ideal generated by $\rho((xy))$, then this is an ideal and there is a canonical projection $\pi : B \bullet H \to k[x, y, x', y']/(\rho(xy))$. Then the partial comodule datum $(B', B' \bullet H, \pi_{B'}, \rho_{B'})$, where $B' = B$, $B' \bullet H = k[x, y, x', y']/(\rho(xy))$, $\pi_{B'} = \pi \circ \pi_B$ and $\rho_{B'} = \pi \circ \rho_B$ is a partial $k[x, y]$-comodule algebra. Since the partial comodule B' is geometric (being a quotient of a global one), B' is also a geometric comodule algebra.

In the same way, one can turn the partial comodule from Example 4.1.7 into a geometric partial comodule algebra and the partial comdoule from Example 4.1.8 into quasi partial comodule algebra.

5.2.2. Partial comodules in the category of algebras. Recall that the category of algebras $\mathsf{Alg}(\mathcal{C})$ in a braided monoidal category $\mathcal{C}$, is again a monoidal category. Furthermore, a coalgebra H in $\mathsf{Alg}(\mathcal{C})$ is exactly a bialgebra in $\mathcal{C}$ and a comodule over H in $\mathsf{Alg}(\mathcal{C})$ is exactly an H-comodule algebra in $\mathcal{C}$. Following this point of view, we introduce the following definitions.

Definition 5.2.5. Let H be a bialgebra in the braided monoidal category $\mathcal{C}$, and consider H as a coalgebra in the category $\mathsf{Alg}(\mathcal{C})$. A *quasi (resp. lax, geometric) partial algebra-comodule* over H is a quasi (resp. lax, geometric) partial H-comodule $(A, A \bullet H, \pi_A, \rho_A)$ in $\mathsf{Alg}(\mathcal{C})$.

Remark 5.2.6. In the global case, algebra-comodules and comodule-algebras are identical structures. In the partial setting this however is no longer the case. Firstly, given a partial comodule datum $(A, A \bullet H, \pi_A, \rho_A)$ in $\mathsf{Alg}(\mathcal{C})$, then applying the forgetful functor $U : \mathsf{Alg}(\mathcal{C}) \to \mathcal{C}$ yields a partial H-comodule datum $UA = (A, A \bullet H, \pi_A, \rho_A)$ in $\mathcal{C}$, provided that $U(\pi_A)$ is an epimorphism in $\mathcal{C}$. This last condition is not necessarily the case if we take $\mathcal{C} = \mathsf{Mod}_k$ where k is an arbitrary commutative ring, but it holds if k is a field. However, even in case of $\mathcal{C} = \mathsf{Vect}_k$, the forgetful functor $U : \mathsf{Alg}_k \to \mathsf{Vect}_k$ does not preserve pushouts. In other words, the canonical morphism $\Theta_{UA} \to U(\Theta_A)$ is not an isomorphism in general. If A is a quasi partial H-comodule algebra, then coassociativity holds in $U\Theta_A$, but not necessarily in Θ_{UA}, and UA is not necessarily a quasi partial H-comodule.

The next result tells however that conversely, partial comodule-algebras are still algebra-comodules.

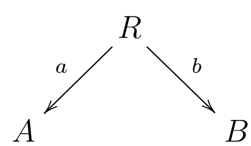
Proposition 5.2.7. *If $(A, A \bullet H, \pi_A, \rho_A)$ is a quasi partial H-comodule algebra, then $(A, A \bullet H, \pi_A, \rho_A)$ is also a quasi partial algebra-comodule over H.*

Proof. It follows from Proposition 5.2.2 that $(A, A \bullet H, \pi_A, \rho_A)$ is a partial comodule datum in the category $\mathsf{Alg}(\mathcal{C})$. If A is a quasi partial comodule, then the coassociativity holds in the sense that

$$\theta_1 \circ (\rho_A \bullet H) \bullet \rho_A = \theta_2 \circ \overline{X \bullet \Delta} \circ \rho_A \tag{5.2}$$

where $(\Theta_A, \theta_1, \theta_2)$ is the coassociativity pushouts in $\mathcal{C}$. On the other hand, we can also consider the coassociativity pushout $(\Theta'_A, \theta'_1, \theta'_2)$ in $\mathsf{Alg}(\mathcal{C})$. Since the forgetful functor $\mathsf{Alg}(\mathcal{C}) \to \mathcal{C}$ does not preserve pushouts, Θ_A and Θ'_A can be non-isomorphic objects in $\mathcal{C}$, but by the universal property of the pushouts $(A \bullet H) \bullet H$, $A \bullet (H \bullet H)$ and Θ_A, we will obtain a morphism $\pi : \Theta_A \to \Theta'_A$. By composing both sides of (5.2) with π, we find that the coassociativity also holds in $\mathsf{Alg}(\mathcal{C})$, and hence A is a quasi partial algebra-comodule. $\square$

The difference between the pushouts in $\mathsf{Alg}(\mathcal{C})$ and $\mathcal{C}$ can be understood very well in the situation where $\mathcal{C} = \mathsf{Vect}_k$. Indeed, consider k-algebra morphisms



where b is surjective. Then we know from Lemma 4.1.4 that the pushout of a and b in Vect is given by the quotient $A/a(\mathsf{Ker}\, b)$. In general (or more precisely, when a is not surjective) $a(\mathsf{Ker}\, b)$ is not an ideal in A, and hence $A/a(\mathsf{Ker}\, b)$ is not an algebra. However, if we denote by I the ideal generated by $a(\mathsf{Ker}\, b)$, then one can easily see that A/I is the pushout of (a, b) in Alg_k.

As a consequence, we find the following.

Corollary 5.2.8. *If $(A, A \bullet H, \pi_A, \rho_A)$ is a geometric partial H-comodule k-algebra, then $(A, A \bullet H, \pi_A, \rho_A)$ is also a geometric partial algebra-comodule over H.*

Proof. By Proposition 5.2.7 we know already that A is a quasi partial algebra-comodule. On the other hand, since A is geometric as comodule algebra, we find that the pushouts $(A\bullet H)\bullet H$ and $A\bullet(H\bullet H)$ are isomorphic in Vect_k. Because of the explicit description of these pushouts recalled above, this means that the following subspaces of $(A\bullet H)\otimes H$ are isomorphic (even identical):

$$(\rho_A\otimes H)(\mathsf{Ker}\,\pi_A) = (\pi_A\otimes H)\circ\Delta(\mathsf{Ker}\,\pi_A)$$

Hence the ideals generated by these subspaces will also be the same, and therefore the corresponding coassociativity pushouts in Alg_k will be isomorphic, which means exactly that A is geometric as a partial algebra-comodule. $\square$

As the following examples illustrate, the converse of the previous corollary does not hold.

Examples 5.2.9. All examples from Example 5.2.4 will give rise to examples of algebra-comodules. Since the examples obtained from Example 4.1.7 and Example 4.1.8 are geometric as comodule-algebra, they are by the previous proposition also geometric as algebra-comodules. We remarked before that the example from Example 4.1.9 is not geometric as comodule-algebra, however we will show now that is does become geometric as algebra-comodule.

Let A be a partial coaction of a Hopf algebra H in the sense of [**21**]. Consider as in Example 4.1.9 the partial comodule datum $(A, A\bullet H,\pi,\rho)$, where $A\bullet H = \{a1_{[0]}\otimes h1_{[1]}\} = (A\otimes H)e$, which is a direct summand of $A\otimes H$ and the left $A\otimes H$-module generated by the idempotent $e=\rho(1)=1_{[0]}\otimes 1_{[1]}$ and which can be seen as the quotient of $A\otimes H$ by the left ideal $(A\otimes H)e'$ where $e' = 1\otimes 1_H - e$ (we denote $1=1_A$ the unit of A). As explained in Example 5.2.4, in order to obtain a partial comodule algebra one has to consider an alternative partial comodule datum, where $A\bullet' H$ is the quotient of $A\otimes H$ by the two-sided ideal $(A\otimes H)e'(A\otimes H)$.

The ideal in $A\otimes H\otimes H$ generated by $(\rho\otimes H)((A\otimes H)e'(A\otimes H))$ is then nothing else than the ideal generated by $(\rho\otimes H)(e')$. Similarly, the ideal in $A\otimes H\otimes H$ generated by the image of $(A\otimes H)e'(A\otimes H)$ under $(\pi\otimes H)\circ A\otimes\Delta$ is the ideal generated by $(\pi\otimes H)\circ(A\otimes\Delta)(e')$. Using axiom (CJ2), we find

$$\begin{aligned}(\rho\otimes H)(e') &= (\rho\otimes H)(1\otimes 1_H - e) = 1_{[0]}\otimes 1_{[1]}\otimes 1_H - 1_{[0][0]}\otimes 1_{[0][1]}\otimes 1_{[1]}\\ &= 1_{[0]}\otimes 1_{[1]}\otimes 1_H - 1_{[0]}1_{[0']}\otimes 1_{[1](1)}1_{[1']}\otimes 1_{[1](2)}\\ &= (\pi\otimes H)\circ A\otimes\Delta(e')\end{aligned}$$

Hence it follows that both elements generate the same ideals, which implies that A is geometric as a partial algebra-comodule.

5.3. Partial Hopf modules and partial Hopf-Galois theory

5.3.1. Partial Hopf modules. Let $\mathcal{C}$ be an oplax monoidal category and denote as in Section 5.2.1 $\otimes_0(\varnothing) = I$ and $\otimes_n(X_1, \dots, X_n) = (X_1 \otimes \dots \otimes X_n)$. Let (A, m, u) be an algebra in $\mathcal{C}$. Then a (right) A-module in $\mathcal{C}$ is an object M endowed with a morphism $\mu_M : (M \otimes A) \to M$ satisfying the following conditions

$$\begin{array}{ccccccc} & & ((M)\otimes(A\otimes A)) & \xrightarrow{\iota\otimes m} & (M\otimes A) & & \\ & \overset{\gamma}{\nearrow} & & & & \overset{\mu_M}{\searrow} & \\ (M\otimes A\otimes A) & & & & & & M \\ & \underset{\gamma}{\searrow} & & & & \underset{\mu_M}{\nearrow} & \\ & & ((M\otimes A)\otimes(A)) & \xrightarrow{\mu_M\otimes\iota} & (A\otimes A) & & \end{array}$$

$$\begin{array}{ccccc} & (M\otimes I) & \xrightarrow{(A\otimes u)} & (M\otimes A) & \\ \overset{\gamma}{\nearrow} & & & & \overset{\mu_M}{\searrow} \\ (M) & & \xrightarrow{\iota_A} & & M \end{array}$$

Then we obtain the following natural definitions.

Definition 5.3.1. Let H be a Hopf k-algebra and $(A, A \bullet H, \pi_A, \rho_A)$ a quasi (resp. geometric) partial H-comodule algebra, i.e. an algebra in the lax monoidal category of quasi (resp. geometric) partial H-comodules. A *quasi (resp. geometric) partial* (A, H)*-relative Hopf module* is a right A-module in the lax monoidal category of quasi (resp. geometric) partial H-comodules.

We will denote by PMod_A^H the category whose objects are quasi partial (A, H)-relative Hopf modules and whose morphisms are morphism of partial H-modules that are at the same time A-linear.

As it is the case for partial comodule-algebras, since the forgetful functors $\mathsf{gPMod}^H \to \mathsf{qPMod}^H \to \mathsf{Vect}_k$ are monoidal, each geometric partial relative Hopf module is also a quasi partial relative Hopf module and a quasi or geometric partial relative Hopf module is also a module for the k-algebra A.

Let us make the previous definition a bit more explicit.

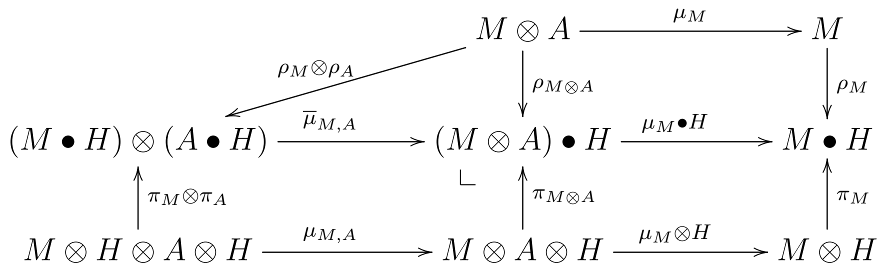
Lemma 5.3.2. *Let $(A, A \bullet H, \pi_A, \rho_A)$ be a quasi partial H-comodule algebra. A quasi partial (A, H)-relative Hopf module is a quasi partial H-comodule $(M, M \bullet H, \pi_M, \rho_M)$ endowed with an A-module structure $\mu_M : M \otimes A \to M$ such that the following compatibility conditions hold:*
[PRHM1] $\mathsf{Ker}\,(\pi_M \otimes \pi_A) \subset \mathsf{Ker}\,(\pi_M \circ \mu_{M\otimes H})$;
[PRHM2] $(ma)_{[0]} \bullet (ma)_{[1]} = m_{[0]}a_{[0]} \bullet m_{[1]}a_{[1]}$ *for all* $m \in M$ *and* $a \in A$.
where

$$\mu_{M\otimes H} : M \otimes H \otimes A \otimes H \to M \otimes H,\ \ \mu_{M\otimes H}((m \otimes h)(a \otimes k)) = ma \otimes hk.$$

is the induced $A \otimes H$-module on $M \otimes H$.

Under these conditions, $M \bullet H$ is a right $A \bullet H$-module.

Proof. Similarly as in the proof of Proposition 5.2.2, consider the following diagram which expresses that the A-action $\mu_M : M \otimes A \to M$ is a morphism of partial comodules, and the construction of $(M \otimes A) \bullet H$ as pushout.



By construction, we know that

$$\mathsf{Ker}\,\pi_{M\otimes A} \ \ = \ \ \mu_{M,A}(\mathsf{Ker}\,(\pi_M \otimes \pi_A))$$

Then by Lemma 4.2.3, in order for the linear map μ_M to be a morphism of partial H-comodules it is needed that $(\mu_M \otimes H)(\mathsf{Ker}\,\pi_{M\otimes A}) \subset \mathsf{Ker}\,\pi_M$. Since $\mu_{M\otimes H} = (\mu_M \otimes H) \circ \mu_{M,A}$, this means furthermore that

$$\pi_M \circ (\mu_M \otimes H)(\mathsf{Ker}\,\pi_{M\otimes A}) = \pi_M \circ \mu_{M\otimes H}(\mathsf{Ker}\,(\pi_M \otimes \pi_A)) = 0$$

or equivalently, $\mathsf{Ker}\,(\pi_M \otimes \pi_A) \subset \mathsf{Ker}\,(\pi_M \circ \mu_{M\otimes H})$, i.e. [PRHM1] holds.

This condition implies that the map

$$\mu_{M\bullet H} = (\mu_A \bullet H) \circ \overline{\mu}_{A,A} : (M \bullet H) \otimes (A \bullet H) \to M \bullet H,$$
$$\mu_{M\bullet H}((m \bullet h)(a \bullet k)) = ma \bullet hk$$

is well-defined and defines an action of $A \bullet H$ on $M \bullet H$. Furthermore, μ_M will be a morphism of partial H-modules if and only if moreover

$$\rho_M \circ \mu_M = \mu_{M\bullet H} \circ (\rho_M \otimes \rho_A)$$

which gives exactly condition [PRHM2]. □

Example 5.3.3. Clearly any algebra in a lax monoidal category is a module over itself, hence any quasi partial H-comodule algebra $(A, A\bullet H, \pi_A, \rho_A)$ is also a quasi partial (A, H)-relative Hopf module.

The following observation will be useful later.

Lemma 5.3.4. *If M is a quasi partial (A, H)-relative Hopf module, then*

$$\pi_M(ma \otimes h) = 0$$

for all

(i) $m \otimes h \in \mathsf{Ker}\, \pi_M$ and $a \in A$;
(ii) $m \in M$ and $a \otimes h \in \mathsf{Ker}\, \pi_A$.

Proof. Since $\mathsf{Ker}\,(\pi_M \otimes \pi_A) = \mathsf{Ker}\, \pi_M \otimes A \otimes H + M \otimes H \otimes \mathsf{Ker}\, \pi_A$, we know that for all $m \otimes h \in \mathsf{Ker}\, \pi_M$ and $a \in A$, $m \otimes h + a \otimes 1_H \in \mathsf{Ker}\,(\pi_M \otimes \pi_A)$. Hence by axiom (PRHM1), we find that $\pi_M(ma \otimes h) = 0$. Similarly, for $m \in M$ and $a \otimes h \in \mathsf{Ker}\, \pi_A$, we have $m \otimes 1_H + a \otimes h \in \mathsf{Ker}\,(\pi_M \otimes \pi_A)$ and we can follow the same reasoning. □

5.3.2. Hopf-Galois theory.

Definition 5.3.5. Let H be a Hopf k-algebra, $(A, A\bullet H, \pi_A, \rho_A)$ a quasi partial H-comodule algebra and $(M, M\bullet H, \pi_M, \rho_M, \mu_M)$ a quasi partial (A, H)-relative Hopf module. The *H-coinvariants* of M are defined as the following equalizer in Vect_k

$$M^{coH} \longrightarrow M \underset{\pi_M\circ(M\otimes\eta_H)}{\overset{\rho_M}{\rightrightarrows}} M \bullet H \tag{5.3}$$

I.e. $M^{coH} = \{m \mid \rho_M(m) = m \bullet 1_H\}$.

Proposition 5.3.6. *Let H be a Hopf k-algebra, $(A, A \bullet H, \pi_A, \rho_A)$ a quasi partial H-comodule algebra.*

(i) The coinvariants A^{coH} of A form a subalgebra of A;
(ii) The coinvariants M^{coH} of a quasi partial (A, H)-relative Hopf module M form a module over A^{coH};
(iii) This induces a functor $(-)^{coH} : \mathsf{PMod}_A^H \to Mod_{A^{coH}}$.

Proof. (i). Since ρ_A and $\pi_A \circ (A \otimes \eta_H)$ are both algebra morphisms and the forgetful functor $U : \mathsf{Alg}_k \to \mathsf{Vect}_k$ creates limits, A^{coH} is a subalgebra of A. Alternatively, this follows by a direct computation

similar to the next part.
(ii). Consider the restriction of the multiplication map $\rho_M : M^{coH} \otimes A^{coH} \to M,\ \rho_M(m \otimes a) = ma$. Then

$$\rho(ma) = m_{[0]}a_{[0]}\bullet m_{[1]}a_{[1]} = (m_{[0]}\bullet m_{[1]})(a_{[0]}\bullet a_{[1]}) = (m\bullet 1_H)(a\bullet 1_H) = (ma\bullet 1_H)$$

hence $\rho_M(ma) \in M^{coH}$.
(iii). This part is easily verified. □

Let us remark that the coinvariant functor is representable.

Lemma 5.3.7. *For any quasi partial (A, H)-relative Hopf module, we have that*

$$M^{coH} \cong \mathsf{Hom}_A^H(A, M),$$

the set of partial H-comodule morphisms from A into M that are right A-linear.

Proof. Let $f : A \to M$ be a right A-linear morphism of partial H-comodules. Then the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & M \\ {\scriptstyle\rho_A}\downarrow & & \downarrow{\scriptstyle\rho_M} \\ A \bullet H & \xrightarrow{\ f\bullet H\ } & M \bullet H \\ {\scriptstyle\pi_A}\uparrow & & \uparrow{\scriptstyle\pi_M} \\ A \otimes H & \xrightarrow{\ f\otimes H\ } & M \otimes H \end{array}$$

Since we know that $\rho_A(1_A) = \pi_A(1_A \otimes 1_H)$, it follows from the commutativity of this diagram that $f(1_A) \in M^{coH}$. Conversely, given any $m \in M^{coH}$ we clearly have that the map $f_m : A \to M,\ f_m(a) = ma$ is right A-linear. Moreover, thanks Lemma 5.3.4(ii), we also know that $(f_m \otimes H)(\mathsf{Ker}\,\pi_A) \subset \mathsf{Ker}\,\pi_M$. Hence the morphism $f_m \bullet H : A \bullet H \to M \bullet H,\ f_m(a \bullet h) = ma \bullet h$ is well-defined. Finally, we find for any $a \in A$ that

$$\begin{aligned} \rho_M(f_m(a)) &= \rho_M(ma) = \rho_M(m)\rho_A(a) \\ &= (m \bullet 1_H)(a_{[0]} \bullet a_{[1]}) = ma_{[0]} \bullet a_{[1]} \\ &= f_m \bullet H(a_{[0]} \bullet a_{[1]}) \end{aligned}$$

I.e. $f_m \in \mathsf{Hom}_A^H(A, M)$. Therefore the above constructions provide well-defined mutual inverses between M^{coH} and $\mathsf{Hom}_A^H(A, M)$. □

Proposition 5.3.8. *For any A^{coH}-module N, the right A-module $N \otimes_{A^{coH}} A$ can be endowed with the structure of a quasi partial (A,H)-relative Hopf module by means of the following partial H-comodule structure*

$$\pi_{N\otimes_{A^{coH}}A} = N \otimes_{A^{coH}} \pi_A : \qquad N \otimes_{A^{coH}} A \otimes H \to N \otimes_{A^{coH}} (A \bullet H) =: (N \otimes_{A^{coH}} A) \bullet H$$
$$\rho_{N\otimes_{A^{coH}}A} = N \otimes_{A^{coH}} \rho_A : \qquad N \otimes_{A^{coH}} A \to N \otimes_{A^{coH}} (A \bullet H)$$

Moreover, this construction yields a functor $-\otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}^H_A$ that is a left adjoint to the coinvariant functor $(-)^{coH} : \mathsf{PMod}^H \to \mathsf{Mod}_{A^{coH}}$.

Proof. The construction of the functor $-\otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}^H_A$ is clear from the statement. To verify the adjunction property, we will define a counit ζ and a unit ν. For any quasi partial (A,H)-relative Hopf module M we define

$$\zeta_M : M^{coH} \otimes_{A^{coH}} A \to M \qquad \zeta_M(m \otimes_{A^{coH}} a) = ma$$

Clearly, ζ_M is a right A-linear map. Let us check that it is also a morphism of partial H-comodules. Firstly, we need to verify that $\pi_M \circ (\zeta_M \otimes H)(\mathsf{Ker}\,\pi_{M^{coH}\otimes_{A^{coH}}A}) = 0$ (see Lemma 4.2.3). Since by construction $\mathsf{Ker}\,\pi_{M^{coH}\otimes_{A^{coH}}A} = M^{coH} \otimes_{A^{coH}} \mathsf{Ker}\,\pi_A$, this follows directly by Lemma 5.3.4(ii). Secondly, we should check that $\rho_M \circ \zeta_M = (\zeta_M \bullet H) \circ (M^{coH} \otimes_{A^{coH}} \rho_A)$, where we know that $\zeta_M \bullet H$ is well-defined by the first part. Indeed, take any $m \otimes a \in M^{coH} \otimes_{A^{coH}A}$ then

$$\begin{aligned}
\rho_M(\zeta_M(m \otimes a)) &= (ma)_{[0]} \bullet (ma)_{[1]} = (m_{[0]} \bullet m_{[1]})(a_{[0]} \bullet a_{[1]}) \\
&= (m \bullet 1_H)(a_{[0]} \bullet a_{[1]}) = ma_{[0]} \bullet a_{[1]} \\
&= \zeta_M(m \otimes_{A^{coH}} a_{[0]}) \bullet a_{[1]} = (\zeta_M \bullet H)(m \otimes_{A^{coH}} \rho_A(a)).
\end{aligned}$$

On the other hand, for any A^{coH}-module N, we define

$$\nu_N : N \to (N \otimes_{A^{coH}} A)^{coH}, \ \nu_N(n) = n \otimes 1_A.$$

Since A is an algebra in the category of partial H-modules we have that $\rho_A(1) \in A^{coH}$ (see (5.1)). It is now easily verified that ζ and ν are indeed the counit and unit for this adjunction. □

Remark 5.3.9. Let $\iota : B \to A^{coH}$ be any ring morphism, then of course the adjunction from Proposition 5.3.8 can be combined with the extension-restriction of scalars functors, to obtain a pair of adjoint functors

$$-\otimes_B A : \mathsf{Mod}_B \rightleftarrows \mathsf{PMod}^H_A : (-)^{coH}.$$

In what follows we will only consider the case $B = A^{coH}$, but our results can be easily generalized to this slightly more general setting.

Definition 5.3.10. Let A be a quasi partial H-comodule algebra and $B \to A^{coH}$ a ring morphism. We call the morphism $\iota : B \to A$ a partial Hopf-Galois extension if and only if the following canonical map is bijective

$$\underline{\mathsf{can}} : A \otimes_B A \to A \bullet H, \underline{\mathsf{can}}(a \otimes_B a') = aa'_{[0]} \bullet a'_{[1]}$$

Remark that here $aa'_{[0]} \bullet a'_{[1]}$ denotes the product $(a \bullet 1_H)(a'_{[0]} \bullet a'_{[1]})$ which is well-defined since $m : A \otimes A \to A$ is a morphism of partial H-comodules.

The following examples show how partial Hopf-Galois extensions can be interpreted as "partial principle bundles".

Example 5.3.11. Let A be a global H-comodule algebra, and suppose that A/A^{coH} is Galois, i.e. the canonical map $A \otimes_{A^{coH}} A \to A \otimes H$ is bijective. Consider a surjective algebra morphism $p : A \to B$ and endow B with the induced structure of a partial comodule algebra. Then we obtain a canonical algebra morphism $A^{coH} \to B^{coH}$ and in fact $B^{coH} \cong A^{coH}/(A^{coH} \cap \mathsf{Ker}\, p)$. We obtain then that the following diagram commutes

$$\begin{CD} A \otimes_{A^{coH}} A @>{\underline{\mathsf{can}}_A}>> A \otimes H \\ @V{p \otimes p}VV @VV{\pi_B \circ (p \otimes H)}V \\ B \otimes_{B^{coH}} B @>{\underline{\mathsf{can}}_B}>> B \bullet H \end{CD}$$

If $\underline{\mathsf{can}}_A$ is an isomorphism, it is clear that $\underline{\mathsf{can}}_B$ is surjective. Moreover, consider any $b \otimes b' \in \mathsf{Ker}\, \underline{\mathsf{can}}_B$. Since $p \otimes p$ is surjective, we can write $b \otimes b' = p(a) \otimes p(a')$, such that $\underline{\mathsf{can}}(a \otimes a') \in \mathsf{Ker}\, \pi_B \circ (p \otimes H)$, but this means exactly that $\underline{\mathsf{can}}(a \otimes a') = u_{[0]} \otimes u_{[1]}$ for some $u \in \mathsf{Ker}\, p$. Since $\underline{\mathsf{can}}$ is bijective, this implies that $a \otimes a' = 1 \otimes u$ and therefore $b \otimes b' = p(1) \otimes p(u) = 0$. So $\underline{\mathsf{can}}_B$ is bijective as well, i.e. B is partially Hopf Galois.

Example 5.3.12. It is known that if an algebraic group G acts strictly transitive on an algebraic space X (i.e. X is a principal homogeneous G-space), then the coordinate algebra $A = \mathcal{O}(G)$ is $\mathcal{O}(G)$-Galois with trivial coinvariants. If we take any subvariety $Y \subset X$ then we know that $\mathcal{O}(Y)$ will be a partial $\mathcal{O}(G)$-comodule algebra. Applying the previous example, we find that that $\mathcal{O}(Y)$ will be partially Hopf-Galois. For example, the partial comodule algebras from Example 4.1.7, Example 4.1.8 (see Example 5.2.4) provide examples of partial principle homogeneous G-spaces (where G is respectively $k[x,y]$ and $k\langle x,y\rangle$).

More generally, X is a principal $\mathcal{O}(G)$-bundel if and only if $\mathcal{O}(X)$ is an $\mathcal{O}(G)$-Galois extension (with possible non-trivial coinvariants), see e.g. [**43**]. Again, any subvariety Y of the principle bundle X will give rise to a partial principle bundle.

In the global case, we know that if A/A^{coH} is Hopf-Galois and A is faithfully flat as left A^{coH}-module (this condition is in fact known to be too strong, see e.g. [**18**]), then the category of relative (A,H)-Hopf modules is equivalent to the category of A^{coH}-modules. Since in the partial setting it follows from earlier observations in this paper that the category of partial comodules is not abelian, the category of partial relative (A,H)-Hopf modules cannot be expected to be equivalent with a module category. Nevertheless, let us show that under the same mild conditions as in the global case, we can characterize when the functor $-\otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}^H_A$ is fully faithful. The following is an adaptation of the approach from [**18**] (see also [**20**]).

Recall that a morphism of left B-modules $f : N \to M$ is called *pure* if and only if for any right B-module P, the map $P\otimes_B f : P\otimes_B N \to P\otimes_B M$ is injective. In particular, if $\iota : B \to A$ be a ring morphism, then ι is said to be pure (as left B-module morphism) if for any right B-module P the map $\iota_P : P \to P\otimes_B A,\ \iota_P(p) = p\otimes_B 1_A$ is injective.

Lemma 5.3.13. *Let $\iota : B \to A$ be a ring morphism. Then the following statements are equivalent:*

(i) f is pure as left B-module morphism
(ii) For any right B-module N, the fork

$$N \xrightarrow{\iota_N} N\otimes_B A \underset{N\otimes_B \iota_A}{\overset{\iota_N\otimes_B A}{\rightrightarrows}} N\otimes_B A\otimes_B A \tag{5.4}$$

is an equalizer in Mod_B.

In particular, if A is faithfully flat as left A-module, then A is left pure.

Proof. <u>$(i)\Rightarrow(ii)$</u> Denote by E the equalizer of (5.4), and define $P = E/\iota_N(N)$. Take any $e\in E$, the we can write $e = n_i\otimes_B a_i$ and $n_i\otimes_B 1_A\otimes_B a_i = n_i\otimes_B a_i\otimes_B 1_A$. Apply $\pi\otimes_B A$ to this identity, then we have that $\iota_P(\pi(e)) = \pi(e)\otimes_B 1_A = \pi(n_i\otimes_B a_i)\otimes_B 1_A = \pi(n_i\otimes_B$

$1_A) \otimes_B a_i = 0$, since $n_i \otimes_B 1_A \in \iota_N(N)$. Since ι_P is injective, it follows that $\pi(e) = \pi(n_i \otimes_B a_i) = 0$ in $P = E/\iota(N)$, hence $n_i \otimes_B a_i \in \iota(N)$.
$\underline{(ii) \Rightarrow (i)}$. Since (5.4) is an equalizer, we have in particular that ι_N is injective. □

Proposition 5.3.14. *Let $\iota : B \to A$ be a partial H-Galois extension, then the functor $- \otimes_B A : \mathsf{Mod}_B \to \mathsf{PMod}_A^H$ is fully faithful if and only if ι is pure.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccc}
N & \xrightarrow{N\otimes_B \iota} & N \otimes_B A & \overset{N\otimes_B \iota \otimes_B A}{\underset{N\otimes_B A \otimes_B \iota}{\rightrightarrows}} & N \otimes_B A \otimes_B A \\
{\scriptstyle \nu_N}\downarrow & & \| & & \downarrow{\scriptstyle N\otimes_B \underline{\mathsf{can}}} \\
(N \otimes_B A)^{coH} & \longrightarrow & N \otimes_B A & \overset{\rho^{N\otimes_B A} = N\otimes_B \rho^A}{\underset{N\otimes_B A \bullet \eta_H}{\rightrightarrows}} & N \otimes_B A \bullet H
\end{array}$$

The lower row is an equalizer by the definition of the coinvariants $(N\otimes_B A)^{coH}$. Since $\underline{\mathsf{can}}$ is an isomorhpism, it then follows that the upper row is an equalizer if and only if ν_N is an isomorphism. The upper row in the above diagram is exactly (5.4). By the previous lemma, this means that $\iota : B \to A$ is pure if and only if the unit ν of the adjunction from Proposition 5.3.8 is a natural isomorphism, i.e. $- \otimes_B A$ is fully faithful. □

As we have remarked before, since partial comodules do not provide an abelian category, one cannot expect that the functor $- \otimes_B A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$ is an equivalence in general. The following observation shows that as soon as H is non-trivial, this functor will never be an equivalence. Indeed, it is clear by construction that any induced partial Hopf module $M = N \otimes_{A^{coH}} A$ satisfies $M \bullet H = M \otimes_A (A \bullet H)$. It follows however from Lemma 5.3.2 that in general we only have an inclusion $M \otimes_A (A \bullet H) \subset M \bullet H$. This motivates the following definition.

Definition 5.3.15. A partial relative Hopf module M is called *minimal* iff $M \bullet H = M \otimes_A (A \bullet H)$.

Example 5.3.16. Let A be a partial H-coaction as in Example 4.1.9, considered as a partial comodule algebra, see Example 5.2.4. Let M be a relative Hopf module in the sense of [**21**], that is M is a right A-module, endowed with a coaction $\rho : M \to M \otimes H,\ \rho(m) = m_{[0]} \otimes m_{[1]}$ satisfying

- $m = m_{[0]}\epsilon(m_{[1]})$;
- $\rho(m_{[0]}) \otimes m_{[1]} = m_{[0]}1_{[0]} \otimes m_{[1](1)}1_{[1]} \otimes m_{[1](2)}$;
- $\rho(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]}$.

Then by defining $M \bullet H = \{m1_{[0]} \otimes h1_{[1]} \mid m \otimes h \in M \otimes H\}$ we find that M can be endowed with the structure of a relative partial Hopf module in the sense defined here. Moreover, one then easily checks that this partial Hopf module is minimal.

Lemma 5.3.17. *Let A be a partial H-comodule algebra.*

(i) There is a canonical epimorphism $(A \bullet H) \bullet H \to (A \bullet H) \otimes_A (A \bullet H)$;
(ii) For any minimal relative partial Hopf module M, we have a canonical epimorphism $p_M : (M \bullet H) \bullet H \to (M \bullet H) \otimes_A (A \bullet H)$.

Consequently, $\mathcal{C} = A \bullet H$ is an A-coring and there is a functor from the category of geometric minimal relative partial Hopf modules to the category of $\mathcal{C}$-comodules.

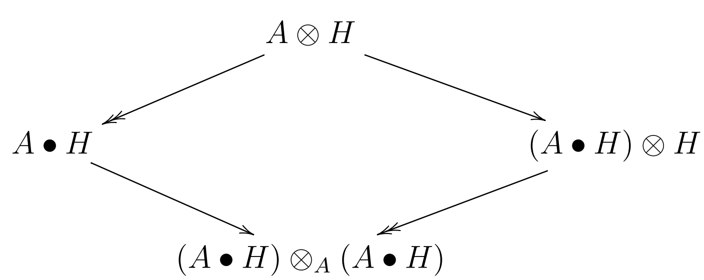
PROOF. (i). Consider the following diagram.

$$\begin{array}{ccccc} (A \bullet H) \otimes H \cong & \xrightarrow{\pi_{A\bullet H}} & (A \bullet H) \bullet H & \longrightarrow & 0 \\ \downarrow \phi & & & & \\ (A \bullet H) \otimes_A (A \otimes H) & \xrightarrow{(A\bullet H)\otimes_A \pi_A} & (A \bullet H) \otimes_A (A \bullet H) & \longrightarrow & 0 \end{array}$$

where ϕ is the isomorphism given by $\phi((a \bullet h) \otimes h) = (a \bullet h) \otimes_A (1_A \otimes h)$. One easily sees that $\mathsf{Ker}\,((A \bullet H) \otimes_A \pi_A) = (A \bullet H) \otimes_A \mathsf{Ker}\,\pi_A$. On the other hand, we know from the earlier sections that $\mathsf{Ker}\,\pi_{A\bullet H} = \rho_A \otimes H(\pi_A)$. Consider any $a \otimes h \in \mathsf{Ker}\,\pi_A$. Then we find that

$$(a_{[1]} \bullet a_{[1]}) \otimes_A (1_A \otimes h) = (1_A \bullet 1_H) \cdot a \otimes_A (1_A \otimes h) = (1_A \bullet 1_H) \otimes_A (a \otimes h)$$

Hence we find that $\phi(\mathsf{Ker}\,\pi_{A\bullet H}) \subset \mathsf{Ker}\,((A \bullet H) \otimes_A \pi_A)$. Consequently, ϕ induces an epimorphism $(A \bullet H) \bullet H \to (A \bullet H) \otimes_A (A \bullet H)$.
(ii). By part (i), we know that the following diagram commutes



Applying the functor $M \otimes_A -$ to this diagram, and using $M \otimes_A (A \bullet H) = M \bullet H$, we find that the following diagram commutes as well

$$\begin{array}{ccccc} & & M \otimes H & & \\ & \swarrow & & \searrow & \\ M \bullet H & & & & (M \bullet H) \otimes H \\ & \searrow & & \swarrow & \\ & & (M \bullet H) \otimes_A (A \bullet H) & & \end{array}$$

Hence, by the universal property of the pushout, we obtain an epimorphism $(M \bullet H) \bullet H \to (M \bullet H) \otimes_A (A \bullet H)$.
For the last statement, it is enough to remark that for a geometric relative partial Hopf module, the coassociativity holds in $M \bullet H \bullet H$. If M is minimal then by the above $(M \bullet H) \otimes_A (A \bullet H)$ is a quotient of $M \bullet H \bullet H$, so coassociativity also holds there. □

Remark 5.3.18. Given a comodule M over the coring $A \bullet H$, one can construct a relative partial comodule datum $(M, M \otimes_A (A \bullet H), M \otimes_A \pi_A, \rho_M)$. However, it is unclear if any such comodule datum provides a (geometric) partial H-comodule.

If A is flat as left B-module, then the functor $- \otimes_B A$ preserves all equalizers. Recall from [**18**] (see [**20**] for a corrected version of this theorem) that the functor $- \otimes_B A$ preserves the equalizers of the form (5.3) provided A is pure as left A-module and B lies in the center of A.

Proposition 5.3.19. *Let M be a relative partial Hopf module M.*

(i) If ζ_M (counit of the adjunction Proposition 5.3.8) is an isomorphism, then M is minimal.
(ii) If A/B is Hopf-Galois and ζ_M is a monomorphism, then M is minimal.
(iii) If M is minimal and geometric, A is partially Hopf Galois and the functor $- \otimes_B A$ preserves in partial the equalizers of the form (5.3) (e.g. A is flat as left B-module, or A is pure as left A-module and $B \subset Z(A)$), then ζ_M is an isomorphism.

Proof. (i) If ζ_M is an isomorphism of partial Hopf modules, then we find a composition of isomorphisms

$$M \otimes_A (A \bullet H) \xrightarrow{\zeta_M^{-1} \otimes_A (A \bullet H)} (M^{coH} \otimes_B A) \otimes_A (A \bullet H) \cong M^{coH} \otimes_B (A \bullet H) = (M^{coH} \otimes_B A) \bullet H \xrightarrow{\zeta_M \bullet H} M \bullet H$$

and hence M is minimal.

<u>(ii)</u>. Consider the following commutative diagram

$$\begin{array}{ccc} M^{coH} \otimes_B A & \longrightarrow & M \otimes_B A \\ \downarrow{\scriptstyle \zeta_M} & & \downarrow \\ M & \xrightarrow{\rho_M} & M \bullet H \end{array}$$

If A is flat as left B-module then the upper horizontal arrow is injective, and the lower horizontal arrow is injective as it has a left inverse $M \bullet \epsilon$. Since $M \otimes_B A \cong M \otimes_A (A \otimes_B A) \cong M \otimes_A (A \bullet H)$, we also know that the right vertical arrow is an epimorphism (see Lemma 5.3.2). Hence if ζ_M is injective, we find that $M \otimes_B A \to M \bullet H$ is an isomorphism, i.e. M is minimal.

<u>(iii)</u>. Since A/B is Hopf-Galois, we find obtain an isomorphism

$$M \otimes_B A \xrightarrow{\cong} M \otimes_A (A \otimes_B A) \xrightarrow{M \otimes_A \mathsf{can}} M \otimes_A (A \bullet H)$$

And therefore, if M is minimal we find that $M \otimes_B A \cong M \otimes_A (A \bullet H) = M \bullet H$. Consider now the following diagram

$$\begin{array}{ccccc} M^{coH} \otimes_B A & \longrightarrow & M \otimes_B A & \overset{\rho_M \otimes_B A}{\underset{\pi_M \circ (M \otimes \eta_H) \otimes_B A}{\rightrightarrows}} & M \bullet H \otimes_B A \\ {\scriptstyle \zeta'_M}\uparrow\downarrow{\scriptstyle \zeta_M} & & \downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle p_M} \\ M & \xrightarrow{\rho_M} & M \bullet H & \overset{\rho_M \bullet H}{\underset{\overline{M \bullet \Delta}}{\rightrightarrows}} & M \bullet H \bullet H \end{array}$$

By assumption, the upper row in this diagram is an equalizer. Since M is geometric, the fork on the lower row splits and hence is also an equalizer. The surjective morphism p_M is obtained from Lemma 5.3.17 and induces the morphism ζ'_M. Then a diagram chasing argument shows that ζ_M and ζ'_M are mutual inverses. □

The following result subsumes the Hopf-Galois theory for partial coactions in the sense of Caenepeel-Janssen [**21**].

Corollary 5.3.20. *Suppose that A a partial H-comodule algebra that is geometric as partial comodule. If A/B is a partial Hopf-Galois extension and either*

- *A is pure as left B-module and $B \subset Z(A)$;*
- *A is faithfully flat as left B-module;*

then $\mathsf{Mod}_{A^{coH}}$ is equivalent to the category full subcategory of PMod_A^H consisting of minimal geometric relative partial Hopf modules.

PROOF. Since A is geometric as partial comodule, the functor $-\otimes_B A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$ lands in the category of minimal geometric parial Hopf modules. By Proposition 5.3.14 and Proposition 5.3.19 we then obtain the stated equivalence of categories. □

As we have remarked before, the functor $- \otimes_B A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$ cannot be expected to become an equivalence of categories. More precisely, it follows from Proposition 5.3.19 that we cannot expect that the functor $(-)^{coH}$ is full whenever it is applied to non-minimal partial Hopf modules. We will finish our work by characterizing under which conditions this functor remains however faithful.

Proposition 5.3.21. *Under the same conditions as in Corollary 5.3.20, A is a generator in PMod_A^H if and only if the functor $(-)^{coH} : \mathsf{PMod}_A^H \to \mathsf{Mod}_{A^{coH}}$ is faithful.*

PROOF. Recall that A is a generator in PMod_A^H if and only if for any object M in PMod_A^H the canonical morphism

$$\phi_M : \coprod_{\mathsf{Hom}_A^H(A,M)} A \to M, \phi(a_f) = \sum_f f(a_f)$$

is surjective. Here $\coprod_{\mathsf{Hom}_A^H(A,M)} A$ denotes a coproduct of copies of A indexed by the set $\mathsf{Hom}_A^H(A, M)$.

Recall from Lemma 5.3.7 that $M^{coH} = \mathsf{Hom}_A^H(A, M)$ for any object M in PMod_A^H. Hence we obtain a well-defined morphism

$$\alpha_M : \coprod_{\mathsf{Hom}_A^H(A,M)} A \to M^{coH} \otimes_{A^{coH}} M,\ \alpha_M(a_f) = f(1_A) \otimes_{A^{coH}} a_f,$$

which is clearly surjective.

One now easily sees that $\phi_M = \zeta_M \circ \alpha_M$. Hence ϕ_M is surjective if and only if ζ_M is surjective. Finally, it is well-know that a right adjoint functor is faithful if and only if the counit of the adjunction is a natural epimorphism. □

Remark 5.3.22. Suppose that the conditions of Proposition 5.3.21, we know that for an object M in PMod_A^H, the partial Hopf module morphism $\zeta_M : M^{coH} \otimes_B A \to M$ is surjective. Hence, we find that $M \cong M \otimes_B A/\mathsf{Ker}\,\zeta_M$ as right A-module. However, as we remarked earlier, $\mathsf{Ker}\,\zeta_M$ is not necessarily a partial H-comodule. We then know from Example 4.1.6 that $M \otimes_B A/\mathsf{Ker}\,\zeta_M$ is a geometric partial comodule. Then ζ_M induces a morphism of partial Hopf modules $\zeta'_M : M \otimes_B$

$A/\mathsf{Ker}\,\zeta_M \to M$, such that the underlying A-module morphism is an isomorphism. However, ζ'_M is not necessarily an isomorphism of partial Hopf modules, since in general $(M \otimes_B A/\mathsf{Ker}\,\zeta_M) \bullet H$ and $M \bullet H$ can be different. Therefore consider the following definition.

Definition 5.3.23. Let $(X, X \bullet H, \pi_X, \rho_X)$ and $(Y, Y \bullet H, \pi_Y, \rho_Y)$ be two partial H-comodule data. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then consider the pushout

$$\begin{array}{ccccc} X \otimes H & \xrightarrow{f \otimes H} & Y \otimes H & \xrightarrow{\pi_Y} & Y \bullet H \\ \downarrow{\scriptstyle \pi_X} & & & & \downarrow{\scriptstyle p_Y} \\ X \bullet H & & \xrightarrow{\quad p_X \quad} & & P_{X,Y} \end{array}$$

With this notation, f is said to be a *weak morphism* of partial comodules, if the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \Big\downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho_Y} \\ & & Y \bullet H \\ & & \downarrow{\scriptstyle p_Y} \\ X \bullet H & \xrightarrow{\quad p_X \quad} & P_{X,Y} \end{array}$$

Then the A-linear inverse of ζ'_M will be a weak morphism of partial comodules that is moreover a 2-sided inverse of the (strong) morphism ζ'_M. This motivates that weak morphisms of (geometric) partial comodules might be better behaved that the strong morphisms we studied in this work. We will investigate this further in future work.

Let us finish by proving result which completely characterizes the image of the functor $- \otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$.

Theorem 5.3.24. *(i) If A and $A \bullet H$ are flat as left A^{coH}-module (e.g. A is flat as left A^{coH}-module and A/A^{coH} is H-Galois), then the functor $- \otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$ preserves equalizers.*

(ii) If A is faithfully flat as left A^{coH}-module then the functor $- \otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$ reflects isomorphisms.

(iii) If A is faithfully flat as left A^{coH}-module and A/A^{coH} is partially Hopf-Galois, then the category of A^{coH}-modules is equivalent to the Eilenberg-Moore category $(\mathsf{PMod}_A^H)^{\mathbb{C}}$, where $\mathbb{C}$ is the comonad associated to the adjoint pair of Proposition 5.3.8.

PROOF. (i). Consider the following equalizer diagram in $\mathsf{Mod}_{A^{coH}}$:

$$E \xrightarrow{\quad e \quad} N \underset{g}{\overset{f}{\rightrightarrows}} M$$

By the flatness of A as a left A^{coH}-module, we then know that $(E\otimes_{A^{coH}} A, e\otimes_{A^{coH}} A)$ is the equalizer of the pair $(f\otimes_{A^{coH}} A, g\otimes_{A^{coH}} A)$ in Mod_A. However, we have to show that this is also an equalizer in PMod_A^H. To this end, consider any partial relative Hopf module T with a morphism $t : T \to E \otimes_{A^{coH}} A$ such that $(f \otimes_{A^{coH}} A) \circ t = (g \otimes_{A^{coH}} A) \circ t$. Then we can apply the forgetful functor $\mathsf{PMod}_A^H \to \mathsf{Mod}_A$ and we find that there exists a unique right A-linear map $u : T \to E \otimes_{A^{coH}} A$ such that $t = e \otimes_{A^{coH}} A \circ u$. We will be done if we can show that u is a morphism of partial H-comodules. Firstly we will verify that $\pi_{E\otimes_{A^{coH}} A} \circ (u \otimes H)(\mathsf{Ker}\, \pi_T) = 0$ (cf. Lemma 4.2.3). Since $A \bullet H$ is flat as a left A-module and e is an injective map (being an equalizer in a module category), it is equivalent to check that

$$(e \otimes_{A^{coH}} (A \bullet H)) \circ \pi_{E\otimes_{A^{coH}} A} \circ (u \otimes H)(\mathsf{Ker}\, \pi_T) = 0.$$

Since $(e \otimes_{A^{coH}} (A \bullet H)) = (e \otimes_{A^{coH}} A) \bullet H$ (functoriality of $- \otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$) and e is a morphism of partial comodules we can rewrite the left hand side of the last equality as

$$\pi_{X\otimes_{A^{coH}} A}\circ(e\otimes_{A^{coH}} A\otimes H)\circ(u\otimes H)(\mathsf{Ker}\, \pi_T) = \pi_{X\otimes_{A^{coH}} A}\circ(t\otimes H)(\mathsf{Ker}\, \pi_T) = 0$$

where the second equality is the defining property of u and the last equality follows from the fact that t is a morphism of partial comodules. Hence the map $u \bullet H : T \to E \otimes_{A^{coH}} (A \bullet H)$ is well defined and the unique map satisfying $u \bullet H \circ \pi_T = \pi_{E\otimes_{A^{coH}}} \circ u \otimes H$. Then using $t = e \otimes_{A^{coH}} A \circ u$ and the surjectivity of π_T, we find that also $(t \bullet H) = (e \otimes_{A^{coH}} A \bullet H) \circ (u \bullet H)$. For u to be a partial comodule morphism, it remains to check that $(u \bullet H) \circ \rho_T = \rho_{E\otimes_{A^{coH}} A} \circ u$. Again, using the flatness of $A \bullet H$ a left A-module and the injectivity of e it is sufficient to check that the compositions of the se maps with $e \otimes_{A^{coH}} (A \bullet H)$ are equal. Using the fact that t and e are partial module morphisms, we can indeed prove that

$$\begin{aligned}
(e \otimes_{A^{coH}} (A \bullet H)) \circ (u \bullet H) \circ \rho_T &= (t \bullet H) \circ \rho_T = \rho_{X\otimes_{A^{coH}}} \circ t \\
&= \rho_{X\otimes_{A^{coH}}} \circ e \otimes_{A^{coH}} A \circ u \\
&= (e \otimes_{A^{coH}} (A \bullet H)) \circ \rho_{E\otimes_{A^{coH}} A} \circ u
\end{aligned}$$

Hence u lives already in PMod_A^H and therefore $(E \otimes_{A^{coH}} A, e \otimes_{A^{coH}} A)$ satisfies the universal property of the equalizer in PMod_A^H.
(ii). Let $f : M \to N$ be a morphism in $\mathsf{Mod}_{A^{coH}}$ such that $f \otimes_{A^{coH}} A$ is an isomorphism in PMod_A^H. Then $f \otimes_{A^{coH}} A$ is also an isomorphism in Mod_A, and since A is faithfully flat as left A-module, we find that f is an isomorphism in $\mathsf{Mod}_{A^{coH}}$.
(iii). This follows immediately from the previous two parts by the dual of Beck's monadicity theorem, see e.g. [**32**, Theorem 2.7] □

Remark 5.3.25. The proof of part (i) in the previous theorem can be adapted to show that the functor $- \otimes_{A^{coH}} A : \mathsf{Mod}_{A^{coH}} \to \mathsf{PMod}_A^H$ preserves arbitrary limits.

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