# Weakening Transferable Utility: the Case of Non-intersecting Pareto Curves


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# Weakening transferable utility: the case of non-intersecting Pareto curves

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**Abstract**

Transferable utility (TU) is a widely used assumption in economics. In this paper, we weaken the TU property to the setting where distinct Pareto frontiers have empty intersections. We call this the no-intersection property (NIP). We show that the NIP is strictly weaker than TU, but still maintains several desirable properties. We discuss the NIP property in relation to several models where TU has turned out to be a key assumption: models of assortative matching, the Coase theorem and Becker's Rotten Kid theorem. We also investigate classes of utility functions for which the NIP holds uniformly.

**Keywords**: Pareto efficiency, Transferable utility, Kaldor-Hicks compensation criterion, Assortative matching, Coase theorem, Rotten Kid theorem

**JEL**: C78; D13; D60; D61; D62.

## 1 Introduction

Transferable utility (TU) is a widely used assumption in economics. A model is said to have transferable utility when an agent can transfer part of her utility to another agent in a lossless manner. The main attractive feature of the TU assumption stems from its desirable aggregation properties. Under TU, individual members may affect the location of the Pareto frontier but not its shape which is, up to normalization, a straight line with slope minus one. This means that, under Pareto optimality, the distribution of utilities over the agents does not affect the group's decision which will be to pick the Pareto frontier farthest from the origin. Results that exploit this property are the famous Coase theorem

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(Coase, 1960) and Becker's Rotten Kid theorem (Becker, 1974). The TU assumption is also frequently used in other economic environments like one-to-one matching models.

It is well known that if TU does not hold, Pareto frontiers may intersect. Such intersections lead to several paradoxes, in particular, a potential failure of the Coase and Rotten Kid theorems and cycles in the Kaldor-Hicks compensation criterion. In this respect, there appears to be a discontinuity from the rather restrictive transferable utility setting where aggregation is very easy, and the general setting with intersecting Pareto curves, where it's very difficult to obtain clear-cut results. In this paper, we weaken the notion of transferable utility, but retain the condition that distinct Pareto frontiers have empty intersections. We call this the No-Intersection Property (NIP). As we show in the paper, many of the properties attributed to TU also hold under the NIP. To the best of our knowledge, the NIP has not yet been studied as a separate property.

It is a well known fact that if a model has transferable utility then distinct Pareto frontiers have empty intersections. As such, any model that satisfies TU will also satisfy the NIP. It can be hypothesized whether the converse is also true. In this paper, we show that the answer is negative. Towards this result, we first show that the NIP is equivalent to the existence of a *Pareto (aggregation) function* which combines the utility levels of all agents into a single number and a *surplus function* that associates with every Pareto frontier a number that indicates its distance from the origin. A profile of utility values (one for each agent) belongs to a certain Pareto frontier if the value of the Pareto function equals the value of the surplus function. As such, higher numbers for the Pareto function correspond to profiles of utilities that lie on Pareto curves further from the origin. We show that TU is obtained if and only if this Pareto function is *additively separable*.

Next, we revisit some results from the literature that heavily rely on the TU assumption and we look at them from the perspective of the weaker NIP assumption. As a first exercise, we look at models of (two-sided) matching. We refer to Chade, Eeckhout, and Smith (2017) and Chiappori (2017) for a recent overview of matching and search theory. In much of this literature, attention has been devoted to either cases where there are no transfers allowed between agents (the so called non-transferable utility (NTU) case, initiated by Gale and Shapley (1962)) or cases with perfectly transferable utility (the TU case, initiated by Shapley and Shubik (1972)). For the latter, the seminal contribution by Becker (1973) showed that, when the match surplus function is supermodular, then any pairwise stable matching exhibits *positive assortative matching*. The fact that matching patterns under TU can be easily analysed in terms of the surplus function makes it a very attractive model for theoretical and empirical applications.[1]

The relative simplicity of analysing matching patterns under TU stems from the fact that the problem of distributing the surplus within a couple and maximizing the total surplus over all possible matches can be treated separately. Essentially, our NIP condition can be seen as positing the potential for a similar separation, but then in the context of an

[1] See for example the empirical framework of Choo and Siow (2006) or the more complex strategic settings in which agents perform pre-match investments in education and then decide on partner choice as in Chiappori, Costa Dias, and Meghir (forthcoming).

imperfectly transferable utility framework, which is an intermediate setting between TU and NTU. We show that under the NIP, if the surplus function satisfies some monotonicity condition with respect to the types of the agents, then the submodularity of the Pareto function and the supermodularity of the surplus function, with at least one of these conditions strict, is sufficient for positive assortative matching patterns to arise in equilibrium. We also provide combinatorial and differential sufficient conditions for positive assortative matching. This can be contrasted with Legros and Newman (2007), who provide sufficient conditions on the Pareto frontiers for assortative matching to occur, in a more general setting where Pareto frontiers are also allowed to intersect.

As a second exercise, we revisit the Coase theorem. We present a formal setting along the lines of Hurwicz (1995) and Bergstrom (2017) and show that, in our setting, the Coase theorem holds if and only if the NIP holds. The final exercise reconsiders the so-called Becker's Rotten Kid theorem (Becker, 1974). The name 'Rotten Kid' originates from the usual description of the two-stage game, where after the kids take certain actions which determine the level of the public goods, the parent has to divide the remaining resources over the kids. The Rotten Kid theorem states that in such setting, any Pareto efficient level of public goods can be implemented as a subgame perfect equilibrium of this two-stage game. This result is remarkable and important for the theory of incentives and household economics (e.g. Chiappori and Mazzocco (2017)). Essentially, it provides conditions such that there is no need for complex incentive schemes to implement Pareto optimal allocations. A particular way to retrieve this result is when the kids in the first stage have preferences that lead to Pareto frontiers that are representable as a simplex (i.e. there is TU), joint with a 'benevolent' (altruistic) parent (Bergstrom, 1989). We show that the NIP condition, together with a relatively mild assumption on the selection rule of the optimal allocations by the parent, recovers the Rotten Kid theorem.

In a final part of the paper, we try to identify classes of utility functions that lead to utility possibility sets satisfying the NIP. Focussing on a similar public goods setting as for the Coase and Rotten Kid theorems, we define the notion of a maximal NIP admissible class of utility functions. These are classes of utility functions over which the NIP condition holds uniformly. We show that there is a trade off between the size of such maximal admissible class and the restrictions one is willing to impose on the set of feasible allocations. We characterize maximal admissible NIP classes for three such restrictions. In settings where the feasible allocations are determined by a technology which is linear in the private good, we show that maximal admissible NIP classes are generated by the generalized quasi-linear (GQL) utility specifications. GQL utility functions have been widely studied in connection to the TU property (Bergstrom, 1989; Chiappori, 2010; Cherchye, Demuynck, and De Rock, 2015; Chiappori and Gugl, 2015). Indeed, in this case, we find that the NIP condition is equivalent to TU. When we weaken the feasibility constraint to be homothetic over the private commodities the maximal admissible NIP classes become smaller. In this setting, all agents have identical preferences that are generated by utility functions that are proportional in the amount of the private good. In such cases, however, the NIP and the TU properties do not necessarily coincide. Finally, we look at the case where the feasibility constraint has a Leontief type specification over the private commodities. In such setting, we again find that all agents should have identical preferences but now there are no restrictions on the shape of this preference ordering.

The structure of the paper is as follows. Section 2 introduces notation, definitions and assumptions. Section 3 introduces the NIP condition and provides the representation theorems. Section 4 relates the NIP with TU. Section 5 discusses the NIP in relation to models of assortative matching, the Coase theorem and Becker's Rotten Kid theorem. Section 6 characterizes several classes of utility functions that lead to the NIP. Section 7 contains a conclusion. All proofs are contained in the appendix.

# 2 Notation and definitions

We consider a setting with $N$ agents. The utility level of agent $i \leq N$ is denoted by $u_i \in \mathbb{R}$. A *utility profile* $\mathbf{u} = (u_1, \ldots, u_N) \in \mathbb{R}^N$ associates a utility level to each agent $i$. Utility profiles are denoted by bold letters $(\mathbf{u}, \mathbf{v}, \mathbf{w})$. We use vector inequalities $\mathbf{u} \geq \mathbf{v}$ if $u_i \geq v_i$ for all $i \leq N$, we write $\mathbf{u} > \mathbf{v}$ if $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$ and we write $\mathbf{u} \gg \mathbf{v}$ if $u_i > v_i$ for all $i$.

A function $g : \mathbb{R}^N \to \mathbb{R}$ is *weakly increasing (decreasing)* if $\mathbf{u} \gg \mathbf{v}$ implies $g(\mathbf{u}) > (<)\, g(\mathbf{v})$ and $\mathbf{u} \geq \mathbf{v}$ implies $g(\mathbf{u}) \geq (\leq) g(\mathbf{v})$. We call the function *increasing (decreasing)* if $\mathbf{u} > \mathbf{v}$ implies $g(\mathbf{u}) > (<) g(\mathbf{v})$.

We assume that each agent has an outside option that gives a utility level $\underline{u_i}$ which we normalize, for convenience, to zero, i.e. $\underline{u_i} = 0$. Let $\Phi \subseteq \mathbb{R}^k$ be a compact set containing the possible values of the environmental variables relevant to the model under consideration. A vector $\phi \in \Phi$ specifies all variables that determine the utility possibility set of the $N$-agent model. For each value $\phi \in \Phi$, we can associate a *utility possibility set* that contains all feasible and individual rational utility profiles that are attainable when the environmental variables take on the value $\phi$. We denote this utility possibility set by $C(\phi)$.

$$C(\phi) = \left\{\mathbf{u} \in \mathbb{R}^N_+ : \mathbf{u} \text{ is attainable in situation } \phi\right\}.$$

We denote by $\mathcal{D} = \bigcup_{\phi \in \Phi} C(\phi)$ the set of all relevant utility profiles, i.e. the utility profiles that are in some utility possibility set.

Throughout this paper we will impose the following assumption on the collection $(C(\phi))_{\phi \in \Phi}$.

**Assumption A.** *For all $\phi \in \Phi$, $C(\phi)$ is non-empty, closed, bounded and comprehensive, i.e. if $\mathbf{u} \in C(\phi)$ and $\mathbf{0} \leq \mathbf{v} \leq \mathbf{u}$ then $\mathbf{v} \in C(\phi)$.*

We denote the *Pareto frontier* of $C(\phi)$ by $\partial C(\phi)$. It is defined as the upper boundary of the set $C(\phi)$.

$$\partial C(\phi) = \{\mathbf{u} \in C(\phi) : \forall \mathbf{v} \gg \mathbf{u}, \mathbf{v} \notin C(\phi).\}.$$

Given assumption A this set is well defined and non-empty. Let us provide two examples.

**Example 1.** *Consider a group with $N$ members who obtain utility from leisure and the consumption of a public good, which is produced within the group. We denote by $\ell_i$ the* *leisure of group member $i$ and by $Q$ the amount of the public good. The total time available by each member is normalized to unity. It is assumed that the group members have identical utility functions, $u_i(\ell_i, Q)$ given by,*

$$u_i(\ell_i, Q) = \alpha(Q)\ell_i,$$

*where $\alpha : \mathbb{R}_+ \to \mathbb{R}_{++}$ is an increasing function.*

*The amount of the public good $Q$ is determined by the amounts of labour $1 - \ell_i$ of the group members. This is summarized by a continuous, decreasing function $g : \mathbb{R}^N \to \mathbb{R}_+$,*

$$g(\ell_1, \ldots, \ell_N) = Q.$$

*To avoid corner solutions, we assume that if for some member $i$, $\ell_i = 1$, then $g(\ell_1, \ldots, \ell_N) \leq 0$. If we solve for $\ell_i$ in terms of $u_i$ and $Q$ and substitute this into the function $g$, we can determine for each level of public good $Q$ the corresponding utility possibility set $C(Q)$,*

$$C(Q) = \left\{(u_1, \ldots, u_N) \in \mathbb{R}^N_+ : g\left(\frac{u_1}{\alpha(Q)}, \ldots, \frac{u_N}{\alpha(Q)}\right) \geq Q\right\}.$$

*This fits our framework where $\Phi = [0, \overline{Q}]$ is the set of possible public good levels and $\overline{Q} = g(0, \ldots, 0)$ is the maximal level of the public good.*

**Example 2.** *Consider a set of employees $E$ and a set of firms $F$. Every employee $e \in E$ has a type, $\delta$ from a set $\Delta$ and every firm $f \in F$ has a type $\xi$ from a set $\Xi$. If an employee $e$ and a firm $f$ are matched, they can produce at most a single unit of output which is sold at a price $P$. If the good is not produced, then the utility level of the employee and profits of the firm are both zero. If the good is produced, then the utility of the employee is given by an increasing utility function $u(\ell, w)$ and the utility (profit) of the firm $f$ is equal to $\pi = P - w$, where $\ell$ is the leisure of the employee and $w$ is the wage (i.e. amount of money transferred from the firm to the employee).*

*Each employee has a total time endowment of 1. We assume that the good is produced if and only if,*

$$1 - \ell \geq \omega(\delta, \xi).$$

*where $\omega$ is some threshold function that depends on the type of both firm and employee. We can define the set of environmental variables as $\Phi = \Delta \times \Xi$. Inverting the utility function $u = u(\ell, w)$ with respect to $\ell$ gives a function $\ell(u, w)$ which is increasing in $u$ and decreasing in $w$. Next, substituting the wage $w = P - \pi$ into this function gives the following formulation of the utility possibility set for given types $(\delta, \xi) \in \Phi$,*

$$C(\delta, \xi) = \left\{(u, \pi) \in \mathbb{R}^2_+ : \ell(u, P - \pi) \leq 1 - \omega(\delta, \xi)\right\}.$$

# 3 The no intersection property

A collection of utility possibility sets satisfies the *no intersection property* if distinct Pareto frontiers have empty intersections.

**Definition 3.1** (No-Intersection Property (NIP))**.** *Assume that the collection of utility possibility sets* $(C(\phi))_{\phi\in\Phi}$ *satisfies Assumption A. Then* $(C(\phi))_{\phi\in\Phi}$ *satisfies the no intersection property (NIP), if for all* $\phi, \psi \in \Phi$*,*

$$\textit{if } \partial C(\phi) \cap \partial C(\psi) \neq \emptyset, \textit{ then } C(\phi) = C(\psi).$$

It will often be convenient to use an equivalent definition of the NIP. For two subsets $A, B \subseteq \mathbb{R}^N_+$ we write $A \sqsubset B$ if for all vectors $\mathbf{u} \in A$ there exists a vector $\mathbf{v} \in B$ such that $\mathbf{v} \gg \mathbf{u}$. We write $A \sqsubseteq B$ if $A = B$ or $A \sqsubset B$.

The following theorem shows that a collection of utility possibility sets $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP, if and only if these sets are completely ordered by the binary relation $\sqsubseteq$.

**Theorem 3.2.** *Let the collection of utility possiblity sets* $(C(\phi))_{\phi\in\Phi}$ *satisfy Assumption A. Then* $(C(\phi))_{\phi\in\Phi}$ *satisfies the NIP if and only if the binary relation* $\sqsubseteq$ *is a complete ordering (reflexive and transitive) on* $(C(\phi))_{\phi\in\Phi}$ *with asymmetric part* $\sqsubset$*, i.e.* $A \sqsubset B$ *if and only if* $A \sqsubseteq B$ *and not* $B \sqsubseteq A$*.*

In order to state our next representation result, we need to introduce two assumptions. The first requires a richness condition on the collection of utility possiblity sets.

**Assumption B.** *For all utility vectors* $\mathbf{u} \in \mathcal{D} = \cup_{\phi\in\Phi} C(\phi)$*, there is a value* $\phi \in \Phi$ *such that* $\mathbf{u} \in \partial C(\phi)$*.*

A second assumption imposes continuity on the utility possibility correspondence.

**Assumption C.** *The set of environmental variables* $\Phi$ *is compact and the utility possibility correspondence* $C : \Phi \twoheadrightarrow \mathbb{R}^N_+$ *is continuous, i.e. both upper and lower hemicontinuous at all* $\phi \in \Phi$*.*[2]

Theorem 3.2 above shows that the NIP is equivalent to having a ranking on the utility possibility sets by the binary relation $\sqsubseteq$. Given this, one might try to give each Pareto frontier a numerical value that reflects this ranking. The intuition is similar to a setting with indifference curves where the utility values of the indifference curves represent the preference ordering. Drawing the analogy further, in a consumption setting we usually introduce utility functions that associates to any bundle of goods the numerical value of the indifference curve through this bundle. In our setting, bundles are represented by utility profiles $\mathbf{u}$. Similarly, we would like to given number to utility profiles that correspond to the values of the Pareto frontier through this profile. The following theorem is a formalization of this intuition.

**Theorem 3.3.** *Let the collection of utility possibility sets* $(C(\phi))_{\phi\in\Phi}$ *satisfy Assumption A.*

[2] In particular, given Assumption A upper hemicontinuity requires that for all $\phi^t \to \phi$ and $\mathbf{u}^t \in C(\phi^t)$ for all $t$, then there is a subsequence $(\phi^{t_i})_{i\in\mathbb{N}}$ such that $\mathbf{u}^{t_i} \to \mathbf{u}$ and $\mathbf{u} \in C(\phi)$. Lower hemicontinuity requires that if $\phi^t \to \phi$ and $\mathbf{u} \in C(\phi)$ then there is exists a number $N \in \mathbb{N}$ and a sequence $(\mathbf{u}^t)_{t\geq N}$ such that $\mathbf{u}^t \to \mathbf{u}$ and $\mathbf{u}^t \in C(\phi^t)$ for all $t \geq N$.

1. *The collection of utility possibility sets* $(C(\phi))_{\phi\in\Phi}$ *satisfies the NIP if and only if there exists a function* $\rho : \Phi \to \mathbb{R}$ *such that for all* $\phi, \psi \in \Phi$,

$$C(\phi) \sqsubseteq C(\psi) \leftrightarrow \rho(\phi) \leq \rho(\psi).$$

2. *If* $(C(\phi))_{\phi\in\Phi}$ *satisfies Assumption B. Then* $(C(\phi))_{\phi\in\Phi}$ *satisfies the NIP if and only if there exists a weakly increasing function* $h : \mathcal{D} \to \mathbb{R}$ *and a function* $\rho : \Phi \to \mathbb{R}$ *such that for all* $u \in \mathcal{D}, \phi \in \Phi$,

$$u \in C(\phi) \leftrightarrow h(u) \leq \rho(\phi).$$

3. *If* $(C(\phi))_{\phi\in\Phi}$ *satisfies Assumptions B and C, then the functions* $\rho$ *and* $h$ *from part 2 can be chosen to be continuous.*

4. *If there exists a continuous function* $h : \mathbb{R}^N \to \mathbb{R}$ *and a function* $\rho : \Phi \to \mathbb{R}$ *such that for all* $u \in \mathbb{R}^N, \phi \in \Phi$,

$$h(u) \leq \rho(\phi) \leftrightarrow u \in C(\phi),$$

*then* $(C(\phi))_{\phi\in\Phi}$ *satisfies the NIP.*

Parts 1, 2, 3 of Theorem 3.3 provide various representation results for the NIP condition for varying assumptions on the utility possibility sets. If $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP, then we call the function $h$ of this representation a *Pareto (aggregation) function* and we call $\rho(\phi)$ a *surplus function*. Utility profiles $\mathbf{u}$ with higher values of $h(\mathbf{u})$ lie on Pareto frontiers further away from the origin, i.e. Pareto frontiers $\partial C(\phi)$ with larger values of $\rho(\phi)$. Observe that both of these functions are only unique up to a (continuous) monotone transformation. Part 4 of Theorem 3.3 gives a useful sufficient condition for a collection of utility possibility sets to satisfy the NIP.

Theorem 3.3 shows that (given Assumptions A and B) any Pareto frontier $\partial C(\phi)$ can be formalized as the set of vectors $\mathbf{u} \in \mathcal{D}$ for which,

$$h(\mathbf{u}) = \rho(\phi).$$

If the Pareto frontier is strictly decreasing, it is possible to invert this identity with respect to $u_1$ and derive the following representation of the Pareto frontier $\partial C(\phi)$,

$$u_1 = f(u_2, \ldots, u_N; \rho(\phi)).$$

Here, $f(u_2, \ldots, u_N; \rho(\phi)) \equiv h^{-1}(u_2, \ldots, u_N; \rho(\phi))$ is decreasing in $u_2, \ldots, u_N$ and increasing in $\rho(\phi)$. In general, absent the NIP condition, any strictly decreasing Pareto frontier can be represented via a function

$$u_1 = g(u_2, \ldots, u_N; \phi),$$

which is also decreasing in $u_2, \ldots, u_N$. This shows that for the NIP condition to hold, there should exist a function $f$ (decreasing in its first $N-1$ arguments and increasing in its last argument) and a surplus function $\rho$ such that,

$$g(u_2, \ldots, u_N; \phi) = f(u_2, \ldots, u_N; \rho(\phi)).$$

This requires that the Pareto frontier function $g(u_2, \ldots, u_N; \phi)$ is weakly separable in $\phi$. A necessary condition for weak separability is that, for all $u_2, \ldots, u_N, v_2, \ldots, v_N$ and all $\phi$ and $\psi$,

$$\text{if } g(u_2, \ldots, u_N; \phi) \geq g(u_2, \ldots, u_N, \psi) \text{ then}, g(v_2, \ldots, v_N; \phi) \geq g(v_2, \ldots, v_N; \psi).$$

If the function $g$ is $C^2$, (i.e. 2 times continuously differentiable). then a local condition is given by the Leontief-Sono conditions (Leontief, 1947; Sono, 1961): for all $\ell, j \leq k$ and all $i = 2, \ldots, N$:[3]

$$\frac{\partial}{\partial u_i}\left(\frac{\frac{\partial g(u_2,\ldots,u_N;\phi)}{\partial \phi_\ell}}{\frac{\partial g(u_2,\ldots,u_N;\phi)}{\partial \phi_j}}\right) = 0.$$

where $\phi_\ell$ and $\phi_j$ are the $i$th and $j$th component of the vector $\phi$. We refer to Blackorby, Primont, and Russell (1978) for an in depth study of the property of (weak) separability.

We now return to our previous two examples and show that, under certain conditions, the NIP holds.

**Example** (example 1 continued). *Let's recall that the utility possibilities sets, $C(Q)$ were implicitly given by*

$$\mathbf{u} \in C(\phi) \leftrightarrow -g\left(\frac{u_1}{\alpha(Q)}, \ldots, \frac{u_N}{\alpha(Q)}\right) \leq -Q.$$

*If we assume that the function $-g$ is homothetic, i.e. there is an increasing function $r : \mathbb{R} \to \mathbb{R}$ and a homogeneous function $s : \mathbb{R}^N_+ \to \mathbb{R}$ such that*[4]

$$-g(v_1, \ldots, v_N) = r(s(v_1, \ldots, v_n)).$$

*then we obtain,*

$$\begin{aligned}
\mathbf{u} \in C(\phi) &\leftrightarrow -g\left(\frac{u_1}{\alpha(Q)}, \ldots, \frac{u_N}{\alpha(Q)}\right) \leq -Q, \\
&\leftrightarrow r\left(\frac{1}{(\alpha(Q))^t} s(u_1, \ldots, u_N)\right) \leq -Q, \\
&\leftrightarrow s(u_1, \ldots, u_N) \geq r^{-1}(-Q)\ (\alpha(Q))^t.
\end{aligned}$$

*where $t$ is the degree of homogeneity of the function $s$. If we define $h(u_1, \ldots, u_N) = s(u_1, \ldots, u_N)$ and $\rho(Q) = r^{-1}(-Q)\ (\alpha(Q))^t$, we see that, by part 4 of Theorem 3.3, $(C(Q))_{Q \in [0, \overline{Q}]}$ satisfies the NIP.*

**Example** (example 2 continued). *For all values $(\delta, \xi) \in \Phi$ we had that $(u, \pi) \in C(\delta, \xi)$ if and only if,*

$$\ell(u, P - \pi) \leq 1 - \omega(\delta, \xi),$$

*Setting $h(u, \pi) = \ell(u, P - \pi)$ and $\rho(\delta, \xi) = 1 - \omega(\delta, \xi)$ shows that, by part 4 of Theorem 3.3, $(C(\delta, \xi))_{(\delta,\xi) \in \Phi}$ satisfies the NIP.*

[3] These conditions will be used in the proofs of Theorems 6.4 and 6.7.

[4] The function $s$ is homogeneous of degree $t$ if for all $\alpha > 0$, $s(\alpha u_1, \ldots, \alpha u_N) = \alpha^t s(u_1, \ldots, u_N)$.

**The Kaldor-Hicks criterion** As an alternative to Theorem 3.2, we can give a characterisation of the NIP in terms of the collections of utility possibility sets that gives a consistent ranking according to the Kaldor-Hicks compensation criterion (Hicks, 1939; Kaldor, 1939). The Kaldor-Hicks criterion states that a change from one situation to another is an improvement if the agents that gain from the switch could, in theory, compensate those who are harmed by the change. It is well-known that the Kaldor-Hicks compensation criterion may lead to inconsistent judgements when Pareto curves intersect. However, as we will show below, this is the only case where this can occur. In other words, the criterion leads to consistent judgement for a collection of utility possibility sets if and only if the NIP is satisfied.

Let $\mathbf{u}, \mathbf{v} \in \mathcal{D}$ be two utility profiles. The Kaldor-Hicks criterion determines $\mathbf{u}$ better than $\mathbf{v}$ if there exists a profile $\mathbf{w}$ which is for all agents an improvement over $\mathbf{v}$ and $\mathbf{w}$ was feasible whenever $\mathbf{u}$ is feasible. We formalize this in the following way.

**Definition 3.4.** *Let $(C(\phi))_{\phi\in\Phi}$ be a collection of utility possibility sets and let $\mathbf{u}, \mathbf{v} \in \mathcal{D}$. If $\mathbf{u} \in \partial C(\phi)$, then $\mathbf{u}$ is a strict Kaldor-Hicks improvement over $\mathbf{v} \in \mathcal{D}$ if there exists a $\mathbf{w} \in \mathcal{D}$ with $\mathbf{w} \gg \mathbf{v}$ and $\mathbf{w} \in C(\phi)$. We denote this by $\mathbf{u} \succ_{KH} \mathbf{v}$.*

*The profile $\mathbf{u}$ is a weak Kaldor-Hicks improvement over $\mathbf{v} \in \mathcal{D}$ if there exists a $\mathbf{w} \in \mathcal{D}$ with $\mathbf{w} \geq \mathbf{u}$ and $\mathbf{w} \in C(\phi)$. We denote this by $\mathbf{u} \succeq_{KH} \mathbf{v}$.*

Before we give the result, we introduce one additional assumption.

**Assumption D.** *If $C(\phi) \neq C(\psi)$ then,*

$$[(C(\phi) \setminus C(\psi)) \cup (C(\psi) \setminus C(\phi))] \cap \mathbb{R}^N_{++} \neq \emptyset.$$

Assumption D, which is very weak, rules out some pathological cases where two different utility possibility sets only differ on the boundary of $\mathbb{R}^N_+$.

The following, rather unsurprising result, states that the NIP is equivalent to the requirement that the Kaldor-Hicks improvement relation is an ordering. In other words, the NIP property characterizes the set of environments where the Kaldor-Hicks criterium provides consistent judgements

**Theorem 3.5.** *If Assumptions A, B and D are satisfied, then $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP if and only if $\succeq_{KH}$ is an ordering (complete and transitive relation) with asymmetric part $\succ_{KH}$.*

# 4 Transferable utility

Transferable utility (TU) is a widely used assumption in economics. A model is said to have the TU property whenever Pareto frontiers are parallel lines with slope equal to minus one. Of course, utility values are ordinal, so we should allow this property to hold after taking some continuous and increasing transformation of the utilities. The following gives a formal definition.

**Definition 4.1** (Transferable utility (TU))**.** *The collection of utility possibility sets $(C(\phi))_{\phi\in\Phi}$ satisfies transferable utility if and only if there exist continuous and increasing functions $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $\kappa : \Phi \to \mathbb{R}$ such that for all $\phi \in \Phi$,*

$$\mathbf{u} \in C(\phi) \leftrightarrow \sum_{i=1}^{N} g_i(u_i) \leq \kappa(\phi).$$

If TU holds, the equation for the Pareto frontier $\partial C(\phi)$ can be written as,

$$\sum_{i=1}^{N} y_i = \kappa(\phi), \text{ with } y_i = g_i(u_i).$$

This defines a hyperplane in the transformed utility space, $(y_1, \ldots, y_N)$, which is orthogonal to the unit vector.

As expected, the TU property requires that the Pareto frontiers do not intersect.

**Lemma 4.2.** *If a collection of Pareto possibility sets $(C(\phi))_{\phi\in\Phi}$ satisfies TU then it satisfies the NIP.*

It may be conjectured that the NIP condition is necessary and sufficient for TU. The following example shows that this is not the case.

**Example 3.** *Let $a, b \in \mathbb{R}_+$ with $a < b$ and set $\Phi = [a, b]$. Define,*

$$C(\phi) = \{(u_1, u_2) \in \mathbb{R}^2_+ : u_1 + u_2 + u_1(u_2)^2 \leq \phi\}.$$

*If we define $h(u_1, u_2) = u_1 + u_2 + u_1(u_2)^2$ and $\rho(\phi) = \phi$, we see that, by part 4 of Theorem 3.3, $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP. However, the collection $(C(\phi))_{\phi\in\Phi}$ does not satisfy TU. If it does, then there should exist a function $\kappa$ and two increasing, continuous functions $g_1$ and $g_2$, such that,*

$$\kappa(\phi) = \kappa(u_1 + u_2 + u_1(u_2)^2) = g_1(u_1) + g_2(u_2),$$

*This is equivalent to the condition that the function $h(u_1, u_2) = u_1 + u_2 + u_2(u_2)^2$ is additive separable. A necessary condition for this to happen is that the Thomson (or double cancellation axiom) holds (see, for example, (Debreu, 1960)),[5] i.e. for all $u_1, u_2, v_1, v_2, z_1, z_2$,*

$$\begin{aligned}&\textit{if } h(u_1, u_2) \geq h(v_1, v_2) \textit{ and } h(v_1, z_2) \geq h(z_1, u_2),\\ &\textit{then } h(u_1, z_2) \geq h(z_1, v_2).\end{aligned}$$

[5] An alternative, local differentiable condition is the Sono condition

$$\frac{\partial}{\partial u_2}\left(\ln\left(\frac{\frac{\partial h(u_1,u_2)}{\partial u_1}}{\frac{\partial h(u_1,u_2)}{\partial u_2}}\right)\right) = k(u_2),$$

for some function $k$ of $u_2$ only.

*This condition is violated for the function $h(u_1, u_2)$ above, e.g. choose $u_1 = 1, u_2 = 1, v_1 = 3, v_2 = 0, z_1 = 8$ and $z_2 = 2$.*[6]

Above example suggests that additive separability is important in order to relate the NIP to TU.

**Definition 4.3.** *The function $h : \mathcal{D} \to \mathbb{R}$ is additive separable if there exist increasing, continuous functions $g_i : \mathbb{R}_+ \to \mathbb{R}$ and an increasing function $g : \mathbb{R} \to \mathbb{R}$ such that for all $\mathbf{u} \in D$,*

$$h(\mathbf{u}) = g\left(\sum_{i=1}^{N} g_i(u_i)\right).$$

The following theorem (together with Lemma 4.2) shows that TU is satisfied if and only if the NIP holds and the Pareto aggregation function is additive separable.

**Theorem 4.4.** *Let $(C(\phi))_{\phi\in\Phi}$ satisfy Assumptions A and B. If $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP, then TU is satisfied if and only if the Pareto aggregation function $h : \mathcal{D} \to \mathbb{R}$ is additive separable.*

**Example** (Example 1 continued)**.** *We showed that if the function $-g$ was homothetic, then NIP was satisfied with Pareto aggregation function $h(u_1, \ldots, u_N) = s(u_1, \ldots, u_N)$ and surplus function $\rho(Q) = r^{-1}(-Q)(\alpha(Q))^t$.*

*In order for TU to hold, we additionally need that $h$ is additive separable. A setting for which this holds is when $s$ is additive separable, i.e.*

$$s(u_1, \ldots, u_N) = \gamma(s_1(u_1) + \ldots, s_N(u_N)),$$

*for some increasing function $\gamma$ and continuous, increasing functions $s_i$. In this case, we have that,*

$$h(u_1, \ldots, u_N) = \gamma\left(\sum_{i=1}^{N} s_i(u_i)\right).$$

*However, it is possible that $h$ is not additive separable, even when it is homogeneous. An example is*

$$h(u_1, u_2) = (u_1)^3 + (u_2)^3 + u_1(u_2)^2.$$

*This function is homogeneous of degree 3 but not additive separable as it violates the Thomson double cancellation condition, e.g. when $u_1 = 1, u_2 = 1, v_1 = 3^{\frac{1}{3}}, v_2 = 0, z_1 = 13^{\frac{1}{3}} + \varepsilon$, $z_2 = 2$ and where $\varepsilon > 0$ is sufficiently small.*

**Example** (Example 2 continued)**.** *We determined $h(u, \pi) = \ell(u, P - \pi)$. A sufficient condition for $h$ to be additive separable is that, $u(\ell, w)$ is additive separable. In this case,*

$$u(\ell, w) = k_1(\ell) + k_2(w),$$

[6] This counterexample can also be found in Bergstrom (2016).

*for some increasing, continuous functions $k_1$ and $k_2$. Then,*

$$\ell(u, w) = k_1^{-1}(u - k_2(w)),$$

*so,*

$$h(u, \pi) = k_1^{-1}(u - k_2(P - \pi)),$$

*which is additive separable.*

# 5 Applications

In this section, we look at several models for which TU turned out to be a key assumption, and we discuss their relationship to the NIP property.

## 5.1 Two-sided matching

Let $M$ and $W$ be two sets of men and women. A matching is a function $\sigma : M \cup W \to M \cup W$ such that,

1. for all $m \in M$, $\sigma(m) \in W \cup \{m\}$ and for all $w \in W$, $\sigma(w) \in M \cup \{w\}$.
2. if $w = \sigma(m)$ then $m = \sigma(w)$.

If $\sigma(m) = m$ or $\sigma(w) = w$ we say that the man $m$ or woman $w$ is single. To each man $m \in M$, we assign a type $\delta_m \in \Delta$ and to each women $w \in W$ we assign a type $\xi_w \in \Xi$. Let $\Phi = \Delta \times \Xi$ be the set of possible type combinations. When the man $m$ of type $\delta_m$ matches to a women $w$ of type $\xi_w$, we assume that the possible utility values $u$ of the man $m$ and $v$ for women $w$ are in the utility possibility set $C(\delta_m, \xi_w)$. We use the standard notion of stability.

**Definition 5.1.** *A matching $\sigma : M \cup W \to M \cup W$ is stable if there are utility values $(u_m, v_w)_{m \in M, w \in W}$ such that for all $m \in M, w \in W$,*

- *$u_m, u_w \geq 0$,*
- *if $\sigma(m) = m$ then $u_m = 0$ and if $\sigma(w) = w$ then $u_w = 0$.*
- *if $\sigma(m) = w$, then $(u_m, v_w) \in \partial C(\delta_m, \xi_w)$.*
- *For all men $m$ of type $\delta_m$ and all women $w$ of type $\xi_w$, if $\sigma(m) \notin w$, then there is no $(u, v) \gg (u_m, v_w)$ such that $(u, v) \in C(\delta_m, \xi_w)$.*

The first conditions imposes individual rationality on the utility profile. The second condition normalizes the utility of being single to zero. The third condition imposes Pareto optimality on the allocation within a couple. Finally, the last condition, also known as the no-blocking pairs condition, states that if $m$ and $w$ are not matched to each other, then it is impossible for both of them to obtain a higher level of utility by forming a couple. Existence of stable matchings has been established under very general conditions (Kaneko, 1982).

Throughout this section, we assume that the collection $(C(\delta,\xi))_{(\delta,\xi)\in\Phi}$ satisfies Assumptions A, B and the NIP condition, so we have the representation in terms of a Pareto function $h$ and a surplus function $\rho$. We would like to determine when stable matchings are assortative in the sense that men and women of higher types are matched together. Towards this end, we assume that there is a ranking on the types of both men and women. Moreover, we assume that the corresponding utility possibility sets satisfy the following monotonicity assumption on $\rho$.

**Assumption E.** *There is a partial (asymmetric and transitive) relation $\succ_m$ on $\Delta$ and $\succ_w$ on $\Xi$ such that for all $\overline{\delta},\underline{\delta}\in\Delta$ and all $\xi\in\Xi$,*

$$\textit{if } \overline{\delta}\succ_m \underline{\delta}, \textit{ then } \rho(\overline{\delta},\xi)>\rho(\underline{\delta},\xi),$$

*and for all $\overline{\xi},\underline{\xi}\in\Xi$ and all $\delta\in\Delta$,*

$$\textit{if } \overline{\xi}\succ_w \underline{\xi}, \textit{ then } \rho(\delta,\overline{\xi})>\rho(\delta,\underline{\xi}),$$

Observe that the relations $\succ_m$ and $\succ_w$ are not necessarily able to rank all types (i.e. the relation is not required to be total).

A first result shows that Assumption E implies that for all stable matchings, higher type men or women must receive higher utilities.

**Theorem 5.2.** *Assume that $(C(\delta,\xi))_{(\delta,\xi)\in\Phi}$ satisfies the NIP condition and Assumptions A and B. Further assume that the surplus function $\rho$ satisfies Assumption E and let $(u_m,v_w)_{m\in M,w\in W}$ be a utility profile associated with a stable matching $\sigma$. If $\delta_m\succ_m\delta_{m'}$ then $u_m\geq u_{m'}$ with strict inequality whenever $m'$ is matched. Likewise, if $\xi_w\succ_w\xi_{w'}$, then $v_w\geq v_{w'}$ with strict inequality whenever $w'$ is matched.*

We are interested in the additional conditions on the utility possibility sets $(C(\delta,\xi))_{(\delta,\xi)\in\Phi}$ that lead to *positive assortatitve matching* (PAM) in the sense that higher type women are matched with higher type men.

**Definition 5.3.** *A matching $\sigma$ has positive assortative matching (PAM) if for all $m,m'$: if $w=\sigma(m)$, $w'=\sigma(m')$ and $\delta_m\succ_m\delta_{m'}$ then it is not the case that $\xi_{w'}\succ_w\xi_w$.*

Consider the following condition,

**Assumption F.** *For all $\overline{\delta},\underline{\delta}\in\Delta$ with $\overline{\delta}\succ_m\underline{\delta}$, all $\overline{\xi},\underline{\xi}\in\Xi$ with $\overline{\xi}\succ_w\underline{\xi}$, and all $\underline{u},\underline{v},\overline{u},\overline{v}\in\mathbb{R}_+$,*

$$\textit{if } \left\{\begin{array}{l} h(\underline{u},\underline{v})=\rho(\underline{\delta},\underline{\xi}),\\ h(\underline{u},\overline{v})=\rho(\underline{\delta},\overline{\xi}),\\ h(\overline{u},\underline{v})=\rho(\overline{\delta},\underline{\xi})\end{array}\right\} \textit{ then, } h(\overline{u},\overline{v})<\rho(\overline{\delta},\overline{\xi}).$$

Assumption F is reformulation of the Generalized Increasing Difference (GID) condition of Legros and Newman (2007).[7] The GID condition of Legros and Newman (2007) characterizes PAM for a larger variety of settings compared to the setting considered in this article. In particular, it is also valid if Pareto curves intersect. Consequentially, the condition is also sufficient in our case. This is formalized in the following corollary (in order to be self contained, we include a proof in the appendix).[8]

**Corollary 5.4** (Follows from Legros and Newman (2007))**.** *Assume that $(C(\delta, \xi))_{(\delta,\xi)\in\Phi}$ satisfies Assumptions A, B and the NIP. If the surplus and Pareto functions satisfy Assumptions E and F, then any stable matching satisfies PAM.*

The next theorem shows that in the NIP setting, supermodularity of the surplus function, together with submodularity of the Pareto aggregator function is sufficient for positive assortative matching.

**Theorem 5.5.** *Let $(C(\delta, \xi))_{(\delta,\xi)\in\Phi}$ satisfy the NIP condition and assumptions A, B. Assume that the surplus function satisfies Assumption E. If the Pareto function $h$ is submodular and if the surplus function $\rho$ is supermodular with one being strict.[9] Then they will also satisfy Assumption F. In particular, any stable matching will satisfy PAM.*

A special case of this theorem is where there is TU. In this case, the Pareto aggregation function takes the form $h(u_1, u_2) = u_1 + u_2$, which is obviously submodular. Then, PAM is obtained whenever $\rho$ is strictly supermodular.

In the non-transferable utility (i.e. when there are no transfers possible between spouses) we have that $h(u, v) = \max\{u, v\}$. In this case, Assumption F is satisfied whenever Assumption E is satisfied. As such, in this non-transferable utility, Assumption E is sufficient for PAM.

**Example** (Example 2 continued)**.** *We had that $h(u, \pi) = \ell(u, P - \pi)$ and $\rho(\delta, \xi) = 1 - \omega(\delta, \xi)$. The function $\rho(\delta, \xi)$ is supermodular if $\omega(\delta, \xi)$ is submodular. Next, the function $h$ is submodular if,*

$$\frac{\partial^2 h(u, \pi)}{\partial u \partial \pi} \leq 0,$$

[7] In particular, under NIP, the GID condition as defined by Legros and Newman (2007) and Assumption E together imply Assumption F.

[8] It can be remarked that our setting is slightly different from Legros and Newman (2007) in the sense that we do not require Pareto frontiers to be strictly decreasing but only require comprehensiveness of the utility possibility sets.

[9] Mathematically, speaking we have that for all $\overline{\delta} \succ_m \underline{\delta}, \overline{\xi} \succ_w \underline{\xi}$,

$$\rho(\underline{\delta}, \underline{\xi}) + \rho(\overline{\delta}, \overline{\xi}) \geq (>) \rho(\overline{\delta}, \underline{\xi}) + \rho(\underline{\delta}, \overline{\xi}),$$

and for all $(\underline{u}, \underline{v}) \ll (\overline{u}, \overline{v})$,

$$h(\underline{u}, \overline{v}) + h(\overline{u}, \underline{v}) \geq (>) h(\overline{u}, \overline{v}) + h(\underline{u}, \underline{v}),$$

with at least one inequality strict.

*It can be verified that this is equal to the following condition on the utility function,*

$$\frac{\partial^2 u(\ell, w)}{\partial \ell \partial w} \leq \frac{\partial^2 u(\ell, w)}{\partial \ell^2} \left( \frac{\frac{\partial u(\ell,w)}{\partial w}}{\frac{\partial u(\ell,w)}{\partial \ell}} \right).$$

*If the utility function is concave, then the right hand side will be negative. As such, this condition holds only if the cross partial derivative of $u(\ell, w)$ is sufficiently negative.*

If the types, $\delta$ and $\xi$ are real numbers and if the surplus function $\rho$ is $C^2$ then supermodularity of $\rho$ can be replaced by the condition that the cross partial derivative $\frac{\partial^2 \rho(\delta,\xi)}{\partial \delta \partial \xi}$ is always positive. This gives an alternative sufficient condition for PAM in the TU case. The next theorem gives a differential condition in the NIP setting.

**Theorem 5.6.** *Let $\Delta, \Xi \subseteq \mathbb{R}$. Assume that $\succ_m$ and $\succ_w$ coincide with the usual order, $>$ on $\mathbb{R}$. Assume that $(C(\delta, \xi))_{(\delta,\xi)\in\Delta\times\Xi}$ satisfies NIP and Assumptions A, B and E.*

*If both Pareto functions $\rho$ and the Pareto aggregation function $h$ are $C^2$ and if for all $(\underline{\delta}, \underline{\xi}), (\delta, \xi) \in \Phi$ with $\delta \geq \underline{\delta}, \xi \geq \underline{\xi}$, and for all $(\underline{u}, \underline{v}), (u, v) \in \mathbb{R}^2_+$ with $h(u, v) = \rho(\delta, \psi)$, $h(\underline{u}, \underline{v}) = \rho(\underline{\delta}, \underline{\xi})$ and $(u, v) \geq (\underline{u}, \underline{v})$,*

$$\frac{\dfrac{\partial^2 h(u, v)}{\partial u \partial v}}{\dfrac{\partial h(\underline{u}, v)}{\partial v} \dfrac{\partial h(u, \underline{v})}{\partial u}} < \frac{\dfrac{\partial^2 \rho(\delta, \xi)}{\partial \delta \partial \xi}}{\dfrac{\partial \rho(\underline{\delta}, \xi)}{\partial \xi} \dfrac{\partial \rho(\delta, \underline{\xi})}{\partial \delta}},$$

*Then Assumption F is satisfied. In particular, any stable matching has PAM.*

Observe that under Assumption E the denominators of the fractions in Theorem 5.6 are always positive. If $\rho$ is supermodular and if $h$ is submodular, then the cross partial derivative of $\rho$ is positive and the cross partial derivative of $h$ is negative. In this case above condition is automatically satisfied, which shows that Theorem 5.5 is also valid in this case. Additionally, if TU holds (i.e. $h(u, v) = u + v$), then the left hand side equals zero, so the condition is satisfied whenever $\frac{\partial^2 \rho(\delta,\xi)}{\partial \delta \partial \xi} > 0$.

## 5.2 The Coase theorem

One interpretation of the Coase theorem (Coase, 1960) is that any Pareto efficient level of an externality (either positive or negative) should be independent of the way liabilities or rights are assigned to the various agents in the economy. It is also known that this result does not hold in general. One particular case, where it is known to hold, absent transaction costs, is when there is TU (Hurwicz, 1995; Bergstrom, 2017).[10]

We analyse this interpretation of the Coase theorem using a variation of the frameworks used by Hurwicz (1995) and Bergstrom (2017). There are $N$ agents who consume one

[10] In section 6 we further elaborate on the difference and connections between the results in this section and the ones in the literature.

private good and $m(\geq 2)$ public goods. Agent $i$ has an initial endowment $\omega_i > 0$ of the private good. The final amount of the private good going to individual $i$ is denoted by $x_i$. We represent by the vector $\mathbf{x} \in \mathbb{R}^N_+$, the allocation of the private good over the $N$ agents. The vector $\mathbf{y} \in \mathbb{R}^m_+$ specifies the vector of public goods.

Each agent $i \leq N$ has a utility function $u_i(x_i, \mathbf{y})$ that depends on the own amount of the private and the total amount of public goods. For an allocation $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{N+m}_+$ we denote by $\mathbf{u}(\mathbf{x}, \mathbf{y})$ the utility profile $\mathbf{v}$ where for all agents $i$: $v_i = u_i(x_i, \mathbf{y})$. In other words, $\mathbf{u}(\mathbf{x}, \mathbf{y})$ is the utility vector resulting from the allocation $(\mathbf{x}, \mathbf{y})$.

Technology is captured by a transformation function $f$ that determines the feasible set of allocations of private and public goods,

$$f(\mathbf{x}, \mathbf{y}) \leq 0.$$

The following provides a commonly used specification of such transformation function.

**Example 4.** *The private good $x$ represents money and there is a total amount $W$ of money available in the economy which can be used for private or public consumption. There is also a function $c : \mathbb{R}^m_+ \to \mathbb{R}_+$ that specifies the amount of money $c(\mathbf{y})$ necessary to produce the public goods vector $\mathbf{y}$. The feasibility constraint can then be written as,*

$$\sum_{i=1}^{N} x_i + c(\mathbf{y}) \leq W.$$

*In this case, we have the specification $f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N} x_i + c(\mathbf{y}) - W$.*

We denote by $\mathcal{Y}$ the set of possible values for the public goods,

$$\mathcal{Y} = \{\mathbf{y} \in \mathbb{R}^m : \exists \mathbf{x} \in \mathbb{R}^N_+, f(\mathbf{x}, \mathbf{y}) \leq 0 \text{ and } \mathbf{u}(\mathbf{x}, \mathbf{y}) \geq \mathbf{0}\}.$$

For each $\mathbf{y} \in \mathcal{Y}$, we can define the utility possibility set $C(\mathbf{y})$ as set set of all feasible and individual rational utility utility profiles, given that the vector of public goods is equal to $\mathbf{y}$,

$$C(\mathbf{y}) = \{\mathbf{v} \in \mathbb{R}^N_+ : \exists \mathbf{x} \in \mathbb{R}^N_+, f(\mathbf{x}, \mathbf{y}) \leq 0 \text{ and } \mathbf{v} \leq \mathbf{u}(\mathbf{x}, \mathbf{y})\}.$$

Observe that individual rationality is guaranteed by the assumption that $\mathbf{v} \in \mathbb{R}^N_+$. Finally, we denote by $\mathcal{P}|_Y$ the set of all Pareto efficient allocations with the level of public goods restricted to lie in $Y$.

$$\mathcal{P}|_Y = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{N+m}_+ : \begin{array}{c} \mathbf{y} \in Y, \mathbf{u}(\mathbf{x}, \mathbf{y}) \in C(\mathbf{y}), \\ \nexists \mathbf{v} \in \bigcup_{y \in Y} C(\mathbf{y}), \mathbf{v} \gg \mathbf{u}(\mathbf{x}, \mathbf{y}) \end{array} \right\}.$$

The Coase theorem describes a setting where all Pareto efficient allocations have the same levels of public goods. We define this in the following way.

**Definition 5.7.** *The collection of utility possibility sets $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies the Coase property if for all subsets $Y \subseteq \mathcal{Y}$ and all $\mathbf{y} \in Y$, if $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}|_Y$ and $(\mathbf{x}', \mathbf{y}) \in \mathcal{P}|_{\{\mathbf{y}\}}$, then $(\mathbf{x}', \mathbf{y}) \in \mathcal{P}|_Y$.*

The definition has the following intuition. Let $Y$ be a subset of $\mathcal{Y}$ and consider an allocation $(\mathbf{x}, \mathbf{y})$ that is Pareto efficient in the set of all allocations for which the public goods are restricted to the set $Y$. Now, take any other allocation $(\mathbf{x}', \mathbf{y})$ with the same amount of public goods which is Pareto efficient but now only over the set of feasible allocations where the level of public goods is restricted to be equal to $\mathbf{y}$. Then if the Coase property holds, we have that $(\mathbf{x}', \mathbf{y})$ is also Pareto efficient over the bigger set of allocations.

Negating this condition, we obtain that if an allocation $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}|_{\{y\}}$ is not Pareto efficient over the set $\bigcup_{\mathbf{y}\in Y} C(\mathbf{y})$, then none of the allocations with public goods equal to $\mathbf{y}$ is Pareto efficient over this set. Effectively, every Pareto efficient allocation should have the same amounts of public goods.

To establish our result, we impose the following conditions on the utility functions, $u_i$, and the transformation function, $f$.

**Assumption G.**

1. *The utility functions $u_i : \mathbb{R}_+^{1+m} \to \mathbb{R}$ are continuous, increasing and are normalized by setting $u_i(\omega_i, \mathbf{0}) = \underline{u_i} = 0$.*

2. *The transformation function $f : \mathbb{R}_+^{N+m} \to \mathbb{R}$ is continuous and increasing.*

3. *For all agents $i \leq N$, there is an amount $\overline{x_i}$ of private goods such that $f(\overline{x_i}\,\mathbf{e}_i, \mathbf{0}) > 0$, where $\mathbf{e}_i$ is the $N$ dimensional vector with a 1 at place $i$ and zeros everywhere else.*

4. *For all $\mathbf{y} \in \mathcal{Y}$, $u_i(0, \mathbf{y}) < u_i(\omega_i, \mathbf{0}) = 0$.*

Assumptions G.1 and G.2 impose some regularity conditions on the utility and transformation functions. Assumption G.3 requires that there is a level of private consumption for agent $i$ that is not feasible, even when the consumption of all other goods is zero. It guarantees that the amounts of private goods $\mathbf{x}$ that are feasible are bounded from above. Finally, Assumption G.4 states that if an agent receives no private goods, then she prefers to break down the negotiations. Also, the condition implies that every feasible and individual rational allocation $(\mathbf{x}, \mathbf{y})$ (i.e. with $f(\mathbf{x}, \mathbf{y}) \leq 0$ and $\mathbf{u}(\mathbf{x}, \mathbf{y}) \geq \mathbf{0}$) must have strictly positive amounts of the private goods, i.e. $\mathbf{x} \gg \mathbf{0}$.

The following theorem shows that the Coase theorem is equivalent to the the NIP property.

**Theorem 5.8.** *Assume that Assumption G is satisfied. Then, the collection of utility possibility sets $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies the NIP if and only if the Coase property is satisfied.*

Lemma A.3 in the appendix shows that for our model, the collection $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies Assumption A which explains why we don't need to list it as a separate assumption in Theorem 5.8. From Theorem 3.3, NIP guarantees the existence of a surplus function

$\rho : \mathcal{Y} \to \mathbb{R}$. In addition, it is easy to show that, given Assumption G, the utility possibility correspondence $C(.)$ is upper hemicontinuous.[11] This property allows us to choose $\rho(.)$ to be upper semicontinuous. Given this, for any compact subset $Y \subseteq \mathcal{Y}$, we can define the set,

$$Y' = \arg\max_{\mathbf{y}\in Y} \rho(\mathbf{y}).$$

It then follows that

$$\mathcal{P}|_Y = \bigcup_{y\in Y'} \mathcal{P}|_{\{\mathbf{y}\}} = \mathcal{P}|_{Y'}.$$

In other words, the efficient levels of public goods can be obtained by maximising the surplus function $\rho(\mathbf{y})$.

## 5.3 Becker's Rotten kid Theorem

The Rotten Kid theorem (Becker, 1974; Bergstrom, 1989; Cornes and Silva, 1999; Chiappori and Werning, 2002) is a variation on the Coase theorem. It describes the following two stage game. In the first period, each individual from a group of agents (kids) takes an action. This profile of actions determines the level of public goods, $\mathbf{y}$. In stage two of the game, another agent, called the parent, takes the actions of the kids as given and allocates to each kid a feasible amount of the private good. The objective function of the parent is given by a welfare function which is strictly increasing in the utility levels of all the kids. The Rotten Kid theorem describes a situation where each allocation (over private and public goods) that is optimal for the parent can be implemented as a subgame perfect Nash equilibrium of this two stage game. In other words, the level of public goods, collectively chosen by the kids, coincides with the optimal level of public goods desired by the parent.

In order to model this setting, we borrow the notation and concepts developed in the previous section. In addition, we introduce an agent, the parent, whose objective function is given by a welfare function $W : \mathbb{R}^N_+ \to \mathbb{R}$ that takes as arguments the utility profiles for the kids.

The games rotten kids play is the following. First each kid selfishly chooses an action $a_i \in A_i$ where $A_i$ is a set of feasible actions for kid $i$. Let us denote the chosen profile of actions by $\mathbf{a} \in \prod_{i=1}^N A_i$. The action profile where kid $i$ chooses $a_i$ and all other kids choose according to the profile $\mathbf{a}$ is denoted by $(a_i, \mathbf{a}_{-i})$. We assume that the profile of actions uniquely determines the level of public goods $\mathbf{y}$ via some surjective function $g : \prod_{i=1}^N A_i \to \mathcal{Y}$,

$$g(\mathbf{a}) = \mathbf{y}.$$

For a given profile of actions $\mathbf{a}$ and an associated level of public goods $\mathbf{y} = g(\mathbf{a})$, the parent chooses a feasible vector of private quantities $\mathbf{x} \in \mathbb{R}^n_+$ that optimizes total utility of the

[11] A proof of this is available from the authors upon request.

parent,

$$\Gamma(\mathbf{a}) = \arg\max_{\mathbf{x}} W(\mathbf{u}(\mathbf{x}, g(\mathbf{a})) \text{ s.t. } f(\mathbf{x}, g(\mathbf{a})) \leq 0, \text{ and } \mathbf{u}(\mathbf{x}, g(\mathbf{a})) \geq \mathbf{0}.$$

As $\Gamma(\mathbf{a})$ might be multivalued, we assume that the parent uses a decision rule $\gamma : \prod_i A_i \to \mathbb{R}^N_+$ that picks for any $\mathbf{a}$, a single element from $\Gamma(\mathbf{a})$, i.e. $\gamma(\mathbf{a}) \in \Gamma(\mathbf{a})$.

We concentrate on subgame perfect Nash equilibria. Using backwards induction, this means that in the first stage of the game, each kid takes the choice rule of the parent as given and picks a strategy $a_i \in A_i$ that maximizes her utility function given the actions taken by the other kids and the decision rule of the parent. In particular, the equilibrium strategy profile $\mathbf{a} \in \prod_{i=1}^N A_i$ is such that for all kids $i \leq N$, and all actions $a_i \in A_i$,

$$u_i(\gamma_i(\mathbf{a}), g(\mathbf{a})) \geq u_i(\gamma_i(a_i, \mathbf{a}_{-i}), g(a_i, \mathbf{a}_{-i})).$$

If $\mathbf{a}$ is a subgame perfect Nash equilibrium, the final allocation is determined by $(\gamma(\mathbf{a}), g(\mathbf{a}))$.

Above optimization problem of the parent takes the level of public goods $g(\mathbf{a})$ as given. Ideally, however, the parent would like to decide on both the allocation of $\mathbf{x}$ and the strategy profile $\mathbf{a}$, which would give the following first best problem,

$$\Gamma = \arg\max_{\mathbf{x},\mathbf{a}} W(\mathbf{u}(\mathbf{x}, g(\mathbf{a}))) \text{ s.t. } f(\mathbf{x}, g(\mathbf{a})) \leq 0 \text{ and } \mathbf{u}(\mathbf{x}, g(\mathbf{a})) \geq \mathbf{0}.$$

We say that the Rotten Kid property holds if any first best allocation in $\Gamma$ can also be reached as a subgame perfect Nash equilibrium. In our setting, this leads to the following definition.

**Definition 5.9.** *Let $\mathcal{A} \subseteq \prod_{i=1}^N A_i$ be the set of subgame perfect Nash equilibria. We say that the Rotten Kid property holds if, for all $(\mathbf{x}, \mathbf{a}) \in \Gamma$, $\mathbf{a} \in \mathcal{A}$.*

In order to obtain our result, we impose the following property on the decision rule $\gamma$,

**Assumption H.** *If $C(g(\mathbf{a})) \sqsubseteq C(g(\mathbf{a}'))$. Then,*

$$\mathbf{u}(\gamma_i(\mathbf{a}'), g(\mathbf{a}')) \geq \mathbf{u}(\gamma_i(\mathbf{a}), g(\mathbf{a})).$$

This assumption states that if the utility possibility set expands then the optimal choice of the parent is to (weakly) increase the utility level of all kids. In the TU setting, i.e. where the sets $C(g(\mathbf{a}))$ are hyperplanes with slope $-1$, this assumption boils down to the requirement that the utility of each kid is a normal good for the parent (this is the assumption made by Bergstrom (1989); Cornes and Silva (1999) and Chiappori and Werning (2002)). The following result shows that, under this assumption, NIP is a sufficient condition for the Rotten Kid property to hold.

**Theorem 5.10.** *Assume that Assumptions G and H are satisfied. If $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies the NIP, then the Rotten Kid property holds.*

# 6 Maximal admissible NIP classes

In this section, we look for classes of utility functions that lead to utility possibility sets that satisfy the NIP. Towards this end, we continue with the public goods setting from Sections 5.2 and 5.3.

In general, the shape of the utility possibility sets, $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$, will depend on the particular profile of utility functions $(u_1(.),\ldots,u_N(.))$, and the transformation function $f(.)$. In this sense, it is possible that the NIP holds for some profile of utility and transformation functions but not for others. The aim of this section is to find classes of such utility and transformation functions over which the NIP holds uniformly.

Let us denote by $\mathcal{U}$ the collection of increasing, $C^2$ utility functions $u:\mathbb{R}^{1+N}\to\mathbb{R}$ and let us denote by $\mathcal{F}$ the collection of possible transformation functions $f:\mathbb{R}_+^{N+m}\to\mathbb{R}$.

Consider a subset $F\subseteq\mathcal{F}$. We say that a subset of utility functions $U\subseteq\mathcal{U}$ is admissible for $F$, if every utility profile selected from $U$ leads to a collection of utility possibility sets that satisfy the NIP uniformly over all transformation functions in $F$. This concept is inspired by Hurwicz (1995) who analysed uniform classes of utility functions that lead to TU.

**Definition 6.1.** *A class of utility functions $U\subseteq\mathcal{U}$ is called an admissible NIP class for $F\subseteq\mathcal{F}$ if for all profiles of utility functions $\mathbf{u}(.)\in\prod_{i=1}^N U$ and all $f\in F$, the collection of utility possibility sets $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies the NIP.*

*The admissible NIP class $U\subseteq\mathcal{U}$ is a maximal admissible NIP class for $F$ if there is no admissible NIP class for $F$, say $V\subseteq\mathcal{U}$ such that $U\subseteq V$ and $U\neq V$.*

Observe that an admissible NIP class not only requires the NIP to hold uniformly over all profiles of utility functions in $U$ but also over all transformation functions $f\in F$. Obviously, there will be a trade off between the size of a maximal admissible NIP class $U$ and the restrictions imposed on the class $F$. In particular the smaller the set of transformation functions, the larger the size of $U$. Vice versa, if we impose weaker restrictions on $F$, then the collection of utility functions that lead to utility possibility sets that satisfy the NIP will be smaller.

As a first instance, we look at the class of transformation functions that are additive separable in the private goods.

**Definition 6.2.** *The class $F_A\subseteq\mathcal{F}$ is defined as the set that contains all transformation functions $f\in\mathcal{F}$ that take the form,*

$$f(\mathbf{x},\mathbf{y})=\sum_{i=1}^N\delta_i x_i+c(\mathbf{y}).$$

*for some $\delta_1,\ldots,\delta_N>0$ and some $C^1$ function $c:\mathbb{R}_+^m\to\mathbb{R}$.*

For all transformation functions in $F_A$, the rate of substitution between $x_i$ and $x_j$ is fixed, i.e. private goods are perfect substitutes. If the private good represents money, then the value of $c(\mathbf{y})$ can be interpreted as the cost of producing the public goods $\mathbf{y}$.

The class of utility functions that will provide the maximal admissible NIP classes, will be formed by the generalized quasi-linear (GQL) utility functions.

**Definition 6.3.** *Let $\alpha : \mathbb{R}^m_+ \to \mathbb{R}^m_{++}$ be an increasing $C^1$ function. Then the GQL class $U_\alpha \subseteq \mathcal{U}$ is defined as the set of all utility functions $u \in U_\alpha$ such that,*

$$u(x, \mathbf{y}) \geq u(x', \mathbf{y}') \leftrightarrow \alpha(\mathbf{y})x + \beta(\mathbf{y}) \geq \alpha(\mathbf{y}')x' + \beta(\mathbf{y}').$$

*for some increasing $C^1$ function $\beta : \mathbb{R}^m_+ \to \mathbb{R}$.*

Observe that for all $u \in U_\alpha$, the function $\alpha(\mathbf{y})$ is the same while $\beta(\mathbf{y})$ may differ. The GQL class of utility functions generalizes the quasi-linear specification by allowing the marginal utility of $x$ to depend on the level of public goods, as long as this dependence is identical for all agents.

**Theorem 6.4.** *Assume that Assumption G.4 is satisfied. Then for all $C^1$ and increasing functions $\alpha : \mathbb{R}^m_+ \to \mathbb{R}^m_{++}$, the set $U_\alpha$ is a maximal admissible NIP class for $F_A$. In this case, the resulting utility possibility sets $(C(\mathbf{y}))_{\mathbf{y} \in \mathcal{Y}}$ will satisfy TU.*

This theorem makes the connection with several known results from the literature. It is well known (e.g., Chiappori (2010); Cherchye et al. (2015); Chiappori and Gugl (2015)) that there is a close connection between the GQL class of utility functions and the TU property if the transformation function is additive in $\mathbf{x}$, (e.g. $f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^N x_i + c(\mathbf{y})$). Moreover, it is also well know that for this technological specifications, the Coase and Rotten Kid theorems are closely connected to the class of GQL utility functions on the one hand and to TU on the other hand (Hurwicz, 1995; Bergstrom, 1989, 2017).

We are now able to provide further insights to these connections. Theorems 5.8 and 5.10 show that the Coase and Rotten Kid theorems are related to the NIP, which on first sight is weaker than TU. Theorem 6.4 on the other hand shows that if the transformation function is linear in the private goods, the NIP, the TU property and the GQL class of utility functions are closely interconnected. Remark, however, that in order to link the TU property to the Coase and Rotten Kid theorems, is is crucial to have the linear technology specification.

Let us now relax the condition on the transformation functions. Instead of additive separability of $f$ in $\mathbf{x}$ we now only assume that the transformation function is homothetic in $\mathbf{x}$.

**Definition 6.5.** *The class $F_H \subseteq \mathcal{F}$ is defined as the set that contains all functions $f \in F_H$ that take the form,*

$$f(\mathbf{x}, \mathbf{y}) = w(\mathbf{x}) + c(\mathbf{y}),$$

*for some increasing, homothetic, $C^1$ function $w : \mathbb{R}^N_+ \to \mathbb{R}$, and a $C^1$ function $c : \mathbb{R}^m_+ \to \mathbb{R}$.*

The class $F_H$ is considerably larger than $F_A$. Given this, it is reasonable to expect that the class of maximal admissible utility functions associated with $F_H$ will be smaller than those associated with $F_A$. This is indeed the case.

**Definition 6.6.** *Let $\alpha : \mathbb{R}^m_+ \to \mathbb{R}^m_{++}$ be an increasing $C^1$ function. Then the set $V_\alpha \subseteq \mathcal{U}$ is defined as the set that contains all functions $u \in \mathcal{U}$ such that,*

$$u(x, \mathbf{y}) \geq u(x', \mathbf{y}') \leftrightarrow \alpha(\mathbf{y})x \geq \alpha(\mathbf{y})x.$$

Observe that different utility functions in $U_\alpha$ are not only ordinal equivalent but the (rescaled) utility functions are also proportional in the private good.

**Theorem 6.7.** *Assume that Assumption G.4 is satisfied. Then for all increasing $C^1$ functions $\alpha : \mathbb{R}^m_+ \to \mathbb{R}^m_{++}$, the set $V_\alpha$ is a maximal admissible NIP class for $F_H$. In this case, the utility possibility sets $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ need not satisfy TU.*

As a final specification, we consider the class of transformation functions where the private goods are perfect complements.

**Definition 6.8.** *The class $F_M \subseteq \mathcal{F}$ is defined as the set that contains all transformation functions $f \in \mathcal{F}$ that take the form,*

$$f(\mathbf{x}, \mathbf{y}) = \max_{i \leq N} x_i + c(\mathbf{y}).$$

*for some function $c : \mathbb{R}^m_+ \to \mathbb{R}$.*

It turns out that the maximal admissible NIP classes of $F_M$ are generated by single utility functions.

**Definition 6.9.** *Let $v \in \mathcal{U}$. The class $W_v \subseteq \mathcal{U}$ is such that $u \in W_v$ if and only if,*

$$u(x, \mathbf{y}) \geq u(x', \mathbf{y}') \leftrightarrow v(\mathbf{x}, \mathbf{y}) \geq v(\mathbf{x}', \mathbf{y}').$$

Utility functions selected from $W_v$ are ordinal equivalent. However, there are no constraints on the shape of this utility function.

**Theorem 6.10.** *Assume that Assumption G.4 is satisfied. Then for all $v \in \mathcal{U}$, the set $W_v$ is a maximal admissible NIP class for $F_M$. The utility possibility sets $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ will not satisfy TU.*

# 7 Conclusion

This paper focusses on a weakening of the well-known transferable utility property, namely the case where two distinct Pareto curves have an empty intersection. We call this the no-intersection property (NIP). We showed that under some regularity condition any setting with NIP has a representation in the form of a Pareto aggregation function and a surplus function. We showed that the TU property holds if and only if this Pareto function is additive separable.

Next, we revisited some models where the TU property has turned out to be an important. We showed that in a two-sided matching model, positive assortative matching is obtained if some monotonicity condition holds, if the Pareto aggregation function is submodular and the surplus function is supermodular. Next, we considered a public goods setting and showed that the Coase and Rotten Kid theorem hold under the weaker condition of NIP (compared to the usual TU assumption). Finally, we characterized various classes of utility functions and corresponding classes of transformation functions for which the NIP condition holds uniformly. We found a clear trade off between the assumptions imposed on the technology and the class of utility functions that give rise to the NIP. In particular, we found that NIP is equivalent to TU in settings where the transformation function is linear in the private goods.

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# A Proofs

## Proof of Theorem 3.2

We first proof the following lemma.

**Lemma A.1.** *Given Assumption A, a collection of utility possibility sets* $(C(\phi)_{\phi\in\Phi}$ *satisfies the NIP if and only if for all* $\phi, \psi \in \Phi$, $C(\phi) \setminus C(\psi) \neq \emptyset$ *implies* $C(\psi) \sqsubset C(\phi)$.

*Proof.* $(\to)$ Assume that $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP. Towards a contradiction, let $C(\phi) \setminus C(\psi) \neq \emptyset$ and not $C(\psi) \sqsubset C(\phi)$.

Let us first show that there exist utility profiles $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathbf{u} \in C(\phi) \setminus C(\psi), , \tag{1}$$
$$\mathbf{v} \in C(\psi) \setminus C(\phi). \tag{2}$$

As $C(\phi) \setminus C(\psi) \neq \emptyset$ we know there is an $\mathbf{u} \in C(\phi) \setminus C(\psi)$ which shows (1). Also, as $C(\psi) \not\sqsubset C(\phi)$ we know that there is a $\mathbf{w} \in C(\psi)$ such that for all $\mathbf{z} \gg \mathbf{w}$, $\mathbf{z} \notin C(\phi)$. There are several cases to consider,

- $\mathbf{w} \notin C(\phi)$.
  In this case, (2) is satisfied by taking $\mathbf{v} = \mathbf{w}$.
- $\mathbf{w} \in C(\phi)$.
  - If there is a $\mathbf{z} \gg \mathbf{w}$ such that $\mathbf{z} \in C(\psi)$, then (2) is satisfied with $\mathbf{v} = \mathbf{z}$.
  - If for all $\mathbf{z} \gg \mathbf{w}$, $\mathbf{z} \notin C(\psi)$.
    Then $\mathbf{w} \in \partial C(\phi)$ and $\mathbf{w} \in \partial C(\psi)$. As such, from the NIP, we have that $C(\phi) = C(\psi)$ which contradicts (1).

For a vector $\mathbf{w} > \mathbf{0}$, define the correspondence

$$\Gamma(\mathbf{w}, \phi) = \{\gamma \in \mathbb{R}_+ : \gamma\, \mathbf{w} \in \partial C(\phi)\}.$$

In Appendix B, we show that $\Gamma(\mathbf{w}, \phi)$, as a function of $\mathbf{w}$, is non-empty, convex valued and upper hemicontinuous on $\mathcal{D} \setminus \{\mathbf{0}\}$.

From (1) and (2), we have that, $\mathbf{u}, \mathbf{v} > \mathbf{0}$ and that 1 is a lower bound for $\Gamma(\mathbf{u}, \phi)$ and that 1 is an upper bound for $\Gamma(\mathbf{v}, \phi)$. Also, 1 is an upper bound for $\Gamma(\mathbf{u}, \psi)$ and 1 is a lower bound for $\Gamma(\mathbf{v}, \psi)$. Let

$$G(\theta) = \Gamma((1-\theta)\mathbf{u} + \theta\mathbf{v}, \phi) - \Gamma((1-\theta)\mathbf{u} + \theta\mathbf{v}, \psi).$$

(Here, for two sets $A$ and $B$, $A - B$ contains all elements $c = a - b$ where $a \in A$ and $b \in B$.) The correspondence $G(\theta)$ is non-empty, upper hemicontinuous and convex valued on $[0, 1]$. Also, $G(0)$ is bounded from below by 0 and $G(1)$ is bounded from above by 0.

By the intermediate value theorem for correspondences (see de Clippel (2008, Lemma 2)), there should is a $\theta^*$ such that,

$$0 \in G(\theta^*).$$

Let $\mathbf{w} = (1-\theta^*)\mathbf{u} + \theta^*\mathbf{v}$. Then, we have that there is a $\gamma$ such that, $\gamma\mathbf{w} \in \partial C(\phi) \cap \partial C(\psi)$. By the NIP, we have that $C(\phi) = C(\psi)$, a contradiction.
($\leftarrow$) Assume that $\{C(\phi)\}_{\phi\in\Phi}$ satisfies the second part of the lemma. Towards a contradiction, assume that the NIP does not hold, i.e. there is a utility profile $\mathbf{u} \in \partial C(\phi) \cap \partial C(\psi)$ and $C(\phi) \neq C(\psi)$. Without loss of generality, assume that $C(\phi) \setminus C(\psi) \neq \emptyset$. Then, we have that $C(\psi) \sqsubset C(\phi)$. As $\mathbf{u} \in C(\psi)$, there must be a profile $\mathbf{w} \gg \mathbf{v}$ such that $\mathbf{w} \in C(\phi)$. This contradicts the fact that $\mathbf{u} \in \partial C(\phi)$. □

Let us now prove Theorem 3.2.
($\rightarrow$) Let $(C(\phi))_{\phi\in\Phi}$ satisfy the NIP and let $\phi, \psi \in \Phi$. If $C(\phi) = C(\psi)$ then immediately, $C(\phi) \sqsubseteq C(\psi)$. If not, by Lemma A.1, either $C(\phi) \setminus C(\psi) \neq \emptyset$ or $C(\psi) \setminus C(\phi) \neq \emptyset$. In the first case $C(\psi) \sqsubseteq C(\phi)$. In the second $C(\psi) \sqsubseteq C(\phi)$ which shows that $\sqsubseteq$ is complete.

For transitivity, let $C(\phi) \sqsubseteq C(\psi) \sqsubseteq C(\xi)$. If either the first or the second comparison is an equality, then immediately $C(\phi) \sqsubseteq C(\xi)$. If not, then $C(\phi) \sqsubset C(\psi) \sqsubset C(\xi)$. Then for all $\mathbf{u} \in C(\phi)$ there is a $\mathbf{v} \gg \mathbf{u}$ such that $\mathbf{v} \in C(\psi)$ and for all $\mathbf{v} \in C(\psi)$ there is a $\mathbf{w} \gg \mathbf{v}$ such that $\mathbf{w} \in C(\xi)$. This immediately implies that for all $\mathbf{u} \in C(\phi)$ there is a $\mathbf{w} \gg \mathbf{u}$ such that $\mathbf{w} \in C(\xi)$, so $C(\phi) \sqsubset C(\xi)$. This shows that $\sqsubseteq$ is transitive.

For the last part, assume, towards a contradiction, that $C(\phi) \sqsubseteq C(\psi)$ and $C(\psi) \sqsubset C(\phi)$. The latter comparison implies that for all $\mathbf{v} \in C(\psi)$ there is an $\mathbf{u} \gg \mathbf{v}$ such that $\mathbf{u} \in C(\phi)$. Taking $\mathbf{v} \in \partial C(\psi)$ shows that $C(\phi) \neq C(\psi)$. The first comparison then implies that $C(\phi) \sqsubset C(\psi)$. But then there is a $\mathbf{w} \gg \mathbf{u} \gg \mathbf{v}$ such that $\mathbf{w} \in C(\psi)$, in contradiction with $\mathbf{v} \in \partial C(\psi)$.
($\leftarrow$) Let $\sqsubseteq$ be an ordering on $(C(\phi))_{\phi\in\Phi}$ with asymmetric part $\sqsubset$. Let $\mathbf{u} \in \partial C(\phi) \cap \partial C(\psi)$ and assume that $C(\phi) \neq C(\psi)$. Without loss of generality, let $C(\phi) \setminus C(\psi) \neq \emptyset$. Then, as $\sqsubset$ is complete, $C(\psi) \sqsubset C(\phi)$. Given that $\mathbf{u} \in C(\psi)$ there should be a $\mathbf{v} \gg \mathbf{u}$ such that $\mathbf{v} \in C(\phi)$. However, $\mathbf{u} \in \partial C(\phi)$ which gives the desired contradiction.

## Proof of Theorem 3.3

### Part 1

Assume that $(C(\phi))_{\phi\in\Phi}$ satisfies Assumption A.
($\rightarrow$) Define the function,

$$\rho(\phi) = \max\{\|\mathbf{u}\| : \mathbf{u} \in C(\phi).\}.$$

Here $\|.\|$ is the usual $L^2$ norm: $\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{N}(u_i)^2}$. The function $\rho(\phi)$ is well defined as $C(\phi)$ is compact and non-empty (by Assumption A).

Let $\rho(\phi) \geq \rho(\psi)$ and assume, towards a contradiction, that $C(\psi) \not\sqsubseteq C(\phi)$. By Theorem 3.2, $C(\phi) \sqsubset C(\psi)$. Also, Let $\mathbf{u} \in C(\phi)$ such that $\rho(\phi) = \|\mathbf{u}\|$. Then as $C(\phi) \sqsubset C(\psi)$ there is a $\mathbf{v} \gg \mathbf{u}$ with $\mathbf{v} \in C(\psi)$. However, this implies that $\rho(\phi) = \|\mathbf{u}\| < \|\mathbf{v}\| \leq \rho(\psi)$, a contradiction.

For the reverse, assume, towards a contradiction, that $C(\psi) \sqsubseteq C(\phi)$ and $\rho(\phi) < \rho(\psi)$. Let $\mathbf{v} \in C(\psi)$ such that $\rho(\psi) = \|\mathbf{v}\|$. Then, as $\rho(\phi) < \|\mathbf{v}\|$ it must be that $\mathbf{v} \in C(\psi) \setminus C(\phi)$. By Lemma A.1 it follows that $C(\phi) \sqsubset C(\psi)$, a contradiction.
($\leftarrow$) Let $\mathbf{u} \in C(\phi) \setminus C(\psi)$. By Lemma A.1 it suffices to show that $C(\psi) \sqsubset C(\phi)$. First observe that $C(\phi) \not\sqsubseteq C(\psi)$. As such $\rho(\psi) < \rho(\phi)$. Then $C(\psi) \sqsubseteq C(\phi)$. Conclude that $C(\phi) \neq C(\psi)$ and $C(\psi) \sqsubseteq C(\phi)$ which shows that $C(\psi) \sqsubset C(\phi)$.

**Part 2**

($\rightarrow$) We construct the function $\rho$ as above in part 1. Let $\mathbf{u} \in \mathcal{D}$. By Assumption B, we know that there is a $\phi \in \Phi$ such that $\mathbf{u} \in \partial C(\phi)$. Define,

$$h(\mathbf{u}) = \rho(\phi).$$

The function $h$ is well defined. Indeed, if,

$$\mathbf{u} \in \partial C(\phi) \cap \partial C(\psi).$$

then by the NIP, $C(\phi) = C(\psi)$ and therefore, $\rho(\phi) = \rho(\psi)$.

Let $\mathbf{u} \in C(\phi)$. If $\mathbf{u} \in \partial C(\phi)$ then immediately, by definition of $h$, $h(\mathbf{u}) = \rho(\phi)$. If $\mathbf{u} \notin \partial C(\phi)$ then there is a $\mathbf{v} \gg \mathbf{u}$ such that $\mathbf{v} \in C(\phi)$. Let $\psi$ be such that $\mathbf{u} \in \partial C(\psi)$. Then $\mathbf{v} \in C(\phi) \setminus C(\psi)$, so by Lemma A.1, $C(\psi) \sqsubset C(\phi)$ and therefore $h(\mathbf{u}) = \rho(\psi) < \rho(\phi)$ as was to be shown.

For the reverse, let $h(\mathbf{u}) \leq \rho(\phi)$. Let $\psi$ be such that $h(\mathbf{u}) = \rho(\psi)$ (i.e. $\mathbf{u} \in \partial C(\psi)$). Then $\rho(\psi) \leq \rho(\phi)$ so $\mathbf{u} \in C(\psi) \subseteq C(\phi)$.
($\leftarrow$) Let $v \in \partial C(\phi) \cap C(\psi)$. Then $h(u) = \rho(\phi) = \rho(\psi)$. As such, $u \in C(\phi)$ if and only if $h(u) \leq \rho(\phi) = \rho(\psi)$ which means that $u \in C(\psi)$. Conclude that $C(\phi) = C(\psi)$.

**Part 3**

Assume that additionally Assumption C is satisfied. By definition of $\rho$ above, we see that if $C(.)$ is continuous, then $\rho$ is also continuous (by Berge's maximum theorem).

Let $\mathbf{u} \in \mathcal{D}$. In order to show that $h$ is continuous at $\mathbf{u}$ it suffices to show that all sequences $(\mathbf{u}^t)_{t\in\mathbb{N}}$ in $\mathcal{D}$ with $\mathbf{u}^t \to \mathbf{u}$ have a subsequence, say $(\mathbf{w}^t)_{t\in\mathbb{N}}$ such that $h(\mathbf{w}^t) \to h(\mathbf{u})$.

Let $(\mathbf{u}^t)_{t\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $\mathbf{u}^t \to \mathbf{u}$. By Assumption B and the definition of $h$ above, there exists a corresponding sequence $(\phi^t)_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$,

$$h(\mathbf{u}^t) = \rho(\phi^t).$$

As $\Phi$ is a compact subset of $\mathbb{R}^k$ (by Assumption C), the sequence $(\phi^t)_{t\in\mathbb{N}}$ has a convergent subsequence, say, $(\psi^t)_{t\in\mathbb{N}}$ with $\psi^t \to \psi \in \Phi$. As $\rho$ is continuous, we have that $\rho(\psi^t) \to \rho(\psi)$.

Let $(\mathbf{w}^t)_{t\in\mathbb{N}}$ be the subsequence of $(\mathbf{u}^t)_{t\in\mathbb{N}}$ that corresponds to the subsequence $(\psi^t)_{t\in\mathbb{N}}$ of $(\phi^t)_{t\in\mathbb{N}}$, i.e. $h(\mathbf{w}^t) = \rho(\psi^t)$. We have that,

$$h(\mathbf{w}^t) = \rho(\psi^t) \to \rho(\psi).$$

Let us show that $\rho(\psi) = h(\mathbf{u})$, thereby completing the proof.

First, observe that for all $t \in \mathbb{N}$, $\mathbf{w}^t \in C(\psi^t)$. Given that $\psi^t \to \psi$ and upper hemicontinuity of $C(.)$, there should be a convergent subsequence of $(\mathbf{w}^t)_{t\in\mathbb{N}}$ whose limit is in $C(\psi)$. This subsequence is also a subsequence of $(\mathbf{u}^t)_{t\in\mathbb{N}}$, so its limit is equal to $\mathbf{u}$. Therefore $\mathbf{u} \in C(\psi)$ which means that $h(\mathbf{u}) \leq \rho(\psi)$.

We still need to show that $h(\mathbf{u}) \geq \rho(\psi)$. Assume towards a contradiction that $h(\mathbf{u}) < \rho(\psi)$. Then $\mathbf{u} \notin \partial C(\psi)$, so there is a $\mathbf{z} \gg \mathbf{u}$ such that $\mathbf{z} \in C(\psi)$. Take $\varepsilon > 0$ small enough such that $\mathbf{z} \gg \mathbf{u} + \varepsilon\,\mathbf{e}$.

We have that $\psi^t \to \psi$ and $\mathbf{z} \in C(\psi)$, so by lower hemicontinuity of $C(.)$ there is a $T$, and a sequence $(\mathbf{z}^t)_{t\geq T}$ such that $\mathbf{z}^t \to \mathbf{z}$ and for all $t \geq T$, $\mathbf{z}^t \in C(\psi^t)$. Let $t$ be large enough such that for all agents $j \leq N$: $|z_j^t - z_j| < \varepsilon/2$. Also, let $t$ be large enough such that for all agents $j \leq N$: $|w_j^t - u_j| < \varepsilon/2$. Then, for all agents $j \leq N$ and all such $t$ large enough

$$\begin{aligned} z_j^t &\geq z_j - \varepsilon/2, \\ &> u_j + \varepsilon/2, \\ &\geq w_j^t. \end{aligned}$$

This shows that for $t$ large enough $\mathbf{z}^t \gg \mathbf{w}^t$. However, this contradicts with the fact that for all $t \in \mathbb{N}$: $\mathbf{z}^t \in C(\psi^t)$ and $\mathbf{w}^t \in \partial C(\psi^t)$. Conclude that $h(\mathbf{u}) = \rho(\psi)$.

**Part 4**

Let us first show that if $\mathbf{u} \in \partial C(\phi)$, then $h(\mathbf{u}) = \rho(\phi)$. We know that $h(\mathbf{u}) \leq \rho(\phi)$. If, towards a contradiction $h(\mathbf{u}) < \rho(\phi)$ then, by continuity of $h$, there exists a $\varepsilon > 0$ such that,

$$h(\mathbf{u} + \varepsilon\mathbf{e}) < \rho(\phi).$$

But then, $\mathbf{u} \ll \mathbf{u} + \varepsilon\mathbf{e} \in C(\phi)$, contradicting $\mathbf{u} \in \partial C(\phi)$.

Now, assume that $\mathbf{u} \in \partial C(\phi) \cap \partial C(\psi)$. Then by the above, $h(\mathbf{u}) = \rho(\phi) = \rho(\psi)$, so if $\mathbf{v} \in C(\phi)$ we have that $h(\mathbf{v}) \leq \rho(\phi) = \rho(\psi)$ which means that $\mathbf{v} \in C(\psi)$. Conclude that $C(\phi) = C(\psi)$, so the NIP condition is satisfied.

## Proof of Theorem 3.5

($\to$) Assume that $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP. Let $\mathbf{u}, \mathbf{v} \in \mathcal{D}$. Given Assumption B there are $\phi, \psi \in \Phi$ with $\mathbf{u} \in \partial C(\phi)$ and $\mathbf{v} \in \partial C(\psi)$. Let us show that,

$$\begin{aligned} \mathbf{u} \succ_{KH} \mathbf{v} &\leftrightarrow C(\psi) \sqsubset C(\phi), \\ \mathbf{u} \succeq_{KH} \mathbf{v} &\leftrightarrow C(\psi) \sqsubseteq C(\phi), \end{aligned}$$

If $\mathbf{u} \succ_{KH} \mathbf{v}$ then there is a $\mathbf{w} \in C(\phi)$ such that $\mathbf{w} \gg \mathbf{v}$. Given that $\mathbf{v} \in \partial C(\psi)$ we have that $\mathbf{w} \notin C(\psi)$. Then, by Lemma A.1, we have that $C(\psi) \sqsubset C(\phi)$. For the reverse, let

$C(\psi) \sqsubset C(\phi)$. Then for $\mathbf{v} \in \partial C(\psi)$ we know, by definition, that there is a $\mathbf{w} \gg \mathbf{v}$ such that $\mathbf{w} \in C(\phi)$, so $\mathbf{u} \succ_{KH} \mathbf{v}$.

For the second part. Let $\mathbf{u} \succeq_{KH} \mathbf{v}$, let $u \in \partial C(\phi)$ and $v \in \partial C(\psi)$. Then there is a $\mathbf{w} \geq \mathbf{v}$ such that $\mathbf{w} \in C(\phi)$. If $\mathbf{w} \notin C(\psi)$, then by Lemma A.1, we immediately have that $C(\psi) \sqsubset C(\phi)$. So take the case where $\mathbf{w} \in C(\psi)$. This means that $\mathbf{w} \in \partial C(\psi)$. If also $\mathbf{w} \in \partial C(\phi)$ then $C(\phi) = C(\psi)$ by the NIP and $C(\psi) \sqsubseteq C(\phi)$. Else, if $\mathbf{w} \notin \partial C(\phi)$ then there is a $\mathbf{z} \gg \mathbf{w}$ such that $\mathbf{z} \in C(\phi)$. Also $\mathbf{z} \notin C(\psi)$ so $C(\psi) \sqsubset C(\phi)$. For the reverse, let $C(\psi) \sqsubseteq C(\phi)$. If $C(\psi) \sqsubset C(\phi)$ we already know that $\mathbf{u} \succ_{KH} \mathbf{v}$, so assume that $C(\psi) = C(\phi)$. In this case, $\mathbf{v} \in \partial C(\phi)$ so we also have $\mathbf{u} \succeq_{KH} \mathbf{v}$.

Given that the Kaldor-Hicks relation agrees with the relation $\sqsubset$ and that this relation gives an ordering whenever the NIP is satisfied, we have that $\succeq_{KH}$ is also an ordering.
($\leftarrow$) Assume that $\succeq_{KH}$ is an ordering with asymmetric part $\succ_{KH}$. Assume, towards a contradiction, that the NIP is not satisfied. In other words, there is a $\mathbf{u} \in \partial C(\phi) \cap \partial C(\psi)$ and $C(\phi) \neq C(\psi)$. Then by Assumption D, we can assume that, without loss of generality, there is a utility profile $\mathbf{v} \in C(\phi) \setminus C(\psi)$ with $\mathbf{v} \gg \mathbf{0}$.

Consider a $\gamma$ such that $\gamma\mathbf{v} \in \partial C(\psi)$. Such vector exists. Moreover $\gamma < 1$ as otherwise $\mathbf{v} \in C(\psi)$. As such, $\mathbf{v} \gg \gamma\mathbf{v}$ and $\mathbf{u} \succ_{KH} \gamma\mathbf{v}$. On the other hand $\gamma\,\mathbf{v} \in \partial C(\psi)$ so,

$$\gamma\,\mathbf{v} \succeq_{KH} \mathbf{u},$$

a contradiction with the fact that $\succ_{KH}$ is the asymmetric part of $\succeq_{KH}$.

## Proof of Lemma 4.2

Assume that $\mathbf{u} \in \partial C(\phi)$ and that $(C(\phi))_{\phi\in\Phi}$ satisfies TU. Let us first show that $\sum_i g_i(u_i) = \kappa(\phi)$. We know that $\sum_i g_i(u_i) \leq \kappa(\phi)$. If, towards a contradiction $\sum_i g_i(u_i) < \kappa(\phi)$, we can, by continuity of $g_i$ find a $\varepsilon > 0$ such that $\sum_i g_i(u_i + \varepsilon) < \kappa(\phi)$, so $\mathbf{u} + \varepsilon\mathbf{e} \in C(\phi)$. This contradicts the assumption that $\mathbf{u} \in \partial C(\phi)$.

Now if $\mathbf{u} \in \partial C(\phi) \cap \partial C(\psi)$, then

$$\sum_i g_i(u_i) = \kappa(\phi) = \kappa(\psi),$$

so for all $\mathbf{v} \in \mathbb{R}^N_+$, we have that $\mathbf{v} \in C(\phi)$ (i.e. $\sum_i g_i(v_i) \leq \kappa(\phi)$) if and only if $\mathbf{v} \in C(\psi)$ (i.e., $\sum_i g_i(v_i) \leq \kappa(\psi)$) which shows that $C(\phi) = C(\psi)$.

## Proof of Theorem 4.4

($\rightarrow$) Assume that TU is satisfied. Then we know that $(C(\phi))_{\phi\in\Phi}$ satisfies the NIP. Let $\rho$ and $h$ be the surplus and Pareto functions from the NIP condition. It is easily verified that $\rho(\phi) \geq \rho(\phi')$ if and only if $\kappa(\phi) \geq \kappa(\phi')$. As such, we have that $\rho(\phi) = g(\kappa(\phi))$ for some strictly increasing function $g(.)$. Then, for $\mathbf{u} \in \partial C(\phi)$ we have that,

$$h(\mathbf{u}) = \rho(\phi) = g(\kappa(\phi)) = g\left(\sum_i^N g_i(u_i)\right),$$

so $h$ is additive separable.
($\leftarrow$) Let $h$ be additive separable. Then by Theorem 3.3. $\mathbf{u} \in C(\phi)$ if and only if,

$$g\left(\sum_{i=1}^{N} g_i(u_i)\right) = h(\mathbf{u}) \leq \rho(\phi).$$

which shows that $(C(\phi))_{\phi\in\Phi}$ satisfies TU with $\kappa(\phi) = g^{-1}(\rho(\phi))$.

## Proof of Theorem 5.2

Assume that $C(\delta,\xi)_{(\delta,\xi)\in\Phi}$ satisfies the NIP and Assumptions A, B and E. Also assume that $\delta_m \succ_m \delta_{m'}$. If $m'$ is single, then $u_{m'} = 0$ so immediately, $u_m \geq u_{m'}$. Next assume that $m'$ is matched, e.g. $\sigma(m') = w'$. If, towards a contradiction, $u_m \leq u_{m'}$, then,

$$h(u_m, v_{w'}) \leq (u_{m'}, v_{w'}) = \rho(\delta_{m'}, \xi_{w'}).$$

Also, by Assumption E: $\rho(\delta_{m'}, \xi_{w'}) < \rho(\delta_m, \xi_{w'})$. This implies that $m, w'$ can form a blocking pair.

## Proof of Corollary 5.4

Assume, towards a contradiction, that Assumptions E and F are satisfied, that $\sigma(.)$ is a stable matching but that it does not satisfy PAM.

In particular there are types $\delta_m \succ_m \delta_{m'}, \xi_w \succ_w \xi_{w'}$, with $\sigma(m) = w'$ and $\sigma(m') = w$. Given that the matching is stable, it must be that $h(u_m, v_{w'}) = \rho(\delta_m, \xi_{w'})$, $h(u_{m'}, v_w) = \rho(\delta_{m'}, \xi_w)$ and there are no blocking pairs, in particular, $h(u_{m'}, v_{w'}) \geq \rho(\delta_{m'}, \xi_{w'})$ and $h(u_m, v_m) \geq \rho(\delta_m, \xi_w)$.

If $h(u_{m'}, v_{w'}) = \rho(\delta_{m'}, \xi_{w'})$ then by Assumption F, $h(u_m, v_w) < \rho(\delta_m, \xi_w)$ which shows that $(m, w)$ forms a blocking pair, which gives a contradiction. As such, it must be that $h(u_{m'}, v_{w'}) > \rho(\delta_{m'}, \xi_{w'})$.

Using Assumption A, we can show the existence of a vector $(\underline{u}, \underline{v}) \in \partial C(\delta_{m'}, \xi_{w'})$ such that $(\underline{u}, \underline{v}) < (u_{m'}, v_{w'})$. Again by Assumption A, we can find a value $\overline{u}$ and $\overline{v}$ such that $(\underline{u}, \overline{v}) \in \partial C(\delta_{m'}, \xi_w)$ and $(\overline{u}, \underline{v}) \in \partial C(\delta_m, \xi_{w'})$. Assumption E guarantees that $\overline{v} \geq \underline{v} \geq v_w$ and $\overline{u} \geq \underline{u} \geq u_m$. Also, by Assumption F: $h(\overline{u}, \overline{v}) < \rho(\delta_m, \xi_w)$. But this shows that $h(u_m, v_w) < \rho(\delta_m, \xi_w)$, a contradiction.

## Proof of Theorem 5.5

Let $\overline{\delta} \succ_m \underline{\delta}$, $\overline{\xi} \succ_w \underline{\xi}$, and,

$$\begin{aligned}
h(\underline{u}, \underline{v}) &= \rho(\underline{\delta}, \underline{\xi}), \\
h(\underline{u}, \overline{v}) &= \rho(\underline{\delta}, \overline{\xi}), \\
h(\overline{u}, \underline{v}) &= \rho(\overline{\delta}, \underline{\xi}).
\end{aligned}$$

From Assumption E, we see that $\overline{u} > \underline{u}$ and $\overline{v} > \underline{v}$. As such,

$$\begin{aligned} h(\overline{u},\overline{v}) + h(\underline{u},\underline{v}) &\leq h(\overline{u},\underline{v}) + h(\underline{u},\overline{v}), \\ &= \rho(\overline{\delta},\underline{\xi}) + \rho(\underline{\delta},\overline{\xi}), \\ &\leq \rho(\overline{\delta},\overline{\xi}) + \rho(\underline{\delta},\underline{\xi}). \end{aligned}$$

where at least one of the two inequalities is strict. This shows that $h(\overline{u},\overline{v}) < \rho(\overline{\delta},\overline{\xi})$ so Assumption F is satisfied.

## Proof of Theorem 5.6

It suffices to show that Assumption F is satisfied. Define $\overline{u},\overline{v},\overline{\delta},\overline{\xi}$ such that,

$$\begin{aligned} h(\underline{u},\underline{v}) &= \rho(\underline{\delta},\underline{\xi}), \\ h(\underline{u},\overline{v}) &= \rho(\underline{\delta},\overline{\xi}), \text{ and }, \\ h(\overline{u},\underline{v}) &= \rho(\overline{\delta},\underline{\xi}). \end{aligned}$$

We need to show that $h(\overline{u},\overline{v}) < \rho(\overline{\delta},\overline{\xi})$. Now,

$$\begin{aligned} h(\overline{u},\overline{v}) &= h(\overline{u},\overline{v}) - h(\overline{u},\underline{v}), \\ &\quad + h(\underline{u},\overline{v}) - h(\underline{u},\underline{v}), \\ &\quad + h(\overline{u},\underline{v}) - h(\underline{u},\overline{v}) + h(\underline{u},\underline{v}), \\ &= \int_{\underline{u}}^{\overline{u}} \int_{\underline{v}}^{\overline{v}} \frac{\partial^2 h(u,v)}{\partial u \partial v} du\, dv + \rho(\overline{\delta},\underline{\xi}) - \rho(\underline{\delta},\overline{\xi}) + \rho(\underline{\delta},\underline{\xi}), \\ &< \int_{\underline{\delta}}^{\overline{\delta}} \int_{\underline{\xi}}^{\overline{\xi}} \frac{\partial^2 \rho(\delta,\psi)}{\partial \delta \partial \xi} d\delta\, d\xi + \rho(\overline{\delta},\underline{\xi}) - \rho(\underline{\delta},\overline{\xi}) + \rho(\underline{\delta},\underline{\xi}), \\ &= \rho(\overline{\delta},\overline{\xi}). \end{aligned}$$

The inequality uses the change of variables $u(\delta) = u$ where $h(u(\delta),\underline{v}) = \rho(\delta,\underline{\xi})$ and $v(\xi) = v$ with $h(\underline{u}, v(\xi)) = \rho(\underline{\delta},\xi)$.

## Proof of Theorem 5.8

We begin by proving the following Lemma.

**Lemma A.2.** *If* $\mathbf{y} \in Y$ *and* $(\mathbf{x},\mathbf{y}) \in \mathcal{P}|_Y$ *then* $\mathbf{u}(\mathbf{x},\mathbf{y}) \in \partial C(\mathbf{y})$. *On the other hand if* $\mathbf{v} \in \partial C(\mathbf{y})$ *then there is an allocation* $(\mathbf{x},\mathbf{y}) \in \mathcal{P}|_{\{\mathbf{y}\}}$ *such that* $\mathbf{v} = \mathbf{u}(\mathbf{x},\mathbf{y})$.

*Proof.* The first part is obvious from the definition of $\mathcal{P}|_Y$. For the second part, let $\mathbf{v} \in \partial C(\mathbf{y})$. Then, by definition, there is a $(\mathbf{x},\mathbf{y})$ such that $f(\mathbf{x},\mathbf{y}) \leq 0$ and $\mathbf{v} \leq \mathbf{u}(\mathbf{x},\mathbf{y})$. Let us show that $\mathbf{v} = \mathbf{u}(\mathbf{x},\mathbf{y})$. Given that $\mathbf{v} \in \partial C(\mathbf{y})$ it is impossible that $\mathbf{v} \ll \mathbf{u}(\mathbf{x},\mathbf{y})$. So if $\mathbf{v} \neq \mathbf{u}(\mathbf{x},\mathbf{y})$ then $\mathbf{v} < \mathbf{u}(\mathbf{x},\mathbf{y})$ and there should be an agent $j$ such that $v_j < u_j(x_j,\mathbf{y})$.

We have that $x_j > 0$ so, by continuity of $f$ there should exist $\varepsilon > 0$ and $\delta > 0$ such that, $v_j < u_j(x_j - \varepsilon, \mathbf{y})$, and

$$f\left(\mathbf{x} - \varepsilon\,\mathbf{e}_j + \sum_{i \neq j} \delta\,\mathbf{e}_i, \mathbf{y}\right) \leq 0.$$

Here $\mathbf{e}_i$ is the vector that has a one at position $i$ and zeros otherwise. let $\mathbf{x}' = \mathbf{x} - \varepsilon\,\mathbf{e}_j + \sum_{i \neq j} \delta\,\mathbf{e}_i$. Observe that $f(\mathbf{x}', \mathbf{y}) \leq 0$ and $\mathbf{u}(\mathbf{x}', \mathbf{y}) \geq \mathbf{0}$. As such $\mathbf{u}(\mathbf{x}', \mathbf{y}) \in C(\mathbf{y})$. The contradiction follows from the fact that $\mathbf{v} \ll \mathbf{u}(\mathbf{x}', \mathbf{y})$. □

**Lemma A.3.** *The collection $(C(\mathbf{y}))_{\mathbf{y} \in \mathcal{Y}}$ satisfies Assumption A.*

*Proof.* Let $\mathbf{y} \in \mathcal{Y}$. Then, by definition, there is a $(\mathbf{x}, \mathbf{y})$ such that $f(\mathbf{x}, \mathbf{y}) \leq 0$ and $\mathbf{u}(\mathbf{x}, \mathbf{y}) \geq \mathbf{0}$, so $\mathbf{u}(\mathbf{x}, \mathbf{y}) \in C(\mathbf{y})$ which means that $C(\mathbf{y})$ is non-empty.

Next, we need to show that $C(\mathbf{y})$ is bounded. Let $\mathbf{v} \in C(\mathbf{y})$. Then there is a vector $\mathbf{x}$, such that $f(\mathbf{x}, \mathbf{y}) \leq 0$ and for all agents $\mathbf{v} \leq \mathbf{u}(\mathbf{x}, \mathbf{y})$. Also, for all agents $i \leq N$, there is a $\overline{x_i}$ such that $f(\overline{x_i}\,\mathbf{e}_i, 0) > 0$. This implies that $x_i \leq \overline{x_i}$ and consequentially, $\mathbf{v} \leq \mathbf{u}(\overline{\mathbf{x}}, \mathbf{y})$. Then,

$$\|\mathbf{v}\| \leq \sqrt{\sum_{i=1}^{N} (u_i(\overline{x}_i, \mathbf{y}))^2},$$

which shows that $C(\mathbf{y})$ is bounded.

To finalize assumption A we have to show that $C(\mathbf{y})$ is closed, let $(\mathbf{v}^t)_{t \in \mathbb{N}}$ be a sequence in $C(\mathbf{y})$ with $\mathbf{v}^t \to \mathbf{v}$. Then there is a sequence $(\mathbf{x}^t)_{t \in \mathbb{N}}$ such that for all $t \in \mathbb{N}$, $f(\mathbf{x}^t, \mathbf{y}) \leq 0$ and $\mathbf{v}^t \leq \mathbf{u}(\mathbf{x}^t, \mathbf{y})$. The sequence $(\mathbf{x}^t)_{t \in \mathbb{N}}$ takes values in a bounded set (each $\mathbf{x}^t$ is bounded from above by $\overline{\mathbf{x}}$), so it has a convergent subsequence $\mathbf{x}^{t_j} \to \mathbf{x}$. Now, $\mathbf{v}^{t_j} \leq \mathbf{u}(\mathbf{x}^{t_j}, \mathbf{y})$. Also, the functions $u_i(x_i, \mathbf{y})$ are continuous, so $\mathbf{v} \leq \mathbf{u}(\mathbf{x}, \mathbf{y})$. Next, $f(\mathbf{x}^{t_j}, \mathbf{y}) \leq 0$ for all $j$, so, given that $f$ is continuous, $f(\mathbf{x}, \mathbf{y}) \leq 0$ which shows that $\mathbf{v} \in C(\mathbf{y})$. □

Let us now prove Theorem 5.8.

($\rightarrow$) Towards a contradiction, assume that the Coase property is satisfied but the NIP is violated. This means that there is a $\mathbf{v} \in \partial C(\mathbf{y}) \cap \partial C(\mathbf{y}')$ and $C(\mathbf{y}) \neq C(\mathbf{y}')$. Without loss of generality, assume that there is a $\mathbf{w} \in C(\mathbf{y}) \setminus C(\mathbf{y}')$. As $\mathbf{v} \in \partial C(\mathbf{y}') \cap \partial C(\mathbf{y})$, by Lemma A.2, there should be an $\mathbf{x}^1$ and $\mathbf{x}^2$ such that $\mathbf{v} = \mathbf{u}(\mathbf{x}^1, \mathbf{y}) = \mathbf{u}(\mathbf{x}^2, \mathbf{y}')$, $(\mathbf{x}^1, \mathbf{y}) \in \mathcal{P}|_{\{\mathbf{y}\}}$ and $(\mathbf{x}^2, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y}'\}}$.

**Step 1:** $(\mathbf{x}^2, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y}, \mathbf{y}'\}}$

If not, there is a $(\mathbf{x}^3, \mathbf{y}'')$ such that $\mathbf{y}'' \in \{\mathbf{y}, \mathbf{y}'\}$, $f(\mathbf{x}^3, \mathbf{y}'') \leq 0$ and

$$\mathbf{u}(\mathbf{x}^3, \mathbf{y}'') \gg \mathbf{u}(\mathbf{x}^2, \mathbf{y}') = \mathbf{u}(\mathbf{x}^1, \mathbf{y}) = \mathbf{v}.$$

If $\mathbf{y}'' = \mathbf{y}'$, then $\mathbf{v} \notin \partial C(\mathbf{y}')$ and if $\mathbf{y}'' = \mathbf{y}$ then $\mathbf{v} \notin \partial C(\mathbf{y})$, a contradiction.

**Step 2: Let $\alpha^* = \max\{\alpha \geq 0 : \alpha\mathbf{w} \in C(\mathbf{y}')\}$. Then there is an allocation $(x^4, \mathbf{y}')$ such that $\mathbf{u}(\mathbf{x}^4, \mathbf{y}') = \alpha^*\mathbf{w}$ and $(\mathbf{x}^4, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y},\mathbf{y}'\}}$**

Observe that $\alpha^*$ is well defined by Lemma A.3. Let $\mathbf{z} = \alpha^*\mathbf{w}$. We have that $\mathbf{z} \in \partial C(\mathbf{y}')$ and that $\mathbf{z} < \mathbf{w}$. The latter follows from the fact that $\mathbf{w} \notin C(\mathbf{y}')$, so $\alpha^* < 1$. Using Lemma A.2 and $\mathbf{z} \in \partial C(\mathbf{y}')$, we know that there is an allocation $(\mathbf{x}^4, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y}'\}}$ such that $\mathbf{u}(\mathbf{x}^4, \mathbf{y}') = \mathbf{z}$. As such, we have that by Step 1 above, $(\mathbf{x}^2, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y},\mathbf{y}'\}}$ and $(\mathbf{x}^4, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y}'\}}$. By the Coase property, it follows that $(\mathbf{x}^4, \mathbf{y}') \in \mathcal{P}|_{\{\mathbf{y},\mathbf{y}'\}}$.

**Step 3: There is an allocation $(\mathbf{x}^5, \mathbf{y})$ such that $\mathbf{u}(\mathbf{x}^5, \mathbf{y}) \gg \mathbf{u}(\mathbf{x}^4, \mathbf{y}')$ and $f(\mathbf{x}^5, \mathbf{y}) \leq 0$.**

From Step 2 above we know that $\alpha^*\mathbf{w} = \mathbf{z} < \mathbf{w}$. Also, $\mathbf{w} \in C(\mathbf{y})$ so, $\mathbf{u}(\mathbf{x}^4, \mathbf{y}') \leq \mathbf{w} \leq \mathbf{u}(\mathbf{x}^6, \mathbf{y})$ for some allocation $(\mathbf{x}^6, \mathbf{y})$ with $f(\mathbf{x}^6, \mathbf{y}) \leq 0$. Also, there is at least one agent $j$ such that $u_j(x_j^4, \mathbf{y}') < u_j(x_j^6, \mathbf{y})$.

As $x_j^6 > 0$, by continuity of the utility functions and $f$, there are numbers $\varepsilon > 0$ and a $\delta > 0$ small enough such that,

$$\begin{aligned} &u_j(x_j^6, \mathbf{y}') < u_j(x_j^6 - \varepsilon, \mathbf{y}), \\ &f\left(\mathbf{x}^6 - \varepsilon\, \mathbf{e}_j + \sum_{i \neq j} \delta\, \mathbf{e}_i, \mathbf{y}\right) \leq 0. \end{aligned}$$

Let $\mathbf{x}^5 = \mathbf{x}^6 - \varepsilon\, \mathbf{e}_j + \sum_{i \neq j} \delta\, \mathbf{e}_i$. Then $\mathbf{u}(\mathbf{x}^5, \mathbf{y}) \gg \mathbf{u}(\mathbf{x}^4, \mathbf{y}')$ and $f(\mathbf{x}^5, \mathbf{y}) \leq 0$ as was to be shown.

From Step 3, we find that $(\mathbf{x}^4, \mathbf{y}') \notin \mathcal{P}|_{\{\mathbf{y},\mathbf{y}'\}}$ which contradicts with Step 2.

($\leftarrow$) Assume that the NIP is satisfied and let $Y$ be a subset of $\mathcal{Y}$. As $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies Assumptions A, we know from the NIP that the relation $\sqsubseteq$ is a complete ordering over $\mathcal{Y}$ with asymmetric part $\sqsubset$.

Now, let $(\mathbf{x}^1, \mathbf{y}) \in \mathcal{P}|_Y$ and assume that $(\mathbf{x}^2, \mathbf{y}) \in \mathcal{P}|_{\{\mathbf{y}\}}$. We need to show that $(\mathbf{x}^2, \mathbf{y}) \in \mathcal{P}|_Y$. Let $\mathbf{v} = \mathbf{u}(\mathbf{x}^1, \mathbf{y})$ and $\mathbf{w} \in \mathbf{u}(\mathbf{x}^2, \mathbf{y})$. From Lemma A.2, we know that $\mathbf{v}, \mathbf{w} \in \partial C(\mathbf{y})$.

Now, assume, towards a contradiction, that $(\mathbf{x}^2, \mathbf{y}) \notin \mathcal{P}|_Y$, then there is a $\mathbf{y}' \in Y$ and an allocation $(\mathbf{x}^3, \mathbf{y}')$ such that $\mathbf{u}(\mathbf{x}^3, \mathbf{y}') \gg \mathbf{u}(\mathbf{x}^2, \mathbf{y}) = \mathbf{w}$ and $f(\mathbf{x}^3, \mathbf{y}') \leq 0$. But then $\mathbf{u}(\mathbf{x}^3, \mathbf{y}') \gg \mathbf{w}$ and, by Lemma A.2 $\mathbf{u}(\mathbf{x}^3, \mathbf{y}') \in C(\mathbf{y}')$. By definition, $C(\mathbf{y}') \not\sqsubseteq C(\mathbf{y})$. By completeness of the relation $\sqsubseteq$, we have that $C(\mathbf{y}) \sqsubset C(\mathbf{y}')$. As $\mathbf{v} \in C(\mathbf{y})$, it follows that there is a profile $\mathbf{z} \gg \mathbf{v} = \mathbf{u}(\mathbf{x}^1, \mathbf{y})$ such that $\mathbf{z} \in C(\mathbf{y}')$. This contradicts with the assumption that $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}|_Y$.

## Proof of Theorem 5.10

Assume $(C(\mathbf{y}))_{\mathbf{y}\in\mathcal{Y}}$ satisfies the NIP. Let $(\mathbf{x}, \mathbf{a}) \in \Gamma$. Then clearly, $\mathbf{x} \in \Gamma(\mathbf{a})$ and by Assumption H, $\mathbf{u}(\mathbf{x}, g(\mathbf{a})) \in \partial C(g(\mathbf{a}))$. Also,

$$W(\mathbf{u}(\mathbf{x}, g(\mathbf{a}))) = W(\mathbf{u}(\gamma(\mathbf{a}), g(\mathbf{a}))).$$

Now assume that $\mathbf{a}$ is not a subgame perfect Nash equilibrium. Then there is an individual $i \leq N$ and a strategy $a_i' \in A_i$ such that

$$u_i(\gamma_i(a_i', \mathbf{a}_{-i}), g(a_i', \mathbf{a}_{-i})) > u_i(\gamma_i(\mathbf{a}), g(\mathbf{a})).$$

This means that,

$$\mathbf{u}(\gamma(\mathbf{a}), g(\mathbf{a})) \not\geq \mathbf{u}(\gamma(a_i', \mathbf{a}_{-i}), g(a_i', \mathbf{a}_{-i})).$$

From Assumption H, we have $C(g(a_i', \mathbf{a}_{-i})) \not\sqsubseteq C(g(\mathbf{a}))$. As the NIP is satisfied, we have that $\sqsubseteq$ is complete with asymmetric part $\sqsubset$, so $C(g(\mathbf{a})) \sqsubset C(g(a_i', \mathbf{a}_{-i}))$.

As $\mathbf{u}(\gamma(\mathbf{a}), g(\mathbf{a})) \in C(g(\mathbf{a}))$, this means that there is a $\mathbf{v} \in C(g(a_i', \mathbf{a}_{-i}))$, such that $\mathbf{v} \gg \mathbf{u}(\gamma(\mathbf{a}), g(\mathbf{a}))$. So there is a $\mathbf{x}'$, such that $f(\mathbf{x}', g(a_i', \mathbf{a}_{-i})) \leq 0$ and,

$$\mathbf{u}(\mathbf{x}', g(a_i', \mathbf{a}_{-i})) \gg \mathbf{u}(\gamma(\mathbf{a}), g(\mathbf{a})).$$

As $W$ is strictly increasing in all components, it follows that

$$W(\mathbf{u}(\mathbf{x}', g(a_i, \mathbf{a}_{-i})) > W(\mathbf{u}(\gamma(\mathbf{a}), g(\mathbf{a}))) = W(\mathbf{u}(\mathbf{x}, g(\mathbf{a}))).$$

As such, $(\mathbf{x}, g(\mathbf{a})) \notin \Gamma$, a contradiction.

## Proof of Theorem 6.4

We can normalize all utility functions such that for all $i$, $u_i(x_i, \mathbf{y}) = \alpha(\mathbf{y})x_i + \beta_i(\mathbf{y})$. Assumption G.4 guarantees that for any individual rational allocation $x_i > 0$. As such, inverting this utility function with respect to $x_i$ gives,

$$x_i(u_i, \mathbf{y}) = \frac{u_i}{\alpha(\mathbf{y})} - \frac{\beta_i(\mathbf{y})}{\alpha(\mathbf{y})}.$$

The requirement that $\alpha(\mathbf{y}) > 0$ shows that this is well defined. Observe that $x_i(u_i, \mathbf{y})$ is strictly increasing in $u_i$. As such, the condition for the Pareto frontier can be written as,

$$(u_1, \ldots, u_N) \in C(\mathbf{y}) \leftrightarrow \sum_{i=1}^{N} \delta_i u_i \leq -\alpha(\mathbf{y})c(y) + \sum_{i=1}^{N} \delta_i \beta_i(\mathbf{y}).$$

Setting $h(u_1, \ldots, u_N) = \sum_{i=1}^{N} \delta_i u_i$ and $\rho(\mathbf{y}) = -\alpha(\mathbf{y})c(y) + \sum_{i=1}^{N} \delta_i \beta_i(\mathbf{y})$ shows that the NIP is satisfied (Theorem 3.3, part 4). As such $U_\alpha$ is an admissible NIP class. In fact, $h$ is additively separable, so TU is also satisfied.

To prove maximality, let $u_i \in U_\alpha$ for all $i \neq j$ and $u_j \in \mathcal{U}$. Further, assume that NIP is satisfied for all $f \in F_A$. We will show that $u_j \in U_\alpha$, thereby showing that $U_\alpha$ is maximal.

If we invert $u_j(x_j, \mathbf{y}) = u_j$ with respect to $x_j$, we obtain a function $x_j = x_j(u_j, \mathbf{y})$, which is strictly increasing in $u_j$. Then, for any Pareto optimal allocation, we have,

$$f(\mathbf{x}, \mathbf{y}) = \frac{1}{\alpha(\mathbf{y})} \sum_{i \neq j} \delta_i u_i + \delta_j x_j(u_j, \mathbf{y}) - \sum_{i \neq j} \frac{\delta_i \beta_i(\mathbf{y})}{\alpha(\mathbf{y})} + c(\mathbf{y}) = 0$$

Equivalently,

$$u_1 = -\sum_{i\neq 1,j} \frac{\delta_i}{\delta_1} u_i - \frac{\delta_j}{\delta_1}\alpha(\mathbf{y})x_j(u_j,\mathbf{y}) + \sum_{i\neq j} \frac{\delta_i}{\delta_1}\beta_i(\mathbf{y}) - \frac{\alpha(\mathbf{y})}{\delta_1}c(\mathbf{y}).$$

Define $r(u_j,\mathbf{y}) = -\frac{\delta_j}{\delta_i}\alpha(\mathbf{y})x_j(u_j,\mathbf{y})$ and $\tilde{r}(\mathbf{y}) = \sum_{i\neq j}\frac{\delta_i}{\delta_1}\beta_i(\mathbf{y}) - \frac{\alpha(\mathbf{y})}{\delta_1}c(\mathbf{y})$. For NIP to hold, we know from Section 3 that the right hand side has to be separable in $\mathbf{y}$. Given that the right hand side is differentiable, we can use the Leontief-Sono condition, which gives that for all public goods $k$ and $\ell$,

$$\frac{\partial}{\partial u_j}\left(\frac{\frac{\partial r}{\partial y_k} + \frac{\partial \tilde{r}}{\partial y_k}}{\frac{\partial r}{\partial y_\ell} + \frac{\partial \tilde{r}}{\partial y_\ell}}\right) = 0.$$

This gives the condition,

$$\frac{\partial^2 r}{\partial y_k \partial u_j}\left(\frac{\partial r}{\partial y_\ell} + \frac{\partial \tilde{r}}{\partial y_\ell}\right) = \frac{\partial^2 r}{\partial y_\ell \partial u_j}\left(\frac{\partial r}{\partial y_k} + \frac{\partial \tilde{r}}{\partial y_k}\right).$$

As this condition has to hold for transformation functions in $F_A$ (i.e. for varying functions $c(\mathbf{y})$), it must be that for all public goods $k$,

$$\frac{\partial^2 r}{\partial y_k \partial u_j} = 0.$$

This implies the existence of a strictly increasing function $\gamma$ and a function $\omega$ such that,

$$r(u_j,\mathbf{y}) \equiv -\alpha(\mathbf{y})x_j(u_j,\mathbf{y}) = -\gamma(u_j) + \omega(\mathbf{y}).$$

Inverting gives,

$$u_j(x_j,\mathbf{y}) = \gamma^{-1}(\alpha(\mathbf{y})x_j + \omega(\mathbf{y})),$$

This shows that $u_j(.) \in U_\alpha$ as was to be shown.

## Proof of Theorem 6.7

To show that that $V_\alpha$ is an admissible NIP class consider the normalization $u_i(x_i,\mathbf{y}) = \alpha(\mathbf{y})x_i$. Given Assumption G.4, the amount $x_i > 0$ for any individual rational allocation. As such, we can invert this function with respect to $x_i$ to obtain $x_i(u_i,\mathbf{y}) = \frac{u_i}{\alpha(\mathbf{y})}$, which is strictly increasing in $u_i$. Substituting into the feasibility constraint, we see that any Pareto efficient allocation satisfies,

$$\begin{aligned}
&w\left(\frac{u_1}{\alpha(\mathbf{y})}, \dots, \frac{u_N}{\alpha(\mathbf{y})}\right) + c(\mathbf{y}) = 0,\\
\to &m\left(\frac{1}{(\alpha(\mathbf{y}))^t} s(u_1,\dots,u_N)\right) = -c(\mathbf{y}),\\
\to &s(u_1,\dots,u_N) = m^{-1}(-c(\mathbf{y}))(\alpha(\mathbf{y}))^t,
\end{aligned}$$

where $m : \mathbb{R} \to \mathbb{R}$ is increasing and $s : \mathbb{R}^N \to \mathbb{R}$ is homogeneous of degree $t$. Setting $h(\mathbf{u}) = s(\mathbf{u})$ and $\rho(\mathbf{y}) = m^{-1}(c(\mathbf{y}))(\alpha(\mathbf{y}))^t$ shows that, by part 4 of Theorem 3.3, this satisfies the NIP.

Next for maximality, let us assume that $u_i(x_i, \mathbf{y}) = \alpha(\mathbf{y})x_i \in V_\alpha$ for all $i \neq j$ and let $u_j \in \mathcal{U}$. It suffices to show that $u_j \in V_\alpha$. Inverting the utility functions,

$$u_i(x_i, \mathbf{y}) = \alpha(\mathbf{y})x_i,$$

with respect to $x_i$ gives,

$$x_i = \frac{u_i}{\alpha(\mathbf{y})}.$$

Also, if we invert $u_j(x_j, \mathbf{y}) \in U$ with respect to $x_j$ we obtain a function $x_j = x_j(u_j, \mathbf{y})$, which is strictly increasing in $u_j$. Then, for any Pareto optimal allocation, we have,

$$\begin{aligned} w(x_1, \ldots, x_N) + c(\mathbf{y}) &= w(u_1/\alpha(\mathbf{y}), \ldots, x_j(u_j, \mathbf{y}), \ldots, u_N/\alpha(\mathbf{y})) + c(\mathbf{y}), \\ &= m\left(\frac{1}{(\alpha(\mathbf{y}))^t} s(u_1, \ldots, x_j(u_j, \mathbf{y})\alpha(\mathbf{y}), \ldots, u_N)\right) + c(\mathbf{y}) = 0 \end{aligned}$$

Inverting with respect to $m$ gives,

$$s(u_1, \ldots, x_j(u_j, \mathbf{y}))\alpha(\mathbf{y}), \ldots, u_N) = m^{-1}(-c(\mathbf{y}))(\alpha(\mathbf{y}))^t.$$

Finally, inverting $s$ with respect to its first component gives,

$$u_1 = z(u_2, \ldots, u_{j-1}, \alpha(\mathbf{y})x_j(u_j, \mathbf{y}), u_{j+1}, \ldots, u_N; m^{-1}(-c(\mathbf{y}))(\alpha(\mathbf{y}))^t).$$

where $z$ is the inverse of $s$.

Define $r(\mathbf{y}, u_j) = \alpha(\mathbf{y})\, x_j(u_j, \mathbf{y})$ and $\tilde{r}(\mathbf{y}) = m^{-1}(-c(\mathbf{y}))(\alpha(\mathbf{y}))^t$. Then,

$$u_1 = z(u_2, \ldots, u_{j-1}, r(\mathbf{y}, u_j), u_{j+1}, \ldots, u_N, \tilde{r}(\mathbf{y})).$$

Similar to Theorem 6.4, the right hand side has to be separable in $\mathbf{y}$. Using the Leontief-Sono conditions, we require that for all public goods $k$ and $\ell$,

$$\frac{\partial}{\partial u_j}\left(\frac{\frac{\partial z}{\partial r}\frac{\partial r}{\partial y_k} + \frac{\partial z}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial y_k}}{\frac{\partial z}{\partial r}\frac{\partial r}{\partial y_\ell} + \frac{\partial z}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial y_\ell}}\right) = 0.$$

After some algebra, we can show that this gives the following second order differential equation,

$$\begin{aligned} &\left(\frac{\partial^2 z}{\partial r^2}\frac{\partial r}{\partial u_j}\frac{\partial r}{\partial y_k} + \frac{\partial z}{\partial r}\frac{\partial^2 r}{\partial y_k \partial u_j} + \frac{\partial^2 z}{\partial \tilde{r}\partial r}\frac{\partial \tilde{r}}{\partial y_k}\frac{\partial r}{\partial u_j}\right) \times \left(\frac{\partial z}{\partial r}\frac{\partial r}{\partial y_\ell} + \frac{\partial z}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial y_\ell}\right) \\ &= \left(\frac{\partial^2 z}{\partial r^2}\frac{\partial r}{\partial u_j}\frac{\partial r}{\partial y_\ell} + \frac{\partial z}{\partial r}\frac{\partial^2 r}{\partial y_\ell \partial u_j} + \frac{\partial^2 z}{\partial \tilde{r}\partial r}\frac{\partial \tilde{r}}{\partial y_\ell}\frac{\partial r}{\partial u_j}\right) \times \left(\frac{\partial z}{\partial r}\frac{\partial r}{\partial y_k} + \frac{\partial z}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial y_k}\right). \end{aligned}$$

Rearranging terms, we obtain,

$$\frac{\partial r}{\partial u_j}\left(\frac{\partial^2 z}{\partial r^2}\frac{\partial z}{\partial \tilde{r}} - \frac{\partial^2 z}{\partial r \partial \tilde{r}}\frac{\partial z}{\partial r}\right) \times$$
$$\left[\frac{\partial r}{\partial y_k}\frac{\partial \tilde{r}}{\partial y_\ell} - \frac{\partial r}{\partial y_\ell}\frac{\partial \tilde{r}}{\partial y_k}\right]$$
$$+ \left(\frac{\partial z}{\partial r}\right)^2 \left[\frac{\partial^2 r}{\partial u_j \partial y_k}\frac{\partial r}{\partial y_\ell} - \frac{\partial^2 r}{\partial u_j \partial y_\ell}\frac{\partial r}{\partial y_k}\right]$$
$$+ \frac{\partial z}{\partial r}\frac{\partial z}{\partial \tilde{r}}\left[\frac{\partial^2 r}{\partial y_k \partial u_j}\frac{\partial \tilde{r}}{\partial y_\ell} - \frac{\partial^2 r}{\partial y_\ell \partial u_j}\frac{\partial \tilde{r}}{\partial y_k}\right] = 0$$

Now, since $F_A \subseteq F_H$, it is necessary that the maximal admissible class is contained in the class of generalized quasi-linear functions $U_\alpha$. This means that $\frac{\partial^2 r}{\partial u_j \partial y_k} = 0$ for all public goods $k$. As such, the two last terms are equal to zero. This gives that,

$$\frac{\partial r}{\partial u_j}\left(\frac{\partial^2 z}{\partial r^2}\frac{\partial z}{\partial \tilde{r}} - \frac{\partial^2 z}{\partial r \partial \tilde{r}}\frac{\partial z}{\partial r}\right)\left[\frac{\partial r}{\partial y_k}\frac{\partial \tilde{r}}{\partial y_\ell} - \frac{\partial r}{\partial y_\ell}\frac{\partial \tilde{r}}{\partial y_k}\right] = 0.$$

The first term is different from zero iff

$$\frac{\partial^2 s}{\partial x_j^2}\frac{\partial s}{\partial x_i} - \frac{\partial^2 s}{\partial x_i \partial x_j}\frac{\partial s}{\partial x_j} \neq 0,$$

which will be the case if $s$ is not additive separable. If so, the expression is zero for varying functions $c$ only if $\frac{\partial r}{\partial y_k} = 0$ for all $k = 1, ..., m$. This means that there is a strictly increasing function $\gamma$ such that,

$$r(u_j, \mathbf{y}) = \alpha(\mathbf{y})x_j(u_j, \mathbf{y}) = \gamma(u_j),$$

In other words, $u_j \in V_\alpha$.

## Proof of Theorem 6.10

First assume that all agents have the same utility function, $v(x_i, \mathbf{y})$. Let us show that $W_v$ is an admissible NIP class. Let $x(v, \mathbf{y})$ be the inverse function of $v(x, \mathbf{y})$ with respect to $x$. Then $(u_1, \ldots, u_N) \in C(\mathbf{y})$ iff

$$\max_{i \leq N} x(u_j, \mathbf{y}) + c(\mathbf{y}) \leq 0.$$

We have,

$$\begin{aligned}
&\max_{i \leq N} x(u_j, \mathbf{y}) + c(\mathbf{y}) \leq 0,\\
\leftrightarrow &\max_{i \leq N} x(u_j, \mathbf{y}) \leq -c(\mathbf{y}),\\
\leftrightarrow &\forall j : x(u_j, \mathbf{y}) \leq -c(\mathbf{y}),\\
\leftrightarrow &\forall j : u_j \leq v(-c(\mathbf{y}), \mathbf{y}),\\
\leftrightarrow &\max_{i \leq N} u_j \leq \rho(\mathbf{y}).
\end{aligned}$$

where we defined $\rho(\mathbf{y}) = v(-c(\mathbf{y}), \mathbf{y}))$. Now, let $h(u_1, \ldots, u_N) = \max_{i \leq N} u_j$. By part 4 of Theorem 3.3, we see that the NIP is satisfied.

For maximality, assume that for all $i \neq j$, $u_i(.) = v(.)$ and let $u_j(.) \in \mathcal{U}$. We will show that if the NIP is satisfied for all $f \in F_M$, then $u_j(.) = v(.)$. Denote the inverse function of $u_j(x_j, \mathbf{y})$ with respect to $x_j$ by $x_j(u_j, \mathbf{y})$ and let $x(v, \mathbf{y})$ be the inverse of $v(x, \mathbf{y})$ with respect to $x$.

Let $\overline{v}(\mathbf{y})$ satisfy $x(\overline{v}(\mathbf{y}), \mathbf{y}) = -c(\mathbf{y})$ and let $\overline{u}_j(\mathbf{y})$ satisfy $x_j(\overline{u_j}(\mathbf{y}), \mathbf{y}) = -c(\mathbf{y})$, i.e. $\overline{v}(\mathbf{y}) = v(-c(\mathbf{y}), \mathbf{y})$ and $\overline{u}_j(\mathbf{y}) = u_j(-c(\mathbf{y}), \mathbf{y})$. Then,

$$\begin{aligned} \mathbf{u} \in C(\mathbf{y}) &\leftrightarrow \max\{x(u_1, \mathbf{y}), \ldots, x_j(u_j, \mathbf{y}), \ldots, x(u_N, \mathbf{y})\} \leq -c(\mathbf{y}), \\ &\leftrightarrow \max\{u_1 - \overline{v}(\mathbf{y}), \ldots, u_j - \overline{u}_j(\mathbf{y}), \ldots, u_N - \overline{v}(\mathbf{y})\} \leq 0. \end{aligned}$$

As such, the utility possibility set exists of all vectors $\mathbf{u}$ that are smaller or equal than the vector,

$$(\overline{v}(\mathbf{y}), \ldots, \overline{v}(\mathbf{y}), \overline{u}_j(\mathbf{y}), \overline{v}(\mathbf{y}), \ldots, \overline{v}(\mathbf{y})).$$

If the NIP is to be satisfied, then all these vectors should be ordered (for different values of $\mathbf{y}$). Now, assume that $u_j(.) \neq v(.)$. Then there are bundles $(x, \mathbf{y})$ and $(x', \mathbf{y}')$ such that,

$$\begin{aligned} &w \equiv v(x, \mathbf{y}) < v(x', \mathbf{y}') \equiv w', \\ &w'_j \equiv u_j(x', \mathbf{y}') < u_j(x, \mathbf{y}) \equiv w_j. \end{aligned}$$

Inverting gives,

$$\begin{aligned} &x = x(w, \mathbf{y}) = x_j(w_j, \mathbf{y}), \\ &x' = x(w', \mathbf{y}') = x_j(w'_j, \mathbf{y}'). \end{aligned}$$

Choose a function $c(\mathbf{y})$ such that $-c(\mathbf{y}) = x(w, y)$ and $-c(\mathbf{y}') = x_j(w'_j, \mathbf{y}')$. We see that,

$$\begin{aligned} &-c(\mathbf{y}) = x(w, \mathbf{y}) = x_j(w_j, \mathbf{y}), \\ &-c(\mathbf{y}') = x(w', \mathbf{y}') = x_j(w'_j, \mathbf{y}'). \end{aligned}$$

As such,

$$\begin{aligned} &\overline{v}(\mathbf{y}) = w, \\ &\overline{u}_j(\mathbf{y}) = w_j, \\ &\overline{v}(\mathbf{y}') = w', \\ &\overline{u}_j(\mathbf{y}') = w'_j. \end{aligned}$$

Then,

$$\begin{aligned} &C(\mathbf{y}) = \{\mathbf{v} : \forall i \neq j, v_i \leq w \text{ and } v_j \leq w_j\}, \\ &C(\mathbf{y}') = \{\mathbf{v} : \forall i \neq j, v_i \leq w' \text{ and } v_j \leq w'_j\}. \end{aligned}$$

However, if NIP is satisfied, this means that the vectors,

$$(w, \ldots, w, w_j, w, \ldots, w) \text{ and } (w', \ldots, w', w'_j, w', \ldots, w'),$$

should be ordered, which is not the case. Conclude that $v$ and $u_j$ should represent the same preference relation.

# B Construction and properties of $\Gamma(\mathbf{u}, \phi)$

Assume that $(C(\phi))_{\phi\in\Phi}$ satisfies Assumption A. Define the correspondence $\Gamma : \mathbb{R}^N_{+,0} \times \Phi \rightrightarrows \mathbb{R}_{++}$, such that,

$$\Gamma(\mathbf{u}, \phi) = \{\gamma \geq 0 : \gamma \cdot \mathbf{u} \in \partial C(\phi)\}.$$

**Fact 1:** $\Gamma(\mathbf{u}, \phi)$ is non-empty, convex valued.
Let $\gamma = \sup\{\gamma \geq 0 : \gamma \cdot \mathbf{u} \in \partial C(\phi)\}$. By Assumption A, $\partial C(\phi)$ is non-empty, which means that $\mathbf{0} \in C(\phi)$. As such, $\gamma = 0$ is a feasible solution to this problem. Next, from Assumption A, we know that $C(\phi)$ is bounded so for all $\mathbf{u} > \mathbf{0}$, there is a $\gamma > 1$ such that $\gamma\mathbf{u} \notin C(\phi)$. This value is an upper-bound for $\{\gamma \geq 0 : \gamma \cdot \mathbf{u} \in C(\phi)\}$. As such, we know that

$$\gamma^* = \sup\{\gamma \geq 0 : \gamma \cdot \mathbf{u} \in C(\phi)\},$$

exists. Also, as $\gamma^*$ is the supremum, we know that for all $t \in \mathbb{N}$ there is a $\gamma^t \in \{\gamma \geq 0 : \gamma \cdot \mathbf{u} \in C(\phi)\}$ such that,

$$\gamma_t + \frac{1}{t} \geq \gamma^* \geq \gamma^t.$$

Observe that $(\gamma^t \cdot \mathbf{u}) \to (\gamma \cdot \mathbf{u})$ and that $(\gamma^t \cdot \mathbf{u}) \in C(\phi)$ for all $t$. By Assumption A, we have that $\gamma^* \cdot \mathbf{u} \in C(\phi)$.

Let $\mathbf{u}^* = \gamma^* \cdot \mathbf{u}$. If $\mathbf{u}^* \notin \partial C(\phi)$, then there is a $\mathbf{v} \gg \mathbf{u}^*$ such that $\mathbf{v} \in C(\phi)$. Let $\gamma' = \min_i\{v_i/u_i^*\} > 1$. Then $\mathbf{u}^* < \gamma'\mathbf{u}^* \leq \mathbf{v}$ so by comprehensiveness $\gamma'\mathbf{u}^* \in C(\phi)$. But then,

$$\gamma' \cdot \gamma^* > \gamma^*,$$

a contradiction with the optimality of $\gamma^*$. This shows that $\gamma^*\mathbf{u} \in \partial C(\phi)$ so $\Gamma(\mathbf{u}, \phi)$ is non-empty.

For convex-valuedness, let $\gamma, \gamma' \in \Gamma(\mathbf{u}, \phi)$ and assume without loss of generality that $\gamma' > \gamma$. Pick $\theta \in [0, 1]$, and set $\gamma_\theta = \theta\gamma + (1-\theta)\gamma'$. Then,

$$\gamma\mathbf{u} \leq \gamma_\theta\mathbf{u} \leq \gamma'\mathbf{u}.$$

As $\gamma'\mathbf{u} \in C(\phi)$ it must be, by comprehensiveness that $\gamma_\theta\mathbf{u} \in C(\phi)$. If $\gamma_\theta\mathbf{u} \notin \partial C(\phi)$ it then follows that there is a $\mathbf{v} \gg \gamma_\theta\mathbf{u}$ such that $\mathbf{v} \in C(\phi)$. But then, also $\mathbf{v} \gg \gamma\mathbf{u}$ which contradicts the assumption that $\gamma\mathbf{u} \in \partial C(\phi)$.

**Fact 2:** $\Gamma(\mathbf{u}, \phi)$ is upper hemicontinuous in $\mathbf{u}$ on $\mathcal{D} \setminus \{\mathbf{0}\}$.

Non-emptyness and compactness of $\Gamma(\mathbf{u}, \phi)$ was shown above. Now, take any sequence $\mathbf{u}^t \to \mathbf{u}$ and any sequence $\gamma^t \in \Gamma(\mathbf{u}^t, \phi)$.

**Step 1: The sequence $\gamma^t$ is bounded**
As $\mathbf{u}^t \to \mathbf{u}$ we know that for any $\varepsilon > 0$, there is a $T$ such that for all $t \geq T$,

$$(1-\varepsilon)\mathbf{u} < \mathbf{u}^t.$$

Let $\gamma^* = \sup\{\gamma \in \Gamma(\mathbf{u}, \phi)\}$. Next, choose $\gamma > \gamma^*$ such that,

$$\gamma(1 - \varepsilon) > \gamma^*.$$

Then, $\gamma(1-\varepsilon)\mathbf{u} > \gamma^*\mathbf{u}$, so $\gamma(1-\varepsilon)\mathbf{u} \notin C(\phi)$.

Now, $\gamma(1-\varepsilon)\mathbf{u} < \gamma\mathbf{u}^t$, which means that $\gamma\mathbf{u}^t \notin C(\phi)$. This means that $\gamma > \gamma^t$ for all $\gamma^t \in \Gamma(\mathbf{u}^t, \phi)$ and all $t \geq T$. Taking the maximum of $\gamma$ and $\cup_{t \leq T}\Gamma(\mathbf{u}^t, \phi)$ gives the desired upper bound.

**Step 2: $\gamma^t$ has a convergent subsequence that converges to $\gamma \in \Gamma(\mathbf{u}, \phi)$.**
Given that $(\gamma^t)_{t\in\mathbb{N}}$ is bounded, it has a convergent subsequence $\gamma^{t_i} \to \gamma$. Consequently $\gamma^{t_i}\mathbf{u}^{t_i} \to \gamma\mathbf{u}$. If we can show that $\gamma\mathbf{u} \in \partial C(\phi)$, then we are done as this means that $\gamma \in \Gamma(\mathbf{u}, \phi)$.

Now, for all $t_i$, $\gamma^{t_i}\mathbf{u}^{t_i} \in \partial C(\phi) \subseteq C(\phi)$. Given that $C(\phi)$ is closed, we have that $\gamma\mathbf{u} \in C(\phi)$. If, towards a contradiction $\gamma\mathbf{u} \in C(\phi) \setminus \partial C(\phi)$ then there is a $\mathbf{v} \gg \gamma\mathbf{u}$ such that $\mathbf{v} \in C(\phi)$. Then, for $t_i$ large enough, $\gamma^{t_i}\mathbf{u}^{t_i} \ll \mathbf{v}$ which contradicts the assumption that $\gamma^{t_i}\mathbf{u}^{t_i} \in \partial C(\phi)$ for all $t_i$.