ECARES

Approval Voting and Shapley Ranking

Pierre Dehez
CORE, Université catholique de Louvain, Belgium

Victor Ginsburgh
ECARES, Université libre de Bruxelles
CORE, Université catholique de Louvain, Belgium

April 2018

ECARES working paper 2018-09

ECARES
ULB - CP 114/04
50, F.D. Roosevelt Ave., B-1050 Brussels BELGIUM
www.ecares.org

Approval Voting and Shapley Ranking

Pierre Dehez
CORE, Université catholique de Louvain, Belgium

Victor Ginsburgh
ECARES, Université Libre de Bruxelles and
CORE, Université catholique de Louvain, Belgium

April 2018

Abstract

Approval voting allows voters to list any number of candidates. Their scores are obtained by summing the votes cast in their favor. Fractional voting instead follows the *One-person-one-vote* principle by endowing voters with a single vote that they may freely distribute among candidates. In this paper, we show that to be fair, such a ranking requires a uniform distribution. It corresponds to Shapley ranking that was introduced to rank wines as the Shapley value of a cooperative game with transferable utility. We analyze the properties of these "ranking games" and provide an axiomatic foundation to Shapley ranking. We also analyze Shapley ranking as a social welfare function and compare it to approval ranking.

The authors are grateful to Claude d'Aspremont for helpful comments and suggestions.

> "Which candidate ought to be elected in a single-member constituency *if all we take into account* is the order in which each of the electors ranks the candidates? ... At the very outset of the argument we try to move from the *is* to the *ought* and to jump the unbridgeable chasm between the universe of science and that of morals."
>
> (Duncan Black, 1958, p. 55)

Introduction

Approval voting is a method that was first formally studied in the 1970's by Weber (1977) and Brams and Fishburn (1978).[1] Given a set of candidates, voters have the possibility to list any number of candidates whom they consider as "good for the job." The method simply consists in assigning to each candidate a score equal to the number of voters who have listed him.[2] The winners are those with the largest score. Beyond being a voting method, rational collective preferences are derived from approval voting, like in Borda count and other scoring methods.

Approval voting has its supporters, starting with Brams and Fishburn, but also its opponents like Saari and van Newenhizen (1988).[3] The fact is that approval voting has not been much implemented except in some large scientific societies such as the *American Mathematical Society* or the *Institute of Electrical and Electronics Engineers*. There are a few other exceptions such as the election of the UN Secretary General.[4] Several experiments have been conducted, in particular by Baujard and Igersheim (2010) following the 2002 French presidential election. Approval voting in practice and in experiments, shows that the words used (approved vs disapproved, or simply yes or no) do matter.

Although voters have preferences over candidates, what they actually reveal under approval voting is much simplified. Each voter orders the candidates from the most preferred to the least preferred and draws a line somewhere to partition the set of candidates into two sublists. We only know that the candidates above the line are strictly preferred to those below. Voting by approval inevitably introduces cardinal considerations into the analysis. Indeed, two voters with identical preferences may well draw the line at different places. Furthermore, the candidates listed by a voter are in some sense relatively "close" to each other and if a voter had incomplete preferences, the candidates that she cannot rank would just not be listed.[5] Even if approval voting is compatible with incomplete preferences, we retain the assumption that

---

[1] See also Brams and Fishburn (1983, 2005), Weber (1995), Brams (2008) and the *Handbook of approval voting* edited by Laslier and Remzi Sanver (2010).

[2] Convention: we use "she" for voters and "he" for candidates.

[3] See the ensuing discussion in the issue of *Public Choice* where their paper was published. See also Saari (2008) who turns out to be a strong advocate of the linear scoring method introduced by Borda (1781), known as Borda count.

[4] See Brams and Fishburn (2010).

[5] Alcantud and Laruelle (2014) study and characterize a voting rule that allows voters to divide candidates into three classes, approved, disapproved and indifferent, thereby allowing for incomplete preferences.

preferences are complete, leaving open the possibility of dichotomous preferences where voters are indifferent within and outside their approval sets.

In what follows, we make a distinction between voting, ranking and ordering. Voting is the procedure that consists for voters to submit ballots. Ranking aggregates the voters' choices in order to associate a score to each candidate. Hence, ranking is cardinal and, in some context such as wine competitions, rankings matter. A ranking leads to an ordering and, in case of an election, the winners are the top candidates in the ordering.

Under the extreme assumption of dichotomous preferences, approval voting induces a non-dictatorial social welfare function that satisfies the Pareto and Condorcet principles, as well as Monotonicity and Independence of irrelevant alternatives. It does not satisfy the axiom of Unrestricted domain, otherwise it would contradict Arrow's impossibility theorem.[6] The assumption of dichotomous preferences is however far too strong. Assuming indifference is not plausible and this is even more the case when applied to the candidates that a voter does not list. In the present paper, we only assume that voters prefer the candidates they list to those they do not list, an assumption that is an integral part of the definition of a ballot. Notice that, under dichotomous preferences, approval voting is equivalent to the Borda method that gives 1 to a candidate that each voter has listed and 0 to the others. The absence of information on voters' preferences is a fundamental difficulty when voters are required to name candidates without ordering them. It is so for approval voting and for any method that limits the number of candidates a voter can list, including plurality voting.

In approval voting, there is no limit to the number of candidates a voter is allowed to list and there is no direct "cost" in listing additional candidates. Fractional voting, introduced by Nambiar (1989), follows the One-person-one-vote principle often use by those who advocate political equality by endowing each voter with a single unit vote that she can freely distribute among any number of candidates. For instance (1/5, 0, 2/5, 0, 1/10, 3/10) is an admissible ballot in the case of six candidates. Fractional voting differs from approval voting also because voters are allowed to rank candidates by attributing a score to each candidate. In this paper, we show that a fair ranking requires a *uniform* distribution. If a voter lists three candidates, each one gets 1/3 instead of 1.

Under approval voting, the fact that a voter adds a candidate to her ballot has no impact on the scores of the other candidates and voters who list several candidates carry more power. This is no longer so under uniform fractional voting since their vote weights less if they list a larger number of candidates. They have an incentive to limit the size of their ballots because adding a candidate reduces the chance that those already present be elected. If the objective of

[6] The reference is Arrow 1951's famous book. The 1963 edition reproduces the first edition and adds a chapter reviewing the developments in social choice theory since 1951.

a voter is to see elected one of the candidates that she places high in her preferences, she will tend to submit a limited list of candidates among which she is relatively indifferent.[7]

Uniform fractional voting corresponds to the concept of *Shapley ranking* introduced by Ginsburgh and Zang (2012) in their paper on ranking wines. They show that it comes out as the Shapley value of a transferable utility game, called ranking game. Here, we extend their analysis by characterizing the set of ranking games. We then study the properties of Shapley ranking in two different ways. We first analyze the axioms that underlie the Shapley value and translate in terms of ballot profiles. This provides an axiomatic foundation to Shapley ranking where the axiom of Efficiency translates into the One-person-one-vote principle. We then move from ranking to ordering and look at the properties of Shapley ranking as a social welfare function that translate individual preferences into collective preferences.

The paper is organized as follows. Approval and fractional ranking are introduced in Section 1 using the concept of ballot profile that specifies, for each subset of candidates, the number of voters who support it. Section 2 first covers transferable utility games, the Shapley value and its axiomatization. Ranking games are then defined and analyzed. In Section 3, we show that their Shapley value coincides with uniform fractional ranking. We then proceed with an axiomatization of Shapley ranking using axioms expressed in terms of ballot profiles. Section 4 looks at the properties of the ordering derived from Shapley ranking, from a social choice perspective. The last section is devoted to concluding remarks.

1. Approval, fractional, plurality and majority voting

Consider a set N of n candidates with $n \geq 2$.[8] There can be any number of voters. Voters have preferences over candidates: $i \succ_h j$ reads "voter h prefers candidate i to candidate j" and $i \sim_h j$ reads "voter h is indifferent between candidates i and j." The weak preference relation $\succsim_h$ represents the preferences of voter h. Preferences are assumed to be complete and rational: $\succsim_h$ is a transitive and reflexive binary relation (a preorder) over N. A preference profile specifies a preference ordering for each voter.

1.1 Approval voting

Under approval voting, voters are asked to list the candidates they approve. We denote by $N_h \subset N$ the *approval set* or *ballot* of voter h, that is the set of candidates voter h lists. We assume that $N_h \neq \varnothing$ and do not exclude $N_h = N.$ The choice of voter h can be identified to an n-tuple $q_h \in \{0,1\}^n$ with $q_{ih} = 1$ if and only if $i \in N_h.$ If M denotes the set of voters, a ballot profile can be arranged in a $n \times m$ matrix whose rows are attributed to candidates and columns

[7] A variant of approval voting consists in limiting to the size of ballots. Actually, Condorcet (1788) proposed a form of approval voting, imposing a size to ballots, in which case, approval voting and uniform fractional voting coincide.

[8] *Notation*: Upper-case letters are used to denote finite sets and subsets, and the corresponding lower-case letters are used to denote the number of their elements.

to voters. A ballot profile is a mapping π that associates to each subset $S \subset N$ of candidates, the number of voters whose approval set coincides with *S*:

$$\pi(S) = \left| \left\{ h \in M \mid N_h = S \right\} \right|. \qquad (1)$$

The set of all ballot profiles on the set *N* is given by $\Pi(N) = \mathbb{N}^{2^n - 1}$ and $\sum_{S \subset N} \pi(S) = m.$

Example 1 Consider the voting situation with four candidates described by the following ballot profile: [9]

$$\pi(1) = \pi(1,2) = \pi(2,3) = \pi(3,4) = \pi(2,3,4) = 1,$$
$$\pi(2) = \pi(3) = \pi(4) = \pi(1,3) = \pi(1,4) = \pi(2,4) = 0,$$
$$\pi(1,2,3) = \pi(1,2,4) = \pi(1,3,4) = \pi(1,2,3,4) = 0.$$

There are five voters and their approval sets are $N_1 = \{1\}$, $N_2 = \{1,2\}$, $N_3 = \{2,3\}$, $N_4 = \{3,4\}$ and $N_5 = \{2,3,4\}$. The associated matrix *Q* is given by:

$Q =$

1	1	0	0	0
0	1	1	0	1
0	0	1	1	1
0	0	0	1	1

The *approval score* of candidate *i* is the number of voters who have listed him:

$$AR_i(N,\pi) = \left| \left\{ h \in M \mid i \in N_h \right\} \right| = \sum_{S: i \in S} \pi(S) \quad (i = 1,..., n). \qquad (2)$$

It is obtained by summing along each row of the representative matrix *Q*. In Example 1, the approval ranking is (2, 3, 3, 2). It leads to the following ordering $2 \sim 3 \succ 1 \sim 4$.

1.2 Fractional voting

Fractional voting was introduced by Nambiar (1989). Each voter is endowed with one unit vote that she can freely distribute among candidates according to her preferences. This allows voters to grade candidates, a possibility we want to exclude. In what follows, we therefore restrict fractional voting to be uniform: The unit vote is uniformly distributed and each of the candidates listed by voter *h* receives a fraction $1/n_h$. This is actually the concept of Shapley ranking introduced by Ginsburgh and Zang (2012). The scores are obtained by summing the fractions allocated to each candidate:

$$SR_i(N,\pi) = \sum_{S: i \in S} \frac{1}{s} \pi(S) = \sum_{h: i \in N_h} \frac{1}{n_h} \quad (i = 1,..., n). \qquad (3)$$

[9] When there is no ambiguity, braces will be omitted.

Under fractional voting, candidates are strongly interdependent. The score of a candidate depends on the other candidates listed by voters. Table 1 shows the approval scores (*AR*) and Shapley scores (*SR*) obtained in Example 1.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	0	0	0	2	1.50
2	0	1	1	0	1	3	1.33
3	0	0	1	1	1	3	1.33
4	0	0	0	1	1	2	0.83

Table 1: Approval and Shapley scores of Example 1

The resulting ordering $1 \succ 2 \sim 3 \succ 4$ places candidate 1 on top and, not surprisingly, it differs from the ordering $2 \sim 3 \succ 1 \sim 4$ that results from approval ranking. Approval and Shapley orderings obviously coincide in the case of two candidates. Example 1 shows that they may not coincide beyond two candidates. The following example illustrates a situation where the two orderings coincide.

Example 2 Consider a voting situation involving three candidates and five voters whose approval sets are $N_1 = \{1\}$, $N_2 = \{1,2\}$, $N_3 = \{2,3\}$, $N_4 = \{1,3\}$ and $N_5 = \{1,2,3\}$. Approval and Shapley scores are given in Table 2. In both cases, candidate 1 comes first while the other two obtain the same score: $1 \succ 2 \sim 3$.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	0	1	1	4	2.33
2	0	1	1	0	1	3	1.33
3	0	0	1	1	1	3	1.33

Table 2: Approval and Shapley scores of Example 2

Notice that by normalizing Shapley scores, we obtain the probabilities that a particular candidate is elected if the *random dictator* procedure were adopted.[10] Each of the *m* voters is asked to identify a subset of candidates, knowing that a voter will first be chosen at random and the winner will then be chosen at random in her approval set. The resulting probabilities are then proportional to the fractional scores:

$$\text{Prob}[\, i \text{ is elected} \,] = \frac{1}{m} \sum_{h:\, i \in N_h} \frac{1}{n_h} = \frac{1}{m} SV_i(N, \pi).$$

[10] This is the terminology used by Bogolmania et al. (2005).

In Example 1 (see Table 1), the probabilities are given by (0.30, 0.27, 0.27, 0.17). In example 3 (see Table 2), the probabilities are given by (0.47, 0.27, 0.27).

1.3 Plurality and majority voting

A number of voting rules are special cases of approval (as well as fractional) voting in which each voter submits a single candidate i.e. $n_h = 1$ for all *h*. This is the case of plurality and majority voting. These methods are well defined only in the absence of indifference in individual preferences. Each voter has then a unique most preferred candidate and candidates are ordered according to their approval scores given by (2). In plurality voting, the winners are the candidates with the largest approval score. In majority voting, the winner is the candidate with an approval score exceeding half the number of voters. The latter is therefore not a decisive method. The following example shows that a candidate who appears first in a majority of voters' preferences may be defeated under approval voting and fractional voting. It illustrates how voting by approval reveals some information on voters' intensities of preferences.

Example 3 Consider a voting situation with four candidates and five voters whose preferences are given by:

$$1 \succ_1 3 \succ_1 2 \succ_1 4, 1 \succ_2 2 \succ_2 3 \succ_2 4, 1 \succ_3 2 \succ_3 4 \succ_3 3, 2 \succ_4 3 \succ_4 4 \succ_4 1 \text{ and } 2 \succ_5 4 \succ_5 1 \succ_5 3.$$

The first candidate has a majority.[11] Now assume that the voters submit the following ballots: $N_1 = \{1,3\}$, $N_2 = \{1,2\}$, $N_3 = \{1,2,4\}$, $N_4 = \{2,3\}$ and $N_5 = \{2,4\}$. Table 3 shows the approval and Shapley scores. Candidate 2 gets the largest approval score as well as the largest Shapley score.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	1	0	0	3	1.33
2	0	1	1	1	1	4	1.83
3	1	0	0	1	0	2	1
4	0	0	1	0	1	2	0.83

Table 3: Approval and Shapley scores of Example 3

When indifference between candidates is not ruled out, plurality and majority voting are not well defined because voters may have several most preferred candidates. If this is the case, voters have to make a selection. We could assume that the name a voter submits is drawn at random among her top candidates. Denoting by N_h the subset of most preferred candidates of voter *h*, each one is assigned a probability equal to $1/n_h$ and the score of a candidate is given by the sum of the probabilities that his name is submitted. In this case, plurality voting and Shapley voting give rise to the same result.

[11] He is therefore also the unique Condorcet winner (see Section 5).

2. Ranking games and their Shapley value

Before defining ranking games, we first recall the definition of a transferable utility game and various related concepts, including the Shapley value.

2.1 Transferable utility games

A *cooperative game with side payments* is defined by a finite set N of n players and a function v (called *characteristic function*) that associates a real number to each subset of players. $v(S)$ is referred as the "worth" of coalition S. It could be a surplus or a cost that measures what a coalition can generate without the participation of the other players. $v(N)$ measures what the "grand coalition" can generate when all players best coordinate themselves.[12] By convention, $v(\varnothing)=0.$ Because a characteristic function on N is defined by a list of $2^n - 1$ real numbers, the set $G(N)$ of all characteristic functions on N can be identified to $\mathbb{R}^{2^n-1}$ the real vector space of dimension $2^n - 1$. The *dual* of a characteristic function v is defined by

$$v^*(S) = v(N) - v(N \setminus S), \tag{4}$$

i.e. $v^*(S)$ is the contribution of coalition S to the grand coalition.[13] Alternatively, it is what remains for coalition S once the complementary coalition has collected its worth. Given a subset $T \subset N,$ the unanimity game (N, u_T) is defined by:

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

The $2^n - 1$ unanimity games form a basis of the vector space $G(N)$. Given any characteristic function v on N, there exists a *unique* collection $(\alpha_T \,|\, T \subset N, T \neq \varnothing)$ of $2^n - 1$ real numbers such that:

$$v(S) = \sum_{T \subset N} \alpha_T(N,v) u_T(S) = \sum_{T \subset S} \alpha_T(N,v). \tag{5}$$

The coefficients α_T are known as the *Harsanyi dividends* (dividends for short).[14] They are obtained from the following recursive formula deduced from (5), starting with $\alpha_\varnothing = 0$: [15]

$$\alpha_T(N,v) = v(T) - \sum_{S \subsetneq T} \alpha_S(N,v) \tag{6}$$

As a consequence, there is a one-to-one relation between games and dividends.

For a given a game $(N,v),$ the *marginal contribution* of player i to a coalition S is defined by $v(S) - v(S \setminus i).$ Players i and j are substitutable in a game (N,v) if their marginal

[12] Games with transferable utility and the notion of characteristic function have been introduced by John von Neumann and Oskar Morgenstern in their 1944 book.

[13] It is easily verified that $(v^*)^* = v$.

[14] See Harsanyi (1959).

[15] Notation: The symbol $\subsetneq$ is used to denote a strict inclusion.

contributions to coalitions to which they both belong are identical. Equivalently, *i* and *j* are substitutable if:

$$v(S \setminus i) = v(S \setminus j) \text{ for all } S \subset N \text{ such that } i, j \in S.$$

Obviously, players who are substitutable in a game are substitutable in the dual. Using (6), it is easily verified that players *i* and *j* are substitutable if $\alpha_{S\setminus i}(N,v) = \alpha_{S\setminus j}(N,v)$ for all $S \subset N$ such that $i, j \in S$. Player *i* is *null* in a game (N,v) if his marginal contributions are all equal to zero: $v(S) = v(S \setminus i)$ for all $S \subset N$. Player *i* is *dummy* in a game (N,v) if he systematically contributes his worth: $v(S) - v(S \setminus i) = v(i)$ for all $S \subset N$ such that $i \in S$. A null (resp. dummy) player in a game is null (resp. dummy) in the dual. Player *i* is null if $\alpha_T(N,v) = 0$ for all $T \ni i$.

The *sum* of two games (N,v_1) and (N,v_2) on a common set of players is the game (N,w) defined by $w(S) = v_1(S) + v_2(S)$. The resulting dividends are equal to the sum of the dividends associated to the two games.

2.2 The Shapley value

The central question addressed by the theory of cooperative games is the allocation among players of the amount resulting from their cooperation. A *solution* of a game (N,v) specifies a payoff x_i for each player *i* and an *allocation rule* is a mapping $\varphi : G(N) \to \mathbb{R}^n$ that associates solutions to games. The Shapley value is the rule that allocates uniformly the dividends of every coalition to its members:

$$SV_i(N,v) = \sum_{T:i\in T} \frac{1}{t}\,\alpha_T(N,v) \quad (i = 1,..., n). \tag{7}$$

Shapley (1953) proves that there is a unique allocation rule $\varphi : G(N) \to \mathbb{R}^n$ that satisfies the following four axioms.

Efficiency: The worth of the grand coalition is exactly allocated, not more nor less.

Anonymity: If players' names are permuted, the solution should be permuted accordingly.

Null player: A zero amount is allocated to null players.

Additivity: The value of a sum of games is the sum of the values.

Anonymity is sometimes replaced by Symmetry, an axiom requiring that an identical amount should be allocated to players who are substitutable. The Null player axiom can be replaced by the Dummy player axiom that requires that dummy players get their stand alone worth. The Shapley value has many other properties. In particular, it is a *self-dual* concept. The value of a game coincides with the value of its dual: $SV(N,v^*) = SV(N,v)$.

2.3 Ranking games and their duals

In a voting context, the candidates are the players although we don't view voting as games properly speaking. Given a ballot profile (N,π) as defined in (1), we look for a definition of a characteristic function *v* that is an equivalent representation of that ballot profile and such that $v(N)$ equals the number of voters. To each subset *S* of candidates, we associate the number of voters whose approval set is *included* in *S* :

$$\underline{v}(S)=\left|\left\{h\in M \mid N_h\subset S\right\}\right|. \tag{8}$$

The function $\underline{v}$ defines a cooperative game with side payments $(N,\underline{v})$ such that $\underline{v}(N)=m$ and $\underline{v}(i)$ is the number of voters who have only listed candidate *i*: $N_h=\{i\}$. It is the concept of *ranking game* introduced by Ginsburgh and Zang (2012). A solution of a ranking game provides a ranking of candidates by specifying for each of them a score equal to a fraction of the total number of voters. The characteristic function associated to Example 1 is given by:

$$\begin{aligned}
&\underline{v}(1)=1,\ \underline{v}(2)=0,\ \underline{v}(3)=0,\ \underline{v}(4)=0,\\
&\underline{v}(1,2)=2,\ \underline{v}(1,3)=1,\ \underline{v}(1,4)=1,\ \underline{v}(2,3)=1,\ \underline{v}(2,4)=0,\ \underline{v}(3,4)=1,\\
&\underline{v}(1,2,3)=3,\ \underline{v}(1,2,4)=2,\ \underline{v}(1,3,4)=2,\ \underline{v}(2,3,4)=3,\\
&\underline{v}(1,2,3,4)=5.
\end{aligned}$$

We observe that a ranking game can be written as the sum of the unanimity games associated to voters:

$$\underline{v}(S)=\sum\nolimits_h u_{N_h}(S) \tag{9}$$

where the game (N,u_{N_h}) can be viewed as the ranking game associated to voter *h*. (8) can alternatively be written in terms of the ballot profile:

$$\underline{v}(S)=\sum_{T\subset S}\pi(T) \tag{10}$$

Following (5) and (10), there is a one-to-one relationship between ballot profiles and ranking games.

Proposition 1 The dividends of the ranking game $(N,\underline{v})$ coincide with its underlying ballot profiles: $\alpha_T(N,\underline{v})=\pi(T)$ for all $T\subset N$.

The subset $RG(N)\subset G(N)$ of all ranking games on a set *N* generated by the set of ballot profiles $\Pi(N)$ forms a remarkable class of games. The characteristic functions defining ranking games are *monotonic* (increasing) and take values in $\mathbb{N}$. They are sums of unanimity games and, because unanimity games are convex, they are *convex*.[16] Ranking games are *positive* in

[16] Convex games have been introduced and studied by Shapley (1971).

the sense that their dividends are non-negative.[17] Furthermore, the set $RG(N)$ is *closed under addition*. Starting from any two voting situations (N,π') and (N,π'') on a common set of candidates, and their associated ranking games $(N,\underline{v}')$ and $(N,\underline{v}'')$, the ranking game $(N,\underline{v}'+\underline{v}'')$ is associated to the voting situation $(N,\pi'+\pi'')$.

Counting the number of voters whose approval set *intersects* S leads to another characteristic function:

$$\overline{v}(S)=\left|\left\{h\in M \mid N_h\cap S\neq\varnothing\right\}\right|. \tag{11}$$

It is such that $\overline{v}(N)=m$ and $\overline{v}(i)$ equals the number of voters who have included candidate i in their approval set i.e. $\overline{v}(i)=AV_i$ as defined by (2). The characteristic functions $\underline{v}$ associated to Example 1 is given by:

$$\overline{v}(1)=2,\overline{v}(2)=3,\overline{v}(3)=3,\overline{v}(4)=2,$$
$$\overline{v}(1,2)=4,\overline{v}(1,3)=5,\overline{v}(1,4)=4,\overline{v}(2,3)=4,\overline{v}(2,4)=4,\overline{v}(3,4)=3,$$
$$\overline{v}(1,2,3)=5,\overline{v}(1,2,4)=5,\overline{v}(1,3,4)=5,\overline{v}(2,3,4)=4,$$
$$\overline{v}(1,2,3,4)=5.$$

The following proposition states that there is a one-to-one relationship between the games $(N,\underline{v})$ and $(N,\overline{v})$. Comparing the two games, we observe that $\underline{v}(S)\leq\overline{v}(S)$ for all $S\subset N$. While $\underline{v}(S)$ is the number of voters who are *exclusively* supporting some candidates in *S*, $\overline{v}(S)$ is the number of voters who are supporting *some* candidates in *S* and it can be seen as the approval score of coalition *S*.

Proposition 2 The games $(N,\underline{v})$ and $(N,\overline{v})$ whose characteristic functions are defined by (8) and (11) are *dual* of each other: $\overline{v}=\underline{v}^*$ and $\underline{v}=\overline{v}^*$.

Proof. To show that $\underline{v}=\overline{v}^*$, consider some subset *S*. The two sets $\left\{h\in M \mid N_h\subset S\right\}$ and $\left\{h\in M \mid N_h\not\subset S\right\}$ form a partition of *M* where

$$\left\{h\in M \mid N_h\not\subset S\right\}=\left\{h\in M \mid N_h\cap(N\setminus S)\neq\varnothing\right\}.$$

Hence, we have $\left|\left\{h\in M \mid N_h\subset S\right\}\right|=m-\left|\left\{h\in M \mid N_h\cap(N\setminus S)\neq\varnothing\right\}\right|$. The proposition thne follows from the definition of dual games (4). ♦

As a consequence, a ballot profile (N,π) can be described in two *equivalent* ways, by a ranking game $(N,\underline{v})$ or by its dual $(N,\overline{v})$. The two characteristic functions coincide for $n=2$. Considering the two extreme voting situations where voters list either a single candidate or all candidates, the ranking game and its dual coincide in the first situation while in the second situation, $\overline{v}(S)=m$ for all $S\subset N$ $(S\neq\varnothing)$, and $\underline{v}(S)=0$ for all $S\neq N$.

[17] Positive games form a particular subclass of convex games on which the core coincides with the Harsanyi set, the set of all distributions of dividends and with the Shapley set, the set of all weighted Shapley values. See Dehez (2017) for details.

3. Shapley ranking

We first confirm that uniform fractional ranking associated to a ballot profile coincides with the Shapley value of the corresponding ranking game. We then proceed with an axiomatization expressed in terms of ballot profiles.

3.1 The Shapley value of a ranking game

Using the axioms that underlie the Shapley value and the additive decomposition of ranking games given by (9), Ginsburgh and Zang (2012) prove that the Shapley value of a ranking game is given by the uniform fractional ranking defined by (3).

Proposition 3 The Shapley ranking associated to a ballot profile (N,π) is the (common) Shapley value of the corresponding dual games $(N,\overline{v})$ and $(N,\underline{v})$:

$$SR_i(N,\pi) = SV_i(N,\overline{v}) = SV_i(N,\underline{v}) \quad (i = 1,...,n). \tag{12}$$

Proof. The Shapley value is additive. Hence, using (9), the Shapley value of the game $(N,\underline{v})$ is the sum of the values of unanimity games (N,u_{N_h}). Candidates outside N_h receive zero by the Null player axiom and candidates inside N_h receive an equal amount by the axiom of Anonymity. Hence, by the axiom of Efficiency, we have:

$$SV_i(N,u_{N_h}) = \begin{cases} \dfrac{1}{n_h} & \text{if } i \in N_h \\ 0 & \text{if } i \notin N_h \end{cases}$$

Adding these values results in (3) and (12) follows by self-duality. ♦

This is the original proof given by Ginsburgh and Zang (2012). It is confirmed by (7).

3.2 Axiomatization of Shapley ranking

A candidate is *null* if he is included in no ballot: $\pi(S) = 0$ for all coalition S of which he is a member. His approval score is then equal to zero. A candidate is *dummy* if he is null or, if some voters list him, they only list him: $i \in N_h \Rightarrow n_h = 1$. Approval score and Shapley score then coincide for dummy candidates. The sum of two ranking games on a common set of candidates is the ranking game associated to the sum of the ballot profiles.

Example 4 Consider the voting situation involving three candidates and five voters whose approval sets are $N_1 = \{1\}, N_2 = \{4\}, N_3 = \{3,4\}, N_4 = \{2,4\}$ and $N_5 = \{2,3\}$. Here, candidate 1 is dummy while candidates 2 and 3 are substitutable. Indeed, we have:

$$\pi(1,3) = \pi(1,4) = 0,\ \ \pi(2,3) = \pi(2,4) = 1 \text{ and } \pi(1,2,3) = \pi(1,2,4) = 0.$$

The approval and Shapley scores are given in Table 4.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	0	0	0	0	1	1
2	0	0	0	1	1	2	1
3	0	0	1	0	1	2	1
4	0	1	1	1	0	3	2

Table 4: Approval and Shapley scores associated to Example 4

The original Shapley's uniqueness proof applies to the set $G(N)$ of all games on *N*. Because ranking games are sums of unanimity games and the set of ranking games *RG*(*N*) is closed under addition, we can actually restrict the axioms to the set of ranking games and express them in terms of ballot profiles:

One-person-one-vote The scores add-up to the number of voters.

Neutrality If candidates' names are permuted, scores are permuted accordingly.

Null candidate Candidates appearing in no ballot get a zero score.

Additivity Given two ballot profiles π' and π'', the scores associated to $\pi' + \pi''$ are equal to the sum of the scores associated to π' and π''.

These four axioms are natural requirements and characterize Shapley ranking.

Proposition 4 Shapley ranking (3) is the unique ranking rule $\varphi : \Pi(N) \to \mathbb{R}^n$ that satisfies One-person-one-vote, Neutrality, Null candidate and Additivity.

Proof Shapley ranking obviously satisfies the four axioms. Now, consider a ranking rule φ satisfying all four axioms. Any ballot profile π on *N* can be decomposed as a sum of elementary ballot profiles $\pi = \sum \pi_T$ where

$$\pi_T(S) = \pi(T) \quad \text{if } S = T$$
$$= 0 \quad \text{if } S \neq T$$

By the Null candidate axiom, we have:

$$\varphi(N, \pi_T) = 0 \quad \text{for all } i \notin T.$$

and by the Neutrality axiom, we have:

$$\varphi(N, \pi_T) = \frac{\pi(T)}{t} \quad \text{for all } i \in T.$$

We then recover Shapley ranking using Additivity. ♦

4. From ranking to ordering

Approval and Shapley orderings are derived from approval and Shapley rankings and they generally differ, as in Example 1. They coincide in the two extreme voting situations where either $n_h = 1$ for all *h* or $n_h = n$ for all *h*. In the first situation, analogous to plurality voting, the game $(N, \bar{v})$ is additive and $AR_i = SR_i$ for all *i*. In the second situation, $AR_i = m$ and $SR_i = m / n$ for all *i* where *m* is the number of voters.

Referring to the underlying individual preferences, approval and Shapley orderings are two social welfare functions that assign collective preferences to individual preferences. What are their properties and how do they compare to each other? Not surprisingly, we will see that little can be said outside the particular case of dichotomous preferences.

4.1 Individual preferences

Concerning individual preferences, several assumptions are however possible:

A1 $i \in N_h$ and $j \notin N_h$ implies $i \succ_h j$

A2 $i, j \in N_h$ implies $i \sim_h j$

A3 $i, j \in N \setminus N_h$ implies $i \sim_h j$

A1 is part of the definition of approval voting. The other two assumptions are less natural and far too restrictive, especially A3. The three assumptions together characterize *dichotomous preferences*. Voters are then indifferent between candidates within their approval sets as well as outside.[18]

4.2 From individual to collective preferences

The validity of four axioms will be considered, *Pareto*, *Independence of irrelevant alternatives*, *Condorcet* and *Monotonicity* on the basis of the approval sets $(N_1, ..., N_m)$ submitted by voters.

The Pareto principle requires that unanimity should be reflected in collective preferences. It requires that if a candidate *i* is preferred to candidate *j* by all voters, then *i* must be collectively preferred to *j*. If preferences are dichotomous, candidate *i* is preferred to candidate *j* by all voters if and only if $i \in N_h$ and $j \notin N_h$ for all $h \in M$. Clearly, approval and Shapley ranking both satisfy the Pareto principle under dichotomous preferences. Assuming only A1 requires a modified version of the Pareto principle that allows for a situation where *i* and *j* both end up on top of the collective preferences.

Pareto principle If candidate *i* is preferred to candidate *j* by all voters, *j* cannot be collectively preferred to *i*.

[18] See Maniquet and Mongin (2015) for a complete analysis of approval voting under dichotomous preferences.

Proposition 5 Under A1, approval and Shapley orderings satisfy the modified Pareto principle.

Proof Consider two candidates *i* and *j* such that $i \succ_h j$ for all $h \in M$. For each voter *h*, there are three cases that define a partition of the set of voters:

(a) $i \in N_h$ and $j \notin N_h \rightarrow h \in M_1$,

(b) $i, j \in N_h \rightarrow h \in M_2$,

(c) $i, j \notin N_h \rightarrow h \in M \setminus (M_1 \cup M_2)$

The difference in approval scores is then equal to $m_1 \geq 0$. Indeed, nothing excludes a situation where M_1 is empty. The difference in Shapley scores is non-negative as well. Indeed, referring to the representative matrix *Q* that describes voter's ballots, we have:

$$SR_i - SR_j = \sum_{h \in M_1} \frac{q_{hi}}{b_h}$$

where $b_h = \sum_{l \in N} q_{hl} > 0$ for all $h \in M$ by assumption. ♦

Arrow (1951) introduced the axiom of independence of irrelevant alternatives (IIA for short), a property that appears to be a natural requirement although it is generally not satisfied in the absence of restrictions on individual preferences. Consider two preference profiles with a common set of candidates and a common set of voters, and two candidates *i* and *j*.

IIA If voters have the same preferences regarding *i* and *j* in both profiles, the collective preferences regarding *i* and *j* derived from the two profiles must be identical.

Arrow's impossibility theorem states that, without restrictions on preferences (axiom of Unrestricted domain), dictatorship is the only social welfare function that satisfies the Pareto principle and IIA. If preferences are dichotomous, it is easy to show that approval voting satisfies IIA.[19] Consider two ballot profiles $(N_1,...,N'_m)$ and $(N''_1,...,N''_m)$, and the associated representative matrices *Q'* and *Q''*. If voters have the same preferences regarding *i* and *j*, the rows *i* and *j* of the matrices *Q'* and *Q''* are identical and therefore $AR'_i = AR''_i$ and $AR'_j = AR''_j$. This does not contradict Arrow's theorem simply because preferences are assumed to be dichotomous: *Unrestricted domain* is not satisfied. Shapley ranking instead does not satisfy IIA, whether preferences are dichotomous or not, because of the strong interdependence that characterizes it, as the following example confirms.

Example 5 Consider two ballot profiles on a set of three candidates and five voters, represented by the matrices *Q'* and *Q''*. Assume that there are five voters who have the same preferences regarding candidates 1 and 2.

[19] This is acknowledged by Brams and Fishburn (2007, p.137).

$$Q' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline \end{array} \qquad Q'' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

In Q', candidate 2 has a higher Shapley score than candidate 1. The order is reversed in Q''.

The Condorcet principle is a property that is often considered to be desirable. A candidate is a *Condorcet winner* if he never loses in pairwise contests. There may be no Condorcet winner and, if such a candidate exists, one could argue that he should be elected.

Condorcet principle Condorcet winners, if any, should be on top of the collective preferences.

In general, no aggregation method satisfies this principle. Of course, if individual preferences were known (like in Borda count), one could first check whether there is a Condorcet winner and elect him if he exists. If preferences are assumed to be dichotomous, there may be several winners and the result of a pairwise contest between two candidates depends on their approval scores. A candidate is then a Condorcet winner if he has the highest approval score.[20] Shapley ranking instead does not satisfy the Condorcet principle, as shown in Example 1 where candidates 2 and 3 are Condorcet winners but none of them is elected under Shapley ranking. Remember that in case of duels, the two rankings coincide.

What happens to collective preferences when the preferences of a single voter change? This is the object of the Monotonicity axiom.

Monotonicity Consider a preference profile such that candidate *i* is *collectively* preferred to *j*. If a voter who prefers *j* to *i* changes his mind in favor of candidate *i*, candidate *i* must remain collectively preferred to candidate *j*.

Proposition 6 Under assumption A1 both approval ordering and Shapley ordering satisfy Monotonicity.

Proof. Assume that $i \succ j$ while $j \succ_k i$ for some $k \in M$. There are three possible cases:

(a) $j \in N_k$ and $i \notin N_k$,

(b) $i, j \in N_k$,

(c) $i, j \notin N_k$.

Assume that voter *k* changes his mind and now prefer *i* to *j*. We denote by N'_k his modified approval set. In case (a), we have three possible cases:

(a1) $N'_k = N_k \setminus j$,

(a2) $N'_k = N_k \cup i$,

(a3) $N'_k = (N_k \cup i) \setminus j$.

[20] This is Theorem 3.1 in Brams and Fishburn (2007).

In case (b), there are two possible cases:

(b1) $N'_k = N_k \setminus j,$

(b2) $N'_k = N_k.$

In case (c), there are also two possible cases:

(c1) $N'_k = N_k \cup i,$

(c2) $N'_k = N_k.$

Consider first approval voting. Initially, we have $AR_i > AR_j$. In cases (b2) and (c2), AR_i and AR_j remain unchanged. In cases (a1) and (b1), AR_i is unaffected while AR_j decreases by 1. In cases (a2) and (c1), AR_j is unaffected while AR_i increases by 1. In case (a3), AR_i increases by 1 and AR_j decreases by 1. Hence, $AR'_i > AR'_j$. Consider now Shapley ranking. Initially, we have $SR_i > SR_j$. In cases (b2) and (c2), SR_i and SR_j remain unchanged. In cases (a1), SR_i is unaffected while SR_j decreases by $1/n_k$. In case (a2), SR_j decreases by $1/n_k(1+n_k)$ while SR_i increases by $1/(1+n_k)$. In case (b1), SR_j decreases by $1/n_k$ while SR_i increases by $1/n_k(n_k-1)$. In case (c1), SR_j is unaffected while SR_i increases by $1/(1+n_k)$. In case (a3), SR_i increases by $1/n_k$ and SR_j decreases by $1/n_k$. Hence, $SR'_i > SR'_j$. ♦

5. Concluding remarks

Approval ranking has its advantages and drawbacks like any other preference aggregation methods, although most of its advantages cannot be formalized. The same applies to Shapley ranking. In particular both methods fail to satisfy the Independence of irrelevant alternatives and the Condorcet principle. However, we have shown that Shapley ranking comes naturally out of approval voting as the Shapley value of a transferable utility game that is an equivalent representation of voters' ballots: Uniform vote splitting is the correct aggregation method under the One-person-one-vote principle. Our analysis extends to the case where a maximum size is imposed to ballot sizes. In the case where a ballot size is imposed, approval and Shapley ranking coincide.

Under Shapley voting, voters know that they have a single vote to distribute among candidates. As a consequence, they will tend to limit the number of candidates they submit, maybe only one, and if a voter retains several candidates, it is likely that they will be "close" to each other. Assumption A2 then becomes more plausible.

References

Alcantud J.C.R. and A. Laruelle (2014), Dis&approval voting: A characterization, *Social Choice and Welfare* 43, 1-10.

Arrow, K. (1951), *Social choice and individual values,* Wiley, New York (2nd and last edition dated 1963).

Banzhaf, J.F (1965), Weighted voting does not work: A mathematical analysis, *Rutgers Law Review* 19, 317-343.

Baujard, A. and H. Igersheim (2010), Framed field experiments on approval voting: Lessons from the 2002 and 2007 French presidential elections, in Laslier J.-F. and M.R. Remzi Sanver (Eds, 2010), 357-395.

Black, D. (1958), *The theory of committees and elections*, Cambridge University Press, Cambridge.

Bogolmania, A., H. Moulin and R. Stong (2005), Collective choice under dichotomous preferences, *Journal of Economic Theory* 122, 165-184.

Borda, J.C. (1781), Mémoire sur les élections au scrutin, *Mémoires de l'Académie des Sciences*, Paris, 657-664.

Brams, S.J. (2008), *The mathematics of democracy*. *Designing better voting and fair-division procedures*, Princeton University Press, Princeton.

Brams, S.J. and P.C. Fishburn (1978), Approval voting, *American Political Science Review* 72, 831-847.

Brams, S.J. and P.C. Fishburn (1983), *Approval voting*, Birkhaüser, Boston (2nd edition dated 2007, Springer-Verlag, Berlin).

Brams, S.J. and P.C. Fishburn (2005), Going from theory to practice: the mixed success of approval voting, *Social Choice and Welfare* 25, 457-474 (reproduced in Laslier J.-F. and M.R. Remzi Sanver (Eds., 2010), 19-37.)

Condorcet, N. (1785), *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Imprimerie Royale, Paris.

Dehez, P. (2017), On Harsanyi dividends and asymmetric values, *International Game Theory Review* 19/3.

Ginsburgh, V. and I. Zang (2012), Shapley ranking of wines, *Journal of Wine Economics* 7, 169-180.

Harsanyi, J.C. (1959), A bargaining model for cooperative *n*-person games, in A.W. Tucker and R.D. Luce (Eds.), *Contributions to the theory of games* IV, Princeton University Press, Princeton, 325-355.

Laslier J.-F. and M.R. Remzi Sanver (2010), *Handbook on approval voting*, Springer-Verlag, Berlin.

Maniquet, F. and Ph. Mongin, (2015), Approval voting and Arrow’s impossibility theorem, *Social Choice and Welfare* 44, 519-532.

Nambiar, K.K. (1989), Fractional voting system: A voting system to circumvent Arrow's paradox, *Current Science* 58, 1076-1081.

Saari, D.G. and J. van Newenhizen (1988), The problem of indeterminacy in approval, multiple, and truncated voting systems, *Public Choice* 59, 101–120.

Saari, D.G. (2008), *Disposing dictators, demystifying voting paradox*, Cambridge University Press, Cambridge.

Shapley, L.S. (1953), A value for *n*-person games, in Kuhn H. and Tucker A.W. (eds.), *Contributions to the Theory of Games II*, Annals of Mathematics Studies 24, Princeton University Press, Princeton, 307-317. Reprinted in A. Roth (Ed. 1988), *The Shapley Value. Essays in Honor of Lloyd Shapley*, Cambridge University Press, Cambridge, 31-40.

Shapley, L. S. (1971), Cores of convex games, *International Journal of Game Theory* 1, 11-26.

von Neumann, J. and O. Morgenstern (1944), *Theory of games and economic behavior,* Princeton University Press, Princeton (3rd and last edition dated 1953).

Weber, R.J. (1977), Comparison of voting systems, mimeo (reproduced in Comparison of public choice systems, *Cowles Foundation Discussion Paper* 498, 1979, Yale University.)

Weber, R.J. (1995), Approval voting, *Journal of Economic Perspectives* 9, 39-49.

Young, P.Y. (1985), Monotonic solutions of cooperative games. *International Journal of Game Theory* 14, 65-72.