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Exploring continuous-variable entropic uncertainty relations and separability criteria in quantum phase space

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Abstract

The uncertainty principle lies at the heart of quantum physics. It exhibits one of the key divergences between a classical and a quantum system: it is impossible to define a quantum state for which the values of two observables that do not commute are simultaneously specified with infinite precision. A paradigmatic example is given by Heisenberg's original formulation of the uncertainty principle expressed in terms of variances of two canonically-conjugate variables, such as position x and momentum p, which was later generalized to a symplectic-invariant form by Schrödinger and Robertson. A different kind of uncertainty relations, originated by Białynicki-Birula and Mycielski, again for canonically-conjugate variables, relies on Shannon entropy instead of variances as a measure of uncertainty. In this thesis, we suggest several improvements of these entropic uncertainty relations and highlight the fact that they are better formulated in terms of entropy power, a notion borrowed from the information theory of real-valued signals. Our first novel entropic uncertainty relation takes x-p correlations into account and is consequently saturated by all pure Gaussian states in an arbitrary number of modes, improving on the original formulation by Białynicki-Birula and Mycielski. Our second main result is the derivation of an entropic uncertainty relation that holds for any n-tuples of not-necessarily canonically conjugate variables based on the matrix of their commutators. We then define a general form of the entropic uncertainty principle that combines both previous results. It expresses the incompatibility between two arbitrary variable n-uples and is saturated by all pure Gaussian states. Interestingly, we can also deduce from it the most general form of the Robertson uncertainty relation based on the covariance matrix of n variables.

This line of research underlines the interest of defining an entropic uncertainty relation that is intrinsically invariant under symplectic transformations. Then, as a first attempt to reach this goal, we conjecture a symplectic-invariant uncertainty relation that is based on the joint differential entropy of the Wigner function. This conjecture is, however, only legitimate for states with a non-negative Wigner function. We also suggest a complex extension of this so-called Wigner entropy, which could provide the way towards an extension (and proof) of the above conjecture for all states. As a second attempt, we introduce the notion of multi-copy uncertainty observables, exploiting a connection with the algebra of angular momenta. Expressing the posi-

tivity of the variance of our multi-copy observable coincides with the Schrödinger-Robertson uncertainty relation, which suggests that the discrete Shannon entropy of such an uncertainty observable provides a new symplectic-invariant measure of uncertainty.

Currently available separability criteria for continuous-variable systems imply a necessary and sufficient condition for a two-mode Gaussian state to be separable, but leave many entangled non-Gaussian states undetected. In this thesis, we introduce two improved separability criteria that enable a stronger entanglement detection. The first improved condition is based on the knowledge of an additional parameter, namely the degree of Gaussianity, and exploits a connection with Gaussianity-bounded uncertainty relations by Mandilara and Cerf. We exhibit families of non-Gaussian entangled states whose entanglement remains undetected by the Duan-Simon criterion. The second improved separability criterion is based on our improved entropic uncertainty relation that takes x-p correlations into account, and has the main advantage over the one proposed by Walborn *et al.* that it does not require any optimization procedure.

Titre

Étude des relations d'incertitude entropiques à variables continues et des critères de séparabilité dans l'espace des phases quantique.

Résumé

Le principe d'incertitude se situe au cœur de la physique quantique. Il représente l'une des différences majeures entre des systèmes classiques et quantiques, soit qu'il est impossible de définir un état quantique pour lequel deux observables qui ne commutent pas auraient des valeurs spécifiées simultanément et avec une précision infinie. La formulation originale du principe d'incertitude est due à Heisenberg et est exprimée en termes des variances de deux variables canoniquement conjuguées, telles que la position x et l'impulsion p. Cela fut par la suite généralisé par Schrödinger et Robertson qui ont donné au principe d'incertitude une forme invariante sous transformations symplectiques. Si l'incertitude est mesurée à l'aide de l'entropie différentielle de Shannon plutôt que des variances, il est alors possible de définir d'autres types de relations d'incertitude. Originellement introduites par Białynicki-Birula et Mycielski, elles expriment également l'incompatibilité entre deux variables canoniquement conjuguées. Dans cette thèse, nous proposons différentes améliorations de ces relations d'incertitude entropiques et mettons particulièrement l'accent sur le fait qu'elles s'expriment mieux sous forme de puissances entropiques, une notion empruntée à la théorie de l'information. En premier lieu, nous introduisons une nouvelle relation d'incertitude entropique qui tient compte des corrélations x-p et qui est par conséquent saturée par tous les états purs Gaussiens, ce qui représente une amélioration par rapport à la formulation originale de Białynicki-Birula et Mycielski. En second lieu, nous dérivons une relation d'incertitude entropique valide pour tous les n-uplets de variables non nécessairement canoniquement conjuguées et basée sur la matrice de leurs commutateurs. Nous définissons ensuite une forme plus générale du principe d'incertitude entropique qui combine les deux résultats précédents. Il exprime l'incompatibilité entre deux n-uplets arbitraires de variables et est saturé par tous les états purs Gaussiens. Notons que de ce principe d'incertitude entropique, nous pouvons déduire la forme la plus générale de la relation d'incertitude de Robertson, basée sur la matrice de covariance de n variables.

Les résultats précédents soulignent un des points essentiels de notre axe de recherche: définir une relation d'incertitude entropique intrinsèquement invariante sous transformations symplectiques. Afin d'atteindre cet objectif, notre première tentative est de conjecturer une relation d'incertitude — invariante sous transformations symplectiques — basée sur l'entropie différentielle jointe de la fonction de Wigner. Cette conjecture n'est cependant légitime que pour des états décrits par une fonction de Wigner non-négative. Nous proposons aussi une extension complexe de cette entropie dite entropie de Wigner, qui pourrait ouvrir la voie vers une extension (et une preuve) de la conjecture proposée ci-dessus qui serait alors valide pour tous les états quantiques. Comme seconde tentative, en exploitant une connexion avec l'algèbre des moments angulaires, nous introduisons la notion d'observables d'incertitude agissant sur plusieurs copies d'un état. Exprimer la positivité de la variance de notre observable coïncide avec la relation d'incertitude de Schrödinger-Robertson, ce qui suggère que l'entropie discrète de Shannon d'une telle observable fournit une nouvelle mesure de l'incertitude. Cette relation d'incertitude est invariante sous transformations symplectiques.

Les critères de séparabilité actuellement disponibles pour les variables continues donnent une condition nécessaire et suffisante afin qu'un état Gaussien bimodal soit séparable, mais laissent de nombreux états intriqués non-Gaussiens non détectés. Dans cette thèse, nous introduisons deux nouveaux critères de séparabilité qui permettent une meilleure détection de l'intrication. La première nouvelle condition est basée sur la connaissance d'un paramètre supplémentaire, à savoir le degré de Gaussianité de l'état, et exploite une connexion avec les relations d'incertitude de Mandilara et Cerf bornées par ce degré de Gaussianité. En particulier, nous donnons l'exemple de familles d'états intriqués non Gaussiens dont l'intrication est détectée par notre critère, mais pas par celui de Duan-Simon. Le second critère de séparabilité entropique que nous proposons est basé sur notre nouvelle relation d'incertitude entropique qui tient compte des corrélations x-p. Son principal avantage par rapport au critère de Walborn *et al.* est de ne nécessiter aucune procédure d'optimisation.

List of publications

The present thesis is based on the following publications

Published in peer-reviewed scientific journals

a) *Detection of non-Gaussian entangled states with an improved continuous-variable separability criterion*, Anaelle Hertz, Evgueni Karpov, Aikaterini Mandilara and Nicolas J. Cerf, Phys. Rev. A **93**, 032330 (2016).
doi:10.1103/PhysRevA.93.032330, arXiv:1511.06621.

b) *Entropy-power uncertainty relations: towards a tight inequality for all Gaussian pure states*, Anaelle Hertz, Michael G. Jabbour and Nicolas J. Cerf, J. Phys. A **50**, 385301 (2017).
doi:10.1088/1751-8121/aa852f, arXiv:1702.07286.

c) *Multidimensional entropic uncertainty relations based on a commutator matrix in position and momentum spaces*, Anaelle Hertz, Luc Vanbever and Nicolas J. Cerf, Phys. Rev. A **97**, 012111 (2018).
doi:10.1103/PhysRevA.97.012111, arXiv:1711.04566.

In preparation

d) *Tight entropic uncertainty relations for arbitrary variables in quantum phase space*, Anaelle Hertz and Nicolas J. Cerf.

e) *Symplectic-invariant entropic uncertainty relation based on a multi-copy uncertainty observable*, Anaelle Hertz, Ognyan Oreshkov and Nicolas J. Cerf.

Publication not related to the present thesis

f) *Higher order nonclassicality from nonlinear coherent states for models with quadratic spectrum*, Anaelle Hertz, Sanjib Dey, Véronique Hussin and Hichem Eleuch, Symmetry **8**, 36 (2016).
doi:10.3390/sym8050036, arXiv:1606.00107.

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1 Introduction, motivation and scope

In quantum mechanics, everything is about probabilities. While classically a particle is localized at a very specific point, in quantum mechanics, it will be described by a wave function $\psi(x,t)$ and we will find the particle at the point x_0, at time t_0 with probability density $|\psi(x_0,t_0)|^2$. This statistical interpretation of quantum mechanics generates the notion of uncertainty. If one measures the position of a quantum particle, one cannot know with certainty the outcome of the measurement, but will only get a statistical distribution of all possible results. Yet, the notion of uncertainty goes even further. If there exists some situations when one knows precisely the position of a particle, then, quantum mechanics tells us that the momentum of this very same particle cannot been known exactly. This principle, at the foundation of quantum mechanics, is known as the uncertainty principle. It was first expressed by Heisenberg, in 1927, for position and momentum [1] and formalized by Kennard [2] as

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4} \tag{1.1}$$

where σ_x^2 and σ_p^2 denote the variance of the position and momentum, respectively, and $\hbar$ is the reduced Planck constant. Shortly after, it was generalized to any pair of observables[1] that do not commute [3, 4]. The uncertainty principle then states that their values cannot be both sharply defined[2].

This fundamental, yet quite mysterious uncertainty principle is still studied extensively today. First of all simply because it is at the root of this branch of physics which is quantum mechanics, but also because it has many important implications. For example, in the last years we have witnessed the rise of quantum information theory, a field which brings together quantum mechanics, quantum optics and information theory. Probably one of its forefront application is quantum cryptography. Without going into details, it is important to know that the security of cryptographic protocols

[1] By observable, we mean any variable that can be measured.

[2] Note that there are actually two views on the uncertainty principle. It can either be seen as a property of the state itself or refer to the disturbance that is due to the measurement process. More details are given in Chapter 4.

relies on the uncertainty principle. Typically, the uncertainty on the data acquired by an eavesdropper is conjugated, in the sense of the uncertainty principle[3], to the data that flows to the authorized receiver. This means that by comparing (a subset of) his data with the original data as sent by the emitter, the receiver can estimate the error (i.e. the uncertainty) that is due to the eavesdropper's interception, and infer from it the error on the eavesdropper's side, hence estimating an upper limit on the tapped information [5, 6]. Studying and developing new uncertainty relations can only help developing new protocols or improving security proofs.

Uncertainty relations find other applications, for example, in the context of separability criteria, that is criteria that enable us to distinguish between entangled and non-entangled states. Indeed, the most famous separability criteria, the positive-partial-transposed criterion [7] and its equivalent form for continuous variables [8, 9] are based on uncertainty relations. In short, they say that a state is entangled when its partial transposed state is not physical. And how do we check this condition? We use uncertainty relations. The more the uncertainty relation is tight, the more we can detect entangled states. It is thus obvious that seeking new uncertainty relations leads to better separability criteria.

We said that quantum information is at the crossroad of quantum optics and information theory. It means that entropies must play a key role because they are the natural quantity of interest in this area. They are of relevant importance in many different facets of quantum information, but in particular for uncertainty relations. Indeed, entropy is another way of measuring uncertainty and, in 1957 Hirschman stated the first entropic uncertainty relation [10], which was proven by Białynicki-Birula and Mycielski in 1975 [11]. This result is interesting not only because it highlights the fact that notions of information theory can help better understand fundamental concepts of quantum mechanics, but it also opened the way to a new and fruitful formulation of uncertainty relations. Why such a success? We believe that an entropic formulation of uncertainty is more robust. In particular, it can be shown that the entropic uncertainty relation implies the one of Heisenberg and so is stronger. It seems thus more natural to try developing entropy-based rather than variance-based uncertainty relations. Moreover, entropic uncertainty relations can be generalized in a way such that (non classical) correlations with the environment are taken into account [12]. Typically, entanglement between a system and its environment can be exploited in order to reduce uncertainty. If an observer has access to a quantum memory system, the entropic formulation also allows to establish more accurate uncertainty relations and this is particularly useful in quantum key distribution, an important protocol of quantum cryptography.

The first steps of quantum information were mainly focused on qubits, that is discrete (binary) variables such as the spin state of an electron or the polarization state of a

[3] About measurement.

photon. Of course, it seems natural since it is simply the quantum analog of the classical bits. However, quantum information can also be expressed in terms of continuous variables and can sometimes even be more interesting. Indeed, if qubits seem easier to manipulate on a theoretical level, they are in fact harder to use in experimental setups because it is difficult to isolate and manipulate single particles. Continuous-variable quantum information, on the other hand, offers a framework which is more easily accessible in a lab as no single-particle source or detection are needed. We will then talk about observables with continuous spectrum and variables will be encoded in the quadratures of the electromagnetic field. Using the basics of quantum optics, many protocols can now be realized. Interestingly, continuous-variable quantum information does not give only experimental advantage, but also suggests a different way to study quantum information. We will use a phase-space representation and give prominent place to Gaussian states as they exhibit very interesting and useful properties. The basic notions of quantum optics in phase space will be explained in Chapter 2, while Chapter 3 will be devoted to information theory. Note that we do not aim to give an exhaustive review of those fields, but simply introduce the main concepts that will be useful to this thesis. We also assume that the reader is familiar with quantum mechanics. In Chapter 4 and 5 we explain, in more detail, uncertainty relations and separability criteria, respectively. Even though we do not introduce any new notions, we will sometimes present some elements in a way different from the usual path taken by standard textbooks because we believe it will help the reader better understand the contribution of this thesis. Those four introductory chapters form the first part of the thesis. Parts II and III are devoted to new results.

In Part II, we focus on entropic uncertainty relations and give several improvements. As it will be explained in Chapter 4, the original formulation of the entropic uncertainty relation of Białynicki-Birula and Mycielski exhibits some weaknesses since it is not invariant under Gaussian transformations and is not saturated by all pure Gaussian states, two conditions that are respected by the variance-based uncertainty relations[4]. To remedy to the second point, we propose in Chapter 6 an entropic uncertainty relation that takes x-p correlations into account. This chapter is the result of a publication in Journal of Physics A [b]. We first show that a proper expression of the uncertainty relation for a pair of canonically conjugate continuous variables relies on entropy power, a standard notion in Shannon information theory for real-valued signals. The resulting entropy-power uncertainty relation is equivalent to the entropic formulation of the uncertainty relation due to Białynicki-Birula and Mycielski, but can be further extended to rotated variables. Hence, based on two reasonable assumptions, we give a proof of a tighter form of the entropy-power uncertainty relation taking correlations into account. Interestingly, it implies the generalized (rotation-invariant) Robertson-Schrödinger uncertainty relation exactly as the original entropic uncertainty relation implies Heisenberg relation. It is saturated for all

[4]We are not talking here about the Heisenberg uncertainty relation, but about its extension by Robertson-Schrödinger.

Gaussian pure states, in contrast with hitherto known entropic formulations of the uncertainty relations.

Another improvement of the entropic uncertainty relation is given in Chapter 7. The uncertainty relation for continuous variables due to Białynicki-Birula and Mycielski expresses the complementarity between two n-tuples of canonically conjugate variables $(x_1, x_2, \cdots, x_n)$ and $(p_1, p_2, \cdots, p_n)$ in terms of Shannon differential entropy. Here, we consider the generalization to variables that are not canonically conjugate and derive an entropic uncertainty relation expressing the balance between any two n-variable Gaussian projective measurements. The bound on entropies is expressed in terms of the determinant of a matrix of commutators between the measured variables. This uncertainty relation also captures the complementarity between any two incompatible linear canonical transforms, the bound being written in terms of the corresponding symplectic matrices in phase space. We also extend this uncertainty relation to Rényi entropies. The results of this chapter form a publication accepted in Physical Review A [c].

As a logical continuation, we propose in Chapter 8 to combine both results in order to suggest the most general entropic uncertainty relations which combines both of the previous results, that is, our entropic uncertainty relation is defined for any n-tuples of quadrature observables and is saturated by all pure Gaussian states. We also prove that the generalized version of the Robertson uncertainty relation based on the covariance matrix can be deduced from this entropic uncertainty relations. This work is in preparation and will be soon submitted to a peer-reviewed scientific journal [d].

At this point, we have solved the first one of the weaknesses of the original entropic uncertainty relation, but not the second one: we are still looking for a relation invariant under Gaussian transformations. In Chapter 9 we make an attempt and suggest a conjecture which answers this problem. However, this is only applicable to states with a positive Wigner function. In order to consider all quantum states, we make a complex extension of the definition of the joint differential entropy. We then study the properties of this so-called Wigner entropy, but were not able to find any new entropic uncertainty relation. When the quantum state has a positive Wigner function, we recover our conjecture, which is verified numerically.

As a second attempt, in Chapter 10, we introduce two multi-copy uncertainty observables. First, we define a 2-copy observables denoted $\hat{L}_z$ that acts on two identical replica of a state and takes on integer or half-integer values from $-n/2$ to $n/2$ for a n-boson state. It is invariant under any symplectic transformation (rotation and squeezing), and vanishes with probability one if and only if it is applied onto a minimum-uncertainty state (Gaussian pure state). The obvious condition that its variance must be positive actually translates into the usual Robertson-Schrödinger uncertainty relation based on the determinant of the covariance matrix. The Shan-

non entropy of this two-copy observable provides a new measure of uncertainty. However, the framework we develop only works for state that are centered at the origin in phase space. To overcome this problem, we then define a 3-copy observable denoted $\hat{L}^*$. Its spectrum is one half of the spectrum of an angular momentum and, here too, the positivity of its variance coincides with the variance-based uncertainty relation. We therefore derive an entropic uncertainty relation based on the Shannon entropy of this 3-copy observable. For Gaussian states, the entropy of both multi-copy observables are equal. This work too will be soon submitted to a peer-reviewed scientific journal [e].

The third part of this thesis is dedicated to separability criteria, which, as we already mentioned, directly depend on uncertainty relations. We thus show how improvements in uncertainty relations lead to better entanglement detection. Chapter 11 was actually the first result we obtained during the realization of this thesis and it is the subject of a publication in Physical Review A [a]. We introduce an improved separability criterion based on an additional parameter of the state: its degree of Gaussianity. Entanglement detection is improved thanks to the fact that the degree of Gaussianity allows to establish more accurate uncertainty relations. We exhibit families of non-Gaussian entangled states whose entanglement remains undetected by the Duan-Simon criterion. In Chapter 12, we use the improved entropic uncertainty relation of Chapter 6 to enhance the already existing entropic separability criteria. Our main contribution here is that it is no longer necessary to make an optimization to find the optimal correlations and possibly detect entanglement since it is already included in the uncertainty relation.

Note that in the conclusion of this thesis (Chapter 13), we provide the reader with two summarizing tables. Table 13.1 gives an overview of all the entropic uncertainty relations encountered throughout the thesis while Table 13.2 exhibits the separability criteria and their associated uncertainty relations.

Part I | Basics of quantum information in phase space

2 Quantum optics in phase space

Quantum optics is a field of research that uses quantum mechanics to describe phenomena involving light, whose energy is quantized according to an integer number of particles known as photons. Roughly speaking, one makes the transition from classical to quantum mechanics by turning the position and momentum observables into non-commuting Hermitian operators. In the following chapter, we give a brief introduction to quantum optics. First, we show how one can quantize the electromagnetic field and give the phase-space representation of a quantum state. Then, we introduce Gaussian unitaries and Gaussian states as well as the symplectic formalism. This chapter is not exhaustive and we decided to focus on the notions in quantum optics that will be useful to understand the results of the present thesis. For more details, we suggest references [13, 14, 15, 16, 17, 18]. In fact, most of the definitions of this chapter come from those references. Note that for simplicity, we will fix $\hbar = 1$ throughout the rest of this thesis.

2.1 Quantization of the electromagnetic field

Classically, the electromagnetic field is described by Maxwell equations. Solving them allows us to describe the electromagnetic field as

$$\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} E_{\mathbf{k}} \mathbf{e}_{\mathbf{k}}{}^{(\lambda)} \left[\alpha_{\mathbf{k},\lambda} e^{i(\mathbf{kr}-\omega_k t)} + \alpha^*_{\mathbf{k},\lambda} e^{-i(\mathbf{kr}-\omega_k t)} \right], \tag{2.1}$$

where $\mathbf{k}$ is the index of the mode, $E_{\mathbf{k}}$ a constant containing all the dimensional prefactors, λ the polarization, $\omega_{\mathbf{k}}$ the angular frequency, $\mathbf{e}_{\mathbf{k}}{}^{(\lambda)}$ the polarization vector and $\alpha_{\mathbf{k}}$ and $\alpha^*_{\mathbf{k}}$ are the complex amplitudes. To quantize the electromagnetic field, we replace the complex amplitudes by the annihilation and creation operators, namely

$$\begin{aligned} \alpha_{\mathbf{k},\lambda} &\rightarrow \hat{a}_{\mathbf{k},\lambda}, \\ \alpha^*_{\mathbf{k},\lambda} &\rightarrow \hat{a}^\dagger_{\mathbf{k},\lambda}. \end{aligned} \tag{2.2}$$

Those *mode operators* satisfy the commutation relation of bosons

$$\begin{aligned}\left[\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] &= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}, \\ \left[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}\right] &= 0, \\ \left[\hat{a}^\dagger_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] &= 0.\end{aligned} \tag{2.3}$$

For brevity, let us study one single mode with fixed polarization and thus drop the subscripts $\mathbf{k}$ and λ. From the single-mode operators $\hat{a}$ and $\hat{a}^\dagger$, we can define the (dimensionless) *quadratures* of the electromagnetic field

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) \qquad \text{and} \qquad \hat{p} = -\frac{i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger), \tag{2.4}$$

which are equivalent to the position and momentum of a harmonic oscillator. It is easy to see that the quadratures satisfy the commutation relation

$$\left[\hat{x}, \hat{p}\right] = i \tag{2.5}$$

and therefore the uncertainty relation [1]

$$\sigma_x^2 \sigma_p^2 \geq \frac{1}{4}. \tag{2.6}$$

In terms of those quadratures, the Hamiltonian of the harmonic oscillator reads

$$\hat{H} = \frac{p^2 + x^2}{2}, \tag{2.7}$$

or, in terms of the creation and annihilation operators,

$$\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2} = \hat{N} + \frac{1}{2} \tag{2.8}$$

where $\hat{N}$ is called the *number operator*. The eigenvectors of $\hat{N}$ are called *Fock states* and denoted by $|n\rangle$. The associated eigenvalue will be $n = 0, 1, 2, \cdots$ and we say that a Fock state $|n\rangle$ contains n photons (or quanta of light), with the state containing no photons at all being the vacuum state $|0\rangle$. From this, the mode operators can be understood as operators that add (creation operator) or remove (annihilation operator) a photon from the state. Formally, we define them as

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle \qquad \text{and} \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle. \tag{2.9}$$

Remark that $\hat{N}$ has a discrete spectrum while $\hat{x}$ and $\hat{p}$, in contrast, have continuous spectra. Indeed, their eigenvectors are defined as

$$\hat{x}|x\rangle = x|x\rangle \qquad \text{and} \qquad \hat{p}|p\rangle = p|p\rangle \tag{2.10}$$

[1] The uncertainty relation is expressed as $\sigma_x^2 \sigma_p^2 \geq \frac{1}{4}|\langle [x, p] \rangle|^2$. More details are given in chapter 4.

with $x, p \in \mathbb{R}$ and $\{|x\rangle\}$, $\{|p\rangle\}$ representing two bases connected by a Fourier transform

$$\begin{aligned} |x\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixp}|p\rangle dp, \\ |p\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp}|x\rangle dx. \end{aligned} \tag{2.11}$$

These bases are orthogonal

$$\langle x|x'\rangle = \delta(x-x'), \qquad \langle p|p'\rangle = \delta(p-p'), \tag{2.12}$$

and complete

$$\int_{-\infty}^{\infty} |x\rangle\langle x|\, dx = \mathbb{1}, \qquad \int_{-\infty}^{\infty} |p\rangle\langle p|\, dp = \mathbb{1}. \tag{2.13}$$

The wave function of any given state is $\psi(x) = \langle x|\psi\rangle$ and its Fourier transform is given by $\phi(p) = \langle p|\psi\rangle$.

A *continuous-variable* system is a quantum system living in an infinite-dimensional Hilbert space and described by observables with continuous spectra. In this chapter (and the rest of this thesis), a continuous-variable system is represented by n modes of the quantized electromagnetic field. It can be understood as a system corresponding to n quantum harmonic oscillators. We will usually associate n modes to a tensor product of n Hilbert spaces. Each mode is associated with a Hilbert space spanned by the Fock basis and has its own mode operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ which verify the commutation relations (2.3).

2.2 Wigner function

If one follows a classical particle in a phase space, it is represented by a single point since a classical particle can be ascribed a definite position and momentum. However, if we want to do the same for a quantum particle, due to the uncertainty relations[2], we will not have a point anymore since position and momentum cannot be known precisely, simultaneously. Rather, the quantum particle will be described by a quasi-probability distribution[3] in phase space called the *Wigner function* after Eugene Wigner [19]. There is an exhaustive literature on the Wigner function, but we mainly follow the paper of Case [20] and all the following formulas are taken from this article[4].

[2] More details are given in Chapter 4 since it is its main subject.

[3] To be more accurate, we should call it a probability density function, since we deal with continuous variable, but in the thesis, and like it is done in many papers on the subject, we choose to call it probability distribution, having in mind that we talk about a continuous distribution.

[4] Note however, that we do not use the same convention so the equations differ slightly between this thesis and the paper.

Why is the Wigner function called a "quasi"-probability distribution? Because even though its integration over the entire space gives one — like any probability distribution — the Wigner function can have negative parts. Some will say that this is a signature of the quantum character of the state [21]. Note however that some states with Wigner function positive everywhere can be entangled[5], which is clearly a quantum behavior, while there are states with negative Wigner function that exhibit no specific quantum behavior.

Every quantum state ρ (of n modes) is associated to one Wigner function[6] defined as

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} d\mathbf{y}\; e^{-i\mathbf{p}\cdot\mathbf{y}}\langle \mathbf{x}+\mathbf{y}/2|\rho|\mathbf{x}-\mathbf{y}/2\rangle \tag{2.14}$$

and which is normalized to one

$$\int_{-\infty}^{\infty} d\mathbf{x}d\mathbf{p}W(\mathbf{x},\mathbf{p}) = 1. \tag{2.15}$$

Here the vectors $\mathbf{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$ and $\mathbf{p} = (\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n)$ contain the position and momentum quadratures of all modes.

For pure states, Eq. (2.14) can be simplified and the Wigner function is computed through the wave function $\psi(\mathbf{x}) = \langle x|\psi\rangle$,

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} d\mathbf{y}\; e^{-i\mathbf{p}\cdot\mathbf{y}}\psi(\mathbf{x}+\mathbf{y}/2)\psi^*(\mathbf{x}-\mathbf{y}/2). \tag{2.16}$$

Equivalently, the Wigner function can be defined in the momentum representation

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} d\mathbf{q}\; e^{i\mathbf{x}\cdot\mathbf{q}}\langle \mathbf{p}+\mathbf{q}/2|\rho|\mathbf{p}-\mathbf{q}/2\rangle. \tag{2.17}$$

Another explanation for the name “quasi-probability distribution” is that, unlike a classical probability distribution, the Wigner function cannot take arbitrary large values. Indeed, using the inequality of Cauchy-Schwarz on a one-mode Wigner function, we have

$$\begin{aligned} |W(x,p)| &= \left|\frac{1}{2\pi}\int dy e^{-ipy}\psi(x+y/2)\psi^*(x-y/2)\right| \\ &\leq \frac{1}{2\pi}\sqrt{\int |\psi(x+y/2)|^2dy\int |\psi^*(x-y/2)|^2dy} \\ &= \frac{1}{2\pi}\sqrt{\int 2|\psi(z)|^2dz\int 2|\psi^*(z')|^2dz'}. \end{aligned} \tag{2.18}$$

[5] More details about entanglement will be given in Chapter 5.
[6] The proof of uniqueness is given in [22].

Since the wave functions are normalized, we thus get

$$|W(x,p)| \leq \frac{1}{\pi}. \tag{2.19}$$

If the Wigner function is integrated over p or over x, we obtain the *marginals* of the Wigner function which are the probability distributions of the position and momentum quadratures:

$$\begin{aligned} \int_{-\infty}^{\infty} d\mathbf{p}\, W(\mathbf{x},\mathbf{p}) &= W(\mathbf{x}) = \langle \mathbf{x}|\rho|\mathbf{x}\rangle, \\ \int_{-\infty}^{\infty} d\mathbf{x}\, W(\mathbf{x},\mathbf{p}) &= W(\mathbf{p}) = \langle \mathbf{p}|\rho|\mathbf{p}\rangle. \end{aligned} \tag{2.20}$$

Those probability distributions are true classical distributions. In particular, they are always positive and normalized to 1.

It is also possible to retrieve the wave function given a Wigner function using the following formula

$$\psi(x) = \frac{1}{\psi^*(0)} \int W(x/2,p) e^{ipx} dp. \tag{2.21}$$

The Wigner function can also be used to compute the mean value of an operator $\hat{A}$ in a quantum state ρ

$$\langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A}) = \int_{-\infty}^{\infty} d\mathbf{x} d\mathbf{p}\, W(\mathbf{x},\mathbf{p}) \tilde{A}(\mathbf{x},\mathbf{p}) \tag{2.22}$$

where $\tilde{A}(\mathbf{x},\mathbf{p})$ is the Weyl transform of the operator $\hat{A}$,

$$\tilde{A}(\mathbf{x},\mathbf{p}) = \int d\mathbf{y}\; e^{-i\mathbf{p}\cdot\mathbf{y}} \langle \mathbf{x}+\mathbf{y}/2|\hat{A}_s|\mathbf{x}-\mathbf{y}/2\rangle, \tag{2.23}$$

and $\hat{A}_s$ is the symmetrized version of the operator $\hat{A}$ which is also a polynomial in $\mathbf{x}$ and $\mathbf{p}$ (see [23, 24] for more details). As an example, Table 2.1 lists some Weyl transforms of useful polynomials of $\hat{x}$ and $\hat{p}$.

Operator $\hat{A}$	Weyl transform $\tilde{A}(\mathbf{x},\mathbf{p})$
$\hat{x}$	x
$\hat{p}$	p
$\hat{x}^2$	x^2
$\hat{p}^2$	p^2
$(\hat{x}\hat{p}+\hat{p}\hat{x})/2$	xp

Table 2.1: Example of some Weyl transforms.

Finally, we are often interested in the *purity* $\mu = \mathrm{Tr}(\rho^2)$ of a state ρ which describes its mixedness. A pure state will have $\mu = 1$ while mixed state will have $0 < \mu < 1$. In terms of the Wigner function, the purity of a state is given by

$$\mu = \mathrm{Tr}(\rho^2) = (2\pi)^n \int_{-\infty}^{\infty} d\mathbf{x} d\mathbf{p}\, W^2(\mathbf{x}, \mathbf{p}). \tag{2.24}$$

Fock states

As an example, and because they will be useful in the following chapters, especially for numerical simulations, we give here the wave functions $\psi_n(x)$ and the Wigner functions $W_n(x, p)$ of Fock states $|n\rangle$, with $n = 0, 1, 2, \cdots$. They are expressed as

$$\psi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(x) e^{-x^2/2} \tag{2.25}$$

where $H_n(x)$ are the Hermite polynomials and

$$W_n(x, p) = \frac{(-1)^n}{\pi} e^{-x^2-p^2} L_n\left(2\left(x^2 + p^2\right)\right) \tag{2.26}$$

where $L_n(x)$ are the Laguerre polynomials. On Figure 2.1 we plotted the Wigner function of the vacuum $|0\rangle$ as well as the Fock states $|1\rangle$ and $|5\rangle$. In the two last examples, we see clearly that the Wigner functions have some negative parts.

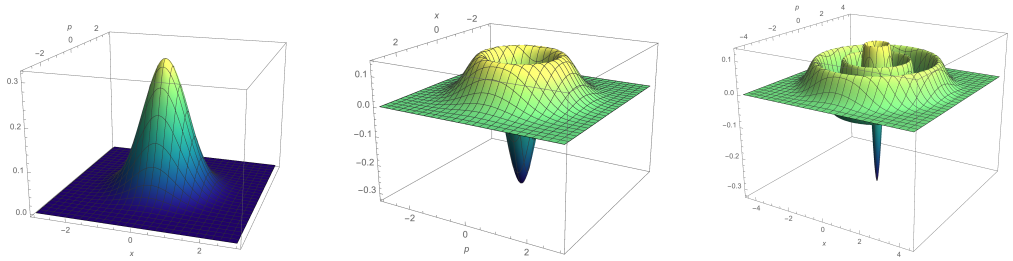

Figure 2.1: Wigner functions of different Fock states. From left to right: the vacuum $|0\rangle$, $|1\rangle$ and $|5\rangle$.

Let us mention that the Wigner function of Fock states is invariant under rotations. Fock states have no covariance, but their variances[7] increase with the number of photon since they are given by $\sigma_x^2 = \sigma_p^2 = n + 1/2$. Note also that for all Fock states, the value of the Wigner function at the origin is given by

$$W_n(0, 0) = \frac{(-1)^n}{\pi}. \tag{2.27}$$

[7] Variances and covariance are introduced formally in the next section.

2.3 Gaussian states

In quantum information with continuous variables, we focus particularly on one class of states: the *Gaussian states*. They inherited their name from their Wigner quasi-distribution function which, itself, is a Gaussian distribution. A Gaussian state is completely characterized by its *mean value vector* and its *covariance matrix*. The mean-value vector of a state ρ, also called the displacement vector, is defined as

$$\langle \mathbf{r} \rangle = \mathrm{Tr}(\mathbf{r}\rho) \tag{2.28}$$

where $\mathbf{r} = \{\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \cdots, \hat{x}_n, \hat{p}_n\}$ is the quadratures vector, and, throughout this thesis, $\langle \cdot \rangle$ stands for the expectation value $\mathrm{Tr}(\cdot\rho)$. The second moments make up the covariance matrix γ whose elements are given by

$$\gamma_{ij} = \frac{1}{2}\langle \{\hat{r}_i, \hat{r}_j\} \rangle - \langle \hat{r}_i \rangle \langle \hat{r}_j \rangle. \tag{2.29}$$

Here $\{\cdot, \cdot\}$ stands for the anti-commutator. For one-mode states, in the present thesis, we will usually express the covariance matrix as

$$\gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_p^2 \end{pmatrix} \tag{2.30}$$

where σ_x^2 and σ_p^2 are the variances of position and momentum and σ_{xp} is the covariance. Remark that a covariance matrix is a real, symmetric, and positive semi-definite matrix [25].

As we already pointed out, the Wigner function of a Gaussian state is a Gaussian distribution given by

$$W_G(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^n \sqrt{\det \gamma}} e^{-\frac{1}{2}(\mathbf{r}-\langle \mathbf{r} \rangle)^T \gamma^{-1} (\mathbf{r}-\langle \mathbf{r} \rangle)}. \tag{2.31}$$

Once again, we see that the only information needed is the displacement vector and the covariance matrix. It is interesting to note that we can easily compute the purity μ of a Gaussian state through its Wigner function (see Eq. (2.24)) and we simply find that

$$\mu_G = \frac{1}{2^n \sqrt{\det \gamma}}. \tag{2.32}$$

As we will see in Section 4.1.2, not all real symmetric positive semi-definite $2n \times 2n$ matrices can be a legitimate covariance matrix of a quantum state. Indeed, they must respect the uncertainty relation $\gamma + i\Omega/2 \geq 0$ (see Eq. (4.9)) which is the only necessary and sufficient constraint γ has to fulfill to be the covariance matrix of a physical state [25].

In 1974, Hudson [26] proved that the Wigner function of a pure state is everywhere positive if and only if the state has a (complex) Gaussian wave function (thus a Gaussian Wigner function). In other words, the only pure states whose Wigner functions are positive everywhere are Gaussian states. As we will see in Chapter 4, Gaussian states are also the only ones to saturate the uncertainty principle. This will imply that the only pure states that have $\det \gamma = 1/4^n$ are necessarily Gaussian.

The simplest example of a Gaussian state is the *vacuum* state $|0\rangle$. Its Gaussian Wigner function is plotted in Figure 2.1. The vacuum state has a mean value vector equal to 0 and its covariance matrix is given by

$$\gamma_{vac} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{2.33}$$

which means that the uncertainty on the quadratures x and p is equal to $1/2$ as it can be seen in Figure 2.2 where we plotted a projection of the Wigner function in phase space.

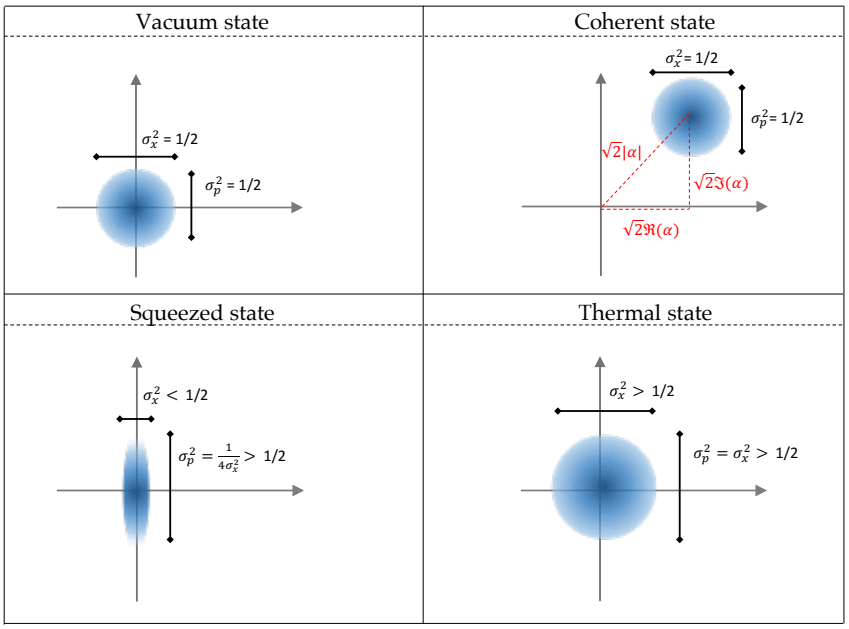

Figure 2.2: Examples of Gaussian states in phase space.

Of course, the vacuum state is not the only Gaussian state. Other examples are the coherent and squeezed states, the two-mode squeezed state or the thermal state. We will talk about them more in details in section 2.5, but we first introduce Gaussian unitaries that are used to create[8] those Gaussian states.

[8] Starting usually from the vacuum $|0\rangle$.

2.4 Gaussian unitaries

Let us consider a unitary transformation $U = e^{-i\hat{H}}$ where $\hat{H}$ is the Hamiltonian of the system (and we conventionally consider a unit time interval). Obviously, since the transformation is unitary, we have $U^\dagger = U^{-1}$ and it transforms a state as

$$\hat{\rho} \to U\hat{\rho}\,U^\dagger. \tag{2.34}$$

If the unitary is such that it maps a Gaussian state onto another Gaussian state, it is called a *Gaussian unitary*. They divide into two categories: the *passive* Gaussian unitaries that preserve the mean number of photons $\langle n \rangle$ and the *active* Gaussian unitaries, which do not. All Gaussian unitaries are generated by Hamiltonians which are second-order polynomials in the mode operators. Namely, if we define the vector of mode operators $\hat{\boldsymbol{a}} = \{\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n\}$, the Hamiltonian has to be of the form

$$\hat{H} = \hat{\boldsymbol{a}}^\dagger \boldsymbol{\alpha} + \hat{\boldsymbol{a}}^\dagger \mathcal{F} \hat{\boldsymbol{a}} + \hat{\boldsymbol{a}}^\dagger \mathcal{G} \hat{\boldsymbol{a}}^\dagger + h.c. \tag{2.35}$$

(*h.c.* stands for hermitian conjugate) where $\boldsymbol{\alpha}$ is a complex vector and $\mathcal{F}$, $\mathcal{G}$ are some symmetric, complex $n \times n$ matrices.

In the Heisenberg picture, Gaussian unitaries correspond to a Bogoliubov transformation

$$\hat{\boldsymbol{a}} \to U^\dagger \hat{\boldsymbol{a}} U = \mathcal{A}\hat{\boldsymbol{a}} + \hat{\boldsymbol{a}}^\dagger \mathcal{B} + \boldsymbol{\alpha} \tag{2.36}$$

where $\mathcal{A}$, $\mathcal{B}$ are matrices that satisfy $\mathcal{A}\mathcal{B}^T = \mathcal{B}\mathcal{A}^T$ and $\mathcal{A}\mathcal{A}^T = \mathcal{B}\mathcal{B}^T + \mathbb{1}$ (in order to preserve the commutation relations). However, rather than expressing this linear transformation at the mode operators level, we will usually prefer to analyze it at the quadratures level.

2.4.1 Symplectic transformations

In terms of the quadrature operators, a Gaussian unitary is more simply described. Let us define the quadrature vector $\mathbf{r} = \{\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n\}$. Then, a Gaussian unitary is defined as

$$\hat{\mathbf{r}} \to \mathcal{S}\hat{\mathbf{r}} + \mathbf{d} \tag{2.37}$$

where $\mathbf{d}$ is a real vector of dimension $2n$ and $\mathcal{S}$ is a real $2n \times 2n$ matrix. Once again, the commutation relations have to be preserved and this is respected if the matrix $\mathcal{S}$ is *symplectic*, that is if

$$\mathcal{S}\Omega\mathcal{S}^T = \Omega \tag{2.38}$$

with

$$\Omega = \bigoplus_{k=1}^{n} \omega, \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.39}$$

and Ω being the symplectic form. Note that $\Omega^T = \Omega^{-1} = -\Omega$ and $\Omega^2 = -\mathbb{1}$.

Be aware that this definition of symplectic matrices is linked to the definition of $\mathbf{r}$ i.e. the order of the entries in $\mathbf{r}$. If one choses to define $\mathbf{r} = \{\hat{x}_1, \cdots, \hat{x}_n, \hat{p}_1, \cdots, \hat{p}_n\}$[9], then the matrix $\mathcal{S}$ is symplectic if

$$\mathcal{S}J\mathcal{S}^T = J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{2.40}$$

Here too, note that $J^T = J^{-1} = -J$ and $J^2 = -\mathbb{1}$.

Clearly, the eigenvalues of $\mathbf{r}$ must also follow the same transformation rule and so the quadratures transform as

$$\mathbf{r} \to \mathcal{S}\mathbf{r} + \mathbf{d} \tag{2.41}$$

under a symplectic transformation. Regarding the mean values and the covariance matrix, the transformation rule is

$$\langle \mathbf{r} \rangle \to \mathcal{S}\langle \mathbf{r} \rangle + \mathbf{d} \qquad \text{and} \qquad \gamma \to \mathcal{S}\gamma\mathcal{S}^T. \tag{2.42}$$

In addition, any symplectic matrix has the following properties:

- For any symplectic transformation $\mathcal{S}$, $\mathcal{S}^T$, $\mathcal{S}^{-1}$ and $-\mathcal{S}$ are also symplectic.
- The inverse of $\mathcal{S}$ is given by $\mathcal{S}^{-1} = -\Omega\mathcal{S}^T\Omega$ (or $\mathcal{S}^{-1} = -J\mathcal{S}^T J$, depending on the definition of $\mathbf{r}$).
- $\det \mathcal{S} = 1$, which implies that $\det \gamma$ is conserved by any symplectic transformation.
- If $\mathbf{r} = \{\hat{x}_1, \cdots, \hat{x}_n, \hat{p}_1, \cdots, \hat{p}_n\}$ and $\mathcal{S} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $\mathcal{S}J\mathcal{S}^T = J$ implies that ab^T and cd^T are symmetric matrices and $ad^T - bc^T = \mathbb{1}$.
- We said earlier that a Gaussian unitary is passive if it conserves the mean photon number. In term of symplectic transformations, a Gaussian unitary will be passive if and only if

  $$\mathbf{d} = 0 \qquad \text{and} \qquad \mathcal{S}^T\mathcal{S} = \mathbb{1}, \tag{2.43}$$

  which means that the symplectic matrix must be orthogonal.

Williamson's theorem

An important result of the symplectic analysis is *Williamson's theorem* [27] which states that, after the appropriate symplectic transformation, every positive real matrix of even dimension can be brought to a diagonal form $\gamma^{\oplus}$, with its *symplectic values*

[9] As it will be done in chapter 7.

ν_k on the diagonal. In other words, there exists a symplectic matrix $\mathcal{S}$ such that[10]

$$\gamma = \mathcal{S}\gamma^{\oplus}\mathcal{S}^T, \qquad \text{where} \qquad \gamma^{\oplus} = \bigoplus_{k=1}^{n} \nu_k \mathbb{1}_{2\times 2}. \tag{2.44}$$

Obviously, since the determinant of a symplectic matrix is equal to 1, γ and $\gamma^{\oplus}$ have the same determinant. Therefore, for a one-mode state, its symplectic value is simply equal to $\sqrt{\det\gamma}$. For a two-mode state, the two symplectic values $\nu_{\pm}$ can be found using the following formula [28]

$$\nu_{\pm} = \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4\det(\gamma)}}{2}} \tag{2.45}$$

where the covariance can be written in the block form

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \tag{2.46}$$

and $\Delta = |A| + |B| + 2|C|$. In general, one can find the symplectic values by diagonalizing the matrix $i\Omega\gamma$ and taking the absolute value of its eigenvalues. Indeed, let us define the matrix $M = \Omega\gamma$. Using the fact that $\Omega = \mathcal{S}^T\Omega\mathcal{S}$ for all symplectic matrices $\mathcal{S}$, we can write

$$\begin{aligned} \mathcal{S}^T M \mathcal{S}^{-T} &= \mathcal{S}^T\Omega\gamma\mathcal{S}^{-T} \\ &= \Omega\mathcal{S}^{-1}\gamma\mathcal{S}^{-T} \\ &= \Omega\gamma^{\oplus} \\ &= \begin{pmatrix} 0 & \nu_1 & & & \\ -\nu_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \nu_n \\ & & & -\nu_n & 0 \end{pmatrix}. \end{aligned} \tag{2.47}$$

where $\mathcal{S}^{-T}$ means $(\mathcal{S}^T)^{-1}$. The eigenvalues of the matrix $\mathcal{S}^T M\mathcal{S}^{-T}$ are thus given by $\{\pm i\nu_i\}$. But M and $\mathcal{S}^T M\mathcal{S}^{-T}$ have the same eigenvalues since $\mathcal{S}$ is symplectic[11] and so $\{\pm i\nu_i\}$ are also the eigenvalues of $\Omega\gamma$.

As we will see in chapter 4, the uncertainty relation $\gamma + i\Omega/2 \geq 0$ can also be expressed in terms of symplectic values. A state is physical if and only if all its symplectic values satisfy $\nu_i \geq 1/2$ for $i = 1, \cdots, n$.

[10] We use here the definition $\mathbf{r} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)$.

[11] To prove this, we have to know that every symplectic matrix can be decomposed into a product of one orthogonal matrix (a rotation), one diagonal matrix (corresponding to N single-mode squeezing transformations) and another orthogonal matrix. Since the spectrum of a matrix is conserved under an orthogonal or a diagonal matrix transformation, it is conserved under a symplectic transformation.

2.5 Examples of Gaussian unitaries and Gaussian states

2.5.1 Coherent states and displacement operator

Let us start by discovering some single-mode Gaussian states. If we take a suitable superposition of Fock states, we can create a family of minimal uncertainty states in the sense that they saturate the Heisenberg uncertainty relation $\sigma_x^2\sigma_p^2 = 1/4$ (see Eq. (2.6)). Moreover, if we take both variances σ_x^2 and σ_p^2 to be equal, those states are called *coherent states* $|\alpha\rangle$. A first way to define a coherent state is as an eigenstate of the annihilation operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{2.48}$$

with α complex. A coherent state can also be seen as a displaced vacuum state. It means that we can apply a *displacement operator*

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{2.49}$$

to the vacuum state $|0\rangle$ so that

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.50}$$

Since we applied a Gaussian unitary to a Gaussian state (the vacuum), the output state which is our coherent state is also Gaussian. Note that $D(\alpha)$ is unitary so that

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha). \tag{2.51}$$

Under the action of $D(\alpha)$, the creation and annihilation operators transform as

$$D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha \qquad D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) = \hat{a}^\dagger + \alpha^* \tag{2.52}$$

while the quadratures transform as

$$\begin{aligned} D^\dagger(\alpha)\hat{x}D(\alpha) &= \hat{x} + \sqrt{2}\Re(\alpha) \\ D^\dagger(\alpha)\hat{p}D(\alpha) &= \hat{p} + \sqrt{2}\Im(\alpha) \end{aligned} \tag{2.53}$$

where $\Re(\alpha)$ and $\Im(\alpha)$ stand for the real and imaginary part of the complex number α. The two last equations clearly show that the displacement operator translates the vacuum state in phase space as illustrated in Figure 2.2. In other words, a displacement operator will displace the mean value vector, but will not modify the covariance matrix, which will be the one of the vacuum state, that is

$$\langle \mathbf{r}\rangle = \sqrt{2}\begin{pmatrix}\Re(\alpha)\\ \Im(\alpha)\end{pmatrix} \qquad \text{and} \qquad \gamma = \gamma_{vac} = \frac{1}{2}\mathbb{1}. \tag{2.54}$$

Therefore, the Wigner function of the coherent state is the same as the one of the vacuum given in Figure 2.1, but not centered at the origin anymore.

In contrast to a Fock state, a coherent state has an undefined number of photons. Following definition (2.50), we see that the probability of measuring n photons follows a Poisson distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}, \tag{2.55}$$

where the mean value and the variance of the distribution are both given by $|\alpha|^2$.

Let us mention that coherent states are not orthogonal

$$\langle\beta|\alpha\rangle = e^{-|\alpha-\beta|^2/2}, \tag{2.56}$$

but are over complete

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \mathbb{1} \tag{2.57}$$

which means they form an over-complete basis.

2.5.2 Squeezed states and squeezing operator

Coherent states are part of the family of states that minimizes the uncertainty relation, but have the particularity that the variances of both quadratures are equal. If we still want to minimize the uncertainty but allow different values of σ_x^2 and σ_p^2, we can introduce the *squeezed states*. They can be generated using the squeezing operator

$$S(z) = e^{\frac{1}{2}(z^*\hat{a}^2 - z\hat{a}^{\dagger 2})} \tag{2.58}$$

where $z = re^{i\phi}$ is a complex number, r is the squeezing parameter and ϕ the squeezing angle. Thus, a general squeezed state $|\alpha, z\rangle$ is defined as

$$|\alpha, z\rangle = D(\alpha)S(z)|0\rangle. \tag{2.59}$$

When there is no displacement, the squeezed vacuum state will be written as

$$|z\rangle = S(z)|0\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}e^{in\phi}(\tanh r)^n|2n\rangle. \tag{2.60}$$

Remark, once again, that the squeezed state is Gaussian since we applied a Gaussian unitary to a Gaussian state. Note also that the squeezed vacuum contains only Fock states with an even number of photons.

Just like for the displacement operator, it also holds for the squeezing operator that

$$S^\dagger(z) = S^{-1}(z) = S(-z). \tag{2.61}$$

The action of $S(z)$ on the mode operators is given by

$$\begin{aligned} S^\dagger(z)\hat{a}S(z) &= \hat{a}\cosh r - \hat{a}^\dagger e^{i\phi}\sinh r \\ S^\dagger(z)\hat{a}^\dagger S(z) &= \hat{a}^\dagger\cosh r - \hat{a}e^{-i\phi}\sinh r. \end{aligned} \tag{2.62}$$

Now, if we fix $\phi = 0$ so that $z = r$, the action of $S(r)$ on the quadratures is given by

$$\begin{aligned} S^\dagger(r)\hat{x}S(r) &= e^{-r}\hat{x} \\ S^\dagger(r)\hat{p}S(r) &= e^{r}\hat{p}. \end{aligned} \tag{2.63}$$

The last two equations clearly show that, this time, the squeezing operator squeezes one quadrature and anti-squeezes the other one in phase space, as it is shown in Figure 2.2. Therefore, the associated symplectic transformation is

$$\mathcal{S} = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix} \tag{2.64}$$

and the covariance matrix of a squeezed state is given by

$$\gamma = \mathcal{S}\gamma_{vac}\mathcal{S}^T = \frac{1}{2}\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}. \tag{2.65}$$

Later in this thesis, we will need to use squeezed states for numerical computations. We thus give here the wave function of a squeezed state with squeezing along an arbitrary axis, characterized by the angle ϕ,

$$\begin{aligned} \psi_s(x) &= \left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4}\exp\left\{-\frac{x^2}{2}\frac{\cos\phi - ie^{-2r}\sin\phi}{e^{-2r}\cos\phi - i\sin\phi}\right\} \\ &= \left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4}\exp\left\{-\frac{x^2}{2}\frac{\cosh r + e^{2i\phi}\sinh r}{\cosh r - e^{2i\phi}\sinh r}\right\}. \end{aligned} \tag{2.66}$$

The associated covariance matrix is then given by

$$\begin{aligned} \gamma &= \frac{1}{2}\begin{pmatrix} e^{2r}\sin^2\phi + e^{-2r}\cos^2\phi & \cos\phi\sin\phi(e^{-2r} - e^{2r}) \\ \cos\phi\sin\phi(e^{-2r} - e^{2r}) & e^{-2r}\sin^2\phi + e^{2r}\cos^2\phi \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cosh 2r - \cos 2\phi\ \sinh 2r & -\sin 2\phi\ \sinh 2r \\ -\sin 2\phi\ \sinh 2r & \cosh 2r + \cos 2\phi\ \sinh 2r \end{pmatrix}. \end{aligned} \tag{2.67}$$

Obviously, if the squeezing is along the axis so that $\phi = 0$, we fall back on the covariance matrix Eq. (2.65).

2.5.3 Phase-shift operator

The free evolution of an harmonic oscillator induces the unitary

$$R(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}} = e^{-i\theta\hat{N}} \tag{2.68}$$

called the *phase-shift operator*. Its name comes from the fact that it adds a phase to the mode operator

$$\hat{a} \to \hat{a}e^{-i\theta}. \tag{2.69}$$

In phase space, it has the effect of rotating the quadratures by an angle θ

$$\begin{aligned} \hat{x} &\to \hat{x}\cos\theta + \hat{p}\sin\theta \\ \hat{p} &\to -\hat{x}\sin\theta + \hat{p}\cos\theta. \end{aligned} \tag{2.70}$$

The associated symplectic matrix is then simply given by the rotation matrix

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.71}$$

2.5.4 Beam splitter

We now introduce unitary transformations which transform two-mode states. The first one is induced by the *beam splitter operator*

$$B(\beta) = e^{\beta(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2)}. \tag{2.72}$$

The transmissivity of the beam splitter is given by $\tau = \cos^2\beta$ and its value is between 0 and 1. In the Heisenberg picture, the beam splitter operator transforms the mode operators as

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \to \begin{pmatrix} \sqrt{\tau} & \sqrt{1-\tau} \\ -\sqrt{1-\tau} & \sqrt{\tau} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{2.73}$$

while the quadratures transform according to

$$\begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \hat{x}_2 \\ \hat{p}_2 \end{pmatrix} \to \mathcal{S}_{BS} \begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \hat{x}_2 \\ \hat{p}_2 \end{pmatrix} \tag{2.74}$$

where

$$\mathcal{S}_{BS} = \begin{pmatrix} \sqrt{\tau} & 0 & \sqrt{1-\tau} & 0 \\ 0 & \sqrt{\tau} & 0 & \sqrt{1-\tau} \\ -\sqrt{1-\tau} & 0 & \sqrt{\tau} & 0 \\ 0 & -\sqrt{1-\tau} & 0 & \sqrt{\tau} \end{pmatrix} \tag{2.75}$$

is a symplectic matrix.

An interesting property of the beam splitter is that if the input is composed of two coherent states, the output will remain a two-mode coherent state, but with different values of displacements (thus behaving as classical light fields).

2.5.5 Two-mode squeezer

Another important two-mode unitary is induced by the *two-mode squeezing operator*[12]

$$S_{TMS} = e^{\frac{r}{2}(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger)}. \tag{2.76}$$

The two-mode squeezing transformation can be seen as a combination of a balanced beam splitter ($\tau = 1/2$), a one-mode squeezer and a one-mode antisqueezer, and another balanced beam splitter, as depicted in Figure 2.3.

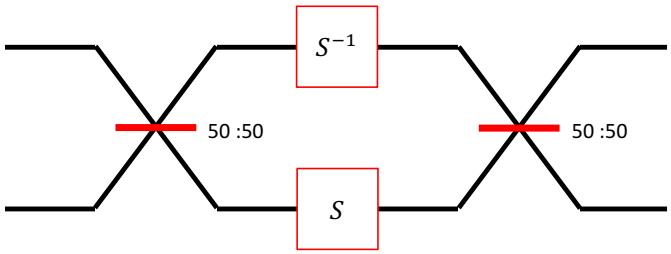

Figure 2.3: Physical realization of a two-mode squeezer.

Using the symplectic matrices for the beam splitter and the one-mode squeezing operator, we can compute the symplectic transformation of the two-mode squeezing operator

$$\begin{aligned}
\mathcal{S}_{TMS} &= \mathcal{S}_{BS}^T(\mathcal{S}^{-1}\otimes\mathcal{S})\mathcal{S}_{BS} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^r & 0 & 0 & 0 \\ 0 & e^{-r} & 0 & 0 \\ 0 & 0 & e^{-r} & 0 \\ 0 & 0 & 0 & e^r \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix}.
\end{aligned} \tag{2.77}$$

[12] To be precise, the two-mode squeezing operator $S_{TMS}(z)$ also depends on a squeezing angle ϕ such that $z = re^{i\phi}$, but by simplicity, we chose to consider $\phi = 0$. Note that one can always add some rotations before and/or after the squeezing in order to change the squeezing orientation axis.

Therefore, the transformation of the quadratures is given by

$$\begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \hat{x}_2 \\ \hat{p}_2 \end{pmatrix} \rightarrow \mathcal{S}_{TMS} \begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \hat{x}_2 \\ \hat{p}_2 \end{pmatrix}. \tag{2.78}$$

When we apply the two-mode squeezing operator on a vacuum state, we obtain a well known Gaussian state: the *two-mode squeezed vacuum state*. Also called the *EPR state* (for Einstein-Podolski-Rosen[13]), it is defined as

$$|EPR\rangle = S_{TMS}(r)|0,0\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (\tanh r)^n |n,n\rangle. \tag{2.79}$$

Note that we will sometimes prefer the following notation

$$|EPR\rangle = \sqrt{1-\lambda^2} \sum_{n=0}^{\infty} \lambda^n |n,n\rangle \tag{2.80}$$

where we simply used $\lambda = \tanh r$. The covariance matrix of this state is given by

$$\gamma = S_{TMS}\, \gamma_{vac}\, S_{TMS}^T = \frac{1}{2} \begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix}. \tag{2.81}$$

The EPR state is famous because in the limit of infinite squeezing ($r \rightarrow \infty$), it represents a maximally entangled state: we obtain perfect correlations between the $\hat{x}$ quadratures and perfect anti-correlations between the $\hat{p}$ quadratures as in the original EPR state.

2.5.6 Thermal states

If we trace out one of the modes of the two-mode squeezed vacuum state, we obtain the mixed state

$$\rho_{th} = \frac{1}{(\cosh r)^2} \sum_{n=0}^{\infty} (\tanh r)^{2n} |n\rangle\langle n| \tag{2.82}$$

called the *thermal state*. We can compute its mean photon number

$$\langle n \rangle = \langle \hat{N} \rangle = \sinh^2 r \tag{2.83}$$

and using this equivalence, we can rewrite the thermal state as

[13] See Chapter 5.

$$\rho_{th} = \sum_{n=0}^{\infty} \frac{\langle n \rangle^n}{(\langle n \rangle + 1)^{n+1}} |n\rangle\langle n|. \tag{2.84}$$

This last expression is precisely the formulation of a thermal state in thermodynamics. The thermal state is another example of a Gaussian state and so is completely described by its covariance matrix, given by

$$\gamma_{th} = \begin{pmatrix} \langle n \rangle + \frac{1}{2} & 0 \\ 0 & \langle n \rangle + \frac{1}{2} \end{pmatrix}. \tag{2.85}$$

As we can see in Figure 2.2, the thermal state is similar to the vacuum state in the sense that the uncertainties are the same for both quadratures. However, it is not a minimal uncertainty state anymore since the product of the variances is greater than $1/4$ (see Eq. (2.6)). Note that for integer $\langle n \rangle$, a thermal state with $\langle n \rangle$ mean number of photons has the same covariance matrix as the Fock state $|n\rangle$.

It is worth to remark that the symplectic diagonal form of the covariance matrix, Eq. (2.44), can be seen as the covariance matrix of a tensor product of thermal states, each of them with a mean number of photon $\langle n \rangle = \nu_i - 1/2$.

2.6 Passive states

Finally, even if they are not Gaussian states[14], we would like to introduce here the *passive states* [29]. They are defined as a mixture of Fock states with decreasing weights for increasing photon number

$$\rho_{passive} = \sum_{i=0}^{\infty} c_i |i\rangle\langle i| \qquad \text{with} \quad c_0 \geq c_1 \geq \cdots \geq c_n \geq \cdots \tag{2.86}$$

so their Wigner function is given by

$$W_{passive}(x,p) = \sum_{i=0}^{\infty} c_i W_i(x,p) \tag{2.87}$$

where $W_i(x,p)$ is the Wigner function of the Fock state $|i\rangle$. Interestingly the Wigner function of a passive state is positive everywhere [29]. We can also define the *extremal passive states*, i.e. passive states with equal weights up to a certain number of photon

$$\rho_{extremal} = \frac{1}{n+1} \sum_{i=0}^{n} |i\rangle\langle i|. \tag{2.88}$$

Note that all passive states can be expressed as a convex mixture of these extremal passive states.

[14] Except when $c_0 = 1$ and $c_i = 0$ for all $i \neq 0$ since it is then the vacuum state.

3 | Shannon information theory

Shannon information theory owes its name to Claude E. Shannon who developed this theory in 1948 [30]. One of the principal quantities in information theory is entropy. But, not only is this quantity central to the information theory field, it is also one of the main subject of this thesis since entropy allows us to measure the uncertainty of a random variable. Information theory is a large field, but we chose in this chapter to present only the elements that will be relevant to this thesis. For more information, we refer the reader to the book of Cover and Thomas [31] from which most of the equations of this chapter are taken.

Shannon information theory was first developed for discrete variables, using discrete probability distributions and was then extended to continuous variables. Therefore, even though we are interested in the latter, we will first introduce some important formulas for the discrete case.

3.1 Discrete variables

3.1.1 Shannon entropy

Let X be a random discrete variable, $x \in \mathcal{X}$ the possible values that X can take and $p(x)$ its probability distribution. The *Shannon entropy* of X, $H(X)$ is defined as

$$H(X) \equiv H(p) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x). \tag{3.1}$$

Note that the logarithm is in base 2 so the entropy is expressed in bits. From this definition, we understand that the entropy is a measure of the uncertainty in the random variable X. We can understand $H(X)$ as the number of bits, on average, required to describe an instance of the random variable.

For example, let us compute the entropy of a fair coin toss. In this case, both probabilities to obtain head ($X = 0$) or tail ($X = 1$) are equal. Thus, $H(X) = 1$ which means that 1 bit of information is needed to describe the variable X. This one bit can

also be understood as the amount of information gained by flipping the coin.

Intuitively, adding terms with zero probability should not change the entropy. Therefore, we use the convention $0 \log 0 = 0$, which is justified by the limit of $x \log x$ when x tends to 0. An important property of the discrete entropy is that $H(X) \geq 0$, since $0 \leq p(x) \leq 1$. As we will see, this property will no longer be true for continuous variables.

Let us mention that the entropy $H(p)$ is a concave function which means that

$$H(\lambda p_1 + (1-\lambda)p_2) \geq \lambda H(p_1) + (1-\lambda)H(p_2). \tag{3.2}$$

3.1.2 Joint entropy

Equation (3.1) can be generalized to n variables $\{X_1, \cdots, X_n\}$. We then speak of *joint entropy* and it is defined as

$$H(X_1, \cdots, X_n) = -\sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_n \in \mathcal{X}_n} p(x_1, \cdots, x_n) \log_2 p(x_1, \cdots, x_n). \tag{3.3}$$

where $p(x_1, \cdots, x_n)$ is the joint probability distribution and each variable X_i can take discrete values $x_i \in \mathcal{X}_i$. Since, the variables X_i might be correlated, the joint entropy can only be lower than the entropies of the individual variables. We say that the joint entropy is subadditive:

$$H(X_1, \cdots, X_n) \leq \sum_i^n H(X_i) \tag{3.4}$$

The equality is obtained if and only if all variables are independent.

3.1.3 Relative entropy

If one desires to measure the distance between two distributions p and q, one can use the *relative entropy* $D(p||q)$. The relative entropy is a measure of how different two distributions are. To better understand this notion, let us suppose we want to construct a code for a variable X with its true probability distribution $p(x)$. We would need $H(p)$ bits on average to describe the variable. Suppose now that, instead, we use the probability distribution $q(x)$, we would now need $H(p) + D(p||q)$ bits on average to describe the random variable.

The relative entropy, also called the *Kullback–Leibler distance* between two probability distributions $p(x)$ and $q(x)$ is defined as

$$D(p||q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}. \tag{3.5}$$

An important property of the relative entropy is that $D(p||q) \geq 0$ and we reach the equality if and only if $p(x) = q(x),\ \forall x$. Note that $D(p||q)$, however, is not a genuine distance in the mathematical sense.

In contrast to the Shannon entropy, the relative entropy $D(p||q)$ is a convex function since

$$D(\lambda p_1 + (1-\lambda)p_2||\lambda q_1 + (1-\lambda)q_2) \leq \lambda D(p_1||q_1) + (1-\lambda)D(p_2||q_2). \tag{3.6}$$

3.1.4 Mutual information

Another quantity which is worth mentioning is the *mutual information*. It gives the amount of information that one random variable contains about another random variable. In other words, due to the knowledge of one variable, the mutual information is the amount by which we are able to reduce the uncertainty of the other variable. For two random variables X and Y, the mutual information is defined as

$$I(X:Y) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)} = D(p(x,y)||p(x)p(y)). \tag{3.7}$$

From the second equality, we understand that the mutual information is simply the relative entropy between the joint distribution $p(x,y)$ and the product of marginal distributions $p(x)p(y)$. Therefore, we can conclude that $I(X:Y) \geq 0$ with equality if and only if X and Y are independent. The mutual information can also be expressed in terms of the joint and individual entropies:

$$I(X:Y) = H(X) + H(Y) - H(X,Y). \tag{3.8}$$

3.2 Continuous variables

3.2.1 Shannon differential entropy

Let us now introduce similar concepts for continuous random variables. We will thus speak of *differential Shannon entropy*. In general, it is similar to the discrete case, in particular in the sense that the differential entropy still represents the uncertainty of the random variable. However, as we will see, some important differences arise in some properties of the differential entropies.

The differential entropy of a continuous variable X with probability distribution $p(x)$ is defined as

$$h(X) \equiv h(p) = -\int_{-\infty}^{\infty} dx\, p(x)\ln p(x). \tag{3.9}$$

The definition is the continuous extension of the discrete case, where we simply re-

placed the sum by an integral.[1] To distinguish them, we will use a small letter for the differential entropy $h(X)$ and a capital letter for the discrete entropy $H(X)$.

As we mentioned earlier, the discrete entropy cannot take negative values. This is no longer true for the continuous entropy which can take any real value and can thus be negative. Indeed, let us, for example, take a random variable distributed uniformly on an interval from 0 to $1/2$. Its probability density is thus

$$p(x) = \begin{cases} 2 & \text{if } x \in [0, 1/2] \\ 0 & \text{elsewhere.} \end{cases} \tag{3.10}$$

The computation of its entropy then gives $h(x) = -\int_0^{1/2} dx\, 2 \ln 2 = -\ln 2$ which is negative.

3.2.2 Joint entropy, relative entropy, mutual information and properties

Here too, if we have a probability distribution of n continuous variables $p(x_1, \cdots, x_n)$, we can define the *joint differential entropy*

$$h(X_1, \cdots, X_n) = -\int dx_1 \cdots dx_n\, p(x_1, \cdots, x_n) \ln p(x_1, \cdots, x_n). \tag{3.11}$$

The relative entropy between two probability distributions p and q as well as the mutual information have definitions similar to the discrete case, where we basically replace the summation by an integration

$$D(p||q) = \int dx\, p(x) \ln \frac{p(x)}{q(x)} \tag{3.12}$$

$$I(X:Y) = \int dxdy\, p(x,y) \ln \frac{p(x)p(y)}{p(x,y)}. \tag{3.13}$$

In addition,

$$I(X:Y) = h(X) + h(Y) - h(X,Y) \tag{3.14}$$

still holds and the properties of $D(p||q)$ and $I(X:Y)$ are the same as in the discrete case. In particular, they are both always positive and $D(p||q) = 0$ if and only if $p(x) = q(x),\ \forall x$. The subadditivity of the joint entropy is also still true

$$h(X_1, \cdots, X_n) \leq \sum_i h(X_i). \tag{3.15}$$

Under a translation, the value of the differential entropy does not change

$$h(X + c) = h(X), \tag{3.16}$$

[1] To be more precise, $h(X)$ is the limit of $H(X^\Delta) + \log \Delta$ when $\Delta \to 0$ and $H(X^\Delta)$ is the quantized entropy of the variable X. More details can be found in [31].

however, under a dilation the differential entropy changes as

$$h(aX) = h(X) + \ln|a|. \tag{3.17}$$

For a multivariate distribution, the equivalent expression is

$$h(A\mathbf{X}) = h(\mathbf{X}) + \ln|\det A| \tag{3.18}$$

where A is an invertible matrix that transforms the vector $\mathbf{X}$.

3.2.3 Entropy of Gaussian distributions

In the previous chapter, we introduced Gaussian states and their properties. In quantum information theory, Gaussian states, or we should say for now, Gaussian distributions also have interesting properties. Let X be a Gaussian distributed variable,

$$X \sim p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \tag{3.19}$$

and let us compute its entropy

$$\begin{aligned} h(X) &= -\int dx\, p(x) \ln p(x) \\ &= -\int dx\, \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\right) \\ &= \ln\sqrt{2\pi\sigma^2} \int dx\, \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} + \int dx\, \frac{x^2}{2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \\ &= \ln\sqrt{2\pi\sigma^2} + \frac{1}{2} \\ &= \frac{1}{2}\ln(2\pi e\sigma^2). \end{aligned} \tag{3.20}$$

In general, for n variables, the Gaussian distribution is given by

$$p_G(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det\gamma}} e^{-\frac{1}{2}(\mathbf{x}-\langle\mathbf{x}\rangle)^T \gamma^{-1} (\mathbf{x}-\langle\mathbf{x}\rangle)} \tag{3.21}$$

and the entropy is equal to $\frac{1}{2}\ln((2\pi e)^n \det\gamma)$ where γ is the covariance matrix.

What is the main interest of those Gaussian distributions? If we compare the entropy of all distributions with same covariance matrix, the maximum is given by the Gaussian distribution, that is

$$h(p) \leq h(p_G) = \frac{1}{2}\ln((2\pi e)^n \det\gamma). \tag{3.22}$$

One can prove this relation by evaluating the relative entropy between a Gaussian distribution $p_G(\mathbf{x})$ and any other distribution $p(\mathbf{x})$ with same covariance and then

using the fact that $D(p(\mathbf{x})||p_G(\mathbf{x})) \geq 0$. Note that the equality is reached only if $p(\mathbf{x})$ is Gaussian too.

From the subadditivity of the entropy applied to a multivariate Gaussian distribution, we can derive an interesting inequality named *Hadamard's inequality*. Indeed, if we apply Eq. (3.15) to the multivariate Gaussian distribution Eq. (3.21) we find

$$\det \gamma \leq \prod_i^n \gamma_{ii}. \tag{3.23}$$

3.2.4 Entropy power

Sometimes it is useful to speak of *entropy powers* instead of entropies. The entropy power of a distribution of n random variables $\mathbf{X}$ is defined as

$$N(\mathbf{X}) = \frac{1}{2\pi e} e^{\frac{2}{n}h(\mathbf{X})}. \tag{3.24}$$

It is the variance (power) of a set of n independent Gaussian variables that produce the same entropy as $\mathbf{X}$.

In particular, we can derive the well-known *entropy power inequality*

$$N(\mathbf{X}+\mathbf{Y}) \geq N(\mathbf{X}) + N(\mathbf{Y}). \tag{3.25}$$

In Chapter 6, we will show that entropy powers are more suitable to express entropic uncertainty relations. In particular, the latter are then written in a form similar to the variance-based uncertainty relations.

3.2.5 Rényi entropy

Since the beginning of this section, we only talked about Shannon differential entropies. However, they belong to a wider family called *Rényi entropies*. The Rényi entropy $h_\alpha(X)$ of order α is defined as

$$h_\alpha(X) = \frac{1}{1-\alpha} \log \left[\int_{-\infty}^{\infty} dx\, p^\alpha(x) \right]. \tag{3.26}$$

For $\alpha = 1$, we need to take the limit of this expression and we fall back on the definition of the Shannon entropy. Rényi entropies are monotonically decreasing as a function of α. Note that they also satisfy the scaling property so that

$$h_\alpha(A\mathbf{X}) = h_\alpha(\mathbf{X}) + \ln|\det A| \tag{3.27}$$

when $\mathbf{X}$ transforms as $A\mathbf{X}$. Rényi entropies are Schur-concave[2] but they do not satisfy subadditivity.

3.2.6 Wehrl entropy

Finally, let us mention the *Wehrl entropy* [32] defined as

$$h_W(x,p) = -\int dxdp\, Q(x,p) \ln Q(x,p) \tag{3.28}$$

where

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle \tag{3.29}$$

is the *Husimi Q-representation* [33]. Just as the Wigner function, the Q-representation is normalized. However, in contrast to the Wigner function, it is always positive for all quantum states. Indeed, $Q(\alpha)$ can be seen as proportional to the probability of finding the system in the coherent state $|\alpha\rangle$ and so $0 \leq Q(\alpha) \leq 1/\pi$. In particular, this implies that the Wehrl entropy is always positive, unlike the Shannon differential entropy. In his paper, Wehrl actually conjectured that $h_W(x,p) \geq 1$ and this was proven a bit later by Lieb [34]. The equality occurs if and only if the density matrix ρ is a pure state projector onto any coherent state, i.e. $\rho = |\alpha\rangle\langle\alpha|$.

[2] A Schur-convex function is a function such that for all x, y such that x is majorized by y, f satisfies $f(x) \leq f(y)$. A function f is Schur-concave if $-f$ is Schur-convex.

4 | Uncertainty relations

The uncertainty principle lies at the heart of quantum physics. It exhibits one of the key divergences between a classical and a quantum system. Classically, it is in principle possible to specify the precise value of all measurable quantities simultaneously in a given state of a system. In contrast, whenever two quantum observables do not commute, it is impossible to define a quantum state for which their values are simultaneously specified with infinite precision. First formulated by Heisenberg for position and momentum, the uncertainty principle has been generalized to variables which are non canonically conjugate and has been extended to include correlations. In addition, while it was first formulated in terms of variances, uncertainty relations have been extended to an entropic formulation. In this chapter, we give an overview of uncertainty relations and review some of the most important formulations.

4.1 Variance-based uncertainty relations

4.1.1 Heisenberg uncertainty relation

In 1927, Heisenberg was the first to express an uncertainty relation between the position and momentum of a particle. In his paper [1], he exposed a thought experiment — known as the Heisenberg's microscope — for measuring the position of an electron. The idea was to send a γ-ray on the particle and to measure the position of the scattered photon. We can then deduce from it the position of the electron, but with a small indeterminacy δx due to the wave property of the photon. According to Compton's effect [35], we can also compute the momentum of the scattered photon, but we cannot know precisely its direction. This generates an indeterminacy δp on the measurement of the momentum of the particle. From this experiment, Heisenberg explained that there is a trade-off about how precisely both the position and the momentum can be measured, and it is expressed as

$$\delta x \delta p \sim h \tag{4.1}$$

where h is the Planck constant. Shortly after, Kennard [2] mathematically formalized the uncertainty relation and proved that

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4} \tag{4.2}$$

where σ_x^2 and σ_p^2 represent the variances of the position and momentum of a quantum particle and $\hbar = h/2\pi$ is the reduced Plank constant.

Note that, as expressed by Kennard, the uncertainty relation is actually a property of Fourier transforms and thus has many applications in classical physics. What makes it quantum then? It is the wave description of the particle. More precisely, the uncertainty relation connects the position x of the particle and the wavelength λ of its associated wave, but thanks to de Broglie formula $p = h/\lambda$ [36], the wavelength is related to the momentum p which implies the uncertainty relation for position and momentum.

Remark also that Heisenberg made a statement about measurements, while Kennard's formulation is really expressing an intrinsic property of the state. Indeed, Kennard uses the variances which only depend on the state itself. His inequality does not concern any trade-off relation between the knowledge on the position and disturbance on the momentum due to the quantum measurement. Following Heisenberg's view, where the uncertainty originates from the measurement, many works have focused on finding an appropriate definition for measurement uncertainties (see [37] for a review). In particular, let us mention Ozawa [38] who derived an inequality about error-disturbance and claimed that this was a rigorous version of Heisenberg's formulation of the uncertainty principle. Nevertheless, this claim is still a matter of debate (for more details, see for example [39, 40]). Nowadays, most textbooks adopt the view of Kennard, even though Eq. (4.2) is widely called the *Heisenberg's uncertainty relation*.

If the uncertainty relation was first formulated for position and momentum, it is well-known that it holds not just for the position and momentum of a particle, but for any pair of canonically-conjugate variables[1]. In fact, in 1928, Robertson [41] extended the formulation of the uncertainty principle to two arbitrary observables A and B as

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} |\langle \psi | [A, B] | \psi \rangle|^2 \tag{4.3}$$

where $[\cdot, \cdot]$ stands for the commutator. Obviously, if $A = x$ and $B = p$, since $[x, p] = i$, we retrieve the Heisenberg uncertainty relation. Note that being aware that uncertainty relations are expressed in terms of $\hbar$, for simplicity, we now fix $\hbar = 1$ throughout the chapter.

Relation (4.2) is invariant under (x, p)-displacements in phase space, since it only

[1] That is variables related to each other by a Fourier transform.

depends on central moments (esp. second-order moments of the deviations from the means). Furthermore, it is saturated by all pure Gaussian states provided that they are squeezed in the x or p direction only. More precisely, if we define the covariance matrix

$$\gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_p^2 \end{pmatrix} \tag{4.4}$$

as in Eq. (2.30), we note that the Heisenberg relation is saturated for pure Gaussian states provided the principal axes of γ are aligned with the x- and p-axes, namely $\sigma_{xp} = 0$. The principal axes are the x_θ- and p_θ-axes for which $\sigma_{x_\theta\, p_\theta} = 0$, where

$$\hat{x}_\theta = \cos\theta\, \hat{x} + \sin\theta\, \hat{p} \qquad\qquad \hat{p}_\theta = -\sin\theta\, \hat{x} + \cos\theta\, \hat{p} \tag{4.5}$$

are obtained by rotating x and p by an angle θ as shown in Figure 4.1.

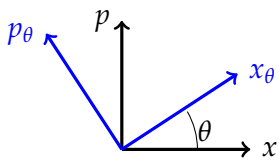

Figure 4.1: Principal axes (x_θ, p_θ) of the covariance matrix γ, defined in such a way that $\sigma_{x_\theta\, p_\theta} = 0$.

4.1.2 Robertson-Schrödinger uncertainty relation

The fact that Eq. (4.2) is saturated only by certain Gaussian states is linked to the fact that this uncertainty relation is not invariant under rotation. The problem of invariance was solved in 1930 by Schrödinger [3] and Robertson [4] who added an anticommutator to the relation (4.3). The new uncertainty relation, for any two arbitrary observables now reads

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4}\Big|\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle\Big|^2 + \frac{1}{4}\Big|\langle[A,B]\rangle\Big|^2 \tag{4.6}$$

and its proof is given in Section 4.1.3. In the special case of position and momentum, $A = x$ and $B = p$ and the Robertson-Schrödinger uncertainty relation reads

$$\det(\gamma) \geq \frac{1}{4}. \tag{4.7}$$

This uncertainty relation is now invariant under symplectic transformations and so is saturated by all pure Gaussian states, regardless of the orientation of the principal axes of the covariance matrix. Indeed, in Section 2.4.1, we mentioned that under a symplectic transformation $\mathcal{S}$, the new covariance matrix is given by $\gamma' = \mathcal{S}\gamma\mathcal{S}^T$. Now, remember that the determinant of a symplectic matrix is equal to 1, so that

$$\det(\gamma') = \det(\mathcal{S})\det(\gamma)\det(\mathcal{S}) = \det(\gamma) \tag{4.8}$$

which shows that Eq. (4.7) is invariant under symplectic transformations, hence under all Gaussian unitary transformations (since it is also invariant under displacements).

In n modes, the generalization of the Robertson-Schrödinger uncertainty relation for position and momentum is due to Simon et al. [42]. It is formulated as an inequality for the covariance matrix γ

$$\gamma + \frac{i}{2}\Omega \geq 0 \tag{4.9}$$

where

$$\Omega = \bigoplus_{k=1}^{n} \omega, \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.10}$$

as already defined in Eq. (2.39). For one mode, Eq. (4.9) reduces to the Robertson-Schrödinger uncertainty relation, but in general, we can understand Eq. (4.9) as n inequalities that must be satisfied in order for a covariance matrix to represent a physical state. Remember that according to Williamson's theorem (see Section 2.4.1) we can always diagonalize γ in its symplectic form $\gamma^{\oplus}$ with the symplectic values ν_i on the diagonal. Therefore, if γ is the covariance matrix of a physical state, it satisfies Eq. (4.9) and so must $\gamma^{\oplus}$ since

$$\begin{aligned} & \gamma + \tfrac{i}{2}\Omega && \geq 0 \\ \Leftrightarrow \quad & \mathcal{S}\gamma\mathcal{S}^\dagger + \tfrac{i}{2}\mathcal{S}\Omega\mathcal{S}^\dagger && \geq 0 \\ \Leftrightarrow \quad & \gamma^{\oplus} + \tfrac{i}{2}\Omega && \geq 0 \end{aligned} \tag{4.11}$$

where $\mathcal{S}$ is a symplectic matrix and Ω is, by definition, invariant under a symplectic transformation. This means that the eigenvalues of the matrix

$$\begin{pmatrix} \nu_1 & i/2 & & & & & \\ -i/2 & \nu_1 & & & & 0 & \\ & & \nu_2 & i/2 & & & \\ & & -i/2 & \nu_2 & & & \\ & & & & \ddots & & \\ & 0 & & & & \nu_n & i/2 \\ & & & & & -i/2 & \nu_n \end{pmatrix} \tag{4.12}$$

must all be positive. In other words, Eq. (4.9) is equivalent to

$$\nu_i \geq \frac{1}{2} \qquad \text{for } i = 1, \cdots, n. \tag{4.13}$$

In Section 5.3.1 we give an example where we check the physicality of a state with the help of the symplectic values.

Among others, an inequality easy to derive is

$$\det(\gamma^{\oplus}) = \prod_{i=1}^{n} \nu_i^2 \geq \left(\frac{1}{4}\right)^n. \tag{4.14}$$

More interestingly, since the covariance matrix is invariant under symplectic transformations, $\det(\gamma) = \det(\gamma^{\oplus})$ and we thus have, for all quantum states, the following inequality

$$\det(\gamma) \geq \left(\frac{1}{4}\right)^n \tag{4.15}$$

which is the straightforward generalization of Robertson-Schrödinger uncertainty relation.

Finally, we mention that in 1934, Robertson [43] introduced a covariance-based uncertainty relation for N observables which generalizes Eq. (4.6). If we define the vector $\mathbf{R} = (R_1, \cdots, R_N)$ containing N observables, then the uncertainty relation is expressed as

$$\det(\sigma(\mathbf{R})) \geq \det(C(\mathbf{R})) \tag{4.16}$$

where $\sigma(\mathbf{R})$ is a covariance matrix and $C(\mathbf{R})$ the commutator matrix. Their elements are defined as

$$\sigma_{ij} = \frac{1}{2}\langle R_i R_j + R_j R_i\rangle - \langle R_i\rangle\langle R_j\rangle \quad \text{and} \quad C_{ij} = -\frac{i}{2}\langle [R_i, R_j]\rangle, \tag{4.17}$$

respectivelly. We can easily see that for $N = 2$, we retrieve Eq. (4.6). Surprisingly, when N is odd, $\det C = 0$. Indeed, C is an antisymmetric matrix since $C_{ij} = -C_{ji}$ and so

$$C = -C^T \quad \Leftrightarrow \quad \det C = (-1)^n \det C^T \quad \Leftrightarrow \quad \det C = (-1)^N \det C \tag{4.18}$$

which implies that $\det C = 0$ when N is odd. This uncertainty relation is thus not interesting for an odd number of variables. In 2014, Kechrimparis and Weigert [44] proved that for the three pairwise canonical observables $\hat{p}$, $\hat{x}$ and $\hat{r} = -\hat{x} - \hat{p}$ (which satisfy the commutation relations $[\hat{p}, \hat{x}] = [\hat{x}, \hat{r}] = [\hat{r}, \hat{p}] = -i$), the bound on the product of variances is actually given by

$$\sigma_x^2 \sigma_p^2 \sigma_r^2 \geq \left(\frac{1}{\sqrt{3}}\right)^3. \tag{4.19}$$

Very recently, Dodonov proposed a more general relation involving any triple or quadruple of observables [45].

4.1.3 Proof of the Robertson-Schrödinger uncertainty relation

The proof of Robertson-Schrödinger, Eq. (4.6) is easy. Since it is at the root of this thesis, we chose to present it here.

The variance of an observable A in a state $|\psi\rangle$ is defined by $\sigma_A^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$. Thus,

$$\begin{aligned}\sigma_A^2 &= \langle\psi|(A - \langle A\rangle)(A - \langle A\rangle)|\psi\rangle \\ &= \langle(A - \langle A\rangle)^\dagger\psi|(A - \langle A\rangle)\psi\rangle \\ &= \langle(A - \langle A\rangle)\psi|(A - \langle A\rangle)\psi\rangle \end{aligned} \tag{4.20}$$

where, in the last line, we used the fact that an observable is an Hermitian operator. We do the same for σ_B^2. Let us now define $\sigma_A^2 = \langle f|f\rangle$ and $\sigma_B^2 = \langle g|g\rangle$. Using Cauchy-Schwartz inequality, we obtain

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2. \tag{4.21}$$

Let us thus compute $\langle f|g\rangle$:

$$\begin{aligned}\langle f|g\rangle &= \langle(A - \langle A\rangle)\psi|(B - \langle B\rangle)\psi\rangle \\ &= \langle\psi|(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle \\ &= \langle\psi|(AB - A\langle B\rangle - \langle A\rangle B + \langle A\rangle\langle B\rangle)|\psi\rangle \\ &= \langle AB\rangle - \langle A\rangle\langle B\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle \\ &= \langle AB\rangle - \langle A\rangle\langle B\rangle, \end{aligned} \tag{4.22}$$

and similarly $\langle g|f\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle$.

For any complex number, we have the property

$$|z|^2 = \Re(z)^2 + \Im(z)^2 = \left|\frac{z + z^*}{2}\right|^2 + \left|\frac{z - z^*}{2}\right|^2. \tag{4.23}$$

Therefore, we can develop $|\langle f|g\rangle|^2$ as

$$\begin{aligned}|\langle f|g\rangle|^2 &= \left|\frac{\langle f|g\rangle + \langle g|f\rangle}{2}\right|^2 + \left|\frac{\langle f|g\rangle - \langle g|f\rangle}{2}\right|^2 \\ &= \left|\frac{\langle AB\rangle - \langle A\rangle\langle B\rangle + \langle BA\rangle - \langle A\rangle\langle B\rangle}{2}\right|^2 \\ &\qquad + \left|\frac{\langle AB\rangle - \langle A\rangle\langle B\rangle - \langle BA\rangle + \langle A\rangle\langle B\rangle}{2}\right|^2 \\ &= \frac{1}{4}\Big|\langle\{A, B\}\rangle - 2\langle A\rangle\langle B\rangle\Big|^2 + \frac{1}{4}\Big|\langle[A, B]\rangle\Big|^2 \end{aligned} \tag{4.24}$$

which gives, using Eq. (4.21),

$$\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\Big|\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle\Big|^2 + \frac{1}{4}\Big|\langle[A,B]\rangle\Big|^2. \tag{4.25}$$

This completes the proof.

4.1.4 Gaussianity-bounded uncertainty relation

Degree of Gaussianity

We understood by now that Gaussian states play a prominent role in continuous-variable quantum information. However, non-Gaussian states should not be put aside as several protocols necessarily require the use of non-Gaussian states, such as for example entanglement distillation [46, 47, 48] or quantum error correction [49, 50]. With the increasing importance of non-Gaussian states, the question of measuring the Gaussian character of a state has naturally arisen. Several Gaussianity measures have been introduced, see e.g. [51, 52, 53, 54], but we find it more convenient in this thesis to use the degree of Gaussianity g introduced in [55]. In particular, this definition will allow us to present another uncertainty relation which will serve in detecting entangled states in Chapter 11.

Consider a two-mode state ρ. Its first- and second-order moments, denoted respectively as $\mathbf{d}$ and γ, are expressed from the vector of quadratures $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$. As defined in Chapter 2, the elements of the coherent vector are given by $d_j = \langle \hat{r}_j \rangle$, while the elements of the covariance matrix are defined as $\gamma_{ij} = \frac{1}{2}\langle \hat{r}_i\hat{r}_j + \hat{r}_j\hat{r}_i\rangle - d_i d_j$. The degree of Gaussianity g of state ρ is defined as

$$g = \frac{\mathrm{Tr}(\rho\rho^G)}{\mathrm{Tr}(\rho^G\rho^G)} \tag{4.26}$$

where ρ^G is the Gaussian state characterized by the first-order moment $\mathbf{d}$ and covariance matrix γ of state ρ.

Properties of the degree of Gaussianity

One may easily verify that $g = 1$ for Gaussian states, since $\rho = \rho^G$. Note, however, that the converse is not true. Indeed, here are some counter examples. Consider ρ a non-Gaussian state being the mixture of two Fock states

$$\rho = \frac{1}{2\sqrt{2}}|2\rangle\langle 2| + \left(1 - \frac{1}{2\sqrt{2}}\right)|0\rangle\langle 0|. \tag{4.27}$$

The covariance matrix of ρ,

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \qquad \text{with} \qquad a = \frac{1+\sqrt{2}}{2}, \tag{4.28}$$

determines a Gaussian (thermal) state

$$\rho^G = \frac{\sqrt{2}}{1+\sqrt{2}} \sum_j \left(\frac{1}{1+\sqrt{2}} \right)^j |j\rangle\langle j|. \tag{4.29}$$

It is then easy to see that

$$\mathrm{Tr}[\rho^G \rho^G] = \mathrm{Tr}[\rho \rho^G] = \frac{1}{1+\sqrt{2}}. \tag{4.30}$$

This obviously gives $g = 1$, although ρ is a non-Gaussian state. Other counterexamples of non-Gaussian states with $g = 1$ may be found among the states of the form

$$\rho = p\,|n\rangle\langle n| + (1-p)\,|0\rangle\langle 0|. \tag{4.31}$$

Given n, the real roots of the equation

$$(1+2np)(np)^n - (1+2np-n)(1+np)^n = 0 \tag{4.32}$$

satisfying $0 < p < 1$ provide $g = 1$. Note that Eq. (4.32) is polynomial of degree n, therefore, the number of its roots providing counter examples is expected to increase with n.

Other important properties of the degree of Gaussianity are that g is real and is invariant under Gaussian unitary transformations, transposition, and partial transposition. The proofs are given below. Note that definition (4.26) holds for a n-modal state, but in this thesis, we will focus on $n = 1$ and $n = 2$, because this is what we will need to define the separability criterion of Chapter 11.

- g is invariant under Gaussian unitary operations (i.e. symplectic transformations and displacements).

  Proof: Consider an arbitrary state ρ and corresponding Gaussian state ρ^G. A Gaussian unitary operator U^G transforming Gaussian states to Gaussian states transforms ρ to $\rho' = U^G \rho (U^G)^\dagger$. The operator similarly transforms the Gaussian state ρ^G to $\rho'^G = U^G \rho^G (U^G)^\dagger$. The latter transformation is equivalent to a symplectic transformation of the corresponding covariance matrices. By construction of ρ^G, its covariance matrix is also the covariance matrix of ρ. This covariance matrix is transformed by the symplectic transformation into the covariance matrix of ρ'^G. Therefore, ρ'^G is the Gaussian state corresponding to ρ'.

Then a simple calculation gives us the desired result

$$\begin{aligned} g' &= \frac{\mathrm{Tr}(\rho'\rho'^G)}{\mathrm{Tr}(\rho'^G\rho'^G)} \\ &= \frac{\mathrm{Tr}(U^G\rho(U^G)^\dagger U^G\rho^G(U^G)^\dagger)}{\mathrm{Tr}(U^G\rho^G(U^G)^\dagger U^G\rho^G(U^G)^\dagger)} \\ &= \frac{\mathrm{Tr}(\rho\rho^G)}{\mathrm{Tr}(\rho^G\rho^G)} \\ &= g, \end{aligned} \tag{4.33}$$

where we used the invariance of the trace under cyclic permutations. □

- g is invariant under partial transposition.

  Proof: Partial transposition implies sign-flip of one of the two momentum quadratures (say $p_2 \to -p_2$), i.e. one of the arguments of the Wigner function describing a two-mode state. Then we have

$$\begin{aligned} g' &= \frac{\mathrm{Tr}(\rho'\rho'^G)}{\mathrm{Tr}(\rho'^G\rho'^G)} \\ &= \frac{(2\pi)^2\int dx_1dp_1dx_2dp_2W_{\rho'}(x_1,p_1,x_2,p_2)W_{\rho'^G}(x_1,p_1,x_2,p_2)}{(2\pi)^2\int dx_1dp_1dx_2dp_2W_{\rho'^G}(x_1,p_1,x_2,p_2)W_{\rho'^G}(x_1,p_1,x_2,p_2)} \\ &= \frac{(2\pi)^2\int dx_1dp_1dx_2dp_2W_{\rho}(x_1,p_1,x_2,-p_2)W_{\rho^G}(x_1,p_1,x_2,-p_2)}{(2\pi)^2\int dx_1dp_1dx_2dp_2W_{\rho^G}(x_1,p_1,x_2,-p_2)W_{\rho^G}(x_1,p_1,x_2,-p_2)} \\ &= \frac{\mathrm{Tr}(\rho\rho^G)}{\mathrm{Tr}(\rho^G\rho^G)} \\ &= g \end{aligned} \tag{4.34}$$

  where at the last step we make the change of variables $-p_2 \to p_2$. □

- g is invariant under transposition.

  Proof: The proof follows the same steps as for the partial transposition, but in this case, we have both $p_2 \to -p_2$, and $p_1 \to -p_1$. This does not change the conclusion. □

- g is real.

  Proof: $\mathrm{Tr}(\rho\rho^G)$ and $\mathrm{Tr}(\rho^G\rho^G)$ can both be seen as the expectation value of an Hermitian operator (a density matrix is Hermitian). Since the expectation value of an Hermitian operator is real, so is g. □

Gaussianity-bounded uncertainty relation

The aim of defining the degree of Gaussianity is to present the Gaussianity-bounded uncertainty relation found by Mandilara and Cerf [55]. They show that the knowledge of g gives a tighter bound on the uncertainty relation. Moreover, remarkably,

all Fock states — not just the vacuum — appear as minimum uncertainty states.

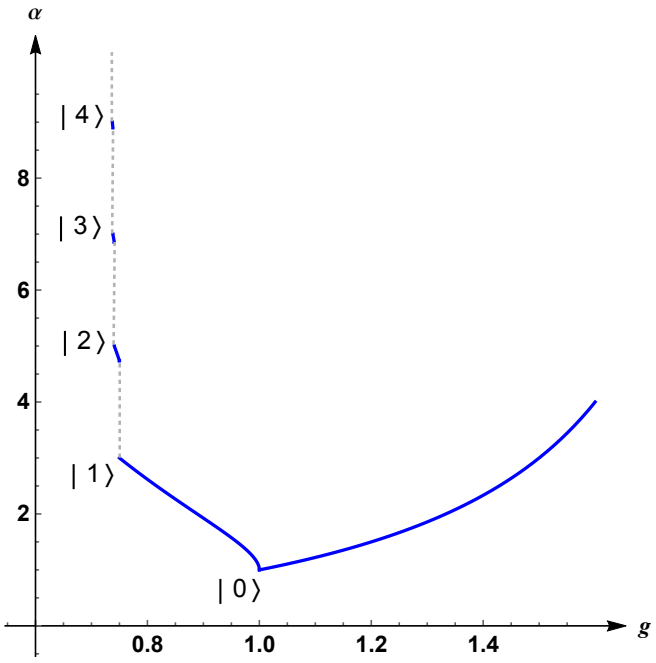

Figure 4.2: Gaussianity-bounded uncertainty relation. Plot of α, threshold value for a state to be physical, as a function of the degree of Gaussianity g. All physical states lie on or above this curve.

The Gaussianity-bounded uncertainty relation appear in Figure 4.2. All physical states lie above the curve determined by the following equations. Let us define[2]

$$\alpha = 2\sqrt{\det(\gamma)}. \tag{4.35}$$

If $g \geq 1$, the curve is given by

$$\alpha = \frac{g}{2-g}. \tag{4.36}$$

If $g < 1$, the parametric equations of the curve are given by

$$\begin{aligned} \alpha &= 2n + 3 - 2r, \\ g &= \frac{2\alpha(\alpha-1)^n}{(\alpha+1)^{n+1}}\left(\frac{(\alpha-1)(1-r)}{\alpha+1} + r\right), \end{aligned} \tag{4.37}$$

where $n \in \mathbb{N}$ and $r \in [0,1[$. The latter curve consists of consecutive segments corresponding each to a binary mixture of nearest-neighbor Fock states $|n\rangle$ and $|n+1\rangle$. Note that for $g < 1$, the curve exhibits some discontinuities.

4.1.5 Purity-bounded uncertainty relation

Finally, before ending the section about variance-based uncertainty relation, we would like to mention another uncertainty relation whose bound is now based on the purity

[2]This α has nothing to do with the α used to describe a coherent state (see Section 2.5).

of the state. Indeed, Dodonov [29] presented the following uncertainty inequality

$$\det(\gamma) \geq \Phi(\mu) \tag{4.38}$$

where $\Phi(\mu)$ is a function of $\mu = \mathrm{Tr}(\rho^2)$, the purity of the state. This function does not have a simple analytical form, but details can be found in [29]. Note, however, that in the case of pure states, we fall back on Robertson-Schrödinger uncertainty principle. In [56], Mandilara et al. pushed the work a bit further by deriving an uncertainty relation bounded by both, the purity and the degree of Gaussianity of the state.

4.2 Entropy-based uncertainty relations

4.2.1 Continuous-variable entropic uncertainty relations

As we mentioned in Chapter 3, uncertainty can also be measured with entropy. So an obvious question arises: can we also express an uncertainty principle in terms of entropies. The first to ask and answer this question was Hirschman [10] in 1957. He conjectured the first entropic uncertainty relation for the position and momentum observables which was then proved by Beckner [57] and Białynicki-Birula and Mycielski [11] in 1975. The entropic uncertainty relation reads

$$h(\mathbf{x}) + h(\mathbf{p}) \geq n \ln(\pi e \hbar) \tag{4.39}$$

where $h(\cdot)$ is the differential entropy defined in Eq. (3.9) and n is the number of modes. The proof of this inequality is given in Section 4.2.2. We want to point out that Eq. (4.39) may look wrong at first sight as we take the logarithm of a quantity with dimension $\hbar$. This may be viewed as a feature of the differential entropy, since we have a similar issue in the definition Eq. (3.9) itself, but the problem actually cancels out in Eq. (4.39) since we have dimension $\hbar$ on both sides of the equality. More rigorously, Eq. (4.39) may be understood as the limit of a discretized version of the entropic uncertainty relation, with a discretization step tending to zero [58]. This problem was absent in the original expression of this uncertainty relation [11] because the variable $k = p/\hbar$ was considered instead of p, giving $h(x) + h(k) \geq \ln(\pi e)$. Being aware of this slight abuse of notation, we now prefer to keep $\hbar = 1$ for simplicity.

This entropic formulation of the uncertainty principle has recently attracted much attention in quantum information sciences because entropies are the natural quantities of interest in this area. In particular, an extended version of the entropic uncertainty relation was derived, where some available quantum side-information (e.g., a quantum memory) is taken into account [59, 60]. It expresses the tradeoff between the information that two parties may have on non-commuting observables, which is of particular relevance to quantum key distribution. A variant version of this uncer-

tainty relation formulated in terms of smooth entropies [61] indeed provides a useful tool for finite-key security analysis [62], going beyond asymptotic proofs. In the special case of continuous-variable quantum key distribution, the original entropic uncertainty relation [11] was first applied to proving the optimality of Gaussian individual attacks at the asymptotic key limit [5]. More recently, a finite-key analysis for certain continuous-variable protocols was performed based on the smooth-entropy formalism extended to infinite dimensions [6].

An interesting fact about inequality (4.39), already proven in the original paper [11], is that the entropic uncertainty relation is stronger than, and hence implies, the uncertainty relation of Heisenberg, Eq. (4.2). Indeed, we saw in Chapter 3 that for a fixed variance σ_x^2, the maximum entropy is given by the Gaussian distribution, with entropy given by $\frac{1}{2}\ln(2\pi e\sigma_x^2)$. This is true for any variable, and so is valid for p too. Therefore, we can write

$$\begin{aligned} \frac{1}{2}\ln(2\pi e\sigma_x^2) + \frac{1}{2}\ln(2\pi e\sigma_p^2) &\geq h(x) + h(p) \\ &\geq \ln(\pi e) \end{aligned} \tag{4.40}$$

from which we deduce that $\sigma_x^2\sigma_p^2 \geq 1/4$ which is precisely Heisenberg's uncertainty principle (with $\hbar = 1$). Note that since $h(x) = \frac{1}{2}\ln(2\pi e\sigma_x^2)$ if and only if x has a Gaussian distribution, the entropic uncertainty relation is strictly stronger than the variance-based one for non-Gaussian states. This pushes us to believe that entropic uncertainty relations are more fundamental or at least preferable over the variance-based ones. This is also the reason why the second part of this thesis is entirely devoted to improved entropic uncertainty relations.

Just as the Heisenberg uncertainty relation, Eq. (4.39) is only saturated by pure Gaussian states whose principal axes are aligned with the x- and p-axes (i.e., $\sigma_{xp} = 0$). This means that there is room for an improved entropic uncertainty relation that would be saturated by all pure Gaussian states, in analogy with the Robertson-Schrödinger uncertainty relation Eq. (4.7). This will indeed be the subject of Chapter 6.

Remark that we may also re-express the entropic uncertainty relation in terms of relative entropies. More precisely, using a measure of non-Gaussianity that relies on the relative entropy [52], we have

$$D(x||x_G) = h(x_G) - h(x) \geq 0, \qquad D(p||p_G) = h(p_G) - h(p) \geq 0 \tag{4.41}$$

so that the entropic uncertainty relation is equivalent to

$$D(x||x_G) + D(p||p_G) \leq \ln\left(\frac{\sigma_x\sigma_p}{1/2}\right). \tag{4.42}$$

We see immediately that if the Heisenberg relation is saturated, $\sigma_x^2\sigma_p^2 = 1/4$, then $D(x||x_G) = D(p||p_G) = 0$, which means that the x- and p-quadratures must both

be Gaussian distributed. Thus, as emphasized in ref. [63], the entropic uncertainty relation may also been viewed as an improved version of the Heisenberg relation where the lower bound is lifted up by exploiting an entropic measure of the non-Gaussianity of the state, namely

$$\sigma_x^2 \sigma_p^2 \geq \frac{1}{4} e^{2D(x||x_G)+2D(p||p_G)}. \tag{4.43}$$

Finally, let us mention that we can write an uncertainty relation for Rényi entropies defined in Eq. (3.26) [64]

$$h_\alpha(\mathbf{x}) + h_\beta(\mathbf{p}) \geq n \ln(\pi) + \frac{n \ln(\alpha)}{2(\alpha - 1)} + \frac{n \ln(\beta)}{2(\beta - 1)} \tag{4.44}$$

where α and β satisfy

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2. \tag{4.45}$$

4.2.2 Proof of the entropic uncertainty relation of Białynicki-Birula and Mycielski

Here, we would like to present the proof of the entropic uncertainty relation for continuous variables, Eq. (4.39). Since the proof exploits the fact that position and momentum are related by a Fourier transform, it will provide us with intuition in Chapter 7 where we will derive an entropic uncertainty relation for variables related by a fractional Fourier transform.

Let $\psi(\mathbf{x})$ be the wave function of a pure state and $\phi(\mathbf{p})$ its Fourier transform. The probability distributions of the position and momentum are give by $W_x(\mathbf{x}) = |\psi(\mathbf{x})|^2$ and $W_p(\mathbf{p}) = |\phi(\mathbf{p})|^2$ respectively. Here, $W_x(\mathbf{x})$ and $W_p(\mathbf{p})$ are the marginals of the Wigner function.

It was shown by Babenko [65] and Beckner [57] that a Fourier transformation satisfies the following (q, q')-norm inequality

$$\|\phi(\mathbf{p})\|_{q'} \leq k(q, q') \|\psi(\mathbf{x})\|_q \tag{4.46}$$

where

$$\|\psi(\mathbf{x})\|_q = \left(\int dx^n |\psi(\mathbf{x})|^q \right)^{1/q} \tag{4.47}$$

is the q-norm, $q \geq 2$,

$$\frac{1}{q} + \frac{1}{q'} = 1, \tag{4.48}$$

n is the dimension of $\mathbf{x}$ and $\mathbf{p}$, and

$$k(q,q') = \left(\frac{2\pi}{q'}\right)^{n/2q'} \left(\frac{2\pi}{q}\right)^{-n/2q}. \tag{4.49}$$

We can rewrite Eq. (4.46) in the form

$$\chi(q') \equiv k(q,q')\|\psi(\mathbf{x})\|_q - \|\phi(\mathbf{p})\|_{q'} \geq 0 \tag{4.50}$$

where q can be treated as a function of q' according to Eq. (4.48).

In the specific case where $q = q' = 2$, Eq. (4.46) is known as *Parseval-Plancherel theorem* and $k(q,q') = 1$, so that $\chi(2) = 0$. Now, since $\chi(q') \geq 0$ and $\chi(2) = 0$, the derivative of $\chi(q')$ at $q' = 2$ cannot be negative. In other words,

$$\left.\frac{d\chi(q')}{dq'}\right|_{q'=2} \geq 0 \tag{4.51}$$

is equivalent to

$$\frac{nN}{4}\ln\frac{1}{\pi e} - \frac{1}{2N}\int d^n x|\psi(\mathbf{x})|^2 \ln|\psi(\mathbf{x})| - \frac{1}{2N}\int d^n p|\phi(\mathbf{p})|^2 \ln|\phi(\mathbf{p})| + N\ln N \geq 0 \tag{4.52}$$

where $N = \|\psi(\mathbf{x})\|_2 = \|\psi(\mathbf{p})\|_2 = 1$ since the wave functions of position and momentum are normalized. Using the definition (3.9) of the differential entropy, this completes the proof for pure states. Since the differential entropy is concave, pure states are the most restrictive states, which means that the uncertainty relation also holds verified for mixed states.

4.2.3 Discrete-variable entropic uncertainty relations

Entropic uncertainty relations were first introduced for continuous variables, but were later also developed for discrete observables of finite-dimensional systems. The first relation is due to Deutsch in 1983 [66], but was then improved by Maassen and Uffink in 1988 [67] who proved a conjecture by Krauss [68]. Using the Shannon entropy as defined in Eq. (3.1), the discrete entropic uncertainty relation reads

$$H(A) + H(B) \geq \log\frac{1}{\max\limits_{a,b}|\langle a|b\rangle|^2} \tag{4.53}$$

where $|a\rangle$ and $|b\rangle$ are the eigenstates of the observables A and B. Therefore, the bound is given by the maximum overlap between two eigenstates of A and B. Interestingly, the bound is state-independent, contrary to the Robertson-Schrödinger uncertainty relation.

Discrete entropic uncertainty relations can also be derived from a majorization the-

ory approach [69]. One can consider the product of two probability distributions and express a majorization inequality from which it is possible to derive an entropic uncertainty relation. In short, if there exists a majorization relation of the form

$$p \otimes q \prec w \tag{4.54}$$

where p, q and w are probability distributions, from the Schur-concave property of Rényi entropies, we can deduce that

$$H_\alpha(p) + H_\alpha(q) \geq H_\alpha(w) \tag{4.55}$$

where $H_\alpha(\cdot)$ is a Rényi entropy. This idea was first introduced by Partovi [70] and was then extended by Puchała *et al.* [71] and Friedland *et al.* [72].

I will not give any more details about discrete entropic uncertainty relations, because it is outside the scope of this thesis, but more information can be found in Coles *et al.*'s review [12] on entropic uncertainty relations.

5 | Separability criteria

Another distinctive difference between classical and quantum mechanics is the nature of the correlations between two systems as in the quantum world, correlations go beyond the ones observed classically. We talk here about *entanglement*. The name was coined by Schrödinger, but the phenomenon was first pointed out by Einstein, Podolsky and Rosen in 1935 [73]. In their famous paper, they proposed a thought experiment which was supposed to break down the theory of quantum mechanics. Their goal was to show that quantum mechanics was incomplete and inconsistent with causality. Ironically, it is the same paper which is now one of the fundamentals of the theory.

To introduce the concept of entanglement, let us present a simplified version of the EPR argument. Consider a two-mode squeezed state (see Eq. (2.79)), in the unphysical limit of infinite squeezing, i.e. $r \to \infty$ so that

$$|EPR\rangle \propto \sum_{n=0}^{\infty} |n,n\rangle. \tag{5.1}$$

Using the identity operator $\mathbb{1} = \int dx_1 dx_2 |x_1,x_2\rangle\langle x_1,x_2|$, we can rewrite the state as

$$|EPR\rangle \propto \int dx_1 dx_2 \ |x_1,x_2\rangle \left(\sum_{n=0}^{\infty} \langle x_1|n\rangle \langle x_2|n\rangle \right). \tag{5.2}$$

We see from Eq. (2.25) that the wave function of a Fock state is real, meaning that $\langle x_2|n\rangle = \langle n|x_2\rangle$. Hence,

$$\begin{aligned} |EPR\rangle &\propto \int dx_1 dx_2 \ |x_1,x_2\rangle \left(\sum_{n=0}^{\infty} \langle x_1|n\rangle \langle n|x_2\rangle \right) \\ &= \int dx_1 dx_2 \, |x_1,x_2\rangle \, \langle x_1| \left(\sum_{n=0}^{\infty} |n\rangle\langle n| \right) |x_2\rangle \\ &= \int dx \, |x,x\rangle \end{aligned} \tag{5.3}$$

since $\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1}$ and $\langle x_1|x_2\rangle = \delta(x_1 - x_2)$. This is the state introduced in the EPR paper. The authors then argued as follows. Suppose that one part of the state is given to Alice and the second to Bob and both of them are situated at distant loca-

tions, far enough so that they cannot communicate. Imagine now that Alice measures the position of her part of the state and she obtains x_0 as a result. According to the law of quantum mechanics, we know that their shared state collapses to $|x_0, x_0\rangle$ so that Bob should now on measures x_0 too, meaning that the half of Bob has a well defined position. Suppose instead that Alice measures the momentum of her half of the state and obtained p_0. Using the Fourier transform relation between position and momentum, we can show that the EPR state can also be written as $|EPR\rangle \propto \int dp|p, -p\rangle$ so that the quantum state will collapse to $|p_0, -p_0\rangle$. Once again, it now means that the half of Bob has a well-defined momentum. Remember that we want Bob and Alice to be far away meaning that Alice cannot modify the physical state of Bob's half. Therefore, since Bob is oblivious of Alice's measurement choices, the half of Bob must have been in a state with well-defined position and momentum from the beginning. However, this is not possible, because it violates the Heisenberg uncertainty relation. That is why Einstein, Podolsky and Rosen claimed that quantum mechanics was incomplete or inconsistent.

It was surely not their first motivation, but this argument is famous as the authors were actually the first ones to realize that quantum correlations can go beyond the classical ones. Today, we understand much better this phenomenon, in particular thanks to Bell and his inequalities. He showed in 1964 that the predictions of quantum mechanics cannot be reproduced by a theory of local hidden variables [74]. There is still some holes to fill in order to have a totally coherent interpretation of quantum mechanics, nevertheless, we are not mystified by entanglement anymore and we focus rather on its applications, as entanglement proves to be a useful resource in quantum protocols such as teleportation. One important problem nowadays, however, is to determine whether a state is entangled or separable (that is, not entangled). Indeed, when it comes to mixed states, it is provably a hard decision problem [75]. This is why many scientists are working on separability criteria, i.e. they seek conditions on a state allowing us to say with certainty whether this state is entangled or not. In this chapter, we review important separability criteria for both discrete and continuous variables. This will serve as a starting point to the improved separability criteria presented in the third part of this thesis.

5.1 Entangled states

Let us first give a mathematical definition of separable and entangled states. The definitions below are valid for bipartite states, that is for states that live in a tensor product of two Hilbert spaces.

A pure state $|\psi\rangle$ is *separable* if and only if it can be written as the tensor product of two states

$$|\psi\rangle = |\psi_1\rangle|\psi_2\rangle \tag{5.4}$$

where $|\psi_1\rangle$ lives in the first Hilbert space and $|\psi_2\rangle$ in the second. Otherwise it is *entangled*. A mixed state ρ is *separable* if and only if it can be written as a convex combination of pure product states

$$\rho = \sum_i p_i \rho_1 \otimes \rho_2 = \sum_i p_i |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| \tag{5.5}$$

where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. Otherwise it is *entangled*.

5.2 Discrete-variable separability criteria

The bipartite entanglement of pure states is easy to handle. Indeed, we know that we can define an orthonormal base for each subsystem of any pure bipartite state, $|u_i\rangle$ and $|v_i\rangle$, so that the total state can be written in the *Schmidt decomposition* [76] as

$$|\psi\rangle = \sum_i c_i |u_i\rangle|v_i\rangle \tag{5.6}$$

where $c_i \geq 0$ and $\sum_i c_i^2 = 1$. The number of non-zero values c_i is called the *Schmidt number* of the state. A pure state will be entangled if and only if it has a Schmidt number strictly greater than 1. This implies that two subsystems that partition a pure state are entangled if and only if their reduced states are mixed states. With this in mind, we can understand that the *von Neumann entropy* [77] is suitable to measure entanglement in a bipartite state. Indeed, for a state ρ, its von Neumann entropy is defined as

$$S(\rho) = -\mathrm{Tr}\,(\rho \ln \rho) \tag{5.7}$$

and has the property to be equal to zero if and only if ρ represents a pure state. From the Schmidt decomposition, we have that

$$S(\rho_1) = -\sum_i c_i^2 \ln c_i^2 \tag{5.8}$$

where $\rho_1 = \mathrm{Tr}_2 |\psi\rangle\langle\psi|$ is the reduced state of the first system. $S(\rho_1)$ will be equal to zero if only if the Schmidt number is equal to one, that is if $|\psi\rangle$ is a product state of two pure states (so is not entangled).

When it comes to mixed states, the problem of detecting entanglement is much more complicated. In discrete variables, a widely used separability criterion is the Peres–Horodecki criterion [7, 78], often called PPT criterion for *positive partial transpose*. Its name is due to the fact that the PPT criterion states that if a state is separable its density matrix remains positive after a partial transposition. This condition is generally only necessary and becomes sufficient only for systems of dimension 2×2 and 2×3. Note that the positivity of the density matrix after the partial transposition is simply another way to say that the state remains a physical state since we know that a den-

sity matrix ρ describes a physical state only if $\rho \geq 0$. We will see in Section 5.3 that the idea of verifying the physicality of the partial transposed state will be applied in continuous-variable separability criteria.

Mathematically, the PPT criterion takes the following form. If the initial state is described by the density matrix

$$\rho = \sum_{ijkl} p_{ijkl} |i,j\rangle\langle k,l| \tag{5.9}$$

the partial transposition of the density matrix is given by

$$\rho^{PT} = \sum_{ijkl} p_{ijkl} |i,l\rangle\langle k,j| = \sum_{ijkl} p_{ilkj} |i,j\rangle\langle k,l| \tag{5.10}$$

where we understand that the *partial* transposition means that only part of the state is transposed. Note that we can equivalently partially transpose the first or the second system. The PPT criterion then states that if ρ is separable, then ρ^{PT} has only non-negative eigenvalues (since it must represent a physical state). In other words, if ρ^{PT} has at least one negative eigenvalue, we know that ρ is entangled.

Of course, there exist also other sufficient criteria to detect entanglement. For example, in [79, 80] it was shown that if a state ρ_{AB} is separable, its von Neumann entropy verify

$$S(\rho_{AB}) \geq S(\rho_A) \qquad \text{and} \qquad S(\rho_{AB}) \geq S(\rho_B) \tag{5.11}$$

so that, once again, the violation of these inequalities is a sign of entanglement.

Nielsen and Kempe [81] also developed a condition based on the theory of majorization. They showed that if ρ_{AB} is separable, then $\lambda(\rho_{AB}) \prec \lambda(\rho_A)$ and $\lambda(\rho_{AB}) \prec \lambda(\rho_B)$ where $\lambda(\rho)$ is the eigenvalues vector of ρ and $x \prec y$ means that y majorizes x. As mentioned in the title of their paper, this majorization separability criterion emphasizes the fact that each part of an entangled system may exhibit more disorder than the system as a whole. Note that this criterion is stronger than the one based on the von Neumann entropy since it implies it.

5.3 Continuous-variable separability criteria

5.3.1 Variance-based separability criterion

In the context of continuous-variable systems, the most widespread necessary criterion for the separability of any two-mode state has been derived by Duan *et al.* [8] and Simon [9] simultaneously in 2000. This criterion results from translating the PPT condition, which had been established for finite-dimensional discrete systems,

to continuous-variable (infinite-dimensional) systems. In continuous variables, the partial transposition of a state is simply expressed as a sign inversion of the momentum of one of the subsystems. More precisely, from the definition of the Wigner function, it is possible to show the following equivalence

$$\rho \to \rho^{PT} \quad \Leftrightarrow \quad W(x_1, p_1, x_2, p_2) \to W(x_1, p_1, x_2, -p_2). \tag{5.12}$$

The partial transposition can therefore be understood as a mirror reflection in the phase space or a time reversal on mode 2, which inverts only the p_2 coordinate. As stated by Simon [9], the PPT criterion can thus be translated as follows in the continuous-variable framework: "if ρ is separable, then its Wigner distribution necessarily goes over into a Wigner distribution under the phase space mirror reflection $(x_1, p_1, x_2, p_2) \to (x_1, p_1, x_2, -p_2)$".

Following the notations of Duan *et al.* [8], the separability criterion expresses the fact that if a two-mode state is separable, then its so-called EPR variance Δ complies with the following inequality[1]

$$\Delta \equiv \langle (\Delta \hat{u})^2 \rangle + \langle (\Delta \hat{v})^2 \rangle \geq \alpha^2 + \frac{1}{\alpha^2} \tag{5.13}$$

for any real (nonzero) α, where the operators

$$\hat{u} = |\alpha| \hat{x}_1 + \frac{1}{\alpha} \hat{x}_2 \qquad \text{and} \qquad \hat{v} = |\alpha| \hat{p}_1 - \frac{1}{\alpha} \hat{p}_2 \tag{5.14}$$

are functions of the quadratures components $\hat{x}$ and $\hat{p}$ of modes 1 and 2. Thus, if a state violates inequality (5.13) for at least one value of α, it is entangled.

Both formulations of the criterion, the one of Duan *et al.* and the one of Simon are equivalent. Note that Eq. (5.13) was derived without using the PPT criterion. Nevertheless, one can derive this inequality by applying the PPT criterion on the parametric form of the uncertainty relation introduced in [8] and thus establish the correspondence with the results in [9]. Indeed, let us define the two following operators

$$\hat{u} = |\alpha| \hat{x}_1 + \frac{1}{\alpha} \hat{x}_2 \qquad \text{and} \qquad \hat{v}' = |\alpha| \hat{p}_1 + \frac{1}{\alpha} \hat{p}_2. \tag{5.15}$$

The first one is the same as defined by Duan and the second one has been modified, replacing the minus sign by a plus. From the Robertson uncertainty relation Eq. (4.3), we know that every physical state must obey the following inequality

$$\sqrt{\left\langle (\Delta \hat{u})^2 \right\rangle \left\langle (\Delta \hat{v}')^2 \right\rangle} \geq \frac{1}{2} \left| \left\langle [\hat{u}, \hat{v}'] \right\rangle \right| = \frac{1}{2} \left(\alpha^2 + \frac{1}{\alpha^2} \right). \tag{5.16}$$

[1] Here $\langle (\Delta \hat{u})^2 \rangle = \langle \hat{u}^2 \rangle - \langle \hat{u} \rangle^2$ is the usual variance. The difference in notation between $\langle (\Delta \hat{u})^2 \rangle$ and $\sigma_{\hat{u}}^2$ is simply here to distinguish between separability conditions and elements of the covariance matrix. It is also a notation used in many papers.

In addition, using the identity $x^2 + y^2 \geq 2xy$ (since $(x-y)^2 \geq 0$), we can conclude that any ρ satisfies

$$\left\langle (\Delta\hat{u})^2_\rho \right\rangle + \left\langle (\Delta\hat{v}')^2_\rho \right\rangle \geq 2\sqrt{\left\langle (\Delta\hat{u})^2_\rho \right\rangle \left\langle (\Delta\hat{v}')^2_\rho \right\rangle} \geq \alpha^2 + \frac{1}{\alpha^2}. \tag{5.17}$$

We can now make use of the PPT criterion. If a state ρ is separable, then its partial transposed state ρ^{PT} must also be physical and so, it must verify the same inequality

$$\left\langle (\Delta\hat{u})^2_{\rho^{PT}} \right\rangle + \left\langle (\Delta\hat{v}')^2_{\rho^{PT}} \right\rangle \geq \alpha^2 + \frac{1}{\alpha^2}. \tag{5.18}$$

Doing a partial transpose, on the second mode, simply means that $\hat{p}_2 \to -\hat{p}_2$. Therefore, we have $\left\langle (\Delta\hat{u})^2_{\rho^{PT}} \right\rangle = \left\langle (\Delta\hat{u})^2_\rho \right\rangle$ while $\left\langle (\Delta\hat{v}')^2_{\rho^{PT}} \right\rangle = \left\langle (\Delta\hat{v})^2_\rho \right\rangle$ ($\hat{v}$ is the operator defined by Duan) which gives us the result of equation (5.13).

Remarkably, this separability condition becomes necessary *and* sufficient in the case of Gaussian states. One can show that every covariance matrix of a Gaussian state can be transformed by symplectic transformations into the *standard form* [8]

$$\gamma = \begin{pmatrix} n_1 & & c_1 & \\ & n_2 & & c_2 \\ c_1 & & m_1 & \\ & c_2 & & m_2 \end{pmatrix} \tag{5.19}$$

where

$$\begin{aligned} \frac{n_1 - 1}{m_1 - 1} &= \frac{n_2 - 1}{m_2 - 1}, \\ |c_1| - |c_2| &= \sqrt{(n_1 - 1)(m_1 - 1)} - \sqrt{(n_2 - 1)(m_2 - 1)}. \end{aligned} \tag{5.20}$$

Then, by introducing the following operators, related to the covariance matrix

$$\hat{u} = \alpha_0 \hat{x}_1 - \frac{c_1}{|c_1|}\frac{1}{\alpha_0}\hat{x}_2, \qquad \hat{v} = \alpha_0 \hat{p}_1 - \frac{c_2}{|c_2|}\frac{1}{\alpha_0}\hat{p}_2 \tag{5.21}$$

where

$$\alpha_0^2 = \sqrt{\frac{m_1 - 1}{n_1 - 1}} = \sqrt{\frac{m_2 - 1}{n_2 - 1}} \tag{5.22}$$

we can express a necessary and sufficient separability criterion: a Gaussian state is separable if and only if

$$\Delta \geq \alpha_0^2 + \frac{1}{\alpha_0^2} \qquad \Leftrightarrow \qquad \alpha_0^2 \frac{n_1 + n_2}{2} + \frac{m_1 + m_2}{2\alpha_0^2} - |c_1| - |c_2| \geq \alpha_0^2 + \frac{1}{\alpha_0^2}. \tag{5.23}$$

This separability condition can even be extended to (N+M)-mode Gaussian states [82].

Note that from Eq. (5.16), it is possible to derive another separability criterion. In-

deed, using again that $\left\langle (\Delta\hat{u})^2_{\rho^{PT}} \right\rangle = \left\langle (\Delta\hat{u})^2_{\rho} \right\rangle$ and $\left\langle (\Delta\hat{v}')^2_{\rho^{PT}} \right\rangle = \left\langle (\Delta\hat{v})^2_{\rho} \right\rangle$ after a partial transposition, we can write that any separable state must verify the inequality

$$\Delta' \equiv 2\sqrt{\left\langle (\Delta\hat{u})^2_{\rho} \right\rangle \left\langle (\Delta\hat{v})^2_{\rho} \right\rangle} \geq \alpha^2 + \frac{1}{\alpha^2}. \tag{5.24}$$

This separability criterion was obtain by Mancini *et al.* in 2002 [83]. According to Eq. (5.17), this criterion is stronger than the one of Duan *et al.* Nevertheless, both criteria must coincide for Gaussian states since the Duan-Simon criterion is necessary and sufficient in this case. To see it, let us take a Gaussian state. If $\Delta' < \alpha^2 + \frac{1}{\alpha^2}$ then we know that the state is entangled. If however, $\Delta' \geq \alpha^2 + \frac{1}{\alpha^2}$, according to Eq. (5.17), we can also say that the sum of the variances Δ is greater than $\alpha^2 + \frac{1}{\alpha^2}$. Yet, for Gaussian states, we showed that $\Delta \geq \alpha^2 + \frac{1}{\alpha^2}$ means that the state is separable (see Eq. (5.23)). Condition (5.24) is thus necessary and sufficient for Gaussian states.

What we must remember from those separability criteria is that a state is entangled if its partial transposed state is not a valid quantum state. This means that the main point is to check the physicality of the partial transposed state through uncertainty relations. Duan *et al.* used the Robertson uncertainty relation (4.3), while Simon based his argument on the uncertainty relation Eq. (4.9), but we saw that it boils down to verify that Eq. (4.13), based on the symplectic values, is respected. Let us study the specific cases of two families of Gaussian states whose covariance matrix is written in a second standard form [8]

$$\gamma = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}. \tag{5.25}$$

Of course, it is always possible to study the symplectic values of this matrix, but in the specific cases when 1) $a = b$ and $c_1 = \pm c_2 = c$; 2) $c_1 = \pm c_2 = c$, they take a simple form which allows us to easily study the separability of the underlying states.

Example 1: Symmetric Gaussian states, $a = b$ and $c_1 = \pm c_2 = c$.

A Gaussian state with covariance matrix

$$\gamma = \begin{pmatrix} \gamma_A & C \\ C^T & \gamma_B \end{pmatrix} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & \pm c \\ c & 0 & a & 0 \\ 0 & \pm c & 0 & a \end{pmatrix} \tag{5.26}$$

(we choose $c \geq 0$ without loss of generality) will be physical if the symplectic values are greater than $1/2$. Note that the local covariance matrices must also represent a physical state, meaning that $\det \gamma_A = \det \gamma_B = a^2 \geq 1/4$. Since covariance matrices

must be such that $\gamma \geq 0$ (i.e. the eigenvalues must be positive), we need to impose $a \geq 1/2$. The eigenvalues of γ are $a \pm c$. They are always non-negative if $a \geq c$. The symplectic values are computed with the help of Eq. (2.45) and we find

$$\nu_{\pm} = \begin{cases} a \pm c & \text{if } \det C \geq 0 \\ \sqrt{a^2 - c^2} & \text{if } \det C \leq 0 \end{cases} \tag{5.27}$$

so that γ is the covariance matrix of a valid quantum state if $\nu_{\pm} \geq 1/2$.

Let us see now if the state is separable or not. Note that if $\det C \geq 0$, the covariance matrix of the partial transposed state is also given by Eq. (5.26) but with $\det C \leq 0$. In this case, the state will be separable if the symplectic values of the partial transposed state $\nu_{\pm}^{PT}$ are greater than $1/2$ (remember that the criterion is necessary and sufficient for Gaussian states), i.e. if $\nu_{\pm}^{PT} = \sqrt{a^2 - c^2} \geq 1/2$ (according to Eq. (5.27) for $\det C \leq 0$). From Eq. (5.27), we see that $\nu_+\nu_- = a^2 - c^2$ where $\nu_{\pm}$ are the symplectic values of the states before the partial transposition. Therefore,

$$\nu_{\pm}^{PT} = \sqrt{a^2 - c^2} = \sqrt{\nu_+\nu_-} \geq \nu_- \geq \frac{1}{2}. \tag{5.28}$$

This state is thus always separable.

If now $\det C \leq 0$, the covariance matrix of the partial transposed state is also given by Eq. (5.26) but with $\det C \geq 0$. In this case, the state is separable if $\nu_{\pm}^{PT} = a \pm c \geq 1/2$ or simply if $a - c \geq 1/2$, since $a + c \geq 1/2$ is always true. If $a - c \leq 1/2$ the state is entangled.

Example 2: Non-symmetric Gaussian states, $c_1 = \pm c_2 = c$.

We now take a Gaussian state with γ_A and γ_B different so that the covariance matrix is given by

$$\gamma = \begin{pmatrix} \gamma_A & C \\ C^T & \gamma_B \end{pmatrix} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & \pm c \\ c & 0 & b & 0 \\ 0 & \pm c & 0 & b \end{pmatrix} \tag{5.29}$$

(we choose $c \geq 0$ without loss of generality). Once again, the local covariance matrices must also represent a physical state, so we need to impose $a \geq 1/2$ and $b \geq 1/2$. The eigenvalues of γ needing to be positive, we have the additional constraint $ab \geq c^2$. Finally, The symplectic values are given by

$$\nu_{\pm} = \begin{cases} \frac{1}{2}(a + b \pm \sqrt{(a-b)^2 + 4c^2}) & \text{if } \det C \geq 0 \\ \frac{1}{2}(\sqrt{(a+b)^2 - 4c^2} \pm (a - b) & \text{if } \det C \leq 0 \end{cases} \tag{5.30}$$

so that γ is the covariance matrix of a valid quantum state if $\nu_\pm \geq 1/2$. This adds the physicality conditions

$$\begin{aligned} (a-1)(b-1) &\geq c \qquad \text{if } \det C \geq 0 \\ (a+1)(b-1) &\geq c \qquad \text{if } \det C \leq 0. \end{aligned} \tag{5.31}$$

As in the previous example, if $\det C \geq 0$, the covariance matrix of the partial transposed state is given by Eq. (5.29) but with $\det C \leq 0$ and vice versa. One can then check that if $\det C \geq 0$, the state is always separable since $\tilde{\nu}_\pm \geq 1/2$ implies $(a+1)(b-1) \geq c$ which is always respected since $(a+1)(b-1) \geq (a-1)(b-1)$ and $(a-1)(b-1) \geq c$ according to Eq. (5.31) (for a state with $\det C \geq 0$). If $\det C \leq 0$, the state is separable if $(a-1)(b-1) \geq c$ and entangled otherwise.

5.3.2 Entropy-based separability criterion

Another important separability criterion is due to Walborn *et al.* [84] in 2009. The idea, once again, is to check if the state we obtain after a partial transposition is still a physical (valid) state. However, while the previous criteria used the variance-based uncertainty relations to see if the covariance matrix is physical or not, Walborn *et al.* base their criterion on the entropic uncertainty relation of Białynicki-Birula and Mycielski, Eq. (4.39). Since, as we showed, this entropic uncertainty relation is stronger than the variance-based one (see Chapter 4), this separability criterion is stronger in the sense that it allows us to detect more non-Gaussian entangled states.

Following the notation in [84], we define

$$r_j = \cos\theta_j x_j + \sin\theta_j p_j \qquad \text{and} \qquad s_j = \cos\theta_j p_j - \sin\theta_j x_j \tag{5.32}$$

for $j = 1, 2$ as well as

$$r_\pm = r_1 \pm r_2 \qquad \text{and} \qquad s_\pm = s_1 \pm s_2. \tag{5.33}$$

To each of these observables, we can associate a Shannon differential entropy (Eq. (3.9))

$$h(r) = -\int dr R(r) \ln R(r) \tag{5.34}$$

where $R(r)$ is the probability distribution of r. In particular, we have

$$h(r_\pm) = -\int dr_\pm R_\pm(r_\pm) \ln R_\pm(r_\pm) \tag{5.35}$$

with

$$\begin{aligned} R_{\pm}(r_{\pm}) &= \frac{1}{2}\int dr_{\mp} R_1\left(\frac{r_+ + r_-}{2}\right) R_2\left(\frac{r_+ - r_-}{2}\right) \\ &= \int dr R_1(r) R_2(\mp r \pm r_{\pm}) \end{aligned} \tag{5.36}$$

and similarly for $h(s_{\pm})$. The separability criterion reads thus as follows: any separable state must verify the inequality

$$h(r_{\pm}) + h(s_{\mp}) \geq \ln(2\pi e) \tag{5.37}$$

meaning once again that if this relation is violated, the state is entangled. This separability criterion is equivalent to the Duan-Simon one for Gaussian states.[2]

Note that the power of this entropic separability criteria lies in the fact that one can optimize over the θ_j in order to find the most restrictive inequality and so to detect a maximum of entangled states. However it is double-edged since it requires more complex numerical computations. In Chapter 12, we suggest a new separability criterion which allows us to detect the same amount of entangled states, but without the use of any optimization.

5.3.3 Other separability criteria

With the exception of the special case of Gaussian states, separability criteria are only necessary conditions which means that many entangled states are left undetected by the previous criteria. Earlier work has aimed at improving the actual separability criterion for arbitrary states. Let us mention, for example, Shchukin and Vogel who derived a hierarchy of inequalities involving higher-order moments of the quadrature components [85] (the Duan-Simon criterion only depends on the second-order moments). However, many entangled state are still left undetected. In the third part of this thesis, we propose two improved separability criteria. In Chapter 11, the detection of entangled states is enhanced by taking into account an additional parameter, namely the degree of Gaussianity of the state, while in Chapter 12, the improvement is due to the fact that we use our improved entropic uncertainty relation of Chapter 6.

[2] To see it, one need to use the fact that $h(r_{\pm}) = \ln(2\pi e\sigma^2_{r_{\pm}})/2$ for a Gaussian state (and similarly for $h(s_{\mp})$) and then isolate the variances. We then retrieve Eq. (5.24) which we proved to be equivalent to the Duan-Simon criterion for Gaussian states.

Part II | Continuous-variable uncertainty relations

6 | Tight entropic uncertainty relation for canonically conjugate variables

This chapter is the subject of the following article:
A. Hertz, M. G. Jabbour, and N. J. Cerf, J. Phys. A **50** 385301 (2017) [b].

In Chapter 4, we mentioned that the entropic uncertainty relation due to Białynicki-Birula and Mycielski [11]

$$h(x) + h(p) \geq \ln(\pi e) \tag{6.1}$$

is not invariant under all symplectic transformations and is not saturated by all pure Gaussian states. In this chapter, we investigate whether tighter entropic uncertainty relations can be derived, which, by taking correlations into account, are saturated for *all* Gaussian pure states. The idea is to make an analogy with the Schrödinger-Robertson uncertainty relation which is saturated by all pure Gaussian states and is invariant under all symplectic transformations. To reach this goal, we make use of *entropy powers*, Eq. (3.24), which, we believe are more suitable to describe entropic uncertainty relations and especially to emphasize the homology with variance-based uncertainty relations.

In the next section, we first review variance- and entropy-based uncertainty relations, and then define what we coin the *entropy-power uncertainty relation* for a pair of canonically-conjugate variables, namely $N_x\, N_p \geq 1/4$, where N_x and N_p are entropy powers. It trivially implies the Heisenberg relation as a simple consequence of the definition of entropy power (actually, they coincide for Gaussian states). Then, we suggest an extended form of the entropy-power uncertainty relation, which would be stronger than the regular form for rotated variables as it builds on the covariance matrix γ. It reads

$$N_x\, N_p \geq \frac{1}{4}\, \frac{\sigma_x^2\, \sigma_p^2}{\det \gamma} \tag{6.2}$$

where σ_x^2 and σ_p^2 are variances. This inequality is proven by making use of variational calculus, conditionally on two assumptions: pure Gaussian states are the global minimizers of the uncertainty functional and this uncertainty functional is concave. We also suggest an extended version of the above entropy-power (or entropic) uncertainty relation that is valid for n modes and is saturated for all n-mode Gaussian pure states. Finally, we also conduct extensive numerical tests in order to illustrate the validity of our extended uncertainty relations and conjectured concavity.

6.1 Entropy-power uncertainty relations

Variance-based and entropic uncertainty relations were already introduced in Chapter 4. Here, we will show that it is possible to rewrite the entropic uncertainty relation (6.1) in a form similar to the one expressed in terms of variances, provided we make use of the notion of entropy power. We remember from Eq. (3.24) that entropy powers of the x- and p-quadratures are defined as

$$N_x = \frac{1}{2\pi e} e^{2h(x)}, \qquad N_p = \frac{1}{2\pi e} e^{2h(p)}, \tag{6.3}$$

and we have $N_x = \sigma_x^2$ and $N_p = \sigma_p^2$ if and only if the x- and p-quadratures are Gaussian distributed. Thus, Eq. (6.1) can be simply reexpressed as

$$N_x N_p \geq \frac{1}{4} \tag{6.4}$$

which is what we call an *entropy-power uncertainty relation* for a pair of canonically-conjugate variables, as presented in the introduction. It closely resembles the Heisenberg relation (4.2), but with entropy powers instead of variances.

Since $N_x \leq \sigma_x^2$ and $N_p \leq \sigma_p^2$, which reflects the fact that the Gaussian distribution maximizes the entropy for a fixed variance, we have the chain of inequalities

$$\sigma_x^2 \sigma_p^2 \geq N_x N_p \geq \frac{1}{4}. \tag{6.5}$$

Hence, the entropy-power uncertainty relation implies the Heisenberg uncertainty relation, and they coincide for Gaussian x- and p-distributions (we mentioned in Section 4.2 that this implication was already derived in [11]). This can also be connected to relative entropies as a measure of non-Gaussianity. From the definition of N_x and N_p, we get

$$h(x) = \frac{1}{2}\ln(2\pi e N_x), \qquad h(p) = \frac{1}{2}\ln(2\pi e N_p) \tag{6.6}$$

which implies that

$$D(x||x_G) = \frac{1}{2}\ln\left(\frac{\sigma_x^2}{N_x}\right), \qquad D(p||p_G) = \frac{1}{2}\ln\left(\frac{\sigma_p^2}{N_p}\right) \tag{6.7}$$

or equivalently

$$\sigma_x^2 = N_x\, e^{2\,D(x||x_G)}, \qquad \sigma_p^2 = N_p\, e^{2\,D(p||p_G)}\,. \tag{6.8}$$

It is clear that Eq. (6.4) becomes more stringent than Heisenberg's uncertainty relation, Eq. (4.2) as soon as we deviate from a Gaussian state.

6.2 Extended forms of entropic uncertainty relations

6.2.1 Motivation

Our goal is to address the problem that, unlike the Schrödinger-Robertson uncertainty relation, the entropic uncertainty relation (6.1) – or equivalently the entropy-power uncertainty relation (6.4) – is not saturated by all pure Gaussian states but only by those whose principal axes are aligned with the x- and p-axes. In other words, we would like to make Eq. (6.1) or (6.4) depend on the possible correlations between x and p (as witnessed, for instance, by $\sigma_{xp} \neq 0$). Ideally, the new inequality should have the property of being invariant under all Gaussian unitary transformations (displacements and symplectic transformations) and being saturated by all pure Gaussian states, regardless of the orientation of the principal axes.

A first natural idea is to make use of the joint differential entropy, which is defined as

$$h(x,p) = -\int f(x,p)\ln f(x,p)\,\mathrm{d}x\,\mathrm{d}p \tag{6.9}$$

where $f(x,p)$ is the joint probability density of the random variables x and p. As mentioned in Chapter 3, the joint entropy can also be expressed as $h(x,p) = h(x) + h(p) - I(x{:}p)$ where $I(x{:}p) \geq 0$ is the mutual information. Thus, one may think of improving the entropic uncertainty relation (6.1) by replacing $h(x)+h(p)$ with $h(x,p)$. Moving the mutual information $I(x{:}p)$ on the right-hand side of the inequality, it thus corresponds to an improvement of the lower bound. Moreover, $h(x,p)$ has the desired property of invariance under symplectic transformations. Indeed, if we transform the coordinates according to $(x'\ p')^T = S\cdot(x\ p)^T$, where S is the transformation matrix, the joint differential entropy transforms as[1]

$$h(x',p') = h(x,p) + \ln|\det S|. \tag{6.10}$$

[1] See Section 3.2.2.

Thus, if S corresponds to a symplectic transformation, $|\det S| = 1$, then the joint differential entropy remains invariant. Furthermore, $h(x,p)$ is also invariant under (x,p)-displacement, so it looks like a good uncertainty functional.

However, we deal with quantum states, so the Wigner function $W(x,p)$ is not a genuine probability density and may admit negative values. Hence, the joint differential entropy of $W(x,p)$ is not always defined (one would need to compute the logarithm of negative values), and so is the mutual information $I(x{:}p)$. Nevertheless, in Chapter 9 we will conjecture an entropic uncertainty relation, based on the joint probability distribution, valid only for states with positive Wigner functions:

$$h(x,p) \geq \ln(\pi e) \qquad \forall \text{ states s.t. } W(x,p) \geq 0. \tag{6.11}$$

This conjecture can equivalently be written as

$$h(x) + h(p) \geq \ln(\pi e) + I(x{:}p) \qquad \forall \text{ states s.t. } W(x,p) \geq 0. \tag{6.12}$$

which corresponds to lifting up the lower bound of the ordinary relation (6.1).

6.2.2 Tight uncertainty relation saturated by all pure Gaussian states

Equations (6.11) or (6.12) are not valid for states with negative Wigner functions, but they give us a hint on how to proceed in order to derive an entropic uncertainty relation that is valid for all states and takes correlations into account. While the joint entropy and mutual information are not defined for all states, they are well defined for Gaussian states (since their Wigner function is always positive). In particular, the Gaussian mutual information is expressed as a function of the covariance matrix,

$$I_G(x{:}p) = \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right) \geq 0. \tag{6.13}$$

We obtain our tight[2] entropic uncertainty relation simply by substituting $I(x{:}p)$ with $I_G(x{:}p)$ in Eq. (6.11) or (6.12), namely

$$h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right) \geq \ln(\pi e). \tag{6.14}$$

This inequality is meaningful for all states, regardless of whether the Wigner function is positive everywhere or not. Unlike Eq. (6.11), however, it is not invariant under rotations. In the next section, conditionally on two assumptions, we will prove inequality (6.14) using a variational method. Assumptions concern the concavity of the uncertainty functional and the fact that Gaussian pure states are global minima. They are not proven but are reasonable assumptions since they hold for the usual entropic

[2]By tight, we mean that the inequality is saturated by all pure Gaussian states.

uncertainty relation, Eq. (6.1). Moreover, we have numerical evidence confirming they should hold.

Note that $I_G(x{:}p)$ vanishes if the principal axes of the covariance matrix are the x- and p-axes, i.e. $\sigma_{xp} = 0$, so that Eq. (6.14) reduces to the regular entropic uncertainty relation (6.1) in this case.

As before, it is useful to rewrite our new relation in terms of entropy powers, resulting in

$$N_x\, N_p \geq \frac{1}{4}\,\frac{\sigma_x^2\,\sigma_p^2}{\det\gamma} \tag{6.15}$$

which can be viewed as an improved version of the entropy-power uncertainty relation (6.4), where the lower bound $1/4$ is lifted up when the principal axes differ from the x- and p-axes ($\sigma_{xp} \neq 0$). If the principal axes correspond to the x- and p-axes, we recover Eq. (6.4). Alternatively, we may also reexpress our new relation as

$$\frac{N_x\, N_p}{\sigma_x^2\,\sigma_p^2}\ \det\gamma \geq \frac{1}{4}. \tag{6.16}$$

Then, using $N_x \leq \sigma_x^2$ and $N_p \leq \sigma_p^2$, we see that our tight entropy-power inequality (6.15) implies the Robertson-Schrödinger uncertainty relation, namely

$$\det\gamma \geq \frac{N_x\, N_p}{\sigma_x^2\,\sigma_p^2}\ \det\gamma \geq \frac{1}{4}. \tag{6.17}$$

These two inequalities coincide for Gaussian x- and p-distributions. Furthermore, they are both saturated for pure Gaussian states regardless of the orientation of the principal axes (since $\det\gamma = 1/4$ and $N_x = \sigma_x^2$, $N_p = \sigma_p^2$).

In addition, we may reexpress Eq. (6.15) as

$$\det\gamma \geq \frac{1}{4}\,\frac{\sigma_x^2\,\sigma_p^2}{N_x\, N_p} \tag{6.18}$$

which can be viewed as an improved version of the Robertson-Schrödinger uncertainty relation where the lower bound $1/4$ is lifted up when the x- and p-distributions deviate from Gaussian distributions. In terms of non-Gaussianity measures based on relative entropies, it transforms into

$$D(x||x_G) + D(p||p_G) \leq \ln\left(\frac{\sqrt{\det\gamma}}{1/2}\right). \tag{6.19}$$

which is the counterpart of Eq. (4.42) but having replaced $\sigma_x^2\,\sigma_p^2$ with $\det\gamma$, just as we do when going from the Heisenberg to the Robertson-Schrödinger relation. It also

corresponds to a stronger version of Eq. (4.43), which reads

$$\det \gamma \geq \frac{1}{4} e^{2D(x||x_G)+2D(p||p_G)}. \tag{6.20}$$

To be complete, let us mention that we can express our tight entropic uncertainty relation (6.14) as

$$h(x) + h(p) \geq h(x_G) + h(p_G) + \ln(\mu_G) \tag{6.21}$$

where x_G (p_G) is Gaussian distributed with variance σ_x^2 (σ_p^2) and $\mu_G = \mathrm{tr}\rho_G^2$ is the purity of the Gaussian state ρ_G associated to the covariance matrix γ.

6.2.3 Numerical evidence of relation (6.14)

Before presenting the proof of our conjecture, let us mention that we have conducted numerical tests in order to verify the correctness of the tight entropic uncertainty relation. This was especially important since our proof relies on two assumptions that are not proven.

For numerical purposes, it was simpler to consider the uncertainty relation in its form with differential entropies, Eq. (6.14). First, we have considered random pure states, which we generated by applying a random unitary transformation to the vacuum state. In Figure 6.1, each blue dot corresponds to $h(x) + h(p)$ as computed for a random state generated with a 4×4 unitary matrix (each state belongs to the space spanned by the Fock states $|n\rangle$ with $n = 0, 1, 2, 3$). The dashed red curve represents the previous bound $\ln(\pi e)$ while the plain red curve represents the improved lower bound on $h(x) + h(p)$ that results from Eq. (6.14), namely $\ln(\pi e) + I_G(x{:}p)$. Here, the Gaussian mutual information is expressed as

$$I_G(x : p) = -\frac{1}{2} \ln\left(1 - \rho^2\right) \tag{6.22}$$

where $\rho = \sigma_{xp}/(\sigma_x \sigma_p)$ stands for the correlation parameter. We clearly see that all points lie above the improved lower bound, corroborating the new entropic uncertainty relation (6.14). Note that other tests have been carried out with unitary transformations of greater dimensions, but this generally yields states with greater values of $h(x) + h(p)$, which are less interesting for verification purposes.

As a more stringent test, we have computed $h(x) + h(p)$ for some slightly non-Gaussian pure states lying in the neighborhood of the Gaussian pure states that saturate the uncertainty relation. To do so, we have generated states of the form $|\psi\rangle \propto (|s\rangle + \epsilon|\phi\rangle)$ where $|s\rangle$ is a squeezed state, $|\phi\rangle$ is any other pure state and $\epsilon \ll 1$. In Figure 6.2, we have chosen $|\phi\rangle$ as some random pure state generated by the above method, $\epsilon = 0.01$, and a squeezed state $|s\rangle$ along an axis rotated by an angle of $\theta = \pi/4$ with the x-axis (with a squeezing parameter $s \equiv e^r = 1.5$). Its wave function

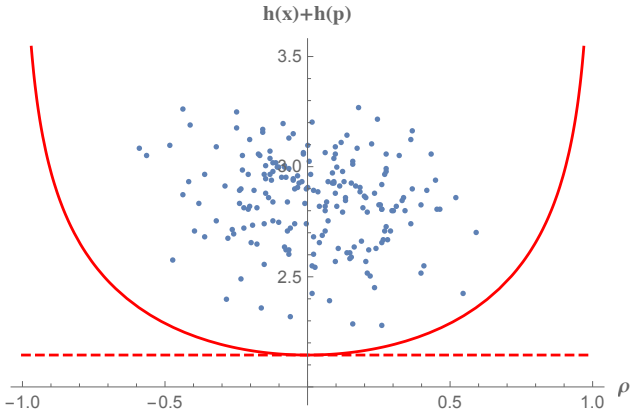

Figure 6.1: Test of the tight entropic uncertainty relation (6.14) for random pure states generated by applying a 4×4 random unitary onto the vacuum state. The blue dots correspond to $h(x) + h(p)$, the plain red curve represents the improved lower bound $\ln(\pi e) + I_G(x{:}p)$ and the dashed red curve shows the previous bound $\ln(\pi e)$. All quantities are plotted as a function of the correlation coefficient ρ.

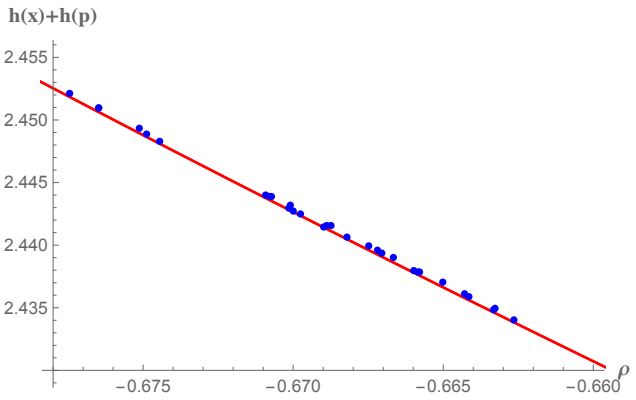

Figure 6.2: Test of the tight entropic uncertainty relation (6.14) for slightly non-Gaussian states of the form $|\psi\rangle \propto (|s\rangle + \epsilon|\phi\rangle)$ where $|s\rangle$ is a squeezed state (with $s = 1.5$) along an axis rotated by an angle of $\theta = \pi/4$, $|\phi\rangle$ is a random pure state as in Figure 6.1, and $\epsilon = 0.01$. The blue dots correspond to $h(x) + h(p)$, while the red curve represents the improved lower bound $\ln(\pi e) + I_G(x{:}p)$. All quantities are plotted as a function of the correlation coefficient ρ. A zoom in of the interesting region is shown in this figure .

has the form

$$\langle x|s\rangle = \sqrt[4]{\frac{2s^2}{\pi\left(s^4+1\right)}}\exp\left(\frac{i\left(s^2+i\right)x^2}{2\left(s^2-i\right)}\right) \tag{6.23}$$

which is non-Gaussian, implying that it cannot saturate the ordinary entropic uncertainty relation (6.1). Be aware that the Wigner function of $|s\rangle$ is of course still Gaussian, so that this state is Gaussian. We have verified that, even if they lie very close to the boundary, all states $|\psi\rangle$ verify the tight entropic uncertainty relation. Similar simulations have also been performed with squeezed states of different parameters and with different values of ϵ, yet no counterexample was found.

6.3 Conditional proof of relation (6.14)

We will now give a conditional proof of our tight entropic uncertainty relation (6.14). We first prove it for Gaussian states, since no hypothesis is needed in this case and then prove it for all states by using a variational method, in analogy to the procedure used in Ref. [86, 87]. This proof, however, will be based on two assumptions. More precisely, we will prove that any squeezed vacuum state rotated by an arbitrary angle is an extremum of the uncertainty functional

$$F(\hat{\rho}) = h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right). \tag{6.24}$$

Since $F(\hat{\rho})$ is invariant under (x,p)-displacements, it follows that all Gaussian pure states are extrema too. The first assumption is that these extrema are global minima of our functional. The second assumption is that the uncertainty function $F(\hat{\rho})$ is concave. This allows us to conclude that relation (6.14) is also valid for mixed states. We know that both assumptions prevail for the regular entropic uncertainty relation (6.1) as well as for our conjectured relation (6.11), so the above assumptions are very natural.

6.3.1 Special case of Gaussian states

In the case of Gaussian states, we can prove Eq. (6.14) without any assumptions.

Theorem 1. *Let ρ^G be a Gaussian state with covariance matrix $\gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_p^2 \end{pmatrix}$ and differential entropies $h(x) = \frac{1}{2}\ln(2\pi e\sigma_x^2)$ and $h(p) = \frac{1}{2}\ln(2\pi e\sigma_p^2)$. Then it satisfies the entropic uncertainty relation, Eq. (6.14) and the saturation is obtained if and only if ρ^G is pure.*

Proof. By simple calculation we have

$$\begin{aligned} h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right) &= \frac{1}{2}\ln(2\pi e\sigma_x^2) + \frac{1}{2}\ln(2\pi e\sigma_p^2) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right) \\ &= \ln(\pi e) + \frac{1}{2}\ln(4\det\gamma) \\ &\geq \ln(\pi e) \end{aligned} \tag{6.25}$$

since $\det\gamma \geq 1/4$ according to Robertson-Schrödinger uncertainty relation, Eq. (4.7). This proves Eq. (6.14) for Gaussian states. This inequality is saturated if and only if ρ^G is pure since $\det\gamma = 1/4$ only for pure Gaussian states. □

6.3.2 General case

The two assumptions

The proof of Eq. (6.14) for any state relies on two assumptions that we cannot prove. We however explain here why we believe they are natural assumptions.

Assumption 1. *Pure Gaussian states are global minimizers of the uncertainty functional $F(\hat{\rho})$, Eq. (6.24).*

First of all, note that since $F(\hat{\rho})$ is invariant under (x, p)-displacements, it is enough to consider only squeezed vacuum state rotated by an arbitrary angle.

This hypothesis is verified in Theorem 1 for Gaussian states since indeed the minimum is attained by all pure Gaussian states. We know that this hypothesis also holds for the original entropic uncertainty relation of Białynicki-Birula and Mycielski, Eq. (6.1). Finally, the numerical evidence of Section 6.2.3 also confirms that pure Gaussian states are global minima of the uncertainty functional.

Assumption 2. *The uncertainty functional $F(\hat{\rho})$, Eq. (6.24) is concave.*

This assumptions is important as it allows us to extend our proof to mixed states. As an example, the original entropic uncertainty relation (6.1) was first proven for pure states in [11]. However, since the differential entropy is a concave function of the probability distribution, it is valid for mixed states as well (as mentioned in [11]). Decomposing a mixed state into pure states, the concavity implies that pure states are the "worst cases", i.e., the lowest value of the functional $h(x) + h(p)$.

Naturally, we also need to investigate the concavity of our new uncertainty functionals. Unfortunately, it seems hard to prove the concavity of the uncertainty functional $F(\hat{\rho})$ of Eq. (6.24) which appears on the left-hand side of the tight entropic uncertainty relation (6.14). This is because while $h(x)$ and $h(p)$ are concave, $I_G(x{:}p)$ is not convex. And while it is known that $\log(\det\gamma)$ is concave [31], nothing can be said about $\log(\sigma_x^2\sigma_p^2)$. Nevertheless, numerical tests corroborate the fact that $F(\hat{\rho})$ is a concave function of the state. As an example, we have analyzed mixtures of two pure states of the form $\lambda|\psi_1\rangle\langle\psi_1| + (1-\lambda)|\psi_2\rangle\langle\psi_2|$, with $0 \leq \lambda \leq 1$. In Figure 6.3, we have numerically verified that $F(\lambda|\psi_1\rangle\langle\psi_1| + (1-\lambda)|\psi_2\rangle\langle\psi_2|) \geq \lambda F(|\psi_1\rangle\langle\psi_1|) + (1-\lambda)F(|\psi_2\rangle\langle\psi_2|)$.

Interestingly, we can prove the concavity of $F(\hat{\rho})$ in some special cases by using the expression of the entropic uncertainty relation in terms of non-Gaussianity measures based on relative entropies, Eq. (6.19). We consider the mixture of two states that have the same first- and second-order moments. Hence, the right-hand side term of

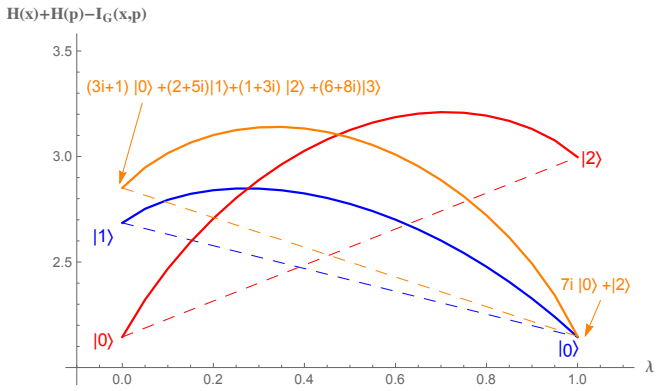

Figure 6.3: Test of the concavity of the uncertainty functional $F(\hat{\rho})$ defined in relation (6.14). We consider three different binary mixtures tuned by parameter λ: $\lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|$, $\lambda|2\rangle\langle 2| + (1-\lambda)|0\rangle\langle 0|$, and $\lambda|\psi\rangle\langle\psi| + (1-\lambda)|\phi\rangle\langle\phi|$, where $|\psi\rangle = 7i|0\rangle + |2\rangle$ and $|\phi\rangle = (1+3i)|0\rangle + (2+5i)|1\rangle + (1+3i)|2\rangle + (6+8i)|3\rangle$.

Eq. (6.19) is constant and we need to prove that

$$\begin{aligned} D(\lambda x_1 + (1-\lambda)x_2 \,||\, [\lambda x_1 + (1-\lambda)x_2]_G) \\ \leq \lambda D(x_1 \,||\, [x_1]_G) + (1-\lambda) D(x_2 \,||\, [x_2]_G) \end{aligned} \tag{6.26}$$

where $[x]_G$ means that we take the Gaussian distribution that leads to the same variance as the probability distribution of x (of course, we have an identical inequality for p). By comparison, the convexity of the relative entropy implies that

$$\begin{aligned} D(\lambda x_1 + (1-\lambda)x_2 \,||\, \lambda[x_1]_G + (1-\lambda)[x_2]_G) \\ \leq \lambda D(x_1 \,||\, [x_1]_G) + (1-\lambda) D(x_2 \,||\, [x_2]_G) \end{aligned} \tag{6.27}$$

which is equivalent to the previous inequality since we mix distributions with the same first- and second-order moments.

Note that the uncertainty relation (6.14) is invariant under displacements so that, without loss of generality, we only need to consider states with zero mean values. Thus, we have proven the concavity of $F(\hat{\rho})$ when two states with the same covariance matrix are mixed. Yet, in the general case, we have not been able to prove concavity.

Proof conditional on the assumptions

Based on the two previous assumptions, we can now state our main theorem.

Theorem 2. *If Assumptions 1 and 2 hold, then any state ρ with covariance matrix $\gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{xp} & \sigma_p^2 \end{pmatrix}$ and differential entropies $h(x)$ and $h(p)$ satisfies the entropic uncertainty relation*

$$h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right) \geq \ln(\pi e). \tag{6.28}$$

Proof. We first prove the theorem for pure states. Let us seek a pure state $|\psi\rangle$ that minimizes the functional $F(|\psi\rangle\langle\psi|)$. For this, we use the Lagrange multiplier method and insert the normalization of $|\psi\rangle$ as a constraint. Since $F(|\psi\rangle\langle\psi|)$ is invariant under displacements, we may also impose without loss of generality the constraint that mean values vanish, $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$. We define

$$J = F(|\psi\rangle\langle\psi|) + \lambda(\langle\psi|\psi\rangle - 1) + \mu\langle\psi|\hat{x}|\psi\rangle + \nu\langle\psi|\hat{p}|\psi\rangle \tag{6.29}$$

where λ, μ and ν are Lagrange multipliers. Since we impose the state $|\psi\rangle$ to be normalized and centered at the origin, we can express the second-order moments as $\sigma_x^2 = \langle\psi|\hat{x}^2|\psi\rangle$, $\sigma_p^2 = \langle\psi|\hat{p}^2|\psi\rangle$, and $\sigma_{xp} = \frac{1}{2}\langle\psi|\{\hat{x},\hat{p}\}|\psi\rangle$, so that we may replace the functional $F(|\psi\rangle\langle\psi|)$ in J by

$$\tilde{F}(|\psi\rangle\langle\psi|) = h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\langle\psi|\hat{x}^2|\psi\rangle\langle\psi|\hat{p}^2|\psi\rangle}{\langle\psi|\hat{x}^2|\psi\rangle\langle\psi|\hat{p}^2|\psi\rangle - \frac{1}{4}\langle\psi|\{\hat{x},\hat{p}\}|\psi\rangle^2}\right). \tag{6.30}$$

Now, in order to solve the variational equation

$$\frac{\partial J}{\partial\langle\psi|} = 0 \tag{6.31}$$

we start by expressing the variational derivative of each term of J separately. The first term gives

$$\begin{aligned}\frac{\partial h(x)}{\partial\langle\psi|} &= \frac{\partial}{\partial\langle\psi|}\left(-\int W_x(x)\ln W_x(x)\mathrm{d}x\right)\\ &= \frac{\partial}{\partial\langle\psi|}\left(-\int\langle\psi|x\rangle\langle x|\psi\rangle\ln(\langle\psi|x\rangle\langle x|\psi\rangle)\mathrm{d}x\right)\\ &= -\left(\ln W_x(\hat{x}) + 1\right)|\psi\rangle\end{aligned} \tag{6.32}$$

where we used the relations $\int dx|x\rangle\langle x| = \mathbb{1}$ and $\int dx f(x)|x\rangle\langle x| = f(\hat{x})$. Similarly, the second term gives

$$\frac{\partial h(p)}{\partial\langle\psi|} = -\left(\ln W_p(\hat{p}) + 1\right)|\psi\rangle. \tag{6.33}$$

For the third term, we use

$$\begin{aligned}\frac{\partial}{\partial\langle\psi|}\ln\left(\frac{\langle\psi|\hat{x}^2|\psi\rangle\langle\psi|\hat{p}^2|\psi\rangle}{\langle\psi|\hat{x}^2|\psi\rangle\langle\psi|\hat{p}^2|\psi\rangle - \frac{1}{4}\langle\psi|\{\hat{x},\hat{p}\}|\psi\rangle^2}\right)\\ = \left[\frac{\hat{x}^2}{\sigma_x^2} + \frac{\hat{p}^2}{\sigma_p^2} - \frac{\hat{x}^2\sigma_p^2 + \hat{p}^2\sigma_x^2 - \{\hat{x},\hat{p}\}\sigma_{xp}}{\det\gamma}\right]|\psi\rangle\end{aligned} \tag{6.34}$$

while the last terms give

$$\frac{\partial}{\partial\langle\psi|}\left(\lambda(\langle\psi|\psi\rangle - 1) + \mu\langle\psi|\hat{x}|\psi\rangle + \nu\langle\psi|\hat{p}|\psi\rangle\right) = (\lambda + \mu\hat{x} + \nu\hat{p})|\psi\rangle. \tag{6.35}$$

Putting all this together, the variational equation (6.31) can be rewritten as an eigen-

value equation for $|\psi\rangle$,

$$\Big[-\ln W_x(\hat{x}) - \ln W_p(\hat{p}) - 2 + \lambda + \mu\hat{x} + \nu\hat{p} \\ - \frac{\hat{x}^2}{2\sigma_x^2} - \frac{\hat{p}^2}{2\sigma_p^2} + \frac{\hat{x}^2\sigma_p^2 + \hat{p}^2\sigma_x^2 - \{\hat{x},\hat{p}\}\sigma_{xp}}{2\det\gamma}\Big]|\psi\rangle = 0. \tag{6.36}$$

Let us check that Eq. (6.36) is verified by $|\psi\rangle = \hat{S}|0\rangle$, that is, by a squeezed vacuum state with $\hat{S} = \exp\{\frac{1}{2}(z^*\hat{a}^2 - z\hat{a}^{\dagger 2})\}$, where $z = re^{i\phi}$ is a complex number[3]. For such a state, the marginals of the Wigner functions are given by

$$W_x(x) = (2\pi\sigma_x^2)^{-\frac{1}{2}}\, e^{-\frac{x^2}{2\sigma_x^2}}, \qquad W_p(p) = (2\pi\sigma_p^2)^{-\frac{1}{2}}\, e^{-\frac{p^2}{2\sigma_p^2}}, \tag{6.37}$$

so that

$$\ln W_x(\hat{x}) + \ln W_p(\hat{p}) = -\ln(2\pi\sigma_x\sigma_p) - \frac{\hat{x}^2}{2\sigma_x^2} - \frac{\hat{p}^2}{2\sigma_p^2}. \tag{6.38}$$

Hence, we can simplify the eigenvalue equation as

$$\left[\ln(2\pi\sigma_x\sigma_p) - 2 + \lambda + \mu\hat{x} + \nu\hat{p} + \hat{A}\right]|\psi\rangle = 0 \tag{6.39}$$

where we have defined the operator

$$\hat{A} = \frac{\hat{x}^2\sigma_p^2 + \hat{p}^2\sigma_x^2 - \{\hat{x},\hat{p}\}\sigma_{xp}}{2\det\gamma} = \frac{1}{2}\begin{pmatrix}\hat{x} & \hat{p}\end{pmatrix}\gamma^{-1}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}. \tag{6.40}$$

Let us now compute the action of $\hat{A}$ on the squeezed vacuum state, that is,

$$\hat{A}|\psi\rangle = \hat{A}\hat{S}|0\rangle = \hat{S}(\hat{S}^\dagger\hat{A}\hat{S})|0\rangle \tag{6.41}$$

where we used the fact that $\hat{S}^\dagger = \hat{S}^{-1}$ (see Eq. (2.61)). For this, we use the canonical transformation of $\hat{x}$ and $\hat{p}$ in the Heisenberg picture, namely

$$\begin{pmatrix}\hat{S}^\dagger\hat{x}\hat{S}\\ \hat{S}^\dagger\hat{p}\hat{S}\end{pmatrix} = \mathcal{M}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix} \tag{6.42}$$

where $\mathcal{M}$ is a symplectic matrix defined as

$$\begin{aligned}\mathcal{M} &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{-r} & 0\\ 0 & e^{r}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\ &= \begin{pmatrix}\cosh r - \cos\phi\,\sinh r & -\sin\phi\,\sinh r\\ -\sin\phi\,\sinh r & \cosh r + \cos\phi\,\sinh r\end{pmatrix}\end{aligned} \tag{6.43}$$

with $\phi = 2\theta$. The covariance matrix γ of state $|\psi\rangle$ can be expressed with transforma-

[3]Note that the squeezing angle ϕ allows to apply squeezing in any direction. We thus consider all pure Gaussian states, no matter along which angle the squeezing is applied.

tion $\mathcal{M}$ applied to the covariance matrix of the vacuum state γ_{vac}, namely

$$\gamma = \mathcal{M}\gamma_{\text{vac}}\mathcal{M}^T. \tag{6.44}$$

Using Eqs. (6.42) and (6.44), we get

$$\begin{aligned}
\hat{A}|\psi\rangle &= \frac{1}{2}\hat{S}\begin{pmatrix}\hat{x} & \hat{p}\end{pmatrix}\mathcal{M}^T\gamma^{-1}\mathcal{M}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}|0\rangle \\
&= \frac{1}{2}\hat{S}\begin{pmatrix}\hat{x} & \hat{p}\end{pmatrix}\mathcal{M}^T(\mathcal{M}^T)^{-1}\gamma_{\text{vac}}^{-1}\mathcal{M}^{-1}\mathcal{M}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}|0\rangle \\
&= \frac{1}{2}\hat{S}\begin{pmatrix}\hat{x} & \hat{p}\end{pmatrix}\gamma_{\text{vac}}^{-1}\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}|0\rangle \\
&= \hat{S}(\hat{x}^2+\hat{p}^2)|0\rangle \\
&= \hat{S}(2\hat{N}+1)|0\rangle \\
&= \hat{S}|0\rangle \\
&= |\psi\rangle
\end{aligned} \tag{6.45}$$

where $\hat{N}$ is the number operator. This calculation implies that the squeezed vacuum state $|\psi\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue 1. Therefore, the eigenvalue equation can be written as

$$\left[\ln(2\pi\sigma_x\sigma_p) - 1 + \lambda + \mu\hat{x} + \nu\hat{p}\right]|\psi\rangle = 0. \tag{6.46}$$

We can determine the value of λ by multiplying this equation on the left by $\langle\psi|$ and using the constraints $\langle\psi|\psi\rangle = 1$ and $\langle\psi|\hat{x}|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle = 0$, namely

$$\langle\psi|\left[\ln(2\pi\sigma_x\sigma_p) - 1 + \lambda + \mu\hat{x} + \nu\hat{p}\right]|\psi\rangle = \ln(2\pi\sigma_x\sigma_p) - 1 + \lambda = 0. \tag{6.47}$$

Therefore, state $|\psi\rangle$ is indeed a solution of our extremization problem, if we set $\lambda = 1 - \ln(2\pi\sigma_x\sigma_p)$. We are left with equation

$$\left[\mu\hat{x} + \nu\hat{p}\right]|\psi\rangle = 0 \tag{6.48}$$

which is satisfied if we set $\mu = \nu = 0$. Summing up, we have proven that, with the appropriate choice of λ, μ and ν, the squeezed vacuum states (with arbitrary squeezing and rotation) are solutions of Eq. (6.36), so they are extrema of our uncertainty functional $F(|\psi\rangle\langle\psi|)$. Since $F(|\psi\rangle\langle\psi|)$ is invariant under displacements, the displaced squeezed states are also solutions, so this result includes all pure Gaussian states. Using Assumption 1, we consider that pure Gaussian states are not just extrema of our functional, but are the global minima. We find the minimal value $\ln(\pi e)$ simply by evaluating F for any of these states.

Remark that in our proof, we consider $\langle\psi|$ and $|\psi\rangle$ as independent. This can be understood as follows. When expressing the variation of a functional of $\psi(x)$, we

should take into account that it is complex valued so we must consider separately the variation in the real part and in the imaginary part. As we will see, this is equivalent in making variation in $\psi(x)$ and $\psi^*(x)$. Let us consider a real function $f(z, z^*)$ with z a complex number. We can thus define its real and imaginary parts

$$z_r = \Re(z) = \frac{z + z^*}{2} \qquad z_i = \Im(z) = \frac{z - z^*}{2i}. \tag{6.49}$$

According to the chain rule we have

$$\begin{aligned} \frac{\partial f}{\partial z}\Big|_{z^*} &= \frac{\partial f}{\partial z_r}\frac{\partial z_r}{\partial z} + \frac{\partial f}{\partial z_i}\frac{\partial z_i}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial z_r} - i\frac{\partial f}{\partial z_i}\right), \\ \frac{\partial f}{\partial z^*}\Big|_{z} &= \frac{\partial f}{\partial z_r}\frac{\partial z_r}{\partial z^*} + \frac{\partial f}{\partial z_i}\frac{\partial z_i}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f}{\partial z_r} + i\frac{\partial f}{\partial z_i}\right). \end{aligned} \tag{6.50}$$

Since, $f(z, z^*)$ is real, $\frac{\partial f}{\partial z_r}$ and $\frac{\partial f}{\partial z_i}$ will also be real. Equating both of the above equations to zero then yields to $\frac{\partial f}{\partial z_r} = 0$ and $\frac{\partial f}{\partial z_i} = 0$, which proves the equivalence. Generalizing this to a real functional, it thus means that one has to consider

$$\frac{\partial J[\langle\psi|, |\psi\rangle]}{\partial\langle\psi|} = 0 \quad \text{and} \quad \frac{\partial J[\langle\psi|, |\psi\rangle]}{\partial|\psi\rangle} = 0 \tag{6.51}$$

where in the first equation we keep $|\psi\rangle$ constant while in the second we keep $\langle\psi|$ constant. Now, the functional has the property that $J[\langle\psi|, |\psi\rangle] = J[|\psi\rangle, \langle\psi|]^*$. Therefore, $\frac{\partial J[\langle\psi|, |\psi\rangle]}{\partial|\psi\rangle} = 0$ is simply the complex conjugate of $\frac{\partial J[\langle\psi|, |\psi\rangle]}{\partial\langle\psi|} = 0$ and we do not learn any new information from the second equation. It is thus sufficient to solve $\frac{\partial J}{\partial\langle\psi|} = 0$.

In order to extend the proof to mixed states, we use Assumption 2 about the concavity of the functional. □

We conjecture that Theorem 2 holds even without the assumptions.

6.4 Generalization to n modes

In ref. [11], Białynicki-Birula and Mycielski also extended the entropic uncertainty relation to n modes, namely

$$h(\mathbf{x}) + h(\mathbf{p}) \geq n\,\ln(\pi e) \tag{6.52}$$

where the joint differential entropies $h(\mathbf{x})$ and $h(\mathbf{p})$ are computed from the marginals of the Wigner functions $W_x(\mathbf{x})$ and $W_p(\mathbf{p})$, with $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{p} = (p_1, \cdots, p_n)$.

Naturally, our entropic uncertainty relation (6.14) can also be extended to n modes as

$$h(\mathbf{x}) + h(\mathbf{p}) - \frac{1}{2}\ln\left(\frac{\det(\gamma_x)\det(\gamma_p)}{\det(\gamma)}\right) \geq n\,\ln(\pi e) \tag{6.53}$$

where the covariance matrix γ is defined as $\gamma_{ij} = \mathrm{Tr}[\hat{\rho}\,\{r_i, r_j\}]/2 - \mathrm{Tr}[\hat{\rho}\,r_i]\mathrm{Tr}[\hat{\rho}\,r_j]$ and γ_x (γ_p) is the reduced covariance matrix of the x (p) quadratures.

Just before presenting the conditional proof of this relation which is obtained by following a variational method based on Assumptions 1 and 2 — as in the one-mode case — let us mention that equation (6.53) is again saturated by all n-mode Gaussian pure states. More precisely, we have the following theorem for all Gaussian states:

Theorem 3. *Let ρ^G be a n-modal Gaussian state with covariance matrix γ defined as $\gamma_{ij} = \mathrm{Tr}[\hat{\rho}\,\{r_i, r_j\}]/2 - \mathrm{Tr}[\hat{\rho}\,r_i]\mathrm{Tr}[\hat{\rho}\,r_j]$ and differential entropies $h(\mathbf{x}) = \frac{1}{2}\ln((2\pi e)^n \det(\gamma_x))$ and $h(\mathbf{p}) = \frac{1}{2}\ln((2\pi e)^n \det(\gamma_p))$. Then it satisfies the entropic uncertainty relation (6.53) and the saturation is obtained if and only if ρ^G is pure.*

Proof. By simple calculation we have

$$\begin{aligned} h(\mathbf{x}) + h(\mathbf{p}) - \frac{1}{2}\ln\left(\frac{\det(\gamma_x)\det(\gamma_p)}{\det(\gamma)}\right) &= n\ln(\pi e) + \frac{1}{2}\ln(4^n \det\gamma) \\ &\geq n\ln(\pi e) \end{aligned} \tag{6.54}$$

since $\det\gamma \geq 1/4^n$ according to the n-modal version of Robertson-Schrödinger uncertainty relation, Eq. (4.15). This proves Eq. (6.53) for Gaussian states. This inequality is saturated if and only if ρ^G is pure since $\det\gamma = 1/4^n$ only for pure Gaussian states. □

As an example, we can see that relation (6.53) is thus saturated by the two-mode squeezed vacuum state with covariance matrix

$$\gamma = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix}, \tag{6.55}$$

obtained by injecting an x-squeezed state and a p-squeezed state (both with a squeezing parameter r) on a balanced beam splitter. This is easy to check by computing the entropies with $h(\mathbf{x}) = \frac{1}{2}\ln((2\pi e)^n \det(\gamma_x))$ and $h(\mathbf{p}) = \frac{1}{2}\ln((2\pi e)^n \det(\gamma_p))$ and using $\det(\gamma_x) = \det(\gamma_p) = (1/2)^2$ and $\det(\gamma) = (1/2)^4$. However, the regular entropic uncertainty relation (6.52) is already saturated for this state, which is expected since the state exhibits no x-p correlations. More interestingly, the state resulting from two rotated squeezed states (one being rotated by $\pi/4$, the other by $-\pi/4$) injected on a balanced beam splitter still saturates relation (6.53), while it does not any more

saturate relation (6.52). Indeed, the covariance matrix of this state reads

$$\gamma = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & 0 & -\sinh 2r \\ 0 & \cosh 2r & -\sinh 2r & 0 \\ 0 & -\sinh 2r & \cosh 2r & 0 \\ -\sinh 2r & 0 & 0 & \cosh 2r \end{pmatrix}. \tag{6.56}$$

so that we get $h(\mathbf{x}) + h(\mathbf{p}) = 2\ln(\pi e \cosh 2r) > 2\ln(\pi e)$. But since

$$\det(\gamma_x) = \det(\gamma_p) = (1/2)^2 \cosh^2 2r \tag{6.57}$$

and $\det(\gamma) = (1/2)^4$, we get

$$-\frac{1}{2}\ln\left(\frac{\det(\gamma_x)\det(\gamma_p)}{\det(\gamma)}\right) = -2\ln(\cosh 2r), \tag{6.58}$$

implying that relation (6.53) is saturated by this state.

In this context, it is also interesting to rewrite the tight entropic uncertainty relation (6.53) in term of entropy powers, defined this time for the joint entropy in n dimensions, namely

$$N_x^{(n)} = \frac{1}{2\pi e} e^{\frac{2}{n}h(\mathbf{x})} \qquad N_p^{(n)} = \frac{1}{2\pi e} e^{\frac{2}{n}h(\mathbf{p})} \tag{6.59}$$

Equation (6.52) then transforms into a n-mode entropy-power uncertainty relation

$$N_x^{(n)} N_p^{(n)} \geq \frac{1}{4}, \tag{6.60}$$

which has the same form as relation (6.4) but for n modes, while our new equation (6.53) transforms into a tight version of the n-mode entropy-power uncertainty relation

$$N_x^{(n)} N_p^{(n)} \geq \frac{1}{4}\left(\frac{\det(\gamma_x)\ \det(\gamma_p)}{\det(\gamma)}\right)^{1/n}, \tag{6.61}$$

which is the n-mode counterpart of Eq. (6.15).

Here too, we can use the fact that the maximum entropy for a fixed covariance matrix is given by the Gaussian distribution, which implies that $N_x^{(n)} \leq (\det \gamma_x)^{1/n}$ and $N_p^{(n)} \leq (\det \gamma_p)^{1/n}$. Rewriting Eq. (6.61) as

$$\frac{\left(N_x^{(n)} N_p^{(n)}\right)^n}{\det(\gamma_x)\ \det(\gamma_p)} \det(\gamma) \geq \left(\frac{1}{4}\right)^n, \tag{6.62}$$

we then see that our new n-mode entropy-power uncertainty relation implies some (variance-based) n-mode uncertainty relation (see Eq. (4.15)), namely

$$\det(\gamma) \geq \frac{\left(N_x^{(n)} N_p^{(n)}\right)^n}{\det(\gamma_x)\ \det(\gamma_p)} \det(\gamma) \geq \left(\frac{1}{4}\right)^n. \tag{6.63}$$

6.4.1 Conditional proof of the n-mode generalization

We now prove the n-modal version of Theorem 2 using the same variational method and conditionally to the same assumptions. It means that we will prove that any n-mode squeezed vacuum state is an extremum of the uncertainty functional

$$F(\hat{\rho}) = h(\mathbf{x}) + h(\mathbf{p}) - \frac{1}{2}\ln\left(\frac{\det(\gamma_x)\det(\gamma_p)}{\det(\gamma)}\right) \tag{6.64}$$

Since $F(\hat{\rho})$ is invariant under $(\mathbf{x},\mathbf{p})$-displacements, it will imply that all Gaussian pure states are similarly extrema. Note that we keep the same two assumptions, that is that pure Gaussian states are global minima of our uncertainty functional (Assumption 1) and that the uncertainty functional $F(\hat{\rho})$ is concave in $\hat{\rho}$ (Assumption 2), so that (6.53) is valid for mixed states as well.

Theorem 4. *If Assumptions 1 and 2 hold, then any state ρ with covariance matrix γ defined as $\gamma_{ij} = \mathrm{Tr}[\hat{\rho}\,\{r_i, r_j\}]/2 - \mathrm{Tr}[\hat{\rho}\,r_i]\mathrm{Tr}[\hat{\rho}\,r_j]$ and differential entropies $h(\mathbf{x})$ and $h(\mathbf{p})$ satisfies the entropic uncertainty relation*

$$h(\mathbf{x}) + h(\mathbf{p}) - \frac{1}{2}\ln\left(\frac{\det(\gamma_x)\det(\gamma_p)}{\det(\gamma)}\right) \geq n\,\ln(\pi e). \tag{6.65}$$

Proof. Once again, we start by proving the theorem for pure states. We seek an n-mode pure state $|\psi\rangle$ that minimizes the functional $F(|\psi\rangle\langle\psi|)$ with constraints on the normalization of $|\psi\rangle$ and mean values of $\mathbf{x}$ and $\mathbf{p}$ quadratures. We use the Lagrange multiplier method with

$$\begin{aligned} J = h(\mathbf{x}) \;&+\; h(\mathbf{p}) - \frac{1}{2}\ln\det(\gamma_x) - \frac{1}{2}\ln\det(\gamma_p) + \frac{1}{2}\ln\det(\gamma) + \lambda(\langle\psi|\psi\rangle - 1) \\ &+\; \sum_{i=1}^{2n} \mu_i \langle\psi|\hat{r}_i|\psi\rangle. \end{aligned} \tag{6.66}$$

Here, λ and μ_i are Lagrange multipliers, while the elements of the covariance matrix γ can be expressed as $\gamma_{ij} = \langle\psi|\hat{r}_i\hat{r}_j + \hat{r}_j\hat{r}_i|\psi\rangle/2$ since the states are normalized and centered on 0. As in the one-mode case, we solve the variational equation $\frac{\partial J}{\partial\langle\psi|} = 0$, so we write the derivative of each term

$$\begin{aligned} \frac{\partial h(\mathbf{x})}{\partial\langle\psi|} &= -\left(\ln W_x(\mathbf{x}) + 1\right)|\psi\rangle \\ \frac{\partial h(\mathbf{p})}{\partial\langle\psi|} &= -\left(\ln W_p(\mathbf{p}) + 1\right)|\psi\rangle. \end{aligned} \tag{6.67}$$

For the three terms involving the derivative of the determinant of a matrix, we use

Jacobi's formula so that

$$\begin{aligned}
\frac{\partial}{\partial\langle\psi|}\ln\det(\gamma_x) &= \frac{1}{\det(\gamma_x)}\frac{\partial}{\partial\langle\psi|}\det(\gamma_x)\\
&= \frac{1}{\det(\gamma_x)}\mathrm{Tr}\left[\det(\gamma_x)\gamma_x^{-1}\frac{\partial\gamma_x}{\partial\langle\psi|}\right]\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^{-1}_{x_{ik}}\frac{\partial\gamma_{x_{ki}}}{\partial\langle\psi|}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^{-1}_{x_{ik}}\frac{(\hat{x}_k\hat{x}_i+\hat{x}_i\hat{x}_k)}{2}|\psi\rangle\\
&= \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\hat{x}_k\gamma^{-1}_{x_{ik}}\hat{x}_i}{2}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\hat{x}_i\gamma^{-1}_{x_{ik}}\hat{x}_k}{2}\right]|\psi\rangle\\
&= \mathbf{x}^T\gamma_x^{-1}\mathbf{x}\,|\psi\rangle. \qquad (6.68)
\end{aligned}$$

where we used the fact that $\gamma^{-1}_{ik}=\gamma^{-1}_{ki}$ since the matrix is symmetric. Similarly, we find

$$\frac{\partial}{\partial\langle\psi|}\ln\det(\gamma_p)=\mathbf{p}^T\gamma_p^{-1}\mathbf{p}\,|\psi\rangle \qquad \frac{\partial}{\partial\langle\psi|}\ln\det(\gamma)=\mathbf{r}^T\gamma^{-1}\mathbf{r}\,|\psi\rangle. \qquad (6.69)$$

Finally, the last terms give

$$\frac{\partial}{\partial\langle\psi|}\left(\lambda(\langle\psi|\psi\rangle-1)+\sum_{i=1}^{2n}\mu_i\langle\psi|\hat{r}_i|\psi\rangle\right)=\left(\lambda+\sum_{i=1}^{2n}\mu_i\hat{r}_i\right)|\psi\rangle. \qquad (6.70)$$

so that the variational equation can be rewritten as an eigenvalue equation for $|\psi\rangle$,

$$\left[-\ln W_x(\mathbf{x})-\ln W_p(\mathbf{p})-2+\lambda+\sum_{i=1}^{2n}\mu_i\hat{r}_i-\frac{1}{2}\mathbf{x}^T\gamma_x^{-1}\mathbf{x}-\frac{1}{2}\mathbf{p}^T\gamma_p^{-1}\mathbf{p}+\frac{1}{2}\mathbf{r}^T\gamma^{-1}\mathbf{r}\right]|\psi\rangle=0. \qquad (6.71)$$

We now check that Eq. (6.71) is verified by $|\psi\rangle=\hat{S}|0\rangle$, that is, by any n-mode squeezed vacuum state. For such a state, the marginals of the Wigner functions are given by

$$\begin{aligned}
W_x(\mathbf{x}) &= \left((2\pi)^n\det(\gamma_x)\right)^{-\frac{1}{2}}e^{-\frac{1}{2}\mathbf{x}^T\gamma_x^{-1}\mathbf{x}},\\
W_p(\mathbf{p}) &= \left((2\pi)^n\det(\gamma_p)\right)^{-\frac{1}{2}}e^{-\frac{1}{2}\mathbf{p}^T\gamma_p^{-1}\mathbf{p}}, \qquad (6.72)
\end{aligned}$$

so that

$$\ln W_x(\mathbf{x})+\ln W_p(\mathbf{p})=-\ln\left((2\pi)^n\sqrt{\det(\gamma_x)\det(\gamma_p)}\right)-\frac{1}{2}\mathbf{x}^T\gamma_x^{-1}\mathbf{x}-\frac{1}{2}\mathbf{p}^T\gamma_p^{-1}\mathbf{p} \qquad (6.73)$$

We apply $\frac{1}{2}\,\mathbf{r}^T\,\gamma^{-1}\,\mathbf{r}$ on the squeezed vacuum state $|\psi\rangle$ by using the canonical trans-

formation of $\mathbf{r}$ in the Heisenberg picture, namely $\hat{S}^\dagger \mathbf{r}\, \hat{S} = \mathcal{M}\mathbf{r}$, we find

$$\begin{aligned} \frac{1}{2}\,\mathbf{r}^T\,\gamma^{-1}\,\mathbf{r}\,|\psi\rangle &= \frac{1}{2}\,\mathbf{r}^T\,\gamma^{-1}\,\mathbf{r}\,\hat{S}|0\rangle \\ &= \frac{1}{2}\hat{S}\,\mathbf{r}^T\,M^T\gamma^{-1}M\mathbf{r}\,|0\rangle \\ &= \frac{1}{2}\hat{S}\,\mathbf{r}^T\,\gamma_{\text{vac}}^{-1}\mathbf{r}\,|0\rangle \\ &= \hat{S}|0\rangle \\ &= |\psi\rangle \end{aligned} \tag{6.74}$$

since the covariance matrix γ of state $|\psi\rangle$ can be expressed as $\gamma = \mathcal{M}\gamma_{\text{vac}}\mathcal{M}^T$. This implies that state $|\psi\rangle$ is an eigenvector of $\frac{1}{2}\,\mathbf{r}^T\,\gamma^{-1}\,\mathbf{r}$ with eigenvalue 1. Therefore, using this result together with equation (6.73), the eigenvalue equation for $|\psi\rangle$ can be written as

$$\left[\ln\left((2\pi)^n\sqrt{\det(\gamma_x)\det(\gamma_p)}\right) - 1 + \lambda + \sum_{i=1}^{2n}\mu_i\hat{r}_i\right]|\psi\rangle = 0. \tag{6.75}$$

The value of λ is found by multiplying this equation on the left by $\langle\psi|$ and by using the constraints $\langle\psi|\psi\rangle = 1$ and $\langle\psi|\hat{r}_i|\psi\rangle = 0$ for all i, namely

$$\begin{aligned} 0 &= \langle\psi|\left[\ln\left((2\pi)^n\sqrt{\det(\gamma_x)\det(\gamma_p)}\right) - 1 + \lambda + \sum_{i=1}^{2n}\mu_i\hat{r}_i\right]|\psi\rangle \\ \Rightarrow\quad \lambda &= 1 - \ln\left((2\pi)^n\sqrt{\det(\gamma_x)\det(\gamma_p)}\right). \end{aligned} \tag{6.76}$$

We are now left with equation

$$\left[\sum_{i=1}^{2n}\mu_i\hat{r}_i\right]|\psi\rangle = 0 \tag{6.77}$$

which is satisfied if we set all the $\mu_i = 0$.

In conclusion, we have proven that, with the appropriate choice of λ and μ_i, the n-mode squeezed vacuum states are solutions of Eq. (6.71), so they extremize our uncertainty functional $F(|\psi\rangle\langle\psi|)$. Since $F(|\psi\rangle\langle\psi|)$ is invariant under displacements, the displaced squeezed vacuum states are also solutions, so this extremization result encapsulates all pure Gaussian states. Using Assumption 1, we consider that pure Gaussian states are not just extrema of our functional, but are the global minima. We find the minimum value $n\ln(\pi e)$ by evaluating F for any of these states.

In order to extend the proof to mixed states, we use Assumption 2 about the concavity of the functional. □

We conjecture that Theorem 4 also holds even without the assumptions.

6.5 Attempts towards a full proof

We would like to complete this chapter by mentioning some proof attempts of the entropic uncertainty relation Eq. (6.14). Indeed, we explained that the proof given in Section 6.3 is only partial since we had to make two (reasonable) assumptions.[4] Naturally, we would have preferred a complete proof and thus tried different other techniques, but none of them was successful. To prevent the reader to lose time on trying those different methods, we list the most promising ones and explain briefly why there were not successful.

Gaussian functional

Here, the idea is to start with a pair of canonically conjugate quadratures x_θ and p_θ which correspond to the principal axes of the covariance matrix, so that $\sigma_{x_\theta p_\theta} = 0$. Then, we apply a rotation in the phase space, resulting in the pair of canonically conjugate quadratures x and p. This rotation conserves the determinant of the covariance matrix $\det \gamma$. If we could prove that

$$\frac{N_x\, N_p}{\sigma_x^2\, \sigma_p^2}\, \det \gamma \geq \frac{N_{x_\theta}\, N_{p_\theta}}{\sigma_{x_\theta}^2\, \sigma_{p_\theta}^2}\, \det \gamma = N_{x_\theta} N_{p_\theta} \tag{6.78}$$

where we used the fact that $\det \gamma = \sigma_{x_\theta}^2\, \sigma_{p_\theta}^2$, then the Białynicki-Birula and Mycielski inequality $N_{x_\theta} N_{p_\theta} \geq 1/4$ would imply our entropic uncertainty relation in its entropy-power form, Eq. (6.2). Thus, we wished to prove the second inequality in

$$1 \geq \frac{N_x\, N_p}{\sigma_x^2\, \sigma_p^2} \geq \frac{N_{x_\theta}\, N_{p_\theta}}{\sigma_{x_\theta}^2\, \sigma_{p_\theta}^2}. \tag{6.79}$$

Translated back into entropies, this gives

$$0 \leq h(x^G) + h(p^G) - h(x) - h(p) \leq h(x_\theta^G) + h(p_\theta^G) - h(x_\theta) - h(p_\theta) \tag{6.80}$$

where the letter G indicates that the variable has a Gaussian distribution (with the same covariance matrix). In terms of relative entropies it can also be written as

$$0 \leq D(x||x^G) + D(p||p^G) \leq D(x_\theta||x_\theta^G) + D(p_\theta||p_\theta^G) \tag{6.81}$$

which would have mean that the sum of the non-Gaussianity in the two canonically conjugate quadratures can only decrease when we rotate away from the principal axes.

Since $h(x^G, p^G) = h(x_\theta^G, p_\theta^G)$ (the joint differential entropy is invariant under rota-

[4] We supposed that the extremum we found is a global minimum and that our functional is concave.

tions), we can write, for Gaussian states, that

$$h(x^G) + h(p^G) - \frac{1}{2}\ln\left(\frac{\sigma^2_{x^G}\sigma^2_{p^G}}{|\gamma|}\right) = h(x^G_\theta) + h(p^G_\theta) \tag{6.82}$$

and thus inequality (6.80) can be rewritten as

$$h(x) - h(x_\theta) + h(p) - h(p_\theta) \geq \frac{1}{2}\ln\left(\frac{\sigma^2_x\sigma^2_p}{|\gamma|}\right) \tag{6.83}$$

since $\sigma^2_x = \sigma^2_{x^G}$ and $\sigma^2_p = \sigma^2_{p^G}$. The r.h.s. of Eq. (6.83) corresponds to the value taken by the l.h.s when all variables are replaced by their Gaussian counterpart.

Finally, we can also define the functional

$$F(\rho) \equiv h(x) + h(p) - h(x_\theta) - h(p_\theta) \tag{6.84}$$

and then, proving Eq. (6.83) or the second inequality in Eq. (6.80) is equivalent to show that the functional $F(\rho)$ is minimized for Gaussian states, that is

$$F(\rho) \geq F(\rho_G). \tag{6.85}$$

We wanted to use the paper of Wolf *et al.* [88] who presented a general method to prove that a functional satisfying some specific conditions is minimized by Gaussian states. Unfortunately, our inequality is too strong and we found numerically some counter-examples[5] to Eq. (6.85).

Entropy-power inequality

Here too, the idea was to compare the x, p quadratures with the ones in the principal axes: x_θ and p_θ. We wanted to use the entropy power inequality [6] which states that $N_z \geq N_x + N_y$ when $z = x + y$ to first express two inequalities of the form

$$\begin{aligned} N_x &\geq \cos^2\theta N_{x_\theta} + \sin^2\theta N_{p_\theta} \\ N_p &\geq \sin^2\theta N_{x_\theta} + \cos^2\theta N_{p_\theta} \end{aligned} \tag{6.86}$$

and then to find an inequality on the product $N_x N_p$. However, this is not possible because the entropy power inequality only applies if the variables are independent. Here, it is not because $\sigma_{x_\theta p_\theta} = 0$ that x_θ and p_θ are necessarily independent.

[5] Counter-examples were obtained with some slightly non-Gaussian states.
[6] See Section 3.2.4.

Fractional Fourier transforms

We will give more details about fractional Fourier transforms in Section 7.1, but in short, it allows us to connect the wave function of x_θ to the one of x. If $\theta = \pi/2$, it is the usual Fourier transform. The idea was thus to connect $h(x)$ and $h(x_\theta)$ through the use of the fractional Fourier transforms in order to prove something of the form

$$h(x) + h(p) \geq h(x_\theta) + h(p_\theta) + C \tag{6.87}$$

where C would hopefully be the Gaussian mutual information. We would then have used Białynicki-Birula and Mycielski entropic uncertainty relation for $h(x_\theta) + h(p_\theta)$, since, along the principal axis, it already gives the best possible bound. Unfortunately, we do not know how to bound the difference $h(x) - h(x_\theta)$. As we will see in Chapter 7, we can indeed use the fractional Fourier transforms, but to find a bound on the sum $h(x) + h(x_\theta)$.

Gaussian channel

Another idea was to use a Gaussian channel and to compare the functional of the input and output states. Indeed this technique was successful to prove that for a phase-insensitive bosonic channel, the minimum output entropy is achieved by coherent input states [89, 90]. In short, we were hoping to prove that the functional

$$F(\rho) = h(x) + h(p) - \frac{1}{2} \ln \left(\frac{\sigma_x^2 \sigma_p^2}{\det \gamma} \right) \tag{6.88}$$

can only decrease when ρ goes through an infinitesimal pure loss channel.[7] We thus wanted to prove that

$$\frac{d}{dt} F(\rho) \leq 0 \tag{6.89}$$

where we would have used the Lindblad equation to describe the infinitesimal evolution of ρ. Since the fixed point of a pure loss channel is the vacuum, we would have proven that for any state, $F(\rho) \geq F(|0\rangle) = \ln(\pi e)$. Unfortunately, we found some counter-examples. For some states, the functional $F(\rho)$ increases before decreasing towards the asymptotic value $\ln(\pi e)$. Indeed, let us take for example a Fock state $|1\rangle$. At the output of a pure-loss channel of transmissivity η we obtain the state

$$\eta |1\rangle\langle 1| + (1 - \eta) |0\rangle\langle 0|. \tag{6.90}$$

If we now compare the functional $F(\rho)$ for both input and output states we can see in Figure 6.4 that the entropy increases when η is slightly greater than 0.

[7] A pure-loss channel of transmissivity η is represented by a beam splitter of transmissivity η where the auxiliary input mode is the vacuum state. This mode is traced out at the output of the beam splitter and we then only consider the in signal mode.

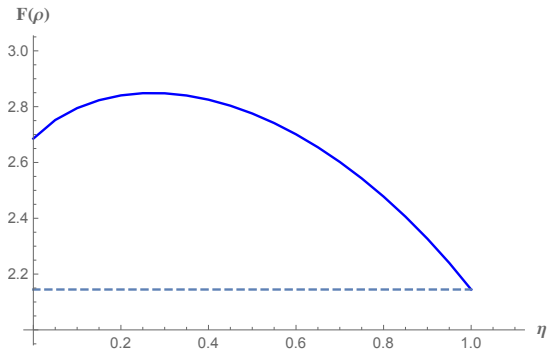

Figure 6.4: Functional $F(\rho)$ for a state $\rho = \eta|1\rangle\langle 1| + (1-\eta)|0\rangle\langle 0|$. The dashed line represents $F(|0\rangle) = \ln(\pi e)$.

6.6 Conclusion

We have shown that the entropic uncertainty relation derived by Białynicki-Birula and Mycielski can be expressed as an entropy-power uncertainty relation, which makes a straightforward connection with Heisenberg uncertainty relation: the variances in the latter are simply replaced with entropy powers in the former. Moreover, the entropic version of the uncertainty relation implies the variance-based one as a consequence of the fact that the entropy power of a variable cannot exceed its variance. Then, we suggested a tighter form of the entropic uncertainty relation, which takes the correlation between the x- and p-variables into account. It can also be expressed as a tighter entropy-power uncertainty relation, Eq. (6.2), and is saturated for all pure Gaussian states. It is the entropic counterpart of the Schrödinger-Robertson uncertainty relation, which it implies. We have provided a proof of Eq. (6.2) based on variational calculus and conditionally on two reasonable assumptions, and have provided numerical evidence that it is correct. Interestingly, this tighter entropic and entropy-power uncertainty relations can be extended to n modes, and all the above-mentioned properties remain true.

Possible applications of these new entropic uncertainty relations include the elaboration of stronger separability criteria for continuous-variable systems, as we will see in Chapter 12. Both variance- and entropy-based uncertainty relations can be translated into a sufficient entanglement condition (a necessary and sufficient condition for Gaussian states) as they can be used to express a condition on the physicality of the partially-transposed state [8, 9, 84]. For example, in Chapter 11, we will show that an uncertainty relation that is tight for all Fock states [55] yields an entanglement criterion that enables the detection of certain non-Gaussian entangled states whose entanglement remains undetected by the Duan-Simon criterion. Thus, a natural direction is now to exploit our tighter entropic uncertainty relations in order to improve our tools for discriminating entangled from separable states in continuous-variable quantum systems.

7 | Entropic uncertainty relations for arbitrary quadratures

This chapter is the subject of the following article:
A. Hertz, L. Vanbever, and N. J. Cerf, arXiv:1711.04566 (2017) [c].

Traditionally, continuous-variable entropic uncertainty relations were formulated for the position and momentum quadratures or, more precisely, for quadratures related by a Fourier transform. However, in 2011, Huang [91] generalized the entropic uncertainty relation to a pair of observables that are not canonically conjugate[1]. Defining the observables

$$\hat{A} = \sum_{i=1}^{n} (a_i \hat{x}_i + a'_i \hat{p}_i), \quad \hat{B} = \sum_{i=1}^{n} (b_i \hat{x}_i + b'_i \hat{p}_i), \tag{7.1}$$

he showed that

$$h(\hat{A}) + h(\hat{B}) \geq \ln(\pi e |[\hat{A}, \hat{B}]|) \tag{7.2}$$

where $h(\cdot)$ is the Shannon differential entropy and $[\hat{A}, \hat{B}]$ (which is a scalar) is the commutator between both observables. Obviously, if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, this inequality reduces to the entropic uncertainty relation of Białynicki-Birula and Mycielski[2].

$$h(\hat{x}) + h(\hat{p}) \geq \ln(\pi e). \tag{7.3}$$

In addition, a similar result had earlier been obtained by Guanlei *et al.* [92] in the special case where $n = 1$, namely

$$h(\hat{x}_\theta) + h(\hat{x}_\phi) \geq \ln(\pi e |\sin(\theta - \phi)|) \tag{7.4}$$

where $\hat{x}_\theta = \hat{x}\cos\theta + \hat{p}\sin\theta$ and $\hat{x}_\phi = \hat{x}\cos\phi + \hat{p}\sin\phi$ are two rotated quadratures.

In this chapter, we introduce a generalization of the uncertainty relation of Białynicki-Birula and Mycielski, which is stated in the form of our Theorem 5. It addresses the

[1] That is variables that are not related to each other by a Fourier transform.
[2] See Section 4.2.

situation where n arbitrary quadratures are jointly measured on n modes, expressing the balance between two such joint measurements (see Figure 7.2). In other words, we state an entropic uncertainty relation between two arbitrary n-modal Gaussian projective measurements (or, equivalently, two n-mode Gaussian unitaries U_A and U_B). The lower bound of our uncertainty relation, Eq. (7.31), depends on the determinant of an $n \times n$ matrix formed with the commutators between the n measured quadratures in both cases. In contrast, Eq. (7.3) is restricted to the case of measuring either all x quadratures or all p quadratures on n modes, while Refs. [91, 92] treat the balance between two single-mode measurements only.

Interestingly, the probability distribution of the measured quadratures is given by the squared modulus of the linear canonical transform (LCT) associated with U_A or U_B, so that our entropic uncertainty relation also captures the complementarity between two incompatible n-dimensional LCTs as expressed by our Lemma 1. It simply reduces to the usual entropic uncertainty relation, Eq. (4.39) when the two LCTs are connected by an n-dimensional Fourier transform, mapping $\mathbf{x} = (x_1, x_2, \cdots x_n)^T$ onto $\mathbf{p} = (p_1, p_2, \cdots p_n)^T$.

In the next section, we define general n-dimensional LCT's and give some useful properties. Then, we present our results on uncertainty relations for n modes. First, we derive a generalized entropic uncertainty relation based on Shannon differential entropies (our Theorem 5, with its extension to a larger-dimensional space), then we extend it to Rényi entropies (our Theorem 6), and finally we exhibit a covariance-based uncertainty relation (our Theorem 7). In the last section, we conclude and suggest a conjecture for a generalized entropic uncertainty relation in the case where the commutators differ from scalars.

7.1 Linear canonical transforms

Before deriving our uncertainty relations, we need to properly define fractional Fourier transforms (FRFTs) along with their generalization to LCTs Some early papers on FRFTs appeared in the 1920's, but this topic became investigated in depth only more recently in the fields of signal processing and quantum optics (see, e.g., [93, 94, 95, 96, 97] for more details). In one dimension, the FRFT of a wave function $f(x)$ can be understood as the new wave function obtained when the Wigner function corresponding to $f(x)$ undergoes a rotation of angle α in phase space. If $\alpha = \pi/2$, then the FRFT simply coincides with the usual Fourier transform, connecting the time and frequency domains in the field of signal processing or the canonically conjugate x- and p-quadratures in quantum optics. Mathematically, the one-dimensional FRFT of the function $f(x)$ is defined as

$$\mathcal{F}_\alpha(y) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{\frac{i}{2}y^2\cot\alpha} \int e^{\frac{-iyx}{\sin\alpha}} e^{\frac{i}{2}x^2\cot\alpha} f(x)\, dx. \tag{7.5}$$

The one-dimensional FRFT can be generalized to one-dimensional LCTs by including all affine linear transformations in phase space (x, p), going beyond rotations. Accordingly, the LCT of the wave function $f(x)$ is the new wave function obtained when the corresponding Wigner function undergoes a symplectic transformation $\mathcal{S}$. The one-dimensional LCT of $f(x)$ is defined as

$$\mathcal{F}_{\mathcal{S}}(y) = \sqrt{\frac{1}{2\pi i b}}\, e^{\frac{id}{2b}y^2} \int e^{\frac{-iyx}{b}} e^{\frac{ia}{2b}x^2} f(x)\, dx \tag{7.6}$$

where $\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a symplectic matrix with a, b, c, and d being real parameters, and $b \neq 0$.

The notion of a LCT can readily be extended to n dimensions, the resulting transformation also being sometimes called an n-dimensional FRFT. The physical interpretation is straightforward: a LCT is the transformation of an n-dimensional wave function $f(\mathbf{x})$ that is affected by any symplectic transformation in the $2n$-dimensional phase space of variables $(x_1, x_2, \cdots x_n)$ and $(p_1, p_2, \cdots p_n)$. We write the symplectic matrix $\mathcal{S}$ as

$$\begin{aligned} \mathcal{S} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} & 0 \\ db^{-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ b^{-1}a & \mathbb{1} \end{pmatrix} \end{aligned} \tag{7.7}$$

where a, b, c, and d are $n \times n$ real matrices, $\mathbb{1}$ is the $n \times n$ identity matrix, and $\det(b) \neq 0$. Since $\mathcal{S}$ is symplectic, we saw in Section 2.4.1 that it must obey the constraint

$$\mathcal{S} J \mathcal{S}^T = J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{7.8}$$

being the symplectic form, so that $\det(S) = 1$. This also implies that ab^T and cd^T are symmetric matrices, and $ad^T - bc^T = \mathbb{1}$. The corresponding symplectic transformation in phase space is

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{q} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}, \tag{7.9}$$

where $\mathbf{y} = (y_1, y_2, \cdots y_n)^T$ and $\mathbf{q} = (q_1, q_2, \cdots q_n)^T$ form a new pair of canonically conjugate n-tuples. In state space, the LCT of $f(\mathbf{x})$ can be written as

$$\begin{aligned} \mathcal{F}_{\mathcal{S}}[f(\mathbf{x})](\mathbf{y}) &= \frac{1}{\sqrt{(2\pi)^n |\det(b)|}} \int d\mathbf{x} f(\mathbf{x})\, e^{-i\, b^{-1}\mathbf{y}\cdot\mathbf{x}}\, e^{\frac{i}{2}\left[\mathbf{x}^T(b^{-1}a)\mathbf{x} + \mathbf{y}^T(db^{-1})\mathbf{y}\right]} \\ &= C_{db^{-1}} D_{b^{-1}} \mathcal{F} C_{b^{-1}a}[f(\mathbf{x})](\mathbf{y}) \end{aligned} \tag{7.10}$$

with

$$\begin{aligned} C_r[f](\mathbf{x}) &= e^{\frac{i}{2}\mathbf{x}^T r\mathbf{x}} f(\mathbf{x}) \\ D_b[f](\mathbf{x}) &= \sqrt{|\det(b)|} f(b\,\mathbf{x}) \\ \mathcal{F}[f](\mathbf{x}) &= \frac{1}{(2\pi)^{n/2}} \int d\mathbf{y} f(\mathbf{y}) e^{-i\,\mathbf{x}\cdot\mathbf{y}} \end{aligned} \tag{7.11}$$

where C_r represents the chirp multiplication, D_b the squeezing (or dilation) operator, and $\mathcal{F}$ the usual Fourier transform. These operators are directly related to the decomposition of $\mathcal{S}$ in Eq. (7.7) and we show an example of the action of each transformation on some Wigner function in Figure 7.1. Note also that the chirp multiplication (in one dimension) can be expressed as a product of the other two operators, namely $C_r = R_{\theta-\pi/2} \cdot D_{\tan\theta} \cdot R_\theta$ where R_θ represent the rotation and $r = \tan\theta - \cot\theta$. Finally, note that the set of LCTs in phase space is in one-to-one correspondence with the set of Gaussian unitaries in state space [17], which can indeed be decomposed into passive linear-optics operations (phase shifters and beam splitters, i.e., rotations in phase space) and active squeezing operations (i.e., area-preserving dilations in phase space).

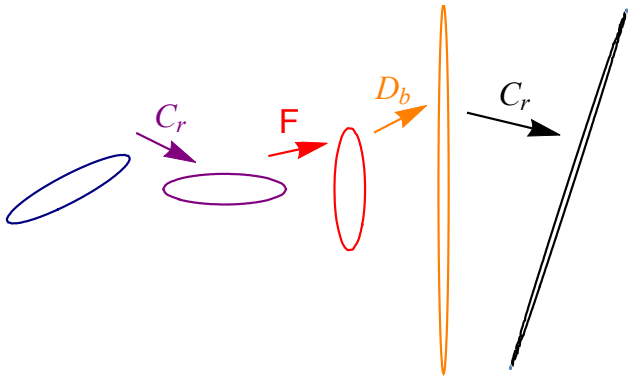

Figure 7.1: Example of the action of the different operations involved in a linear canonical transform (see Eq. 7.11) on the projection of some Wigner function.

Here are some properties of LCTs that will be useful to prove our results in the next section

1. $\mathcal{F}_\mathcal{A}\mathcal{F}_\mathcal{B} = \mathcal{F}_{\mathcal{AB}}$
2. $D_{b_1} D_{b_2} = D_{b_1 b_2}$
3. $\mathcal{F}^{-1} = D_{-1}\mathcal{F}$ where $\mathcal{F}^{-1}$ is the inverse Fourier transform.
4. $|C_r f| = |f|$

Proof.

1. Using Eq. (7.10) and the corresponding representation in phase space, Eq. (7.7), we see that $\mathcal{F}_\mathcal{A}\mathcal{F}_\mathcal{B}$ is represented by the matrix $\mathcal{AB}$. Since the symplectic matri-

ces form a group, the product of two symplectic matrices is a symplectic matrix, which also admits decomposition (7.7) and thus represents the linear canonical transform $\mathcal{F}_{\mathcal{AB}}$.

Proofs of *2*, *3* and *4* are straightforward. □

7.2 Multidimensional uncertainty relations

7.2.1 Entropic uncertainty relation between two linear canonical transforms

Let $|\psi\rangle$ be an arbitrary n-mode state. We wish to express the complementarity between two incompatible LCTs corresponding to two Gaussian unitaries (U_A or U_B) applied onto $|\psi\rangle$. As shown in Figure 7.2, we measure in both cases the n-tuple of output x-quadratures, which corresponds to applying two possible n-modal Gaussian projective measurements on $|\psi\rangle$. The vectors of measurement outcomes are noted, respectively, $\mathbf{y} = (y_1, \cdots, y_n)^T$ or $\mathbf{z} = (z_1, \cdots, z_n)^T$. Denoting as $\mathcal{A}$ and $\mathcal{B}$ the symplectic transformations associated with U_A and U_B, and writing the $2n$-dimensional vector of input quadratures as $\mathbf{r} = (\hat{x}_1, \cdots, \hat{x}_n, \hat{p}_1, \cdots, \hat{p}_n)^T$, we may express the corresponding vectors of output quadratures as

$$\mathbf{r}_A = \mathcal{A}\,\mathbf{r} \equiv \begin{pmatrix} \mathbf{y} \\ \mathbf{q} \end{pmatrix}, \quad \mathbf{r}_B = \mathcal{B}\,\mathbf{r} \equiv \begin{pmatrix} \mathbf{z} \\ \mathbf{o} \end{pmatrix} \tag{7.12}$$

where $\mathbf{q}$ (resp. $\mathbf{o}$) is the vector of quadratures that are canonically conjugate with $\mathbf{y}$ (resp. $\mathbf{z}$). The probability distributions for $\mathbf{y}$ and $\mathbf{z}$ are thus given by the squared moduli of the LCTs associated with U_A and U_B, i.e. $|\mathcal{F}_{\mathcal{A}}[\psi(\mathbf{x})](\mathbf{y})|^2$ and $|\mathcal{F}_{\mathcal{B}}[\psi(\mathbf{x})](\mathbf{z})|^2$. In order to find an entropic uncertainty relation for $\mathbf{y}$ and $\mathbf{z}$, we first express the complementarity between $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$ in the following Lemma.

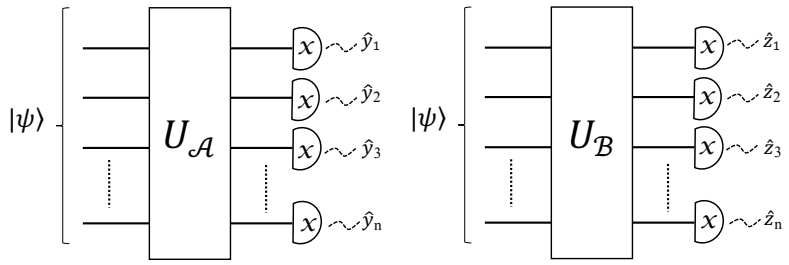

Figure 7.2: Schematic of two n-modal Gaussian projective measurements applied onto state $|\psi\rangle$, resulting in the n quadratures $\hat{y}_i$'s or $\hat{z}_i$'s. These measurements can be implemented by applying a Gaussian unitary (U_A or U_B) onto $|\psi\rangle$ and measuring the $\hat{x}$-quadratures of the n modes. Our uncertainty relation, Eq. (7.31), expresses the complementarity between the $\hat{y}_i$'s and $\hat{z}_i$'s, or equivalently between the linear canonical transforms associated with U_A and U_B.

Lemma 1. *Let $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$ be two LCTs of a function $f(\mathbf{x})$ with $\mathbf{x} = (x_1, \cdots x_n)$. Then, their squared moduli satisfy the entropic uncertainty relation*

$$h(|\mathcal{F}_\mathcal{A}|^2) + h(|\mathcal{F}_\mathcal{B}|^2) \geq \ln\left((\pi e)^n |\det(\mathcal{B}_b \mathcal{A}_a^T - \mathcal{B}_a \mathcal{A}_b^T)|\right) \tag{7.13}$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_a & \mathcal{A}_b \\ \mathcal{A}_c & \mathcal{A}_d \end{pmatrix} \quad \textit{and} \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_a & \mathcal{B}_b \\ \mathcal{B}_c & \mathcal{B}_d \end{pmatrix} \tag{7.14}$$

are the symplectic matrices associated with $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$ [acting on the quadrature operators as in Eq. (7.12)] and $h(\cdot)$ denotes Shannon differential entropy.

Proof. Let us define the function

$$G(\mathbf{x}) = C_{-b^{-1}a}\, \mathcal{F}_\mathcal{A}(\mathbf{x}) = e^{-\frac{i}{2}\mathbf{x}^T(b^{-1}a)\mathbf{x}}\, \mathcal{F}_\mathcal{A}(\mathbf{x}). \tag{7.15}$$

The inverse Fourier transform of $G(\mathbf{x})$ is

$$g(\mathbf{p}) = \mathcal{F}^{-1}[G(\mathbf{x})](\mathbf{p}) = [D_{-1}\mathcal{F}\, G](\mathbf{p}), \tag{7.16}$$

where the second equality results from property 3. Since $|G(\mathbf{x})|^2 = |\mathcal{F}_\mathcal{A}(\mathbf{x})|^2$, the probability distributions are equal, so that $h\left(|G(\mathbf{x})|^2\right) = h\left(|\mathcal{F}_\mathcal{A}(\mathbf{x})|^2\right)$. Then, we may apply the Białynicki-Birula and Mycielski entropic uncertainty relation, Eq. (4.39), to G and g, which gives

$$h\left(|\mathcal{F}_\mathcal{A}(\mathbf{x})|^2\right) + h\left(|g(\mathbf{p})|^2\right) \geq n\ln(\pi e). \tag{7.17}$$

With the change of variables $\mathbf{p} \to b^{-1}\mathbf{p}$, we have

$$\begin{aligned} h\left(|g(\mathbf{p})|^2\right) &= -\int |g(\mathbf{p})|^2 \ln|g(\mathbf{p})|^2\, d\mathbf{p} \\ &= -\int \left|g\left(b^{-1}\mathbf{p}\right)\right|^2 \ln\left|g\left(b^{-1}\mathbf{p}\right)\right|^2 \frac{d\mathbf{p}}{|\det(b)|}. \end{aligned} \tag{7.18}$$

By using the above properties of LCTs, we have

$$\begin{aligned} \left|g\left(b^{-1}\mathbf{p}\right)\right|^2 &= \left|\sqrt{\det(b)}\, D_{b^{-1}}[g](\mathbf{p})\right|^2 \\ &= |\det(b)|\, |[D_{b^{-1}}D_{-1}\mathcal{F}\, G](\mathbf{p})|^2 \\ &= |\det(b)|\, |[D_{b^{-1}}D_{-1}\mathcal{F}\, C_{-b^{-1}a}\mathcal{F}_\mathcal{A}](\mathbf{p})|^2 \\ &= |\det(b)|\, |[C_{-db^{-1}}D_{-b^{-1}}\mathcal{F}\, C_{-b^{-1}a}\mathcal{F}_\mathcal{A}](\mathbf{p})|^2 \\ &= |\det(b)|\, \left|\left[\mathcal{F}_{\mathcal{S}_\ominus}\mathcal{F}_\mathcal{A}\right](\mathbf{p})\right|^2 \\ &= |\det(b)|\, \left|\left[\mathcal{F}_{\mathcal{S}_\ominus\mathcal{A}}\right](\mathbf{p})\right|^2, \end{aligned} \tag{7.19}$$

where we have defined $\mathcal{S}_\ominus = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$, so by plugging it into Eq. (7.18), we get

$$h\left(|g\left(\mathbf{p}\right)|^2\right) \;=\; h\left(|\mathcal{F}_{\mathcal{S}_\ominus\mathcal{A}}\left(\mathbf{p}\right)|^2\right) - \ln|\det(b)| \tag{7.20}$$

since $|\mathcal{F}_{\mathcal{S}_\ominus\mathcal{A}}|^2$ is normalized. Now, replacing $h\left(|g\left(\mathbf{p}\right)|^2\right)$ in Eq. (7.17), we obtain

$$h\left(|\mathcal{F}_{\mathcal{A}}\left(\mathbf{x}\right)|^2\right) + h\left(|\mathcal{F}_{\mathcal{S}_\ominus\mathcal{A}}\left(\mathbf{p}\right)|^2\right) \geq \ln((\pi e)^n|\det(b)|). \tag{7.21}$$

The last step is simply to define $\mathcal{B} = \mathcal{S}_\ominus\mathcal{A}$ or equivalently $\mathcal{S}_\ominus = \mathcal{B}\mathcal{A}^{-1}$. Since symplectic matrices form a group and $\mathcal{A}$ and $\mathcal{B}$ are symplectic, $\mathcal{S}_\ominus$ is necessarily symplectic too. The property that $\mathcal{A}$ is symplectic translates into

$$\mathcal{A}^{-1} = J\mathcal{A}^T J^T = \begin{pmatrix} \mathcal{A}_d^T & -\mathcal{A}_b^T \\ -\mathcal{A}_c^T & \mathcal{A}_a^T \end{pmatrix}, \tag{7.22}$$

hence $b = \mathcal{B}_a\mathcal{A}_b^T - \mathcal{B}_b\mathcal{A}_a^T$. Replacing b into Eq. (7.21) completes the proof of Eq. (7.13), which thus provides a n-dimensional entropic uncertainty relation for any two incompatible LCTs. □

Note that in the special case of one mode ($n = 1$), we recover the result obtained by Guanlei *et al.* [98] for one-dimensional LCTs, namely

$$h(|\mathcal{F}_{\mathcal{A}}|^2) + h(|\mathcal{F}_{\mathcal{B}}|^2) \geq \ln\left(\pi e|ab' - a'b|\right) \tag{7.23}$$

for two 2×2 matrices $\mathcal{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\mathcal{B} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ (see Eq. (20) in Ref. [98]). Furthermore, for one-dimensional FRFTs (when $\mathcal{A}$ and $\mathcal{B}$ are simply rotations), we recover Eq. (7.4) (see Eq. (15) in Ref. [92]). Now, returning to the n-mode case, if we choose $\mathcal{A} = \mathbb{1}$ and $\mathcal{B}$ being the direct sum of $\pi/2$ rotations on each modes (i.e., the usual n-dimensional Fourier transform)[3], then $\mathcal{A}_a = \mathbb{1}$, $\mathcal{A}_b = 0$, $\mathcal{B}_a = 0$, and $\mathcal{B}_b = \mathbb{1}$, so that $\mathcal{B}_b\mathcal{A}_a^T - \mathcal{B}_a\mathcal{A}_b^T = \mathbb{1}$. Hence, we get back to the original entropic uncertainty relation of Białynicki-Birula and Mycielski, Eq. (4.39). Finally, if we consider twice the same measurement, i.e., $\mathcal{A} = \mathcal{B}$, then

$$\begin{aligned} \mathcal{S}_\ominus \;&=\; \mathcal{A}\mathcal{A}^{-1} = \begin{pmatrix} \mathcal{A}_a & \mathcal{A}_b \\ \mathcal{A}_c & \mathcal{A}_d \end{pmatrix}\begin{pmatrix} \mathcal{A}_d^T & -\mathcal{A}_b^T \\ -\mathcal{A}_c^T & \mathcal{A}_a^T \end{pmatrix} \\ &=\; \begin{pmatrix} \mathcal{A}_a\mathcal{A}_d^T - \mathcal{A}_b\mathcal{A}_c^T & -\mathcal{A}_a\mathcal{A}_b^T + \mathcal{A}_b\mathcal{A}_a^T \\ \mathcal{A}_c\mathcal{A}_d^T - \mathcal{A}_d\mathcal{A}_c^T & -\mathcal{A}_c\mathcal{A}_b^T + \mathcal{A}_d\mathcal{A}_a^T \end{pmatrix}. \end{aligned} \tag{7.24}$$

But since $\mathcal{S}_\ominus = \mathbb{1}$, we have $\mathcal{A}_b\mathcal{A}_a^T - \mathcal{A}_a\mathcal{A}_b^T = 0$, so that the lower bound in Eq. (7.13) is $-\infty$. This means that we have no lower limit on the entropy $h(|\mathcal{F}_{\mathcal{A}}|^2)$ so the probability distribution $|\mathcal{F}_{\mathcal{A}}|^2$ can be arbitrarily narrow, as expected.

Interestingly, in the special case where $\mathcal{A} = \mathbb{1}$, it is possible to find a simpler alterna-

[3] To be more precise, according to our definition of the quadrature vector $\mathbf{r}$, $\mathcal{B} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$.

tive proof of Lemma 1. We define

$$\mathcal{S} = \begin{pmatrix} \mathcal{B}_a & \mathcal{B}_b \\ -(\mathcal{B}_b^{-1})^T & 0 \end{pmatrix} \tag{7.25}$$

and may easily check that $\mathcal{S}$ is a symplectic matrix by verifying that $\mathcal{S}J\mathcal{S}^T = J$. Indeed

$$\mathcal{S}J\mathcal{S}^T = \begin{pmatrix} -\mathcal{B}_b\mathcal{B}_a^T + \mathcal{B}_a\mathcal{B}_b^T & \mathcal{B}_b\mathcal{B}_b^{-1} \\ -(\mathcal{B}_b^{-1})^T\mathcal{B}_b^T & 0 \end{pmatrix}, \tag{7.26}$$

is equal to J since $\mathcal{B}_a\mathcal{B}_b^T$ is a symmetric matrix and $\det(\mathcal{B}_b) \neq 0$ (as $\mathcal{B}$ is also a symplectic matrix). Thus, $\mathcal{S}$ transforms $\hat{r}$ into a new vector of quadratures,

$$\mathcal{S}\hat{r} = \begin{pmatrix} \mathbf{z} \\ -(\mathcal{B}_b^{-1})^T\mathbf{x} \end{pmatrix} \tag{7.27}$$

where $\mathbf{z}$ is the vector of position quadratures in $\mathbf{r}_B$ [see Eq. (7.12)]. Since $\mathcal{S}$ is symplectic, $\mathbf{z}$ and $-(\mathcal{B}_b^{-1})^T\mathbf{x}$ are two vectors of canonically conjugate quadratures, which we may plug into the usual entropic uncertainty relation, Eq. (4.39), giving

$$h\left(-(\mathcal{B}_b^{-1})^T\mathbf{x}\right) + h(\mathbf{z}) \geq n\ln(\pi e). \tag{7.28}$$

By using the scaling property of the differential entropy (see Section 3.2.2), we have $h(-(\mathcal{B}_b^{-1})^T\mathbf{x}) = h(\mathbf{x}) + \ln(|\det((\mathcal{B}_b^{-1})^T)|)$, so that Eq. (7.28) becomes

$$h(\mathbf{x}) + h(\mathbf{z}) \geq \ln\left((\pi e)^n |\det(\mathcal{B}_b)|\right). \tag{7.29}$$

Since the probability distribution of $\mathbf{x}$ is $|\mathcal{F}_{\mathbb{1}}|^2$ and that of $\mathbf{z}$ is $|\mathcal{F}_{\mathcal{B}}|^2$, we recover Lemma 1 when $\mathcal{A} = \mathbb{1}$.

7.2.2 Entropic uncertainty relation based on a commutators matrix

Lemma 1 provides an entropic uncertainty relation for any two n-dimensional LCTs, $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$. As we show in the following theorem, this uncertainty relation can also be expressed in terms of a matrix of commutators between the measured variables. This is our main result.

Theorem 5. *Let $\mathbf{y} = (\hat{y}_1, \cdots \hat{y}_n)^T$ be a vector of commuting quadratures and $\mathbf{z} = (\hat{z}_1, \cdots \hat{z}_n)^T$ be another vector of commuting quadratures. Let each of the components of $\mathbf{y}$ and $\mathbf{z}$ be written as a linear combination of the $(\hat{x}, \hat{p})$ quadratures of an n-modal system, namely*

$$\begin{aligned} \hat{y}_i &= \sum_{k=1}^{n} a_{i,k}\,\hat{x}_k + \sum_{k=1}^{n} a'_{i,k}\,\hat{p}_k \qquad (i = 1, \cdots n) \\ \hat{z}_j &= \sum_{k=1}^{n} b_{j,k}\,\hat{x}_k + \sum_{k=1}^{n} b'_{j,k}\,\hat{p}_k \qquad (j = 1, \cdots n). \end{aligned} \tag{7.30}$$

Then, the probability distributions of the vectors of jointly measured quadratures $\hat{y}_i$'s or $\hat{z}_j$'s

satisfy the entropic uncertainty relation

$$h(\mathbf{y}) + h(\mathbf{z}) \geq \ln\left((\pi e)^n |\det K|\right) \tag{7.31}$$

where $K_{ij} = [\hat{y}_i, \hat{z}_j]$ *denotes the* $n \times n$ *matrix of commutators (which are scalars) and* $h(\cdot)$ *denotes Shannon differential entropy.*

Proof. Since the quadratures $\hat{y}_i$ commute, $[\hat{y}_i, \hat{y}_j] = 0$, $\forall i, j$, they can be jointly measured, and similarly for the $\hat{z}_j$'s. Thus, the n measured quadratures correspond here to the output of $\mathcal{F}_\mathcal{A}$ or $\mathcal{F}_\mathcal{B}$ described by the symplectic matrix $\mathcal{A}$ or $\mathcal{B}$, as defined in Eq. (7.14). We simply have to compute the commutator between quadratures $\hat{y}_i$ (at the output of $\mathcal{F}_\mathcal{A}$) and $\hat{z}_j$ (at the output of $\mathcal{F}_\mathcal{B}$):

$$\begin{aligned}
K_{ji} &= [\hat{y}_j, \hat{z}_i] \\
&= \sum_{k=1}^{2n}\sum_{m=1}^{2n} \mathcal{A}_{jk}\mathcal{B}_{im}[\hat{r}_k, \hat{r}_m] \\
&= i\sum_{m=1}^{2n}\left(\sum_{k=1}^{n}\mathcal{A}_{jk}\mathcal{B}_{im}\delta_{m,k+n} - \sum_{k=n+1}^{2n}\mathcal{A}_{jk}\mathcal{B}_{im}\delta_{m,k-n}\right) \\
&= i\left(\sum_{k=1}^{n}\mathcal{A}_{jk}\mathcal{B}_{i,k+n} - \sum_{k=n+1}^{2n}\mathcal{A}_{jk}\mathcal{B}_{i,k-n}\right) \\
&= i\left(\sum_{k=1}^{n}\mathcal{A}_{jk}\mathcal{B}_{i,k+n} - \mathcal{A}_{j,k+n}\mathcal{B}_{ik}\right) \\
&= i\left(\sum_{k=1}^{n}(\mathcal{A}_a)_{jk}(\mathcal{B}_b)_{ik} - (\mathcal{A}_b)_{jk}(\mathcal{B}_a)_{ik}\right) \\
&= i\left(\mathcal{B}_b\mathcal{A}_a^T - \mathcal{B}_a\mathcal{A}_b^T\right)_{ij}
\end{aligned} \tag{7.32}$$

Using Lemma 1, we know that the probability distributions $|\mathcal{F}_\mathcal{A}(\mathbf{y})|^2$ and $|\mathcal{F}_\mathcal{B}(\mathbf{z})|^2$ satisfy the entropic uncertainty relation, Eq. (7.13). Since $\mathcal{B}_b\mathcal{A}_a^T - \mathcal{B}_a\mathcal{A}_b^T = -iK^T$, we have $|\det(\mathcal{B}_b\mathcal{A}_a^T - \mathcal{B}_a\mathcal{A}_b^T)| = |\det(K)|$, which concludes the proof of Eq. (7.31). □

We now show that this result holds even if we jointly measure n quadratures on a larger-dimensional system.

Theorem 5. (Extended version.) *Let* $\mathbf{y}_n = (\hat{y}_1, \cdots \hat{y}_n)^T$ *be a vector of commuting quadratures and* $\mathbf{z}_n = (\hat{z}_1, \cdots \hat{z}_n)^T$ *be another vector of commuting quadratures. Let each of the components of* $\mathbf{y}_n$ *and* $\mathbf{z}_n$ *be written as a linear combination of the* $(\hat{x}, \hat{p})$ *quadratures of a* N*-modal system with* $N > n$*, namely*

$$\begin{aligned}
\hat{y}_i &= \sum_{k=1}^{N} a_{i,k}\,\hat{x}_k + \sum_{k=1}^{N} a'_{i,k}\,\hat{p}_k \qquad (i = 1, \cdots n) \\
\hat{z}_j &= \sum_{k=1}^{N} b_{j,k}\,\hat{x}_k + \sum_{k=1}^{N} b'_{j,k}\,\hat{p}_k \qquad (j = 1, \cdots n)
\end{aligned} \tag{7.33}$$

Then, the probability distributions of the vectors of jointly measured quadratures $\hat{y}_i$'s or $\hat{z}_j$'s satisfy the entropic uncertainty relation

$$h(\mathbf{y}_n) + h(\mathbf{z}_n) \geq \ln\left((\pi e)^n |\det K|\right) \tag{7.34}$$

where $K_{ij} = [\hat{y}_i, \hat{z}_j]$ denotes the $n \times n$ matrix of commutators (which are scalars) and $h(\cdot)$ denotes Shannon differential entropy.

Proof. The N-dimensional vectors $\mathbf{y}$ and $\mathbf{z}$ can be decomposed as

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_n \\ \mathbf{y}_> \end{pmatrix}, \qquad \mathbf{z} = \begin{pmatrix} \mathbf{z}_n \\ \mathbf{z}_> \end{pmatrix}, \tag{7.35}$$

with $\mathbf{y}_n = (y_1, \cdots y_n)^T$ or $\mathbf{z}_n = (z_1, \cdots z_n)^T$ being the n measured quadratures, while $\mathbf{y}_> = (y_{n+1}, \cdots y_N)^T$ or $\mathbf{z}_> = (z_{n+1}, \cdots z_N)^T$ is being traced over. We write

$$\hat{y}_i = \sum_{k=1}^{2N} \mathcal{A}_{ik}\, \hat{r}_k \qquad \hat{z}_j = \sum_{k=1}^{2N} \mathcal{B}_{jk}\, \hat{r}_k \tag{7.36}$$

with $i, j = 1, \cdots n$, which generalizes Eq. (7.12) in the case where $\hat{r}$ is a $2N$-dimensional vector and $\mathcal{A}$ and $\mathcal{B}$ are $2N \times 2N$ symplectic matrices (we only need to specify the upper block of size $n \times 2N$ of $\mathcal{A}$ and $\mathcal{B}$, which defines $\hat{y}_i$ or $\hat{z}_j$, and complete the matrices by ensuring that they remain symplectic).

We first note that the right-hand side term of Eq. (7.34) is invariant under symplectic transformations (if both symplectic matrices $\mathcal{A}$ and $\mathcal{B}$ are multiplied by the same symplectic matrix). Indeed, the commutation relations are preserved along symplectic transformations and the determinant is invariant under permutations (the order of the quadratures is irrelevant), hence $\det(K)$ is invariant. Thus, we may always apply some symplectic transformation on the N modes so that the measured quadratures in the first case are $y_i = x_i$, with $i = 1, \cdots, n$. The two upper blocks of matrix $\mathcal{A}$ are then given by

$$\mathcal{A}_a = \begin{pmatrix} \mathbb{1}_{n\times n} & 0_{n\times(N-n)} \\ \cdots & \cdots \end{pmatrix}_{N\times N} \tag{7.37}$$

and

$$\mathcal{A}_b = \begin{pmatrix} 0_{n\times n} & 0_{n\times(N-n)} \\ \cdots & \cdots \end{pmatrix}_{N\times N} \tag{7.38}$$

where we do not need to specify the matrix elements denoted with a dot.

Next, we may assume with no loss of generality that the two upper blocks of $\mathcal{B}$ are given by

$$\mathcal{B}_a = \begin{pmatrix} B_{n\times n} & C_{n\times(N-n)} \\ D_{(N-n)\times n} & \mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix}_{N\times N} \tag{7.39}$$

and

$$\mathcal{B}_b = \begin{pmatrix} B'_{n\times n} & C'_{n\times(N-n)} \\ 0_{(N-n)\times n} & \mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix}_{N\times N}, \tag{7.40}$$

with B and C containing all $b_{j,k}$ entries for $j = 1, \cdots, n$ and $k = 1, \cdots, N$, and B' and C' containing all $b'_{j,k}$ entries for $j = 1, \cdots, n$ and $k = 1, \cdots, N$. This is the case because the last $N - n$ quadratures $\mathbf{z}_>$ are traced over, so they may be chosen arbitrarily as long as $\mathcal{B}$ remains symplectic. This means that we must check that $\mathcal{B}_a \mathcal{B}_b^T$ is symmetric, which implies that

$$D = \left(C - C'\right)^T \left(B'\right)^{-T} \tag{7.41}$$

where $(\cdot)^{-T}$ stands for the inverse of the transpose of the matrix. Thus, the matrix D can always be chosen in order to ensure that $\mathcal{B}$ is symplectic. It is easy to write the $n \times n$ restricted matrix of commutators of the measured quadratures $\mathbf{y}_n$ and $\mathbf{z}_n$ (the first n quadratures of $\mathbf{y}$ and $\mathbf{z}$), giving $|\det K| = |\det B'|$, so the inverse of B' is well defined as long as $\det K \neq 0$.

At this point, we only need to prove Eq. (7.34) in the case the upper blocks of $\mathcal{A}$ and $\mathcal{B}$ are defined as above and $|\det K|$ is replaced by $|\det B'|$. As before, we define the symplectic matrix

$$\mathcal{S} = \begin{pmatrix} \mathcal{B}_a & \mathcal{B}_b \\ -(\mathcal{B}_b)^{-T} & 0 \end{pmatrix}. \tag{7.42}$$

It transforms the vector of quadratures $\mathbf{r}$ into

$$\mathcal{S}\,\hat{r} = \begin{pmatrix} \mathcal{B}_a\,\mathbf{x} + \mathcal{B}_b\,\mathbf{p} \\ -(\mathcal{B}_b)^{-T}\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_n \\ \mathbf{z}_> \\ -(B')^{-T}\mathbf{x}_n \\ (C')^T(B')^{-T}\mathbf{x}_n - \mathbf{x}_> \end{pmatrix} \tag{7.43}$$

where $\mathbf{x}_n = (x_1, \cdots x_n)^T$, $\mathbf{x}_> = (x_{n+1}, \cdots x_N)^T$. This implies that $\mathbf{z}_n$ and $-(B')^{-T}\mathbf{x}_n$ are canonically conjugate n-tuples, so that we may apply Eq. (4.39) on the reduced state of the first n modes, giving

$$h\left(-(B')^{-T}\mathbf{x_n}\right) + h(\mathbf{z_n}) \geq n\ln(\pi e). \tag{7.44}$$

Using the scaling property of the differential entropy

$$h(-(B')^{-T}\mathbf{x_n}) = h(\mathbf{x_n}) + \ln(|\det(B')^{-T}|), \tag{7.45}$$

we obtain

$$h(\mathbf{x}_n) + h(\mathbf{z}_n) \geq \ln((\pi e)^n |\det B'|) \tag{7.46}$$

This implies Eq. (7.34), thus completing the proof of the extended version of Theorem 5. □

Interestingly, Eq. (7.34) coincides with Eq. (7.2) in the special case $n = 1$. Thus, our Theorem 5 can be viewed as an extension of the result by Huang [91] when we measure more than one mode ($n > 1$). As already mentioned, we can check that if $\mathcal{A} = \mathbb{1}$ and $\mathcal{B}$ is a direct sum of $\pi/2$-rotations on all modes (i.e., the usual Fourier transform), then $K = -i\mathbb{1}$ and we recover the Białynicki-Birula and Mycielski relation, Eq. (4.39).

7.2.3 Extension to Rényi entropies

We already saw in Chapter 3 that the Shannon differential entropy is a special case of the family of Rényi differential entropies defined as

$$h_\alpha(|f(\mathbf{x})|^2) = \frac{1}{1-\alpha} \ln\left(\int d\mathbf{x}\, (|f(\mathbf{x})|^2)^\alpha \right) \tag{7.47}$$

when $\alpha \to 1$. Let us now derive generalized entropic uncertainty relations for these entropies.

Theorem 6. *Let $\mathbf{y} = (\hat{y}_1, \cdots \hat{y}_n)^T$ be a vector of commuting quadratures, $\mathbf{z} = (\hat{z}_1, \cdots \hat{z}_n)^T$ be another vector of commuting quadratures, and let each of the components of these vectors be written as a linear combination of the $(\hat{x}, \hat{p})$ quadratures of an N-modal system ($N \geq n$). Then, the probability distributions of the vectors of jointly measured quadratures $\hat{y}_i$'s or $\hat{z}_j$'s satisfy the Rényi entropic uncertainty relation*

$$h_\alpha(\mathbf{y}) + h_\beta(\mathbf{z}) \;\; \geq \;\; n\ln(\pi) + \frac{n\ln(\alpha)}{2(\alpha-1)} + \frac{n\ln(\beta)}{2(\beta-1)} + \ln|\det K|\,. \tag{7.48}$$

where

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2, \qquad \alpha > 0, \qquad \beta > 0, \tag{7.49}$$

$K_{ij} = [\hat{y}_i, \hat{z}_j]$ is the matrix of commutators (which are scalars), and $h_\alpha(\cdot)$ is the Rényi differential entropy as defined in Eq. (7.47).

Proof. The proof follows exactly the same steps as the proof of Lemma 1 and Theorem 5 (both versions) except that Eq. (4.39) is replaced by its counterpart for Rényi entropies (see Eq. (4.44))

$$h_\alpha(\mathbf{x}) + h_\beta(\mathbf{p}) \geq n\ln(\pi) + \frac{n\ln(\alpha)}{2(\alpha-1)} + \frac{n\ln(\beta)}{2(\beta-1)} \tag{7.50}$$

for (α, β) satisfying Eq. (7.49). Note that Rényi entropies also verify the scaling property $h_\alpha(S\mathbf{x}) = h_\alpha(\mathbf{x}) + \ln|S|$ (see Eq. 3.27). □

As expected, in the limit where $\alpha \to 1$ and $\beta \to 1$, we recover our uncertainty relations for Shannon differential entropies. In addition, in the one-dimensional case ($N = n = 1$), Eq. (7.48) coincides with the result found in [98].

7.2.4 Corresponding covariance-based uncertainty relation

Finally, by exploiting Theorem 5, it is also possible to derive an uncertainty relation in terms of covariance matrices. This can been viewed as a n-dimensional extension of the usual Robertson uncertainty relation in position and momentum spaces where, instead of expressing the complementarity between observables $\hat{A}$ and $\hat{B}$ (which are linear combinations of quadratures), namely

$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{|[\hat{A},\hat{B}]|}{2} \tag{7.51}$$

with $[\hat{A},\hat{B}]$ being a scalar, we consider the complementarity between two n-tuples of commuting observables.

Theorem 7. *Let $\mathbf{y} = (\hat{y}_1, \cdots \hat{y}_n)^T$ be a vector of commuting quadratures, $\mathbf{z} = (\hat{z}_1, \cdots \hat{z}_n)^T$ be another vector of commuting quadratures, and let each of the components of these vectors be written as a linear combination of the $(\hat{x},\hat{p})$ quadratures of an N-modal system ($N \geq n$). Let $\gamma^{\mathcal{A}}_{ij} = \langle\{\hat{y}_i,\hat{y}_j\}\rangle/2 - \langle\hat{y}_i\rangle\langle\hat{y}_j\rangle$ and $\gamma^{\mathcal{B}}_{ij} = \langle\{\hat{z}_i,\hat{z}_j\}\rangle/2 - \langle\hat{z}_i\rangle\langle\hat{z}_j\rangle$ be the (reduced) covariance matrices of the $\hat{y}_i$ and $\hat{z}_i$ quadratures. Then*

$$\left(\det\gamma^{\mathcal{A}}\right)^{\frac{1}{2}}\left(\det\gamma^{\mathcal{B}}\right)^{\frac{1}{2}} \geq \frac{|\det K|}{2^n} \tag{7.52}$$

where $K_{ij} = [\hat{y}_i,\hat{z}_j]$ denotes the commutator matrix.

Proof. Let us define the entropy powers of $\mathbf{y}$ and $\mathbf{z}$ as

$$N_{\mathcal{A}} = \frac{1}{2\pi e}e^{\frac{2}{n}h(\mathbf{y})}, \qquad N_{\mathcal{B}} = \frac{1}{2\pi e}e^{\frac{2}{n}h(\mathbf{z})}, \tag{7.53}$$

which allows us to rewrite Eq. (7.31) as an entropy-power uncertainty relation (see Chapter 6)

$$N_{\mathcal{A}}N_{\mathcal{B}} \geq \frac{|\det K|^{2/n}}{4}. \tag{7.54}$$

Since the maximum entropy for a fixed covariance matrix is reached by the Gaussian distribution, we have that $N_{\mathcal{A}} \leq (\det\gamma^{\mathcal{A}})^{1/n}$ and $N_{\mathcal{B}} \leq (\det\gamma^{\mathcal{B}})^{1/n}$. Combining these inequalities with Eq. (7.54), we prove our theorem. □

In the one-mode case, we obtain $\Delta\hat{y}_1\Delta\hat{z}_1 \geq |[\hat{y}_1,\hat{z}_1]|/2$ which is Robertson uncertainty relation applied to the two quadratures $\hat{y}_1$ and $\hat{z}_1$, as already mentioned. Thus, Theorem 7 extends this relation to two joint measurements of n modes and accounts for the correlations between the $\hat{y}_i$'s via the term $\det\gamma^{\mathcal{A}}$ (as well as between the $\hat{z}_j$'s via the term $\det\gamma^{\mathcal{B}}$). Note, however, that this covariance-based uncertainty relation is less strong than the entropic uncertainty relation since Theorem 7 follows from Theorem 5.

7.3 Conclusion

We have derived an entropic uncertainty relation which applies to any two n-dimensional LCTs $\mathcal{F}_{\mathcal{A}}(\mathbf{y})$ and $\mathcal{F}_{\mathcal{B}}(\mathbf{z})$ or any two n-modal Gaussian projective measurements resulting in outcomes $\mathbf{y}$ and $\mathbf{z}$. As implied by our Theorem 5, the sum of the entropy of the probability distributions for $\mathbf{y}$ and $\mathbf{z}$ is lower bounded by a quantity that depends on the determinant of the matrix of commutators $[\hat{y}_i, \hat{z}_j]$, a quantity that is invariant under symplectic transformations. This is a generalization of the usual entropic uncertainty relation due to Białynicki-Birula and Mycielski in the case of any two n-dimensional observables that are not canonically conjugate but are connected by an arbitrary LCT.

Theorem 5 can also be viewed as a natural extension of the uncertainty relation (7.2) due to Huang [91]. As shown in Figure 7.2, the two considered measurements can be realized by applying a Gaussian unitary U_A or U_B before measuring the $\hat{x}$ quadratures. If we restrict ourselves to measuring the $\hat{x}$ quadrature of the first mode only, then the resulting quadrature is $\hat{A}$ or $\hat{B}$ as defined in Eq. (7.1). Thus, our entropic uncertainty relation generalizes Huang's setup by including the measurement of any number of modes instead of the first one only. It naturally accounts for the correlations between the measured y_i's (as well as z_j's) via the use of joint entropies. Following the same scheme, we also recover the usual entropic uncertainty relation by applying either the identity ($U_{\mathcal{A}} = \mathbb{1}$) or a tensor product of $\pi/2$ rotations on each mode ($U_{\mathcal{B}} = R_{\pi/2}^{\otimes n}$) before measuring all x quadratures.

Our results still hold true (with some adaptations) when Shannon entropies are replaced by Rényi entropies, as proven in Theorem 6. They also imply a generalized version of Robertson uncertainty relation expressing the complementarity between two n-tuples of quadrature observables in terms of the determinant of a commutator matrix, see Theorem 7.

As a final note, it must be stressed that we have restricted ourselves to observables that are linear combinations of the $\hat{x}$ and $\hat{p}$ quadratures throughout this work, which implies that all commutators $[\hat{y}_i, \hat{z}_j]$ are scalars, as well as $\det K$. However, we believe that it should be possible to extend Theorem 5 to general vectors of commuting Hermitian operators $\mathbf{A}$ and $\mathbf{B}$. Then, all commutators would be replaced by their mean values, in analogy to the usual Robertson relation. We therefore suggest the following conjecture:

Conjecture 1. *Let $\mathbf{A} = (A_1, \cdots A_n)$ be a vector of commuting observables, $\mathbf{B} = (B_1, \cdots B_n)$ be another vector of commuting observables and $|\psi\rangle$ be the state of the system. The probability distributions of the jointly measured observables A_i's or B_j's in state $|\psi\rangle$ satisfy the entropic uncertainty relation*

$$h(\mathbf{A}) + h(\mathbf{B}) \geq \ln\left((\pi e)^n |\det\langle\psi|K|\psi\rangle|\right) \tag{7.55}$$

where $K_{ij} = [A_i, B_j]$

This would be a further generalization of the entropic uncertainty relation, also implying an extended Robertson relation involving a matrix of mean values of commutators instead of Eq. (7.52). Investigating this conjecture is an interesting topic of future work.

8 | Tight entropic uncertainty relation for arbitrary quadratures

In Chapter 6, we showed that it is possible to define an entropic uncertainty relation which is saturated by all pure Gaussian states, in contrast to the usual entropic uncertainty of Białynicki-Birula and Mycielski which is saturated only by pure Gaussian states with vanishing covariance. This uncertainty relation is expressed as (see Eq. (6.14))

$$h(\mathbf{x}) + h(\mathbf{p}) - \frac{1}{2}\ln\left(\frac{\det(\gamma_x)\det(\gamma_p)}{\det(\gamma)}\right) \geq n\ln(\pi e) \tag{8.1}$$

where γ_x and γ_p are the reduced covariance matrices of the $\hat{x}$ and $\hat{p}$ quadratures and γ is the full covariance matrix. We remember that the goal was to establish an analogy with the Robertson-Schrödinger uncertainty relation $\det\gamma \geq 1/4$ which is saturated by all pure Gaussian states.

In Chapter 7, we gave another improvement of the entropic uncertainty relation, this time by expressing an uncertainty relation valid for any n-tuples of quadrature observables $\mathbf{y}$ and $\mathbf{z}$, which are not necessarily canonically conjugate. It is expressed as (see Eq. (7.31))

$$h(\mathbf{y}) + h(\mathbf{z}) \geq \ln\left((\pi e)^n |\det K|\right) \tag{8.2}$$

where $K_{ij} = [\hat{y}_i, \hat{z}_j]$. The idea behind this uncertainty relation was also to generalize the Robertson-Schrödinger uncertainty relation, but in a different way than Eq. (8.1). Indeed, we saw in Chapter 4 that $\det\gamma \geq 1/4$ actually comes from the relation

$$\sigma_A^2\sigma_B^2 \geq \left|\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2 + \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|^2 \tag{8.3}$$

which expresses the uncertainty of any operators $\hat{A}$ and $\hat{B}$ which are not necessary canonically conjugate. The important point is that the lower bound on the uncertainty depends on the commutator between the two observables (the anticommutator appears in the definition of the covariance matrix γ) and we thus generalized the entropic uncertainty in the same way, that is by showing that the lower bound of the

uncertainty of non-canonically conjugate n-tuples of observables actually depends on their commutators.

Since we just presented two improvements of the entropic uncertainty relation, both arising from the idea of translating the variance-based uncertainty relation of Schrödinger-Robertson to the entropic framework, it seems natural to combine both relations to create a more general one. In particular, relation (8.2), like the Białynicki-Birula and Mycielski one, is not saturated by all pure Gaussian states. To see this, let us simply evaluate the one-mode case of Eq. (8.2)[1]

$$h(x) + h(x_\theta) \geq \ln(\pi e |\sin\theta|) \tag{8.4}$$

for a squeezed state with covariance matrix

$$\gamma = \frac{1}{2}\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}. \tag{8.5}$$

Since the entropy of a Gaussian distributed variable is given by $h(x) = \ln(2\pi e\sigma_x^2)/2$, Eq. (8.4) is saturated when

$$\begin{aligned} h(x) + h(x_\theta) = \frac{1}{2}\ln((2\pi e)^2\sigma_x^2\sigma_{x_\theta}^2) &= \ln(\pi e|\sin\theta|) \\ \Leftrightarrow \qquad 4\sigma_x^2\sigma_{x_\theta}^2 &= \sin^2\theta. \end{aligned} \tag{8.6}$$

From the covariance matrix, we can compute the variances of the x and x_θ-quadratures

$$\sigma_x^2 = \frac{e^{-2r}}{2}, \qquad \sigma_{x_\theta}^2 = \frac{e^{-2r}\cos^2\theta}{2} + \frac{e^{2r}\sin^2\theta}{2}. \tag{8.7}$$

Thus, Eq. (8.4) can be saturated only if

$$e^{-4r}\cos^2\theta + \sin^2\theta = \sin^2\theta \tag{8.8}$$

which happens only if $\theta = \pm\pi/2$ — which is nothing else than the Białynicki-Birula situation since $x_{\pi/2} = p$ — or in the limit of infinite squeezing. But in general, for an arbitrary angle θ, the squeezed state with covariance matrix (8.5) does not saturate the entropic uncertainty relation (8.2). The latter is thus not saturated by all pure Gaussian states. In fact, we will see in Section 8.2 that Eq. (8.2) is only saturated by pure Gaussian states with vanishing covariance between the y and z quadratures.

Therefore, the idea of combining equations (8.1) and (8.2) is to get an entropic uncertainty relation valid for any n-tuples of quadrature observables and saturated by *all* pure Gaussian states because we now take the y-z correlations into account.

In the next sections we start by introducing our new entropic uncertainty relation and then show that it is saturated by all pure Gaussian states. We then give a conditional

[1] This also corresponds to Eq. (7.4).

proof of this result based on two assumptions, as we did in Chapter 6. We end by showing that this entropic uncertainty relation implies the generalized version of the Schrödinger-Robertson uncertainty relation.

8.1 A general entropic uncertainty relation saturated by all pure Gaussian states

Let $|\psi\rangle$ be an arbitrary n-mode state. We wish to express the complementarity between n-tuples of quadrature observables which can be represented by the output x-quadrature measurements of two incompatible linear canonical transforms (see Section 7.1) corresponding to two Gaussian unitaries (U_A or U_B) applied onto $|\psi\rangle$. The vectors of measurement outcomes are denoted by $\mathbf{y} = (y_1, \cdots, y_n)^T$ and $\mathbf{z} = (z_1, \cdots, z_n)^T$ respectively. $\mathcal{A}$ and $\mathcal{B}$ denoting the symplectic transformations associated with U_A and U_B, and writing $\mathbf{r} = (\hat{x}_1, \cdots, \hat{x}_n, \hat{p}_1, \cdots, \hat{p}_n)^T$ for the $2n$-dimensional vector of input quadratures, we may express the corresponding vectors of output quadratures as

$$\mathbf{r}_A = \mathcal{A}\,\mathbf{r} \equiv \begin{pmatrix} \mathbf{y} \\ \mathbf{q} \end{pmatrix}, \quad \mathbf{r}_B = \mathcal{B}\,\mathbf{r} \equiv \begin{pmatrix} \mathbf{z} \\ \mathbf{o} \end{pmatrix} \tag{8.9}$$

where $\mathbf{q}$ (resp. $\mathbf{o}$) is the vector of quadratures that are canonically conjugate with $\mathbf{y}$ (resp. $\mathbf{z}$). As we do for the x, p-quadratures, it is possible to define a covariance matrix Γ for the y, z quadratures. Its elements are expressed as

$$\Gamma_{ij} = \frac{1}{2}\langle R_i R_j + R_j R_i \rangle - \langle R_i \rangle \langle R_j \rangle \tag{8.10}$$

with $\mathbf{R} = (y_1, ..., y_n, z_1, ..., z_n)$. Using definition (8.10) of the covariance matrix Γ, we introduce the general form of the entropic uncertainty relation.

$$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\ln\left(\frac{\det(\Gamma_y)\det(\Gamma_z)}{\det(\Gamma)}\right) \geq \ln\left((\pi e)^n |\det K|\right) \tag{8.11}$$

where $h(\cdot)$ denotes the Shannon differential entropy, Γ_y and Γ_z are the reduced covariance matrices of the $\hat{y}_i$ and $\hat{z}_i$ quadratures and $K_{ij} = [\hat{y}_i, \hat{z}_j]$ defines the elements of the $n \times n$ matrix of commutators (which are scalars).

We will give the (conditional) proof of Eq. (8.11) in Section 8.4, but first, remark that this uncertainty relation is invariant under displacements. Indeed, we know that any differential entropy is invariant under displacements [31] and so are Γ, Γ_y and Γ_z as it is obvious to see from their definitions. From now on, we can thus restrict to states centered at the origin.

8.2 Explicit entropy calculation for Gaussian states

The understanding of the proof of Eq. (8.11) is easier if we first study how this entropic uncertainty relation applies to pure Gaussian states. The Wigner function of a *pure* n-modal Gaussian state is given by [2]

$$W^G(x,p) = \frac{1}{\pi^n} e^{-\frac{1}{2}\mathbf{r}^T\gamma^{-1}\mathbf{r}} \tag{8.12}$$

with $\mathbf{r} = (x_1, ..., x_n, p_1, ..., p_n)$ and its covariance matrix is expressed as

$$\gamma = \begin{pmatrix} \gamma_x & \gamma_{xp} \\ \gamma_{xp} & \gamma_p \end{pmatrix}_{2n\times 2n} \tag{8.13}$$

where γ_x and γ_p are the reduced covariance matrices of the position and momentum quadratures. Note that since the state is pure and Gaussian, $\det(\gamma) = (1/4)^n$.

To evaluate Eq. (8.11) we need to find the covariance matrix Γ, Eq. (8.10), for the y,z-quadratures. Since we know how to obtain y and z from x and p through the symplectic transformations $\mathcal{A}$ and $\mathcal{B}$, we can compute the elements of Γ. For example, if we evaluate Γ_{ij} for $1 \leq i,j \leq n$, we have

$$\begin{aligned}
\Gamma_{ij} &= \frac{1}{2}\langle R_i R_j + R_j R_i\rangle - \langle R_i\rangle\langle R_j\rangle \\
&= \frac{1}{2}\langle \sum_{k=1}^{2n}\mathcal{A}_{ik}r_k \sum_{l=1}^{2n}\mathcal{A}_{jl}r_l + \sum_{l=1}^{2n}\mathcal{A}_{jl}r_l \sum_{k=1}^{2n}\mathcal{A}_{ik}r_k\rangle - \langle\sum_{k=1}^{2n}\mathcal{A}_{ik}r_k\rangle\langle\sum_{l=1}^{2n}\mathcal{A}_{jl}r_l\rangle \\
&= \sum_{k=1}^{2n}\sum_{l=1}^{2n}\mathcal{A}_{ik}\mathcal{A}_{jl}\gamma_{kl} \\
&= \sum_{k=1}^{2n}\sum_{l=1}^{2n}\mathcal{A}_{ik}\gamma_{kl}\mathcal{A}^T_{lj} \\
&= (\mathcal{A}\gamma\mathcal{A}^T)_{ij}.
\end{aligned} \tag{8.14}$$

In the same way, if we evaluate $\Gamma_{i+n,j+n}$ for $1 \leq i,j \leq n$, we have

$$\begin{aligned}
\Gamma_{i+n,j+n} &= \frac{1}{2}\langle R_{i+n}R_{j+n} + R_{j+n}R_{i+n}\rangle - \langle R_{i+n}\rangle\langle R_{j+n}\rangle \\
&= \frac{1}{2}\langle \sum_{k=1}^{2n}\mathcal{B}_{ik}r_k \sum_{l=1}^{2n}\mathcal{B}_{jl}r_l + \sum_{l=1}^{2n}\mathcal{B}_{jl}r_l \sum_{k=1}^{2n}\mathcal{B}_{ik}r_k\rangle - \langle\sum_{k=1}^{2n}\mathcal{B}_{ik}r_k\rangle\langle\sum_{l=1}^{2n}\mathcal{B}_{jl}r_l\rangle \\
&= \sum_{k=1}^{2n}\sum_{l=1}^{2n}\mathcal{B}_{ik}\mathcal{B}_{jl}\gamma_{kl} \\
&= \sum_{k=1}^{2n}\sum_{l=1}^{2n}\mathcal{B}_{ik}\gamma_{kl}\mathcal{B}^T_{lj} \\
&= (\mathcal{B}\gamma\mathcal{B}^T)_{ij}
\end{aligned} \tag{8.15}$$

[2] See Section 2.3.

and if we evaluate $\Gamma_{i,j+n}$ for $1 \leq i, j \leq n$, we have

$$\begin{aligned}
\Gamma_{i,j+n} &= \frac{1}{2}\langle R_i R_{j+n} + R_{j+n} R_i \rangle - \langle R_i \rangle \langle R_{j+n} \rangle \\
&= \frac{1}{2}\langle \sum_{k=1}^{2n} \mathcal{A}_{ik} r_k \sum_{l=1}^{2n} \mathcal{B}_{jl} r_l + \sum_{l=1}^{2n} \mathcal{B}_{jl} r_l \sum_{k=1}^{2n} \mathcal{A}_{ik} r_k \rangle - \langle \sum_{k=1}^{2n} \mathcal{B}_{ik} r_k \rangle \langle \sum_{l=1}^{2n} \mathcal{A}_{jl} r_l \rangle \\
&= \sum_{k=1}^{2n} \sum_{l=1}^{2n} \mathcal{A}_{ik} \mathcal{B}_{jl} \gamma_{kl} \\
&= \sum_{k=1}^{2n} \sum_{l=1}^{2n} \mathcal{A}_{ik} \gamma_{kl} \mathcal{B}^T_{lj} \\
&= (\mathcal{A}\gamma\mathcal{B}^T)_{ij}. \qquad (8.16)
\end{aligned}$$

Since the covariance matrix is symmetric, we can now write

$$\Gamma = \begin{pmatrix} \Gamma_y & \Gamma_{yz} \\ \Gamma_{yz} & \Gamma_z \end{pmatrix} = \begin{pmatrix} (\mathcal{A}\gamma\mathcal{A}^T)_{i,j=1,\ldots,n} & (\mathcal{A}\gamma\mathcal{B}^T)_{i,j=1,\ldots,n} \\ (\mathcal{B}\gamma\mathcal{A}^T)_{i,j=1,\ldots,n} & (\mathcal{B}\gamma\mathcal{B}^T)_{i,j=1,\ldots,n} \end{pmatrix}. \qquad (8.17)$$

Pay attention to the fact that matrices $\mathcal{A}\gamma\mathcal{A}^T$, $\mathcal{A}\gamma\mathcal{B}^T$, $\mathcal{B}\gamma\mathcal{A}^T$ and $\mathcal{B}\gamma\mathcal{B}^T$ all have dimensions $2n \times 2n$ but we truncate them to keep only the reduced matrices with indexes running from 1 to n. Therefore, Γ_y, Γ_z and Γ_{yz} have dimension $n \times n$ while Γ is a $2n \times 2n$ matrix.

As we did in the previous chapter, we can use a block matrix representation of the symplectic transformations,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_a & \mathcal{A}_b \\ \mathcal{A}_c & \mathcal{A}_d \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_a & \mathcal{B}_b \\ \mathcal{B}_c & \mathcal{B}_d \end{pmatrix} \qquad (8.18)$$

so that we can simplify the expression of the covariance matrix. Indeed, we have, for example

$$\begin{aligned}
\mathcal{A}\gamma\mathcal{A}^T &= \begin{pmatrix} \mathcal{A}_a & \mathcal{A}_b \\ \mathcal{A}_c & \mathcal{A}_d \end{pmatrix} \begin{pmatrix} \gamma_x & \gamma_{xp} \\ \gamma_{xp} & \gamma_p \end{pmatrix} \begin{pmatrix} \mathcal{A}_a^T & \mathcal{A}_c^T \\ \mathcal{A}_b^T & \mathcal{A}_d^T \end{pmatrix} \qquad (8.19) \\
&= \begin{pmatrix} \mathcal{A}_a\gamma_x\mathcal{A}_a^T + \mathcal{A}_a\gamma_{xp}\mathcal{A}_b^T + \mathcal{A}_b\gamma_{xp}\mathcal{A}_a^T + \mathcal{A}_b\gamma_p\mathcal{A}_b^T & \cdots \\ \cdots & \cdots \end{pmatrix}
\end{aligned}$$

where we do not need to specify the matrix elements with the dots since all we need is

$$(\mathcal{A}\gamma\mathcal{A}^T)_{i,j=1,\ldots,n} = \mathcal{A}_a\gamma_x\mathcal{A}_a^T + \mathcal{A}_a\gamma_{xp}\mathcal{A}_b^T + \mathcal{A}_b\gamma_{xp}\mathcal{A}_a^T + \mathcal{A}_b\gamma_p\mathcal{A}_b^T. \qquad (8.20)$$

By doing the same calculations for the other reduced matrices of Γ we finally obtain that Γ can be written as the product of the three following matrices

$$\Gamma = \begin{pmatrix} \mathcal{A}_a & \mathcal{A}_b \\ \mathcal{B}_a & \mathcal{B}_b \end{pmatrix} \begin{pmatrix} \gamma_x & \gamma_{xp} \\ \gamma_{xp} & \gamma_p \end{pmatrix} \begin{pmatrix} \mathcal{A}_a & \mathcal{A}_b \\ \mathcal{B}_a & \mathcal{B}_b \end{pmatrix}^T. \qquad (8.21)$$

In particular, it means that the determinant of Γ is given by

$$\det\Gamma = \det\gamma \det\left[\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}\right]^2. \tag{8.22}$$

It is already an elegant expression, but it can be further simplified. First note that $\mathcal{B}$ represents a symplectic transformation and hence satisfies $\mathcal{B}\left(\begin{smallmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{smallmatrix}\right)\mathcal{B}^T = \left(\begin{smallmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{smallmatrix}\right)$ (see Section 2.4.1). In particular, this means that[3] $\mathcal{B}_a\mathcal{B}_b^T = \mathcal{B}_b\mathcal{B}_a^T$ or $\mathcal{B}_a = \mathcal{B}_b\mathcal{B}_a^T\mathcal{B}_b^{-T}$. Second, let us consider a block matrix M of size $(n+m)\times(n+m)$ written as

$$M = \begin{pmatrix}A_{n\times n} & B_{n\times m}\\ C_{m\times n} & D_{m\times m}\end{pmatrix}. \tag{8.23}$$

It is easy to see that the following equality holds (we assume that D is invertible[4])

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ -D^{-1}C & \mathbb{1}\end{pmatrix} = \begin{pmatrix}A-BD^{-1}C & B\\ 0 & D\end{pmatrix} \tag{8.24}$$

and if we take the determinant of this equation we obtain

$$\det(M) = \det(A-BD^{-1}C)\det(D) \tag{8.25}$$

where we have exploited the fact that the determinant of a block triangular matrix is given by the product of the determinants of its diagonal blocks [99]. Using this relation together with the symmetry of the matrix $\mathcal{B}_a\mathcal{B}_b^T$ we can compute the following determinant

$$\begin{aligned}\det\left[\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}\right] &= \det(\mathcal{A}_a-\mathcal{A}_b\mathcal{B}_b^{-1}\mathcal{B}_a)\det\mathcal{B}_b\\ &= \det(\mathcal{A}_a-\mathcal{A}_b\mathcal{B}_b^{-1}\mathcal{B}_b\mathcal{B}_a^T\mathcal{B}_b^{-T})\det\mathcal{B}_b^T\\ &= \det(\mathcal{A}_a-\mathcal{A}_b\mathcal{B}_a^T\mathcal{B}_b^{-T})\det\mathcal{B}_b^T\\ &= \det(\mathcal{A}_a\mathcal{B}_b^T-\mathcal{A}_b\mathcal{B}_a^T)\\ &= \det(\mathcal{B}_b\mathcal{A}_a^T-\mathcal{B}_a\mathcal{A}_b^T)^T\\ &= \det(\mathcal{B}_b\mathcal{A}_a^T-\mathcal{B}_a\mathcal{A}_b^T).\end{aligned} \tag{8.26}$$

Therefore, the determinant of Γ can now be written as

$$\det\Gamma = \det\gamma\left(\det(\mathcal{B}_b\mathcal{A}_a^T-\mathcal{B}_a\mathcal{A}_b^T)\right)^2. \tag{8.27}$$

But this expression can still be simplified. Indeed, we proved in Chapter 7 that $|\det(\mathcal{B}_b\mathcal{A}_a^T-\mathcal{B}_a\mathcal{A}_b^T)| = |\det K|$ were $K_{ij} = [\hat{y}_i,\hat{z}_j]$ are the elements of the matrix

[3] $(\cdot)^{-T}$ denotes the transpose of the inverse.
[4] If D is not invertible, Eq. (8.24) can be written in a similar way in terms of A^{-1}.

of the commutators as defined in Eq. (8.11)). Thus,

$$\det \Gamma = \det \gamma |\det K|^2. \tag{8.28}$$

Note that this equation is actually true for any state, Gaussian or not. In our specific example however, since we are dealing with a pure Gaussian state, we know that the determinant of its covariance matrix, is equal to $1/4^n$ so that

$$\det \Gamma^G = \frac{1}{4^n} |\det K|^2. \tag{8.29}$$

We now know the covariance matrix Γ and its determinant so that the last step before evaluating Eq. (8.11) is to compute the differential entropies of the y and z quadratures. We remember that those quadratures are obtained after applying some symplectic transformations. This means that the Wigner function of our initial state, which is a Gaussian, might be squeezed or rotated, but will remain Gaussian. The probability distributions of the jointly measured quadratures $\hat{y}_i$ or $\hat{z}_j$ are thus given by the following Gaussian distributions

$$P(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^n \det \Gamma_y}} e^{-\frac{1}{2}\mathbf{y}^T \Gamma_y^{-1} \mathbf{y}}, \qquad P(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n \det \Gamma_z}} e^{-\frac{1}{2}\mathbf{z}^T \Gamma_z^{-1} \mathbf{z}} \tag{8.30}$$

and we easily evaluate the differential entropies

$$h(\mathbf{y}) = \frac{1}{2} \ln \left((2\pi e)^n \det \Gamma_y \right), \qquad h(\mathbf{z}) = \frac{1}{2} \ln \left((2\pi e)^n \det \Gamma_z \right). \tag{8.31}$$

Inserting these quantities together with the value of $\det(\Gamma^G)$ in Eq. (8.11) yields to the bound $\ln((\pi e)^n |\det(K)|)$ so that all Gaussian pure states saturate this entropic uncertainty relation, as desired.

Let us emphasize the fact that the previous entropic uncertainty relation, which does not take correlations into account (see Eq. (8.2)) is only saturated by pure states with vanishing covariance Γ_{yz}. Indeed if we evaluate Eq. (8.2) for our pure Gaussian state we find

$$h(\mathbf{y}) + h(\mathbf{z}) = \ln \left((2\pi e)^n \sqrt{\det \Gamma_y \det \Gamma_z} \right). \tag{8.32}$$

This quantity will be equal to the lower bound of Eq. (8.2), $\ln((\pi e)^n |\det(K)|)$ only when

$$\begin{aligned} & 2^n \sqrt{\det \Gamma_y \det \Gamma_z} &&= |\det(K)| \\ \Leftrightarrow \quad & 2^n \sqrt{\det \Gamma_y \det \Gamma_z} &&= \sqrt{\det \Gamma}\, 2^n \\ \Leftrightarrow \quad & \det \Gamma_y \det \Gamma_z &&= \det \Gamma \end{aligned} \tag{8.33}$$

where we used Eq. (8.29). Obviously this is true only when $\Gamma_{yz} = 0$, i.e. when there is no correlation between the y_i and z_i quadratures. This calculation thus confirms

that the initial entropic uncertainty relation, based on the commutators matrix, is not saturated by all pure Gaussian states.

If we now consider a general mixed Gaussian state, Eq. (8.11) is easy to prove and we can actually state the following theorem:

Theorem 8. *Let ρ^G be a Gaussian state and let $\mathbf{y}$ and $\mathbf{z}$ be two vectors of commuting quadratures, as defined in Eq. (8.9). Then, the probability distributions of the vectors of jointly measured quadratures $\hat{y}_i$'s or $\hat{z}_j$'s satisfy the entropic uncertainty relation (8.11). The saturation is obtained if and only if ρ^G is pure.*

Proof. For any Gaussian state, not necessarily pure, its entropies are given by Eq. (8.31). Therefore,

$$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\left(\frac{\det(\Gamma_y)\det(\Gamma_z)}{\det(\Gamma)}\right) = \ln\left((2\pi e)^n\sqrt{\det(\Gamma)}\right). \tag{8.34}$$

Robertson-Schrödinger uncertainty relation tells us that[5] $\det\gamma \geq 1/4^n$ and from Eq. (8.28) we can thus deduce that

$$\sqrt{\det\Gamma} \geq \frac{|\det K|}{2^n} \tag{8.35}$$

which means that

$$\ln\left((2\pi e)^n\sqrt{\det(\Gamma)}\right) \geq \ln\left((2\pi e)^n\frac{|\det K|}{2^n}\right) = \ln((\pi e)^n|\det K|) \tag{8.36}$$

which completes the proof. We thus proved that inequality (8.11) is true for all Gaussian states and moreover, as we saw previously, it is saturated when the Gaussian states are pure. □

8.3 Other formulations

We emphasized in Eq. (8.28) the relation between $\det(\Gamma)$ and $\det(K)$. Using this equivalence, we can rewrite our entropic uncertainty relation (8.11) as

$$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\ln\left(\frac{\det\Gamma_y\det\Gamma_z}{\det\gamma}\right) \geq \ln\left((\pi e)^n\right). \tag{8.37}$$

Remark that γ is the covariance matrix for the x, p-quadratures while Γ_y and Γ_z are the reduced covariance matrices of the y, z quadratures. Interestingly, with this formulation we do not need the matrix of commutators anymore. Indeed, if we know γ and the symplectic transformations $\mathcal{A}$ and $\mathcal{B}$, it is easy to access Γ_y and Γ_z through Eq. (8.21) which makes the computation of this entropic uncertainty relation easier.

[5] See Eq. (4.15).

Note also that this formulation takes a form similar to Eq. (8.1), the first entropic uncertainty relation we introduced in Chapter 6.

8.4 Conditional proof of Eq. (8.11)

As we did in Chapter 6, we will now give a conditional proof of our entropic uncertainty relation (8.11). We use a variational method (see Ref. [86, 87]), but conditionally on two assumptions. More precisely, we seek a state ρ which extremizes our uncertainty functional

$$F(\rho) = h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\ln\left(\frac{\det\Gamma_y \det\Gamma_z}{\det\Gamma}\right) \tag{8.38}$$

and we will show that any pure Gaussian state attains an extremum. The steps of this proof are similar to the ones developed in Chapter 6, with the main difference that we will have to make a change of variables going from x, p to y, z-quadratures. The assumptions are also the same (see Section 6.3.2) that is :

1. Pure Gaussian states are global minima of the uncertainty functional, Eq. (8.38).
2. The uncertainty functional, Eq. (8.38) is concave[6].

The second assumption allows us to conclude that relation (8.11) is also valid for mixed states. We know that both assumptions prevail for the regular entropic uncertainty relation (4.39) as well as for the entropic uncertainty relation (8.2), so the above assumptions are very natural.

Theorem 9. *Let* $\mathbf{y}$ *and* $\mathbf{z}$ *be two vectors of commuting quadratures, as defined in Eq. (8.9). If Assumptions 1 and 2 hold, then the probability distributions of the vectors of jointly measured quadratures* $\hat{y}_i$*'s or* $\hat{z}_j$*'s satisfy the entropic uncertainty relation (8.11).*

Proof. We start by proving the theorem for pure states. As it was already mentioned, $F(|\psi\rangle)$ is invariant under displacements, so that we can restrict our search of extrema to states centered on 0. We also require our extremal state to be normalized. Including those constraints to the functional, by using the method of Lagrange multipliers, we want to solve $\frac{\partial J}{\partial\langle\psi|} = 0$ with

$$J = h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\ln\left(\frac{\det\Gamma_y \det\Gamma_z}{\det\Gamma}\right) + \lambda(\langle\psi|\psi\rangle - 1) + \sum_{i=1}^{2n}\mu_i\langle\psi|\hat{R}_i|\psi\rangle. \tag{8.39}$$

Here, λ and μ_i are Lagrange multipliers. Note that, as explained in Section 6.3, it is not necessary to consider $\frac{\partial J}{\partial|\psi\rangle} = 0$ since no additional information will be obtained.

[6] We have numerical evidence of the concavity, but choose not to present any graph since there are similar to the one of Figure 6.3.

Let us evaluate each term independently. To begin, the derivative of $h(\mathbf{y})$ gives

$$\begin{aligned}\frac{\partial h(\mathbf{y})}{\partial\langle\psi|} &= \frac{\partial}{\partial\langle\psi|}\left(\int P(\mathbf{y})\ln P(\mathbf{y})d\mathbf{y}\right)\\ &= \frac{\partial}{\partial\langle\psi|}\left(\int\langle\psi|\mathbf{y}\rangle\langle\mathbf{y}|\psi\rangle\ln(\langle\psi|\mathbf{y}\rangle\langle\mathbf{y}|\psi\rangle)d\mathbf{y}\right)\\ &= -\left(\ln P(\mathbf{y})+1\right)|\psi\rangle \end{aligned} \tag{8.40}$$

and similarly for $h(\mathbf{z})$. With the help of Jacobi's formula [100], the derivatives of the determinant of the three covariance matrices give

$$\begin{aligned}\frac{\partial}{\partial\langle\psi|}\ln\det\Gamma_y &= \frac{1}{\det\Gamma_y}\frac{\partial}{\partial\langle\psi|}\det\Gamma_y\\ &= \frac{1}{\det\Gamma_y}\mathrm{Tr}\left[\det\Gamma_y\,\Gamma_y^{-1}\frac{\partial\Gamma_y}{\partial\langle\psi|}\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^{-1}_{y_{ik}}\frac{\partial\Gamma_{y_{ki}}}{\partial\langle\psi|}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^{-1}_{y_{ik}}\frac{(\hat{y}_k\hat{y}_i+\hat{y}_i\hat{y}_k)}{2}|\psi\rangle\\ &= \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\hat{y}_k\Gamma^{-1}_{y_{ik}}\hat{y}_i}{2}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\hat{y}_i\Gamma^{-1}_{y_{ik}}\hat{y}_k}{2}\right]|\psi\rangle\\ &= \mathbf{y}^T\Gamma_y^{-1}\mathbf{y}\,|\psi\rangle. \end{aligned} \tag{8.41}$$

and similarly

$$\begin{aligned}\frac{\partial}{\partial\langle\psi|}\ln\det\Gamma_z &= \mathbf{z}^T\Gamma_z^{-1}\mathbf{z}\,|\psi\rangle\\ \frac{\partial}{\partial\langle\psi|}\ln\det\Gamma &= \mathbf{R}^T\Gamma^{-1}\mathbf{R}\,|\psi\rangle. \end{aligned} \tag{8.42}$$

Finally, the last terms give

$$\frac{\partial}{\partial\langle\psi|}\left(\lambda(\langle\psi|\psi\rangle-1)+\sum_{i=1}^{2n}\mu_i\langle\psi|\hat{R}_i|\psi\rangle\right)=\left(\lambda+\sum_{i=1}^{2n}\mu_i\hat{R}_i\right)|\psi\rangle \tag{8.43}$$

so that the variational equation can be rewritten as an eigenvalue equation for $|\psi\rangle$,

$$\begin{aligned}\Big[-\ln P(\mathbf{y})-\ln P(\mathbf{z})-2+\lambda &+ \sum_{i=1}^{2n}\mu_i\hat{R}_i-\frac{1}{2}\mathbf{y}^T\Gamma_y^{-1}\mathbf{y}\\ &- \frac{1}{2}\mathbf{z}^T\Gamma_z^{-1}\mathbf{y}+\frac{1}{2}\mathbf{R}^T\Gamma^{-1}\mathbf{R}\Big]|\psi\rangle=0.\end{aligned} \tag{8.44}$$

The states extremizing our functional are thus the eigenstates of the equation above. However, instead of looking for all eigenstates, we show that pure Gaussian states are solution of this eigenvalue equation.

We already found in the preceding section the probability distributions $P(\mathbf{y})$ and $P(\mathbf{z})$ for a pure Gaussian state (see Eq. (8.30)), so we have

$$\ln P(\mathbf{y}) + \ln P(\mathbf{z}) = -\ln\left((2\pi)^n \sqrt{\det\Gamma_y \det\Gamma_z}\right) - \frac{1}{2}\mathbf{y}^T\Gamma_y^{-1}\mathbf{y} - \frac{1}{2}\mathbf{z}^T\Gamma_z^{-1}\mathbf{z} \tag{8.45}$$

and the eigenvalue equation reduces to

$$\left[\ln\left((2\pi)^n \sqrt{\det\Gamma_y \det\Gamma_z}\right) - 2 + \lambda + \sum_{i=1}^{2n} \mu_i \hat{R}_i + \frac{1}{2}\mathbf{R}^T\Gamma^{-1}\mathbf{R}\right]|\psi\rangle = 0. \tag{8.46}$$

Now, in the state space, a pure Gaussian state can be written as $|\psi^G\rangle = \hat{S}|0\rangle$ where $\hat{S}$ is the squeezing operator.[7] Once again, we suppose the state to be centered at the origin so we do not need to apply a displacement operator. Let us now apply $\frac{1}{2}\mathbf{R}^T\Gamma^{-1}\mathbf{R}$ to this Gaussian state. To do so, we use the canonical transformation of $\mathbf{r}$ in the Heisenberg picture, namely $\hat{S}^\dagger \mathbf{r}\, \hat{S} = \mathcal{M}\mathbf{r}$ where $\mathcal{M}$ is a symplectic matrix so that $\gamma^G = \mathcal{M}\gamma_{\text{vac}}\mathcal{M}^T$. However, this expression is true for the x, p-quadratures while we are working with the y, z ones. We thus use the following change of variables

$$\begin{aligned}
\hat{S}^\dagger \begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix} \hat{S} &= \hat{S}^\dagger \begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix} \begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} \hat{S} \\
&= \begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix} \hat{S}^\dagger \begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} \hat{S} \\
&= \begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix} \mathcal{M} \begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix}
\end{aligned} \tag{8.47}$$

where we use the fact that $\hat{S}$ and $\left(\begin{smallmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{smallmatrix}\right)$ commute since they act on two different Hilbert spaces. Therefore, we now find

$$\begin{aligned}
\frac{1}{2}\mathbf{R}^T\Gamma^{-1}\,\mathbf{R}\;|\psi^G\rangle &= \frac{1}{2}\,\mathbf{R}^T\,\Gamma^{-1}\,\mathbf{R}\,\hat{S}|0\rangle \\
&= \frac{1}{2}\hat{S}\hat{S}^\dagger\,\mathbf{R}^T\Gamma^{-1}\,\mathbf{R}\hat{S}\,|0\rangle \\
&= \frac{1}{2}\hat{S}\begin{pmatrix}\hat{S}^\dagger\mathbf{y}\hat{S} & \hat{S}^\dagger\mathbf{z}\hat{S}\end{pmatrix}\Gamma^{-1}\begin{pmatrix}\hat{S}^\dagger\mathbf{y}\hat{S}\\ \hat{S}^\dagger\mathbf{z}\hat{S}\end{pmatrix}|0\rangle \\
&= \frac{1}{2}\hat{S}\begin{pmatrix}\mathbf{x} & \mathbf{p}\end{pmatrix}\mathcal{M}^T\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}^T\Gamma^{-1}\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}\mathcal{M}\begin{pmatrix}\mathbf{x}\\ \mathbf{p}\end{pmatrix} \\
&= \frac{1}{2}\hat{S}\,\mathbf{r}^T\,\gamma_{\text{vac}}^{-1}\,\mathbf{r}\,|0\rangle \\
&= \hat{S}|0\rangle = |\psi^G\rangle.
\end{aligned} \tag{8.48}$$

In the second line, we used the fact that $\hat{S}^\dagger = \hat{S}^{-1}$ (see Eq. (2.61)) and in the fifth line,

[7] See Chapter 2.

we used Eq. (8.21) in order to compute the inverse of Γ as follows

$$\begin{aligned}\Gamma^{-1} &= \left[\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}\gamma^G\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}^T\right]^{-1}\\ &= \left[\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}\mathcal{M}\gamma_{\text{vac}}\mathcal{M}^T\begin{pmatrix}\mathcal{A}_a & \mathcal{A}_b\\ \mathcal{B}_a & \mathcal{B}_b\end{pmatrix}^T\right]^{-1}. \end{aligned} \tag{8.49}$$

The result of calculations (8.48) implies that the Gaussian state $|\psi^G\rangle$ is an eigenvector of $\frac{1}{2}\mathbf{R}^T\Gamma^{-1}\mathbf{R}$ with eigenvalue 1. Eq. (8.46) can thus be simplified again as

$$\left[\ln\left((2\pi)^n\sqrt{\det\Gamma_y\det\Gamma_z}\right) - 1 + \lambda + \sum_{i=1}^{2n}\mu_i\hat{R}_i\right]|\psi\rangle = 0. \tag{8.50}$$

The value of λ is found by multiplying this equation on the left by $\langle\psi^G|$ and by using the normalization constraint $\langle\psi^G|\psi^G\rangle = 1$ as well as the fact that mean values vanish, $\langle\psi|\hat{R}_i|\psi\rangle = 0$ for all i, so that we are now left with

$$\left[\sum_{i=1}^{2n}\mu_i\hat{R}_i\right]|\psi\rangle = 0 \tag{8.51}$$

which is satisfied if we set all the $\mu_i = 0$.

In summary, we showed that there exist appropriate choices of λ and μ_i such that all squeezed vacuum states, with arbitrary squeezing or rotation, are extrema of the uncertainty functional $F(|\psi\rangle)$ and since this functional is invariant under displacement, this remains true for all pure Gaussian states. Using Assumption 1, we consider that pure Gaussian are not just extrema, but global minima of the uncertainty functional. The last step is now simply to evaluate this minimum, which we did in the previous section, and we obtain $\ln((\pi e)^n|\det K|)$. This completes the proof of Eq. (8.11) for pure state.

To complete the proof of Theorem 9 for mixed states, we use Assumption 2 about the concavity of the functional. Indeed, if F is concave and true for pure state, it is automatically true for mixed states too. □

8.5 Attempt to define a purity-bounded entropic uncertainty relation

Let us mention that we thought of adding the purity of the state in the bound of the entropic uncertainty relation in the following way. We saw in Eq. (8.34) that the value

of our tight entropic uncertainty relation, for any Gaussian state, is given by

$$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\left(\frac{\det(\Gamma_y)\det(\Gamma_z)}{\det(\Gamma)}\right) = \ln\left((2\pi e)^n\sqrt{\det(\Gamma)}\right). \tag{8.52}$$

We also saw in Section 2.3 that the purity of a Gaussian state is given by

$$\mu_G = \frac{1}{2^n\sqrt{\det\gamma}}. \tag{8.53}$$

Together with Eq. (8.28) we can thus show that, for any Gaussian state,

$$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\left(\frac{\det(\Gamma_y)\det(\Gamma_z)}{\det(\Gamma)}\right) = \ln\left((\pi e)^n\frac{|\det K|}{\mu_G}\right). \tag{8.54}$$

The idea was therefore to suggest that for any state, the bound of the entropic uncertainty relation might depend on the purity of the state, as it is true for Gaussian states, that is

$$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\left(\frac{\det(\Gamma_y)\det(\Gamma_z)}{\det(\Gamma)}\right) \geq \ln\left((\pi e)^n\frac{|\det K|}{\mu}\right). \tag{8.55}$$

Nevertheless, numerical computations have shown that we can find many counter-examples to this inequality and it is thus not a good suggestion for a tighter entropic uncertainty relation.

8.6 Corresponding covariance-based uncertainty relation

We mentioned in Section 4.1.2 that Robertson generalized Eq. (8.3) for N observables [43]. This uncertainty relation reads

$$\det(\sigma(\mathbf{R})) \geq \det(C(\mathbf{R})) \tag{8.56}$$

where $\mathbf{R} = (R_1, \cdots, R_N)$ is a vector containing all the observables, $\sigma(\mathbf{R})$ is the covariance matrix and $C(\mathbf{R})$ the commutator matrix. Their elements are defined as

$$\sigma_{ij} = \frac{1}{2}\langle R_iR_j + R_jR_i\rangle - \langle R_i\rangle\langle R_j\rangle, \qquad C_{ij} = -\frac{i}{2}\langle[R_i, R_j]\rangle \tag{8.57}$$

and it is easy to see that for $N = 2$, we retrieve Eq. (8.3).

We introduce here this generalized version of the variance-based uncertainty relation because, interestingly, when we consider quadrature observables, it is implied by our generalized entropic uncertainty relation Eq. (8.11)[8].

[8]To be more exact, it only implies a particular case of the Robertson uncertainty relation.

Exactly like we did in Section 7.2.4, we introduce the entropy powers of $\mathbf{y}$ and $\mathbf{z}$ as

$$N_y = \frac{1}{2\pi e} e^{\frac{2}{n}h(\mathbf{y})}, \qquad N_z = \frac{1}{2\pi e} e^{\frac{2}{n}h(\mathbf{z})}. \tag{8.58}$$

This allows us to rewrite Eq. (8.11) as an entropy-power uncertainty relation

$$N_y N_z \geq \frac{|\det K|^{2/n}}{4} \left(\frac{\det \Gamma_y \det \Gamma_z}{\det \Gamma} \right)^{1/n}. \tag{8.59}$$

Since the maximum entropy for a fixed covariance matrix is reached by the Gaussian distribution, we can upper bound $N_y N_z$ by $(\det \Gamma_y \det \Gamma_z)^{1/n}$. Combining this inequality with Eq. (8.59), we obtain the uncertainty relation

$$\det \Gamma \geq \left(\frac{|\det K|}{2^n} \right)^2 \tag{8.60}$$

which is nothing else than a special case of Robertson uncertainty relation (8.56). Indeed, with Eq. (8.10), we see that Γ has the same definition as $\sigma(\mathbf{R})$ in Eq. (8.57) with $N = 2n$. In addition, the matrix of commutators $C(\mathbf{R})$ can be written in terms of $K_{ij} = [\hat{y}_i, \hat{z}_j]$ as

$$C(\mathbf{R}) = -\frac{i}{2} \begin{pmatrix} 0_{n\times n} & K \\ -K & 0_{n\times n} \end{pmatrix} \tag{8.61}$$

since $R_i = y_i$ and $R_{n+i} = z_i$ for $i = 1, \cdots, n$ and $[y_i, y_j] = [z_i, z_j] = 0$ for all i and j. Therefore,

$$\det(C(\mathbf{R})) = \left(-\frac{i}{2} \right)^{2n} \det(K)^2 = \left(\frac{|\det K|}{2^n} \right)^2 \tag{8.62}$$

where we used the fact that K_{ij} are all pure imaginary numbers. This completes the proof that the most general entropic uncertainty relation implies the most general variance-based uncertainty relation.

Let us mention also that this entropy-power formulation helps us to better understand that the general entropic uncertainty relation Eq. (8.11) implies the one of Chapter 7, Eq. (8.2). Indeed, since $\det \Gamma_y \det \Gamma_z \geq \det \Gamma$, we can deduce from Eq. (8.59) (the entropy-power version of Eq. (8.11)) that

$$N_y N_z \geq \frac{|\det K|^{2/n}}{4} \tag{8.63}$$

which is the entropy-power version of (8.2).[9]

[9] See Eq. (7.54).

8.7 Conclusion and perspectives

In this chapter, we derived an entropic uncertainty relation, defined for any n-tuples of quadrature observables and saturated by all pure Gaussian states. We gave a proof based on a variational method and conditionally to two reasonable assumptions. In particular, we mentioned that this entropic uncertainty relation is more general than the ones derived in Chapter 6 and 7. In the previous chapters, we already saw several times that entropic uncertainty relations imply variance-based ones. Here too, we have shown that the general entropic uncertainty relation implies the general variance-based uncertainty relation of Robertson Eq.(8.56). As a matter of fact, to be more exact, it actually only implies a specific case of it. We thus believe that there exists a more general entropic uncertainty relation which we conjecture to be the following

Conjecture 2.

$$h(\xi_1) + h(\xi_2) + \cdots + h(\xi_n) - \frac{1}{2}\ln\left(\frac{\sigma_1^2\sigma_2^2\cdots\sigma_n^2}{\det\Gamma_\xi}\right) \geq \frac{1}{2}\ln((2\pi e)^n \det C) \tag{8.64}$$

where the ξ_i are n observables which do not necessarily commute, σ_i^2 is the variance of ξ_i, Γ_ξ is the covariance matrix of all the observables and C the matrix of the commutators. The elements of Γ_ξ and C are defined as in Eq. (8.57).

Remark that this conjecture is interesting only for an even number of observables. Indeed, as mentioned already in Chapter 4 for the Robertson uncertainty relation, Eq. (8.56), when n is odd, $\det C = 0$ and the bound of our conjecture equals $-\infty$.

In the case of Gaussian states, it is easy to prove the validity of this conjecture. One more time we define the entropy of a Gaussian distributed variable ξ_i as $\ln(2\pi e\sigma_i^2)/2$ and inserting this in Eq. (8.64) we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\frac{1}{2}\ln(2\pi e\sigma_i^2) - \frac{1}{2}\ln\left(\frac{\sigma_1^2\sigma_2^2\cdots\sigma_n^2}{\det\Gamma_\xi}\right) &\geq \frac{1}{2}\ln((2\pi e)^n\det C)\\
\Leftrightarrow \quad \frac{1}{2}\ln\left((2\pi e)^n\det\Gamma_\xi\right) &\geq \frac{1}{2}\ln((2\pi e)^n\det C)\\
\Leftrightarrow \quad \det\Gamma_\xi &\geq \det C
\end{aligned} \tag{8.65}$$

and we know that this equation is true since it is the Robertson uncertainty relation.

Now, for any state, not necessarily Gaussian, we can show that this conjecture actually implies the Robertson uncertainty relation Eq. (8.56) in its general form. The technique is always the same. We first define the entropy power

$$N_i = \frac{1}{2\pi e}e^{2h(\xi_i)} \tag{8.66}$$

to rewrite Eq. (8.64) in its entropy-power form

$$N_1 N_2 \cdots N_n \geq \frac{\sigma_1^2 \sigma_2^2 \cdots \sigma_n^2}{\det \Gamma_\xi} \det C \tag{8.67}$$

and then use the fact that, for a fixed variance, the maximum entropy is given by a Gaussian distribution, or in other words, $N_i \leq \sigma_i^2$. We thus obtain the chain of inequalities

$$\sigma_1^2 \sigma_2^2 \cdots \sigma_n^2 \geq N_1 N_2 \cdots N_n \geq \frac{\sigma_1^2 \sigma_2^2 \cdots \sigma_n^2}{\det \Gamma_\xi} \det C \tag{8.68}$$

from which we derive Eq. (8.56). With those two arguments, we believe that the next step of this research is to prove conjecture (8.64), using probably the same variational method.

9 | Wigner Entropy

We understood by now that an improved version of the entropic uncertainty relation of Białynicki-Birula and Mycielski (see Eq. 4.39) must exist since their relation is not invariant under Gaussian unitaries and so is not saturated by all pure Gaussian states. As already mentioned in Chapter 6, we propose a conjecture for a stronger entropic uncertainty relation that would be invariant under symplectic transformations. This uncertainty relation is based on a quantity that we call Wigner entropy, but is valid only for states with positive Wigner function. Thus, in the second part of this chapter, we propose to extend the definition to the complex plane and give some properties of this Wigner entropy.

9.1 A conjecture for positive Wigner functions

When one thinks about how to improve the original entropic uncertainty relation $h(x) + h(p) \geq \ln(\pi e)$, a first natural idea is to consider the joint differential entropy instead of the sum of the entropies of the marginal distributions. Classically, the joint entropy is defined as

$$h(x,p) = -\int f(x,p) \ln f(x,p)\, \mathrm{d}x\, \mathrm{d}p \tag{9.1}$$

where $f(x,p)$ is the joint probability density of the random variables x and p, as we saw in Chapter 3. In this same chapter, we also mentioned that $h(x,p) \leq h(x) + h(p)$ since $h(x,p) = h(x) + h(p) - I(x{:}p)$ and the mutual information $I(x{:}p)$ is always positive. Therefore, considering the joint entropy instead of the sum of $h(x)$ and $h(p)$ will give a tighter entropic uncertainty relation. Moreover, $h(x,p)$ has the invariance property that we seek. Indeed, remember that if we apply a symplectic transformation $\mathcal{S}$ to the coordinates, they transform as

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \mathcal{S} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{9.2}$$

and the joint differential entropy becomes (see Eq. (3.18))

$$h(x', p') = h(x, p) + \ln |\det(\mathcal{S})|. \tag{9.3}$$

But $\mathcal{S}$ is a symplectic matrix, so $|\det(\mathcal{S})| = 1$, and the joint differential entropy remains invariant under symplectic transformations as rotations or squeezing. Since, $h(x, p)$ is also invariant under (x,p)-displacement, it looks like a good uncertainty functional. Therefore, we propose the following conjecture

Conjecture 3.

$$h(x, p) \geq \ln(\pi e) \qquad \forall \text{ states s.t. } W(x, p) \geq 0. \tag{9.4}$$

The joint entropy of the quantum state $h(x, p)$ is now called the *Wigner entropy* since it is computed through the Wigner function of the states

$$h(x, p) = -\int W(x, p) \ln W(x, p) \, \mathrm{d}x \, \mathrm{d}p. \tag{9.5}$$

Note that the conjecture is defined only for states described by a Wigner function positive everywhere. However, in most cases, the Wigner function exhibits some negative parts. Hence, the Wigner entropy is not always defined since one would need to compute the logarithm of negative values. We thus decide to restrict the conjecture to states with positive Wigner function only.

A difficulty, however, is related to the fact that characterizing the set of states with positive Wigner functions is not an easy task [101]. This suggests that proving this conjecture is hard too. Yet, an idea would be to use the majorization theory for continuous variables. Majorization theory is well known for discrete variables [69] and was already proven to be a powerful tool in the proof of uncertainty relations[1]. In continuous variables, we find almost no literature on the subject, but very recently, Jabbour *et al.* [102] started to build the theory of continuous-variable majorization, in order to prove, among other things, this conjecture. In particular, they have numerical evidence that the Gaussian Wigner distribution of the vacuum continually majorizes all the other Wigner functions, that is

$$W_\rho(x, p) \prec_c W_{|0\rangle\langle 0|}(x, p). \tag{9.6}$$

If they can prove this statement, it will directly imply conjecture (9.4).

Remember, that the differential entropy is a concave function of the probability distribution. Indeed, since we only consider states with non-negative Wigner function, we can apply all the properties of the classical joint entropy to the Wigner entropy, and in particular its concavity. This means that if one succeeds in proving the conjecture for pure states, it will be valid for mixed states as well. Decomposing a mixed

[1] See Section 4.2.3 for uncertainty relations as well as Section 5.2 for an illustration of the use of majorization theory in developing separability criteria.

state into pure states, the concavity implies that pure states are the "worst cases", that is the lowest value of the functional $h(x,p)$.

Finally, note that our conjectured rotation-invariant uncertainty relation (6.11) is saturated by any pure Gaussian state, regardless of the orientation of the principal axes. By taking the exponential of the joint entropy $h(x,p)$, we can define the entropy power of the Wigner entropy

$$N_{xp} = \frac{1}{(2\pi e)^2} e^{2h(x,p)} \tag{9.7}$$

so that the conjecture (9.4) can be rewritten as

$$N_{xp} \geq \frac{1}{4}. \tag{9.8}$$

Once again, we can use the fact that the maximum entropy is reached by the Gaussian distribution with the same covariance matrix, that is $h(x,p) \leq \frac{1}{2}\ln((2\pi e)^2 \det\gamma)$ and we find

$$\det\gamma \geq N_{xp} \geq \frac{1}{4}. \tag{9.9}$$

This means that relation (9.4) implies the Schrödinger-Robertson uncertainty relation (4.7).

Naturally, our conjectured entropic uncertainty relation can also be extended to n modes:

$$h(\boldsymbol{r}) \geq n\,\ln(\pi e) \qquad \forall \text{ states s.t. } W(\boldsymbol{r}) \geq 0 \tag{9.10}$$

where $\boldsymbol{r} = (x_1, p_1, x_2, p_2, ..., x_n, p_n)$. Here, the joint differential entropy $h(\boldsymbol{r})$ is invariant under Gaussian n-modal unitaries (all symplectic transformations and displacements) and our conjectured uncertainty relation (9.10) is saturated by all n-mode Gaussian pure states.

We can also define the n-modal entropy power of the Wigner entropy

$$N_{xp}^{(n)} = \frac{1}{(2\pi e)^2} e^{\frac{2}{n}h(x,p)} \tag{9.11}$$

so that the conjecture (9.10) can be rewritten as

$$N_{xp}^{(n)} \geq \frac{1}{4}. \tag{9.12}$$

Since the maximal entropy is given by the n-modal Gaussian distribution, we have $h_{max} = h(\mathbf{r}_G) = \frac{1}{2}\ln((2\pi e)^{2n}\det\gamma)$, and we can deduce that

$$\det\gamma \geq \left(N_{xp}^{(n)}\right)^n \geq \frac{1}{4^n}. \tag{9.13}$$

This means that the $n-$modal entropic uncertainty relation (9.10) implies a n-modal

version of the Schrödinger-Robertson uncertainty relation (4.7), like in Chapter 6.

In the following section, we run numerics to check the validity of Eq. (9.4), and provide examples of states where Eq. (9.4) gives a slightly better bound than the usual entropic uncertainty relation $h(x) + h(p) \geq \ln(\pi e)$.

9.2 Numerical evidences

We have not been able to find an analytical proof of our conjectured rotation-invariant uncertainty relation (9.4) based on the joint entropy, so we have turned to numerical tests. Since relation (9.4) is restricted to states with positive Wigner functions, we have tested, in particular, passive states of the harmonic oscillator, that is mixtures of Fock states with decreasing weights for increasing photon numbers (see Section 2.6).

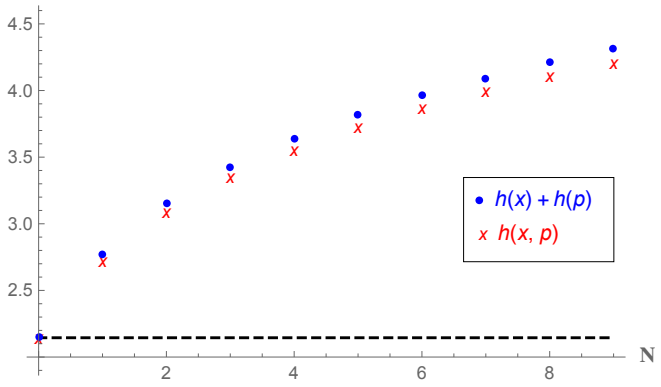

Figure 9.1: Test of the uncertainty relation (9.4) based on the joint entropy for extremal passive states, with N being the highest photon number of the state. The blue dots correspond to $h(x) + h(p)$, the red dots correspond to $h(x, p)$, while the dashed line is the lower bound $\ln(\pi e)$.

In Figure 9.1, we consider *extremal* passive states (passive states with equal weights up to a certain photon number N and vanishing weights for larger photon numbers) and have plotted the joint entropy $h(x, p)$ as a function of N, see red dots. The dashed line is the lower bound $\ln(\pi e)$, so we clearly see that the uncertainty relation (9.4) is obeyed. Since $h(x, p)$ is concave in the state, proving (9.4) for extremal passive states would actually suffice to prove it for all passive states. For comparison with the regular entropic uncertainty relation (4.39), we have also plotted $h(x) + h(p)$, see blue dots, which illustrates that our rotation-invariant uncertainty relation provides an improvement. Although the improvement is minor in this example, it is worth noting that Eq. (9.4) takes into account some x-p correlations that are not visible in the second-order moments. In fact, passive states have a diagonal covariance matrix. This means that the improved entropic uncertainty relation presented in Chapter 6 would not give a better bound (see Eq. (6.14)). In this case, $\sigma_x^2 \sigma_p^2 = \det \gamma$, so that the Gaussian mutual information is equal to zero. Therefore, although correlations

between x and p are not accessible via the second-order moments ($\sigma_{xp} = 0$) they can be accessed via the mutual information $I(x{:}p)$ providing some support to our conjecture Eq. (9.4).

We have also numerically tested other states with positive Wigner functions which are closer to the bound, such as mixtures of two squeezed states, and relation (9.4) was verified in every tested case.

9.3 A complex-valued Wigner entropy

9.3.1 Definition

We emphasized the fact that the previous conjecture is only valid for positive Wigner functions because the Wigner entropy implies the computation of the logarithm of Wigner functions and we know that the logarithm is not defined for negative probability densities in the real domain. Nevertheless, we still would like to extend the Wigner entropy to all states, even with negative Wigner functions. We thus propose to make a complex extension of the Wigner entropy by using the complex logarithm function

$$\mathrm{Ln}\,(z) = \ln|z| + i\,\mathrm{Arg}(z) \tag{9.14}$$

so that the Wigner entropy can be expressed as

$$\begin{aligned} h(x,p) &= -\int_{-\infty}^{\infty} W(x,p)\mathrm{Ln}(W(x,p))dxdp \\ &= -\int_{-\infty}^{\infty} W(x,p)\ln|(W(x,p))|dxdp - i\int_{-\infty}^{\infty}\mathrm{Arg}(W(x,p))dxdp \\ &= \Re(h(x,p)) + i\,\Im(h(x,p)) \end{aligned} \tag{9.15}$$

with the real and imaginary parts defined as

$$\begin{aligned} \Re(h(x,p)) &= -\int W(x,p)\ln|W(x,p)|dxdp \\ \Im(h(x,p)) &= \pi\int \frac{|W(x,p)| - W(x,p)}{2}\,dxdp. \end{aligned} \tag{9.16}$$

Note that $\mathrm{Arg}(W(x,p))$ is equal to 0 if the Wigner function is positive and is equal to π if it is negative (modulo 2π) hence the definition of the imaginary part in Eq. (9.16). The imaginary part can thus be understood as π times the volume of the negative pockets of the Wigner function.

To avoid a multivalued function (since $\mathrm{Arg}(W(x,p))$ is defined modulo 2π), it will be preferable to consider the exponential of the Wigner entropy $e^{h(x,p)}$.

9.3.2 Properties of the complex Wigner entropy

We now list some of the properties of the real and imaginary parts of the Wigner entropy.

Property 1. *Both real and imaginary parts of the Wigner entropy are invariant under any symplectic transformation.*

Proof. Let $\mathcal{S}$ be a symplectic transformation such that the quadratures transform according to $\mathbf{r}' = \mathcal{S}\mathbf{r}$. After this transformation, the new Wigner function will thus be expressed as $W'(x', p') = W(x, p)|\det(\mathcal{S})|$ but since $\mathcal{S}$ is symplectic, its determinant is equal to 1 and so the Wigner function remains the same, but expressed in the new variables. Therefore,

$$\begin{aligned}\Re(h'(x', p')) &= -\int W'(x', p') \ln |W'(x', p')| dx' dp' \\ &= -\int W(x, p) \ln |W(x, p)| \frac{dxdp}{|\det(\mathcal{S})|} \\ &= \Re(h(x, p)) \end{aligned} \tag{9.17}$$

and

$$\begin{aligned}\Im(h(x, p)) &= \pi \int \frac{|W'(x', p')| - W'(x', p')}{2} dx' dp' \\ &= \pi \int \frac{|W(x, p)| - W(x, p)}{2} \frac{dxdp}{|\det(\mathcal{S})|} \\ &= \Im(h(x, p)). \end{aligned} \tag{9.18}$$

Note that this property remains valid when we extend the Wigner function to n modes. □

Property 2. *The real part of the Wigner entropy, $\Re(h(x, p))$ is additive.*

Proof. Let us consider a Wigner function defined on two modes as

$$W(x_1, p_1, x_2, p_2) = W_1(x_1, p_1) W_2(x_2, p_2). \tag{9.19}$$

Then,

$$\begin{aligned}\Re(h(x_1, p_1, x_2, p_2)) &= -\int W(x_1, p_1, x_2, p_2) \ln |W(x_1, p_1, x_2, p_2)| dx_1 dp_1 dx_2 dp_2 \\ &= -\int W_1(x_1, p_1) \ln |W_1(x_1, p_1)| dx_1 dp_1 \int W_2(x_2, p_2) dx_2 dp_2 \\ &\quad -\int W_2(x_2, p_2) \ln |W_2(x_2, p_2)| dx_2 dp_2 \int W_1(x_1, p_1) dx_1 dp_1 \\ &= \Re(h_1(x_1, p_1)) + \Re(h_2(x_2, p_2)) \end{aligned} \tag{9.20}$$

since Wigner functions are normalized to 1. □

Property 3. *The imaginary part of the Wigner entropy, $\Im(h(x,p))$ is superadditive when we choose $Arg(W(x,p)) = \pi$.*

Proof. Once again, $W(x_1,p_1,x_2,p_2) = W_1(x_1,p_1)W_2(x_2,p_2)$ and thus

$$\begin{aligned}\Im(h(x_1,p_1,x_2,p_2)) &= \pi\int\frac{|W(x_1,p_1,x_2,p_2)| - W(x_1,p_1,x_2,p_2)}{2}dx_1dp_1dx_2dp_2\\ &= \pi\int\frac{|W_1W_2| - W_1W_2}{2}dx_1dp_1dx_2dp_2 \end{aligned}\tag{9.21}$$

and

$$\begin{aligned}&\Im(h_1(x_1,p_1)) + \Im(h_2(x_2,p_2)) \\ &= \pi\int\frac{|W_1| - W_1}{2}dx_1dp_1 + \pi\int\frac{|W_2| - W_2}{2}dx_2dp_2\\ &= \pi\int\frac{(|W_1| - W_1)W_2 + (|W_2| - W_2)W_1}{2}dx_1dp_1dx_2dp_2\\ &= \pi\int\frac{-W_1W_2 + |W_1|W_2 + |W_2|W_1 - W_2W_1}{2}dx_1dp_1dx_2dp_2\\ &\quad +\pi\int\frac{|W_1W_2| - |W_1W_2|}{2}dx_1dp_1dx_2dp_2\\ &= \Im(h(x_1,p_1,x_2,p_2)) - \pi\int\frac{(|W_1| - W_1)(|W_2| - W_2)}{2}dx_1dp_1dx_2dp_2.\end{aligned}\tag{9.22}$$

Since the integrand of the last integral is always positive, we have

$$\Im(h(x_1,p_1,x_2,p_2)) \geq \Im(h_1(x_1,p_1)) + \Im(h_2(x_2,p_2)).\tag{9.23}$$

Note that $\Im(h(x,p))$ becomes subadditive if we choose $Arg(W(x,p)) = -\pi$. □

Property 4. *The modulus of the exponential of the Wigner entropy, $\left|e^{h(x,p)}\right|$ is multiplicative.*

Proof. Once again, we consider $W(x_1,p_1,x_2,p_2) = W_1(x_1,p_1)W_2(x_2,p_2)$ and thus

$$\begin{aligned}\left|e^{h(x_1,p_1,x_2,p_2)}\right| &= \left|e^{\Re(h(x_1,p_1,x_2,p_2)) + i\Im(h(x_1,p_1,x_2,p_2)}\right|\\ &= e^{\Re(h(x_1,p_1,x_2,p_2))}\\ &= e^{\Re(h_1(x_1,p_1)) + \Re(h_2(x_2,p_2))}\\ &= e^{\Re(h_1(x_1,p_1))}e^{\Re(h_2(x_2,p_2))}\\ &= \left|e^{\Re(h_1(x_1,p_1))}\right|\left|e^{\Re(h_2(x_2,p_2))}\right|\end{aligned}\tag{9.24}$$

since the real part of the Wigner entropy is additive. □

Property 5. *The exponential of the Wigner entropy, $e^{h(x,p)}$ is supermultiplicative when we choose $Arg(W(x,p)) = \pi$.*

Proof. Once more, we let $W(x_1, p_1, x_2, p_2) = W_1(x_1, p_1)W_2(x_2, p_2)$ and thus

$$\begin{aligned} e^{h(x_1,p_1,x_2,p_2)} &= e^{\Re(h(x_1,p_1,x_2,p_2))+i\Im(h(x_1,p_1,x_2,p_2)} \\ &\geq e^{\Re(h_1(x_1,p_1))+\Re(h_2(x_2,p_2))+i\Im(h_1(x_1,p_1))+i\Im(h_2(x_2,p_2))} \\ &= e^{h_1(x_1,p_1)}e^{h_2(x_2,p_2)} \end{aligned} \tag{9.25}$$

since the real part of the Wigner entropy is additive and its imaginary part is superadditive. Note that $e^{h(x,p)}$ becomes submultiplicative if we choose $Arg(W(x,p)) = -\pi$. □

Property 6. *The imaginary part of the Wigner entropy, $\Im(h(x,p))$ is a convex function of the Wigner function when we choose $Arg(W(x,p)) = \pi$.*

Proof. Let us consider the following mixture $W(x,p) = \lambda W_1(x,p) + (1-\lambda)W_2(x,p)$, where $\lambda \in [0,1]$. Then,

$$\begin{aligned} \Im(h(x,p)) &= \pi \int dxdp \left(\frac{|\lambda W_1(x,p) + (1-\lambda)W_2(x,p)|}{2} \right. \\ &\quad \left. - \frac{\lambda W_1(x,p) + (1-\lambda)W_2(x,p)}{2} \right) \\ &\leq \pi \int dxdp \left(\frac{\lambda|W_1(x,p)| + (1-\lambda)|W_2(x,p)|}{2} \right. \\ &\quad \left. - \frac{\lambda W_1(x,p) + (1-\lambda)W_2(x,p)}{2} \right) \\ &= \lambda\Im(h_1(x,p)) + (1-\lambda)\Im(h_2(x,p)) \end{aligned} \tag{9.26}$$

where we use the triangle inequality in the second line. Note that $\Im(h(x,p))$ becomes a concave function if we choose $Arg(W(x,p)) = -\pi$. □

Unfortunately, we cannot say anything about the concavity of the real part of the Wigner entropy, $\Re(h(x,p))$. Numerical tests show that it is neither concave nor convex. We also had a look into the concavity of the exponential of the Wigner entropy, as well as both, its real and imaginary parts, but once again, we did not reach any conclusion.

9.3.3 An entropic uncertainty relation?

Obviously, the aim of defining a complex extension of the Wigner function is, over all, to discover a new entropic uncertainty relation. Sadly, the goal was not reached. On Figure 9.2 we plotted the real and imaginary parts of both the Wigner entropy (top) and exponential of the Wigner entropy (bottom) in order to avoid the problem of the multivalued function. The gray dots represent random quantum states generated using the same technique as for the graph of Figure 6.1, i.e. by applying a random

unitary transformation to the vacuum state. The blue dots represent random quantum states generated from a small deviation of the vacuum $|0\rangle$. They have the form $|\psi\rangle = |0\rangle + \epsilon|\phi\rangle$ where ϵ is small and $|\phi\rangle$ is a random state. The purple squares are the Fock states and their number of photons increases when they go away from the vacuum point represented by the coordinates $(\ln(\pi e), 0) \approx (2.1, 0)$ on the top graph and by $(\pi e, 0) \approx (8.5, 0)$ on the bottom graph. It seems from the graph of $e^{h(x,p)}$ that all states are going away from the vacuum in a spiral shape. The orange triangles represent a mixture of the vacuum $|0\rangle$ and the Fock state $|1\rangle$, with different weights. Note that there is a discontinuity between values of $h(x, p)$ for the vacuum $|0\rangle$ and for the mixture $(1-\epsilon)|0\rangle + \epsilon|1\rangle$ where $\epsilon \to 0$. The red line is located at $\ln(\pi e)$ (the entropy of a pure Gaussian state) on the top graph and the radius of the red circle is πe (the exponential of the entropy of a pure Gaussian state) on the bottom graph. We were hoping that all points would lie at the right of the red line or outside the circle, which would have meant that

$$\Re(h(x,p)) \geq \ln(\pi e) \tag{9.27}$$

or

$$\left|e^{h(x,p)}\right| \geq \left|e^{\ln(\pi e)}\right| = \pi e. \tag{9.28}$$

since

$$\left|e^{h(x,p)}\right| = \left|e^{\Re(h(x,p))+i\Im(h(x,p))}\right| = e^{\Re(h(x,p))}. \tag{9.29}$$

In particular, we know that both equations are true for states with positive Wigner function, since it reduces to our conjecture 9.4 in this case. When the Wigner function is positive (for example, for all Gaussian states or for passive states), there is no imaginary part and all complex entropies are located on the abscissa, above the point $\ln(\pi e)$ on the first graph and the point $e^{\ln(\pi e)}$ on the second.

Nevertheless, we clearly see in Figure 9.2 that neither relation (9.27) or relation (9.28) hold since there are some points located to the left of the red line or inside the red circle. We thus cannot consider the real part of the Wigner entropy or the norm of the exponential of the Wigner entropy as good candidates for a new entropic uncertainty relation.

If one examines those conjectures a little bit further, it actually makes sense that there are not valid uncertainty relations. Indeed, from the definition of the real part of the Wigner entropy, Eq. (9.16), we do not see why $\Re(h(x,p)) \geq \ln(\pi e)$ would necessarily be respected. In fact, the Wigner function being potentially negative, it is not surprising that the positive and negative parts compensate each other so that the real part of the Wigner entropy would have values inferior to the classical bound, where there is no negative pockets.

Unfortunately, we were not able to define any entropic uncertainty relation based

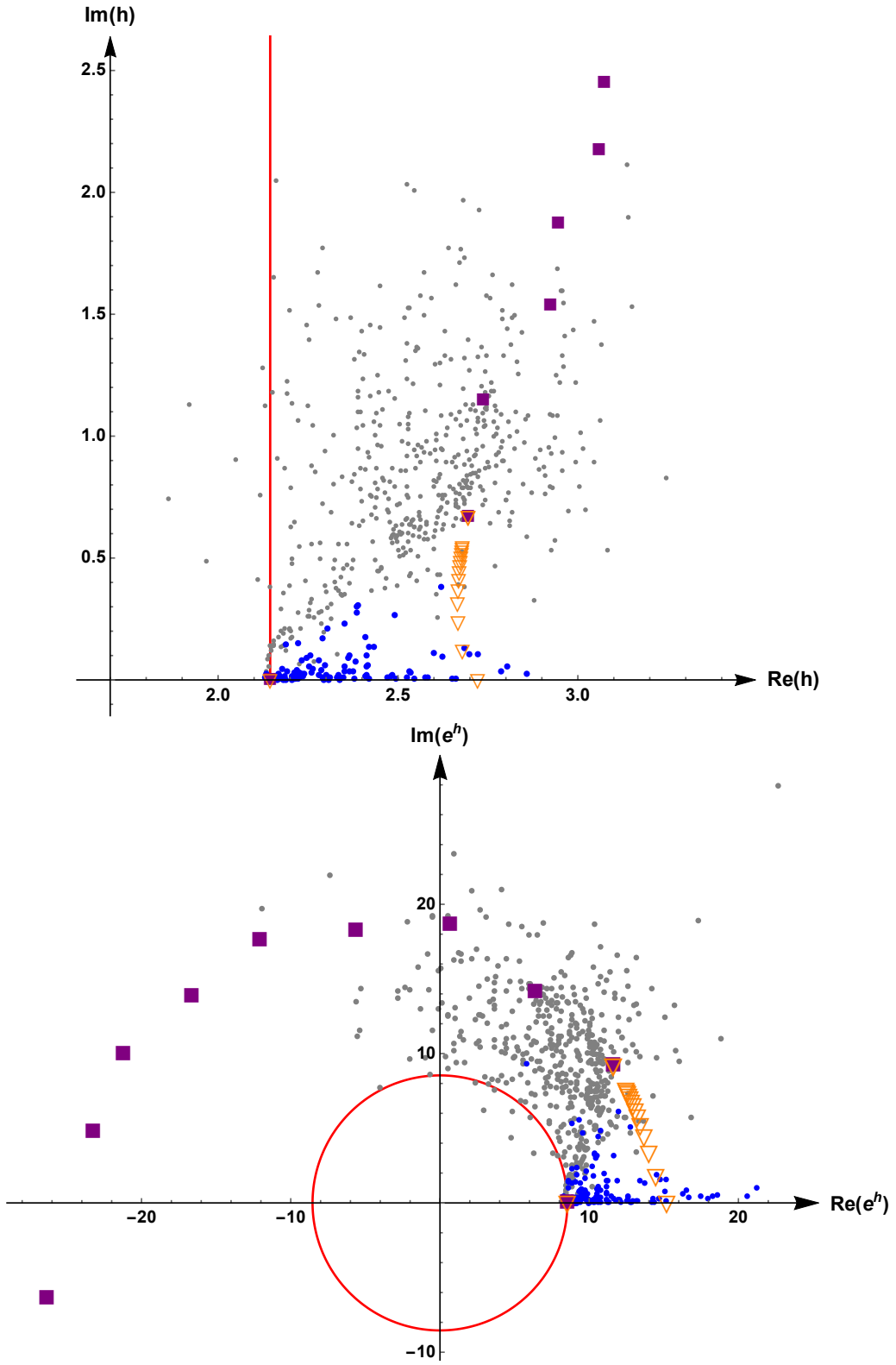

Figure 9.2: Real and imaginary part of the complex Wigner entropy (top) and the exponential of the Wigner entropy (bottom). Purple squares represent the Fock states, orange triangles represent a mixture of $|0\rangle$ and $|1\rangle$, gray dots represent random states generated from applying random unitaries to the vacuum, blue dots represent random states generated from a small deviation of the vacuum, and the radius of the red circle as well as the red line are equal to the (exponential of the) entropy of a pure Gaussian state, that is πe and $\ln(\pi e)$ respectively.

on the Wigner entropy yet, but we still believe that there must exist one. One possibility would still be to consider that there exists a forbidden region on each one of the graphs, but its shape might be much more complicated than a simple disk or line. It should include the abscissas axis from 0 to $\ln(\pi e)$ or πe (depending if we look at $h(x,p)$ or $e^{h(x,p)}$) since this reduces to the case of positive Wigner functions. Another interesting goal would be to find an operational interpretation to this complex-valued Wigner entropy.

10 | Symplectic-invariant entropic uncertainty relation based on a multi-copy uncertainty observable

In this chapter, the motivation is once again to seek a symplectic invariant entropic uncertainty relation. The path we follow is however slightly different from the previous chapters. Here, we develop a new framework that leads to new entropic uncertainty relations based on the measurement of a multi-copy uncertainty operator, that is through the computation of its Shannon entropy.[1] More precisely, we define a 2-copy uncertainty observable $\hat{L}_z$ and its associated Shannon entropy

$$H(\hat{L}_z)_\rho = -\sum_m p_m \ln p_m \tag{10.1}$$

where p_m is the probability of measuring the eigenvalue m of $\hat{L}_z$. Since the Shannon entropy is always positive, our new uncertainty relation reads

$$H(\hat{L}_z)_\rho \geq 0. \tag{10.2}$$

Interestingly, we will see that this coincides with the Robertson-Schrödinger uncertainty relation with respect to the minimum-uncertainty states. However, this works only if we first center the state at the origin. To avoid it, we also propose an extended framework based on a 3-copy uncertainty observable $\hat{L}^*$. From this observable, we can also compute the Shannon entropy and thus define a new entropic uncertainty relation

$$H(\hat{L}^*)_\rho \geq 0 \tag{10.3}$$

which also corresponds to the Robertson-Schrödinger uncertainty relation. Here too the saturation is reached by the pure Gaussian states.

[1] See definition (3.1).

10.1 2-copy uncertainty observable $\hat{L}_z$

10.1.1 Definition of $\hat{L}_z$ and link with the uncertainty relation

We are looking for an observable that simultaneously accesses both x and p quadratures. To reach this goal, we consider a 2-copy observable which is acting on two identical copies of state $|\psi\rangle$. Defining $|\Psi\rangle \equiv |\psi\rangle_1 \otimes |\psi\rangle_2$ as the joint state of systems 1 and 2, we may consider the 2-copy observable $\hat{Z} = \hat{x}_1 \otimes \hat{p}_2$. Its mean value gives

$$\langle\!\langle \hat{Z} \rangle\!\rangle_\Psi \equiv \langle \Psi | \hat{Z} | \Psi \rangle = \langle \psi | \hat{x} | \psi \rangle \, \langle \psi | \hat{p} | \psi \rangle \tag{10.4}$$

where we will use, throughout this chapter the notation $\langle\!\langle \hat{Z} \rangle\!\rangle_\Psi = \langle \psi | \langle \psi | \hat{Z} | \psi \rangle | \psi \rangle$ to express the mean value in two copies of the state. Its second-order moment gives

$$\langle\!\langle \hat{Z}^2 \rangle\!\rangle_\Psi = \langle \psi | \hat{x}^2 | \psi \rangle \, \langle \psi | \hat{p}^2 | \psi \rangle. \tag{10.5}$$

In the special case where the x and p distributions are centered on zero, $\langle\!\langle \hat{Z}^2 \rangle\!\rangle$ thus gives access to the product of the variances in x and p in state $|\psi\rangle$, which is not accessible with a single instance of the state. We may easily verify that the observable $\hat{Z}$ is invariant under a squeezing of the x quadrature with parameter r, that is, under the canonical transformation

$$\hat{x}^{(r)} = e^{-r} \, \hat{x}, \qquad\qquad \hat{p}^{(r)} = e^{r} \, \hat{p}. \tag{10.6}$$

Indeed,

$$\hat{Z}^{(r)} = \hat{x}_1^{(r)} \otimes \hat{p}_2^{(r)} = \hat{x}_1 \otimes \hat{p}_2 = \hat{Z} \tag{10.7}$$

so that measuring $\hat{Z}$ on a state $|\Psi\rangle$ is insensitive to applying a prior squeezing operation along the x (or p) quadrature on the state $|\Psi\rangle$. However, this property does not extend to rotated states since $\hat{Z}$ is not invariant under a rotation.

We define the uncertainty observable as the 2-copy operator

$$\hat{L}_z = \frac{1}{2}\big(\hat{x}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{x}_2\big) \tag{10.8}$$

where we use index z to denote that it is the third component (or z projection) of an angular momentum operator $\hat{L}$. This definition can be motivated by taking a rotation-averaged version of the above operator $\hat{Z} = \hat{x}_1 \otimes \hat{p}_2$. Indeed, since the canonical transformation of a rotation of angle θ is written as

$$\hat{x}^{(\theta)} = \cos\theta \, \hat{x} + \sin\theta \, \hat{p} \qquad\qquad \hat{p}^{(\theta)} = -\sin\theta \, \hat{x} + \cos\theta \, \hat{p} \tag{10.9}$$

we have

$$\frac{1}{2\pi} \int_0^{2\pi} \hat{x}_1^{(\theta)} \otimes \hat{p}_2^{(\theta)} \, d\theta = \frac{1}{2}\big(\hat{x}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{x}_2\big). \tag{10.10}$$

This observable is obviously invariant under a rotation as well as a squeezing operation, so it is invariant under the set of all symplectic transformations. Moreover, $\hat{L}_z$ is also invariant under a real rotation between systems 1 and 2. Indeed, if we define

$$\begin{aligned} \hat{x}_1' &= \cos\theta\hat{x}_1 + \sin\theta\hat{x}_2 \\ \hat{x}_2' &= -\sin\theta\hat{x}_1 + \cos\theta\hat{x}_2 \end{aligned} \tag{10.11}$$

we can easily show that

$$\hat{L}_z' = \frac{1}{2}(\hat{x}_1'\hat{p}_2' - \hat{p}_1'\hat{x}_2') = \frac{1}{2}(\hat{x}_1\hat{p}_2 - \hat{p}_1\hat{x}_2) = \hat{L}_z. \tag{10.12}$$

The expectation value of this observable vanishes for all states $|\psi\rangle$, since

$$\langle\!\langle \hat{L}_z \rangle\!\rangle_\Psi = \frac{1}{2}\Big(\langle\hat{x}\rangle_\psi\langle\hat{p}\rangle_\psi - \langle\hat{p}\rangle_\psi\langle\hat{x}\rangle_\psi\Big) = 0. \tag{10.13}$$

Its second-order moment gives

$$\begin{aligned} \langle\!\langle \hat{L}_z^2 \rangle\!\rangle_\Psi &= \frac{1}{2}\Big(\langle\hat{x}^2\rangle_\psi\langle\hat{p}^2\rangle_\psi - \langle\hat{x}\hat{p}\rangle_\psi\langle\hat{p}\hat{x}\rangle_\psi\Big) \\ &= \frac{1}{2}\Big(\langle\hat{x}^2\rangle_\psi\langle\hat{p}^2\rangle_\psi - \frac{1}{4}\langle\{\hat{x},\hat{p}\}\rangle_\psi^2 + \frac{1}{4}\langle[x,p]\rangle^2\Big) \\ &= \frac{1}{2}\Big(\det\gamma_c + \frac{1}{4}\langle[x,p]\rangle^2\Big) \end{aligned} \tag{10.14}$$

where we have set $\hbar = 1$ and used the fact that

$$\langle xp\rangle = \frac{1}{2}\Big(\langle[x,p]\rangle + \langle\{x,p\}\rangle\Big) \qquad \text{and} \qquad \langle px\rangle = \frac{1}{2}\Big(\langle -[x,p]\rangle + \langle\{x,p\}\rangle\Big). \tag{10.15}$$

In the last line, γ_c represent the covariance matrix of a state $|\psi\rangle$ centered on 0 and is defined as[2]

$$\gamma_c = \begin{pmatrix} \langle x^2\rangle & \frac{1}{2}\langle\{x,p\}\rangle \\ \frac{1}{2}\langle\{x,p\}\rangle & \langle p^2\rangle \end{pmatrix}. \tag{10.16}$$

since $\langle x\rangle = \langle p\rangle = 0$. Thus since the variance of $\hat{L}_z$ is defined as

$$(\Delta\hat{L}_z)^2 = \langle\!\langle \hat{L}_z^2 \rangle\!\rangle - \langle\!\langle \hat{L}_z \rangle\!\rangle^2 = \langle\!\langle \hat{L}_z^2 \rangle\!\rangle \tag{10.17}$$

we just showed that the variance of the 2-copy observable is linked to the determinant of the covariance matrix in the following way

$$(\Delta\hat{L}_z)^2 = \frac{1}{2}\left(\det\gamma_c + \frac{1}{4}\langle[x,p]\rangle^2\right). \tag{10.18}$$

Now we know that a variance is always positive. It means that

$$\det\gamma_c \geq -\frac{1}{4}\langle[x,p]\rangle^2. \tag{10.19}$$

[2]See Eq. (2.29).

If x and p are classical variables, their commutator vanishes and the previous equation simply says that a covariance matrix is always positive. However, if x and p are quantum variables, they do not commute anymore since $[x, p] = i$ and Eq. (10.18) is nothing else than the Robertson-Schrödinger uncertainty relation that is $\det \gamma \geq \frac{1}{4}$[3]. Indeed, we know that the covariance matrix is invariant under displacements which means that $\det \gamma = \det \gamma_c$. Therefore, we can understand the Robertson-Schrödinger principle as the uncertainty relation for a state that we first center at the origin. From this perspective, $\det \gamma \geq \frac{1}{4}$ is simply equivalent to the inequality

$$\langle\!\langle \hat{L}_z^2 \rangle\!\rangle \geq 0 \tag{10.20}$$

where we first need to center the state on 0.

Obviously, we have found here a trivial inequality since we are simply saying that the variance of an operator is positive. However, its equivalence with the Robertson-Schrödinger uncertainty relation suggests a new formulation of uncertainty relations.

10.1.2 Physical realization

To begin, it is useful to give a physical interpretation to the uncertainty operator. Using

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}} \qquad \text{and} \qquad \hat{a}^\dagger = \frac{\hat{x} - i\hat{p}}{\sqrt{2}}, \tag{10.21}$$

we may rewrite the 2-copy observable $\hat{L}_z$ as

$$\hat{L}_z = \frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2). \tag{10.22}$$

With this definition, it is easy to see that the action of L_z on any pure Gaussian state gives 0. Indeed, if $|\psi\rangle = |s\rangle = S(s)|0\rangle$ is a squeezed state and $S(s) = e^{\frac{1}{2}(s^*\hat{a}^2 - s\hat{a}^{\dagger 2})}$ a squeezing operator[4], we have the following equalities (once again, we only take centered states, so we do not need to consider the displacement operator)

$$\begin{aligned}
& a|0\rangle = 0 \\
\Leftrightarrow \quad & S(s)aS^\dagger(s)S(s)|0\rangle = 0 \\
\Leftrightarrow \quad & S(s)aS^\dagger(s)|s\rangle = 0 \\
\Leftrightarrow \quad & (a\cosh r + e^{i\phi}\sinh r\, a^\dagger)|s\rangle = 0 \\
\Leftrightarrow \quad & (a + e^{i\phi}\tanh r\, a^\dagger)|s\rangle = 0.
\end{aligned} \tag{10.23}$$

[3] Here, γ represents the genuine covariance matrix of any state, potentially not centered on the origin.
[4] See Section 2.5.

Therefore,

$$\begin{aligned} L_z|s\rangle|s\rangle &= \frac{i}{2}(a_1 a_2^\dagger - a_1^\dagger a_2)|s\rangle|s\rangle \\ &= \frac{i}{2}\Big((-e^{i\phi}\tanh r a_1^\dagger)a_2^\dagger - a_1^\dagger(-e^{i\phi}\tanh r a_2^\dagger)\Big)|s\rangle|s\rangle \\ &= 0. \end{aligned} \tag{10.24}$$

More interestingly, this formulation of $\hat{L}_z$ highlights the fact that it corresponds to a beam-splitter transformation. As shown in Figure 10.1, if we make a $\pi/2$ rotation of the second mode, $\hat{a}_2 \to \hat{a}_2' = -i\hat{a}_2$, followed by a $50:50$ beam-splitter transformation of the two modes according to

$$\hat{a}_1 = (\hat{b}_1 + \hat{b}_2)/\sqrt{2} \qquad \hat{a}_2' = (\hat{b}_1 - \hat{b}_2)/\sqrt{2} \tag{10.25}$$

we may reexpress the $\hat{L}_z$ observable as

$$\hat{L}_z = \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2). \tag{10.26}$$

Thus, it corresponds (up to a factor $1/2$) to the difference between the photon numbers at the two output modes of the beam splitter. Indeed at the output of the beam splitter, the quadratures are given by

$$\begin{aligned} x_1^{out} &= \frac{1}{\sqrt{2}}(x_1 + x_2') = \frac{1}{\sqrt{2}}(x_1 + p_2) \\ p_1^{out} &= \frac{1}{\sqrt{2}}(p_1 + p_2') = \frac{1}{\sqrt{2}}(p_1 - x_2) \\ x_2^{out} &= \frac{1}{\sqrt{2}}(x_1 - x_2') = \frac{1}{\sqrt{2}}(x_1 - p_2) \\ p_2^{out} &= \frac{1}{\sqrt{2}}(p_1 - p_2') = \frac{1}{\sqrt{2}}(p_1 + x_2) \end{aligned} \tag{10.27}$$

and so

$$\begin{aligned} \hat{n}_1^{out} - \hat{n}_2^{out} &= \frac{1}{2}\left((x_1^{out})^2 + (p_1^{out})^2 - (x_2^{out})^2 - (p_2^{out})^2\right) \\ &= \frac{1}{4}\left((x_1 + p_2)^2 + (p_1 - x_2)^2 - (x_1 - p_2)^2 - (p_1 + x_2)^2\right) \\ &= x_1 p_2 - p_1 x_2 \\ &= 2L_z \end{aligned} \tag{10.28}$$

This provides us with a nice physical interpretation of the observable $\hat{L}_z$.

Remember that a two-mode vacuum squeezed state can be realized with two single-mode squeezed states with orthogonal squeezing orientations. Then, if we start with two copies of an arbitrary pure Gaussian single-mode state (centered on the origin) and rotate one of them by an angle $\pi/2$ before processing both of them through a

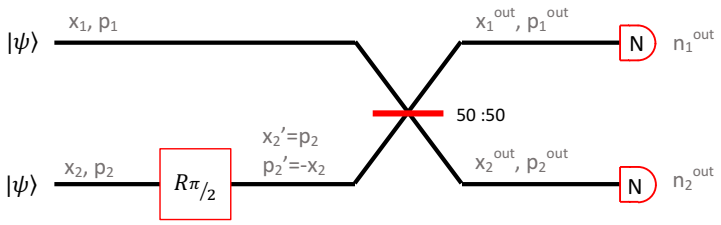

Figure 10.1: Physical realization to measure the 2-copy observable $\hat{L}_z$.

beam splitter, we get precisely a two-mode squeezed vacuum state. Such a state exhibits perfect photon-number correlations since it is written as $\sum_n c_n|n\rangle|n\rangle$, so measuring the photon-number difference gives zero with certainty. This is consistent with the fact that our observable $\hat{L}_z$ takes value 0 and exhibits no uncertainty (no variance) when applied to any pure Gaussian state (centered on the origin).

10.1.3 Ladder operators

By analogy with the algebra of angular momenta, it is possible to define the 2-copy operators $\hat{L}_x$ and $\hat{L}_y$. This will then allow us to define the ladder operators $\hat{L}_+$ and $\hat{L}_-$. Remark that when we define $\hat{L}_z = \frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2)$ as in Eq. (10.22) it can be associated to the Pauli matrix $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ as

$$\hat{L}_z = \frac{1}{2}A^\dagger\sigma_y A \tag{10.29}$$

where $A = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$. Similarly, we can define

$$\hat{L}_y = \frac{1}{2}A^\dagger\sigma_x A, \qquad \hat{L}_x = \frac{1}{2}A^\dagger\sigma_z A, \tag{10.30}$$

where $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$are the other two Pauli matrices. Thus, we obtain the two following definitions

$$\begin{aligned} \hat{L}_y &= \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger) \\ &= \frac{1}{2}(\hat{x}_1\hat{x}_2 + \hat{p}_1\hat{p}_2), \\ \hat{L}_x &= \frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2) \\ &= \frac{1}{4}((\hat{x}_1^2 + \hat{p}_1^2) - (\hat{x}_2^2 + \hat{p}_2^2)) \\ &= \frac{1}{2}(\hat{n}_1 - \hat{n}_2). \end{aligned} \tag{10.31}$$

Since the Pauli matrices respect the commutation relation $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ where ϵ_{ijk} is the Levi-Civita symbol, we have that $[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$. Here is an example of the calculation of the commutator between $\hat{L}_x$ and $\hat{L}_y$, where we use the properties of the Pauli matrices. Using the notation above, we have

$$\begin{aligned}[\hat{L}_x, \hat{L}_y] &= \frac{1}{4}[A^\dagger \sigma_z A, A^\dagger \sigma_x A] \\ &= \frac{1}{4}A^\dagger \Big(\sigma_z A A^\dagger \sigma_x - \sigma_x A A^\dagger \sigma_z\Big) A. \end{aligned} \tag{10.32}$$

We can easily compute

$$AA^\dagger = \hat{L}_0 \mathbb{1} + \hat{L}_y \sigma_x + \hat{L}_z \sigma_y + \hat{L}_x \sigma_z \tag{10.33}$$

where

$$\hat{L}_0 = \frac{\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2}{2} + 1 = \frac{1}{2}A^\dagger A + 1 \tag{10.34}$$

and thus the commutator becomes

$$\begin{aligned}[\hat{L}_x, \hat{L}_y] &= \frac{1}{4}A^\dagger \Big(\sigma_z(\hat{L}_0 \mathbb{1} + \hat{L}_y \sigma_x + \hat{L}_z \sigma_y + \hat{L}_x \sigma_z)\sigma_x \\ &\qquad -\sigma_x(\hat{L}_0 \mathbb{1} + \hat{L}_y \sigma_x + \hat{L}_z \sigma_y + \hat{L}_x \sigma_z)\sigma_z\Big) A \\ &= \frac{1}{4}A^\dagger \big(\hat{L}_0[\sigma_z, \sigma_x] - 2i\hat{L}_z\big) A \\ &= \frac{i}{2}A^\dagger \big(\hat{L}_0 \sigma_y - \hat{L}_z\big) A \\ &= \frac{i}{2}A^\dagger \left(\left(\frac{1}{2}A^\dagger A + 1\right)\sigma_y - \frac{1}{2}A^\dagger \sigma_y A\right) A \\ &= \frac{i}{2}A^\dagger \sigma_y A + \frac{i}{4}\left(A^\dagger \left(A^\dagger A\right)\sigma_y A - A^\dagger \left(A^\dagger \sigma_y A\right) A\right) \\ &= i\hat{L}_z + \frac{i}{4}\left(A^\dagger \left(A^\dagger A\right)\sigma_y A - A^\dagger \left(A^\dagger \sigma_y A\right) A\right). \end{aligned} \tag{10.35}$$

Looking at the quantity we just obtained, we simply want the last term to be equal to zero. However, the calculation is not straightforward, because the matrices do not all have consistent dimensions for multiplications. Nevertheless, we can prove that

$$\begin{aligned} A^\dagger M \left(A^\dagger A\right) N A &= \sum_{ijk} \hat{a}_i^\dagger M_{ij} \left(\sum_l \hat{a}_l^\dagger \hat{a}_l\right) N_{jk} \hat{a}_k \\ &= \sum_l \hat{a}_l^\dagger \left(\sum_{ijk} \hat{a}_i^\dagger M_{ij} N_{jk} \hat{a}_k\right) \hat{a}_l \\ &= A^\dagger \left(A^\dagger M N A\right) A \end{aligned} \tag{10.36}$$

where the objects inside the brackets have the dimension of a scalar and the matrices M and N are composed of scalar numbers so they commute with the mode operators.

If we define $M = \mathbb{1}$ and $N = \sigma_y$ we have

$$A^\dagger \left(A^\dagger A\right) \sigma_y A - A^\dagger \left(A^\dagger \sigma_y A\right) A = 0 \tag{10.37}$$

which completes the computation of the commutator

$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z. \tag{10.38}$$

Since the 2-copy operators respect the commutation relations of an angular momentum operator, we can define the ladder operators

$$\begin{aligned} \hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \frac{1}{2}(\hat{a}_1^\dagger + i\hat{a}_2^\dagger)(\hat{a}_1 + i\hat{a}_2) \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y = \frac{1}{2}(\hat{a}_1^\dagger - i\hat{a}_2^\dagger)(\hat{a}_1 - i\hat{a}_2). \end{aligned} \tag{10.39}$$

We can also define the operator $\hat{L}^2$ as

$$\begin{aligned} \hat{L}^2 &= \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \\ &= \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2}{2} \left(\frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2}{2} + 1\right) \\ &= \frac{\hat{n}_1 + \hat{n}_2}{2} \left(\frac{\hat{n}_1 + \hat{n}_2}{2} + 1\right). \end{aligned} \tag{10.40}$$

Therefore $\frac{n_1+n_2}{2}\left(\frac{n_1+n_2}{2}+1\right)$ are the eigenvalues of L^2. Since, in general, the eigenvalues of the squared modulus of an angular momentum are given by $l(l+1)$, we will define

$$l = \frac{n_1 + n_2}{2}. \tag{10.41}$$

Thus, if we label the eigenvectors of $\hat{L}_z$ by $\|l, m\rangle\!\rangle$ where l represents one half of the total photon number and m is the eigenvalue of $\hat{L}_z$, we can write

$$\begin{aligned} \hat{L}_z \|l, m\rangle\!\rangle &= m\|l, m\rangle\!\rangle \\ \hat{L}^2 \|l, m\rangle\!\rangle &= l(l+1)\|l, m\rangle\!\rangle \\ \hat{L}_+ \|l, m\rangle\!\rangle &= \sqrt{l(l+1) - m(m+1)}\|l, m+1\rangle\!\rangle \\ \hat{L}_- \|l, m\rangle\!\rangle &= \sqrt{l(l+1) - m(m-1)}\|l, m-1\rangle\!\rangle. \end{aligned} \tag{10.42}$$

Let us mention that this definition of operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ is equivalent, from a mathematical point of view, to the model of Schwinger [103] who worked out a connection between the algebra of angular momentum and two uncoupled bosonic oscillators. It is also equivalent to the definition of the Stokes operators [104, 105, 106] which describe the polarization of light.

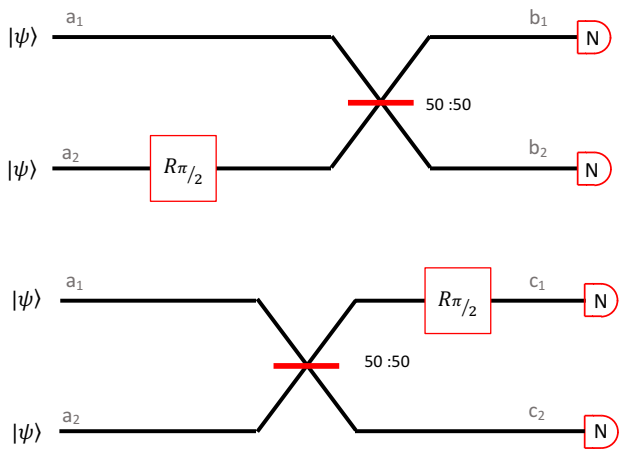

Figure 10.2: Alternative physical realization to measure the 2-copy observable $\hat{L}_z$.

10.1.4 Alternative definitions

In Figure 10.2, the first circuit is the same as in Figure 10.1, but emphasizes the fact that, at the output, we obtain two different mode operators $\hat{b}_1$ and $\hat{b}_2$. The second circuit is slightly different, since we now do the rotation after the beam splitter, and the mode operators $\hat{c}_1$ and $\hat{c}_2$ are obtained at the output. The idea is simply to show that the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ can equivalently be expressed in terms of the $\hat{a}$, $\hat{b}$ or $\hat{c}$ mode operators.

In terms of the mode operator $\hat{a}$, the expressions are given by equations (10.22) and (10.31). According to the first circuit, we already said that $\hat{L}_z$ corresponds to one half of the photon-number difference of the output modes. Using the definition $\hat{b}_i = (\hat{x}_i^{out} + i\hat{p}_i^{out})/\sqrt{2}$ and $\hat{b}_i^\dagger = (\hat{x}_i^{out} + i\hat{p}_i^{out})/\sqrt{2}$ for $i = 1, 2$ and equation (10.27), we can prove that

$$\begin{aligned} \hat{L}_x &= \frac{1}{2}(\hat{b}_1\hat{b}_2^\dagger + \hat{b}_1^\dagger\hat{b}_2) \\ \hat{L}_y &= \frac{i}{2}(\hat{b}_1\hat{b}_2^\dagger - \hat{b}_1^\dagger\hat{b}_2). \end{aligned} \tag{10.43}$$

Using exactly the same method, but based on the second circuit of Figure 10.2, we have $\hat{c}_i = (\hat{x'}_i^{out} + i\hat{p'}_i^{out})/\sqrt{2}$ and $\hat{c}_i^\dagger = (\hat{x'}_i^{out} + i\hat{p'}_i^{out})/\sqrt{2}$ for $i = 1, 2$ and

$$\begin{aligned} x'^{out}_1 &= \frac{1}{\sqrt{2}}(p_1 + p_2) & x'^{out}_2 &= \frac{1}{\sqrt{2}}(x_1 - x_2) \\ p'^{out}_1 &= -\frac{1}{\sqrt{2}}(x_1 + x_2) & p'^{out}_2 &= \frac{1}{\sqrt{2}}(p_1 - p_2). \end{aligned} \tag{10.44}$$

Table 10.1 gives a summary of all possible definitions of the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$.

	$\hat{x}, \hat{p}$	$\hat{a}, \hat{a}^\dagger$	$\hat{b}, \hat{b}^\dagger$	$\hat{c}, \hat{c}^\dagger$
$\hat{L}_x$	$\frac{1}{4}\left((\hat{x}_1^2+\hat{p}_1^2)-(\hat{x}_2^2+\hat{p}_2^2)\right)$	$\frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)$	$\frac{1}{2}(\hat{b}_1\hat{b}_2^\dagger+\hat{b}_1^\dagger\hat{b}_2)$	$\frac{i}{2}(\hat{c}_1\hat{c}_2^\dagger-\hat{c}_1^\dagger\hat{c}_2)$
$\hat{L}_y$	$\frac{1}{2}\left(\hat{x}_1\hat{x}_2+\hat{p}_1\hat{p}_2\right)$	$\frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_1\hat{a}_2^\dagger)$	$\frac{i}{2}(\hat{b}_1\hat{b}_2^\dagger-\hat{b}_1^\dagger\hat{b}_2)$	$\frac{1}{2}(\hat{c}_1^\dagger\hat{c}_1-\hat{c}_2^\dagger\hat{c}_2)$
$\hat{L}_z$	$\frac{1}{2}\left(\hat{x}_1\hat{p}_2-\hat{p}_1\hat{x}_2)\right)$	$\frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger-\hat{a}_1^\dagger\hat{a}_2)$	$\frac{1}{2}(\hat{b}_1^\dagger\hat{b}_1-\hat{b}_2^\dagger\hat{b}_2)$	$\frac{1}{2}(\hat{c}_1\hat{c}_2^\dagger+\hat{c}_1^\dagger\hat{c}_2)$

Table 10.1: All possible definitions of the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$.

10.1.5 Eigensystem of L_z

As we already mentioned, the variance of the operator $\hat{L}_z$ coincides with the uncertainty relation of Schrödinger-Robertson. We will now show that the Shannon entropy of this uncertainty observable given by Eq. (10.1) provides a new measure of uncertainty and non-Gaussianity. For this, we need first to determine the eigensystem of the operator.

We will express the eigenvectors of $\hat{L}_z$ as linear combinations of the 2-mode Fock states $|j,k\rangle$:

$$\|l,m\rangle\rangle = \sum_j \sum_k c_{jk}|j,k\rangle. \tag{10.45}$$

Before looking for their general form, let us start with some examples for specific l. Note that if we fix the value of l, it means that the only non-zero c_{jk} are those such that $j+k=2l$. If we fix l=0, the only eigenvector is

$$\|0,0\rangle\rangle = |0,0\rangle. \tag{10.46}$$

If we fix $l=1/2$, we have two eigenvectors with eigenvalues $m=\pm 1/2$

$$\begin{aligned} \|1/2,-1/2\rangle\rangle &= \frac{1}{\sqrt{2}}(|0,1\rangle+i|1,0\rangle) \\ \|1/2,1/2\rangle\rangle &= \frac{1}{\sqrt{2}}(|0,1\rangle-i|1,0\rangle). \end{aligned} \tag{10.47}$$

If we fix $l=1$, we have three eigenvectors with eigenvalues $m=\{-1,0,1\}$

$$\begin{aligned} \|1,-1\rangle\rangle &= \frac{1}{2}(|2,0\rangle-i\sqrt{2}|1,1\rangle-|0,2\rangle) \\ \|1,0\rangle\rangle &= \frac{1}{\sqrt{2}}(|2,0\rangle+|0,2\rangle) \\ \|1,1\rangle\rangle &= \frac{1}{2}(|2,0\rangle+i\sqrt{2}|1,1\rangle-|0,2\rangle). \end{aligned} \tag{10.48}$$

And so on. For higher values of l, it becomes more and more cumbersome to find the general form of the eigenstates, but we will derive a procedure that allows us

to construct all of them in principle. First, note that given the commutation relation (10.38) (and its cyclic permutations), for every value of l, the possible eigenvalues of $\hat{L}_z$ are $m \in \{-l, l\}$ with integer jumps[5].

For every even total number of photons, i.e. every time l is an integer, we will have an eigenstate that admit an eigenvalue $m = 0$. For those states, the general (not-normalized) form will be

$$\|l,0\rangle\rangle = \beta \frac{1+(-1)^l}{2}|l,l\rangle + \sum_{i=0}^{\lfloor l/2-1/2 \rfloor} \alpha_i \big(|2i, 2l-2i\rangle + |2l-2i, 2i\rangle\big) \tag{10.49}$$

with

$$\begin{aligned} \alpha_i &= \sqrt{\frac{(2l)!!\,(2l-2i-1)!!\,(2i-1)!!}{(2l-2i)!!\,(2l-1)!!\,(2i)!!}} \\ \beta &= \sqrt{\frac{(2l)!!\,(l-1)!!\,(l-1)!!}{(l)!!\,(2l-1)!!\,(l)!!}} \end{aligned} \tag{10.50}$$

where $(\cdot)!!$ denotes the double factorial and the index i has to be an integer. Note that those states are thus written as linear combination of only even Fock states of the form $|2j, 2k\rangle$. Let us now take two copies of a squeezed state $|s\rangle$[6]

$$|s\rangle \otimes |s\rangle = \frac{1}{\cosh r} \sum_{j,k=0}^{\infty} \frac{\sqrt{(2j)!!(2k)!!}}{2^{j+k} j!k!} (\tanh r)^{k+j} |2j, 2k\rangle. \tag{10.51}$$

It is also written as a linear combination of even Fock states of the form $|2j, 2k\rangle$. This means that we are able to write $|s\rangle \otimes |s\rangle$ as a linear combination of the eigenstates $\|l,0\rangle\rangle$ (with l integer). Therefore, applying $\hat{L}_z$ on those states will always give us zero, which confirms the result of Eq. (10.24).

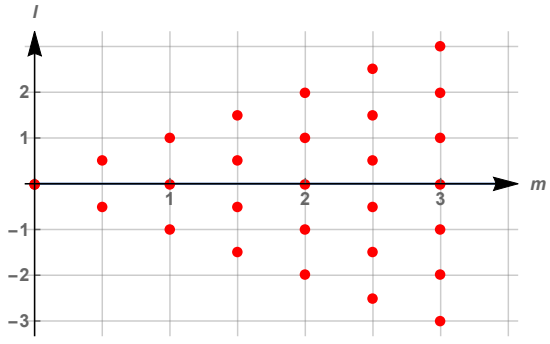

Figure 10.3: Possible eigenvalues of a state with $l/2$ total number of photons.

Now, remember that for every value of l, possible values for the eigenvalues of $\hat{L}_z$ are $m \in \{-l, l\}$ with integer jumps, as it can been seen in Figure 10.3. The eigenstates

[5] Indeed, $[\hat{L}_z, \hat{L}_+] = \hat{L}_+$ and thus $\hat{L}_z\hat{L}_+\|l,m\rangle\rangle = (\hat{L}_+\hat{L}_z + \hat{L}_+)\|l,m\rangle\rangle = (m+1)\hat{L}_+\|l,m\rangle\rangle$ where we assumed that $\hat{L}_z\|l,m\rangle\rangle = m\|l,m\rangle\rangle$.

[6] See Eq. (2.60).

corresponding to the lowest diagonal, i.e., the (not-normalized) eigenstates of the form $\|l,-l\rangle\!\rangle$ are defined as follows

$$\begin{aligned}\|l,-l\rangle\!\rangle &= \sum_{k=0}^{\lfloor l-1/2\rfloor} i^k\sqrt{\binom{2l}{k}}\Big(|k,2l-k\rangle+(-1)^k i^{2l}|2l-k,k\rangle\Big)\\ &\quad+\frac{1+(-1)^{2l}}{2}\,i^l\sqrt{\binom{2l}{l}}|l,l\rangle. \end{aligned}\tag{10.52}$$

We can now apply the ladder operator $\hat{L}_+$, defined in Eq. (10.39), as many times as needed, in order to find all the other eigenstates since

$$\|l,m+1\rangle\!\rangle=\frac{1}{\sqrt{l(l+1)-m(m+1)}}\,\hat{L}_+\,\|l,m\rangle\!\rangle. \tag{10.53}$$

We thus have access to all eigenstates of the operator $\hat{L}_z$.

10.1.6 Symmetry property

Let us briefly mention an interesting property of $\hat{P}$, the operator that exchanges the indexes of two systems. It can be seen as a reflexion along the $x_1=x_2$ line and the $p_1=p_2$ line and acts as follows on $\hat{L}_z$, $\hat{L}_y$ and $\hat{L}_x$

$$\begin{aligned}\hat{P}\hat{L}_z\hat{P} &= \frac{i}{2}\hat{P}(\hat{a}_2^\dagger\hat{a}_1-\hat{a}_1^\dagger\hat{a}_2)\hat{P}=\frac{i}{2}(\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)=-\hat{L}_z\\ \hat{P}\hat{L}_y\hat{P} &= \frac{1}{2}\hat{P}(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1)\hat{P}=\frac{1}{2}(\hat{a}_2^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_2)=\hat{L}_y\\ \hat{P}\hat{L}_x\hat{P} &= \frac{1}{2}\hat{P}(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)\hat{P}=\frac{1}{2}(\hat{a}_2^\dagger\hat{a}_2-\hat{a}_1^\dagger\hat{a}_1)=-\hat{L}_x\end{aligned}\tag{10.54}$$

where we use the fact that $\hat{P}^\dagger=\hat{P}$. Also, we have

$$\begin{aligned}\hat{P}\hat{L}_+\hat{P} &= \hat{P}(\hat{L}_x+i\hat{L}_y)\hat{P}=-\hat{L}_x+i\hat{L}_y=-\hat{L}_-\\ \hat{P}\hat{L}_-\hat{P} &= \hat{P}(\hat{L}_x-i\hat{L}_y)\hat{P}=-\hat{L}_x-i\hat{L}_y=-\hat{L}_+.\end{aligned}\tag{10.55}$$

In particular, we can evaluate the action of the operator $\hat{P}$ on an eigenstate of $\hat{L}_z$. Since $\hat{L}_z\|l,m\rangle\!\rangle=m\|l,m\rangle\!\rangle$, $\hat{L}_z=-\hat{P}\hat{L}_z\hat{P}$ and $\hat{P}^{-1}=\hat{P}$, we have

$$-\hat{P}\hat{L}_z\hat{P}\|l,m\rangle\!\rangle=m\|l,m\rangle\!\rangle\qquad\Leftrightarrow\qquad\hat{L}_z\hat{P}\|l,m\rangle\!\rangle=-m\hat{P}\|l,m\rangle\!\rangle. \tag{10.56}$$

This means that $\hat{P}\|l,m\rangle\!\rangle$ is also an eigenvector of $\hat{L}_z$ with eigenvalue $-m$ so

$$\hat{P}\|l,m\rangle\!\rangle\propto\|l,-m\rangle\!\rangle. \tag{10.57}$$

Thus, from the eigenstate $\|l,m\rangle\!\rangle$, we can find the eigenstate $\|l,-m\rangle\!\rangle$ by interchanging systems 1 and 2. From Eq. (10.57), we also understand that $m=0$ states must be

symmetric under the exchange of both systems as it can be seen in Eq. (10.49).

10.1.7 Entropic uncertainty relation based on $\hat{L}_z$

In Section 10.1.5, we found the eigenvalues and eigenstates of our 2-copy observable. Since we know the whole eigenspectrum of $\hat{L}_z$, we can, in theory, compute its Shannon entropy. We saw in Eq. (10.1) that

$$H(\hat{L}_z)_\rho = -\sum p_m \ln p_m \tag{10.58}$$

where p_m is the probability of measuring the eigenvalue m. From the eigenvectors, we can compute the probability as

$$p_m = \sum_{l=|m|}^{\infty} \langle\!\langle l, m \| \rho \otimes \rho \| l, m \rangle\!\rangle. \tag{10.59}$$

The sum starts at $l = |m|$ since $-l \leq m \leq l$.

We saw in Chapter 3 that Shannon entropy is always positive. We therefore propose the following new uncertainty relation

$$H(\hat{L}_z)_\rho \geq 0. \tag{10.60}$$

In particular, we will see in the next sections that it is saturated by all pure Gaussian states and is invariant under symplectic transformations.

10.1.8 Invariance of $H(\hat{L}_z)_\rho$

Let us go back to our main motivation: looking for an invariant uncertainty relation. Fortunately, $H(L_z)_\rho$ has the property we seek, that is the invariance under any symplectic transformation, so under any Gaussian unitary U (except displacements). Suppose we apply $U \otimes U$ on an eigenstate $\|l, m\rangle\!\rangle$ and then compute the action of $\hat{L}_z$. We obtain a new eigenvector with the same eigenvalue. We thus do not change the eigenspectrum. Indeed, since $\hat{L}_z$ is itself invariant under any Gaussian unitary (except displacements) we have

$$\begin{aligned} \hat{L}_z \|l, m\rangle\!\rangle &= m \|l, m\rangle\!\rangle \\ U^\dagger \otimes U^\dagger \hat{L}_z U \otimes U \|l, m\rangle\!\rangle &= m \|l, m\rangle\!\rangle \\ \hat{L}_z U \otimes U \|l, m\rangle\!\rangle &= m\, U \otimes U \|l, m\rangle\!\rangle \\ \hat{L}_z \|l', m\rangle\!\rangle &= m \|l', m\rangle\!\rangle \end{aligned} \tag{10.61}$$

where we defined $\|l', m\rangle\!\rangle \equiv U \otimes U \|l, m\rangle\!\rangle$ as a new eigenvector of $\hat{L}_z$ with the same eigenvalue m, but a different value of l given by l'. Let us now consider the state

$\|\xi\rangle\!\rangle = \sum_m \sum_l a_{l,m} \|l,m\rangle\!\rangle$ written as a linear combination of the eigenvectors. Applying $U \otimes U$ on this state will give

$$U \otimes U \|\xi\rangle\!\rangle = \sum_m \sum_l a_{m,l} \|l'(l), m\rangle\!\rangle \tag{10.62}$$

which means that we find a new eigenvector, but which still belongs to the same eigenspace of the eigenvalue m. To know the exact value of l' in function of l, we simply apply the Gaussian unitary $\hat{L}^2$ on the eigenstate. However, it is not necessary here. The important point is that the subspace spanned by all states with eigenvalue m is invariant under a Gaussian unitary. In particular, it thus means that the projector associated to the measure of an eigenvalue m

$$\mathbb{P}_m = \sum_{l=|m|}^{\infty} \|l,m\rangle\!\rangle\langle\!\langle l,m\| \tag{10.63}$$

is invariant under Gaussian unitaries and so is the probability p_m of measuring m since

$$p_m = \sum_{l=|m|}^{\infty} \langle\!\langle l,m\|\rho \otimes \rho\|l,m\rangle\!\rangle = \mathrm{Tr}(\rho \otimes \rho \mathbb{P}_m). \tag{10.64}$$

Therefore, the Shannon entropy of $\hat{L}_z$ as well as our uncertainty relation $H(\hat{L}_z)_\rho \geq 0$ are invariant under any Gaussian unitary (except displacements).

10.1.9 Special case of Gaussian states

In general, it is hard to compute the Shannon entropy of $\hat{L}_z$, because one first need to express the state $|\psi\rangle$ as a linear combination of the eigenstates $\|l,m\rangle\!\rangle$ of $\hat{L}_z$. However, it does not require many efforts in the case of Gaussian states. Beforehand, note that according to Williamson theorem[7], every Gaussian state can be brought to a thermal state by applying some Gaussian transformations. In addition, we saw that $H(\hat{L}_z)$ is invariant under any Gaussian unitary. Therefore, computing the Shannon entropy of any thermal state actually gives the Shannon entropy of any Gaussian state with the same symplectic spectrum. Luckily, it is not complicated to evaluate $H(\hat{L}_z)$ for a thermal state

$$\rho_{th} = \sum_{n=0}^{\infty} \frac{\langle n\rangle^n}{(\langle n\rangle + 1)^{n+1}} |n\rangle\langle n| \tag{10.65}$$

because when inserted in the physical realization of Figure 10.1 the probabilities of measuring any value of $\hat{L}_z$ are easy to compute. Indeed, as mentioned before, measuring $\hat{L}_z$ simply corresponds to measuring $\hat{d} = \frac{\hat{n}_1 - \hat{n}_2}{2}$, the difference between the photon numbers at the two outputs. Since a thermal state is invariant under a rotation, after the $\pi/2$ rotation of our physical realization, we still have a thermal state. Moreover, when two copies of a thermal state are inserted in a beam splitter, the out-

[7] See Section 2.4.1.

put as well is formed of the two same thermal states. It is thus straightforward to compute the probability of obtaining a certain value for d.

The probability of measuring n_i photons on the i^{th} output is

$$P(\hat{n}_i = n_i) = \frac{\langle n \rangle^{n_i}}{(\langle n \rangle + 1)^{n_i+1}} \tag{10.66}$$

for $i = 1, 2$. Thus, the probability of measuring $\hat{d}$ is

$$P(\hat{d} = d) = \begin{cases} \sum_{n_2=0}^{\infty} P(\hat{n}_1 = n_2 + 2d) P(\hat{n}_2 = n_2) & d > 0 \\ \sum_{n_1=0}^{\infty} P(\hat{n}_1 = n_1) P(\hat{n}_2 = n_1 - 2d) & d < 0 \\ \sum_{n_1=0}^{\infty} P(\hat{n}_1 = n_1) P(\hat{n}_2 = n_1) & d = 0 \end{cases} \tag{10.67}$$

which gives

$$P(\hat{d} = d) = \frac{1}{2\langle n \rangle + 1} \left(\frac{\langle n \rangle}{\langle n \rangle + 1} \right)^{2d} \quad \forall\, d. \tag{10.68}$$

We remember that d can be half-integer and we will thus sum on $2d \to d$ to simplify the calculations. We can now compute the Shannon entropy of $\hat{L}_z$ in this state:

$$\begin{aligned} H(\hat{L}_z)_{\rho_{th}} &= -\sum_d P(\hat{d} = d) \ln P(\hat{d} = d) \\ &= -\sum_d \frac{1}{2\langle n \rangle + 1} \left(\frac{\langle n \rangle}{\langle n \rangle + 1} \right)^d \ln \frac{1}{2\langle n \rangle + 1} \left(\frac{\langle n \rangle}{\langle n \rangle + 1} \right)^d \\ &= \ln(2\langle n \rangle + 1) - \frac{2\langle n \rangle (\langle n \rangle + 1)}{2\langle n \rangle + 1} \ln \frac{\langle n \rangle}{\langle n \rangle + 1} \\ &= \ln(2\langle n \rangle + 1) + E(\langle n \rangle). \end{aligned} \tag{10.69}$$

Interestingly, if we compute the Shannon differential entropy of a thermal state with[8]

$$W_{\rho_{th}} = \frac{1}{2\pi\sqrt{\det \gamma}} e^{-\frac{1}{2}\mathbf{r}^T \gamma^{-1} \mathbf{r}} = \frac{1}{\pi(2\langle n \rangle + 1)} e^{-\frac{1}{2\langle n \rangle + 1}(x^2 + p^2)}, \tag{10.70}$$

we find

$$h(x, p)_{\rho_{th}} = -\int W_{\rho_{th}}(x, p) \ln W_{\rho_{th}}(x, p) dx = \ln(\pi e) + \ln(2\langle n \rangle + 1) \tag{10.71}$$

which allows us to write

$$H(\hat{L}_z)_{\rho_{th}} = h(x, p)_{\rho_{th}} - \ln(\pi e) + E(\langle n \rangle). \tag{10.72}$$

Note that, as it can be seen in Figure 10.4, $E(\langle n \rangle)$ admits values between 0 and 1 so that, $H(\hat{L}_z)_{\rho_{th}} + \ln(\pi e)$ is very close to $h(x, p)_{\rho_{th}}$.

[8] See Section 2.5.6.

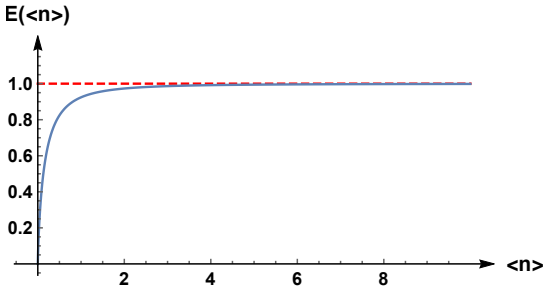

Figure 10.4: Graph of $E(\langle n \rangle)$.

It is also possible to write both expressions in terms of the symplectic values of ρ_{th}, using the fact that, for the thermal state, $\langle n \rangle = \nu - 1/2$:

$$H(\hat{L}_z)_{\rho_{th}} = \ln(2\nu) - \frac{4\nu^2 - 1}{4\nu} \ln \frac{2\nu - 1}{2\nu + 1} \tag{10.73}$$

$$h(x,p)_{\rho_{th}} = = \ln(\pi e) + \ln(2\nu). \tag{10.74}$$

Note that $H(\hat{L}_z)_{\rho_{th}}$ is monotonically increasing in ν.

Once again, we emphasized that any Gaussian state can be expressed as a thermal state — if we apply the appropriate squeezing or rotation — but the only thermal state that gives $H(\hat{L}_z)_{\rho_{th}} = 0$ is the vacuum (since we only consider states centered on the origin). Therefore, since our entropic uncertainty relation $H(\hat{L}_z) \geq 0$ is invariant under symplectic transformations, it is saturated by the same states as the uncertainty relation of Robertson-Schrödinger $\det(\gamma) \geq 1/4$.

We thus presented a new framework that allows us to define a new entropic uncertainty relation which coincides with the covariance-based uncertainty relation with respect to saturation. Note that the quantity $H(\hat{L}_z)$ can also be seen as a measure of pure non-Gaussianity since $H(\hat{L}_z) = 0$ only for pure Gaussian states. Finally, if we only look at Gaussian states, $H(\hat{L}_z)$ defined as in Eq. (10.73) can be understood as a measure of the mixedness of the Gaussian state since the purity of a Gaussian state is given by $\mu = 1/2\nu$.

10.1.10 Example of non-Gaussian states

Let us compute the entropy $H(\hat{L}_z)$ for some examples of non Gaussian states.

Example 1: Let us first consider the Fock state $|1\rangle$. If we insert two copies of $|1\rangle$ in the optical circuit of Figure 10.1 we find the state

$$\frac{1}{\sqrt{2}}(|0\,2\rangle - |2\,0\rangle) \tag{10.75}$$

at the output. Therefore, the photon-number difference will be ± 2, each with probability $1/2$ and the entropy is thus given by

$$H(\hat{L}_z)_{|1\rangle} = -\sum_m p_m \ln p_m = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2. \tag{10.76}$$

As expected, the value is greater than 0 which is in agreement with our entropic uncertainty relation Eq. (10.60).

Example 2: Let us now consider a mixture of $|0\rangle$ and $|1\rangle$

$$\rho = \alpha|0\rangle\langle 0| + (1-\alpha)|1\rangle\langle 1|. \tag{10.77}$$

This time, we compute the entropy not with the help of the optical circuit, but rather with the formula (10.59)[9]

$$p_m = \sum_{l=|m|}^{\infty} \langle\!\langle l, m \| \rho \otimes \rho \| l, m \rangle\!\rangle. \tag{10.78}$$

Since

$$\rho\otimes\rho = \alpha^2|0\,0\rangle\langle 0\,0| + (1-\alpha)^2|1\,1\rangle\langle 1\,1| + \alpha(1-\alpha)|0\,1\rangle\langle 0\,1| + \alpha(1-\alpha)|1\,0\rangle\langle 1\,0| \tag{10.79}$$

we only need to consider states $\|l,m\rangle\!\rangle$ with $l = \{0, \frac{1}{2}, 1\}$ and they are given in Eqs. (10.46)-(10.48). Accordingly, the possible values of m are $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. We can now compute the different probabilities p_m

$$\begin{aligned}
p_0 &= \sum_{l=0}^{1} \langle\!\langle l,0\|\rho\otimes\rho\|l,0\rangle\!\rangle = \langle\!\langle 0,0\|\rho\otimes\rho\|0,0\rangle\!\rangle + \langle\!\langle 1,0\|\rho\otimes\rho\|1,0\rangle\!\rangle = \alpha^2 \\
p_{\pm\frac{1}{2}} &= \sum_{l=0}^{1} \langle\!\langle l,\pm 1/2\|\rho\otimes\rho\|l,\pm 1/2\rangle\!\rangle = \langle\!\langle 1/2,\pm 1/2\|\rho\otimes\rho\|1/2,\pm 1/2\rangle\!\rangle = \alpha(1-\alpha) \\
p_{\pm 1} &= \sum_{l=0}^{1} \langle\!\langle l,\pm 1\|\rho\otimes\rho\|l,\pm 1\rangle\!\rangle = \langle\!\langle 1,\pm 1\|\rho\otimes\rho\|1,\pm 1\rangle\!\rangle = \frac{(1-\alpha)^2}{2}
\end{aligned} \tag{10.80}$$

and the entropy is equal to

$$\begin{aligned}
H(\hat{L}_z)_\rho &= -\alpha^2\ln\alpha^2 - 2\alpha(1-\alpha)\ln\alpha(1-\alpha) - 2\frac{(1-\alpha)^2}{2}\ln\frac{(1-\alpha)^2}{2} \\
&= (1-\alpha)^2\ln 2 - 2\alpha\ln\alpha - 2(1-\alpha)\ln(1-\alpha)
\end{aligned} \tag{10.81}$$

which is always greater than 0 except when $\alpha = 1$ because then ρ is simply equal to the vacuum and we are thus computing the entropy of a pure Gaussian state. If $\alpha = 0$, we find $H(\hat{L}_z)_\rho = \ln 2$ as expected since it is the previous result obtained for the Fock state $|1\rangle$.

[9] Note that there is a slight abuse of notation here since the sum on l takes half integer steps that is $l = \{|m|, |m|+1/2, |m|+1, \cdots\}$.

Note that the Shannon entropy of this mixture is a concave function of α. This suggests that $H(\hat{L}_z)$ is probably a concave function in general.

10.2 3-copy uncertainty observable $\hat{L}^*$

10.2.1 Definition of $\hat{L}^*$ and link with the uncertainty relation

All this framework is interesting, but not ideal since we first need to center the state on the origin. To avoid it, we now propose to define a 3-copy observable instead. The idea comes from a paper from Brun [107] where he shows that every nth-degree polynomial function can be computed as the mean value of a n-copy observable.

We define the general covariance matrix γ, for any state, not necessarily centered on 0, as

$$\gamma = \begin{pmatrix} \langle x^2\rangle - \langle x\rangle^2 & \frac{1}{2}\langle\{x,p\}\rangle - \langle x\rangle\langle p\rangle \\ \frac{1}{2}\langle\{x,p\}\rangle - \langle x\rangle\langle p\rangle & \langle p^2\rangle - \langle p\rangle^2 \end{pmatrix}. \tag{10.82}$$

This definition is valid for both classical or quantum systems. If we compute its determinant, we then have

$$\begin{aligned} \det\gamma &= \left(\langle x^2\rangle - \langle x\rangle^2\right)\left(\langle p^2\rangle - \langle p\rangle^2\right) - \left(\frac{1}{2}\langle\{x,p\}\rangle - \langle x\rangle\langle p\rangle\right)^2 \\ &= \langle x^2\rangle\langle p^2\rangle - \langle x^2\rangle\langle p\rangle\langle p\rangle - \langle p^2\rangle\langle x\rangle\langle x\rangle + \langle x\rangle^2\langle p\rangle^2 \\ &\quad - \left(\frac{1}{4}\langle\{x,p\}\rangle^2 - \langle\{x,p\}\rangle\langle x\rangle\langle p\rangle + \langle x\rangle^2\langle p\rangle^2\right) \\ &= \langle x^2\rangle\langle p^2\rangle - \langle x^2\rangle\langle p\rangle\langle p\rangle - \langle p^2\rangle\langle x\rangle\langle x\rangle - \frac{1}{4}\langle\{x,p\}\rangle^2 + \langle\{x,p\}\rangle\langle x\rangle\langle p\rangle. \end{aligned} \tag{10.83}$$

Let us now define the 3-copy observable. If it was surprising to find out that the 2-copy observable had the form of an angular momentum, it now seems natural to consider the three components $\hat{L}_i^*$ of a genuine angular momentum.[10] To be consistent with the definition of the 2-copy observable, we nevertheless introduce a one half factor and define

$$\begin{aligned} \hat{L}_x^* &= \frac{1}{2}(\hat{x}_2\hat{p}_3 - \hat{p}_2\hat{x}_3) \\ \hat{L}_y^* &= \frac{1}{2}(\hat{x}_3\hat{p}_1 - \hat{p}_3\hat{x}_1) \\ \hat{L}_z^* &= \frac{1}{2}(\hat{x}_1\hat{p}_2 - \hat{p}_1\hat{x}_2). \end{aligned} \tag{10.84}$$

The 3-copy observable reads

$$\hat{L}^* = \frac{1}{\sqrt{3}}(\hat{L}_x^* + \hat{L}_y^* + \hat{L}_z^*). \tag{10.85}$$

[10] The symbol * is here only to differentiate those 3-copy observables from the 2-copy observables $\hat{L}_i$ of the previous section.

Since the 2-copy observable $\hat{L}_z$ is invariant under rotations and squeezing, the $\hat{L}_i^*$ observables having all the same form, are also invariant under those symplectic transformations and so is $\hat{L}^*$. Moreover, $\hat{L}^*$ is also invariant under displacement. Indeed, since we always consider three copies of a same state, the displacement will always be the same in each mode's quadrature. In other words, the displacement in position (or momentum) will always be applied in the direction $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ which is the orientation of the component $\hat{L}^*$ of the angular momentum. Since an angular momentum is invariant under a shift or a kick in its direction, $\hat{L}^*$ is invariant under displacement. This confirms that we do not need to restrict to states centered on the origin anymore.

Interestingly, the variance of $\hat{L}^*$ can be related to the determinant of the covariance matrix, the same way we did for $\hat{L}_z$ in the beginning of this chapter. First, remark that $\langle\!\langle\!\langle \hat{L}^* \rangle\!\rangle\!\rangle_\Psi = 0$ where $\langle\!\langle\!\langle \hat{L}^* \rangle\!\rangle\!\rangle_\Psi$ means that we compute the mean value on three copies of a state $|\psi\rangle$. Indeed

$$\langle\!\langle\!\langle \hat{L}_x^* \rangle\!\rangle\!\rangle_\Psi = \frac{1}{2}\langle\psi|\langle\psi|\langle\psi|\hat{L}_x^*|\psi\rangle|\psi\rangle|\psi\rangle = \frac{1}{2}(\langle x\rangle\langle p\rangle - \langle p\rangle\langle x\rangle) = 0 \tag{10.86}$$

and similarly for $\langle\!\langle\!\langle \hat{L}_y^* \rangle\!\rangle\!\rangle_\Psi$ and $\langle\!\langle\!\langle \hat{L}_z^* \rangle\!\rangle\!\rangle_\Psi$, so that $\langle\!\langle\!\langle \hat{L}^* \rangle\!\rangle\!\rangle_\Psi = 0$. Now, let us compute the second-order moment of $\hat{L}^*$. First, we have

$$(L_x^* + L_y^* + L_z^*)^2 = L_x^{*2} + L_y^{*2} + L_z^{*2} + L_x^*L_y^* + L_y^*L_x^* + L_x^*L_z^* + L_z^*L_x^* + L_y^*L_z^* + L_z^*L_y^* \tag{10.87}$$

with

$$\begin{aligned} L_x^{*2} + L_y^{*2} + L_z^{*2} &= \frac{1}{4}\left(x_2^2p_1^2 + x_3^2p_1^2 + x_1^2p_2^2 + x_3^2p_2^2 + x_1^2p_3^2 + x_2^2p_3^2\right) \\ &\quad -\frac{1}{4}\Big(x_1p_1p_2x_2 + x_2p_2p_3x_3 + x_3p_3p_1x_1 + p_1x_1x_2p_2 \\ &\quad +p_2x_2x_3p_3 + p_3x_3x_1p_1\Big) \\ &= \frac{1}{4}\left(x_2^2p_1^2 + x_3^2p_1^2 + x_1^2p_2^2 + x_3^2p_2^2 + x_1^2p_3^2 + x_2^2p_3^2\right) \\ &\quad -\frac{1}{8}\left(\{x_1,p_1\}\{x_2,p_2\} + \{x_2,p_2\}\{x_3,p_3\} + \{x_3,p_3\}\{x_1,p_1\}\right) \\ &\quad +\frac{1}{8}\left([x_1,p_1][x_2,p_2] + [x_2,p_2][x_3,p_3] + [x_3,p_3][x_1,p_1]\right) \end{aligned} \tag{10.88}$$

and

$$\begin{aligned} L_x^*L_y^* + L_x^*L_z^* + L_y^*L_z^* + L_y^*L_x^* + L_z^*L_x^* + L_z^*L_y^* &= -\frac{1}{2}\left(p_1p_2x_3^2 + p_1p_3x_2^2 + p_2p_3x_1^2\right) \\ &\quad -\frac{1}{2}\left(p_1^2x_2x_3 + p_2^2x_1x_3 + p_3^2x_1x_2\right) \\ &\quad +\frac{1}{4}\Big(\{x_1,p_1\}(x_2p_3 + p_2x_3) \\ &\quad +\{x_2,p_2\}(x_1p_3 + p_1x_3) \\ &\quad +\{x_3,p_3\}(x_1p_2 + p_1x_2)\Big). \end{aligned} \tag{10.89}$$

Therefore, if we take the mean value of $(\hat{L}^*)^2$ on three copies of the state we obtain

$$\begin{aligned} \langle\!\langle\!\langle (\hat{L}^*)^2 \rangle\!\rangle\!\rangle &= \frac{1}{3}\langle\!\langle\!\langle (L_y^* + L_x^* + L_z^*)^2 \rangle\!\rangle\!\rangle \\ &= \frac{1}{12}6\langle x^2\rangle\langle p^2\rangle - \frac{1}{6}3\langle x^2\rangle\langle p\rangle\langle p\rangle - \frac{1}{6}3\langle p^2\rangle\langle x\rangle\langle x\rangle + \frac{1}{12}6\langle\{x,p\}\rangle\langle x\rangle\langle p\rangle \\ &\quad -\frac{1}{24}3\langle\{x,p\}\rangle^2 + \frac{1}{24}3\langle[x,p]\rangle^2 \\ &= \frac{1}{2}\left(\det\gamma + \frac{1}{4}\langle[x,p]\rangle^2\right). \end{aligned} \tag{10.90}$$

If we summarize, we just showed that the variance of $\hat{L}^*$ is related to the determinant of the covariance matrix as

$$(\Delta\hat{L}^*)^2 = \frac{1}{2}\left(\det\gamma + \frac{1}{4}\langle[x,p]\rangle^2\right). \tag{10.91}$$

Once again, since a variance is always positive, we can thus deduce that

$$\det\gamma \geq -\frac{1}{4}\langle[x,p]\rangle^2. \tag{10.92}$$

If x and p are classical, they commute and so the previous relation simply says that a covariance matrix is always positive. However, when x and p are quantum, they do not commute anymore since $[x,p] = i$ and Eq. (10.92) says that $\det\gamma \geq 1/4$ which is nothing else than Robertson-Schrödinger uncertainty relation. Therefore, this suggests that $\hat{L}^*$ is a good uncertainty observable and we would like to define a new entropic uncertainty relation based on the Shannon entropy of this observable:

$$H(\hat{L}^*)_\rho \geq 0. \tag{10.93}$$

Yet, to compute the Shannon entropy we need to know the eigenvectors or $\hat{L}^*$ and their associated measurement probabilities. Since $\hat{L}^* = (\hat{L}_x^* + \hat{L}_y^* + \hat{L}_z^*)/\sqrt{3}$, our 3-copy observable is still the component of an angular momentum, but rotated along the direction $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Its eigenspectrum is thus the one of an angular momen-

tum. More precisely, the eigenvalues of $(\hat{L}^*)^2$ and $\hat{L}_z^*$ will be given, respectively, by

$$\begin{aligned} l^* &= 0 && m = \{0\} \\ l^* &= 1/2 && m = \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\} \\ l^* &= 1 && m = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\} \\ &\text{etc.} \end{aligned} \tag{10.94}$$

Comparing to a genuine angular momentum, the eigenvalues are all divided by two because of the definition of the $\hat{L}_i^*$ (see Eq. (10.84)). Moreover, the jump between two eigenvalue is $1/2$ instead of 1 because the commutation relations are $[\hat{L}_i^*, \hat{L}_j^*] = \frac{i}{2}\epsilon_{ijk}\hat{L}_k^*$[11] (while it is $[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k^*$ for a genuine angular momentum).

The eigenfunctions are the spherical harmonics. Nevertheless, their form is not convenient to manipulate since they have to be written in the associated rotated quadratures. Computing the probabilities of measuring the eigenvalues of $\hat{L}^*$ through the spherical harmonics is then not an easy task and we most likely should find a way to skirt this calculation.

10.2.2 Alternative definitions

Using the relations between the x, p quadratures and the mode operators, we can also write the 3-copy observables as

$$\begin{aligned} \hat{L}_x^* &= \frac{i}{2}(\hat{a}_2\hat{a}_3^\dagger - \hat{a}_2^\dagger\hat{a}_3) \\ \hat{L}_y^* &= \frac{i}{2}(\hat{a}_3\hat{a}_1^\dagger - \hat{a}_3^\dagger\hat{a}_1) \\ \hat{L}_z^* &= \frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2). \end{aligned} \tag{10.95}$$

In particular, this allows us to compute $(\hat{L}^*)^2$ as

$$(\hat{L}^*)^2 = \frac{1}{12}\left((\hat{n}_1 + \hat{n}_2 + \hat{n}_3)(\hat{n}_1 + \hat{n}_2 + \hat{n}_3 + 1) - (\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2} + \hat{a}_3^{\dagger 2})(\hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2)\right) \tag{10.96}$$

where $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$. We remark that it is symmetric, but does not have the form $l(l+1)$ anymore, like we had before.

Another definition is given with the help of the generalization of the Pauli matrices in 3×3 dimensions: the Gell-Mann matrices. There are eight of them, named λ_i, but

[11] As we show in the next section.

if we choose

$$S_x \equiv \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y \equiv -\lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z \equiv \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10.97}$$

and define $\hat{A} = \begin{pmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{pmatrix}^T$ we can write the three operators $\hat{L}_i^*$ as

$$\hat{L}_x^* = \frac{1}{2} A^\dagger S_x A, \qquad \hat{L}_y^* = \frac{1}{2} A^\dagger S_y A, \qquad \hat{L}_z^* = \frac{1}{2} A^\dagger S_z A. \tag{10.98}$$

From any of the above formulations, we can compute the commutation relations of the $\hat{L}_i^*$ observables. They almost obey the ones of an angular momentum, that is

$$[\hat{L}_i^*, \hat{L}_j^*] = \frac{i}{2} \epsilon_{ijk} \hat{L}_k^* \tag{10.99}$$

where the $1/2$ factor comes from our definition of the $\hat{L}_i^*$ which also had an additional one half factor. All the algebraic properties of operators $\hat{L}_x^*$, $\hat{L}_y^*$, $\hat{L}_z^*$ should be describable in a unified form based on (10.98) and the properties of the Gell-Mann matrices.

10.2.3 Physical realization

For $\hat{L}^*$ too, it is possible to find an optical circuit that allows us to measure this observable. The circuit is shown in Figure 10.5. We see that it is similar to the one of the 2-copy observable in the sense that here too, we apply a rotation followed by a beam splitter and then compute the difference of the output mean photon numbers on mode 2 and 3, that is

$$\hat{L}^* = \frac{1}{4}(\hat{n}_2 - \hat{n}_3). \tag{10.100}$$

However, before applying this transformation, we first apply two beam splitting transformations of transmittance $1/2$ and $1/3$. The purpose is to make the appropriate rotation such that we carry out the right change of basis in order to be in a situation similar to the 2-copy observable case. Indeed, after the two first beam splitters, the mode operators are now given by

$$\begin{aligned} \hat{a}_1' &= \frac{1}{\sqrt{3}}(\hat{a}_1 + \hat{a}_2 + \hat{a}_3) \\ \hat{a}_2' &= \frac{1}{\sqrt{2}}(-\hat{a}_1 + \hat{a}_2) \\ \hat{a}_3' &= \frac{1}{\sqrt{6}}(-\hat{a}_1 - \hat{a}_2 + 2\hat{a}_3). \end{aligned} \tag{10.101}$$

In particular, the first mode is equal to the the sum of the three input mode operators. It means that the angular momentum of this mode is equal to $(\hat{L}_x^* + \hat{L}_y^* + \hat{L}_z^*)/\sqrt{3}$

which is equal to $\hat{L}^*$. Therefore, learning from the 2-copy observable, we know that we can measure this observable by applying a $\pi/2$ rotation followed by a beam splitter on the two other modes. Of course, we still have a degree of liberty on the two last modes after the two first beam splitters, since any real rotation between mode 2 and 3 will not affect mode 1 (and so it will still corresponds to $\hat{L}^*$) and this should not change the value of the output measurement.[12]

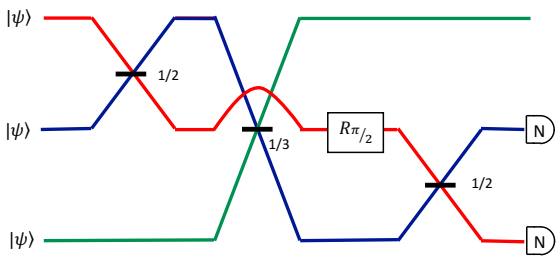

Figure 10.5: Physical realization to measure the 3-copy observable $\hat{L}^*$.

Remark that the invariance under displacement is easy to see from the circuit of Figure 10.5. Indeed, let us insert a state with displacement $D(\alpha)$[13] at each input of the circuit. After the two first beam splitters, the displacement on each mode becomes

$$D(\alpha)^{\otimes 3} \to D(\sqrt{3}\alpha)D(0)D(0). \tag{10.102}$$

It means that no matter what is the value of the initial displacement, the displacement will be 0 on the two last modes before we apply the rotation and the last beam splitter. Therefore, the result of the measurement of the photon-number difference — which gives $\hat{L}^*$ — will not depend on the displacement.

Note also, that measuring the photon-number difference between modes 1 and 2 or modes 3 and 1, we can measure the two other observables that complete this new basis which includes $\hat{L}^*$.

10.2.4 Special case of Gaussian states

Interestingly, if we insert 3 copies of a squeezed state in the optical circuit of Figure 10.5, after the first two beam splitters, we still have the three same squeezed states on each mode. It means that once again, similarly to the 2-copy case, we will have a zero photon-number difference at the output and with probability one. In other words, the entropy of $\hat{L}^*$ is equal to zero for a pure Gaussian state. The new

[12] We can easily verify that adding a beam splitter between mode 2 and 3, right before the rotation, do not change the measurement of $\hat{L}^*$. This is compatible with the fact that the 2-copy observable $\hat{L}_z$ is invariant under a real rotation between both systems.

[13] D is the displacement operator as defined in Section 2.5.1.

entropic uncertainty relation $H(\hat{L}^*) \geq 0$ thus, one more time, coincides with the Robertson-Schrödinger uncertainty relation with respect to saturation.

Moreover, if we insert any Gaussian state, mixed or pure, at the input, we will still have the three same Gaussian states after the two beam splitting transformations.[14] In particular, it means that on modes 2 and 3, right before the $\pi/2$ rotation, we have two copies of the Gaussian state. The situation is thus the same as in the 2-copy optical circuit of Figure 10.1 and all conclusions can be applied here. In particular, the entropy of a Gaussian state will be the same as before

$$H(\hat{L}^*)_{\rho^G} = H(\hat{L}_z)_{\rho^G} = \ln(2\nu) - \frac{4\nu^2 - 1}{4\nu} \ln \frac{2\nu - 1}{2\nu + 1} \tag{10.103}$$

where ν is the symplectic value of the Gaussian state.

10.2.5 Case of non-Gaussian states

Obviously, we however expect the entropy of any non-Gaussian state to be different, hence the interest of defining another entropic uncertainty relation. For example, let us consider the Fock state $|1\rangle$. If we insert three copies of this state in the circuit of Figure 10.5, right before the $\pi/2$ rotation, the state will not be $|1\rangle^{\otimes 2}$ anymore. Therefore, the entropy $H(\hat{L}^*)_{|1\rangle}$ will be different than the entropy of the 2-copy observable, $H(\hat{L}_z)_{|1\rangle}$ (as computed in Section 10.1.10).

[14] When two copies of a Gaussian state are traveling through a beam splitter, we find the same two Gaussian states at the output.

Part III | Continuous-variable separability criteria

11 | Improved continuous-variable separability criterion based on the degree of Gaussianity

This chapter is the subject of the following article:
A. Hertz, E. Karpov, A. Mandilara, and N. J. Cerf, Phys. Rev. A **93** 032330 (2016) [a].

Quantum entanglement is nowadays considered a central resource in the field of quantum information and computation [75]. It is therefore crucial to be able to determine whether a quantum state is separable or entangled, which is provably a hard decision problem when it comes to mixed states. Currently available separability criteria for continuous-variable states are generally based on the covariance matrix. The well-known separability criterion of Duan *et al.* and Simon, for example, gives a necessary and sufficient condition for a two-mode Gaussian state to be separable, but leaves many entangled non-Gaussian states undetected (see Chapter 5). Here, we introduce an improvement of this criterion that enables a stronger entanglement detection. The improved condition is based on the knowledge of an additional parameter, namely the degree of Gaussianity,

$$g = \frac{\mathrm{Tr}(\rho\rho^G)}{\mathrm{Tr}(\rho^G\rho^G)}, \tag{11.1}$$

and exploits a connection with Gaussianity-bounded uncertainty relations introduced in Chapter 4. We exhibit families of non-Gaussian entangled states whose entanglement remains undetected by the Duan-Simon criterion.

In order to follow the same notation used in the publication from which this chapter is taken [a], we define here the covariance matrix as $\gamma_{ij} = \langle \hat{r}_i\hat{r}_j + \hat{r}_j\hat{r}_i\rangle - 2\langle \hat{r}_i\rangle\langle \hat{r}_j\rangle$ where $\mathbf{r} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$. It simply means that the covariance is worth double the definition we had got used to in this thesis. In particular, it means that the Robertson-Schrödinger uncertainty relation now reads $\det\gamma \geq 1$. Note also that we assume with no loss of generality that all states considered in the following have vanishing coher-

ent vectors ($d_j = 0$) since first-order moments are irrelevant as far as entanglement detection is concerned.

In this context, the continuous-variable separability criterion expresses that if a two-mode state is separable, then its so-called EPR variance complies with the following inequality

$$\Delta \equiv \frac{1}{2}\Big(\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle\Big) \geq \frac{1}{2}\Big(\alpha^2 + \frac{1}{\alpha^2}\Big), \tag{11.2}$$

for any real (nonzero) α, where the operators

$$\hat{u} = |\alpha|\hat{x}_1 + \frac{1}{\alpha}\hat{x}_2, \qquad \hat{v} = |\alpha|\hat{p}_1 - \frac{1}{\alpha}\hat{p}_2, \tag{11.3}$$

are functions of the quadratures components $\hat{x}$ and $\hat{p}$ of modes 1 and 2.

It is natural to expect that a stronger criterion can be obtained with more information on the state, but the additional parameter should be chosen carefully. We could think, for example, of using the purity of the state $\mu = Tr[\rho^2]$ as the additional constraint since Dodonov proved that the lower bound of the uncertainty improves for mixed states [29]. However, the purity of the state is not a good candidate, since the Duan-Simon criterion is necessary and sufficient for every Gaussian state of arbitrary purity. As we show in this chapter, the degree of Gaussianity g is however a good choice and we will thus exploit the fact that the knowledge of g gives a tighter bound in the uncertainty relations (see Section 4.1.4), which in turns translates into a stronger condition for detecting entanglement.

In the next section, we suggest a new criterion for a better detection of entangled states employing this degree of Gaussianity, thus improving the Duan-Simon criterion for non-Gaussian states. Then, we provide explicit examples of entangled non-Gaussian states that are left unnoticed by the Duan-Simon criterion, demonstrating the advantage of our new criterion. We also give an analytical way of computing parameter g. The examples are produced from non-Gaussian states belonging both to the set of classical states (with positive Glauber P-function) and genuinely quantum states (Fock states), reflecting the general applicability of our method. Finally, we conclude and discuss possible extensions of our work.

11.1 Improved separability condition

Let us investigate the separability of an arbitrary two-mode state ρ. As mentioned earlier, the PPT criterion consists in verifying the physicality of the partially transposed state ρ^{T_2} (which must hold for any separable two-mode state). Then, an entangled state ρ will be detected if ρ^{T_2} is not physical (it exhibits a negative eigenvalue). Applying a partial transposition (acting on the second mode) in state space is equivalent to a mirror reflection $\hat{p}_2 \to -\hat{p}_2$ in phase space. Thus, in order to detect

entanglement, we need to check the physicality of ρ^{T_2} in phase space, which can be achieved based on the symplectic values of its covariance matrix.

Suppose that a two-mode state ρ has a covariance matrix γ. According to Williamson theorem (see Section 2.4.1), there always exists a unitary transformation U_S mapping the state ρ onto σ such that the associated symplectic transformation S maps γ onto

$$\gamma_\sigma = S\gamma S^T = \begin{pmatrix} \nu_+ \mathbb{1} & 0 \\ 0 & \nu_- \mathbb{1} \end{pmatrix}, \tag{11.4}$$

where $\nu_+(\nu_-)$ is the largest (smallest) symplectic value of γ and $\mathbb{1}$ is the 2×2 identity matrix. The uncertainty relations implie that the inequality $\nu_+ \geq \nu_- \geq 1$ [1] must be respected for any physical state. Applying this condition to the partially transposed state ρ^{T_2}, we understand that the entanglement of states ρ is detected whenever the smallest symplectic value of ρ^{T_2} is strictly smaller than 1, which is the core of the Duan-Simon criterion[2].

Let us now introduce our improved criterion. In order to detect the entanglement of state ρ, we apply a partial transposition on the second mode (which may lead to an unphysical state) followed by a symplectic transformation, which gives access to the symplectic values $\nu_\pm$ of the partial transposed state ρ^{T_2}. This is the "entanglement analyzing box" shown in Figure 11.1.

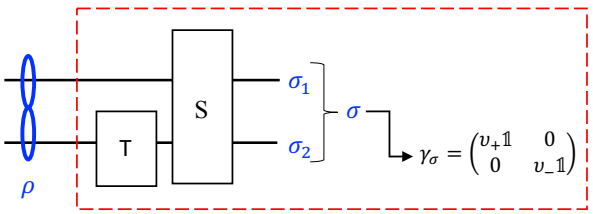

Figure 11.1: "Entanglement analyzing box". A partial transposition (T) and symplectic transformation (S) are applied to state ρ, giving access to the symplectic values $\nu_\pm$ of the partial transposed state. Note that this circuit is not physical since T is antiunitary.

A key observation is that the inequality $\nu_+ \geq \nu_- \geq 1$ boils down to expressing the uncertainty relation for the two modes making σ at the output of the entanglement analyzing box. Indeed, we have $\det(\gamma_{\sigma_1}) = (\nu_+)^2 \geq 1$ and $\det(\gamma_{\sigma_2}) = (\nu_-)^2 \geq 1$, where γ_{σ_1} (γ_{σ_2}) refers to the covariance matrix of the first (second) mode of σ. Furthermore, a tighter lower bound on the uncertainty $\det(\gamma)$ of a mode can be obtained if the degree of Gaussianity g of this mode is known [55] [we use definition (11.1) for a single mode]. Combining these elements implies that we can detect the non-physicality of ρ^{T_2} whenever the lowest symplectic value ν_- lies under the lower bound corresponding to the degree of Gaussianity g_2 of σ_2, as shown in Figure 11.2.

[1] The difference with Eq. (4.13) is due to our definition of the covariance matrix in this chapter.
[2] See Section 5.3.1.

This lower bound is equal to 1 for $g_2 = 1$, but is strictly larger than 1 for non-Gaussian states with $g_2 \neq 1$.

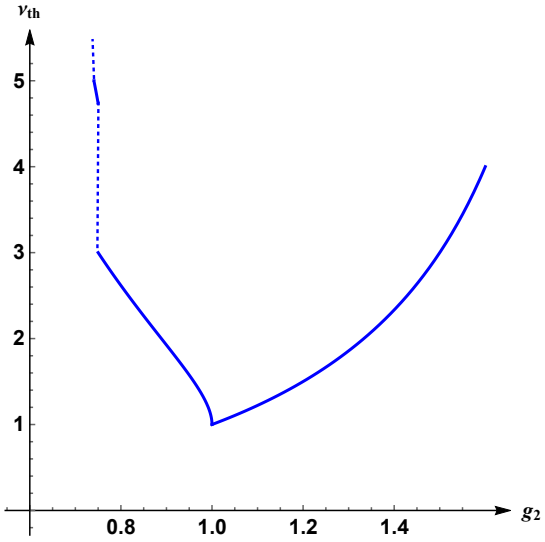

Figure 11.2: Plot of ν_{th}, threshold (minimum allowed) value for ν_- as a function of the degree of Gaussianity g_2 (details are given in [55]). All physical states lie on or above this curve. Note that for $g_2 < 1$, the curve exhibits some discontinuities.

Hence, we obtain an improved separability criterion which works as follows. After applying the "entanglement analyzing box" of Figure 11.1 to state ρ, we detect its entanglement if the symplectic value of the reduced state σ_2 is smaller than a bound, which is a function of the degree of Gaussianity g_2 of σ_2. In other words,

$$\nu_- < \nu_{\text{th}}(g_2) \quad \Rightarrow \quad \text{entanglement} \tag{11.5}$$

where $\nu_{\text{th}}(g_2)$ is the threshold given by the curve in Figure 11.2. As we already saw in Chapter 4, if $g_2 \geq 1$, the curve is given by $\nu_{\text{th}} = g/(2-g)$. If $g_2 < 1$, the parametric equations of the curve are given by

$$\begin{aligned} \nu_{\text{th}} &= 2n + 3 - 2r, \\ g_2 &= \frac{2\nu_{\text{th}}(\nu_{\text{th}}-1)^n}{(\nu_{\text{th}}+1)^{n+1}} \left(\frac{(\nu_{\text{th}}-1)(1-r)}{\nu_{\text{th}}+1} + r \right), \end{aligned} \tag{11.6}$$

where $n \in \mathbb{N}$ and $r \in [0,1[$. The latter curve consists of consecutive segments corresponding each to a binary mixture of nearest-neighbor Fock states $|n\rangle$ and $|n+1\rangle$. In the examples that we will discuss in the next section, the degree of Gaussianity always lies in the segment where $n = 0$, which corresponds to

$$\nu_{\text{th}}(g_2) = \frac{2 - g_2 + 2\sqrt{1-g_2}}{g_2} \qquad \text{if} \quad 3/4 \leq g_2 < 1. \tag{11.7}$$

In order to exploit condition (11.5), the last step is thus to be able to compute the degree of Gaussianity g_2 as given by Eq. (11.1). The analytical computation of g_2

is not trivial for an arbitrary two-mode state (although we give an explicit method for some class of states in the next section), but at least a numerical computation is always feasible based on the Wigner function. First, remark that the denominator of g_2 is simply equal to $1/\nu_-$ since it corresponds to the purity of a Gaussian state (see Eq. (11.12)). To express the numerator of g_2, we use the Wigner function $\tilde{W}_2(x_2, p_2)$ of the second mode σ_2 at the output of the entanglement analyzing box. Starting from $W(x_1, p_1, x_2, p_2)$, namely the Wigner function of the initial two-mode state ρ, we find $W^{T_2}(x_1, p_1, x_2, p_2) = W(x_1, p_1, x_2, -p_2)$ after partial transposition and then $\tilde{W}(\mathbf{r}) = W^{T_2}(S^{-1}\mathbf{r})$ after symplectic transformation, corresponding to a change of variable $\mathbf{r} \to S\mathbf{r}$ with $\mathbf{r} = (x_1, p_1, x_2, p_2)^T$. Finally, we integrate over x_1 and p_1 to have the Wigner function of the second mode σ_2, which gives

$$g_2 = \frac{\mathrm{Tr}(\sigma_2 \sigma_2^G)}{\mathrm{Tr}(\sigma_2^G \sigma_2^G)} = 2\pi\nu_- \int \tilde{W}_2(x_2, p_2)\tilde{W}_2^G(x_2, p_2)dx_2 dp_2 \tag{11.8}$$

where $\tilde{W}_2^G(x_2, p_2)$ is the Wigner function of the Gaussian state with covariance matrix $\nu_- \mathbb{1}$.

Figure 11.3 enables us to visualize how entanglement detection is improved by our method. Three distinct zones are represented, delimited by the curve of Figure 11.2 and by the constant line $\nu_- = 1$. If a state lies in the red (lower) zone, it is an entangled state detected by the Duan-Simon criterion ($\nu_- < 1$), hence it is uninteresting for our purposes here. If it lies in the white (upper) zone, no conclusion can be made because the partially transposed state is physical. But interestingly, if it lies in the blue (intermediate) zone, we detect entanglement which was otherwise unnoticed.

Remark that, since partial transposition and symplectic transformation conserve the Gaussian character of a state (see Chapter 4), if ρ is a Gaussian state then σ as well as the reduced states σ_1 and σ_2 are also Gaussian. Then $\nu_{\mathrm{th}}(1) = 1$, and we recover the (necessary and sufficient) Duan-Simon separability criterion for Gaussian states, as expected.

Let us also mention that our criterion does not improve entanglement detection when the covariance matrix of ρ is diagonal, since the partially transposed state remain physical. For example, the "NOON" states of the form $(|N0\rangle + |0N\rangle)/\sqrt{2}$ have a diagonal covariance matrix for $N \geq 2$. Thus, even though those states are always entangled, we cannot do any better than the Duan-Simon criterion, and entanglement is undetected by our criterion. However, there exist many other interesting cases where our method is useful, as shown in the next section.

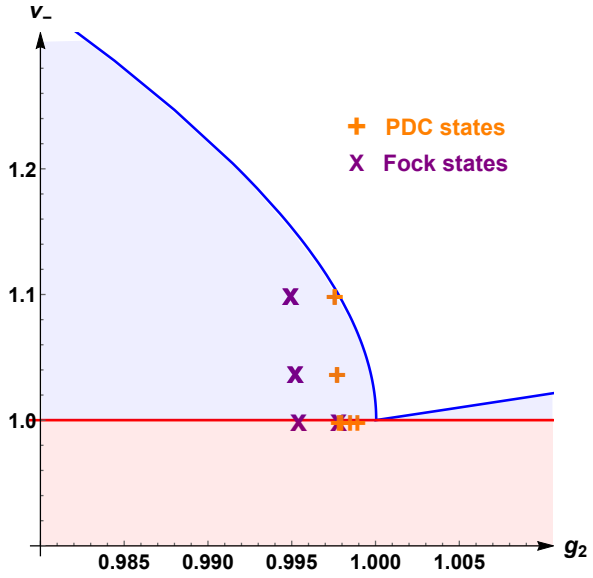

Figure 11.3: Examples of non-Gaussian entangled states generated from ($\times$) Fock or (+) phase-diffused coherent (PDC) states, which are detected by our improved criterion. In general, all entangled states detected by the Duan-Simon criterion ($\nu_- \geq 1$) lie in the red (lower) zone, while the entangled states that are detected by our criterion but remain undetected by the Duan-Simon criterion lie in the blue (intermediate) zone. The white (upper) zone contains either separable or undetected entangled states. The curve $\nu_{\text{th}}(g_2)$ separating the blue and white zones corresponds to the lower bound on ν_- for a fixed degree of Gaussianity g_2.

11.2 Detection of non-Gaussian entangled states

In this section, we apply our new criterion to two types of non-Gaussian states. Those examples have in common that entanglement is not detected on the sole basis of the covariance matrix (using the Duan-Simon criterion), but is detected with our improved criterion exploiting the degree of Gaussianity.

11.2.1 Non-Gaussian states generated from Fock states or phase-diffused coherent states

The first example uses non-Gaussian states as generated by the circuit shown in Figure 11.4. The preparation of the states works as follows. Initially, we have a state $\rho_{in} = \sum_{n=0}^{\infty} \phi_n |n\rangle\langle n|$ of covariance matrix $\gamma_{in} = a\,\mathbb{1}$ in the first mode, where $a \in [1, \infty)$, and the vacuum state in the second mode. Both states are processed through a two-mode squeezer (TMS) of parameter $\lambda \in [0, 1)$. Note that if ρ_{in} is a Fock state $|n\rangle$, at this point of the circuit we have a photon-added EPR state, which we know is always entangled (its entanglement is monotonically increasing with n) [108]. Of course, if we have the vacuum on both modes, the resulting state is simply an EPR state. The next step in the circuit consists in processing each mode of the state

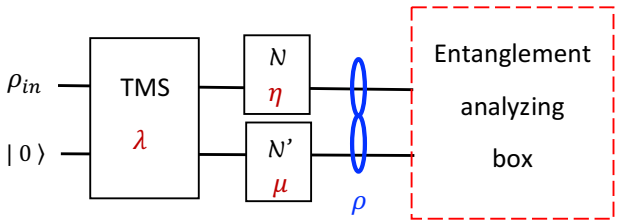

Figure 11.4: Quantum circuit used to prepare the non-Gaussian states ρ with covariance matrix γ. The entanglement of ρ is analyzed by the red box (see Figure 11.1), which pictures our improved criterion.

through two independent Gaussian additive-noise channels $\mathcal{N}$ and $\mathcal{N}'$. The variance of the added noise on the first (second) mode is denoted as η (μ). This construction ensures that the resulting (not necessarily Gaussian) state will always be physical provided that $a \geq 1, 0 \leq \lambda < 1$ and $\eta, \mu \geq 0$.

The state ρ at the output of this circuit has a covariance matrix

$$\gamma = \begin{pmatrix} \left(\frac{a+\lambda^2}{1-\lambda^2} + \eta\right) \mathbb{1} & \frac{(a+1)\lambda}{1-\lambda^2}\sigma_z \\ \frac{(a+1)\lambda}{1-\lambda^2}\sigma_z & \left(\frac{a\lambda^2+1}{1-\lambda^2} + \mu\right)\mathbb{1} \end{pmatrix}, \tag{11.9}$$

where σ_z is the third Pauli matrix. Note that this form for a covariance matrix is actually quite general. Indeed, Duan *et al.* have shown [8] that any covariance matrix of a two-mode state can be transformed into the form

$$\gamma = \begin{pmatrix} n & & c & \\ & n & & d \\ c & & m & \\ & d & & m \end{pmatrix} \tag{11.10}$$

by applying local linear unitary Bogoliubov operations, i.e., combinations of squeezing transformations and rotations. These operations do not influence the separability of the state, and are thus always allowed when studying entanglement. The covariance matrix (11.9) depends on three parameters while the most general form (11.10) has only one additional parameter, which reflects that (11.9) encompasses a wide class of two-mode Gaussian states.

The entanglement of the resulting state ρ is now analyzed as depicted by the red box of Figure 11.1. The resulting state σ has a covariance matrix of the form of Eq. (11.4) and the symplectic values of ρ^{T_2} can be expressed as a function of the different parameters characterizing γ, namely

$$\nu_{\pm} = \frac{1}{2}\left(\frac{(a+1)(1+\lambda^2)}{1-\lambda^2} + \eta + \mu \pm \sqrt{(a-1+\eta-\mu)^2 + \frac{4(a+1)^2\lambda^2}{(1-\lambda^2)^2}}\right). \tag{11.11}$$

Note that this expression is valid regardless of whether ρ is Gaussian or not. At this point, using the Duan-Simon separability criterion ignoring whether ρ is Gaussian or not would detect entangled state only if $\nu_- < 1$. But we can improve on this by taking into account the degree of Gaussianity g_2 at the output of the circuit of Figure 11.1, as explained previously. The calculation of g_2 could be done through the computation of the Wigner function, but this would require some numerical integrations. We now give a different way of calculating g_2 for this specific example. The final expression is not very elegant, but calculations are performed completely analytically.

Computation of the degree of Gaussianity

Let us show how one can perform the computation of the degree of Gaussianity g_2 of the reduced one-mode state corresponding to σ_2 in the "entanglement analyzing box" of Figure 11.1 (this state corresponds to the smallest symplectic value ν_-). The same technique allows computation of the degree of Gaussianity g of the two-mode state ρ (see Figure 11.4) as well.

By our convention the Gaussian state σ_2^G has the same covariance matrix $\gamma_{\sigma_2} = \nu_- \mathbb{1}$ as σ_2. Then, since it corresponds to the purity of σ_2^G, the denominator in the definition of g_2 given by Eq. (11.1) is trivially evaluated as

$$\mathrm{Tr}[\sigma_2^G \sigma_2^G] = \frac{1}{\sqrt{\det(\gamma_{\sigma_2})}} = \frac{1}{\nu_-}. \tag{11.12}$$

The evaluation of the numerator in Eq. (11.1) is more involved. Although after the two-mode squeezer, we have a simple form of the density matrix of the state ρ_{TMS}, the addition of the Gaussian noises makes the density matrix of state ρ (and so the ones of the reduced states σ_1 and σ_2) very hard to express in a simple form. State ρ is obtained as a result of the application of Gaussian additive noise channels Φ_η and Φ_μ to the first and second modes of ρ_{TMS} correspondingly

$$\begin{aligned} \rho &= (\Phi_\eta \otimes \Phi_\mu)[\rho_{TMS}] \qquad (11.13) \\ &= \int dx_1 dp_1 dx_2 dp_2 \, \frac{e^{-\frac{x_2^2+p_2^2}{2\mu} - \frac{x_1^2+p_1^2}{2\eta}}}{(2\pi)^2 \eta\mu} D(x_1,p_1) D(x_2,p_2) \, \rho_{TMS} \, D^\dagger(x_1,p_1) D^\dagger(x_2,p_2), \end{aligned}$$

where $D(x,p)$ is the displacement operator. We will perform calculations avoiding the direct use of the density matrix σ_2. Instead we use the following construction. Let σ be the density matrix of the two-mode state which has σ_1 and σ_2 as reduced states. Then we have the following equivalent representation of the numerator in Eq. (11.1):

$$\begin{aligned} \mathrm{Tr}[\sigma_2 \sigma_2^G] &= \mathrm{Tr}[\sigma(\mathbb{1} \otimes \sigma_2^G)] \\ &= \lim_{V\to\infty} \frac{V+1}{2} \mathrm{Tr}[\sigma(\rho_{th}^V \otimes \sigma_2^G)], \qquad (11.14) \end{aligned}$$

where

$$\rho_{th}^{V} = \frac{2}{V+1} \sum_{n} \left(\frac{V-1}{V+1} \right)^{n} |n\rangle\langle n| \tag{11.15}$$

is a thermal state with a covariance matrix $V\mathbb{1}$. Identity (11.14) holds because this state multiplied by $(V+1)/2$ tends to $\mathbb{1}$ when V tends to infinity. Next we express σ as a result of the transformation of the initial state $\rho_{in} \otimes |0\rangle\langle 0|$ by the circuit in Figure 11.4:

$$\begin{aligned} \mathrm{Tr}[\sigma(\rho_{th}^{V} \otimes \sigma_2^{G})] = &\quad \mathrm{Tr}\left[U_S\, T[(\Phi_\eta \otimes \Phi_\mu)[U_\lambda(\rho_{in} \otimes |0\rangle\langle 0|)U_\lambda^\dagger]]U_S^\dagger\ \rho_{th}^{V} \otimes \sigma_2^{G}\right] \\ = &\quad \mathrm{Tr}\left[(\rho_{in} \otimes |0\rangle\langle 0|)U_\lambda^\dagger(\Phi_\eta \otimes \Phi_\mu)[T[U_S^\dagger \rho_{th}^{V} \otimes \sigma_2^{G} U_S]]U_\lambda\right]. \end{aligned} \tag{11.16}$$

Here U_S is the final symplectic transformation, U_λ describes the action of the two-mode squeezer, T is the partial transposition in the second mode and Φ_η (Φ_μ) denotes the additive Gaussian noise channels with the noise variances η (μ) being applied to first (second) mode. At the final step we used the invariance of the trace under cyclic permutations[3] and the equivalence of the partial transposition and additive noise channel to their duals with respect to the scalar product of operators defined as $\langle\langle A|B\rangle\rangle = \mathrm{Tr}[A^\dagger B]$ on a set of density operators of two-mode states. Let us prove this last two statements.

- The partial transposition as a map defined on the set of density operators is equal to its dual.

  Proof. Let us take a representation of two arbitrary density operators describing bipartite states in some basis

  $$\rho = \sum_{ijkl} c_{ijkl} |ij\rangle\langle kl| \qquad \text{and} \qquad \sigma = \sum_{nmrs} d_{nmrs} |nm\rangle\langle rs| \tag{11.17}$$

  and apply partial transposition T on ρ. Then we have

  $$\begin{aligned} \mathrm{Tr}[\,T[\rho]\,\sigma] &= \mathrm{Tr}\left[\sum_{ijkl} c_{ijkl} |il\rangle\langle kj| \sum_{nmrs} d_{nmrs} |nm\rangle\langle rs|\right] \\ &= \sum_{ijkl} c_{ijkl} d_{kjil} \\ &= \mathrm{Tr}\left[\sum_{ijkl} c_{ijkl} |ij\rangle\langle kl| \sum_{nmrs} d_{nmrs} |ns\rangle\langle rm|\right] \\ &= \mathrm{Tr}[\rho\, T[\sigma]]. \end{aligned} \tag{11.18}$$

  □

- The Gaussian additive noise (product) channel is equal to its dual on the set of density operators of two-mode states.

[3] Note that the invariance of the trace under cyclic permutations holds for bounded operators when at least one is trace-class (which is the case of the density matrix) [109] and both density matrices and Gaussian unitaries are bounded operators [110].

Proof. We prove first the equivalence on the example of the two-mode channel $\Phi_\eta \otimes \Phi_\mu$ applied to ρ_{TMS}:

$$\mathrm{Tr}\left[(\Phi_\eta \otimes \Phi_\mu)[\rho_{TMS}]\rho'\right] = \mathrm{Tr}[\rho_{TMS}(\Phi_\eta \otimes \Phi_\mu)[\rho']]. \tag{11.19}$$

Using Eq. (11.13) and the linearity of the trace we move it inside the integral and then make a cyclic permutation of the displacement operators. Then by applying the expression of the Hermitian conjugate of the displacement operator in the form $D^\dagger(x,p) = D(-x,-p)$ and by changing the variables

$$\begin{aligned} -x_1 &\to x_1 \qquad & -p_1 &\to p_1 \\ -x_2 &\to x_2 \qquad & -p_2 &\to p_2, \end{aligned} \tag{11.20}$$

we come to the desired conclusion.

This proof holds if we replace the state ρ_{TMS} by an arbitrary density operator. □

Following Eq. (11.16) the trace in the right hand side of Eq. (11.14) can be computed as the trace of the product of density matrices when the dual circuit is applied to state $\rho_{th}^V \otimes \sigma_2^G$ taking into account the properties of the dual maps discussed above. The big advantage of doing so is that $\rho_{th}^V \otimes \sigma_2^G$ is a Gaussian state and all the transformations which constitute the circuit (and their duals) preserve the Gaussian character of the state. Thus the state

$$\rho^* = U_\lambda^\dagger(\Phi_\mu \otimes \Phi_\eta)[T[U_S^\dagger \rho_{th}^V \otimes \sigma_2^G U_S]]U_\lambda \tag{11.21}$$

is completely determined by its covariance matrix γ^* which is the result of the application of the dual circuit to the covariance matrix of $\rho_{th}^V \otimes \sigma_2^G$:

$$\gamma^* = S_{TMS}(-\lambda)\left(T\left[S^T\begin{pmatrix} V & 0 \\ 0 & \nu_- \end{pmatrix}S\right] + \begin{pmatrix} \eta & 0 \\ 0 & \mu \end{pmatrix}\right)S_{TMS}^T(-\lambda), \tag{11.22}$$

where S represents the symplectic diagonalization that gives the symplectic values, T the partial transposition (which acts on the two-mode covariance matrices as $p_2 \to -p_2$) and

$$S_{TMS}(\lambda) = \frac{1}{\sqrt{1-\lambda^2}}\begin{pmatrix} \mathbb{1} & \lambda\sigma_z \\ \lambda\sigma_z & \mathbb{1} \end{pmatrix}, \tag{11.23}$$

the two-mode squeezing transformation.

From covariance matrix γ^*, we can easily deduce the associated Wigner function $W^*(x_1,p_1,x_2,p_2)$ of ρ^* (see Eq. (2.31) for the Wigner function of a Gaussian state) and

with its help compute the following trace which is equal to the trace in Eq. (11.16)

$$\begin{aligned}\mathrm{Tr}[\rho_{in}\otimes|0\rangle\langle 0|\,\rho^*] &= (2\pi)^2\int dx_1dp_1dx_2dp_2\,W_{\rho_{in}}(x_1,p_1)\,W_{|0\rangle}(x_2,p_2)W^*(x_1,p_1,x_2,p_2)\\ &= 2\pi\,C\int dx_1dp_1\,W_{\rho_{in}}(x_1,p_1)W^m_{th}(x_1,p_1),\end{aligned}\tag{11.24}$$

where the normalization factor C is obtained by integrating over the variables of the second mode

$$2\pi\int dx_2dp_2W_{|0\rangle}(x_2,p_2)W^*(x_1,p_1,x_2,p_2) = C\,W^m_{th}(x_1,p_1),\tag{11.25}$$

and by stressing out a new Wigner function $W^m_{th}(x_1,p_1) = \frac{1}{\pi m}e^{-(x_1^2+p_1^2)/m}$ corresponding to a thermal state of variance m. Going back to the state space, the computation of the trace can be carried as follows:

$$\begin{aligned}\mathrm{Tr}[\rho_{in}\otimes|0\rangle\langle 0|\,\rho^*] &= C\,\mathrm{Tr}\Big[\sum_n\phi_n|n\rangle\langle n|\rho^m_{th}\Big]\\ &= C\,\frac{2}{m+1}\sum_n\phi_n\Big(\frac{m-1}{m+1}\Big)^n.\end{aligned}\tag{11.26}$$

Both parameters, m and C, depend on V. The explicit formula for m and C are cumbersome and we do not present it here, however, they allow us to carry out the limit $V\to\infty$ which provides the trace in Eq. (11.14). Together with Eq. (11.12) this gives us a value for g_2 following Eq. (11.1).

Numerical results

Now that we have defined a circuit to generate families of non-Gaussian states and detect their entanglement, we will focus on some explicit examples of such states in order to illustrate the usefulness of our improved criterion. In the circuit of Figure 11.4, we start with Fock-diagonal states ρ_{in}, which have a diagonal covariance matrix γ_{in} with variance $a = \sum_{n=0}^{\infty}\phi_n(2n+1)$. We are interested in non-Gaussian states ρ_{in} and will consider two rather extreme cases of such states. The first case is a single Fock state $|n\rangle$ with $n>0$, the parameter of the covariance matrix being thus $a = 2n+1$. This state has clear quantum features, such as negative parts in the Wigner function. Our second choice is a non-Gaussian mixture of coherent states with a random phase, which can be viewed as "classical". This phase-diffused coherent (PDC) state can equivalently be represented as a mixture of Fock states following a Poisson distribution:

$$\rho_{in} = \sum_{k=0}^{\infty}e^{-(a-1)/2}\frac{\left(\frac{a-1}{2}\right)^k}{k!}|k\rangle\langle k|.\tag{11.27}$$

These two examples for ρ_{in} are simple at a theoretical level, and may also be implemented experimentally. For the state ρ to be feasible experimentally, we will focus

on values of the parameter λ of the two-mode squeezer that are smaller than 0.8 ($\approx$ 10 dB). The values of the noise variances η and μ will be chosen smaller than 2 units of vacuum noise because otherwise the state ρ is necessarily separable (regardless of whether it is Gaussian or not). Indeed, each mode of ρ can be seen as the output of a classical Gaussian additive noise channel taking the corresponding mode of ρ_{in} as its input, and it is known that such a channel is entanglement breaking if $\eta \geq 2$ ($\mu \geq 2$) [111].

In Figure 11.3, we exhibit explicit examples of non-Gaussian states ρ that are generated from ρ_{in} being either a Fock state or a phase-diffused coherent state. The corresponding numerical values of the circuit parameters (a, λ, μ, ν) are displayed in Table 11.1. We first choose sets of values of the circuit parameters such that $\nu_- = 1$, implying that the Duan-Simon criterion does not detect entanglement. In this case, entanglement is detected as soon as $g_2 \neq 1$, so all these example states are proven to be entangled with our improved separability criterion. We then extend our search to larger values of ν_-. An entangled state is then detected whenever $\nu_- < \nu_{\text{th}}(g_2)$. Since in our examples, $3/4 \leq g_2 < 1$, function $\nu_{\text{th}}(g_2)$ is given by Eq. (11.7). All points localized in the blue zone are thus examples of non-Gaussian entangled states that are detected by our improved separability criterion but not otherwise. Remark that entangled states can be found with both choices of ρ_{in} (either a highly non-classical Fock states or a classical mixtures of phase-diffused coherent states).

type of ρ_{in}	a	λ	η	μ	ν_-	g_2
Fock	3	0.6	1/13	1	1	0.99541
Fock	3	0.3	0.1	228/757	1	0.99780
Fock	3	0.5	0.1	0.9	1.1	0.99492
Fock	3	0.5	0.1	0.8	1.04	0.99521
PDC	2	0.3	0.1	513/1271	1	0.99798
PDC	2	0.7	0.1	931/677	1	0.99850
PDC	3	0.6	1/13	1	1	0.99781
PDC	3	0.3	0.1	228/757	1	0.99893
PDC	3	0.5	0.1	0.9	1.1	0.99758
PDC	3	0.5	0.1	0.8	1.04	0.99771

Table 11.1: Values of the circuit parameters used to generate the examples of non-Gaussian entangled states that are detected by the improved separability criterion. The corresponding values of ν_- and g_2 are also given.

Interestingly, for all states created with our circuit when Gaussian noise is added on the first mode only (i.e., $\mu = 0$), we found out that the Duan-Simon separability criterion becomes necessary and sufficient, even for non-Gaussian states. Indeed, if $\eta < 2$,

the symplectic value ν_- is smaller than 1 for all values of a and λ, hence the state ρ is entangled. On the contrary, if $\eta \geq 2$, we have an entanglement breaking channel, so we know that the state ρ is necessarily separable. This confirms the validity of our method.

Finally, Figure 11.5 illustrates how the different circuit parameters influence the separability of the state. Starting with a Fock state (or with a phrase-diffused coherent state) with circuit parameters $a = 3$, $\lambda = 0.5$, $\eta = 0.1$, and $\mu = 0.7$, we see that by varying one of the parameters we can always create entangled states that are unnoticed by the Duan-Simon criterion. (Note that there is no curve to plot corresponding to varying a for Fock states since a can only take odd integer values in this case.)

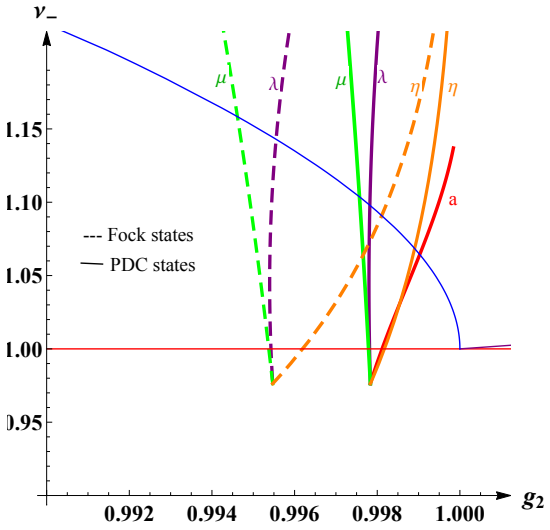

Figure 11.5: Evolution of symplectic eigenvalue ν_- when varying the circuit parameters, illustrating how they influence the separability of the states. For each curve, one of the parameters varies while the others are fixed, starting from $a = 3$, $\lambda = 0.5$, $\eta = 0.1$ and $\mu = 0.7$. Increasing ν_- is achieved by increasing a, η, μ, and decreasing λ. The blue curve is given by $\nu_{\text{th}}(g_2)$.

11.2.2 Squeezed single-photon path-entangled state

As a second example, let us consider a squeezed single-photon path-entangled state, i.e. the non-Gaussian state created from the circuit of Figure 11.6.

A vacuum and single-photon Fock states are both squeezed, with respective squeezing parameters s_- and s_+, and are then coupled with a balanced beam splitter. The wave function of the output (pure) state has the form [84]

$$\psi(x,y) = \frac{(x+y)}{\sqrt{\pi s_- s_+^3}} e^{-\frac{(x+y)^2}{4s_+^2} - \frac{(x-y)^2}{4s_-^2}}. \tag{11.28}$$

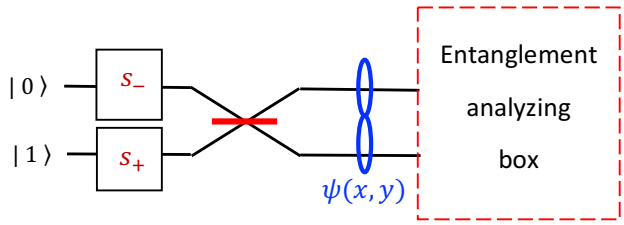

Figure 11.6: Quantum circuit used to prepare a squeezed single-photon path-entangled state $\psi(x,y)$. The entanglement of the state is analyzed by the red box.

This state is obviously entangled for all values of $s_\pm$, but the Duan-Simon separability criterion detects entanglement only for $s_-/s_+ > \sqrt{3}$ or $s_-/s_+ < 1/\sqrt{3}$. However, similarly as what Walborn *et al.* [84] have shown using their entropic entanglement criterion, we can detect entanglement for all values of $s_\pm$ with our improved criterion. Let us suppose that $s_-/s_+ \geq 1$. Applying the "entanglement analyzing box" to this state, we find that

$$\nu_- = \sqrt{3}\,\frac{s_+}{s_-} \quad \text{and} \quad g_2 = \frac{3}{4}\sqrt{\frac{3}{2}}, \tag{11.29}$$

where g_2 is computed with the help of Wigner functions. Therefore, according to Eq. (11.7), $\nu_{\text{th}}(g_2) = 1.7986$ and entanglement is detected if

$$\nu_- < \nu_{\text{th}}(g_2) \quad \Leftrightarrow \quad \frac{s_-}{s_+} > 0.963. \tag{11.30}$$

However, we supposed at the beginning that $s_-/s_+ \geq 1$. Entanglement is thus always detected. The same analysis can be done if $s_-/s_+ < 1$. Although the entropic criterion of Walborn *et al.* also detects the entanglement of this state for all $s_\pm$, we believe that our method is easier to apply since their method require an optimization over different angle to be truly effective (see Section 5.3.2).

11.3 Conclusion

In this Chapter, we have proposed a new continuous-variable separability criterion by considering the degree of Gaussianity of the state, thereby allowing a stronger detection of two-mode non-Gaussian entangled states. Our improved criterion works by verifying the physicality of the symplectic values of the partially transposed state in terms of Gaussianity-bounded uncertainty relations. We demonstrated the advantages of our method by providing explicit examples of states whose entanglement is detected by our criterion but left undetected by the Duan-Simon criterion. We proposed an optical circuit for creating a family of such states and studied the entanglement detection as a function of the parameters of the circuit. The values of those circuit parameters were chosen so that these example states could be experimentally

generated to demonstrate the method. The general applicability of the method is witnessed by the fact that these example states can be generated both from genuinely quantum non-Gaussian states (Fock states) or from classical non-Gaussian mixtures of phase-diffused coherent states (states with a positive P-function). We expect that many more examples of entangled states could be found, first by testing different values of the parameters a, λ, η and μ, second by generalizing the circuit (for example, at the very beginning of the circuit, one can insert a thermal state instead of the vacuum) or simply by devising a new circuit generating other types of non-Gaussian states such as those of our second example.

As mentioned in the introduction, a separability condition such as inequality (11.2) cannot be rewritten as such with a tighter lower bound that would solely depend on purity $\mathrm{Tr}(\rho^2)$. This is because the Duan-Simon criterion is necessary and sufficient for all Gaussian states (of arbitrary purity). Hence, the lower bound in inequality (11.2) cannot be moved upwards without being violated by some mixed Gaussian states that are known to be separable. However, we expect that our separability criterion may be further improved by taking into account both the degree of Gaussianity and purity of the state, and then making use of the purity- and Gaussianity-bounded uncertainty relations [56]. This topic is worth further investigation.

12 | Improved entropic separability criterion for non-Gaussian states

In this chapter, we propose a new separability criterion for bipartite continuous-variable systems. This criterion is more sensitive than the one of Duan *et al* and Simon (see Section 5.3.1), but not than the one of Walborn *et al.* (see Section 5.3.2). In fact, the entropic separability criterion that we introduce will detect exactly the same entangled states as the already existing entropic criterion, but the calculations involved in the process are much easier. In [84], Walborn *et al.* give a criterion based on two parameters which are the angles of rotations θ_1 and θ_2 of the quadratures (see Eq. (5.32)). The idea is then to find the optimal angles, i.e. the ones that highlight the correlations in the state. In other words, one needs to optimize on the angles θ_1 and θ_2 in order to find the most restrictive separability criterion. Walborn *et al.* separability criterion is based on the entropic uncertainty relation of Białynicki-Birula and Mycielski, Eq. (4.39). Here, we propose to use the improved entropic uncertainty relation presented in Chapter 8 since this uncertainty relation already takes the x-p correlations into account. We thus do not need to make any optimization anymore, since it is already included in this entropic uncertainty relation.

12.1 Derivation of the criterion

As in the other separability criteria, let us consider the EPR quadratures

$$x_\pm = x_1 \pm x_2 \qquad \text{and} \qquad p_\pm = p_1 \pm p_2 \tag{12.1}$$

where x_i, p_i are the position and momentum quadratures of mode $i = 1, 2$ and they obey the commutation relation $[x_i, p_j] = i\delta_{i,j}$, so that

$$\begin{aligned} [x_+, p_+] &= 2i & [x_-, p_-] &= 2i \\ [x_+, p_-] &= 0 & [x_-, p_+] &= 0. \end{aligned} \tag{12.2}$$

To each EPR quadratures is associated a probability distributions $X_\pm(x)$ or $P_\pm(p)$ and we can write their Shannon differential entropies (see Eq. (3.9))

$$\begin{aligned} h(x_\pm) &= -\int dx\, X_\pm(x) \ln X_\pm(x) \\ h(p_\pm) &= -\int dp\, P_\pm(p) \ln P_\pm(p). \end{aligned} \tag{12.3}$$

Let us now consider a separable state. Since this state is separable, it must remain physical after a partial transposition, (that is the PPT criterion, see Chapter 5), so it has to respect the entropic uncertainty relation Eq. (8.11)

$$\tilde{h}(x_\pm) + \tilde{h}(p_\pm) - \frac{1}{2}\ln\left(\frac{\tilde{\sigma}^2_{x_\pm}\tilde{\sigma}^2_{p_\pm}}{\tilde{\sigma}^2_{x_\pm}\tilde{\sigma}^2_{p_\pm} - (\tilde{\sigma}_{x_\pm p_\pm})^2}\right) \geq \ln(2\pi e) \tag{12.4}$$

where $(\tilde{\cdot})$ indicates that entropies and variances are computed after the partial transposition and we use Eq. (12.2) to evaluate the commutator $[x_\pm, p_\pm]$.

Under the partial transposition, the quadratures transform as

$$(x_1, p_1, x_2, p_2) \to (x_1, p_1, x_2, -p_2) \tag{12.5}$$

so that the probability distributions of $\tilde{X}_\pm$ and $\tilde{P}_\pm$ are given by

$$\tilde{X}_\pm(x) = X_\pm(x) \qquad \text{and} \qquad \tilde{P}_\pm(p) = P_\mp(p) \tag{12.6}$$

which means that

$$\tilde{h}(x_\pm) = h(x_\pm) \qquad \text{and} \qquad \tilde{h}(p_\pm) = h(p_\mp). \tag{12.7}$$

In the same manner, the variances are now given by

$$\tilde{\sigma}^2_{x_\pm} = \sigma^2_{x_\pm}, \qquad \tilde{\sigma}^2_{p_\pm} = \sigma^2_{p_\mp} \qquad \text{and} \qquad \tilde{\sigma}_{x_\pm p_\pm} = \sigma_{x_\pm p_\mp}, \tag{12.8}$$

so that any separable state must verify

$$h(x_\pm) + h(p_\mp) - \frac{1}{2}\ln\left(\frac{\sigma^2_{x_\pm}\sigma^2_{p_\mp}}{\sigma^2_{x_\pm}\sigma^2_{p_\mp} - (\sigma_{x_\pm p_\mp})^2}\right) \geq \ln(2\pi e). \tag{12.9}$$

In other words, any bipartite state is entangled if Eq. (12.9) is violated. This is our entropic separability criterion[1].

[1] Assuming that Eq. (8.11) is fully proven (see Chapter 8).

12.2 Example of an entangled mixed state

We present here an example that illustrates the utility of our separability criterion Eq. (12.9). Let us consider the dephased cat state [84]

$$\rho = \mathcal{N}\Big(|\alpha,\alpha\rangle\langle\alpha,\alpha| + |-\alpha,-\alpha\rangle\langle-\alpha,-\alpha| - \frac{1}{4}|\alpha,\alpha\rangle\langle-\alpha,-\alpha| - \frac{1}{4}|-\alpha,-\alpha\rangle\langle\alpha,\alpha|\Big) \tag{12.10}$$

where $|\alpha\rangle$ represents a coherent state and $\mathcal{N}$ is a normalization constant. This state is entangled for all values of $\alpha \neq 0$, however, it is never detected by the variance-based separability criterion. Indeed, after a partial transposition, the covariance matrix of the state will be

$$\tilde{\gamma} = \begin{pmatrix} a & 2\Im(\alpha)\Re(\alpha) & \frac{2\Im(\alpha)^2+8e^{4|\alpha|^2}\Re(\alpha)^2}{4e^{4|\alpha|^2}-1} & -2\Im(\alpha)\Re(\alpha) \\ 2\Im(\alpha)\Re(\alpha) & b & 2\Im(\alpha)\Re(\alpha) & \frac{8e^{4|\alpha|^2}\Im(\alpha)^2+2\Re(\alpha)^2}{1-4e^{4|\alpha|^2}} \\ \frac{2\Im(\alpha)^2+8e^{4|\alpha|^2}\Re(\alpha)^2}{4e^{4|\alpha|^2}-1} & 2\Im(\alpha)\Re(\alpha) & a & -2\Im(\alpha)\Re(\alpha) \\ -2\Im(\alpha)\Re(\alpha) & \frac{8e^{4|\alpha|^2}\Im(\alpha)^2+2\Re(\alpha)^2}{41-e^{4|\alpha|^2}} & -2\Im(\alpha)\Re(\alpha) & b \end{pmatrix} \tag{12.11}$$

with

$$\begin{aligned} a &= \frac{4\Im(\alpha)^2 + 4e^{4|\alpha|^2}(4\Re(\alpha)^2+1) - 1}{8e^{4|\alpha|^2}-2} \\ b &= \frac{16\Im(\alpha)^2e^{4|\alpha|^2} + 4e^{4|\alpha|^2} + 4\Re(\alpha)^2 - 1}{8e^{4|\alpha|^2}-2}. \end{aligned} \tag{12.12}$$

We can thus evaluate the symplectic values with the help of Eq. (2.45). The smallest symplectic values of the partial transposed matrix is given by

$$\tilde{\nu}_- = \frac{1}{4}\sqrt{e^{-4|\alpha|^2}\,(8|\alpha|^2-1)+4} \tag{12.13}$$

and this is always greater than $1/2$, for any value of α. This fact is actually obvious when we realize that the symplectic values of the partial transposed state are actually the same as the ones of the initial state. Since we start from a physical state, we will always have $\tilde{\nu}_- = \nu_- \geq 1/2$. It means that the entanglement of the state is never detected by a variance-based separability criterion.

Let us now apply the entropic separability criterion of Walborn *et al.*, Eq. (5.37). We do not give here the explicit values of the Shannon entropies because they are complicated, but for a large range of α, we have $h(x_\pm) + h(p_\mp) < \ln(2\pi e)$[2], when we choose $\theta_1 = \theta_2 = 0$ in Eq. (5.32), which means that the entanglement of this state is detected.

[2]For this state, $h(x_+) + h(p_-) = h(x_-) + h(p_+)$.

Nevertheless, let us now suppose that we have the following cat state

$$\rho' = \mathcal{N}\Big(|\alpha, i\alpha\rangle\langle\alpha, i\alpha| + |-\alpha, -i\alpha\rangle\langle-\alpha, -i\alpha| - \frac{1}{4}|\alpha, i\alpha\rangle\langle-\alpha, -i\alpha| - \frac{1}{4}|-\alpha, -i\alpha\rangle\langle\alpha, i\alpha|\Big). \tag{12.14}$$

It is the same cat state as before, but we applied a rotation[3]

$$\rho' = e^{-i\phi_1\hat{N}_1}e^{-i\phi_2\hat{N}_2}\,\rho\; e^{i\phi_1\hat{N}_1}e^{i\phi_2\hat{N}_2} \tag{12.15}$$

such that the new quadratures can be expressed as

$$x'_j = \cos\phi_j x_j + \sin\phi_j p_j \qquad \text{and} \qquad p'_j = \cos\phi_j p_j - \sin\phi_j x_j. \tag{12.16}$$

In Eq. (12.14) we chose the specific case of $\phi_1 = 0$ and $\phi_2 = -\pi/2$. The rotation does not change the entanglement of the state, so we know that this state is entangled. Let us see if it is detected by the different criteria.

The covariance matrix of the partial transposed state, $\tilde{\gamma}'$ will be different, but the symplectic values will be the same. Indeed, symplectic values do not change under a symplectic transformation like the rotation. It means that entanglement in this state is still not detected by the variance-based separability criteria.

If we now turn to Walborn *et al.* criterion, and we fix $\theta_1 = \theta_2 = 0$, there will now be some cases were the entanglement will not be detected anymore. Indeed, if we fix, for example $\alpha = 4$, with the initial quadratures, we had

$$h(x_\pm) + h(p_\mp) = 2.822 < \ln(2\pi e) \tag{12.17}$$

while with the prime quadratures (12.16) we obtain

$$h(x'_\pm) + h(p'_\mp) = 4.224 > \ln(2\pi e). \tag{12.18}$$

Entanglement is thus not detected anymore.

If we now use our separability criterion Eq. (12.9), we find

$$h(x'_\pm) + h(p'_\mp) - \frac{1}{2}\ln\left(\frac{\sigma^2_{x'_\pm}\sigma^2_{p'_\mp}}{\sigma^2_{x'_\pm}\sigma^2_{p'_\mp} - (\sigma_{x'_\pm p'_\mp})^2}\right) = 2.815 < \ln(2\pi e) \tag{12.19}$$

so that entanglement is now detected. Of course, if we would have used our criteria with the initial state (before the rotation) we would have found

$$h(x_\pm) + h(p_\mp) - \frac{1}{2}\ln\left(\frac{\sigma^2_{x_\pm}\sigma^2_{p_\mp}}{\sigma^2_{x_\pm}\sigma^2_{p_\mp} - (\sigma_{x_\pm p_\mp})^2}\right) = 2.822 < \ln(2\pi e) \tag{12.20}$$

which shows that, here too, we are able to detect entanglement. Remark that the

[3] See Section 2.5.3.

value is the same as in Eq. (12.17). This is because ρ does not exhibit any correlation $\sigma_{x_\pm p_\mp}$ so that both, our criterion and the one of Walborn *et al.* (with $\theta_1 = \theta_2 = 0$) give always the same value for this specific state. In Table 12.1, we give a summary of the entanglement detection according to the different separability criteria.

	Variance-based $\tilde{\nu}_- < 1/2$	Walborn *et al.* Eq. (5.37)	New criterion Eq. (12.9)
ρ	$\times$	✓	✓
ρ'	$\times$	$\times$	✓

Table 12.1: Entanglement detection of the cat states ρ and ρ' by the different separability criteria. The symbol ✓ means that entanglement is detected and $\times$ that it is not.

12.3 Discussion

Of course, if one makes an optimization over all the possible angles, the state of the previous example will be detected by Walborn *et al.* criterion when choosing $\theta_1 = -\phi_1 = 0$ and $\theta_2 = -\phi_2 = \pi/2$. Yet, in general it is not easy to find the optimal angles and the optimization might request some heavy calculations. Also, remark that in our example, we detect entanglement but are very close to the bound (see Eq. (12.19)). Since it is not an obvious violation of the uncertainty relation, Walborn *et al.* might have very easily missed it and never detect entanglement. The power of our entropic separability criterion over the one of Walborn *et al.* is that we never need to make an optimization. We will detect all the entangled states detected by the latter criterion, without the need of looking for the optimal angles showing the maximum of correlations. This optimization is indeed somehow included in the term $\frac{1}{2}\ln\left(\frac{\sigma^2_{x_\pm}\sigma^2_{p_\mp}}{\sigma^2_{x_\pm}\sigma^2_{p_\mp}-(\sigma_{x_\pm p_\mp})^2}\right)$ which takes correlations into account.

We also believe that this separability criterion brings another validation to the new entropic uncertainty relations introduced in Chapters 6 and 8. Indeed, we were not able to prove the concavity of those relations, so that we do not have a formal proof of those entropic uncertainty relations for mixed state (but numerical evidence). Nevertheless, we showed in this chapter that we were able to reproduce results obtained in [84] for mixed states. This gives an additional confirmation that Eqs. (6.14) and (8.11) should be true.

13 | Conclusion

The focus of this thesis was to explore entropic uncertainty relations. The original formulation of Białynicki-Birula and Mycielski is not invariant under Gaussian transformations, which implies that it is saturated only by certain pure Gaussian states. The main motivation was thus to improve this entropic uncertainty relation in order to find a symplectic invariant formulation, that would thus be saturated by all pure Gaussian states. Indeed, the variance-based counterpart, that is the Robertson-Schrödinger uncertainty relation, verifies both conditions. Since we believe that entropic uncertainty relations are more robust, it should be possible to answer those questions in the entropic framework and develop the equivalent entropic uncertainty relations.

In Chapter 6 we suggested a first improvement of the entropic uncertainty relation which solves the saturation problem: the new entropic uncertainty relation, which takes x-p correlations into account, is now saturated by all pure Gaussian states. However, it is not invariant under symplectic transformations. If we consider the joint entropy, like we did in Chapter 9, the new entropic uncertainty relation is now invariant under symplectic transformations and thus, obviously, still saturated by all pure Gaussian states. Nevertheless, we did not completely solve our problem since the uncertainty relation proposed in this chapter, which is only a conjecture, is valid only for states with positive Wigner function. For Wigner functions with negative parts, we showed that it is possible to extend the definition of our so called Wigner entropy, to a complex-valued function. Unfortunately, we were not able to derive any entropic uncertainty relation from the Wigner entropy.

We also extended the entropic uncertainty relation to any arbitrary quadratures, so we are not restricted to canonically-conjugate variables only. This is the subject of Chapter 7. The relation developed here can also be understood as an entropic uncertainty relation expressing the balance between any two n-variable Gaussian projective measurements. Interestingly, the bound of this entropic uncertainty relation depends on a commutator matrix, exactly like the variance-based uncertainty relation does. Nevertheless, similarly to the original entropic uncertainty relation, it is not saturated by all pure Gaussian states. Therefore, in Chapter 8, we proposed a tight formulation of this entropic uncertainty relation. By taking into account the

correlations between all the quadratures — through the covariance matrix — the new entropic uncertainty relation is now saturated by all pure Gaussian states. A small summary of all the entropic uncertainty relations encountered throughout the thesis can be found in the first column of Table 13.1. The symbol ✓ means that the entropic uncertainty relation is proven, † that it is proven conditionally on reasonable assumptions and * that it is still just a conjecture.

Remark that in all those chapters, we always insisted on one point: entropic uncertainty relations are actually better expressed in an entropy-power formulation. Indeed, it resembles more the variance-based uncertainty relations. Moreover, using the fact that for fixed variance, the maximum entropy is given by a Gaussian, we can easily deduce a variance-based uncertainty relation from an entropy-power one. The second column of Table 13.1 shows the equivalent entropy-power formulation of the different entropic uncertainty relations, while we present in the third column the variance-based uncertainty relations implied by the entropy-power ones.

We made significant improvement in entropic uncertainty relations, however, we were still not able to answer the main question: what is the symplectic invariant entropic uncertainty relation? We believe that we should keep looking for this more general uncertainty relation. One possible line of approach is to develop further this concept of Wigner entropy presented in Chapter 9. We need to better understand what a complex-valued joint entropy means and then suggest its entropic uncertainty relation. We believe that this entropic uncertainty relation should coincide with conjecture (9.10) for positive Wigner functions. Indeed, even if it is not proven yet (except for Gaussian states) we have numerical evidences that it is true.

Moreover, if we do not know yet how to access the correlations between the quadratures through the joint entropy, we showed that taking correlations in the covariance matrix into account help to find a tighter entropic uncertainty relation. We thus believe that research should also still be oriented in this direction and in particular, the next step would be to prove Eq. (8.64).

In the idea of finding an entropic uncertainty relation invariant under any Gaussian transformation, we proposed, in Chapter 10, a different framework. We defined two multi-copy observables acting on two and three copies of a state and computed the Shannon entropy of those observables. Since, for both of them, the positivity of the variance of the multi-copy observable is equivalent to the variance-based uncertainty relation of Robertson-Schrödinger, and they are saturated by the same states (the pure Gaussian states) we proposed two new entropic uncertainty relations based on the simple condition that the Shannon entropy is always positive. In particular, we showed that those entropic uncertainty relations are invariant under Gaussian transformations and are saturated by all pure Gaussian states. The 3-copy observable has the advantage to not restrict to states centered on the origin. Nevertheless, we do not know yet the eigenvectors of this observable and thus do not know how to

compute its Shannon entropy in general. For Gaussian states however, both Shannon entropies coincide.

Chapter 11 and 12 highlight one of the main applications of uncertainty relations: separability criteria. In both chapters, we showed that using a tighter uncertainty relation allows us to detect more entangled states. In Chapter 11, the separability condition is based on the Gaussianity-bounded uncertainty relation. By taking into account the degree of Gaussianity of the state, we gave some explicit examples of non-Gaussian entangled states detected by our criterion, but not by the usual one of Duan and Simon *et al.*. In Chapter 12, the detection is improved by the use of the entropic uncertainty relation of Chapter 8. Note that to be more exact, the entanglement detection reaches the same level as with the criterion of Walborn *et al.*, however, the calculations are easier since we showed that our separability criterion does not require any optimization.

In Table 13.2, we present a summary of the separability criteria and their associated uncertainty relations. Each inequality of the second column is a condition that is respected by any separable states. Therefore, an entangled state is detected whenever those inequalities are violated. The tighter the separability condition is, the more entangled states we can detect hence the interest of developing tighter uncertainty relations.

Let us conclude by mentioning that this work is more general than it first appears to be. Indeed, even if everything was presented in the quantum optics framework, the uncertainty relations presented here can actually be applied to any problem that deals with bosonic modes.

Entropic uncertainty relations		Entropy-power uncertainty relations	Variance-based uncertainty relations
$h(x)+h(p)\geq \ln(\pi e)$		$N_x\,N_p\geq 1/4$	$\sigma_x^2\sigma_p^2\geq\frac{1}{4}$
Białynicki-Birula and Mycielski Eq. (4.39)	✓	Eq. (6.4)	Heisenberg Eq. (4.2)
$h(\mathbf{x})+h(\mathbf{p})-\frac{1}{2}\left(\frac{\det\gamma_x\det\gamma_p}{\det\gamma}\right)\geq n\ln(\pi e)$		$\left(N_x^{(n)}N_p^{(n)}\right)^n\frac{\det\gamma}{\det\gamma_x\,\det\gamma_p}\geq\frac{1}{4^n}$	$\det\gamma\geq\frac{1}{4^n}$
Eq. (6.14)	†	Eq. (6.61)	Schrödinger-Robertson-Simon Eq. (4.15)
$h(\mathbf{y})+h(\mathbf{z})\geq\ln\left((\pi e)^n\lvert\det K\rvert\right)$		$(N_{\mathcal{A}}N_{\mathcal{B}})^n\geq\frac{\lvert\det K\rvert^2}{4^n}$	$\det\gamma^{\mathcal{A}}\,\det\gamma^{\mathcal{B}}\geq\frac{\lvert\det K\rvert^2}{4^n}$
Eq. (7.31)	✓	Eq. (7.54)	Eq. (7.52)
$h(\boldsymbol{r})\geq n\,\ln(\pi e)\qquad\forall$ states s.t. $W(\boldsymbol{r})\geq 0$		$\left(N_{xp}^{(n)}\right)^n\geq\frac{1}{4^n}$	$\det\gamma\geq\frac{1}{4^n}$
Eq. (9.10)	*	Eq. (9.13)	Schrödinger-Robertson-Simon Eq. (4.15)
$h(\mathbf{y})+h(\mathbf{z})-\frac{1}{2}\ln\left(\frac{\det\Gamma_y\det\Gamma_z}{\det\Gamma}\right)\geq\ln\left((\pi e)^n\lvert\det K\rvert\right)$		$(N_yN_z)^n\frac{\det\Gamma}{\det\Gamma_y\det\Gamma_z}\geq\frac{\lvert\det K\rvert^2}{4^n}$	$\det\Gamma\geq\frac{\lvert\det K\rvert^2}{4^n}$
Eq. (8.11)	†	Eq. (8.59)	Eq. (8.60)
$h(\xi_1)+h(\xi_2)+\cdots+h(\xi_n)-\frac{1}{2}\ln\left(\frac{\sigma_1^2\sigma_2^2\cdots\sigma_n^2}{\det\Gamma_\xi}\right)\geq\frac{1}{2}\ln((2\pi e)^n\det C)$		$N_1N_2\cdots N_n\frac{\det\Gamma_\xi}{\sigma_1^2\sigma_2^2\cdots\sigma_n^2}\geq\det C$	$\det\Gamma_\xi\geq\det C$
Eq. (8.64)	*	Eq. (8.67)	Robertson Eq. (8.56)

Table 13.1: Summary of all the the entropic uncertainty relations and their entropy-power formulations as well as the variance-based uncertainty relations implied by them. The symbol ✓ means that the entropic uncertainty relation is proven, † that it is proven conditionally on reasonable assumptions and * that it is a conjecture.

Uncertainty relations	Separability criteria
$\nu_i \geq \frac{1}{2}$	$\langle(\Delta\hat{u})^2\rangle + \langle(\Delta\hat{v})^2\rangle \geq \alpha^2 + \frac{1}{\alpha^2}$
Variance-based UR Eq. (4.13)	Duan-Simon *et al.* Eq. (5.13)
$\nu_i \geq f(g)$	$\nu_- \geq \nu_{\text{th}}(g_2)$
Gaussianity-bounded UR Sec. 4.1.4	Eq. (11.5)
$h(\mathbf{x}) + h(\mathbf{p}) \geq n\ln(\pi e)$	$h(r_\pm) + h(s_\mp) \geq \ln(2\pi e)$
Białynicki-Birula and Mycielski EUR Eq. (4.39)	Walborn *et al.* Eq. (5.37)
$h(\mathbf{y}) + h(\mathbf{z}) - \frac{1}{2}\ln\left(\frac{\det(\Gamma_y)\det(\Gamma_z)}{\det(\Gamma)}\right) \geq \ln\left((\pi e)^n \lvert\det K\rvert\right)$	$h(x_\pm) + h(p_\mp) - \frac{1}{2}\ln\left(\frac{\sigma^2_{x_\pm}\sigma^2_{p_\mp}}{\sigma^2_{x_\pm}\sigma^2_{p_\mp} - (\sigma_{x_\pm p_\mp})^2}\right) \geq \ln(2\pi e)$
General EUR Eq. (8.11)	Eq. (12.9)

Table 13.2: Summary of the separability criteria and their associated uncertainty relations. Each inequality of the second column is a condition that is respected by any separable state. In the first column, UR stands for *Uncertainty Relation* and EUR for *Entropic Uncertainty Relation*.

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