Waveguide solutions for a nonlinear Schrödinger equation with mixed dispersion

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Ao nosso amigo Djairo com admiração

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1 Introduction

The standard model for propagation of laser beams is the 2*D* Schrödinger equation with Kerr nonlinearity

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi = 0, \quad \psi(x, y, 0) = \psi_0(x, y).$$

It is well known that this equation can become singular at finite time, see, for instance, [13] and the classical references therein. Karpman and Shagalov [16] studied the regularization and stabilization effect of a small fourth-order dispersion, namely they considered the equation

$$i\partial_t\psi + \Delta\psi + |\psi|^{2\sigma}\psi - \gamma\Delta^2\psi = 0, \tag{1}$$

for some $\gamma > 0$, the equation being now considered in $[0, \infty[\times\mathbb{R}^N$, $N \geq 1$. One of their results shows, by help of some stability analysis and numerical computations, that when $N\sigma \leq 2$, the waveguide solutions are stable for all γ and when $2 < N\sigma < 4$, they are stable for small values of γ. This result shows that when adding a small fourth-order dispersion term, a new critical exponent/dimension appears. In particular, the Kerr nonlinearity becomes subcritical in dimension 2 and 3 which is obviously an important feature of this extended model.

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In [13], Fibich et al. have motivated the study of (1) by recalling that NLS (the nonlinear Schrödinger equation) arises from NLH (the nonlinear Helmholtz equation) as a paraxial approximation. But since NLS can become singular at a finite time, this suggests that some of the small terms, neglected in the paraxial approximation, plays in fact an important role to prevent the blow up. The natural question addressed by Fibich et al. is therefore whether nonparaxiality prevents the collapse. The small fourth-order dispersion coefficient γ is then shown to be part of the nonparaxial correction to NLS.

In [13], Fibich et al. showed the role of the new critical exponent $\sigma = 4/N$ in the global existence in time when applying the arguments of Weinstein [25]. The necessary Strichartz estimates follow from Ben-Artzi et al. [1]. A necessary condition for existence of waveguide solutions is given in [13, Lemma 4.1], see also the Derrick-Pohozahev identity in Section 6.

The purpose of this short note is to show that classical tools, available in the literature, allow to state the existence and some qualitative properties of least energy waveguide solutions. In particular, a small fourth-order dispersion coefficient does not affect the symmetry, uniqueness and nondegeneracy of the least energy waveguide solution at least for a Kerr nonlinearity in dimension $N \le 3$.

From now on, we focus on standing wave solutions of (1), referred to as waveguide solutions in nonlinear optics, namely on solutions of (1) of the form

$$\psi(t,x) = \exp(i\alpha t)u(x).$$

This ansatz yields the semilinear elliptic equation

$$\gamma\Delta^2 u(x) - \Delta u(x) + \alpha u(x) = |u|^{2\sigma}u(x), \quad x \in \mathbb{R}^N. \tag{2}$$

By scaling the solutions as $v(x) = u(\frac{x}{\sqrt[4]{\gamma}})$, it is equivalent to consider the equation

$$\Delta^2 v(x) - \beta\Delta v(x) + \alpha v(x) = |v|^{2\sigma}v(x), \quad x \in \mathbb{R}^N. \tag{3}$$

where $\beta = \frac{1}{\sqrt{\gamma}}$.

It is standard that least energy solutions can be obtained by considering the minimization problem

$$m_{\mathbb{R}^N} := \inf_{u \in M_{\mathbb{R}^N}} J_{\mathbb{R}^N}(u) \tag{4}$$

where

$$J_{\mathbb{R}^N}(u) = \int_{\mathbb{R}^N} (|\Delta u|^2 + \beta|\nabla u|^2 + \alpha|u|^2)\, dx \tag{5}$$

and

$$M_{\mathbb{R}^N} := \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx = 1\}.$$

Indeed, if $u \in M_{\mathbb{R}^N}$ achieves the infimum $m = m_{\mathbb{R}^N}$, then u weakly solves

$$\Delta^2 u - \beta \Delta u + \alpha u = m|u|^{2\sigma} u. \tag{6}$$

Henceforth, if $m > 0$, then $v = (m)^{\frac{1}{2\sigma}} u$ solves (3). Moreover v is a least energy solution in the sense that it minimizes the action functional $E : H^2(\mathbb{R}^N) \to \mathbb{R}$ defined by setting

$$E(u) := \frac{1}{2} J_{\mathbb{R}^N}(u) - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx$$

among the set of (H^2 or smoother) solutions or equivalently within the Nehari manifold

$$\{u \in H^2(\mathbb{R}^N) : E'(u)(u) = 0\}.$$

We then prove the following results.

Theorem 1.1. *Assume $\alpha > 0$, $\beta > -2\sqrt{\alpha}$ and $2 < 2\sigma + 2 < \frac{2N}{N-4}$ if $N \geq 5$. Then problem (3) has a nontrivial least energy solution. If $\beta \geq 2\sqrt{\alpha}$, then any least energy solution does not change sign, is radially symmetric around some point and strictly radially decreasing.*

An existence statement (as well as the information on the sign of the minimizer) is also given in Section 3 and Section 4 when the equation is considered in a bounded domain with Navier boundary conditions. The symmetry properties of the solutions that match the symmetries of the domain are discussed in Section 4.

When β is large, the Laplacian is the driven term in the differential operator in (3) and we therefore expect to recover the uniqueness (up to translations) of the least energy solution. By scaling, we can discuss this issue by looking at least energy solutions of (2) for small γ. As a preliminary observation, we prove the strong convergence in H^1 to the unique least energy solution of NLS.

Theorem 1.2. *Assume $2 < 2\sigma + 2 < \frac{2N}{N-2}$ if $N \geq 3$. If $\gamma_k \to 0$ and u_k is a sequence of least energy solutions of (2), then $(u_k)_k$ converges (after possible translations) in H^1 to u_0, where u_0 is the unique positive (radially symmetric) solution of the limit problem (2) with $\gamma = 0$.*

The positive solution of (2) with $\gamma = 0$ is unique up to translations. To ensure uniqueness, we have assumed that u_0 is the positive solution radially decreasing around 0. For the physical model (2) with $\sigma = 1$ in dimension $N \leq 3$, we can improve this convergence to strong convergence in H^2. The nondegeneracy of the least energy waveguide of NLS allows then to use the Implicit Function Theorem to prove uniqueness for small γ.

Theorem 1.3. *Assume $N \leq 3$ and $\sigma = 1$. Then there exists $\gamma_0 > 0$ such that if $0 < \gamma < \gamma_0$,* (2) *has a unique least energy solution (up to translations). Fixing its maximum at the origin, this solution is radially symmetric and strictly radially decreasing.*

An equivalent statement can be proved for the Navier boundary value problem in a ball (and a weaker statement holds for other bounded domains), see Section 6.

In the H^1 critical or supercritical regime, the least energy solution should disappear at the limit $\gamma \to 0$. In fact, if $\frac{2N}{N-2} \leq 2\sigma + 2 < \frac{2N}{N-4}$, $N \geq 5$, the least energy solutions are unbounded in H^2 when $\gamma \to 0$, see Section 6.

In contrast with Theorem 1.1, when β is small in (3), some of the usual properties of the least energy solution of NLS cannot hold. Namely, if one can prove that any least energy solution is radial in that case, then oscillations arise at infinity. These oscillations were suggested in [13]. We focus again on the model equation (2) with $\sigma = 1$ in dimension $N \leq 3$. We prove that least energy solutions among radial solutions do oscillate at infinity.

Theorem 1.4. *Suppose that $-2\sqrt{\alpha} < \beta < 2\sqrt{\alpha}$ and $N \leq 3$. Then every radial least energy solution of* (3) *with $\sigma = 1$ is sign-changing.*

This statement shows that when $\beta < 2\sqrt{\alpha}$, least energy solutions cannot be radial and monotone in contrast with the case $\beta \geq 2\sqrt{\alpha}$. We point out that on a bounded domain, we are not aware of an equivalent statement.

The paper is organized as follows. Section 2 deals with the functional framework and the formulation of the problem on a bounded domain. In Section 3, we prove the existence of a least energy solution in the whole space as well as in bounded domains. In Section 4, we consider the qualitative properties for large β. Section 5 is dedicated to the proof of Theorem 1.2 and Theorem 1.3 while Section 6 contains the proof of Theorem 1.4. In the last section, we give some concluding remarks.

Notes added in proofs: We thank Jean-Claude Saut for bringing to our attention the reference [5] which deals with an anisotropic mixed dispersion NLS also proposed in [13]. We believe that some arguments from [5] can be used to obtain the exponential decay of the ground state at least in some particular cases.

We also mention the very recent preprint [6] where the first theoretical proof of blow-up is obtained for the biharmonic NLS as well as a new *Fourrier rearrangement* is proposed in the Appendix. This rearrangement decreases the L^2-norm of $(-\Delta u)^s$ for every $s \geq 0$ and is therefore adequate to deal with polyharmonic as well as fractional equations. Applied to our problem, it completes Theorems 1.1 and 1.4 in the following way. Assuming $\beta \geq 0$ and $\sigma \in \mathbb{N}_0$ (including therefore the physical case $\sigma = 1$), there is a ground state solution of (C) which is radially symmetric. As a consequence of Theorem 1.4, assuming $\sigma \in \mathbb{N}_0$ and $0 \leq \beta < 2\sqrt{\alpha}$, this ground state is radially oscillatory at infinity. When σ is not an integer, the radial symmetry remains an open question in the range $\beta < 2\sqrt{\alpha}$ though the natural conjecture is that radial symmetry holds for every σ and every β in the range covered by Theorem 1.1.

2 Functional framework

In this section, we settle the functional setting. The natural space for (2) and (3) is $H^2(\mathbb{R}^N)$ or $H^2(\Omega) \cap H_0^1(\Omega)$ when we consider the boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^N$ with Navier boundary conditions, namely

$$(P_\beta) \qquad \begin{cases} \Delta^2 u - \beta \Delta u + \alpha u = |u|^{2\sigma} u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

We therefore set $H_\Omega := H^2(\Omega) \cap H_0^1(\Omega)$ and $H_{\mathbb{R}^N} := H^2(\mathbb{R}^N)$. We introduce the following conditions on α and β:

$(A1)$ $\alpha > 0$ and $\beta > -2\sqrt{\alpha}$;
$(A1')$ $\alpha > -\beta\lambda_1(\Omega) - \lambda_1^2(\Omega)$ and $-2\lambda_1(\Omega) < \beta$;

where $\lambda_1(\Omega)$ stands for the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ when Ω is a bounded domain. Observe that when $|\Omega|$ is large, $\lambda_1(\Omega)$ is small. If β is negative, $(A1')$ is then more restrictive than $(A1)$. The following lemma follows from standard computations.

Lemma 2.1. *Assume Ω is a bounded smooth domain and* $(A1)$ *or* $(A1')$ *holds. Then H_Ω is a Hilbert space endowed with the inner product defined through*

$$\langle u, v \rangle = \int_\Omega (\Delta u \Delta v + \beta \nabla u \nabla v + \alpha u v)\, dx \quad \forall\, u, v \in H_\Omega.$$

Proof. From H^2 elliptic regularity [14, 18], we know that if $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\|u\|_{H^2} \le C \|\Delta u\|_{L^2}$$

for some $C > 0$ depending on Ω, so that H_Ω is a Hilbert space endowed with the inner product

$$\langle u, v \rangle_{H_\Omega} = \int_\Omega \Delta u \Delta v \, dx \quad \forall\, u, v \in H_\Omega.$$

It will be enough to show that there exists a constant $C > 0$ such that

$$\int_\Omega (|\Delta u|^2 + \beta |\nabla u|^2 + \alpha |u|^2)\, dx \ge C \|u\|_{H_\Omega} \qquad \forall\, u \in H_\Omega. \tag{7}$$

Obviously the inequality (7) holds true if we have $\alpha \ge 0$ and $\beta \ge 0$. For $u \in H_\Omega$, we can apply Young's inequality to obtain

$$\begin{aligned}
\|u\|^2 &= \int_\Omega (|\Delta u|^2 + \beta|\nabla u|^2 + \alpha|u|^2)\,dx \\
&= \int_\Omega (|\Delta u|^2 - \beta u \Delta u + \alpha|u|^2)\,dx \\
&\geq \left(1 + \frac{\beta}{2\epsilon}\right) \int_\Omega |\Delta u|^2\,dx + \left(\alpha + \frac{\beta\epsilon}{2}\right) \int_\Omega |u|^2\,dx \qquad (8)
\end{aligned}$$

for every $\epsilon > 0$. We have to distinguish two cases. If we can choose $\epsilon > 0$ such that both terms in the right-hand side of (8) are positive, then we are done. This ends the proof if $\beta > -2\sqrt{\alpha}$, namely if $(A1)$ holds. If

$$1 + \frac{\beta}{2\epsilon} > 0 \quad \text{and} \quad \alpha + \frac{\beta\epsilon}{2} < 0,$$

we write

$$\|u\|^2 \geq \left(1 + \frac{\beta}{2\epsilon}\right) \left[\int_\Omega |\Delta u|^2\,dx + g(\epsilon) \int_\Omega |u|^2\,dx\right],$$

where

$$g(\epsilon) = \frac{\alpha + \beta\epsilon/2}{1 + \beta/2\epsilon}.$$

Recalling Poincaré inequality

$$\int_\Omega |\Delta u|^2\,dx \geq \lambda_1^2(\Omega) \int_\Omega u^2\,dx \qquad \forall\, u \in H_\Omega,$$

we can complete the proof if

$$g(\epsilon) > -\lambda_1^2(\Omega)$$

for some $\epsilon > 0$. When $\beta > -2\lambda_1(\Omega)$, this condition can be fulfilled if

$$\alpha > -\beta\lambda_1(\Omega) - \lambda_1^2(\Omega)$$

while if $\beta \leq -2\lambda_1(\Omega)$, we recover the condition

$$-2\sqrt{\alpha} < \beta.$$

□

In the case $\Omega = \mathbb{R}^N$, the same arguments show that $(A1)$ implies

$$\langle u, v\rangle = \int_{\Omega} (\Delta u \Delta v + \beta \nabla u \nabla v + \alpha u v)\, dx$$

is a scalar product on $H_{\mathbb{R}^N}$. Elliptic regularity can be used here to ensure that

$$\left(1 + \frac{\beta}{2\epsilon}\right) \int_{\mathbb{R}^N} |\Delta u|^2\, dx + \left(\alpha + \frac{\beta\epsilon}{2}\right) \int_{\mathbb{R}^N} |u|^2\, dx$$

is a norm on $H^2(\mathbb{R}^N)$ as soon as $1 + \frac{\beta}{2\epsilon} > 0$ and $\alpha + \frac{\beta\epsilon}{2} > 0$. This yields the following lemma.

Lemma 2.2. *Assume that* $(A1)$ *holds. Then the bilinear form*

$$\langle u, v\rangle = \int_{\Omega} (\Delta u \Delta v + \beta \nabla u \nabla v + \alpha u v)\, dx \quad \forall\, u, v \in H_{\mathbb{R}^N},$$

is an inner product on $H_{\mathbb{R}^N}$.

3 Existence of minimizers

In this section, we handle the minimization problem (4). We start with the simpler case of a bounded domain. In this case, the minimization problem writes

$$m_{\Omega} := \inf_{u \in M_{\Omega}} J_{\Omega}(u)$$

where

$$J_{\Omega}(u) = \int_{\Omega} (|\Delta u|^2 + \beta |\nabla u|^2 + \alpha |u|^2)\, dx$$

and

$$M_{\Omega} := \{u \in H_{\Omega} : \int_{\Omega} |u|^{2\sigma+2}\, dx = 1\}.$$

In the case of a bounded domain, it is standard to prove that m_{Ω} is achieved when $2\sigma + 2$ is a subcritical exponent because J_{Ω} is the square of a norm on H_{Ω} and we can rely on the compactness of the embedding of H_{Ω} into $L^{2\sigma+2}(\Omega)$. Moreover, since m_{Ω} is clearly positive, we deduce that $v = (m_{\Omega})^{\frac{1}{2\sigma}} u$ solves (P_{β}). Moreover v is a least energy solution in the sense that it minimizes the action functional $E_{\Omega} : H_{\Omega} \to \mathbb{R}$ defined by

$$E_\Omega(u) := \frac{1}{2}J_\Omega(u) - \frac{1}{2\sigma+2}\int_\Omega |u|^{2\sigma+2}\,dx$$

among the set of (H^2 or smoother) solutions or equivalently within the Nehari manifold

$$\{u \in H_\Omega : E'_\Omega(u)(u) = 0\}.$$

Theorem 3.1. *Assume Ω is a bounded smooth domain and $(A1)$ or $(A1')$ holds. Suppose moreover that $2 < 2\sigma + 2 < \frac{2N}{N-4}$ if $N \geq 5$. Then problem (P_β) has a nontrivial least energy solution.*

To handle the case of $\Omega = \mathbb{R}^N$, since we cannot use sign information, nor symmetry, we follow the celebrated method of concentration-compactness of P.L. Lions. We give a sketchy proof since classical arguments apply. All the details can easily be reconstructed from Kavian [17, Chapitre 8 - Exemple 8.5] with minor and obvious modifications with respect to the case treated therein.

Proof (Proof of the existence part in Theorem 1.1.). We introduce

$$M_\lambda = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx = \lambda\}$$

where $\lambda > 0$ is fixed and we consider the minimization problem

$$m_\lambda := \inf_{u\in M_\lambda} J_{\mathbb{R}^N}(u)$$

where $J_{\mathbb{R}^N}(u)$ is defined as in (5).

Let $(u_k)_k \subset M_\lambda$ be such that $J_{\mathbb{R}^N}(u_k) \to m_\lambda$. Then, $(u_k)_k$ is bounded in $H^2(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |u_k|^{2\sigma+2} = \lambda$. Thus, we can apply P.L. Lions' concentration-compactness lemma to the sequence $(\rho_k)_k = (\int_{\mathbb{R}^N} |u_k|^{2\sigma+2})_k$, see [21, Lemma I. 1]. Since $m_\lambda = \lambda^{\frac{1}{\sigma+1}} m_1$, we have $m_\lambda > 0$ for all $\lambda > 0$ and therefore, for all $R > 0$, the sequence

$$Q_k(R) := \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_k(x)|^{2\sigma+2}\,dx$$

does not converge to zero. Namely, vanishing is ruled out.

Since $2\sigma + 2 > 2$, we have, for $0 < \theta < \lambda$,

$$\lambda^{\frac{1}{\sigma+1}} < \theta^{\frac{1}{\sigma+1}} + (\lambda - \theta)^{\frac{1}{\sigma+1}},$$

which yields

$$m_\lambda < m_\theta + m_{\lambda-\theta}, \qquad \forall\, \theta \in]0,\lambda[. \tag{9}$$

Then dichotomy is ruled out using classical truncation arguments.

Therefore, the compactness holds for ρ_k, i.e., going to a subsequence of (u_k) if necessary, there exists a sequence $(y^k) \subset \mathbb{R}^N$ such that for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R(y^k)} |u_k|^{2\sigma+2}\, dx > \lambda - \varepsilon.$$

Setting $w_k(x) := u_k(x + y^k)$, we have that (w_k) is also a minimizing sequence for m_λ. Then, up to a subsequence, w_k weakly converges in $H^2(\mathbb{R}^N)$ to $w \in M_\lambda$ and $J_{\mathbb{R}^N}(w) = m_\lambda$. This concludes the proof of the existence in Theorem 1.1. □

Remark 3.2. When $\beta \geq 2\sqrt{\alpha}$, we can avoid the use of the concentration-compactness lemma. Indeed, take a minimizing sequence $(u_k)_k \subset H^2(\mathbb{R}^N)$ for m. Then, let us set $f_k := -\Delta u_k + \beta u_k/2$ and define $v_k \in H^2(\mathbb{R}^N)$ to be the strong solution of $-\Delta v_k + \beta v_k/2 = |f_k|^*$ in $\mathbb{R}^N$, where $|f_k|^*$ denotes the Schwarz symmetrization of $|f_k|$. Thus for each $k \in \mathbb{N}$, we have $v_k \in H^2_{rad}(\mathbb{R}^N)$ which is the space of H^2 functions that are radially symmetric around the origin. Then a particular case of [3, Lemma 3.4] see also [4] implies

$$\begin{aligned} J\left(\frac{v_k}{|v_k|_{2\sigma+2}}\right) &= \frac{\displaystyle\int_{\mathbb{R}^N} (-\Delta v_k + \beta v_k/2)^2\, dx - (\beta^2/4 - \alpha)\int_{\mathbb{R}^N} v_k^2\, dx}{|v_k|^2_{2\sigma+2}} \\ &\leq \frac{\displaystyle\int_{\mathbb{R}^N} (-\Delta u_k + \beta u_k/2)^2\, dx - (\beta^2/4 - \alpha)\int_{\mathbb{R}^N} u_k^2\, dx}{|u_k|^2_{2\sigma+2}}. \end{aligned}$$

Using the compact embedding of $H^2_{rad}(\mathbb{R}^N)$ into $L^{2\sigma+2}(\mathbb{R}^N)$, see, for instance, [20, Théorème II.1], it follows that $(v_k)_k$ weakly converges in H^2 to some $v \in M$ and the remaining arguments are standard.

4 Sign and symmetry

In order to investigate the symmetry properties of a fourth order equation with Navier boundary conditions or in the whole space, it is natural to ask if the equation may be rewritten as a cooperative system. If this is the case, then the moving plane procedure applies, see the work of Troy [23] in the case of a bounded domain or de Figueiredo-Yang [10] (if we assume exponential decay) and Busca-Sirakov [7] (without assuming exponential decay) when $\Omega = \mathbb{R}^N$. Observe that when $\alpha > 0$ and $|\beta| \geq 2\sqrt{\alpha}$, we can indeed write the equation as a cooperative system

$$-\Delta u + \frac{\beta}{2}u - v = 0, \quad -\Delta v + (\alpha - \frac{\beta^2}{4})u + \frac{\beta}{2}v = |u|^{2\sigma}u.$$

To prove that least energy solutions do not change sign, we use the minimality combined to the classical maximum principle for a single equation. The argument goes back to van der Vorst, see, for instance, [24]. We sketch it for completeness to emphasize the role of the assumption $|\beta| \geq 2\sqrt{\alpha}$.

Lemma 4.1. *Assume that* $|\beta| \geq 2\sqrt{\alpha}$ *and* $-\lambda_1(\Omega) < \beta/2$ *if* Ω *is bounded or* $\beta > 0$ *if* $\Omega = \mathbb{R}^N$. *If* $u \in H_\Omega$ *is a minimizer of* (4)*, then*

$$u > 0 \quad \text{and} \quad -\Delta u + \beta u/2 > 0 \quad \text{in } \Omega,$$

or else

$$u < 0 \quad \text{and} \quad -\Delta u + \beta u/2 < 0 \quad \text{in } \Omega.$$

Proof. Let $w \in H_\Omega$ be such that

$$\begin{cases} -\Delta w + \beta w/2 = |-\Delta u + \beta u/2|, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Then

$$-\Delta(w \pm u) + \beta(w \pm u)/2 \geq 0.$$

Using the strong maximum principle we know that u has a fixed sign if $-\Delta u + \beta/2\, u$ does not change sign. We then argue by contradiction, suppose that $-\Delta u + \beta u/2$ changes sign. Then $|-\Delta u + \beta u/2| \neq 0$ and the strong maximum principle implies that $w > |u|$. For convenience denote by $|\cdot|_{2\sigma+2}$ the $L^{2\sigma+2}$ norm in Ω. Therefore

$$\begin{aligned} J_\Omega\left(\frac{w}{|w|_{2\sigma+2}}\right) &= \frac{\int_\Omega (-\Delta w + \beta w/2)^2\,dx - (\beta^2/4 - \alpha)\int_\Omega w^2\,dx}{|w|^2_{2\sigma+2}} \\ &< \frac{\int_\Omega (-\Delta u + \beta u/2)^2\,dx - (\beta^2/4 - \alpha)\int_\Omega u^2\,dx}{|u|^2_{2\sigma+2}} \end{aligned}$$

which contradicts the minimality of u. Observe that the last inequality holds because the numerator is nonnegative. $\square$

Remark 4.2. In the case of a bounded domain Ω and $0 < \alpha \leq \lambda_1(\Omega)^2$, we then know the sign of the least energy solutions of (P_β) for values of $\beta \in (-2\lambda_1(\Omega), -2\sqrt{\alpha}] \cup [2\sqrt{\alpha}, \infty)$. For Ω bounded, we do not know if the least energy solutions change sign for $\beta \in (-2\sqrt{\alpha}, 2\sqrt{\alpha})$. Section 6 deals with the case $\Omega = \mathbb{R}^N$ under the assumption that the minimizer is radial.

Proof (Proof of Theorem 1.1 continued). Existence was proved in Section 3 while we just proved in Lemma 4.1 that any least energy solution does not change sign.

Writing $f(u,v) = (\frac{\beta^2}{4} - \alpha)u - \frac{\beta}{2}v + |u|^{2\sigma}u$ and $g(u,v) = v - \frac{\beta}{2}u$, the equation is equivalent to the cooperative system

$$\Delta u + g(u,v) = 0, \quad \Delta v + f(u,v) = 0.$$

We are in the setting of Busca-Sirakov [7] and [7, Theorem 2] applies. Observe that clearly u and v must be symmetric with respect to the same point. □

In the case of a bounded domain, we have proved so far the following result for (P_β).

Theorem 4.3. *Assume Ω is a bounded smooth domain and $(A1)$ or $(A1')$ holds. Suppose moreover that $2 < 2\sigma + 2 < \frac{2N}{N-4}$ if $N \geq 5$. Then problem (P_β) has a nontrivial least energy solution. If in addition $|\beta| \geq 2\sqrt{\alpha}$ and $-\lambda_1(\Omega) < \beta/2$, then any least energy solution does not change sign. If Ω is a ball, then any least energy solution is radially symmetric and strictly radially decreasing.*

Proof. Existence has been achieved in Theorem 3.1 while the sign information follows from Lemma 4.1. If Ω is a ball, the symmetry of the minimizer follows from [23, Theorem 1]. □

We point out that the condition $|\beta| \geq 2\sqrt{\alpha}$ is crucial to rewrite the problem (P_β) as a cooperative system. In fact, we can deal more generally with smooth bounded or unbounded domain Ω with some symmetries. Then the symmetry properties of the solutions of constant sign can be deduced from the moving plane method adapted to cooperative systems in [23].

5 The effect of a small fourth order dissipation

In this section, we study the behaviour of minimizers of (4) when the coefficient of fourth order dissipation tends to zero. We assume throughout the section that $\alpha > 0$ and we choose the norm on $H^1(\mathbb{R}^N)$ defined through

$$\|u\|_{H^1}^2 = \int_\Omega (|\nabla u|^2 + \alpha |u|^2)\, dx.$$

We recall that the problem

$$\Delta^2 v(x) - \beta\Delta v(x) + \alpha v(x) = |v|^{2\sigma} v(x), \quad x \in \mathbb{R}^N$$

is equivalent to

$$\gamma\Delta^2 u(x) - \Delta u(x) + \alpha u(x) = |u|^{2\sigma} u(x), \quad x \in \mathbb{R}^N.$$

by scaling the solutions as $u(x) = v(\frac{x}{\sqrt{\beta}})$ where $\gamma = 1/\beta^2$. As before we consider the associated minimization problem

$$m_\gamma = \inf_{u\in M} J_\gamma(u)$$

where

$$M = \{u \in H_\Omega : \int_\Omega |u|^{2\sigma+2}\,dx = 1\}$$

and

$$J_\gamma(u) = \int_\Omega (\gamma|\Delta u|^2 + |\nabla u|^2 + \alpha|u|^2)\,dx.$$

When $\Omega = B_R$ or $\Omega = \mathbb{R}^N$, the results of the previous sections imply that when $\gamma \le \frac{1}{4\alpha}$, any minimizer is radially symmetric and strictly radially decreasing (after a possible translation in the case $\Omega = \mathbb{R}^N$). In the case $\Omega = \mathbb{R}^N$, we assume from now on that the maximum of any minimizer has been translated to the origin.

For $\gamma = 0$, the associated minimization problem is

$$m_0 = \inf_{u\in M_0} J_0(u)$$

where

$$M_0 = \{u \in H_0^1(\Omega) : \int_\Omega |u|^{2\sigma+2}\,dx = 1\}$$

and

$$J_0(u) = \int_\Omega (|\nabla u|^2 + \alpha|u|^2)\,dx.$$

Assume $2 < 2\sigma + 2 < \frac{2N}{N-2}$ if $N \ge 3$, $\Omega = B_R$ or $\Omega = \mathbb{R}^N$ and let u_0 be the unique minimizer of J_0 in M_0. We refer to [9, 15, 19] for the uniqueness property (in the case $\Omega = \mathbb{R}^N$, we fix the maximum of the solution at the origin to achieve uniqueness). We first prove that if $\gamma_k \to 0$, then any sequence $(u_k)_k$ of minimizer of J_{γ_k} converge strongly in H^1 to u_0. A similar statement obviously holds for other bounded domains except that uniqueness of the minimizer does not hold in general so that in the conclusion, we can only state that we have convergence to one minimizer, see Theorem 5.3.

Proposition 5.1. *Assume* $2 < 2\sigma + 2 < \frac{2N}{N-2}$ *if* $N \geq 3$, $\Omega = B_R$ *or* $\Omega = \mathbb{R}^N$. *There exists* $C > 0$ *such that for every* $\gamma > 0$, *we have*

$$m_0 \leq m_\gamma \leq m_0 + C\gamma.$$

Moreover, if $\gamma_k \to 0$ *and* $(u_k)_k$ *is a sequence such that* $J_{\gamma_k}(u_k) = m_{\gamma_k}$, *then* $u_k \to u_0$ *strongly in* H^1.

Proof. The estimate of m_γ is clear since by elliptic regularity, we easily infer that $u_0 \in H^2(\Omega)$. Therefore, we have

$$m_\gamma \leq J_\gamma(u_0) = \gamma \int_\Omega |\Delta u_0|^2 \, dx + J_0(u_0) \leq C\gamma + m_0,$$

whereas taking any minimizer u_γ for m_γ, we get

$$m_\gamma = J_\gamma(u_\gamma) = \gamma \int_\Omega |\Delta u_\gamma|^2 \, dx + J_0(u_\gamma) \geq m_0.$$

Let $\gamma_k \to 0$ and $(u_k)_k$ be a sequence of minimizers for $m_k := m_{\gamma_k}$. Then

$$\int_\Omega (|\nabla u_k|^2 + \alpha |u_k|^2) \, dx \leq m_k \leq m_0 + C\gamma_k \to m_0.$$

Since we know that u_k is a radial function, it follows that u_k is bounded in $H^1_{rad}(\Omega)$ - the space of H^1 functions that are radially symmetric around the origin—so that up to a subsequence, u_k converges weakly in H^1 to some $u \in M$. The strong convergence in $L^{2\sigma+2}$ when $\Omega = \mathbb{R}^N$ follows from the compact embedding of $H^1_{rad}(\mathbb{R}^N)$ into $L^{2\sigma+2}(\mathbb{R}^N)$, see [20, 22].

Now, by weak lower semi-continuity, we have

$$\begin{aligned} m_0 \leq \int_\Omega (|\nabla u|^2 + \alpha |u|^2) \, dx &\leq \liminf_{k\to\infty} \int_\Omega (|\nabla u_k|^2 + \alpha |u_k|^2) \, dx \\ &\leq \limsup_{k\to\infty} \int_\Omega (|\nabla u_k|^2 + \alpha |u_k|^2) \, dx = m_0. \end{aligned}$$

Hence the convergence is strong in H^1 and u is a minimizer for m_0. By uniqueness, $u = u_0$ and the whole sequence converges. □

In the model case with a Kerr nonlinearity in dimension $N \leq 3$, we can improve this convergence.

Proposition 5.2. *Assume $\Omega = \mathbb{R}^N$, $\sigma = 1$ and $N \leq 3$. If $\gamma_k \to 0$ and $(u_k)_k$ is a sequence such that $J_{\gamma_k}(u_k) = m_{\gamma_k}$, then $u_k \to u_0$ strongly in H^2.*

Proof. To fix the ideas, we deal with the case $N = 3$, $N = 2$ being similar. The starting point is an a priori bound in H^1 and the strategy is to end up with an a priori H^4-bound. We already know from Proposition 5.1 that u_k converges to u_0 strongly in H^1. To improve the convergence, we use the Euler-Lagrange equation

$$\gamma_k \Delta^2 u_k - \Delta u_k + \alpha u_k = m_k u_k^3,$$

where $m_k = m_{\gamma_k}$. We can assume $\gamma_k \leq 1$ and $m_k \in [m_0, m_0 + C]$.

Bound in H^1. Since u_k is a minimizer, we can assume

$$\|u_k\|_{H^1} \leq m_0 + C.$$

This also provides an a priori bound in L^q for every $q \in [2, 6]$.

Bound in H^2. We denote $v_k = -\gamma_k \Delta u_k$. Then v_k solves

$$-\Delta v_k + \frac{1}{\gamma_k} v_k = w_k, \tag{10}$$

where $w_k := m_k u_k^3 - \alpha u_k$. Since $J_{\gamma_k}(u_k) \leq m_0 + C$, we infer that $v_k \to 0$ strongly in L^2. In particular, $(v_k)_k$ is bounded in L^2. Observe also that $(w_k)_k$ is a priori bounded in L^2. Now, by elliptic regularity, we infer that $v_k \in H^2(\mathbb{R}^3)$ with a bound that does not depend on k. Indeed, since $\frac{1}{\gamma_k} \geq 1$, we get this a priori bound as in Krylov [18, Chapter 1, Theorems 6.4 & 6.5]. Now, from this a priori H^2-bound on $(v_k)_k$ and the Euler equation

$$-\Delta u_k + \alpha u_k = m_k u_k^3 + \Delta v_k, \tag{11}$$

we deduce that $(u_k)_k$ is a priori bounded in $H^2(\mathbb{R}^3)$ as well.

Bound in H^4. It is straightforward to check that the H^2-bound on u_k implies that $w_k \in L^2(\mathbb{R}^3)$ and $\Delta w_k \in L^2(\mathbb{R}^3)$. Then, elliptic regularity implies w_k is bounded in H^2 as well. Using again (10), we now infer that $v_k \in H^4$ with a bound independent of k, arguing as in Krylov for H^{m+2} regularity [18, Chapter 1, Theorem 7.5 & Corollary 7.6]. Looking at (11) again, we have that the right-hand side is bounded in H^2, whence $u_k \in H^4$ with a bound independent of k.

Conclusion. Observe now that we can use the equation (11) to conclude. Since $-\Delta v_k = \gamma_k \Delta^2 u_k \to 0$ strongly in L^2, we conclude that

$$m_k u_k^3 + \Delta v_k \to m_0 u_0^3$$

strongly in L^2 and elliptic regularity applied to (11) implies that the convergence of u_k to u_0 is actually strong in H^2. $\square$

Now that we have proved the strong convergence in H^2 to the unique minimizer for $\gamma = 0$, we can use its non degeneracy to apply the Implicit Function Theorem. This yields Theorem 1.3.

Proof (Proof of Theorem 1.3). We start by setting $X := H^2_{rad}(\mathbb{R}^3)$ and $Y := H^{-2}(\mathbb{R}^3)$. Let $F : \mathbb{R}^+ \times X \to Y$ be the operator defined (in the sense of distributions) by

$$F(\gamma, u) = \gamma \Delta^2 u - \Delta u + \alpha u - |u|^2 u.$$

Namely, for every $v \in H^2(\mathbb{R}^3)$, we have

$$F(\gamma, u)(v) = \int_{\mathbb{R}^3} (\gamma \Delta u \Delta v + \nabla u \nabla v + \alpha u v - |u|^2 u v)\, dx.$$

Obviously $F(0, \sqrt{m_0} u_0) = 0$. Also, F is continuously differentiable in a neighbourhood of $(0, \sqrt{m_0} u_0)$ with $D_u F(\gamma, u) \in \mathscr{L}(X, Y)$ defined by

$$D_u F(\gamma, u) v = \gamma \Delta^2 v - \Delta v + \alpha v - 3|u| u v, \qquad \forall\ v \in X,$$

i.e.

$$D_u F(\gamma, u) v[w] = \int_{\mathbb{R}^3} (\gamma \Delta v \Delta w + \nabla v \nabla w + \alpha v w - 3|u| u v w)\, dx, \qquad \forall\ v, w \in X.$$

We thus have in the distributional sense

$$L(v) := D_u F(0, \sqrt{m_0} u_0) v = -\Delta v + \alpha v - 3 m_0 u_0^2 v.$$

It is well known that the kernel of L is of dimension 3 when considered in $H^2(\mathbb{R}^3)$ and it is spanned by the partial derivatives of u_0. In particular, the kernel of L restricted to $H^2_{rad}(\mathbb{R}^3)$ is trivial and $L : X \to Y$ is one-to-one. We refer, for instance, to [8, 15, 19]. Moreover, it follows from the Open Mapping Theorem that $L^{-1} : Y \to X$ is continuous.

Since the linear map L is a homeomorphism, we can apply the Implicit Function Theorem. Namely, there exists $\gamma_0 > 0$ and an open set $U_0 \subset X$ that contains $\sqrt{m_0} u_0$ such that for every $\gamma \in [0, \gamma_0[$, the equation $F(\gamma, u) = 0$ has a unique solution $u_\gamma \in U_0$ and the curve

$$\Gamma : [0, \gamma_0[\to H^2(\mathbb{R}^3) : \gamma \mapsto u_\gamma$$

is of class C^1.

Now suppose that the uniqueness of least energy solutions fails in every interval $(0,\gamma)$. We can then construct two sequences in M of least energy solutions along a sequence γ_k converging to 0. We call them $(u_k)_k$ and $(v_k)_k$ whereas m_k is their common energy. By assumption, $u_k \neq v_k$. Since $\gamma_k \to 0$, we know that u_k and v_k are radially symmetric. Since these two sequences converge in H^2 to u_0 as $k \to \infty$, we have

$$\sqrt{m_k}u_k, \sqrt{m_k}v_k \to \sqrt{m_0}u_0,$$

where the convergence is strong in H^2. Then, for k large enough, there exist two solutions of the equation $F(\gamma_k, u) = 0$ in U_0 with $\gamma_k < \gamma_0$. This is a contradiction and ends the proof. □

We now state the counterpart of Theorem 1.3 for the boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^N$ with Navier boundary conditions, namely

$$(P_\gamma) \qquad \begin{cases} \gamma\Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

We assume in the next statement that Ω is smooth. We have not searched to optimize the required regularity of the boundary. At some point, we need to take two partial derivatives into the equation. We assume enough regularity of the boundary so that the solution belongs at least to $H^6(\Omega)$. One could work with interior regularity which requires less regularity on the boundary but since our main motivation is to cover the case of a ball, working with global regularity is fine for our purpose as the ball has the regularity required.

Theorem 5.3. *Assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain of class C^6 and $3 \leq 2\sigma + 2 < \frac{2N}{N-2}$ if $3 \leq N \leq 5$. If $\gamma_k \to 0$ and $(u_k)_k$ is a sequence of least energy solutions of (P_{γ_k}), then, up to a subsequence, u_k converges strongly in H^2 to some minimizer u_0 for m_0. If, in addition, Ω is a ball, then there exists $\gamma_0 > 0$ such that if $0 < \gamma < \gamma_0$, the problem (P_γ) has a unique least energy solution. This solution is radially symmetric and strictly radially decreasing.*

Proof. *Step 1. Global regularity.* Using elliptic regularity [14, Theorems 8.12 & 8.13], we easily infer that the solutions u_k are smooth, namely at least $H^6(\Omega)$. Indeed, one can write the equation as a double Dirichlet problem

$$-\Delta u_k = \phi_k, \quad u_k = 0 \text{ on } \partial\Omega,$$

$$-\gamma_k\Delta\phi_k + \phi_k = m_k|u_k|^{2\sigma}u_k - \alpha u_k, \quad \phi_k = 0 \text{ on } \partial\Omega.$$

Here γ_k stays fixed and we can start with the fact that $u_k \in H^2(\Omega)$, without caring about the dependence on k. Then the term $m_k|u_k|^{2\sigma}u_k - \alpha u_k \in L^2(\Omega)$ as it can be easily checked from the assumption on σ and the embedding of $H^2(\Omega)$ into $L^q(\Omega)$ for every $q \geq 1$ if $N \leq 4$ and $q \in [1, \frac{2N}{N-4}]$ if $N = 5$. We therefore infer from [14, Theorems 8.12] that $\phi_k \in H^2(\Omega)$ which in turn implies that $u_k \in H^4(\Omega)$ by [14, Theorems 8.13]. Now computing $\Delta(m_k|u_k|^{2\sigma}u_k - \alpha u_k)$, we realize that it is an L^2 function and therefore $m_k|u_k|^{2\sigma}u_k - \alpha u_k$ is an H^2 function. Indeed, the condition on σ ensures the required integrability of $|u_k|^{2\sigma-1}|\nabla u_k|^2$ and $|u_k|^{2\sigma}|\Delta u_k|$. We then conclude that ϕ_k belongs in fact at least to H^4 and therefore $u_k \in H^6(\Omega)$.

Step 2. Strong convergence in H^1. Arguing as in the proof of Proposition 5.1, we infer that there exists a minimizer $u_0 \in M_0$ and a subsequence that we still denote $(u_k)_k$ such that $u_k \to u_0$ strongly in H^1. If Ω is a ball, then u_0 is the unique minimizer and the whole sequence converge.

Step 3. Strong convergence in H^2. To improve the convergence, we argue as in the proof of Proposition 5.2. If $2\sigma + 1 \leq \frac{N}{N-2}$, then we can bootstrap using the H^{m+2} regularity theory. Due to the boundary condition, the argument of Krylov [18, Chapter 1] cannot be applied directly to get higher regularity in general, see [18, Chapter 8]. However, in our case, since we deal with Navier condition, we have that $u_k = \Delta u_k = 0$ on the boundary and therefore the equation (P_{γ_k}) tells that $\Delta^2 u_k = 0$ on the boundary as well. By Step 1, we can take the Laplacian inside the equation in (P_{γ_k}) and use the fact that Δu_k solves a boundary problem with Navier boundary conditions, namely

$$\begin{aligned} \gamma_k \Delta^2(\Delta u_k) - \Delta(\Delta u_k) + \alpha(\Delta u_k) &= m_k f(u_k), \qquad \text{in } \Omega, \\ \Delta(\Delta u_k) = \Delta u_k &= 0, \qquad \text{on } \partial\Omega, \end{aligned}$$

where

$$f(u_k) = (2\sigma + 1)\mathrm{sign}(u_k)\left(2\sigma u_k^{2\sigma-1}|\nabla u_k|^2 + u_k^{2\sigma}\Delta u_k\right). \tag{12}$$

Then we can use the H^2 regularity for the Dirichlet problem associated with the systems

$$v_k = -\gamma_k \Delta u_k \qquad -\Delta v_k + \frac{1}{\gamma_k} v_k = w_k, \tag{13}$$

and

$$y_k = \Delta v_k = -\gamma_k \Delta^2 u_k \qquad -\Delta y_k + \frac{1}{\gamma_k} y_k = m_k f(u_k), \tag{14}$$

where $w_k = m_k|u_k|^{2\sigma}u_k - \alpha u_k$ and $f(u_k)$ is defined in (12). Applying [18, Chapter 8, Theorem 8.7] to the second equation of the first system (13), we get an H^2 a priori bound of v_k. Now turning to the Dirichlet problem

$$-\Delta u_k + \alpha u_k = m_k|u_k|^{2\sigma}u_k + \Delta v_k, \quad u_k = 0 \text{ on } \partial\Omega, \tag{15}$$

we deduce that u_k is a priori bounded in H^2 which leads to an L^2 bound for $f(u_k)$. Applying then [18, Chapter 8, Theorem 8.7] on the second equation of the system (14) gives an H^2 a priori bound of Δv_k. Whence v_k is a priori bounded in H^4. This allows to conclude that u_k is a priori bounded in H^4 because the right-hand side of Δ(15), namely

$$-\Delta(\Delta u_k) + \alpha \Delta u_k = m_k f(u_k) + \Delta^2 v_k,$$

is a priori bounded in L^2. The remaining steps are now as in the proof of Proposition 5.2.

If $\frac{N}{N-2} + 1 < 2\sigma + 2 < \frac{2N}{N-2}$, we can only start with a bound in $L^{\frac{2N}{(N-2)(2\sigma+1)}}$ on the right-hand side of

$$-\Delta v_k + \frac{1}{\gamma_k} v_k = w_k,$$

where we still use the notations $v_k = -\gamma_k \Delta u_k$ and $w_k = m_k |u_k|^{2\sigma} u_k - \alpha u_k$. We therefore need to improve this bound first. Arguing as above (still using [18, Chapter 8, Theorem 8.7]), we deduce an a priori bound in $W^{2,q}$ with $q = \frac{2N}{(N-2)(2\sigma+1)}$. Then Sobolev embeddings give a better integrability of w_k and we can bootstrap until we get an L^2 a priori bound on w_k. The strong convergence in H^2 is then achieved as in the proof of Proposition 5.2 taking into account the above remark concerning the way to obtain the higher order elliptic regularity. Observe that even if $\frac{N}{N-2} + 1 < 2\sigma + 2$, no additional bootstrap is necessary to derive the H^4 bound on u_k since once we get an a priori H^2 bound on u_k, the assumption on σ implies that $f(u_k)$ is a priori bounded in L^2.

Uniqueness in the case $\Omega = B_R$. When Ω is a ball, the arguments used in the proof of Theorem 1.3 are available. The nondegeneracy of u_0 allows to apply the Implicit Function Theorem to conclude the local uniqueness (in an H^2 neighbourhood of u_0) for γ small. The remaining arguments are then as in the proof of Theorem 1.3. □

We end up the analysis of the asymptotics for $\gamma \to 0$ by showing that the least energy solution blows up in H^2 when $2\sigma + 2$ is H^1 critical or supercritical. We focus on the case of $\Omega = \mathbb{R}^N$.

We first derive the Derrick-Pohozahev identity for minimizers. If u achieves m_γ in M, then, defining v_λ by $v_\lambda(x) = \lambda^{\frac{N}{2\sigma+2}} u(\lambda x)$, we infer that $f(\lambda) := J_\gamma(v_\lambda)$ achieves a local minimum at $\lambda = 1$. This yields a Derrick-Pohozahev identity

$$\gamma\,(2N - (2\sigma + 2)(N-4)) \int_{\mathbb{R}^N} |\Delta u|^2\,dx + (2N - (2\sigma+2)(N-2)) \int_{\mathbb{R}^N} |\nabla u|^2\,dx$$
$$+\alpha(2N - (2\sigma+2)N) \int_{\mathbb{R}^N} |u|^2\,dx = 0.$$

If $2\sigma + 2 \geq \frac{2N}{N-4}$, then u must be zero which is obviously a contradiction. This shows that m_γ is not achieved for $2\sigma + 2 \geq \frac{2N}{N-4}$.

For $\frac{2N}{N-2} \leq 2\sigma + 2 < \frac{2N}{N-4}$, the first coefficient in the Derrick-Pohozahev identity is positive whereas the other two are nonpositive. We can then write

$$\gamma\,(2N - (2\sigma + 2)(N - 4)) \int_{\mathbb{R}^N} |\Delta u|^2\, dx \geq \alpha(2\sigma N) \int_{\mathbb{R}^N} |u|^2\, dx.$$

Now, from Gagliardo-Nirenberg inequality, we infer that for some $C > 0$,

$$1 = \left(\int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx \right)^{\frac{8}{4(2\sigma+2)-2\sigma N}} \leq C \left(\int_{\mathbb{R}^N} |\Delta u|^2\, dx \right)^{\frac{2N}{4(2\sigma+2)-2\sigma N}} \int_{\mathbb{R}^N} |u|^2\, dx,$$

which implies

$$\gamma\,(2N - (2\sigma + 2)(N - 4)) \left(\int_{\mathbb{R}^N} |\Delta u|^2\, dx \right)^{1+\frac{2\sigma N}{4(2\sigma+2)-2\sigma N}} \geq \alpha(2\sigma N)C.$$

This shows that Δu blows up in $L^2(\mathbb{R}^3)$ when $\gamma \to 0$.

6 Sign-changing radial minimizer

In this section, we show that a radial least energy solution of (3) with $\sigma = 1$ is sign-changing when $-2\sqrt{\alpha} < \beta < 2\sqrt{\alpha}$. We assume $N = 3$ but the arguments apply in dimension $N = 2$ also.

We will require the decay of the radial derivatives. Arguing as in de Figueiredo et al [11, Theorem 2.2], one easily gets the following lemma.

Lemma 6.1. *Let $u \in H^m_{rad}(\mathbb{R}^3)$ and let $v\ :]0,\infty[\to \mathbb{R}$ be the function defined by $v(r) := u(x)$ with $r = |x|$. Then, $v \in H^m(]0,\infty[, r^2)$. Moreover, for a.e. $|x| \in]0,\infty[$ we have*

$$\left|D^j u(x)\right| \geq \left|v^{(j)}(|x|)\right|, \qquad \forall\, j = 0, 1, \ldots, m.$$

In order to prove the Theorem 1.4 we adapt some arguments of Bonheure et al [2, Theorem 6].

Proof (Proof of Theorem 1.4). We suppose $N = 3$, the case $N = 2$ is similar.

Step 1. Classical regularity. We start by observing that by elliptic regularity, we have $u \in H^6(\mathbb{R}^3)$ which implies $u \in C^{4,1/2}(\mathbb{R}^3)$ and the solution can be understood in the classical sense. Indeed, we know that the solution is H^2, so that from the equation

$$-\Delta(-\Delta u) = |u|^2 u - \alpha u + \beta \Delta u,$$

we infer that $-\Delta(-\Delta u) \in L^2(\mathbb{R}^3)$. This implies that $-\Delta u, -\Delta(-\Delta u) \in L^2(\mathbb{R}^3)$ and henceforth $-\Delta u \in H^2(\mathbb{R}^3)$. Since $u \in H^2(\mathbb{R}^3)$, we conclude that $u \in H^4(\mathbb{R}^3)$. Looking again at the equation, we can now use the fact that the right-hand side is an H^2-function. Then $-\Delta u, -\Delta(-\Delta u) \in H^2(\mathbb{R}^3)$ and therefore $-\Delta u \in H^4(\mathbb{R}^3)$. At last, combining the fact that $u \in H^4(\mathbb{R}^3)$ and $-\Delta u \in H^4(\mathbb{R}^3)$, we deduce that $u \in H^6(\mathbb{R}^3)$. Here above, the required elliptic regularity theory can be found in [18, Chapter 1] and since we are in the whole space, this is just a consequence of simple Fourier analysis.

Step 2. Equation in radial coordinates and decay at infinity. Writing now the equation (3) in radial coordinates (the expression is especially simple in dimension $N = 3$), we compute that v, defined by $v(r) := u(x)$ for $r = |x|$, solves

$$v^{iv} + \frac{4}{r}v''' - \beta v'' - \frac{2\beta}{r}v' + \alpha v = |v|^2 v, \qquad r \in]0, \infty[. \tag{16}$$

The $H^5(\mathbb{R}^3)$ regularity yields

$$\lim_{|x|\to\infty} (u(x), \partial_{x_i} u(x), \partial^2_{x_i x_j} u(x), \partial^3_{x_i x_j x_k} u(x)) = (0, 0, 0, 0)$$

whatever $i, j, k \in \{1, 2, 3\}$. Then Lemma 6.1 implies that v satisfies

$$\lim_{r\to\infty} (v(r), v'(r), v''(r), v'''(r)) = (0, 0, 0, 0). \tag{17}$$

Step 3. Asymptotic analysis of the solution of the ordinary differential equation (16).

Claim 1 : Given $R > 0$ we can find $\bar{r} \geq R$ such that $v(\bar{r}) > 0$.

Let $R > 0$ be fixed. Consider the following Cauchy problem

$$(C1) \qquad \begin{cases} w^{iv}(r) - \beta w''(r) + \alpha w(r) = 0, & r > 0, \\ (w(r_0), w'(r_0), w''(r_0), w'''(r_0)) =: w_0, & \end{cases}$$

where $r_0 > 0$ and $w_0 \in \mathbb{R}^4$. By using condition $(A1)$ we have that all the roots of the characteristic equation associated with $(C1)$ are complex, let us say $\pm a \pm ib$. We set $\Delta := 2\pi/b$. Then there exists $c > 0$ such that any solution of $(C1)$ satisfies

$$\sup_{[r_0, r_0+\Delta]} w, \sup_{[r_0, r_0+\Delta]} (-w) \geq c|w_0|. \tag{18}$$

Moreover, there exists $M > 0$ such that any solution of $(C1)$ verifies

$$\|w\|_{C^3([r_0, r_0+\Delta])} \leq M|w_0|.$$

Again, we can also find $N > 0$ such that the solutions of

$$\begin{cases} \psi^{iv}(r) - \beta\psi''(r) + \alpha\psi(r) = h(r), & r > 0, \\ (\psi(r_0), \psi'(r_0), \psi''(r_0), \psi'''(r_0)) =: 0, \end{cases}$$

satisfy

$$\|\psi\|_{C^3([r_0,r_0+\Delta])} \leq N\|h\|_{L^\infty(r_0,r_0+\Delta)}.$$

Let us set $\delta > 0$ so that $c - \frac{MN\delta}{1-N\delta} > 0$. Denote by $v(r) = v(r; r_0, v_0)$ the solution of (16) with initial conditions

$$(v(r_0), v'(r_0), v''(r_0), v'''(r_0)) =: v_0, \qquad \text{where } r_0 > 0.$$

Now, let us fix $r_0 \geq R$ large enough so that $|v_0|$ is small enough to have

$$\sup_{r\in[r_0,r_0+\Delta]} |v(r)|^2, \sup_{r\in[r_0,r_0+\Delta]} \frac{4}{r} \quad \text{and} \quad \sup_{r\in[r_0,r_0+\Delta]} \frac{2\beta}{r} < \delta.$$

We write

$$v = \psi + w,$$

where ψ solves

$$\begin{cases} \psi^{iv} - \beta\psi'' + \alpha\psi = |v|^2 v + \dfrac{2\beta}{r}v' - \dfrac{4}{r}v''', \ r > 0, \\ (\psi(r_0), \psi'(r_0), \psi''(r_0), \psi'''(r_0)) = 0, \end{cases}$$

and w is a solution of

$$\begin{cases} w^{iv}(r) - \beta w''(r) + \alpha w(r) = 0, & r > 0, \\ (w(r_0), w'(r_0), w''(r_0), w'''(r_0)) = v_0. \end{cases}$$

Now, let us choose $\overline{r} \in [r_0, r_0 + \Delta]$ such that

$$w(\overline{r}) \geq c|v_0|.$$

Thus,

$$\|\psi\|_{C^3([r_0,r_0+\Delta])} \leq N\delta\|v\|_{C^3([r_0,r_0+\Delta])},$$

which implies that

$$\|\psi\|_{C^3([r_0,r_0+\Delta])} \leq \frac{N\delta}{1-N\delta}\|w\|_{C^3([r_0,r_0+\Delta])} \leq \frac{MN\delta}{1-N\delta}|v_0|.$$

Then we obtain

$$v(\bar{r}) \geq c|v_0| - \|\psi\|_{L^\infty} \geq \left(c - \frac{MN\delta}{1-N\delta}\right)|v_0| > 0.$$

Claim 2 : Given $R > 0$ *we can find* $\underline{r} \geq R$ *such that* $v(\underline{r}) < 0$*.*

The proof of this claim is similar to that of Claim 1.

Conclusion. We have proved in the last step that u changes sign. In fact, we have even proved that u oscillate as $|x| \to +\infty$. □

7 Comments

This note provides some simple results for the model equation (3) with a Kerr nonlinearity and aims to partially complete the discussion on waveguide solutions in [13, Section 4.1]. The methods we used are standard. On the other hand, since radial solutions present oscillations for $-2\sqrt{\alpha} < \beta < 2\sqrt{\alpha}$, we expect that one needs new arguments to answer the question whether the least energy solutions are radial or not in this case. Also uniqueness is a challenging question if we are not in the asymptotic regime $\beta \to \infty$ (or equivalently $\gamma \to 0$).

We also mention that the important question about the decay at infinity of the least energy solutions will be addressed in a future work. We are only aware of [12] for a result in that direction. The analysis therein relies on the computation of the fundamental solution of the fourth-order operator in (3) with $\beta = 0$.

The analysis of the decay should also allow to extend the statement of Theorem 1.3 to the case $2 < 2\sigma + 2 < \frac{2N}{N-2}$ and $N \geq 3$. Indeed, the arguments we used are just fine for the Kerr nonlinearity whereas some technical adjustments are needed for a general subcritical power. In fact, one checks easily that our arguments apply in dimension $N \leq 4$ if we assume $2 \leq 2\sigma + 1 \leq \frac{N}{N-2}$. The lower inequality on σ implies the required $C^{1,1}$ regularity of the function $s \mapsto |s|^{2\sigma}s$ whereas the upper inequality is used to start the bootstrap with an L^2-bound on $|u|^{2\sigma}u$ (here u is a solution).

The same remark holds for Theorem 1.4 which should be true with less restrictive assumptions. In dimension $N \leq 8$, one can deal with $2 \leq 2\sigma + 1 \leq \frac{N}{N-4}$. The other cases will require more care and will be treated in a forthcoming work.

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