A Finite Element Method for Through-Flow Calculations in Turbomachines

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ABSTRACT

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A rigorous derivation of the pitch-averaged flow equations is presented and the assumption of axisymmetric flow leads, with the introduction of a stream function, to the equation to be solved.

A description is given of the finite element technique which is applied in this problem. The method of solution allows the calculation of transonic stages.

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NOMENCLATURE

b : tangential blockage factor
c : chord
c_p : specific heat at constant pressure
f : r.h.s. of basic quasi-harmonic equation
F : force vector
h : enthalpy
H : stagnation enthalpy
I : rothalpy
J : jacobian matrix
k : coefficient of quasi-harmonic equation
K : stiffness matrix
m : meridional direction
n : normal direction
N : number of blades
N_i : fluctuation terms ; shape functions
p : pressure
r : radius ; gas constant
P : position vector
s : entropy ; blade spacing
t : time ; blade thickness

T : temperature
U : blade velocity
\mathbf{\dot{V}} : absolute velocity vector
\mathbf{\dot{W}} : relative velocity vector
W : weight function
z : axial distance
a : absolute flow angle
\phi : relative flow angle in tangential plane
\phi' : blade angle in tangential plane
\gamma : specific heat ratio
\{\delta\} : vector of unknowns at nodal points
\alpha : lean angle of a blade
\eta,\xi : local coordinates of an isoparametric quadrilateral element
\theta : tangential angular coordinate
\lambda : stagger angle
\mu : under-relaxation coefficient
\rho : specific mass
\phi : flow angle in meridional plane
\psi : streamfunction
\omega : rotor angular velocity

Subscripts
b : blade
f : friction
\bar{z} : mean blade line
p : pressure side
r : radial component
R : relative
s : suction side
t : total
z : axial component
\theta : tangential component

INTRODUCTION

The modern calculation methods for the through-flow in turbomachinery date back to the basic paper of Wu [1], published some 25 years ago. As is well
known, the inviscid three-dimensional flow field is separated in Wu's method in two two-dimensional flows; one of these flows is located in blade-to-blade surfaces (the \$S_1\$ surfaces); the other flow lying on hub-to-shroud surfaces (the \$S_2\$ surfaces). Since the calculation of the flow in one of these surfaces requires the knowledge of the flow in the other one, the complete calculation is clearly iterative. However even in this complicated and lengthy way, the blade-to-blade flow would still not give the complete knowledge needed to estimate performances or to design a stage, since the losses would not be estimated. It is therefore usual to bypass the calculation of the flow in the blade-to-blade surfaces and replace this flow by some experimental correlations for losses and deviations. This is the way which is mostly followed, whereby one is left with the calculation of the flow in the \$S_2\$ surfaces whose form has then to be known since the derivatives occurring in the remaining flow equations are now derivatives along \$S_2\$.

This is however not possible without calculating the \$S_1\$ surfaces and therefore some hypothesis is required. The simplest one is the assumption of axisymmetry, whereby the \$S_2\$ surfaces are identical to the meridional plane (or more precisely, the derivatives with respect to \$r\$ and \$z\$ along \$S_2\$, reduce to the ordinary \$/r\$ and \$/z\$ derivatives).

An important drawback in this method is the lack of knowledge of the degree of approximation which is made by assuming axisymmetry. This is for instance illustrated by the tangential blockage factor appearing in the continuity equation, which is approximated by one minus the thickness over spacing ratio [1], but where no indication can be given on the error which is connected with this assumption. The authors prefer therefore to follow another way in order to obtain the meridional through-flow equations, by deriving them as an averaged part of the pitch-averaged equations. Equations for the averaged velocity components are obtained which contain blade forces and also terms describing the deviations from exact axisymmetry. These terms could even be estimated from the knowledge of the blade-to-blade flow distribution [2] but provide anyway rigorous expressions for the non-axisymmetric contributions.

In particular, it is demonstrated that the above mentioned hypothesis about the tangential blockage factor is exact, even in a non-axisymmetric situation, only for an incompressible flow. Besides, from a physical point of view, it can also be noticed that the averaged flow is precisely the flow which would be seen by a standing averaging instrument (like a pressure probe), behind a rotor. Both methods lead, of course, to the same equations in case of exact axisymmetry.

The numerical techniques used in order to solve the meridional through-flow equations fall into two categories: the method of streamline curvature and the 'matrix' method. The method of streamline curvature, as introduced in papers by L.H. Smith [3] (who derived an exact form for the radial equilibrium equation by circumferential averaging the flow equations), Novak [4], and others (e.g. [5]), is based on the explicit introduction, in the radial flow equation, of the radius of curvature of the meridional projection of the streamlines.

In this way, an ordinary first order non-linear differential equation for the meridional velocity is obtained, which can be solved by classical methods. However, the curvature radius has to be estimated at each point and at each iteration, and this leads to the calculation of a second derivative of a function which is known at a limited number of points. This is not a simple matter [6] and can lead to numerical instabilities in the calculations.

In the 'matrix' method, the streamfunction or the meridional flow component is introduced (Stokes streamfunction) giving a Poisson equation which has to be solved iteratively. This is done through a finite difference approximation, Marsh [7], Davis [8]. Although satisfactory results were reported with this method, great difficulty is found with the finite difference grids in order to match arbitrary hub and shroud geometries and complicated computational "molecules" have to be introduced consisting of 10 to 15 points in order to obtain sufficient accuracy [9].

In this paper a new method is presented on Finite Elements, which does not suffer from the drawbacks of the other two methods. Indeed, the streamline radius of curvature has not to be estimated and arbitrary geometries are treated with equal ease; axial, radial or mixed flow machine geometries, curved hub and shroud boundaries are taken into account in a straightforward way. The basic equation is obtained, like in the matrix method, by introducing a streamfunction, but it is unnecessary here to reduce it artificially to a Poisson equation, since it can be solved under the quasi-harmonic form. Moreover, taking into account the properties of the basic equations, the method is extended up to transonic compressor stages.

Examples of mixed flow geometries illustrate the versatility of the program with respect to geometry. The numerical results show a very satisfactory agreement with experimental data and the presented method appears therefore to offer a valid alternative to the previously used methods with some clear advantages compared to them.

In the first section the circumferential equations are presented, while the second section describes the finite element technique. Results are presented in the following sections for single and multi-stage axial compressors and compared with experimental data. An example of an axisymmetric axial bend is also presented. Detailed derivation of the pitch-averaged equations are presented in an appendix.

THE CIRCUMFERENTIAL AVERAGED EQUATIONS

Consider the three-dimensional flow through an arbitrary turbomachine governed by the equations of conservation of mass, momentum and energy.

General equations

The continuity equation, written in the system relative to a blade row, is

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  
(1)

The momentum conservation law is the Navier-Stokes equation which is written here, for a blade row rotating with angular velocity \( \Omega \), as

\[ \frac{d\mathbf{\Phi}}{dt} + 2 \Omega \mathbf{\omega} \times \mathbf{\Phi} - \nabla \mathbf{\Phi} = -\frac{1}{\rho} \mathbf{p} - \frac{1}{\rho} \mathbf{f} \]  
(2)

\( \mathbf{\Phi} \) is the general friction force equal to the gradient of viscous plus turbulent shear stresses.

These friction forces become important in the boundary layer regions but can be neglected in the large part of the flow volume, if the machine operates in uninstalled regimes. However, since losses are introduced in the calculation of the flow by considering irreversible entropy variations, it would be inconsistent...
to neglect completely the presence of these dissipative forces. This can also be justified by the fact that although the boundary layer and the corresponding wake of a blade are small, since a blade row usually contains a large number of blades, the total volume occupied by all boundary layers or wakes is not negligible with respect to the total flow volume. Therefore, in the circumferential-averaged equations, the friction force term will be maintained and related to the pressure loss coefficient \([3][11]\). The energy equation is written under the general form of the first law of thermodynamics

\[
T \cdot ds = dh - \frac{dp}{\rho}
\]

(3)

The identity (4)

\[
\frac{d\bar{\Psi}}{dt} + \nabla \cdot \bar{\nabla} = \frac{\bar{\Psi}}{\bar{t}} + \frac{\bar{\Psi} \omega^2}{2} - \bar{\Psi} \cdot \bar{\nabla} \cdot \bar{\nabla} \Omega (\bar{\Psi} \cdot \bar{\nabla})
\]

(4)

and the definition of the rothety

\[
I = h + \frac{\bar{\Psi}^2}{2} - \frac{\bar{\Psi} \omega^2}{2} = (H + U_v \bar{V}_g)
\]

(5)

where \(U = \omega r\), applied to equation (3) lead to equation (6)

\[
\frac{\bar{\Psi}}{\bar{t}} - \bar{\Psi} \cdot \bar{\nabla} \cdot \bar{\nabla} \Omega (\bar{\Psi} \cdot \bar{\nabla}) = T \bar{V}_s - \frac{\bar{V}_s^2}{2} + \bar{F}_f^{\rho} / \rho
\]

(6)

Since only adiabatic machines are considered, the entropy production following a fluid particle along its streamline is equal to the work of the dissipative friction forces, giving

\[
T \cdot \bar{V}_s = - \bar{F}_f^{\rho} / \rho
\]

(7)

Therefore, the energy conservation equation takes up the following general form, obtained from equations (6) and (7)

\[
(\bar{\Psi} \cdot \bar{V}) I = - \bar{V} \frac{\bar{\Psi}}{\bar{t}}
\]

(8)

showing, that in a relative steady flow situation, the rothety \(I\) is constant along a relative streamline. It may be observed that the relative steady flow assumption implies that the effects of upstream wakes on the flow in a blade row can be neglected.

In a standing blade row (stator) \(\omega = 0\) and \(I = H\). With the relative velocity \(\bar{V}\) being replaced by the absolute velocity \(\bar{V}\), we have

\[
(\bar{\Psi} \cdot \bar{V}) I = - \bar{\Psi} \frac{\bar{\Psi}}{\bar{t}}
\]

(9)

This equation shows that the total stagnation enthalpy is constant along an absolute streamline in a steady flow.

In order to obtain circumferential-averaged equations, cylindrical coordinates \((r, \theta, z)\) are used. Figure 1 shows the lay-out for an axial compressor. Equations (1) and (2), added together, give the following equation which is best suited for the averaging process.

\[
\frac{3}{\bar{t}} (p \bar{W}_r) + \bar{V}_f (p \bar{W}_r) - \rho \omega^2 \bar{V}_r + 2 \rho \bar{V}_r \cdot \bar{V}_r = - \bar{V}_p + \bar{F}_f
\]

(10)

where \(\bar{V}_f(p \bar{W}_r)\) denotes the dyadic product defined by

\[
\bar{V}_f(p \bar{W}_r) = \bar{V}_f \cdot (\bar{V}_f \rho \bar{W}_r) + \rho (\bar{V}_f \bar{W}_r) \bar{V}_f
\]

(11)

The cylindrical projections of these equations become

\[
\frac{3}{\bar{t}} (p \bar{W}_r) + \frac{1}{r} \frac{3}{\bar{t}} \left( r \rho \bar{W}_r \right) + \frac{1}{r} \frac{3}{\bar{t}} \left( r \rho \bar{W}_r \right) + \frac{3}{\bar{t}} (p \bar{W}_r) = 0
\]

(12)

Fig. 1 Lay-out for an axial compressor

**Averaged equations**

In order to derive exact averaged equations, blade thickness is taken into account as well as blade angles such as the lean angle (angle of the blade with the radial direction).

The circumferential average is defined as the mean value between pressure side of one blade to the suction side of the adjacent blade.

\[
\bar{\bar{A}} = \frac{1}{2 \pi} \int_0^{2\pi} A \, d\theta
\]

(16)

At a radius \(r\), for a row with \(N\) blades

\[
s = 2 \pi r / N
\]

(17)
and defining a tangential blockage factor $b$, through

$$b = 1 - \frac{t}{s}$$

the range of the averaging region is given by

$$\theta_s - \theta_p = \frac{2\pi}{N} b$$

This is clearly seen on figure 2, which is a section of the machine by a plane perpendicular to the axial $z$ direction.

The lean angle $\eta$ is defined by

$$\tan \eta = -r \frac{\partial \phi}{\partial r}$$

while the blade angle is defined by

$$\tan \beta' = r \frac{\partial \phi}{\partial z}$$

$$\begin{align*}
\frac{\partial E}{\partial r} & = \frac{1}{b} \int_{\theta_p}^{\theta_s} \frac{\partial E}{\partial \theta} \, d\theta = \frac{1}{b} \frac{\partial E}{\partial \theta} \bigg|_{\theta_p}^{\theta_s} \\
& = -\frac{1}{2\pi b/N} \left[ \frac{3\theta_b}{3r} - \frac{\partial}{\partial r} \left( \frac{3\theta_b}{3r} \right) \right]
\end{align*}$$

or

$$\frac{\partial E}{\partial r} = \frac{1}{b} \frac{3}{3r} \left( \frac{\partial \phi}{\partial r} \right) - \frac{1}{2\pi b/N} \left[ \frac{3\theta_b}{3r} - \frac{\partial}{\partial r} \left( \frac{3\theta_b}{3r} \right) \right]$$

Fig. 2 Section of a turbomachine

Furthermore, along the mean line $l$ between pressure side and suction side,

$$\theta_p = \theta_s + (1-b) \frac{\pi}{N}$$

$$\theta_s = \theta_s - (1-b) \frac{\pi}{N} + \frac{2\pi}{N}$$

with $b=1$ outside a blade row.

Since the assumption of inviscid flow is considered as valid when the blades are not stalled, the velocity vector $W$ is tangent to the blade. Therefore, one has, by elementary geometrical considerations (as shown by L.H. Smith [3]),

$$\tan \beta' = \tan \beta + \tan \phi \cdot \tan \eta$$

where

$$\tan \phi = \frac{W_r}{W_z}$$

and

$$\tan \beta = \frac{W_r}{W_z}$$

With the definition of equation (16), the following rules for an arbitrary function $g$ will be used

$$\frac{\partial g}{\partial r} = \frac{1}{b} \frac{3}{3r} \left( \frac{\partial \phi}{\partial r} \right) - \frac{1}{2\pi b/N} \left[ \frac{3\theta_b}{3r} - \frac{\partial}{\partial r} \left( \frac{3\theta_b}{3r} \right) \right]$$

or

$$\frac{\partial g}{\partial r} = \frac{1}{b} \frac{3}{3r} \left( \frac{\partial \phi}{\partial r} \right) - \frac{1}{2\pi b/N} \left[ \frac{3\theta_b}{3r} - \frac{\partial}{\partial r} \left( \frac{3\theta_b}{3r} \right) \right]$$

Taking into account the geometrical axisymmetry of the considered flow system,

$$\frac{\partial E}{\partial r} = 0$$

Averaged continuity equation. Application of equations (27) to (30) to the continuity equation (12) gives

$$\frac{1}{br} \frac{\partial}{\partial r} \left( \rho W r \right) + \frac{1}{b} \frac{\partial}{\partial z} \left( \rho W_z \right)$$

$$- \frac{1}{2\pi b/N} \left[ \frac{\partial}{\partial r} \left( \rho W r \right) + \rho W_z \frac{\partial}{\partial z} \left( \rho W_z \right) - \rho \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \right) \right] = 0$$

The last term between square brackets can be written as

$$- \frac{1}{2\pi b/N} \left[ - \rho W_z \tan \phi \cdot \tan \eta + \rho W_z \tan \beta' \right]$$

$$= - \rho W_z \cdot \tan \beta'$$

which is zero from the inviscid slip condition (24). Equivalently, this term can be set to zero, assuming a viscous non-slip condition at the blade walls.

With the definition of the fluctuating part $g'$ of an arbitrary flow function $g$, with respect to its circumferential average $\bar{g}$

$$g = \bar{g} + g' \quad \text{with} \quad \bar{g}' = 0$$

equation (31) becomes

$$\frac{1}{br} \frac{\partial}{\partial r} \left( \rho W r \right) + \frac{1}{b} \frac{\partial}{\partial z} \left( \rho W_z \right)$$

$$= - \frac{1}{br} \frac{\partial}{\partial r} \left( \rho \frac{\partial}{\partial r} \right) - \frac{1}{b} \frac{\partial}{\partial z} \left( \rho \frac{\partial}{\partial z} \right)$$

If it is possible to define a function $B$ through

$$\frac{1}{br} \frac{\partial}{\partial r} \left( \rho \frac{\partial}{\partial r} \right) + \frac{1}{b} \frac{\partial}{\partial z} \left( \rho \frac{\partial}{\partial z} \right) = \rho \left( \frac{\partial}{\partial m} \right)$$

where $\frac{\partial}{\partial m}$ operator is the meridional gradient

$$\frac{\partial}{\partial m} = \frac{\partial}{\partial r} + 

\text{then equation (33) can be written}

$$\frac{1}{r} \frac{\partial}{\partial r} \left( b \rho \frac{\partial}{\partial r} \right) + \frac{1}{b} \frac{\partial}{\partial z} \left( b \frac{\partial}{\partial z} \right) = 0$$

In incompressible flow, $B=1$, since $\rho'=0$. However, when compressibility effects are not negligible, $B$ can be interpreted as a fictitious blockage factor accounting for compressible fluctuating mass transport. This factor can be larger or smaller than one but it is to be expected that the $(\rho-W)$ correlations $(\rho'W'_r)$ and $(\rho'W'_z)$ will be small in most cases, whereby $B=1$ can be considered as a good approximation.
Averaged momentum equation. Detailed derivation of the averaged momentum equations can be found in the appendix. The deviations from exact axisymmetry are described by the N\_i -terms in equations (82), (84) and (85).

The axisymmetry assumption implies that all N\_i -terms are negligible in equations (82), (84), (85) and that B=1 in equation (36).

In particular, equation (84) becomes, under the axi-symmetric flow assumption and outside a blade row,

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = 0 \]  

(37)

showing that r \( \psi \) is constant along a streamline outside a blade row.

The axisymmetric meridional flow is described by equations (36) and (82) with B=1 and N\_i=0. A streamfunction \( \psi \) is introduced in order to satisfy the continuity equation (36)

\[ \frac{\partial \psi}{\partial z} = \rho \frac{r}{r} \frac{\partial W_z}{\partial z} \]  

(38a)

\[ \frac{\partial \psi}{\partial z} = -\rho \frac{r}{r} \frac{\partial W_r}{\partial z} \]  

(38b)

where the overbars are omitted, from now on.

Equation (82) becomes

\[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right) + \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial \psi}{\partial z} \right) = \frac{1}{r} \left( \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) - T \frac{\partial W_z}{\partial r} - \frac{\partial W_r}{\partial r} \right) \]

(41)

which is the basic equation to be solved. In the "matrix" method [7], [6], [12], this equation is written under the form of a Poisson equation for \( \psi \), putting all other terms such as the \( \rho r \)-terms in the right-hand-side. In the finite element method this artificial operation is not necessary, since equation (39) can be solved under its quasi-harmonic form. It is interesting to note that, as shown by C.H. Wu [1], the basic equation (39) can be elliptic or hyperbolic but the limits depend on the way the tangential component \( W_z \) is defined. If the flow angle \( \beta \) is given, equation (39) is elliptic when the relative Mach number is subsonic, while if \( r \psi \) is imposed, equation (39) is elliptic for subsonic meridional velocities.

Extension to radial or mixed flow machines

Equation (39) is particularly adapted when the axial velocity \( W_z \) is large as it is the case in axial flow machines. However in radial or mixed flow machines, this will not be guaranteed and in the radial parts, equation (39) will not be useful, since the axial velocity will be small when not zero. In these situations, the radial components are large and the axial equation (85) can then be considered.

With the introduction of the streamfunction \( \psi \), equation (38a) and (38b), this equation becomes, omitting the overbars and assuming axisymmetry

\[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial \psi}{\partial z} \right) = \frac{1}{r} \left( \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) - \frac{1}{\rho} \frac{\partial P}{\partial r} \right) \]

(40)

The right-hand-sides of equations (39) and (40) are therefore equal and one can choose to compute the first or the second expression according to the respective values of \( W_z \) and \( W_r \). The choice of equation (39) or (40) will depend on which of the velocity components \( W_z \) or \( W_r \) is the largest. Another, more unified way in which both components \( W_z \) and \( W_r \) do not appear in the denominators is proposed in [10].

Taking into account that \( \psi \) is perpendicular to the relative velocity \( \mathbf{V}_{\text{rel}} \) and that \( \psi \) is in the direction of and opposite to \( \mathbf{V}_{\text{rel}} \), the momentum equations (39), (40) and (84) (written in the axisymmetric approximation) are combined in order to obtain the momentum equation in the direction of the vector \( \mathbf{F}_b \wedge \mathbf{V} \). This leads to equation (41):

\[ \frac{3}{3r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{3}{3z} \frac{\partial}{\partial z} \left( r \frac{\partial \psi}{\partial z} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) - \frac{3}{3r} \frac{\partial}{\partial r} \left( r W_z \right) \]

+ \frac{1}{3r} \frac{3}{3z} \left( W_r - W_z \tan \eta \right) - \frac{3}{3z} \frac{\partial}{\partial z} \left( r \psi \right) \cdot \tan \eta \]

(41)

In this equation, the derivative \( \partial \psi / \partial \psi \) is introduced taking advantage from the constancy of I along a streamline. Hence,

\[ \frac{3}{3r} \frac{\partial I}{\partial r} = \frac{\partial I}{\partial \psi} \frac{\partial \psi}{\partial r} \]  

(42)

Equation (41) is seen to be valid in all geometrical situations since, in opposition to equations (39) and (40), neither radial nor axial velocity components appear in the denominators of the r.h.s. terms. A further advantage of this formulation is the absence of any components of blade force or friction force, although these forces are physically present and not neglected.

THE FINITE ELEMENT METHOD

The calculations developed in the previous chapter have lead to an equation of the form:

\[ \frac{3}{3r} \left( k \frac{3\psi}{3z} \right) + \frac{3}{3z} \left( k \frac{3\psi}{3z} \right) + f(r, z) = 0 \]

in the fluid volume \( V \), with boundary conditions

\[ k \frac{3\psi}{3n} + \alpha_1 (\psi - \psi_0) = 0 \]

on the associated exterior surface \( S \).

As it is in general impossible to solve equation (43) exactly, a numerical solution is searched with help of the weighted residual process and the finite element method.

The weighted residual process

Considering equations (43) and (44) written as

\[ \frac{1}{r} \left( k \frac{3\psi}{3z} \right) + \alpha_1 (\psi - \psi_0) = 0 \]

on \( S \) (46)

an approximation \( \psi(r, z) \) of the unknown solution is searched, such that the corresponding weighted residual is zero:

\[ \int_V W(r, z) \cdot P_w (r, z) dV = 0 \]

(47)
where \( W(r,z) \) is the weight (known function); \( R_y \) and \( R_z \) are respectively the volume and surface residuals.

When \( \psi \) is the exact solution of equations (43) and (44), obviously \( R_y = R_z = 0 \) at any point

\[
R_y = -\frac{1}{r} \left[ k \frac{\partial}{\partial r} \left( k \frac{\partial W}{\partial r} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial W}{\partial z} \right) \right]
\]

\( = \) volume residual \hspace{1cm} (48)

\[
R_z = \frac{1}{r} \left[ k \frac{\partial W}{\partial n} + \alpha_1 (\psi - \psi_o) \right]
\]

\( = \) surface residual \hspace{1cm} (49)

Considering a meridional plane, we have with

\( \Omega = \) intersection of \( V \) and the meridional plane

\( C = \) intersection of \( S \) and the meridional plane

and equation (47) becomes

\[
\int_{\Omega} \left[ -W \left( \frac{2}{3} k \frac{\partial W}{\partial r} + \frac{\partial}{\partial z} \left( k \frac{\partial W}{\partial z} + f\right) \right) + \int_C W k \frac{\partial W}{\partial n} + 2 \pi \right] \, d\Omega = 0
\]

Integrating the first term by parts leads to

\[
\int_{\Omega} \left[ k \left( \frac{\partial W}{\partial r} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial W}{\partial z} \right) - f \right] \, W \, d\Omega = 0 \hspace{1cm} (51)
\]

if \( \psi \) is chosen equal to the prescribed value \( \psi_o \) on the corresponding part of the boundary.

This equation (51) will be used with a Galerkin procedure with the weight functions \( W \) chosen equal to the trial functions introduced in the finite element process.

**APPLICATION OF THE FINITE ELEMENT METHOD [13], [14]**

a) The fluid region \( \psi \) is divided into contiguous elements connected at given points (nodes).

b) On each element, the unknown function \( \psi \) is supposed to have a variation of a given form, determined by the nodal values (values of \( \psi \) at the nodal points) and a set of chosen interpolation functions (called trial of shape functions).

\[
\psi = \sum_{i=1}^{n} N_i \psi_i
\]

\( n \) number of nodal points for one element (figure 3: \( n = 8 \))

\( \psi_i \) = nodal value (unknown)

\( N_i \) = nodal shape function (known).

Several types of finite elements are available in the literature [13].

Putting (52) into (51) and taking \( W(r,z) = N_j(r,z) \), we have

\[
\int_E \left\{ k \left[ \frac{\partial N_j}{\partial r} \psi_i + \frac{\partial N_j}{\partial z} \psi_i \right] + \frac{\partial}{\partial z} \left( \frac{\partial N_j}{\partial z} \psi_i \right) \right\} \, d\Omega = 0
\]

\( \text{or} \)

\[
[K]^{e} \{ \phi \}^{e} = \{ f \}^{e}
\]

with

\[
k_{ij}^{e} = \int_E \left[ k \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial r} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right] \, d\Omega
\]

in the following, called "stiffness matrix" (by analogy with stress analysis problems)

and

\[
\alpha_{ij}^{e} = \int_E \left[ \frac{\partial N_i}{\partial r} \frac{\partial N_j}{\partial z} + \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial r} \right] \, d\Omega
\]

equation (54) being written for each element e.

A similar equation is obtained for the entire flow volume by assembling all equations (54)

\[
[K] \{ \phi \} = \{ f \}
\]

**Mesh generation**

The meridional section of the machine is divided into finite elements, which are chosen to be 8-node quadrilateral isoparametric elements. (see figure 3). The corresponding shape functions are therefore bi-quadratic in the local coordinates \( s, n \).

The mesh is generated in the program. Calculation stations are chosen and disposed in the meridional duct section of the machine as well as at the edges and center lines of the blades, (figure 4).

**Fig. 3** Eight-node isoparametric quadrilateral element.

**Fig. 4** Mesh generation

The number of grid points on each station is fixed and the position of the corner nodes is then calculated either with the condition that the mass flow through each element is the same for uniform flow conditions or with an equidistant disposition.

**Solution procedure**

The equations (57) are non-linear and the system has to be solved iteratively using the following procedure.

a) An initial approximation of the streamfunction is assumed (from the assumption of uniform velocity, e.g.).

b) From the given input, all the thermodynamic variables are computed at the grid points on the first inlet station.
c) The thermodynamic variables at the nodal points on station 1 may then be evaluated from the values obtained at the previous station \((i-1)\).

The following relations are used:

\[
\begin{align*}
\frac{\partial H}{\partial m} &= 0 \quad \Rightarrow \quad H = c_p T_t \quad \text{constant along a streamline in a stator} \\
\frac{\partial H_R}{\partial m} &= 0 \quad \Rightarrow \quad H_R = c_p T_{tr} - \frac{\left(\omega R\right)^2}{2} \quad \text{constant along a streamline in a rotor} \\
\frac{\partial (rV_\theta)}{\partial m} &= r \times V_\theta \quad \text{constant along a streamline in a duct region.}
\end{align*}
\]

\(V_\theta\) is the circumferential velocity and \(V_r\) the radial velocity.

Introducing equation (61) in equation (59) an equation is obtained for \(\rho\) which is of the form

\[
\rho = \left(1 - \frac{A}{\rho}\right)^{\gamma-1} \quad (62)
\]

where \(\tilde{\rho} = \rho/\rho_t\)

This equation is solved iteratively at each node. An analysis of equation (62) shows that this equation has always a solution for subsonic velocities relative to the blade when \(tg\ a\) is given. However, when \(V_\theta\) is given this equation has always a solution when the meridional velocity is subsonic. This property allows an extension of the calculation method to supersonic relative velocities.

d) After \(\rho\) has been obtained, the stiffness matrix elements (55) and the vector components (56) are calculated in order to perform the resolution of system (57).

The system of non-linear equations is solved by applying the "constant stiffness" scheme or the "secant stiffness" scheme [15].

In this way, a new solution \(\psi_k\) at all nodes is obtained. However due to the strong non-linearity of the problem an under-relaxation factor \(\mu\) has to be introduced in the form

\[
\psi_k^{(n+1)} = \psi_k^{(n)} + \mu \left(\psi_k^{(n+1)} - \psi_k^{(n)}\right) \quad (64)
\]

or

\[
\delta_k^{(n+1)} = \delta_k^{(n)} + \mu \left(\delta_k^{(n+1)} - \delta_k^{(n)}\right) \quad (65)
\]

where \(\psi_k^{(n+1)}\) is considered as the solution of the \((n+1)\)-th iteration and used to start the next iteration until convergence is obtained.

APPLICATIONS TO AXIAL COMPRESSORS

The method developed in the previous sections has been programmed and applied to various axial flow compressors.

**Task-1 stage [17]**

This single stage transonic compressor has been designed and tested by NASA at the General Electric Company. The rotor (code number 1B) has a design tip speed of \(426.7\) m/s (1400 ft/s), an inlet hub-tip radius ratio of 0.5, a tip solidity of 1.3, aspect ratio of 2.5 and constant chord. The blades are of the multiple circular arc-type. The stator has an aspect ratio of 2.065 and a hub solidity of 2.155. The stator vane sections are of double circular arc-type at the outlet part of the blade blended into arbitrary shaped hub sections especially designed for low suction surface Mach number. The stator chord is not constant. Figure 6 shows the geometry of the compressor where the dotted lines are the boundary of the elements and the continuous lines are the computed streamlines at design speed and mass flow rate of 95.25 kg/s (210 lbm/sec).
Figure 6: TASK-I compressor geometry with element boundaries and computed streamlines at design speed and mass flow rate of 95.25 kg/s (210 lbm/sec).

Figure 7: TASK-I compressor: calculated and experimental relative Mach numbers at design speed.

Figure 8(a): TASK-I compressor: rotor axial velocity distribution.

Figure 8(b): TASK-I compressor: stator axial velocity distribution.

Figure 8(c): TASK-I compressor: absolute flow angle distribution at rotor outlet.

Figure 7 shows a comparison between calculated results and experimental data of the relative Mach number radial variation. Figure 8(a) and 8(b) show the axial velocity variations compared with experimental results, while figure 8(c) is a plot of calculated and experimental absolute rotor outlet angles. The agreement is very satisfactory.

Two-stage highly loaded axial flow fan [18]

This highly loaded two-stage axial flow fan was
Fig. 9  Two-stage fan geometry with element boundaries and calculated streamlines at design speed and mass flow rate of 83.56 kg/s.

Fig. 10  Two-stage fan: relative Mach number and outlet angle variations at 70% of design speed at first rotor outlet. O, X are experimental values respectively at rotor inlet and outlet.

Fig. 11  Two-stage fan: deviations, incidence and loss coefficient distributions at second stator outlet at 70% of design speed. O: experimental incidence, X: experimental deviations and losses.
designed by NASA for 442 m/s (1450 ft/s) tip speed and pressure ratio of 2.8 and design mass flow rate of 83.56 kg/s (184.2 lbm/s).

The tip diameter at the first rotor is 0.787 m (31 in.) with hub-tip ratio at inlet of 0.4. Blades in both rotors and stators are multiple-circular-arc airfoils. The first rotor has 28 blades of 2.48 average aspect ratio, a hub chord of 0.092 m and a tip chord of 0.116 m (4.55 in.).

The second rotor has 60 blades of constant chord of 0.053 m (2.1 in.) and an average aspect ratio of 2.69. The first stator has 46 blades with chords ranging from 0.07 m (2.75 in.) at hub to 0.079 m (3.10 in.) at tip, the average aspect ratio is equal to 2.75. The second stator has 59 blades, chords from 0.056 m (2.22 in.) at hub to 0.062 m (2.45 in.) at tip and 2.20 average aspect ratio.

Figure 9 shows the geometry of the compressor together with the finite element geometry as well as the computed streamlines at design speed. Examples of calculated results are shown in the following figures and compared with experimental data.

Figure 10 shows the relative Mach number and relative angle variations at first rotor outlet while figures 11 and 12 show flow variations at the second stator outlet at 70% of design speed. Figure 11 shows the calculated deviations and loss coefficients from the correlation subroutine. The differences with respect to the experimental values explains the observed differences in figure 12 for flow angles and meridional velocity distributions compared to the experimental data.

The comparison with experimental results shows generally an excellent agreement for flow parameter variations when the deviations and losses calculated from the correlations are in agreement with the experimental values. This is best illustrated by figures 13 to 15 for the first rotor at design speed, where very good agreement is observed up to a tip Mach number of 1.5. Figure 16, at second rotor outlet shows also good agreement, although at this stage the calculated losses were not in good agreement with experimental data, but the calculated deviations were fairly good.

The present results seem to show that the calculation method is quite accurate and reliable and that the discrepancies with experimental values are due either to incorrect losses and deviations or to viscous effects.

9.4 cm diameter six-stage axial compressor [19]

This 9.4 cm (3.7 in.) diameter six-stage axial flow compressor developed for a 10 kW Brayton cycle space electrical power generation system has been tested in Argon at 0.47 bar inlet pressure. The design tip speed is 297.5 m/s (975 ft/s) at first rotor inlet with corresponding relative Mach number of .788. The hub-tip ratio at rotor 1 inlet is .69 and increases to .73 at stator 6 exit. Hub diameter is constant.

Figure 17 shows a comparison between calculated and experimental performance with fairly good agreement. No experimental traverses are reported but a comparison of the calculated flow variable distributions with design variations show very similar trends.

Radial axisymmetric bend

In order to show the versatility of the method with respect to geometrical boundaries, a calculation of a radial bend has been performed. Figure 18 shows the finite element distributions and the calculated streamlines.

Fig. 12 Two-stage fan: flow angle and meridional velocity variations at second stator outlet at 70% of design speed. O: experimental inlet values, X: experimental outlet values.

CONCLUSIONS

The radial equilibrium equations for an arbitrary turbomachinery has been derived by an exact pitch-averaging process, and the usual equation is obtained by assuming axisymmetry.

A new method of solution is presented based on the finite element method which appears to be reliable and accurate. This method is very versatile with respect to geometrical conditions (like curved boundaries) which can be handled in a straightforward way, and with respect to the calculation mesh which does not need to be regular. This method seems to offer a valid alternative to the other methods actually used in practice.

REFERENCES


Fig. 13 Two-stage fan: deviations, incidences and losses at first rotor at design speed.

- O: experimental incidence, X: experimental deviations and losses.


Fig. 14 Two-stage fan: absolute flow angles and meridional velocity distributions at first rotor outlet at design speed. O: experimental values at inlet, X: experimental values at outlet.


Fig. 15 Two-stage fan: relative Mach number and flow angles distribution at first rotor outlet at design speed. O: experimental values at inlet, X: experimental values at outlet.

Fig. 16 Two-stage fan: relative Mach number and flow angles distribution at second rotor outlet at design speed. O: experimental values at inlet, X: experimental values at outlet.

Fig. 17 Total pressure ratio in function of mass flow rate for a 9.4 cm (3.7 in.) diameter six-stage axial compressor.

APPENDIX

Averaged momentum equation

The radial component equation (13) is first considered. The average of the radial pressure gradient can be written in the following form, instead of (27), by using equations (22), (23) and (17)

$$
\frac{dP}{dr} = \frac{2}{r} \bar{p} + t_g \eta_g \frac{P_e - P_s}{bs} + \left[ \bar{p} - \frac{(P_e^2 + P_s^2)}{2} \right] \frac{1}{3} \frac{d}{dr} \tag{66}
$$
Introducing in this way the tangential blade force \( F_{b,\theta} \) defined by
\[
F_{b,\theta} = \frac{P_p - P_p}{ds}
\]  
and the radial blade force component \( F_r \) along the blade mean line \( L \)
\[
F_{b,r} = F_{b,\theta} \tan \eta_k
\]  
Averaging equation (13) in the circumferential direction, taking into account the inviscid slip condition (24), and the decomposition (32), leads to
\[
\frac{1}{br} \frac{\partial}{\partial r} (\overline{W_r} \overline{W_r} b r) + \frac{1}{b} \frac{\partial}{\partial z} (\overline{W_r} \overline{W_z} b) - \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} = - \rho \frac{\partial^2 \overline{W_r}}{\partial r^2} - 2 \omega \rho \overline{W_z} + \overline{W_r} \frac{\partial^2 \overline{W_z}}{\partial z^2} + \overline{W_z} \frac{\partial^2 \overline{W_r}}{\partial z^2} + \overline{W_r} \rho \overline{W_r} \overline{W_z} - \rho \overline{W_z} \overline{W_z}
\]  
where \( \overline{W_k} \) is the sum of the fluctuation averages of the mean radial equilibrium equation (69)
\[
\overline{W_k} = \frac{1}{br} \frac{\partial}{\partial r} (\overline{W_r} \overline{W_r} b r) + \frac{1}{b} \frac{\partial}{\partial z} (\overline{W_r} \overline{W_z} b) - \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} - \overline{W_r} \rho \overline{W_r} \overline{W_z} - \rho \overline{W_z} \overline{W_z}
\]  
Introducing equation (33) multiplied by \( \overline{W_r} \), equation (69) becomes
\[
\overline{W_r} \frac{\partial \overline{W_r}}{\partial r} + \overline{W_z} \frac{\partial \overline{W_z}}{\partial z} = \frac{1}{br} \frac{\partial}{\partial r} (\overline{W_r} \overline{W_r} b r) + \frac{1}{b} \frac{\partial}{\partial z} (\overline{W_r} \overline{W_z} b) - \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} - \overline{W_r} \rho \overline{W_r} \overline{W_z} - \rho \overline{W_z} \overline{W_z}
\]  
where a fluctuation function \( N_{10}^{(r)} \) has been added to the right-hand-side
\[
N_{10}^{(r)} = - \frac{1}{br} \frac{\partial}{\partial r} (\overline{W_r} \overline{W_r} b r) - \frac{1}{br} \frac{\partial}{\partial z} (\overline{W_r} \overline{W_z} b) - \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} - \overline{W_r} \rho \overline{W_r} \overline{W_z} - \rho \overline{W_z} \overline{W_z}
\]  
The circumferentially-averaged radial equation (75) can also be written under another form by introducing the energy equation (3) in the radial direction, after averaging. Applying a decomposition similar to (66), the averaged equation
\[
- \frac{3 \rho}{3r} = \rho \overline{T} \frac{\partial \overline{W_z}}{\partial r} - \rho \frac{\partial h}{\partial r}
\]  
becomes
\[
- \frac{3 \rho}{3r} = \rho \overline{T} \frac{\partial \overline{W_z}}{\partial r} - \rho \frac{\partial h}{\partial r} + \frac{s}{2} \frac{\partial x}{\partial r} \frac{\partial h}{\partial r} \frac{3b}{3r}
\]  
Comparing with equation (66), the tangential force could be set equal to the term in \( \tan \eta_k \)
\[
F_{b,\theta} = \overline{W_z} (s - p) - (h - p) \frac{1}{2br/N} + N_{10}^{(r)}
\]  
and the fluctuation term \( \overline{W_z} \) replaced by
\[
N_{10}^{(r)} = - \overline{W_z} (s - p) - \frac{1}{2br/N} \frac{3b}{3r} \frac{\partial x}{\partial r} \frac{\partial h}{\partial r}
\]  
Combining equation (75) with equation (78) and rearranging terms, leads finally to
\[
\frac{1}{br} \frac{\partial}{\partial r} (\overline{W_r} \overline{W_r} b r) + \frac{1}{b} \frac{\partial}{\partial z} (\overline{W_r} \overline{W_z} b) - \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} - \overline{W_r} \rho \overline{W_r} \overline{W_z} - \rho \overline{W_z} \overline{W_z}
\]  
with \( \overline{W_z} \) given by equation (81), and \( \overline{W_z} \) defined by
\[
\overline{W_z} = \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} = \overline{W_r} \frac{\partial \overline{W_z}}{\partial z} = \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} + \overline{W_z} \frac{\partial \overline{W_r}}{\partial z}
\]  
The tangential projection of the momentum equation (14) can be circumferentially-averaged in the same way, leading to
\[
\rho \overline{W_r} \frac{\partial \overline{W_r}}{\partial r} + \rho \overline{W_z} \frac{\partial \overline{W_z}}{\partial z} = F_{b,\theta} + \overline{W_r} \overline{W_z} - \rho \frac{\partial x}{\partial r} \frac{\partial h}{\partial r} \frac{3b}{3r}
\]  
and the axial equation becomes, after averaging
\[
\overline{W_r} \frac{\partial \overline{W_z}}{\partial r} = \overline{W_r} \frac{\partial \overline{W_z}}{\partial z} \frac{3b}{3r} \frac{\partial h}{\partial r} - \rho \overline{W_z} \frac{\partial \overline{W_z}}{\partial z} = \overline{W_r} \frac{\partial \overline{W_z}}{\partial r} + \overline{W_z} \frac{\partial \overline{W_r}}{\partial z}
\]  
An important relation can be obtained for the blade force \( \overline{F_b} \)
\[
\overline{F_b} = F_{b,\theta} \tan \eta_k \overline{T} = \overline{F_r} \tan \eta_k \overline{T} = \rho \frac{\partial x}{\partial r} \frac{\partial h}{\partial r} \frac{3b}{3r}
\]  
namely
\[
\overline{F_b} = \overline{F_r} = 0
\]  
from equations (2k) → (26).