Some comparison results for finite-time ruin probabilities in the classical risk model

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Abstract

This paper aims at showing how an ordering on claim amounts can influence finite-time ruin probabilities. Until now such a question was examined essentially for ultimate ruin probabilities. Over a finite horizon, a general approach does not seem possible but the study is conducted under different sets of conditions. This primarily covers the cases where the initial reserve is null or large.

Keywords: Ordering of risks, ruin probabilities, convex type orders, asymptotic orders.

1 Introduction

The evaluation of ruin probabilities strongly depends on the distribution of the claim amounts. Given two claim distributions, it is natural to ask which one implies larger values of ruin probabilities. This topic has been investigated for a long time (see e.g. the book of Goovaerts et al. (1990)). Most often, the attention is focused on ruin over an infinite time horizon. Intuitively, one expects that a more variable claim amount increases the ultimate ruin probability. Such a result was first proved by Michel (1987) for the classical risk model using the convex order of claim sizes. Other stochastic orderings have been considered to tackle different situations or model assumptions. A nice paper by Klüppelberg (1993) uses asymptotic orders to compare ruin probabilities when initial reserves are large, for light- and heavy-tailed claim distributions.

To the best of our knowledge, the influence of claim sizes on finite-time ruin probabilities has been very little studied so far. A notable exception is the paper by De Vylder and Goovaerts (1984) who obtained some interesting results for the compound Poisson risk model. In particular, they showed that contrary to the infinite time case, a more dangerous claim amount in the convex order sense does not necessarily imply larger ruin probabilities over finite-time horizons.

The present paper deals also with the classical risk model. Claims occur according to a Poisson process $\{N_t, t \geq 0\}$ of rate $\lambda > 0$ and claim amounts $\{X_i, i \geq 1\}$ that are independent and identically distributed (i.i.d.) positive random variables (distributed as X) with distribution function F and mean μ. So, the aggregate claim amount up to time t is $S_t = \sum_{i=1}^{N_t} X_i$. The company has an initial reserve of level $u \geq 0$ and receives premium at a constant rate c. The probability of ruin (resp. non-ruin) before time $t \geq 0$ is denoted by $\psi(u,t)$ (resp. $\phi(u,t)$). A positive safety loading factor η is defined by writing $c = \lambda\mu(1+\eta)$. It guarantees that the ruin probability over an infinite time horizon, $\psi(u) = 1 - \phi(u)$, is less than 1 and tends to 0 as $u \to \infty$. For an overview of ruin theory, see e.g. the books of Dickson (2005) and Asmussen and Albrecher (2010).

Our aim here is to go further in the analysis of the possible influence of the claim amounts on the finite-time ruin probabilities. A simple unifying approach does not seem possible and the problem will be examined under several sets of conditions. As a mathematical tool, we will use different well-known stochastic orderings. Much of the theory on stochastic orders can be found e.g. in the books of Denuit et al.(2006), Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). Furthermore, we will also use several asymptotic orders, less standard, that were introduced by Klüppelberg (1993).

The paper is organized as follows. In Section 2, we bring some complements to the analysis made by De Vylder and Goovaerts (1984). These concern the special cases where the initial reserve is null. In Section 3, we obtain a comparison result for the stop-loss transform of ruin probabilities. Such a result gives a partial perspective on the comparison of the probabilities themselves. In Section 4, we establish an asymptotic comparison of ruin probabilities as the initial reserve is large. Our study is directly inspired from the approach of Klüppelberg (1993) for the case of an infinite time horizon. In Section 5, we derive a comparison result for the time dependent Lundberg coefficient. This enables us to discuss the situation where the initial reserve and the time horizon are both large.

2 Null initial reserve

We begin by recalling the definitions of some stochastic orders that will be useful in the paper. The reader is referred e.g. to Shaked and Shantikumar (2007), denoted as S-S in the following.

Let $X^{(1)}$ and $X^{(2)}$ be two non-negative random variables with distribution functions $F_1 = 1 - \overline{F}_1$ and $F_2 = 1 - \overline{F}_2$ and finite means μ_1 and μ_2, respectively. One says that $X^{(1)}$ precedes $X^{(2)}$ in the usual stochastic order, denoted as $X^{(1)} \preceq_{\mathrm{st}} X^{(2)}$, if

$$\overline{F}_1(x) \leq \overline{F}_2(x) \quad \text{for all } x \geq 0.$$

The latter is also equivalent to the inequality $E[g(X^{(1)})] \leq E[g(X^{(2)})]$ for any non-decreasing function g such that the expectations exist.

The stochastic order compares the sizes of the risks. On the other hand, the convex order focuses on their variabilities and allows us to compare two risks with identical means. One says that $X^{(1)}$ precedes $X^{(2)}$ in the convex order, denoted as $X^{(1)} \preceq_{\mathrm{cx}} X^{(2)}$, when $\mu_1 = \mu_2$ and

$$\int_x^\infty \overline{F}_1(u)\, du \leq \int_x^\infty \overline{F}_2(u) du \quad \text{for all } x \geq 0. \tag{2.1}$$

The latter is also equivalent to the inequality

$$E[(X^{(1)} - x)_+] \leq E[(X^{(2)} - x)_+] \quad \text{for all } x \geq 0,$$

where, for any real r, r_+ denotes the positive part of r (i.e. $r_+ = r$ if $r \geq 0$ and $r_+ = 0$ if $r < 0$). Equivalently, $X^{(1)} \preceq_{\mathrm{cx}} X^{(2)}$ if and only if $\mu_1 = \mu_2$ and $E[h(X^{(1)})] \leq E[h(X^{(2)})]$ for all convex functions $h : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist.

Only random variables with the same means can be compared by the convex order. The Lorenz order and the increasing convex order combine the aspects of size (as $\preceq_{\mathrm{st}}$) and variability (as $\preceq_{\mathrm{cx}}$). One says that $X^{(1)}$ is smaller than $X^{(2)}$ in the Lorenz order, denoted as $X^{(1)} \preceq_{\mathrm{Lorenz}} X^{(2)}$, when

$$\frac{X^{(1)}}{\mu_1} \preceq_{\mathrm{cx}} \frac{X^{(2)}}{\mu_2}.$$

$X^{(1)}$ is said to be smaller than $X^{(2)}$ in the increasing convex order, denoted as $X^{(1)} \preceq_{\mathrm{icx}} X^{(2)}$, when (2.1) holds true. Equivalently, $X^{(1)} \preceq_{\mathrm{icx}} X^{(2)}$ if and only if $E[h(X^{(1)})] \leq E[h(X^{(2)})]$ for all non-decreasing convex functions $h : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist. Obviously, when $\mu_1 = \mu_2$, the increasing convex order is equivalent to the convex order. The increasing convex order is also named the stop-loss order as $E[(X - x)_+]$ is the expected reinsurance payment under a stop-loss reinsurance treaty with retention x.

Similarly, an increasing concave order, denoted as $X^{(1)} \preceq_{\mathrm{icv}} X^{(2)}$, is defined by requiring $E[h(X^{(1)})] \leq E[h(X^{(2)})]$ for all non-decreasing concave functions $h : \mathbb{R}^+ \to \mathbb{R}$, provided the expectations exist. This condition is equivalent to

$$E[(X^{(1)} - x)_-] \geq E[(X^{(2)} - x)_-] \quad \text{for all } x \geq 0,$$

where, for any real r, r_- denotes the negative part of r (i.e. $r_- = -r$ if $r \leq 0$ and $r_- = 0$ if $r > 0$).

Now, let us compare the finite-time ruin probability for risk models with no initial reserves ($u = 0$). An index j or a superscript (j), $j = 1, 2$, will be added in the notation to distinguish the models. When $u = 0$, it is well-known that the non-ruin probability before time t is simply given by

$$\phi(0,t) = \frac{1}{ct} E\left[(ct - S_t)_+\right] \quad \text{(i.e. also } \frac{1}{ct} E\left[(S_t - ct)_-\right]\text{).} \tag{2.2}$$

This last equation is often referred as the Takàcs formula (see Takács (1967)). From this result, we can establish the following proposition.

Proposition 1. *If $\lambda_1 \leq \lambda_2$ and $X^{(1)}/c_1 \preceq_{icv} X^{(2)}/c_2$, then $\psi_1(0,t) \leq \psi_2(0,t)$ for all $t > 0$.*

Proof. Since $\lambda_1 \leq \lambda_2$, we have $N_t^{(1)} \preceq_{\text{st}} N_t^{(2)}$ and hence $N_t^{(1)} \preceq_{\text{icv}} N_t^{(2)}$ (see Theorem 4.A.34 in S-S). Furthermore, since $X^{(1)}/c_1 \preceq_{\text{icv}} X^{(2)}/c_2$, we get $S_t^{(1)}/c_1 \preceq_{\text{icv}} S_t^{(2)}/c_2$ (see Theorem 4.A.9 in S-S). So, by definition of the increasing concave order, we obtain

$$E[(S_t^{(1)}/c_1 - t)_-] \geq E[(S_t^{(2)}/c_2 - t)_-],$$

and hence $\psi_1(0,t) \leq \psi_2(0,t)$ by virtue of (2.2). □

We note that the condition $X^{(1)}/c_1 \preceq_{\text{icv}} X^{(2)}/c_2$ implies $\mu_1/c_1 \leq \mu_2/c_2$. In particular, if $\eta_1 = \eta_2$, then $\lambda_1 \geq \lambda_2$. In the case where $\mu_1/c_1 = \mu_2/c_2$, $X^{(1)}/c_1 \preceq_{\text{icv}} X^{(2)}/c_2$ is equivalent to $X^{(2)}/c_2 \preceq_{\text{cx}} X^{(1)}/c_1$ (see Theorem 4.A.35 in S-S).

Proposition 1 is a slight generalization of Theorem 4 in De Vylder and Goovaerts (1984) which states that $\psi_1(0,t) \leq \psi_2(0,t)$ if $c_1 = c_2, \lambda_1 = \lambda_2$ and $X^{(2)} \preceq_{\text{cx}} X^{(1)}$. Indeed, these conditions directly imply $X^{(1)}/c_1 \preceq_{\text{icv}} X^{(2)}/c_2$.

Corollary 1. *If $\lambda_1 = \lambda_2$, $\eta_1 = \eta_2$ and $X^{(1)} \preceq_{Lorenz} X^{(2)}$, then $\psi_1(0,t) \geq \psi_2(0,t)$ for all $t > 0$. In addition, if $\mu_1 = \mu_2$ holds too, the ordering condition becomes $X^{(1)} \preceq_{cx} X^{(2)}$.*

Proof. Obviously, the conditions $\lambda_1 = \lambda_2$, $\eta_1 = \eta_2$ and $X^{(1)} \preceq_{\text{Lorenz}} X^{(2)}$ yield $X^{(1)}/c_1 \preceq_{\text{cx}} X^{(2)}/c_2$. This implies $X^{(2)}/c_2 \preceq_{\text{icv}} X^{(1)}/c_1$ and Proposition 1 then gives $\psi_1(0,t) \geq \psi_2(0,t)$ for all $t > 0$. In the particular case where $\mu_1 = \mu_2$, we get $c_1 = c_2$ and the result follows. □

As mentioned in the introduction, Michel (1987) proved that the more a claim size is variable, the more the ultimate ruin probability is large. Such an implication does not hold over a finite-time horizon. Indeed, when $u = 0$, Corollary 1 shows that, on the contrary, the more a claim size is variable, the more the finite-time ruin probability is small. This seems a priori counter-intuitive but a possible explanation is as follows. When $u = 0$, the ruin, if it occurs, is very likely to happen early, at a time when the reserve is still small in comparison with the mean μ of a claim. Now, if the claim size is more variable, it has a significant chance to be smaller than μ and not to cause ruin. If it is larger than μ, this will not much influence the risk of ruin because ruin will be mainly due to the mean μ.

Example 1. In the case where $X^{(1)}$ and $X^{(2)}$ are exponentially distributed, it comes

$$\begin{aligned} E[(X^{(j)}/c_j - x)_-] &= \frac{1}{\mu_j} \int_0^{c_j x} e^{-y/\mu_j} \left(x - y/c_j\right) dy \\ &= -\frac{\mu_j}{c_j} \left(1 - e^{-c_j x/\mu_j}\right) + x, \quad j = 1, 2. \end{aligned}$$

Thus, the condition $X^{(1)}/c_1 \preceq_{\text{icv}} X^{(2)}/c_2$ of Proposition 1 becomes

$$-\frac{\mu_1}{c_1}\left(1-e^{-c_1x/\mu_1}\right)+x \geq -\frac{\mu_2}{c_2}\left(1-e^{-c_2x/\mu_2}\right)+x \quad \text{for all } x \geq 0.$$

Since $\theta\left(1-e^{-x/\theta}\right)$ is increasing in θ, this condition is satisfied when

$$\frac{\mu_1}{c_1} \leq \frac{\mu_2}{c_2},$$

i.e. when $\lambda_1(1+\eta_1) \leq \lambda_2(1+\eta_2)$. In the particular case where $\lambda_1 = \lambda_2$ and $\eta_1 = \eta_2$, it is interesting to notice that $X^{(1)}/c_1 \preceq_{\text{icv}} X^{(2)}/c_2$ and $X^{(2)}/c_1 \preceq_{\text{icv}} X^{(1)}/c_2$, which means that $\psi_1(0,t) = \psi_2(0,t)$ for all t.

3 Stop-loss transform

Let us consider the maximum aggregate loss up to time t, denoted M_t, defined by

$$M_t = \sup_{0\leq s\leq t}\left\{\sum_{i=1}^{N_s} X_i - cs\right\} = \max_{0\leq k\leq N_t}\left\{\sum_{i=1}^{k}(X_i - cT_i)\right\},$$

where T_i denotes the time between the $(i-1)$-th and i-th claim arrivals. Of course, the T_i's are independent exponential random variables of parameter λ. The ruin probability $\psi(u,t)$ can be expressed as the survival function of M_t, i.e.

$$\psi(u,t) = P(M_t > u).$$

The stop-loss transform of the ruin probability $\psi(u,t)$ is defined by

$$SLT(u,t) = \int_u^\infty \psi(v,t)dv. \tag{3.1}$$

This concept (3.1) was used by Robert (2014) for a different problem. It was also discussed for $\psi(u)$ by Cheng and Pai (2003).

In this section, we consider two risk models with respective ruin probabilities $\psi_1(u,t)$ and $\psi_2(u,t)$ and we aim to compare the corresponding stop-loss transforms $SLT_1(u,t)$ and $SLT_2(u,t)$. From a practical viewpoint, the stop-loss transform of $\psi(u,t)$ can be interpreted as the expected maximal deficit encountered by the insurer up to time t. Indeed, a simple integration by parts enables us to write

$$SLT(u,t) = E\left[(M_t - u)_+\right]. \tag{3.2}$$

Hence, while $\psi(u,t)$ focuses on the risk that (at least) one liquidity issue occurs before time t, $SLT(u,t)$ brings information about the risk faced by the insurance company once the business at stake encounters liquidity problems. From another perspective, the stop-loss transform of $\psi(u,t)$ can also be seen as a key risk indicator for a bank lending money to the insurance company as it represents the expected amount of money the bank will have to inject in the insurance business to ensure its viability.

To compare the stop-loss transforms $SLT_1(u,t)$ and $SLT_2(u,t)$, we will need the following lemma, which is a slight generalization of Theorem 4.A.20 in S-S.

Lemma 1. *Consider two families of distribution functions* $\{G_\theta^{(1)}, \theta \in \chi\}$ *and* $\{G_\theta^{(2)}, \theta \in \chi\}$, *where* χ *is an convex subset of* $\mathbb{R}$ *or* $\mathbb{N}$. *Let* $X_1(\theta)$ *and* $X_2(\theta)$ *denote two random variables with distribution functions* $G_\theta^{(1)}$ *and* $G_\theta^{(2)}$, *respectively. Let* Θ_1 *and* Θ_2 *be two random variables with support* χ *and such that*

$$\Theta_1 \preceq_{st} \Theta_2. \tag{3.3}$$

Denote $Y_1 = X_1(\Theta_1)$ *and* $Y_2 = X_2(\Theta_2)$. *If*

$$X_1(\theta) \preceq_{icx} X_2(\theta) \quad \text{for all } \theta \in \chi, \tag{3.4}$$

and if for every non-decreasing convex function h,

$$E\left[h(X_2(\theta))\right] \text{ is increasing in } \theta, \tag{3.5}$$

then

$$Y_1 \preceq_{icx} Y_2.$$

Proof. Let h be a non-decreasing convex function exist, and for $\theta \in \chi$, denote

$$g_j(\theta) = E[h(X_j(\theta))], \quad j = 1, 2.$$

By (3.4), we have $g_1(\theta) \leq g_2(\theta)$ so that

$$E[h(Y_1)] = E[g_1(\Theta_1)] \leq E[g_2(\Theta_1)].$$

Moreover, (3.3) and (3.5) directly yield

$$E[g_2(\Theta_1)] \leq E[g_2(\Theta_2)] = E[h(Y_2)].$$

Thus, we obtain $E[h(Y_1)] \leq E[h(Y_2)]$ which is the announced ordering. □

We note that if the two families of distribution functions $\{G_\theta^{(1)}, \theta \in \chi\}$ and $\{G_\theta^{(2)}, \theta \in \chi\}$ are identical, Lemma 1 amounts to Theorem 4.A.20 in S-S. We are now in a position to prove our main comparison result.

Proposition 2. *If* $\lambda_1 \leq \lambda_2$, $c_1 \geq c_2$ *and* $X^{(1)} \preceq_{icx} X^{(2)}$, *then* $M_t^{(1)} \preceq_{icx} M_t^{(2)}$ *for all* $t > 0$, *i.e.*

$$SLT_1(u,t) \leq SLT_2(u,t) \quad \text{for all } t > 0 \text{ and } u \geq 0.$$

Proof. To begin with, in both models, we suppose that the number of claims up to time t is fixed and equal to n, and that the interarrival times are also fixed and equal to $t_i > 0$, $1 \leq i \leq n$ (with $t_1 + \ldots + t_n < t$). Then, for $j = 1, 2$, we consider the conditional deficits per period

$$L_i^{(j)}(n, t_1, \ldots, t_n) = [X_i^{(j)} - c_j T_i^{(j)} | N_t^{(j)} = n, T_1^{(j)} = t_1, \ldots, T_n^{(j)} = t_n], \quad 1 \leq i \leq n,$$

with $L_0^{(j)}(n, t_1, \ldots, t_n) = 0$. Clearly, since $X^{(1)} \preceq_{\text{icx}} X^{(2)}$, we have

$$L_i^{(1)}(n, t_1, \ldots, t_n) \preceq_{\text{icx}} L_i^{(2)}(n, t_1, \ldots, t_n).$$

Let us now introduce the associated Lindley processes $\{Z_i^{(j)}(n, t_1, \ldots, t_n),\ 0 \leq i \leq n\}$ defined as

$$Z_{i+1}^{(j)}(n, t_1, \ldots, t_n) = [Z_i^{(j)}(n, t_1, \ldots, t_n) + L_{n-i}^{(j)}(n, t_1, \ldots, t_n)]_+, \quad 0 \leq i \leq n-1, \tag{3.6}$$

with $Z_0^{(j)}(n, t_1, \ldots, t_n) = 0$. From Theorem 3.1, Chapter III, of Asmussen and Albrecher (2010), we know that

$$\begin{aligned} Z_n^{(j)}(n, t_1, \ldots, t_n) &=_{st} \max_{0 \leq k \leq n} \left\{ \sum_{i=0}^{k} L_i^{(j)}(n, t_1, \ldots, t_n) \right\} \\ &= [M_t^{(j)} | N_t^{(j)} = n, T_1^{(j)} = t_1, \ldots, T_n^{(j)} = t_n]. \end{aligned} \tag{3.7}$$

We want to prove that

$$Z_i^{(1)}(n, t_1, \ldots, t_n) \preceq_{\text{icx}} Z_i^{(2)}(n, t_1, \ldots, t_n), \quad 0 \leq i \leq n. \tag{3.8}$$

This is obvious for $i = 0$. Proceeding by recurrence, we suppose that (3.8) holds for some $i \leq n-1$. Since $X^{(1)} \preceq_{\text{icx}} X^{(2)}$ and, for each j, the $X_i^{(j)}$'s are independent, we obtain (see Theorem 4.A.8. in S-S)

$$Z_i^{(1)}(n, t_1, \ldots, t_n) + L_{n-i}^{(1)}(n, t_1, \ldots, t_n) \preceq_{\text{icx}} Z_i^{(2)}(n, t_1, \ldots, t_n) + L_{n-i}^{(2)}(n, t_1, \ldots, t_n). \tag{3.9}$$

Note that if f is an increasing convex function, then the function $x \mapsto f(x^+)$ is also increasing convex. From (3.6) and (3.9), we deduce that (3.8) holds for $i+1$ substituted for i. In particular, taking $i = n$ and using (3.7), we get

$$[M_t^{(1)} | N_t^{(1)} = n, T_1^{(j)} = t_1, \ldots, T_n^{(j)} = t_n] \preceq_{\text{icx}} [M_t^{(2)} | N_t^{(2)} = n, T_1^{(j)} = t_1, \ldots, T_n^{(j)} = t_n]. \tag{3.10}$$

Now, for a Poisson property, conditionally to $N_t = n$, the n successive arrival times are known to be distributed as the order statistics of n uniform random variables on $(0, t)$. This implies that the vectors $(T_1, \ldots, T_n)$ are identically distributed in both models. Thus, we may remove the condition on $(T_1, \ldots, T_n)$ in the ordering (3.10), i.e.

$$[M_t^{(1)} | N_t^{(1)} = n] \preceq_{\text{icx}} [M_t^{(2)} | N_t^{(2)} = n].$$

For each j, $[M_t^{(j)} | N_t = n]$ is stochastically increasing in n. Moreover, as $\lambda_1 \leq \lambda_2$, we have $N_t^{(1)} \preceq_{\text{st}} N_t^{(2)}$. So, Lemma 1 is applicable and yields $M_t^{(1)} \preceq_{\text{icx}} M_t^{(2)}$, as desired. □

This last result is not so surprising, and somehow intuitive, in light of the interpretations given in (3.2) for the stop-loss transforms $SLT_1(u, t)$ and $SLT_2(u, t)$.

Let us notice that Proposition 2 directly implies

$$\limsup_{u \to \infty} \frac{\psi_2(u, t)}{\psi_1(u, t)} \geq 1. \tag{3.11}$$

This observation brings us in a natural way to the next section where we consider a large initial reserve.

Remark 1. In their discussion, De Vylder and Goovaerts (1984) considered the regions in the (u,t)-plane where $\psi_1 = \psi_2$, $\psi_1 < \psi_2$ and $\psi_1 > \psi_2$. They suggested that the region $\psi_1 = \psi_2$ could be a simple curve separating the two other regions. This does not seem to be always true, however. Indeed, when t is small, a first order approximation for $\psi(u,t)$ is

$$\psi(u,t) \sim \lambda t[1 - F(u)] \quad \text{as} \quad t \to 0,$$

so that

$$\psi_2(u,t) - \psi_1(u,t) \sim \lambda t[F_1(u) - F_2(u)] \quad \text{as} \quad t \to 0.$$

Thus, if F_1 crosses F_2 more than once, then $\psi_1 = \psi_2$ does not give a unique curve.

4 Large initial reserve

In this section, we will discuss the comparison problem of two risk models when the initial reserve is large ($u \to \infty$). As before, our interest is focused on the finite-time ruin probabilities.

For ruin over an infinite horizon, Klüppelberg (1993) obtained elegant results when the claim size distributions belong to a class, $\mathscr{S}(\gamma)$, of light-tailed distributions for which the Lundberg coefficient does not exist. Other classes of distributions were discussed too. Here, we will consider again the class $\mathscr{S}(\gamma)$ and, before this, a more general (known) class, $\mathscr{L}(\gamma)$, which contains long tailed and an exponential like tailed distributions.

4.1 For claim distributions belonging to $\mathscr{L}(\gamma)$

We first introduce briefly a special class of real functions, $\tilde{\mathscr{L}}(\gamma)$, and an associated class of distributions functions, $\mathscr{L}(\gamma)$. More details can be found in Klüppelberg (1989), (1990) and Tang and Wei (2010).

Definition 1. Let $\gamma \geq 0$. A function $f : [0,\infty) \to [0,\infty)$ belongs to the class $\tilde{\mathscr{L}}(\gamma)$ if $f(x) > 0$ on $[a,\infty)$ for some positive real a and

$$\lim_{x\to\infty} \frac{f(x-y)}{f(x)} = e^{\gamma y}. \tag{4.1}$$

The convergence of the limit in (4.1) is uniform on compact intervals and

$$\gamma = -\lim_{x\to\infty} \frac{\ln f(x)}{x}. \tag{4.2}$$

The class $\tilde{\mathscr{L}}(\gamma)$ is closely related to the class of regularly varying functions with exponent $-\gamma$, denoted $\mathscr{R}(-\gamma)$. Indeed, $f \in \tilde{\mathscr{L}}(\gamma)$ if and only if $f \circ \ln \in \mathscr{R}(-\gamma)$.

Definition 2. A distribution function F belongs to $\mathscr{L}(\gamma)$ if $\overline{F}$ belongs to $\tilde{\mathscr{L}}(\gamma)$.

If a density function f belongs to $\tilde{\mathscr{L}}(\gamma)$, then the corresponding distribution function F belongs to $\mathscr{L}(\gamma)$ and hence (4.1) holds for both f and $\overline{F}$. Moreover, the moment generating function of F, i.e.

$$\hat{f}(s) = \int_0^\infty e^{sx} dF(x),$$

is finite for $s \in [0, \gamma)$ and infinite for $s \in]\gamma, \infty)$, with $\hat{f}(\gamma)$ finite or not.

When $\gamma > 0$, the distributions in the class $\mathscr{L}(\gamma)$ have an exponential like tail. When $\gamma = 0$, the distributions in $\mathscr{L}(0)$ are called long tailed distributions. An example of distributions in $\mathscr{L}(\gamma)$ are the phase-type distributions. A subclass of $\mathscr{L}(0)$ is the well-known subexponential family of distributions.

Proposition 3. *If $F_1 \in \mathscr{L}(\gamma_1)$ and $F_2 \in \mathscr{L}(\gamma_2)$ with $\gamma_1 > \gamma_2 \geq 0$, then*

$$\lim_{u\to\infty} \frac{\psi_1(u,t)}{\psi_2(u,t)} = 0. \tag{4.3}$$

Proof. Obviously, we have

$$P[S_t^{(j)} > u + c_j t] \leq \psi_j(u,t) \leq P[S_t^{(j)} > u], \quad j = 1, 2.$$

Thus, to prove (4.3), it suffices to show that

$$\lim_{u\to\infty} \frac{P[S_t^{(1)} > u]}{P[S_t^{(2)} > u + c_2 t]} = 0. \tag{4.4}$$

Applying the Chernoff bound to the numerator in (4.3), we obtain

$$0 \leq \frac{P[S_t^{(1)} > u]}{P[S_t^{(2)} > u + c_2 t]} \leq \frac{e^{-ru} e^{\lambda_1 (\hat{f}_1(r) - 1)}}{p_1^{(2)} \overline{F}_2(u + c_2 t)} \quad \text{for all } r > 0 \tag{4.5}$$

where $p_1^{(2)} = P(N_t^{(2)} = 1)$. Now, let $\epsilon > 0$ such that $2\epsilon < \gamma_1 - \gamma_2$ and choose $r = \gamma_1 - \epsilon$. We use Lemma 4.1 of Tang and Wei (2010) to bound the denominator in (4.5). So, there exist $u_0 \geq 0$ and $C > 0$ such that for all $u > u_0$,

$$0 \leq \frac{P[S_t^{(1)} > u]}{P[S_t^{(2)} > u + c_2 t]} \leq \frac{e^{-(\gamma_1 - \epsilon)u} e^{\lambda_1 (\hat{f}_1(\gamma_1 - \epsilon) - 1)}}{p_1^{(2)} C e^{-(\gamma_2 + \epsilon)(u + c_2 t)}}. \tag{4.6}$$

As $u \to \infty$, the right hand-side of (4.6) tends to 0, hence (4.4) and (4.3). □

Roughly speaking, when the asymptotic tail function of the claim amounts decreases exponentially more strongly ($\gamma_1 > \gamma_2$), then the ruin probabilities tend to 0 more quickly as $u \to \infty$.

Example 2. Let us consider two gamma distributions $F_1 \sim \Gamma(\alpha_1, \beta_1)$ and $F_2 \sim \Gamma(\alpha_2, \beta_2)$ with parameters $\alpha_1, \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Obviously, we have $F_1 \in \mathscr{L}(\beta_1)$ and $F_2 \in \mathscr{L}(\beta_2)$. Thus, Proposition 3 is applicable and yields the limit result (4.3).

4.2 For convolution equivalent tail claim distributions

Following the same references as above, we now introduce a subclass of functions, $\tilde{\mathscr{S}}(\gamma)$, and an associated class of distribution functions, $\mathscr{S}(\gamma)$.

Definition 3. Let $\gamma \geq 0$. A function $f : [0, \infty) \to [0, \infty)$ belongs to the class $\tilde{\mathscr{S}}(\gamma)$ if $f \in \tilde{\mathscr{L}}(\gamma)$ and

$$\lim_{x \to \infty} \frac{f^{*2}(x)}{f(x)} = 2d < \infty. \tag{4.7}$$

Definition 4. A distribution function F belongs to $\mathscr{S}(\gamma)$ if $\overline{F}$ belongs to $\tilde{\mathscr{S}}(\gamma)$.

If a density function f belongs to $\tilde{\mathscr{S}}(\gamma)$, then the corresponding distribution function F belongs to $\mathscr{S}(\gamma)$ and (4.7) also holds for $\overline{F}$. The distributions in the class $\mathscr{S}(\gamma)$ are called convolution equivalent tail distributions. In particular, $\mathscr{S}(0)$ is the class of subexponential distributions (in that case, $d = 1$). Also, for $F \in \mathscr{S}(\gamma)$, it is known that $\hat{f}(\gamma) < \infty$ and $d = \hat{f}(\gamma)$. For instance, the inverse Gaussian distribution $F \sim IG(\mu, \beta)$ with density function

$$f(x) = \sqrt{\frac{\beta}{2\pi x^3}} e^{-\frac{\beta(x-\mu)^2}{2\mu^2 x}} \tag{4.8}$$

belongs to the class $\mathscr{S}(\gamma)$ with $\gamma = \beta/2\mu^2$ and $d = e^{\beta/\mu}$. Also, if we define $g(x) = e^{-\gamma x} f(x)$ with $\gamma > 0$, then $f \in \tilde{\mathscr{S}}(0)$ if and only if $g \in \tilde{\mathscr{S}}(\gamma)$.

First, we recall an asymptotic approximation obtained by Klüppelberg (1993) for the ultimate ruin probability.

Proposition 4 (Klüppelberg (1993))**.** *If $F \in \mathscr{S}(\gamma)$ with $\gamma > 0$ and $\lambda d < \lambda + \gamma c$, then*

$$\psi(u) \sim \frac{\lambda(c - \lambda\mu)}{\gamma c^2} \left(1 - \frac{\lambda}{\gamma c}(d-1)\right)^{-2} \overline{F}(u) \quad \text{as} \quad u \to \infty. \tag{4.9}$$

We are going to show that a result of similar form holds too for the finite-time ruin probabilities.

Proposition 5. *If $f \in \tilde{\mathscr{S}}(\gamma)$ is bounded, then*

$$\psi(u, t) \sim C(t)\overline{F}(u) \quad \text{as} \quad u \to \infty, \tag{4.10}$$

where

$$C(t) = \lambda t e^{((d-1)\lambda - \gamma c)t} + \gamma c \int_0^t \phi(0, t-\tau) \lambda \tau e^{((d-1)\lambda - \gamma c)\tau} d\tau. \tag{4.11}$$

Proof. By the standard Seal formula, we have

$$\psi(u, t) = \overline{F}(u + ct, t) + \int_0^t \phi(0, t-\tau) f(u + c\tau, \tau) c d\tau, \tag{4.12}$$

where $\overline{F}(., t)$ is the survival function of $S_t = \sum_{i=1}^{N_t} X_i$ and $f(., t)$ is the associated density. From Theorem 2.13 in Cline (1987) and the condition (4.1), we have

$$\overline{F}(u + ct, t) \sim e^{-\gamma ct} E\left(N_t d^{N_t - 1}\right) \overline{F}(u) = \lambda t e^{((d-1)\lambda - \gamma c)t} \overline{F}(u) \quad \text{as} \quad u \to \infty, \tag{4.13}$$

and from Theorem 3.2 of Klüppelberg (1989)

$$f(u+ct,t) \sim E\left(N_t d^{N_t-1}\right) f(u+ct) = \lambda t e^{(d-1)\lambda t} f(u+ct) \quad \text{as} \quad u \to \infty. \tag{4.14}$$

Moreover, from Lemma 3.1 of Klüppelberg (1989), there exists for all $\epsilon > 0$ a constant $K_\epsilon > 0$ such that

$$\frac{f(u+c\tau,\tau)}{\overline{F}(u)} \leq K_\epsilon E\left[(d+\epsilon)^{N_\tau}\right] \frac{f(u+c\tau)}{\overline{F}(u)}. \tag{4.15}$$

Note that this upper bound is integrable on $[0,t]$ for large u since by Lemma 4.4 in Tang and Wei (2010),

$$\lim_{u\to\infty} \frac{f(u+c\tau)}{\overline{F}(u)} = \lim_{u\to\infty} \frac{f(u+c\tau)}{f(u)} \frac{f(u)}{\overline{F}(u)} = \gamma e^{-\gamma c \tau}, \tag{4.16}$$

and the convergence is uniform on compact intervals of τ. Thus, the dominated convergence theorem applies and from (4.14) and (4.16), we get

$$\begin{aligned} \lim_{u\to\infty} \frac{\int_0^t \phi(0,t-\tau) f(u+c\tau,\tau) c d\tau}{\overline{F}(u)} &= \int_0^t \phi(0,t-\tau) \lim_{u\to\infty} \frac{f(u+c\tau,\tau)}{\overline{F}(u)} c d\tau \\ &= \gamma c \int_0^t \phi(0,t-\tau) \lambda \tau e^{((d-1)\lambda - \gamma c)\tau} d\tau. \end{aligned} \tag{4.17}$$

The result (4.10) then follows from (4.12), (4.13) and (4.17). □

We note that in the particular case where $f \in \tilde{\mathscr{S}}(0)$, (4.11) simplifies to $C(t) = \lambda t$ and (4.10) amounts to a well-known result for the subexponential distributions (see Chapter X.4 in Asmussen and Albrecher (2010)). Also, Proposition 5 is consistent with Kluppelberg's formula (4.9) since when $\lambda d < \lambda + \gamma c$, (4.10) becomes (4.9) as $t \to \infty$.

Let us now consider two different risk models based on densities f_1 and f_2.

Corollary 2. *If $f_1, f_2 \in \tilde{\mathscr{S}}(\gamma)$ are bounded and $\lim_{u\to\infty}\left[\overline{F}_2(u)/\overline{F}_1(u)\right] = a$, then*

$$\lim_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} = \frac{C_2(t)}{C_1(t)} a \tag{4.18}$$

where $C_j(t)$ $(j = 1,2)$ are defined as in (4.11). On another hand, if $f_1, f_2 \in \tilde{\mathscr{S}}(0)$ and $\liminf_{u\to\infty}\left[\overline{F}_2(u)/\overline{F}_1(u)\right] \geq \lambda_1/\lambda_2$ (resp. $\liminf_{u\to\infty}\left[\overline{F}_2(u)/\overline{F}_1(u)\right] > \lambda_1/\lambda_2$), then

$$\liminf_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} \geq 1 \qquad \left(\textit{resp.} \ \liminf_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} > 1\right). \tag{4.19}$$

Proof. From Proposition 5, we directly obtain (4.18) since

$$\lim_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} = \lim_{u\to\infty} \frac{\psi_2(u,t)}{\overline{F_2}(u)} \frac{\overline{F_1}(u)}{\psi_1(u,t)} \frac{\overline{F_2}(u)}{\overline{F_1}(u)} = \frac{C_2(t)}{C_1(t)} a.$$

If $f_1, f_2 \in \tilde{\mathscr{S}}(0)$, then $C_1(t) = \lambda_1 t$, $C_2(t) = \lambda_2 t$ and we see that

$$\liminf_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} = \lim_{u\to\infty} \frac{\psi_2(u,t)}{\overline{F_2}(u)} \frac{\overline{F_1}(u)}{\psi_1(u,t)} \liminf_{u\to\infty} \frac{\overline{F_2}(u)}{\overline{F_1}(u)} = \frac{\lambda_2}{\lambda_1} \liminf_{u\to\infty} \frac{\overline{F_2}(u)}{\overline{F_1}(u)},$$

which gives (4.19) when $\liminf_{u\to\infty}\left[\overline{F}_2(u)/\overline{F}_1(u)\right] \geq \lambda_1/\lambda_2$ (resp. $\liminf_{u\to\infty}\left[\overline{F}_2(u)/\overline{F}_1(u)\right] > \lambda_1/\lambda_2$). □

Let us mention that Klüppelberg (1993) worked in terms of a weak tail order $\leq^w$ defined by

$$X^{(1)} \leq^w X^{(2)} \Leftrightarrow \overline{F}_1(x)/\overline{F}_2(x) \leq a_w < \infty \quad \text{for all } x \geq 0.$$

In that framework, Corollary 2 directly implies that if $f_1, f_2 \in \tilde{\mathscr{S}}(\gamma)$ and $X^{(1)} \leq^w X^{(2)}$, then $M_t^{(1)} \leq^w M_t^{(2)}$ for all $t > 0$.

When $\lambda_1 \leq \lambda_2$, $c_1 \geq c_2$ and $X^{(1)} \preceq_{\text{icx}} X^{(2)}$, we have already seen in (3.11) that $\limsup_{u\to\infty} \psi_2(u,t)/\psi_1(u,t) \geq 1$. Combining Propositions 1 and 5, we can now give additional conditions to also have $\liminf_{u\to\infty} \psi_2(u,t)/\psi_1(u,t) \geq 1$.

Corollary 3. *If $f_1, f_2 \in \tilde{\mathscr{S}}(\gamma)$, with $\gamma > 0$, are bounded and $\liminf_{u\to\infty} \left[\overline{F}_2(u)/\overline{F}_1(u)\right] \geq 1$ (resp. $\liminf_{u\to\infty} \left[\overline{F}_2(u)/\overline{F}_1(u)\right] > 1$), and if $\lambda_1 \leq \lambda_2$, $c_1 \geq c_2$ and $X^{(1)} \preceq_{icx} X^{(2)}$, then* (4.19) *holds true.*

Proof. Let us assume that $\lambda_1 = \lambda_2$, $c_1 = c_2$ and $X^{(1)} \preceq_{\text{cx}} X^{(2)}$. The extension to the more general assumptions $\lambda_1 \leq \lambda_2$, $c_1 \geq c_2$ and $X^{(1)} \preceq_{\text{icx}} X^{(2)}$ is obvious. From Proposition 1, as $\lambda_1 = \lambda_2$ and $c_1 = c_2$, $X^{(1)} \preceq_{\text{cx}} X^{(2)}$ implies $\phi_1(0,t) \leq \phi_2(0,t)$. Furthermore, since $X^{(1)} \preceq_{\text{cx}} X^{(2)}$, we also have $d_1 = \hat{f}_1(\gamma) \leq \hat{f}_2(\gamma) = d_2$. Therefore, we get from (4.11) that $C_1(t) \leq C_2(t)$. Using this inequality, it is easily shown that (4.19) still holds true if $f_1, f_2 \in \tilde{S}(\gamma)$ with $\gamma > 0$ and $\liminf_{u\to\infty} \left[\overline{F}_2(u)/\overline{F}_1(u)\right] \geq 1$ (resp. $\liminf_{u\to\infty} \left[\overline{F}_2(u)/\overline{F}_1(u)\right] > 1$). □

Remark 2. In the particular case where $\lambda_1 = \lambda_2$, $c_1 = c_2$ and $X^{(1)} \preceq_{\text{cx}} X^{(2)}$, suppose that $E[(X^{(1)} - x)_+] < E[(X^{(2)} - x)_+]$ for all $x \in (0, a)$ with $a > 0$ and $X^{(1)}$ (or $X^{(2)}$) defined on some finite interval $[0, b]$ with $b > 0$. This assumption is often fulfilled in practice. Then, De Vylder and Goovaerts (1984) proved that the strict inequality $\phi_1(0,t) < \phi_2(0,t)$ holds true for all $t > 0$. In that case, the condition $\liminf_{u\to\infty} \left[\overline{F}_2(u)/\overline{F}_1(u)\right] \geq 1$ in Corollary 3 suffices to get the stronger result $\liminf_{u\to\infty} [\psi_2(u,t)/\psi_1(u,t)] > 1$.

Proposition 6. *If $F_1, F_2 \in \mathscr{S}(\gamma)$ and*

$$\frac{\lambda_1}{\lambda_2} \limsup_{x\to\infty} \frac{\overline{F}_1(x)}{\overline{F}_2(x)} < e^{((d_2-1)\lambda_2-(d_1-1)\lambda_1-\gamma c_2)t}, \tag{4.20}$$

then

$$\liminf_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} > 1. \tag{4.21}$$

Proof. As $F_1, F_2 \in \mathscr{S}(\gamma)$, Theorem 2.13 of Cline (1987) gives the approximations

$$P[S_t^{(1)} > u] \sim E\big(N_t^{(1)} d_1^{N_t^{(1)}-1}\big)\overline{F}_1(u) = \lambda_1 t e^{(d_1-1)\lambda_1 t}\overline{F}_1(u), \tag{4.22}$$

and

$$P[S_t^{(2)} > u + c_2 t] \sim e^{-\gamma c_2 t}\lambda_2 t e^{(d_2-1)\lambda_2 t}\overline{F}_2(u) \quad \text{as} \quad u \to \infty. \tag{4.23}$$

Using the assumption (4.20), we get from (4.22) and (4.23) that

$$\limsup_{u\to\infty} \frac{P[S_t^{(1)} > u]}{P[S_t^{(2)} > u + ct]} < 1. \tag{4.24}$$

Now, we have

$$P[S_t^{(j)} > u + ct] \leq \psi_j(u,t) \leq P[S_t^{(j)} > u], \quad j = 1, 2,$$

so that

$$\frac{\psi_1(u,t)}{\psi_2(u,t)} \leq \frac{P[S_t^{(1)} > u]}{P[S_t^{(2)} > u + ct]}. \tag{4.25}$$

From (4.24) and (4.25), we then deduce the result (4.21). $\square$

Example 3. Let f_1 and f_2 be two inverse Gaussian densities (see (4.8)) with parameters β_1, μ_1 and β_2, μ_2 such that $\mu_1 > \mu_2$ and $\gamma_1 = \gamma_2 = \gamma$ (i.e. $\beta_1/2\mu_1^2 = \beta_2/2\mu_2^2$). In that case, we easily see that

$$\lim_{x\to\infty} \frac{\overline{F}_1(x)}{\overline{F}_2(x)} = \frac{\mu_1 d_1}{\mu_2 d_2} > 1, \tag{4.26}$$

where $d_1 = e^{2\gamma\mu_1}$ and $d_2 = e^{2\gamma\mu_2}$. Thus, applying Proposition 6, we can assert that

$$\frac{\lambda_1}{\lambda_2}\frac{\mu_1 d_1}{\mu_2 d_2} < e^{((d_2-1)\lambda_2-(d_1-1)\lambda_1-\gamma c_2)t} \Rightarrow \liminf_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} > 1, \tag{4.27}$$

and

$$\frac{\lambda_1}{\lambda_2}\frac{\mu_1 d_1}{\mu_2 d_2} > e^{((d_2-1)\lambda_2-(d_1-1)\lambda_1+\gamma c_1)t} \Rightarrow \liminf_{u\to\infty} \frac{\psi_1(u,t)}{\psi_2(u,t)} > 1. \tag{4.28}$$

Now, suppose that λ_1 and λ_2 satisfy the condition

$$\frac{\mu_1 d_1}{\mu_2 d_2} > \frac{\lambda_2}{\lambda_1} > \frac{d_1-1}{d_2-1}. \tag{4.29}$$

We note that $\mu_1 > \mu_2$ ensures that $\mu_1 d_1/\mu_2 d_2 > (d_1-1)/(d_2-1)$. From (4.29), we have $\lambda_1\mu_1 d_1 > \lambda_2\mu_2 d_2$ and $(d_2-1)\lambda_2 - (d_1-1)\lambda_1 + \gamma c_1 > 0$ for any $c_1 \geq 0$. Moreover, let us choose $c_2 \geq 0$ that satisfies the condition

$$(d_2-1)\lambda_2 - (d_1-1)\lambda_1 - \gamma c_2 > 0. \tag{4.30}$$

From (4.27), and using (4.29), (4.30), we deduce that

$$\liminf_{u\to\infty} \frac{\psi_2(u,t)}{\psi_1(u,t)} > 1 \text{ for } t > \tau_2,$$

where

$$\tau_2 = \frac{\log(\lambda_1\mu_1 d_1/\lambda_2\mu_2 d_2)}{(d_2-1)\lambda_2 - (d_1-1)\lambda_1 - \gamma c_2},$$

while from (4.28) and (4.29),

$$\liminf_{u\to\infty} \frac{\psi_1(u,t)}{\psi_2(u,t)} > 1 \text{ for } 0 < t < \tau_1,$$

where

$$\tau_1 = \frac{\log(\lambda_1\mu_1 d_1/\lambda_2\mu_2 d_2)}{(d_2-1)\lambda_2 - (d_1-1)\lambda_1 + \gamma c_1}.$$

5 Large initial reserve and horizon

A classical result for the compound Poisson risk model is the Lundberg inequality

$$\psi(u) \leq e^{-\rho u} \quad \text{for all} \quad u \geq 0, \tag{5.1}$$

and the Cramér-Lundberg approximation

$$\psi(u) \sim Ce^{-\rho u} \quad \text{as} \quad u \to \infty, \tag{5.2}$$

where ρ, called the adjustment coefficient, is the positive solution (if it exists) of the equation

$$\kappa(\rho) = 0 \qquad \text{where} \qquad \kappa(s) = \lambda \hat{f}(s) - \lambda - cs. \tag{5.3}$$

The coefficient ρ exists in most cases where the claim amounts are light-tailed. For the class $\mathscr{L}(\gamma)$, this is true if $d = \hat{f}(\gamma) = \infty$ with γ given by (4.2). For the class $\mathscr{S}(\gamma)$, then $d < \infty$ by (4.7) and ρ exists if $\lambda d > \lambda + c\gamma$.

The problem of ordering of adjustment coefficient has been discussed by several authors. In particular, the following result can be easily proved (see e.g. van Heerwaarden (1991)).

Proposition 7. *If $\lambda_1 = \lambda_2$, $c_1 = c_2$ and $X^{(1)} \preceq_{icx} X^{(2)}$, then $\rho_1 \geq \rho_2$.*

A time-dependent version of Lundberg's inequality is due to Gerber (1973). More precisely, for $y > 0$, let α_y be solution of

$$\kappa'(\alpha_y) = 1/y, \tag{5.4}$$

and define a time dependent coefficient ρ_y by

$$\rho_y = \alpha_y - y\kappa(\alpha_y). \tag{5.5}$$

Let us also denote

$$m_L = 1/\kappa'(\rho). \tag{5.6}$$

Obviously, we have $\rho_{m_L} = \rho$. The quantity $m_L u$ can be seen as the most likely time of ruin because ruin time $T(u) \to m_L u$ in probability as $u \to \infty$. Then, over an horizon t proportional to the initial reserve u, i.e. $t = yu$, Gerber (1973) proved that

$$\begin{aligned} \psi(u, yu) &\leq e^{-\rho_y u} \quad \text{if} \quad y < m_L, \\ \psi(u) - \psi(u, yu) &\leq e^{-\rho_y u} \quad \text{if} \quad y > m_L. \end{aligned} \tag{5.7}$$

In view of (5.1) and (5.7), ρ_y is called the time-dependent adjustment coefficient. The inequalities (5.7) are strengthened by the approximations of Arfwedson (1955), namely as $u \to \infty$,

$$\begin{aligned} \psi(u, yu) &\sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y |\tilde{\alpha}_y| \sqrt{2\pi y \lambda \hat{f}''(\alpha_y)}} . \frac{e^{-\rho_y u}}{\sqrt{u}}, \quad \text{if} \quad y < m_L, \\ \psi(u) - \psi(u, yu) &\sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y \tilde{\alpha}_y \sqrt{2\pi y \lambda \hat{f}''(\alpha_y)}} . \frac{e^{-\rho_y u}}{\sqrt{u}}, \quad \text{if} \quad y > m_L, \end{aligned} \tag{5.8}$$

where $\tilde{\alpha}_y < \alpha_y$ is the solution of $\kappa(\tilde{\alpha}) = \kappa(\alpha_y)$. Moreover, as $u \to \infty$,

$$\frac{\psi(u, yu)}{\psi(u)} \to \begin{cases} 0 & \text{if} \quad y < m_L, \\ 1 & \text{if} \quad y > m_L. \end{cases} \tag{5.9}$$

For a review of these results, see Chapters IV and V of Asmussen and Albrecher (2000).

Now, let us establish a comparison result for the time-dependent adjustment coefficient ρ_y. The conditions will be the same as in Proposition 7.

Proposition 8. *If $\lambda_1 = \lambda_2$, $c_1 = c_2$ and $X^{(1)} \preceq_{icx} X^{(2)}$, then $\rho_y^{(1)} \geq \rho_y^{(2)}$ for all $y > 0$.*

Proof. Since $X^{(1)} \preceq_{\rm icx} X^{(2)}$, we have $\hat{f}_1(r) \leq \hat{f}_2(r)$ and $\hat{f}_1{}'(r) \leq \hat{f}_2{}'(r)$, $r \geq 0$. From (5.3) and as $\lambda_1 = \lambda_2$ and $c_1 = c_2$, this implies that $\kappa_1(r) \leq \kappa_2(r)$, $\kappa_1'(r) \leq \kappa_2'(r)$. Moreover, $\kappa_j''(r) = \lambda_j \hat{f}_j{}''(r) \geq 0$, $j = 1, 2$. From the definition (5.4) of $\alpha_y^{(j)}$, we then deduce that $\alpha_y^{(1)} \geq \alpha_y^{(2)}$. In addition, we get, by convexity of κ_1,

$$\kappa_2(\alpha_y^{(2)}) \geq \kappa_1(\alpha_y^{(2)}) \geq \kappa_1(\alpha_y^{(1)}) + \frac{1}{y}(\alpha_y^{(2)} - \alpha_y^{(1)}),$$

so that

$$\kappa_1(\alpha_y^{(1)}) \leq \kappa_2(\alpha_y^{(2)}) + \frac{1}{y}(\alpha_y^{(1)} - \alpha_y^{(2)}). \tag{5.10}$$

From (5.4) and using (5.5), we deduce that

$$\begin{aligned} \rho_y^{(1)} &= \alpha_y^{(1)} - y\kappa_1(\alpha_y^{(1)}) \\ &\geq \alpha_y^{(1)} - y\kappa_2(\alpha_y^{(2)}) - \alpha_y^{(1)} + \alpha_y^{(2)} = \rho_y^{(2)}, \end{aligned}$$

as announced. □

We note that in Propositions 7 et 8, if $\hat{f}_1(r) < \hat{f}_2(r)$ (instead of $\leq$), then $\rho_1 > \rho_2$ and $\rho_y^{(1)} > \rho_y^{(2)}$ for all $y > 0$.

Example 4. Consider two exponential survival functions $\overline{F}_1(x) = e^{-\delta_1 x}$ and $\overline{F}_2(x) = e^{-\delta_2 x}$ where $\delta_1 > \delta_2 > 0$. Then, we have $X^{(1)} \preceq_{\rm icx} X^{(2)}$. The quantities $m_L^{(j)}$ defined by (5.6) are given by

$$m_L^{(j)} = \frac{\lambda_j}{c_j^2 \delta_j - c_j \lambda_j}, \quad j = 1, 2;$$

see Chapter V of Asmussen and Albrecher (2010). Suppose that $\lambda_1 = \lambda_2$ and $c_1 = c_2$. We then have $m_L^{(1)} < m_L^{(2)}$. Moreover, Propositions 7 and 8 yield $\rho_1 > \rho_2$ and $\rho_y^{(1)} > \rho_y^{(2)}$ for all $y > 0$.

Let us compare the asymptotic ruin probabilities. We are going to show that if $y \neq m_L^{(j)}, j = 1, 2$, then

$$\lim_{u \to \infty} \frac{\psi_1(u, yu)}{\psi_2(u, yu)} = 0.$$

Remember that $m_L^{(1)} < m_L^{(2)}$. Now, from (5.8), (5.9) and (5.2), $\psi_j(u, yu)$ and $\psi_j(u)$, $j = 1, 2$, have an exponential decay. For $0 < y < m_L^{(1)}$, the exponent in $\psi_j(u, yu)$ is $\rho_y^{(j)}$ and we know that $\rho_y^{(1)} > \rho_y^{(2)}$. For $y > m_L^{(2)}$, the exponent in $\psi_j(u, yu)$ is ρ_j and we know that $\rho_1 > \rho_2$. For $m_L^{(1)} < y < m_L^{(2)}$, the exponents in $\psi_1(u, yu)$ and $\psi_2(u, yu)$ are ρ_1 and $\rho_y^{(2)}$ respectively. It remains to prove that $\rho_1 > \rho_y^{(2)}$. From Proposition 8, $\rho_1 = \rho^{(1)}_{m_L^{(1)}} > \rho^{(2)}_{m_L^{(1)}}$. As κ_j is a convex function, $\rho_y^{(j)}$ is a decreasing function for $y < m_L^{(j)}$, so that $\rho^{(2)}_{m_L^{(1)}} \geq \rho_y^{(2)}$ when $m_L^{(1)} < y < m_L^{(2)}$. The desired inequality follows.

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