PhD Thesis

# Unconventional Supersymmetry, Massless Rarita-Schwinger Theory and Strained Graphene

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*To the memory of my grandmother*
*Todavía te extraño, Abuela...*

# Acknowledgments

Even if it is true people in general commit too many injustices by writing the Acknowledgment Section in a PhD Thesis, I am afraid that in my case this sin is worse. Perhaps, for that reason, I wait to the very last moment to write it. There are many people that from one manner or another help me in this achievement.

I would like to thank in the first place, my two advisors. Prof. Jorge Zanelli from Centro de Estudios Científicos (CECs) in Chile, for his patience and for sharing his wisdom, curiosity, and passion about Physics. Thanks also for the supporting and the encouraging. He reassures me that we can do Physics enjoying like a child. Prof. Marc Henneaux from Université Libre de Bruxelles (ULB), for his patience, for the very enlightened discussions with him and the encouraging. I also appreciate a lot the academic freedom he gave me during my enrollment at ULB. Fundamentally, both of them teach me that a very good scientist can also be a very good person.

My gratitude is also with Prof. Alfredo Iorio from Charles University in Prague, which besides I consider him like my third Advisor regarding what I learned from him, he gave me the invaluable lesson of how to take risks and jump to new areas in physics with passion and scientific rigour. To the other CECs Professors, Fabrizio Canfora, for his friendship and for allowing me to participate in his exciting scientific projects. Prof. Ricardo Troncoso and Cristian Martinez for their respective very good courses in my doctoral formation. To Prof. Glenn Barnich from ULB for giving me support and confidence.

Of course, I will like to express all my gratitude to my girlfriend and partner Carolina Vera, for her love and support during all this path. She shined me with her eyes through this enterprise.

To my Chilean PhD student colleagues and friends, Patricio Salgado-Rebolledo, for his friendship and scientific exchange, Marcela Cardenas, and Oscar Fuentealba for hours of study with fun (or vice-versa). To my Uruguayan colleague and friend Prof. Marcela Peláez for sharing her knowledge. To my former advisor Prof. Pablo Mora for all the support he gave me from the beginning, which was a long time ago. Finally, but not less important, to my family in Uruguay for their emotional support in several difficult moments during my career, especially when one is so far from home.

During this work, I was supported by Fondecyt Grant 1140155 and also from the Czech Science Foundation (GAČR), under contract no. 14-07983S. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

# Abstract

In this Thesis, we propose to analyze three different aspects of Fundamental Physics.

The first part is devoted to the detailed study of what is called *unconventional supersymmetry* in three and four dimensions for Abelian and non-Abelian internal groups. We show the dynamical content of the odd-dimensional theory, counting at the same time the local degrees of freedom for some particular sectors of the phase space. In the non-Abelian three-dimensional case, some black hole solutions are presented, including their Killing spinors. In four dimensions, the supersymmetry is broken explicitly and a standard Dirac Lagrangian coupled with the electromagnetic field and the background geometry is obtained.

In the second part, the dynamical content for the free and gauge coupled massless Rarita-Schwinger theory is presented. We are able to do that through the Dirac's Hamiltonian formalism and the Faddeev-Jackiw method, showing at the same time the symmetries of the theory. It is shown that in the gauge extended theory, which includes extra fermionic fields to restore the fermionic symmetries of the free case, the anticommutator of the Rarita-Schwinger field in the canonical quantization is not positive definite in general.

As the graphene has been proposed as an on "table-top laboratory" for some Quantum Gravity scenarios, in the third part of this Thesis we clarify some subtle features of strained graphene in order to manage properly this material. We show particularly that the pseudo-magnetic field induced by the in-plane strain tensor field cannot emerge from a Quantum Field Theory in curved spacetime approach (bottom-up approach) but from the detailed analysis of the tight-binding Hamiltonian of $\pi$ electrons in graphene (top-down approach) instead.

# Resumé

Dans cette Thèse, nous nous proposons d'analyser trois aspects différents de la Physique Fondamentale.

La première partie est consacrée à l'étude détaillée de ce qu'on appelle *supersymétrie non conventionnelle* à trois et quatre dimensions pour des groupes internes abéliens et non abéliens. Nous montrons le contenu dynamique de la théorie de la dimension impaire, comptant en même temps les degrés de liberté locaux pour certains secteurs particuliers de l'espace des phases. Dans le cas tridimensionnel non-abélien, certaines solutions de trous noirs sont présentées, y compris leurs spinors de Killing. En quatre dimensions, la supersymétrie est brisée explicitement et un Lagrangien de Dirac standard couplé à l'électromagnétisme et à la géométrie d'arriére-plan est obtenu.

Dans la deuxième partie, le contenu dynamique de la théorie de Rarita-Schwinger libre et couplée à un champ de jauge sans masse est présenté. Nous sommes en mesure de le faire par le formalisme Hamiltonien de Dirac et la méthode dite de Faddeev-Jackiw, en montrant en même temps les symétries de la théorie. Il est démontré que dans la théorie étendue de jauge, qui comprend des champs fermioniques supplémentaires pour restaurer les symétries fermioniques du cas libre, l'anticommutator du champ Rarita-Schwinger dans la quantification canonique n'est pas définiti positif en général.

Comme le graphène a été proposé comme un "laboratoire de table" pour certains scénarios de gravité quantique, dans la troisième partie de cette Thèse, nous clarifions certaines caractéristiques subtiles du graphène sous tension afin de gérer correctement ce matériel. Nous montrons en particulier que le champ pseudo-magnétique induit par le champ tensoriel de déformation dans le plan ne peut pas émerger d'une théorie de champ quantique dans un espace courbe (approche bottom-up), mais bien à partir de l'analyse détaillée de l'Hamiltonien tight-binding des $\pi$ électrons dans le graphène (approche top-down).

# Resumen

En esta Tesis se propone analizar tres aspectos diferentes de la Física Fundamental.

La primera parte está dedicada al estudio detallado de lo que ha pasado a llamarse *supersimetría no convencional* en tres y cuatro dimensiones para grupos internos abelianos y no abelianos. Se muestra el contenido dinámico en dimensiones impares de la teoría, contando al mismo tiempo los grados de libertad locales para ciertos sectores del espacio de fases. En el caso tridimensional no abeliano, se presentan algunas soluciones de agujeros negros, incluyendo sus espinores de Killing. En cuatro dimensiones, la supersimetría está rota explícitamente y se obtiene un lagrangiano estándar de Dirac acoplado con el campo electromagnético y la geometría de fondo.

En la segunda parte, se presenta el contenido dinámico de la teoría de Rarita-Schwinger libre y con acoplamiento gauge. Esto se puede hacer a través del formalismo hamiltoniano de Dirac y el método de Faddeev-Jackiw, mostrando al mismo tiempo las simetrías de la teoría. Se observa que en la teoría gauge extendida, la cual incluye campos fermiónicos extra para restaurar la simetría fermiónica del caso libre, el anticonmutador del campo de Rarita-Schwinger no es definido positivo en la cuantización canónica.

Ya que el grafeno se ha propuesto como una “mesa de laboratorio” para algunos escenarios de gravedad cuántica, en la tercera parte de esta Tesis se clarifican algunas características sutiles del grafeno extendido con el objetivo de manejar debidamente el material. Se muestra particularmente que el campo seudo-magnético inducido por el campo de tensión planar no puede emerger de una teoría cuántica de campos en espacios curvos (abordaje top-down), sino de un análisis detallado del hamiltoniano tight-binding de los electrones $\pi$ en el grafeno (abordaje bottom-up).

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# Chapter 1

# Introduction

One of the most important and exciting challenges in Theoretical Physics is to construct a theoretical framework which reconciles the four fundamental interactions. Three of these interactions, the electromagnetic, strong and weak forces, are described by the Standard Model (**SM**), a Yang-Mills (**YM**) theory with a particular internal gauge symmetry. Theories with internal gauge symmetries with fiber bundle structure not only restrict the field content and the possible couplings among them but are also instrumental to ensure renormalizability of these theories [1, 2]. Renormalizable theories are those that allow reabsorbing the divergent part calculated from the perturbative expansion of physical measurable quantities (for instance, cross sections and decay rates of elementary particles). This property is crucial to treat consistently the experimental predictions of the theory and is one of the main reasons for the predictive success of the SM, in addition to its enormous precision corroborated in large particle colliders.

The only fundamental interaction, of the four mentioned, which does not qualify as a YM theory is Gravitation. At large scales, Gravitation is described by General Relativity (**GR**), which is invariant under diffeomorphisms of the coordinates on the spacetime base manifold. Although these diffeomorphisms are also local symmetry transformations, general relativity does not have a simple fiber bundle structure as the other gauge interactions and it turns to be non-renormalizable. One could trace the non-renormalizability of GR in the fact that the Newton Gravitation constant $G$ has dimensions of length squared (in natural units, $\hbar = c = 1$). Thus, as more vertices are added, there appear powers of all orders in the ultraviolet momentum cut-off $\Lambda$, making it impossible to absorb all divergences of the theory with a finite number of parameters [1]. This, in particular, makes it very difficult to quantize GR. Nevertheless, the efforts to reconcile Gravity with Quantum Mechanics lead to several sensible models of Quantum Gravity (**QG**).

Since the internal symmetries turn out to be essential in the cancellation of infinities in the SM, it seems plausible that, in order to unify Gravitation with other fundamental interactions, one should formulate a theory invariant under a gauge group containing both internal symmetries and spacetime symmetries. The '*no go*' Coleman-Mandula theorem [3] states that if the scattering matrix $S$ satisfies certain minimum technical requirements, the theory can only be invariant under a group that is the direct product of the Poincaré group $ISO(3,1)$ times the group of internal symmetries $K$. This means that if we want to construct a quantum field theory (**QFT**) with a standard Lie algebra for an internal symmetry group $G$ that contains $ISO(3,1)$ and $K$ as subgroups, then necessarily $G = ISO(3,1) \times K$. The latter does not give us much hope to cancel infinite diagrams for the Gravitation, at least as formulated with the standard GR.

It is possible to circumvent the Coleman-Mandula theorem using instead of a conventional Lie algebra, a graded one, i.e. if along with the commuting generators, anticommuting generators are also included. Such graded algebras are also known as superalgebras [4]. For instance, Haag-Lopuszanski-Sohnius [5] shown how to extend the Poincaré algebra generators $\mathbb{J}_{\mu\nu}$ and $\mathbb{P}_\mu$ including internal symmetries $\mathbb{T}_i$ in a nontrivial way with the addition of fermionic generators to the algebra, i.e., generators $\mathbb{Q}$ that obey anticommutation rules.

If we denote in a generic way $\mathbb{B}$ the bosonic generators $\mathbb{J}_{\mu\nu}$, $\mathbb{P}_\mu$ and $\mathbb{T}_i$, the algebra can be symbolically expanded as

$$\{\bar{\mathbb{Q}}, \mathbb{Q}\} \sim \mathbb{B}, \quad [\mathbb{B}, \mathbb{Q}] \sim \mathbb{Q}, \quad [\mathbb{B}, \bar{\mathbb{Q}}] \sim -\bar{\mathbb{Q}}, \quad [\mathbb{B}, \mathbb{B}] \sim \mathbb{B}.$$

Central extensions with generators $\mathbb{Z}$ can also be included in more general cases. In this way, one theory is a supersymmetric (**SUSY**) theory if its field equations are invariant under the symmetry group associated with a superalgebra of this type.

There are many examples of SUSY theories, from the Wess-Zumino model in four dimensions [6] to the different types of superstrings in $d = 10$ dimensions [7, 8]. In this kind of theories, supersymmetry (**SUSY**) is manifested assuming that both bosonic and fermionic fields must transform in some vector representation under SUSY rotations,

$$\begin{pmatrix} S' \\ F' \end{pmatrix} = Q \begin{pmatrix} S \\ F \end{pmatrix} = \begin{pmatrix} S_{BB} & S_{BF} \\ S_{FB} & S_{FF} \end{pmatrix} \begin{pmatrix} S \\ F \end{pmatrix},$$

where $Q$ represents the exponentiation of fermionic generators $\mathbb{Q}$ with some spinor parameter $\theta$. Because of the superalgebra, this rotation mixes bosonic and fermionic degrees of freedom. The invariance can be done locally, i.e., making $\theta = \theta(x)$, which gives rise to supergravity (**SUGRA**) [9]. These theories were very attractive at the time because it was found that in four dimensions they have cancellations at one-loop order due to opposite signs of the fermionic and bosonic loops, providing a protection mechanism for the hierarchy problem[1] and even bringing the hope for a possible mechanism to renormalize Gravity[2]. Moreover, many SUSY theories guarantee energies bounded from below and absence of tachyon with no more restrictions [4].

Despite the aforementioned advantages of the standard SUSY theories, the fact that for the super-Poincaré group (the minimal SUSY extension of the Poincaré group), $\mathbb{P}_\mu$ commutes with $\mathbb{Q}$ implies that the masses associated with fermionic and bosonic particles must degenerate: $m_F^2 = m_B^2$. This means that if SUSY were present in nature at current energy scales, every elementary bosonic (fermionic) particle in the SM should have its fermionic (bosonic) companion with the same mass. As this has not been confirmed experimentally, even to scales of current energies reached by the Large Hadronic Collider, SUSY must be broken at observed scales and this symmetry breaking makes the mass of the unobserved partners too large to be detected.

It should be noted that there are ways to implement SUSY in odd dimensions, in which SUSY is not manifested in a vector representation as mentioned above, but with both bosonic and fermionic fields as part of the connection and, therefore, they are in the adjoint representation and not in the fundamental one [10, 11]. This type of construction also has some attractive features as the off-shell closure of the SUSY. As we will see below, this kind of SUSY is in fact much more related to the first part of this Thesis. However, the superparticle partners are still present in these kinds of SUSY models.

At this point, one may wonder whether if it is possible to relax some other condition of previous models to allow for the existence of SUSY without the assumption that SUSY is broken at high energies. In [12] this question was explored in $D = 3$ dimensions. A model was built, in which the fields associated with the SUSY algebra transform in the adjoint rather than the fundamental representation and where furthermore, the dreibein is realized in a different way than in standard SUGRA models. This model has nontrivial dynamics and leads to a different scenario, where local SUSY is absent (although there is still diffeomorphism invariance) but where a rigid SUSY can survive for certain background geometries. Because there is no

[1]The *hierarchy problem* can be stated with the following question: why the energy scale of the electroweak force is much lesser than the Planck scale (the energy scale where it is believed the quantum gravity effects are very important)?

[2]However, it is still not clear if such a miracle occurs at higher orders [9].

local SUSY, there are no SUSY pairings. The supersymmetric backgrounds present, however, a great interest in connection with the Part III of this Thesis and the couplings of fermions with bosons are those we found in the SM. This model it is called *unconventional supersymmetry* (**U-SUSY**) . Their detailed dynamical study [13], its extension to non-Abelian internal group [14], as well as $D = 4$ dimensions [15], is revised in Part I.

In standard SUGRA, the hypothetic superpartner of the particle associated with the Gravity interaction (graviton), is the massless 3/2-spin, which is called *gravitino* [16]. The dynamics of such a particle is described by the Rarita-Schwinger action [17]. Although 3/2-spin particle is an important piece of the standard SUGRA theories, it is argued is also important by its own, playing an important role in anomaly cancellation in different QG candidate theories or grand unified models [18, 19].

The description of massive Rarita-Schwinger (**RS**) fields turns out to be problematic both at classical and quantum level [20, 21]. However, recently the non-existence of such problems for the massless case has been claimed [22, 23], due to the presence of a fermionic gauge symmetry in the theory.[3] In Part II, we studied in detail if this symmetry is present for the massless RS theory. In order to do that, we implement the Dirac's Hamiltonian formalism [25, 26] and Faddeed-Jackiw method [27] first for the free case and later for the case of a 3/2-spin field coupled with an Abelian gauge field, i.e., a Maxwell field. We conclude the fermionic gauge symmetry is not present on-shell for the gauged RS action. Finally, we make the same dynamical study of what is called the *extended* massless RS theory, where the fermionic gauge invariance of the free theory is restored by adding spin-1/2 fields to the gauged RS action [28].

As we mentioned above, there are several sensible models of QG, many of them even do not make necessarily use of SUSY [29, 30]. Such models are very attractive but, as is the case for SUSY theories, their predictions are beyond the reach of existing hadronic colliders or from astrophysical observations. Therefore, we are missing an important piece of information coming from Nature: we need experiments or at least indirect observations, to compare and select among different logically well proposed QG theories.

On this respect, Graphene is a very promising table-top laboratory to experimentally probe some of the fundamental mysteries of Nature [31–33]. The low energy regime of its $\pi$ electrons is very well described by an effective

[3] There is, however, a consistent description of massive spin-3/2 fields in $AdS_4$ with its flat limit [24].

theory that shares many of the features of a massless Dirac QFT in the presence of a background space(time).

The $QED$ description for the $\pi$ electrons is for the case of pure flat graphene: there are no deformations, and no defects or impurities. However, real lab graphene is not perfectly planar as it ripples [34] and, even though is very resistant to in-plane deformations due to the strong $\sigma-$electrons bonds, it can be easily bent. In more realistic cases, taking into account these considerations, one can still preserve the Dirac description, by introducing extra coupling fields. Moreover, the honeycomb sheet could be induced to take particular non-trivial geometric forms, as fullerene for positive curvature [35] or Beltrami pseudosphere for negative curvatures [36, 37]. All these ingredients make the graphene sheet a very good experimental tool in order to test different QG predictions for several geometrical backgrounds in a real lab [31, 32].

It is worth noting the $(2+1)$-dimensional U-SUSY mentioned above is a very natural description of $\pi$ electrons in graphene in the linear regime of the dispersion relation, at least in the flat case. This is the case because, as we will see in Chapter 3, the only dynamical local degrees of freedom are the fermionic ones. Even if local bosonic degrees of freedom are not present in $(2+1)$ U-SUSY, the coupling between the fermion with such a fields, like $SU(2)$ spin connection (as we shall see in Chapter 4) or torsion fields, could give rise to interesting phenomena descriptions in different graphene regimes.

In order for graphene to keep its promises, we need to have full control of what sort of fields are there and what they represent in a field theory language. In the vast literature on the gauge fields of graphene (see [38–40], and [41] for a recent review), there are a variety of proposals, sometimes practically valuable for the applications to condensed matter physics, but most of the time unsatisfactory for probing fundamental physics. The landscape of proposals ranges from $SU(2)$ monopole-like gauge fields in the case of graphene membranes with intrinsic curvature (the inflated graphene buckyballs of [42]) to a concurrence of a spin-connection field and a $U(1)$ field, in the case of purely strained graphene [43] (although sometimes non-Abelian fields are evoked in this case as well [44]). Even in the simplest case, that is purely strained graphene, there is some confusion: does the spin connection arising from straining graphene give physical effects or not? And, what is the interpretation of the $U(1)$ field from a fundamental point of view?

We can infer from the literature, the issue of the nature of the gauge field arising during pure strain (which is capable to mimic magnetic fields of more than 300 Tesla [38]), is not settled yet, see [45], and work is constantly produced on it, see the recent [46]. Our goal in Part III is to clarify the geometric nature of all gauge fields emerging from straining graphene, having in mind their use to probe fundamental properties of nature. In fact, we shall not discover any new field, as compared, for instance, to [47–49], but we shall hopefully be able to clarify issues that are important precisely for the use of graphene as a laboratory to realize otherwise unreachable physics, see, e.g., [32] for the Hawking-Unruh radiation effect.

Our focus in this Part is on the deformed graphene membrane whose two-dimensional intrinsic curvature and torsion are zero. This corresponds, in the phenomenology of graphene, to the case of purely strained graphene. We shall take two roads. First, we investigate whether the most straightforward gauge field one would employ to describe a QFT in curved spacetime can indeed explain the pure strain gauge field of graphene. This road starts from the fundamental fully relativistic constructions and sees whether the results, suitably adapted to graphene, can indeed describe the strain. We shall see here the special role of Weyl symmetry. We call this a *top-down* approach. The second road, instead, takes the opposite path, i.e., it is a *bottom-up* approach: we start from the condensed matter non-relativistic theoretical description of graphene, and we look for the fundamental object/gauge field that can describe strain. This latter road is actually necessary because, as we shall see, this kind of gauge field could not be guessed within standard QFT in curved spacetime [50].

The three parts of this Thesis are related in the sense they deal with different aspects we could take into account in our pursuit of a well-defined and, at least indirectly, tested QG theory or theories. However, as these parts are relatively independent each other, the presentation and discussion of each one are included separately.

# Part I

# Unconventional Supersymmetry

# Chapter 2

# Internal $U(1)$ group

Despite some of the appealing advantages presented in Chapter 1, no evidence of SUSY has been found [51], at least in its standard form. This lack of evidence of SUSY in the SM may be due to the fact that bosons and fermions transform in the fundamental representation under standard SUSY transformations. It is known that if the field does not transform in the fundamental but in the adjoint representation, the dynamical content of the theory could be very different. As we shall see, this would lead to supermultiplets composed by matter and gauge fields, with a different number of bosons and fermions, as well as different masses. As the first example of this theory is given in $(2+1)-$dimensions for the supergroup $OSP(2|2)$ [12], which includes as subgroups the internal Abelian $U(1)$ and Lorentz $SO(2,1)$, we shall consider such a case in some detail in this Chapter.

In $(2+1)$ dimensions a very curious similarity occurs between the Abelian Chern-Simons form and the free Dirac Lagrangian, $AdA \sim \overline{\psi}\not{\partial}\psi$. Even though an Abelian connection $A_\mu$ and a complex Dirac spinor $\psi^\alpha$ transform very differently under gauge$-U(1)$ transformations, they can be accommodated as part of the same connection in the following way,

$$\mathbb{A} = \begin{pmatrix} A^\alpha{}_\beta & \psi^\alpha \\ \overline{\psi}_\alpha & 0 \end{pmatrix} , \tag{2.1}$$

where $A^\alpha{}_\beta = A_\mu(\gamma^\mu)^\alpha{}_\beta$, being $\gamma_\mu$ the set of three $2\times 2$ Dirac matrices in three dimensions, and we consider the adjoint row $\overline{\psi}_\alpha = i\psi^\dagger_\beta(\gamma_0)^\beta{}_\alpha$, as is shown in Appendix A.1. The gauge transformation $A'_\mu = A_\mu + \partial_\mu\alpha$, $\psi' = e^{i\alpha}\psi$, and $\overline{\psi}' = e^{-i\alpha}\overline{\psi}$ can be obtained as in the case of non-Abelian gauge transformation $\mathbb{A}' = g^{-1}\mathbb{A}g + g^{-1}\not{d}g$, where $g = \exp[\alpha\mathbb{K}]$, with the

$3 \times 3$ matrices

$$\mathbb{Z} = \begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & \frac{i}{2} & 0 \\ 0 & 0 & i \end{pmatrix} \;,\quad \not{d} = \begin{pmatrix} (\gamma^\mu)^\alpha{}_\beta \partial_\mu & 0 \\ 0 & 0 \end{pmatrix} \;.$$

This observation is pointing out that we can consider $U(1)$ as a subgroup of a larger non-Abelian group, under which $A_\mu$, $\psi$ and $\overline{\psi}$ transform as components of the same connection. As we shall see below, this symmetry also includes rotations which transform $A_\mu$, $\psi_\alpha$ and $\overline{\psi}_\alpha$ into each other, a key feature of SUSY.

## 2.1 Connection, Lagrangian and field equations

One can object that the expression (2.1) is not truly a connection; the components are not all one-forms. This is corrected by introducing the dreibein $e^a_\mu$ matching in a suitable way the tangent space and the base manifold[1]. A more transparent expression is obtained by writing the connection as a linear combination of generators in a $3 \times 3$ representation with field coefficients. The smallest graded Lie algebra containing $\mathfrak{u}(1)$ subalgebra is $\mathfrak{osp}(2|2)$ [12]. Therefore, a natural connection is $\mathbb{A} = \mathbb{A}_\mu dx^\mu$, with[2]

$$\mathbb{A}_\mu = A_\mu \mathbb{Z} + \overline{\psi}_\alpha (\not{e}_\mu)^\alpha{}_\beta \mathbb{Q}^\beta + \overline{\mathbb{Q}}_\beta (\not{e}_\mu)^\beta{}_\alpha \psi^\alpha + \omega^a_\mu \mathbb{J}_a \;, \tag{2.2}$$

where $\mathbb{Z}$, $\mathbb{Q}^\alpha$, $\overline{\mathbb{Q}}_\beta$, and $\mathbb{J}_a$ are the $U(1)$, SUSY and Lorentz $SO(2,1)$ generators, respectively. We use the notation $(\not{e}_\mu)^\alpha{}_\beta = e^a_\mu (\gamma_a)^\alpha{}_\beta$, being $(\gamma_a)^\alpha{}_\beta$ the Dirac matrices in the tangent space. An explicit expression of the $3 \times 3$ matrix generators is shown in Appendix B.1, leading to the following non-vanishing (anti-)commutators of the $\mathfrak{osp}(2|2)$ Lie algebra,

$$\begin{aligned} [\mathbb{J}_a, \mathbb{J}_b] &= \epsilon_{ab}{}^c \mathbb{J}_c \quad , \quad \left\{\mathbb{Q}^\alpha, \overline{\mathbb{Q}}_\beta\right\} = \mathbb{J}_a (\gamma_a)^\alpha{}_\beta - i\delta^\alpha{}_\beta \mathbb{Z} \;, \\ [\mathbb{J}_a, \mathbb{Q}^\alpha] &= -\frac{1}{2}(\gamma_a)^\alpha{}_\beta \mathbb{Q}^\beta \quad , \quad \left[\mathbb{J}_a, \overline{\mathbb{Q}}_\alpha\right] = \frac{1}{2}\overline{\mathbb{Q}}_\beta (\gamma_a)^\beta{}_\alpha \;, \\ [\mathbb{Z}, \mathbb{Q}^\alpha] &= \frac{i}{2}\mathbb{Q}^\alpha \quad , \quad \left[\mathbb{Z}, \overline{\mathbb{Q}}_\alpha\right] = -\frac{i}{2}\overline{\mathbb{Q}}_\alpha \;. \end{aligned} \tag{2.3}$$

[1] Through this work, last greek letters $\mu, \nu, \ldots$ denote base manifold indexes, and lowercase latin letters $a, b, \ldots$ for tangent space indexes. We reserve capital latin letters $A, B, \ldots$ to matrix generator indexes and the first greek letters $\alpha, \beta, \ldots$ to denote spinorial indexes.

[2] As in this Chapter we work in $D = 3$, we use the dual notation for the Lorentz spin connection, $\omega^a = \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}$ and for the Lorentz generator $\mathbb{J}_a = -\frac{1}{2}\epsilon_a{}^{bc}\mathbb{J}_{bc}$.

From the above algebra, it is explicitly seen that the fermion is 'charged' with respect to $U(1)$ and Lorentz subgroup $SO(2,1)$. The geometry of the $(2+1)$ background is given by the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, with the convention $\eta = \mathrm{diag}(-1,1,1)$.

The Chern-Simons ($CS$) form in $D = 3$ dimensions provides a Lagrangian for the connection $\mathbb{A}$ without additional ingredients (for a detailed review of CS forms in SUGRA see [52], and references therein). Therefore[3],

$$L = \langle \mathbb{A} d\mathbb{A} + \frac{2}{3}\mathbb{A}^3 \rangle, \tag{2.4}$$

where $\langle \ldots \rangle$ is a symmetrized supertrace (see Appendix B.1 for more details on bilinear generators supertrace of the $\mathfrak{osp}(2|2)$ superalgebra). Plugging (2.2) into (2.4), and using the algebra conventions, the following Lagrangian is obtained

$$L = \frac{1}{2} A dA + \frac{1}{2}\omega_a d\omega^a + \frac{1}{6}\epsilon_{abc}\omega^a\omega^b\omega^c + \overline{\psi}\not{e}\left[\overleftarrow{D} - \overrightarrow{D}\right]\not{e}\psi, \tag{2.5}$$

where we introduced the notation[4] $\overleftarrow{D} = \overleftarrow{d} + iA - \frac{1}{2}\omega^a\gamma_a$, $\overrightarrow{D} = \overrightarrow{d} - iA + \frac{1}{2}\omega^a\gamma_a$ for the bosonic covariant derivatives, and we suppress spinorial indexes in (2.4) as they are all contracted.

The Lagrangian (2.5) can be written in a more suggestive form if we expand the derivatives acting on the fermion and on the dreibein,

$$\begin{aligned} L &= \frac{1}{2} A dA + \frac{1}{2}\omega_a d\omega^a + \frac{1}{6}\epsilon_{abc}\omega^a\omega^b\omega^c - 2\overline{\psi}\psi e^a T_a \\ &- 2\overline{\psi}\left(\overleftarrow{\not{D}} - \overrightarrow{\not{D}}\right)\psi |e| d^3x\ , \end{aligned} \tag{2.6}$$

where $|e| = \det[e^a_\mu] = \sqrt{-g}$ and $T^a = de^a + \omega^a{}_b e^b$ is the torsion $2-$form. The second line in the above Lagrangian describes a Dirac spinor minimally coupled to a $U(1)$-gauge field $A_\mu$ and to the background geometry through the spin connection $\omega^a_\mu$. The bosonic gauge fields are described by the first line of (2.6) which is the sum of their respective CS forms. These two facts make the Lagrangian (2.6) straightforwardly invariant under $U(1)$ and $SO(2,1)$, up to a boundary term.

The field equations can be obtained from (2.5), as usual, by varying the action with respect to the independent fields,

$$\delta A : \quad F_{\mu\nu} = \epsilon_{\mu\nu\rho} j^\rho\ , \tag{2.7}$$

[3]Exterior product of forms is understood and wedge symbols are omitted, except where ambiguities arise.

[4]For an $m-$form $\Omega$, we define $\Omega\overleftarrow{d} = (-1)^m d\Omega$.

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, $j^\rho = -i\overline{\psi}\gamma^\rho\psi|e|$ is the electromagnetic current density associated with a spin-1/2 particle,

$$\delta\omega : \quad R^{ab} = 2\overline{\psi}\psi e^a e^b \; , \tag{2.8}$$

$$\delta\overline{\psi}_\alpha : \quad -\not{e}\left[\not{T} - \not{e}\overrightarrow{D}\right]^\alpha_{\ \beta}\psi^\beta = 0 \; . \tag{2.9}$$

Although the dreibein was introduced as a field to connect the tangent space with the base manifold, forming a composite field with the fermion, its presence in the Lagrangian is dynamical as much as the other fields,

$$\delta e^a : \quad -2T_a\overline{\psi}\psi = \eta_{ab}e^b\overline{\psi}\left[\overleftarrow{D} - \overrightarrow{D}\right]\psi \; , \tag{2.10}$$

where the derivatives act only on the fermions.

The solutions of field equations (2.7)-(2.10) will be studied in more detail for the $SU(2,1|2)$ group, which contains an additional $SU(2)$ sector. We anticipate that there are purely bosonic solutions for these equations ($\psi = \overline{\psi} = 0$) whose backgrounds are non-trivial [12]. Some of these backgrounds, as $AdS$ and extremal BTZ black holes, contain non-trivial Killing-spinors, as we will see in Chapter 4.

## 2.2 Symmetries

An infinitesimal group element for $OSP(2|2)$ can be written as $g \simeq \mathbb{1} + \Lambda$, with

$$\Lambda = \lambda\mathbb{K} + \overline{\mathbb{Q}}\varepsilon - \overline{\varepsilon}\mathbb{Q} + \lambda^a\mathbb{J}_a \; ,$$

where $\lambda$, $\overline{\varepsilon}$, $\varepsilon$ and $\lambda^a$ are infinitesimal parameters. The transformation $\mathbb{A}' = \mathbb{A} + \delta\mathbb{A}$, where $\delta\mathbb{A} = d\Lambda + [\mathbb{A}, \Lambda\}$, translates to the component fields as

$$\begin{aligned}
\delta A &= d\lambda - i\left(\overline{\psi}\not{e}\varepsilon + \overline{\varepsilon}\not{e}\psi\right) \; , \\
\delta\left(\not{e}\psi^\alpha\right) &= \overrightarrow{D}\varepsilon^\alpha + \frac{i}{2}\lambda\not{e}^\alpha_{\ \beta}\psi^\beta - \frac{1}{2}\lambda^a(\gamma_a)^\alpha_{\ \beta}\not{e}^\beta_{\ \rho}\psi^\rho \; , \\
\delta\left(\overline{\psi}\not{e}_\alpha\right) &= -\overline{\varepsilon}\overleftarrow{D}_\alpha - \frac{i}{2}\overline{\psi}\not{e}_\alpha\lambda + \frac{1}{2}\overline{\psi}_\rho\not{e}^\rho_{\ \beta}(\gamma_a)^\beta_{\ \alpha}\lambda^a \; , \\
\delta\omega^a &= d\lambda^a + \epsilon^a_{\ bc}\omega^b\lambda^c + \left(\overline{\varepsilon}\gamma^a\not{e}\psi + \overline{\psi}\gamma^a\varepsilon\right) \; .
\end{aligned} \tag{2.11}$$

We stress the fact that the fields $\overline{\psi}_\alpha$ and $\psi^\alpha$ are always coupled with the vierbein $e^a_\mu$. Therefore, as we shall see below, we must impose some condition in (2.11) in order to split the transformation of the composite fields $\overline{\psi}\not{e}_\alpha$ and $\not{e}\psi^\alpha$, and in this way to define how these fields transform under SUSY.

By construction, as the CS form is a pseudo-gauge invariant form, the variation of the action (2.4) under the entire group $OSP(2|2)$ is

$$\delta L = d\mathcal{C}_\lambda + d\mathcal{C}_{\overline{\varepsilon},\varepsilon} + d\mathcal{C}_{\lambda^a} ,$$

where these boundary terms are

$$\begin{aligned}
\mathcal{C}_\lambda &= \frac{1}{2}\lambda dA , \\
\mathcal{C}_{\overline{\varepsilon},\varepsilon} &= \overline{\varepsilon} d(\not{e}\psi) - d(\overline{\psi}\not{e})\varepsilon , \\
\mathcal{C}_{\lambda^a} &= \frac{1}{2} d\omega^a \lambda_a .
\end{aligned}$$

The explicit changes of the fields under $U(1)$, $SO(2,1)$ subgroups and SUSY transformations in terms of the infinitesimal parameters are

- $U(1)$ transformations: $\Lambda = \lambda \mathbb{Z}$,

$$\delta A_\mu = \partial_\mu \lambda \ , \ \delta\psi^\alpha = \frac{i}{2}\lambda\psi^\alpha \ , \ \delta\overline{\psi}_\alpha = -\frac{i}{2}\lambda\overline{\psi}_\alpha \ , \delta e^a_\mu = \delta\omega^a_\mu = 0$$

- $SO(2,1)$ transformations: $\Lambda = \lambda^a \mathbb{J}_a$, of course $\delta A_\mu = 0$ is this case. The second and third equations (2.11) determine the transformation laws for $\not{e}\psi^\alpha$ and $\overline{\psi}\not{e}_\alpha$. These are not fundamental spin-3/2 but composite fields, as mentioned above. The product of a spin-1 (the dreibein $e^a_\mu$) and spin-1/2 (either $\psi$ or $\overline{\psi}$) belongs to the reducible representation $1 \otimes 1/2 = 1/2 \oplus 3/2$, according to Clebsch-Gordan coefficients [53]. In this way, $\delta\left(\not{e}\psi^\alpha\right) = \delta e^a (\gamma_a)^\alpha{}_\beta \psi^\beta + e^a (\gamma_a)^\alpha{}_\beta \delta\psi^\beta$ and similarly for $\delta\left(\overline{\psi}\not{e}_\alpha\right)$, with

$$\delta e^a = \epsilon^a{}_{bc} e^b \lambda^c , \tag{2.12a}$$
$$\delta\psi^\alpha = -\frac{1}{2}\lambda^a (\gamma_a)^\alpha{}_\beta \psi^\beta , \tag{2.12b}$$
$$\delta\overline{\psi}_\alpha = \frac{1}{2}\overline{\psi}_\beta (\gamma_a)^\beta{}_\alpha \lambda^a . \tag{2.12c}$$

  The last equation in 2.11 gives us the transformation of the spin connection *omega*$^a$ under $SO(2,1)$ transformations, i.e.,

$$\delta\omega^a = d\lambda^a + \epsilon^a{}_{bc}\omega^b\lambda^c , \tag{2.12d}$$

  which is the covariant derivative acting on a Lorentz vector (see Appendix B.1 for details on this covariant derivative).

- SUSY transformations: $\Lambda = \overline{\mathbb{Q}}\varepsilon - \overline{\varepsilon}\mathbb{Q}$, if this transformation does not affect the dreibein one-form $e^a$, i.e. $\delta e^a = 0$, therefore,

$$\begin{aligned}
\delta A_\mu &= -ie^a_\mu\left(\overline{\psi}\gamma_a\varepsilon + \overline{\varepsilon}\gamma_a\psi\right) , && (2.13a)\\
\delta\psi^\alpha &= \frac{1}{3}\overrightarrow{\not{D}}\varepsilon^\alpha , && (2.13b)\\
\delta\overline{\psi}_\alpha &= -\frac{1}{3}\overline{\varepsilon}\overleftarrow{\not{D}}_\alpha , && (2.13c)\\
\delta\omega^a_\mu &= \left(\overline{\psi}\varepsilon + \overline{\varepsilon}\psi\right)e^a_\mu - \epsilon^a{}_{bc}e^b_\mu\left(\overline{\psi}\gamma^c\varepsilon - \overline{\varepsilon}\gamma^c\psi\right) , && (2.13d)
\end{aligned}$$

We remark here the invariance of $e^a_\mu$ under SUSY transformation is not expected in the standard local form of SUSY, i.e. SUGRA. This is compatible with the particular way the dreibein is introduced in the connection (2.2), not as the coefficient of the translation generators $P_a$ (as in the case of standard SUGRA or CS-SUGRA [10]), but instead as a dictionary connecting the tangent space and the base manifold. This is a special feature of this U-SUSY, for which choosing $\delta e^a = 0$ allows to obtain a linear representation of SUSY acting on the fields, whose consistency implies the appearance of an extra condition, as we will see below. If the theory has nontrivial solutions for this extra condition, this theory has a rigid SUSY.

It can be directly checked that the Lagrangian (2.5) is real and diffeomorphisms invariant by construction. Due to the way in which the fermion is introduced, i.e., always coupled to the dreibein, there is also a local scale symmetry. This means the Lagrangian (2.5) is invariant under local scale transformations of the form

$$e^a_\mu \to e'^a_\mu = \mu(x)e^a_\mu\,,\ \ \psi^\alpha \to \psi'^\alpha = \mu^{-1}(x)\psi^\alpha\,,\ \ \overline{\psi}_\alpha \to \overline{\psi}'_\alpha = \mu^{-1}(x)\overline{\psi}_\alpha\,, \tag{2.14}$$

where $\mu(x)$ is an invertible, but otherwise arbitrary, function on the base manifold. This symmetry will give some interesting features to this theory, as we shall see in Chapter 3 as well.

## 2.3 No-gravitini projection

The condition $\delta e^a = 0$ under SUSY transformation (2.13a), implies the metric $g_{\mu\nu}$ remains invariant under SUSY meaning the absence gravitini in this theory. Taking a closer look to the SUSY transformation in (2.11) expressed in coordinate basis, we have

$$\delta\left(\not{e}\psi^\alpha\right) = \overrightarrow{\not{D}}\varepsilon^\alpha . \tag{2.15}$$

By imposing $\delta e^a_\mu = 0$, this equation becomes $e^a_\mu (\gamma_a)^\alpha{}_\beta \delta\psi^\beta = \left(\overrightarrow{D}\varepsilon\right)^\alpha_\mu$. Multiplying this last expression by $\gamma^\mu$ gives

$$\delta\psi^\alpha = \frac{1}{3}\overrightarrow{\not{D}}\varepsilon^\alpha \ .$$

Now, multiplying this by $\gamma_\nu$, using again (2.15) and the fact that $\delta e^a_\mu = 0$, we end up with the condition

$$\overrightarrow{D}_\nu \varepsilon^\alpha - \frac{1}{3}\gamma_\nu\gamma^\mu \overrightarrow{D}_\mu \varepsilon^\alpha = 0 \ . \tag{2.16}$$

As we mentioned above, a spinor with an extra Lorentz (or spacetime) index, such as $\xi^\alpha_\mu = \gamma_\mu \psi^\alpha$, belongs to the reducible representation $1 \otimes 1/2 = 1/2 \oplus 3/2$ of the Lorentz group. Hence, it can be uniquely decomposed into its irreducible projections as $\xi_\mu = \Psi_\mu + \Phi_\mu$, where $\Psi_\mu = (P_{1/2})_\mu{}^\nu \xi_\nu$ carries spin-1/2, while $\Phi_\mu = (P_{3/2})_\mu{}^\nu \xi_\nu$ is the spin-3/2 part (see Appendix E.2 for details), with

$$(P_{3/2})_\mu{}^\nu = \delta^\nu_\mu - \frac{1}{3}\gamma_\mu\gamma^\nu = \delta^\nu_\mu - (P_{1/2})_\mu{}^\nu . \tag{2.17}$$

In our case, $\xi_\mu = e^a{}_\mu \gamma_a \psi = \gamma_\mu \psi$ and therefore the gravitino contribution vanishes identically, $\Phi_\mu \equiv 0$. Thus, according to (2.15), a SUSY transformation of $\Psi_\mu = e^a{}_\mu \gamma_a \psi$ gives

$$\delta\Psi_\mu = \gamma_a \delta e^a{}_\mu \psi + \gamma_\mu \delta\psi = \overrightarrow{D}_\mu \varepsilon . \tag{2.18}$$

So, the condition (2.16) can be written simply as[5]

$$(P_{3/2})_\mu{}^\nu \overrightarrow{D}_\nu \varepsilon = 0 \ . \tag{2.19}$$

Equation (2.19) can be solved by demanding that $\overrightarrow{D}_\mu\varepsilon$ belongs to the kernel of $P_{3/2}$, that is

$$\overrightarrow{D}_\mu \varepsilon = \gamma_\mu \chi \ , \tag{2.20}$$

for an arbitrary spinor $\chi$. Consistency of (2.20) induces a relation among the field strength $F_{\mu\nu}$, curvature $R^a_{\mu\nu}$ and torsion $T^a_{\mu\nu}$ of the background geometry. This leads to

$$\begin{aligned} \left(\overrightarrow{D}_\mu\overrightarrow{D}_\nu - \overrightarrow{D}_\nu\overrightarrow{D}_\mu\right)\varepsilon &= -\frac{i}{2}F_{\mu\nu}\varepsilon + \frac{1}{2}R^a_{\mu\nu}(\gamma_a)\varepsilon \\ &= T^a_{\mu\nu}(\gamma_a)\chi - (\gamma_\mu D_\nu - \gamma_\nu D_\mu)\chi \ . \end{aligned}$$

On the other hand, from equation (2.20), we can obtain $\chi = \frac{1}{3}\overrightarrow{\not{D}}\varepsilon$. Therefore,

$$-\frac{i}{2}F_{\mu\nu}\varepsilon + \frac{1}{2}R^a_{\mu\nu}(\gamma_a)\varepsilon - \frac{1}{3}T^a_{\mu\nu}(\gamma_a)\overrightarrow{\not{D}}\varepsilon + \frac{1}{3}(\gamma_\mu D_\nu - \gamma_\nu D_\mu)\overrightarrow{\not{D}}\varepsilon = 0 \ . \tag{2.21}$$

[5]For more details about solutions of no spin-3/2 condition see [54] and references therein.

So, only for very particular backgrounds there exists nontrivial fermionic parameters $\varepsilon^\alpha$ which satisfies (2.21). The number of independent globally well defined solutions of this equation depends on the gauge curvatures and on possible topological obstructions. Nevertheless, in a typical experimental setting appropriate to the strained graphene analysed in the Part III of theis Thesis, the curvatures are negligible in the region where the experiment are carried out[6]. The relevant regions in those cases are huge compared with the quantum wavelength of the particles involved, but at the same time are extremely small compared with the local radius of curvature of spacetime and, to a good approximation the curvature can be safely assumed to vanish. Then $\chi$ can be taken equal to zero and $\varepsilon$ approximates to a Killing spinor of the background (we shall find in Chapter 4 some particular backgrounds solutions with their corresponding Killing spinors solutions).

The presence of fermions requires the introduction of a soldering form (the vielbein) in order to project properties of the dynamical fields in the tangent space onto the base manifold. In particular, the fact that fermions belong to a spin-1/2 representation of the Lorentz group is a feature defined on the tangent space. On the other hand, $\mathbb{A}(x)$ is a one-form on the base manifold and, the introduction of the vielbein is required by the presence of fermionic matter so that a theory that includes fermions must also to include a metric structure, as noted long ago by H. Weyl [55].

One of the essential features of gauge theories is the background independence of the gauge symmetry. A gauge transformation has the form

$$\mathbb{A}(x) \to \mathbb{A}'(x) = g^{-1}(x) \left[\mathbb{A}(x) + d\right] g(x), \tag{2.22}$$

independently of the spacetime geometry. In particular, it does not depend on the metric or affine properties of the background. The invariance of the Yang–Mills Lagrangian, $\mathcal{L}_{\rm YM} = (-1/4)\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}\mathrm{Tr}\left(\mathbb{F}_{\mu\nu}\mathbb{F}_{\alpha\beta}\right)$ under (2.22) holds at any spacetime point, irrespective of the coordinates, the metric, the background curvature, torsion, etc. The decoupling between the internal gauge symmetry and the spacetime geometry is also reflected in the fact that the metric itself is invariant under internal gauge transformations.

Gauge invariance of the metric is ensured if under internal gauge symmetries the vielbein is also invariant, $e^a{}_\mu(x) \to e'^a{}_\mu(x) = e^a{}_\mu(x)$. Actually, a weaker condition like $e'^a{}_\mu(x) = \Lambda^a{}_b(x)e^b{}_\mu(x)$, with $\Lambda \in SO(1,2)$, would suffice to render the metric invariant. This is indeed the case when the gauge group includes Lorentz transformations, something that is not often assumed because the Lorentz group is not usually viewed as an internal symmetry.

[6]However, this is not longer true when we are considering the Beltrami pseudosphere made from graphene sheet in order to test Hawking-Unruh effects, as is shown for instance in [31]

The distinction between internal and spacetime symmetries for the case of the Lorentz group, however, is rather semantic.

The decoupling between the gauge transformations (internal symmetries) and the geometric properties of spacetime (external symmetries) guarantees that gauge invariance holds irrespective of the "environment." Mathematically this is reflected in the fact that a fiber bundle is locally a direct product of a vector space and a manifold: the fibers in the bundle are identical copies of the same algebra, regardless of what the base manifold could be. In the same spirit, here we assume the metric structure to be decoupled from SUSY. The vielbein plays the role of a dictionary to translate between the manifold and the tangent space that is not transformed under SUSY. As shown in [12], this corresponds to a projection of the local SUSY algebra on the spin-1/2 subspace, instead of projecting on the spin-3/2 space, as it is usually done in SUGRA [56].

The consequence of this is that there is no spin-3/2 components on the right side of (2.11), i.e., there are no gravitini. As will be confirmed in Chapter 3, the SUSY of the action (2.5) is not there once imposing $\delta e^a_\mu = 0$. At most, for very particular backgrounds, there are solutions of (2.20) implying a global/rigid accidental SUSY.

## 2.4 Summary

Usually, a gauge theory constructed from a connection belonging to a superalgebra (superconnection), defines a new field, the spin-3/2 gravitino (see Part II for more details about spin-3/2 field theory). In the model presented in this Chapter, this is not the case being consistent to the fact that since no spin-3/2 is required, the gravitini can be projected out. However, we must stress here the fact that the SUSY parameter is not arbitrary but must be constrained by the subsidiary condition (2.16). This means, in particular, that the realization of SUSY here is not local, but global/rigid provided there exist nontrivial solutions. The number of independent solutions $\varepsilon$ for this equation, if there were, will depend on the background geometry. The fact that these parameters are not arbitrary will be reaffirmed later in Chapter 3, where we show there is no first-class constraint associated to SUSY transformations. However, there are background examples where there exist solutions to condition (2.16), as for instance in $(2+1)$-black holes sectors, which we shall see in Chapter 4.

The U-SUSY could be useful as a simple field theory for an electromagnetically charged spin-1/2 field interacting with the $(2+1)$-dimensions background geometry. In particular, it fits very well to the massless Dirac description of

$\pi$ electrons in flat graphene (see Chapter 9). It was proposed that graphene with constant negative curvatures could be used to test some features of QFT in curved spacetime, as the Unruh radiation effect [32], as we will mention in Part III. Therefore, this model could be used to describe the $\pi$ electrons in these backgrounds [57]. Even more, if some fixed background sheet of graphene satisfies condition (2.16) (diffeomorphism invariance is explicitly broken, as there is no more dynamical background), this U-SUSY model describing $\pi$ electrons is a theory with a rigid accidental SUSY. It is possible that this SUSY rigid symmetry appearing in this fixed backgrounds could have an observable (fermionic) current associated, which in principle could be measurable in a real lab.

The fact that fermion mass can arise as an effect of the background spacetime through the torsion (which in the case of condensed matter corresponds to the presence of dislocations in the crystal lattice), seems to have been put forward by Weyl long time ago [55]. It could be interesting to explore if this procedure could be used as a mass generation mechanism for the neutrinos, without the necessity to break any symmetry.

This model may also be regarded as an instance that could be extended to higher-dimensional scenarios, as a non-Abelian internal group in Chapter 4, or more realistic $D = 4$ studied in Chapter 5. The crucial point here is that rigid/global SUSY could manifest itself very differently if the fermion fields are accommodated in the adjoint representation and the vielbein does not make the usual role of the translation generator coefficients, as in CS-SUGRA.

# Chapter 3

# Dynamical Contents

It is well known that CS theories in three dimensions for any Lie algebra have no local degrees of freedom [58]. This is true also for CS theories based on graded Lie algebras [59], like in the case of the CS SUGRA for the $\mathfrak{osp}(2|2)$ algebra. By contrast, a massive spin-1/2 field in a fixed three-dimensional background of has $2n$ propagating degrees of freedom, where $n = 1$ for Majorana and $n = 2$ for Dirac spinors [26, 60]. Now, if in the $\mathfrak{osp}(2|2)$ CS theory the gravitino field $\chi^{\alpha}_{\mu}$ is split into a spin-1/2 Dirac spinor $\psi^{\alpha}$ and the vielbein $e^{a}_{\mu}$, the fermionic sector of the reduced theory describes a Dirac fermion in a curved background, minimally coupled to $\mathfrak{u}(1)$ and $\mathfrak{so}(2,1)$ gauge connection one-forms $A = A_{\mu}dx^{\mu}$ and $\omega^{a}{}_{b} = \omega^{a}{}_{b\mu}dx^{\mu}$, respectively [12]. It is therefore only natural to inquire whether this reduced theory has zero local degrees of freedom (**DOF**) as the original CS system, or has four local degrees of freedom of a spin-1/2 Dirac fermion. The question is further complicated by the fact that in the reduced Lagrangian the dreibein are not Lagrange multipliers (their time derivatives $\dot{e}^{a}$ appear explicitly in the action) and therefore $e^{a}_{\mu}$ are in principle dynamical fields as well.

The identification of the local physical DOF can be addressed by direct application of Dirac's analysis of constrained Hamiltonian systems [25], which systematically separates the dynamical fields from the gauge degrees of freedom. In the case of CS theories, however, the separation between first and second-class constraints is a delicate issue, and the system considered here is not an exemption. The action from (2.5) [12] reads

$$I[\psi, e, A, \omega] = \int \frac{1}{2}\Big[2\overline{\psi}\not{e}(\overleftarrow{D} - \overrightarrow{D})\not{e}\psi + AdA + \frac{1}{2}\omega^{a}{}_{b}d\omega^{b}{}_{a} + \frac{1}{3}\omega^{a}{}_{b}\omega^{b}_{c}\omega^{c}_{a}\Big] \,,$$

In addition to the local $U(1) \times SO(2,1)$ symmetry and spacetime diffeomorphisms, this action is invariant under local Weyl rescalings (2.14),

$$e^{a}_{\mu} \to \lambda e^{a}_{\mu} \,, \quad \psi \to \lambda^{-1}\psi \,, \quad \bar{\psi} \to \lambda^{-1}\bar{\psi} \,,$$

where $\lambda(x)$ is a non-vanishing, real and differentiable function. All of these symmetries are in principle associated with first-class constraints that reduce the physical phase space.

Varying the action with respect to $\psi$ yields the Dirac equation with a mass term $m = \frac{1}{2|e|}\epsilon^{\mu\nu\rho}\eta_{ab}e^a_\mu D_\nu e^b_\rho$ (including hermiticity corrections), while varying with respect to $e^a_\mu$ implies the vanishing of the energy-momentum tensor, $\mathcal{T}^{\mu\nu} = \frac{1}{2|e|}\eta^{ab}E^\mu_a \frac{\delta L}{\delta e^b_\nu} + (\mu \leftrightarrow \nu)$, with $E^\mu_a$ the inverse dreibein. In particular, the vanishing of the trace $\mathcal{T}^\mu{}_\mu = 0$ is consistent with the local scale invariance of the action.

For a fixed background, the local degrees of freedom should correspond to the $2n$ independent components of the Dirac field in flat spacetime. A quick analysis suggests that six out of the nine components of the dreibein can be eliminated by the conditions $\mathcal{T}^{\mu\nu} = 0$, while the remaining three can be gauged away via two spatial diffeomorphisms and a Weyl scaling. In CS theories, time diffeomorphisms are not independent, which means their phase space generators are linear combinations of the remaining first-class constraints [58].

As noted in [12], the closure SUSY for (2.5) requires the parameter of the SUSY transformation to satisfy a subsidiary condition to ensure the variation $\delta\psi$ to have spin-1/2, like $\psi$ itself. This subsidiary condition is satisfied if the SUSY parameter is required to be a Killing spinor of the background and, like in the original WZ system, this means that SUSY is a global (rigid) symmetry [15], as we mentioned in Chapter 2. Since this is not a gauge symmetry, it is not generated by a first-class constraint that would further reduce the number of local physical DOF, as we see below.

## 3.1 Hamiltonian analysis

Splitting the fields and their derivatives into time ($t$) and spatial components ($i, j = 1, 2$), the Lagrangian (2.5) can be written, up to a boundary term, as

$$L = \epsilon^{ij}\left[-\eta_{ab}\dot{e}^a_i e^b_j \bar{\psi}\psi - \dot{\bar{\psi}}\gamma_{ij}\psi + \bar{\psi}\gamma_{ij}\dot{\psi} + \frac{1}{2}\eta_{ab}\dot{\omega}^a_i\omega^b_j + \frac{1}{2}\dot{A}_i A_j\right] - e^a_t K_a + \omega^a_t J_a + A_t K \,, \tag{3.1}$$

where we defined $\gamma_{ij} \equiv e^a_i e^b_j \gamma_{ab}$, and

$$K_a \equiv 2\epsilon^{ij}\left[\eta_{ab}T^b_{ij}\bar{\psi}\psi - e^b_i(\bar{\psi}\gamma_a\gamma_b\overrightarrow{D}_j\psi + \bar{\psi}\overleftarrow{D}_j\gamma_b\gamma_a\psi)\right] , \tag{3.2}$$

$$J_a \equiv \epsilon^{ij}\eta_{ab}(\frac{1}{2}R^b_{ij} - \epsilon^b{}_{cd}e^c_i e^d_j\bar{\psi}\psi) \,, \tag{3.3}$$

$$K \equiv \epsilon^{ij}(\partial_i A_j - i\bar{\psi}\gamma_{ij}\psi) \,. \tag{3.4}$$

The Lagrangian (3.1) describes the evolution of $(21+4n)$ coordinate fields: $e^a_\mu$ (nine), $\omega^a_\mu$ (nine), $A_\mu$ (three), $\psi$ (2n) and $\bar{\psi}$ (2n); among them there are seven ($e^a_t$, $\omega^a_t$ and $A_t$), whose time derivatives do not appear in the Lagrangian and are therefore Lagrange multipliers with vanishing canonical momenta. For the remaining components, the Lagrangian contains only first time derivatives and, therefore, each momentum is a function of the coordinate fields. Thus, the following $(14+4n)$ primary constraints are obtained[1] (see Appendix C.1 for notation of the constraints)

$$\begin{aligned} \varphi^i_a &= p^i_a + 2\epsilon^{ij}\eta_{ab}e^b_j\bar{\psi}\psi \approx 0\,, \\ \Omega &= \chi + \epsilon^{ij}\gamma_{ij}\psi \approx 0\,, \\ \bar{\Omega} &= \bar{\chi} - \epsilon^{ij}\bar{\psi}\gamma_{ij} \approx 0\,, \\ \phi^i_a &= \pi^i_a - \frac{1}{2}\epsilon^{ij}\eta_{ab}\omega^b_j \approx 0\,, \\ \phi^i &= \pi^i - \frac{1}{2}\epsilon^{ij}A_j \approx 0\,. \end{aligned} \tag{3.5}$$

The seven combinations $K_a$, $J_a$, $K$ in (3.1) are then secondary constraints associated to the Lagrange multipliers. Moreover, the canonical Hamiltonian weakly vanishes and the total Hamiltonian can be taken as an arbitrary linear combination of all the constraints[2],

$$H_T = \int d^2x \left[e^a_t K_a - \omega^a_t J_a - A_t K + \varphi^i_a\lambda^a_i + \phi^i_a\Lambda^a_i + \bar{\Lambda}\Omega + \bar{\Omega}\Lambda + \lambda_i\phi^i\right]\,. \tag{3.6}$$

It can be proved that the following seven linear combinations are first-class constraints (see Appendix C.1 for more details)

$$\begin{aligned} \tilde{J}_a &\equiv J_a + \epsilon_a{}^b{}_a\varphi^j_b e^c_j + \frac{1}{2}(\bar{\Omega}\gamma_a\psi - \bar{\psi}\gamma_a\Omega) + D_j\phi^j_a\,, \\ \tilde{K} &\equiv K - \frac{i}{2}(\bar{\Omega}\psi - \bar{\psi}\Omega) + \partial_j\phi^j\,, \\ \Upsilon &\equiv -e^b_j\varphi^j_b + \bar{\Omega}\psi + \bar{\psi}\Omega\,, \\ \mathcal{H}_i &\equiv e^a_i K_a - e^a_i D_j\varphi^j_a + T^a_{ij}\varphi^j_a + \bar{\psi}\overleftarrow{D}_i\Omega + \bar{\Omega}\overrightarrow{D}_i\psi - \omega^a_i\tilde{J}_a - A_i\tilde{K} + \phi^j F_{ij} + \phi^j_a R^a_{ij}\,. \end{aligned} \tag{3.7}$$

Here the (spatial) covariant derivative $D_i$ acts on each field according to its transformation properties, as in (B.5). Using (3.2)-(3.5), the generators $\mathcal{H}_i$ can be expressed as

$$\begin{aligned} \mathcal{H}_i &= (\partial_i A_j - \partial_j A_i)\,\pi^j - A_i\partial_j\pi^j + \left(\partial_i\omega^a_j - \partial_j\omega^a_i\right)\pi^j_a - \omega^a_i\partial_j\pi^j_a \\ &\quad + \left(\partial_i e^a_j - \partial_j e^a_i\right)p^j_a - e^a_i\partial_j p^j_a + \partial_i\overline{\psi}\chi + \overline{\chi}\partial_i\psi\,, \end{aligned}$$

[1] We use the symbol $\approx$ to denote weak equality [25].

[2] Hereafter we perform the integrations over the spatial slices $\Sigma$ given by $t = \mathit{constant}$, for which we do not consider a boundary.

which can be readily seen to generate spatial diffeomorphisms on phase space functions $F$ as $\{F, \int \xi^i \mathcal{H}_i\} = \mathcal{L}_\xi F$. This in turn means that

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_i(y)\partial_j^{(y)}\delta^{(2)}(x-y) - \mathcal{H}_j(x)\partial_i^{(x)}\delta^{(2)}(x-y)\,, \tag{3.8}$$

as expected from generators of spatial diffeomorphisms [61]. On the other hand, it can be directly checked that $\tilde{J}_a$, $\tilde{K}$ and $\Upsilon$ generate $SO(2,1)\times U(1)\times Weyl$ transformations over all the fields and momenta. Indeed, they satisfy the (weakly vanishing) Poisson relations (C.4) with all the constraints, and one finds

$$\begin{aligned} \{\tilde{J}_a, \tilde{J}_b\} &= \epsilon_{ab}{}^c \tilde{J}_c\,, \\ \{\tilde{K}, \tilde{K}\} &= \{\Upsilon, \Upsilon\} = \{\tilde{K}, \Upsilon\} = 0\,, \\ \{\tilde{J}_b, \tilde{K}\} &= \{\tilde{J}_b, \Upsilon\} = 0\,. \end{aligned} \tag{3.9}$$

Together with the generators of spatial diffeomorphisms these then form a first-class Poisson algebra.

Note that performing a shift in the Lagrange multipliers of the form

$$\begin{aligned} \lambda_i^a \to \lambda_i'^a &= -ve_i^a + \lambda_i^a\,, \\ \Lambda'^\alpha \to \Lambda'^\alpha &= v\psi^\alpha + \Lambda^\alpha\,, \\ \overline{\Lambda}_\alpha \to \overline{\Lambda}'_\alpha &= v\overline{\psi}_\alpha + \overline{\Lambda}_\alpha\,, \end{aligned} \tag{3.10}$$

produces a shift in the total Hamiltonian (3.6),

$$H_T \to H_T' = H_T + \int v\Upsilon d^2x\,. \tag{3.11}$$

This accounts for the Weyl invariance (2.14) of the system. However, the absence of spatial derivatives in $\Upsilon$ implies that such symmetry is generated by a purely local constraint with no associated asymptotic charges (recent examples of this fact can be found in [62] and references therein, see [63] for a thorough discussion). Weyl symmetry is thus a local redefinition of the fields without any observable effects. The corresponding symmetry breaking, however, leads to physical consequences as we will discuss.

### 3.1.1 Generic scale invariant sector

We now assume that in a generic[3] background the $(14+4n)$ time preservation equations of the primary constraints fix an equal number of Lagrange multipliers (see Appendix C.2 for details). The other seven parameters remain free

[3]Following [58] we understand by *generic* sectors those with a maximum number of degrees of freedom or, equivalently, a minimum number of independent first-class constraints.

in the total Hamiltonian (3.6), to form a linear combination of the first-class constraints. Choosing $\{\mathcal{H}_i,\ \Upsilon,\ \tilde{J}_a,\ \tilde{K}\ \}$ as the basis of these generators, the total Hamiltonian can be written as

$$H_T = \int d^2x \left[\xi^i \mathcal{H}_i + v\Upsilon - \omega_t^a \tilde{J}_a - A_t \tilde{K}\right] , \tag{3.12}$$

Here the Lagrange multipliers $\xi^i$, $v$, $\omega_t^a$ and $A_t$ are real and arbitrary functions on equal footing. As the Hamiltonian is a combination of first-class constraints, the time preservation relations are fulfilled by construction, and no additional (tertiary) constraints are produced in the Dirac algorithm. Note further that for any phase space function $F$ the Poisson bracket $\{F, H_T\}$ coincides with the corresponding Dirac bracket.

Now, the expression (3.12) was obtained from (3.6) by choosing

$$e_t^a = \xi^i e_i^a \, . \tag{3.13}$$

This means that the three components $e_t^a$ are functions of the two free parameters $\xi^i$, while it also implies a degenerate dreibein, $|e| = 0$. Although this may seem puzzling for a metric interpretation, it is dynamically consistent and allows to do the correct counting of the local degrees of freedom (see e.g., [58] and Appendix C.2). The choice (3.13) is equivalent to the gauge $N^\perp = 0$ in gravitation, which is perfectly acceptable as well as generic choices in ordinary gauge systems, i.e., YM [64, 65]. Furthermore, it also allows to write the generator of temporal diffeomorphisms as a linear combination of generators of local spatial diffeomorphisms, rescalings, Lorentz and $U(1)$ transformations[4],

$$\mathcal{H} = \xi^i \mathcal{H}_i + v\Upsilon - \omega_t^a \tilde{J}_a - A_t \tilde{K} \, . \tag{3.14}$$

Note that the degenerate condition $|e| = 0$ remains invariant under local Weyl symmetry. Next, we consider a choice in which the Weyl symmetry is broken and the $e_t^a$ remains arbitrary so that the dreibein need not be degenerate.

### 3.1.2 Pure spin-$1/2$ generic sector

We now examine a specific sector of the theory in which (3.13) is not imposed but the Weyl invariance is fixed instead. We consider a generic sector for the fields $e$ and $\psi$ that restricts the fermionic excitations to have spin-1/2 only. A fermionic field $\chi_a^\alpha$ transforms as a vector in the index $a$ and as a spinor in

[4]It can be explicitly shown that $\{\cdots, \int N\mathcal{H}\} \approx \mathcal{L}_{N\frac{\partial}{\partial t}}(\cdots)$, which is a general property of coordinate invariant systems [60].

the index $\alpha$ and therefore belongs to a representation $1 \otimes 1/2 = 3/2 \oplus 1/2$ of the Lorentz group, as said in Section 2.3. There is a unique decomposition of this field into irreducible representations $\chi_a = \chi_a^{(3/2)} + \chi_a^{(1/2)}$, where

$$[\delta_a^b - \frac{1}{3}\gamma_a \gamma^b]\chi_b^{(1/2)} = 0\,, \tag{3.15}$$

$$\gamma^a \chi_a^{(3/2)} = 0\,. \tag{3.16}$$

In the case of the field $\psi$, the condition that it only carries spin-1/2 requires that $D_\mu \psi$ also belongs to the spin-1/2 representation and should therefore be in the kernel of the spin-3/2 projector, namely,[5]

$$[\delta_\nu^\mu - \frac{1}{3}\gamma_\nu \gamma^\mu] D_\mu \psi = 0\,, \tag{3.17}$$

where $\gamma_\mu \equiv e_\mu^a \gamma_a$. This implies that the system does not generate local spin-3/2 excitations –no gravitini– through parallel transport of the fermion. It may be regarded as a consistency condition for the system (2.5) if it is meant to describe a Dirac spinor. The general solution of (3.17) is, in analogy to the solution of equation (2.19),

$$D_\mu \psi = \gamma_\mu \xi\,, \tag{3.18}$$

where $\xi$ is an arbitrary Dirac spinor.

Next, in order to study the dynamical content of the sector, we perform a partial gauge fixing. As shown in [12], the field equations for the action (2.5) require the torsion to be covariantly constant, $DT^a = 0$, where $D$ is the Lorentz covariant exterior derivative (see Appendix A). The general solution of this equation, with an appropriate local rescaling of the dreiben –using the freedom due to Weyl symmetry– is of the form

$$T^a = \alpha \epsilon^a{}_{bc} e^b e^c\,, \tag{3.19}$$

where $\alpha$ is an arbitrary (dimensionful) constant. Now, inserting (3.18), (3.19) in (3.2) we obtain

$$K_a = 2\epsilon^{ij} e_i^b e_j^c \Big[ 2\alpha \epsilon_{abc} \bar{\psi}\psi - (\bar{\xi}\gamma_a \gamma_b \gamma_c \psi - \bar{\psi}\gamma_c \gamma_b \gamma_a \xi) \Big]\,. \tag{3.20}$$

In order for the constraint condition $K_a \approx 0$ not to introduce additional restrictions on the fields, the right-hand-side of (3.20) must identically vanish.

[5]Formally, if the scale has not been fixed the sector should be defined as the equivalence class of configurations satisfying (3.17) up to Weyl transformations. A manifestly covariant condition can be attained by introducing a gauge field for scale invariance $D_\mu \to D_\mu + W_\mu$, as originally proposed by Weyl [63].

This demands $\xi = \alpha\psi$ and therefore, this selects the sector[6]

$$D_\mu \psi = \alpha \gamma_\mu \psi \, . \qquad (3.21)$$

Multiplying both sides by $\gamma^\mu$, this reduces to the Dirac equation where the mass $m = 3\alpha$ is an integration constant related to the torsion of the background, in complete agreement with [12].

Both (3.19) and (3.21) break local scale invariance, leaving only a global symmetry under $e^a \to \lambda e^a$, $\psi \to \lambda^{-1}\psi$, $m \to \lambda^{-1} m$ for constant $\lambda$. In analogy with SUSY, this rigid symmetry does not interfere with the counting of local DOF. As pointed out in [12, 14], the introduction of a dimensionful mass constant $m$ enables us to finally determine the scale for the theory.

In Appendix C.2, we show that the sector equations (3.19) and (3.21) can be used to consistently solve and preserve the remaining constraints. In fact, in this case one is enabled to explicitly determine the time evolution of $e$ and $\psi$, which is equivalent to the fact that Lagrange multipliers in the total Hamiltonian are also found in closed form (without using the 'degenerate gauge' (3.13)). We now show how the first-class generators arise to recover the residual symmetries of (3.19,3.21). In principle we will only assume the spatial components of these equations to hold, while the temporal parts will be recovered from Hamilton equations. Thus, note first that in this sector the combinations

$$\begin{aligned} \tilde{K}_a &= K_a - D_i \varphi^i_a + 2\alpha \epsilon^b{}_{ac} e^c_i \varphi^i_b + \alpha(\bar{\Omega}\gamma_a \psi - \bar{\psi}\gamma_a \Omega) \\ &\quad + 2i e^b_i \bar{\psi}\gamma_{ab}\psi \phi^i + 2\epsilon^b{}_{ac} e^c_i \bar{\psi}\psi \phi^i_b \, , \end{aligned} \qquad (3.22)$$

are first-class constraints, as can be directly checked computing the Poisson brackets:

$$\{\tilde{K}_a, \Omega\} \approx \{\tilde{K}_a, \bar{\Omega}\} \approx \{\tilde{K}_a, \varphi^i_b\} \approx 0 \, , \qquad (3.23)$$

$$\{\tilde{K}_a, \phi^i_b\} \approx \{\tilde{K}_a, \phi^i\} \approx 0 \, , \qquad (3.24)$$

$$\{\tilde{K}_a, \tilde{K}_b\} \approx 0 \, . \qquad (3.25)$$

These three constraints are the generators of spacetime diffeomorphisms supplemented by gauge transformations and projected on the tangent space. This is seen from the identity

$$\{\cdots, e^a_i \tilde{K}_a\} \approx \{\cdots, \mathcal{H}_i + A_i \tilde{K} + \omega^a_i \tilde{J}_a\} \, . \qquad (3.26)$$

[6] The projector (3.17) is a generalization of the so-called 'twistor operator', which defines conformal Killing spinors (3.18) in the absence of torsion [6, 66]. Equation (3.21) can be regarded as the Killing spinor equation for a curved background [14]. Remarkably, (3.18) and (3.21) are completely equivalent by virtue of the Dirac equation.

We now set the Lagrange multipliers associated to the primary constraints in order to accommodate the seven first-class generators. The total Hamiltonian reads

$$\begin{aligned} H_T &= \int d^2x \left[ e_t^a K_a - \omega_t^a J_a - A_t K + \varphi_a^i \lambda_i^a + \phi_a^i \Lambda_i^a + \bar{\Lambda}\Omega + \bar{\Omega}\Lambda + \lambda_i \phi^i \right] \\ &= \int d^2x \left[ -\omega_t^a \tilde{J}_a - A_t \tilde{K} + e_t^a \tilde{K}_a \right] =: \int d^2x \, \mathcal{H} \, . \end{aligned} \tag{3.27}$$

Note that here we are implicitly fixing the Weyl freedom, i.e., we have assumed $v = 0$ in the shift (3.11). This is required to preserve the sector. Indeed, the time evolution for the fields $(e_i^a, \psi, A_i, \omega_i^a)$, by virtue of the Hamilton equations, leads to

$$D_t \psi = \dot{\psi} - \frac{i}{2} A_t \psi + \frac{1}{2} \epsilon^{abc} \omega_{bt} \gamma_c \psi = \alpha \gamma_t \psi \, , \tag{3.28}$$

$$T_{it}^a = \partial_i e_t^a - \dot{e}_i^a + \omega_{bi}^a e_t^b - \omega_{bt}^a e_i^a = 2\alpha \epsilon_{bc}^a e_i^b e_t^c \, , \tag{3.29}$$

$$F_{it} = \partial_i A_t - \dot{A}_i = 2i e_i^a e_t^b \bar{\psi} \gamma_{ab} \psi \, , \tag{3.30}$$

$$R_{it}^a = \partial_i \omega_t^a - \dot{\omega}_i + \epsilon^{abc} \omega_{bi} \omega_{ct} = 2\epsilon_{bc}^a e_i^b e_t^c \bar{\psi} \psi \, , \tag{3.31}$$

These are readily seen to recover the temporal parts of equations (3.19)-(3.21) and the constrains (3.3,3.4), thus agreeing with the Euler-Lagrange equations.

As stated, an interesting feature of this gauge is that $e_t^a$ is not restricted at all, which is equivalent to the statement that the three constraints $\tilde{K}_a$ are first-class. For regular configurations with $|e| \neq 0$, it is clear that $(\mathcal{H}, \mathcal{H}_i)$ are then three independent constraints generating temporal and spatial diffeomorphisms, respectively. Nevertheless, even for a degenerate dreibein it is possible to define

$$\mathcal{H}_\perp := \epsilon_{bc}^a e_1^b e_2^c \tilde{K}_a \, , \tag{3.32}$$

which corresponds (up to normalization) to the generator of diffeomorphisms normal to the surfaces $t = constant$, modulo gauge transformations. Defining the Lagrange multipliers $e_t^a$, $A_t$ and $\omega_t^a$ as

$$e_t^a = N^\perp \epsilon^a{}_{bc} e_1^b e_2^c + e_i^a N^i \, , \tag{3.33}$$

$$A_t = \lambda - A_i N^i \, , \tag{3.34}$$

$$\omega_t^a = \lambda^a - \omega^a{}_i N^i \, , \tag{3.35}$$

the generator of time evolution takes the more familiar form [61]

$$\mathcal{H} = N^\perp \mathcal{H}_\perp + N^i \mathcal{H}_i - \lambda \tilde{K} - \lambda^a \tilde{J}_a . \tag{3.36}$$

We thus find the expected $SO(2,1) \times U(1) \times$ Diff residual symmetries and their corresponding generators. We anticipate here that even though in this gauge choice there exist a different set of first-class constraints associated to diffeomorphisms, the number of DOF is the same and this is, therefore, a generic sector. This will be discussed in Section 3.2.

### 3.1.3 Bosonic Vacuum

The purely bosonic vacuum $\bar{\psi} = 0 = \psi$ corresponds to a very particular configuration. In principle, it should *not* be regarded as a subsector of the previous case because it acquires additional degeneracies in the Dirac matrix[7] which lead to new first-class constraints. This is a direct consequence of the whole energy-momentum tensor of the Lagrangian formalism vanishes identically and there are no field equations to determine $e_\mu^a$, so the dreibein is a non-propagating gauge field in this case. Nevertheless, some of the first-class constraints found in the previous section turn out to be not functionally independent and therefore compensate the situation. As we will show, the whole picture results into an equal number of DOF, thus we can think of the vacuum as a generic sector.

First note if the fermions vanish, (C.3) and (3.2)-(3.4) imply

$$\begin{aligned} \{\varphi_a^i, \varphi_b^j\} &= \{\varphi_a^i, \Omega\} = \{\varphi_a^i, \bar{\Omega}\} = \{\varphi_a^i, \phi^j\} = \{\varphi_a^i, \phi_b^j\} = 0\,, && (3.37) \\ \{\varphi_a^i, K\} &= \{\varphi_a^i, J_a\} = \{\varphi_a^i, K_a\} = 0\,, && (3.38) \end{aligned}$$

(where we have set $\bar{\psi} = 0 = \psi$ *after* computing the Poisson brackets). Thus, we find six additional first-class constraints $\varphi_a^i \approx 0$, which generate arbitrary changes in the spatial components of the dreibein,

$$\delta e_i^a = \{e_i^a, \int d^2x\, \lambda_j^b \varphi_b^j\} = \lambda_i^a\,. \tag{3.39}$$

As the time component $e_t^a$ is already a Lagrange multiplier, this in turn means that the dreibein is completely arbitrary (in particular it can be chosen to be invertible). In this sector, the first-class constraints (3.7) read

$$\begin{aligned} \tilde{J}_a &= J_a + \epsilon^b{}_{ac}\varphi_b^j e_j^c + D_j\phi_a^j\,, && (3.40) \\ \tilde{K} &= K + \partial_j\phi^j\,, && (3.41) \\ \Upsilon &= -e_j^b\varphi_b^j\,, && (3.42) \\ \mathcal{H}_i &= e_i^a K_a - e_i^a D_j\varphi_a^j + T_{ij}^a\varphi_a^j - \omega_i^a\tilde{J}_a - A_i\tilde{K}\,. && (3.43) \end{aligned}$$

Note that the Weyl invariance has not been fixed so the torsion components $T_{ij}^a$ remain undetermined. In this sector one can also identify $K_a \approx 0$ as a first-class constraint (which is identically fulfilled). However, since (3.2) is quadratic in the fermionic variables, it can be shown that it does not act on the phase space,

$$\{K_a, F\} = 0\,, \tag{3.44}$$

[7]The Dirac matrix is defined as $\Omega_{AB} := \{\phi_A, \phi_B\}$, where the indexes $A, B$ range over all the constraints [25].

| Sector | Gauge | Generators | $F$ | $S$ |
|---|---|---|---|---|
| Any generic | $N^{\perp}=0$ | $\tilde{J}_a, \tilde{K}, \mathcal{H}_i, \Upsilon$ | 7 | $14+4n$ |
| Spin-1/2 | $v=0$ | $\tilde{J}_a, \tilde{K}, \tilde{K}_a$ | 7 | $14+4n$ |
| Vacuum | — | $\tilde{J}_a, \tilde{K}, \varphi^i_a$ | 10 | $8+4n$ |

Table 3.1: The number of local degrees of freedom for different dynamical sectors. In these cases, $n=1$ for real Majorana spinors and $n=2$ for complex Dirac spinors.

for any function of the physical fields. As $K_a \equiv \frac{\partial \mathcal{L}}{\partial e^a_t}$ can be regarded as the $(t-a)$ components of the energy-momentum tensor, (3.44) is a consequence of the fact that the linearized version of $\mathcal{T}^{\mu}_{\nu}=0$ is fulfilled identically. Considering this functional degeneracy of $K_a$, we see that diffeomorphisms (3.43) are composed only of gauge transformations plus certain particular displacements of the vielbein. Moreover, it is clear that the $SO(2,1)\times U(1)\times \mathrm{Diff}\times Weyl$ transformations are generated by a linear combination of the first-class constraints $\tilde{J}_a, \tilde{K}$ and $\varphi^i_a$ only. The remaining constraints, corresponding to $\Omega$, $\bar{\Omega}$, $\phi^j_b$ and $\phi^j$, are second-class as can be checked from their Poisson brackets (C.3).

## 3.2 Degrees of freedom counting

In a theory with $N$ dynamical field components (that is, excluding Lagrange multipliers), $F$ first-class and $S$ second-class constraints, the number of DOF is given by [67]

$$DOF = \frac{2N-2F-S}{2}\,. \tag{3.45}$$

In the system discussed here there are $N=14+4n$ dynamical field components, $A_i, \omega^a{}_i, e^a_i, \psi, \bar{\psi}$. The following table gives the values of $F$ and $S$ in different cases: In all cases, formula (3.45) gives $DOF=2n$, in complete agreement with the naive counting at the beginning of this chapter. Note that the first two sectors share the same number of independent first-class constraints. For the second, one finds an additional diffeomorphism generator instead of the Weyl scaling.

As the possibility of finding another first-class combination cannot be ruled out in general, one could in principle find a sector where all the three diffeomorphism generators and the Weyl scaling (in addition to $\tilde{J}_a$ and $\tilde{K}$) are independent, even though such a configuration would certainly be *non-generic* by definition. However, this would lead to an odd number of second-class constraints and a non-integer result for the $DOF$, according to (3.45).

## 3.3 Analysis

Let us remember the Lagrangian (2.5) is obtained from a CS form for an $\mathfrak{osp}(2|2)$ connection, in which the spinorial component of the connection is a composite field split as $\not{e}^{\alpha}_{\beta}\psi^{\beta}$. This splitting has a number of nontrivial consequences for the dynamical contents of the theory: i) Instead of zero degrees of freedom of a generic CS action, this system has the four propagating DOF of a Dirac spinor; ii) The system acquires a *proper* Weyl rescaling symmetry, i.e., it has no associated Noether charge and can be directly fixed; iii) The metric structures –the dreibein and the induced metric– are invariant under SUSY, and therefore there is no need to include spin-3/2 fields (gravitini); iv) Supersymmetry is reduced from a gauge symmetry to a rigid/global invariance that is contingent on the features of the background geometry and the gauge fields; v) For the vacuum sector the dreibein becomes pure gauge and diffeomorphisms degenerate into $SO(2,1)\times U(1)$ transformations.

The Dirac formalism completely recovers the Lagrangian equations. The equations for the gauge fields $(\omega, A)$ follow from the constraints and the Hamilton equations for these fields. Furthermore, it can be shown that the Dirac equation and equation $\mathcal{T}^{\mu}{}_{\nu}=0$ are respectively equivalent to (C.9,C.10) for an invertible dreibein. In fact, after Weyl fixing and computing the temporal evolution one gets $D_t\psi = e^a_t\zeta_a$ and $T^a_{ti} = e^b_t e^c_i T^a{}_{bc}$. Then, equations (C.9,C.10) together with the constraint (3.2) can be covariantized to give

$$
\begin{aligned}
T^a_{\mu\nu}\bar{\psi}\psi &= \bar{\psi}\gamma^a\gamma_{[\mu}\overrightarrow{D}_{\nu]}\psi + \bar{\psi}\overleftarrow{D}_{[\nu}\gamma_{\mu]}\gamma^a\psi\,, && (3.46)\\
\gamma^\mu D_\mu\psi &= \frac{1}{4}T^a_{\mu\nu}\gamma^{\nu\mu}\gamma_a\psi\,. && (3.47)
\end{aligned}
$$

The degeneracy of these equations follows from the fact that $\mathcal{T}^{\mu}_{\ \mu}$ is proportional to (3.46) and is a combination of the Dirac equation -plus its conjugate-, and therefore identically vanishes for this theory, which is in turn equivalent to Weyl invariance.

It should be stressed that $DOF = 2n$ is an upper bound for the number of local DOF since in non-generic sectors there might be additional accidental first-class constraints and therefore fewer degrees of freedom, as it happens in some sectors of higher-dimensional CS systems [58]. The general counting performed in Section 3.1.1 proceeds under the assumption that this is not the case. The argument given there, using the 'degenerate gauge', even holds for the spin-1/2 sector of Section 3.1.2, but for that configuration, it is illustrative to explicitly use the Weyl fixing instead (see the end of Appendix C.2).

In that sense, the purpose of choosing a specific sector such as the spin-1/2 is twofold: On the one hand, the Lagrange multipliers can be readily solved, allowing for an explicit solution of (C.9,C.10) leading to a full realization of the first-class constraints. On the other, the Weyl symmetry is "gauged away" in this case, providing a symmetry breaking mechanism. One is left with a global version of the scale invariance which is broken by fixing the fermion mass or the normalization of the dreibein.

In this system, SUSY seems to play a marginal role. It starts out as part of the gauge invariance of the action (2.5), then it is seen as a global (rigid) symmetry without first-class constraints associated to it, contingent on the existence of some spacetime symmetry, which need not occur in every spacetime background. The action and the equations are invariant under

$$\begin{aligned}
\delta\psi &= \frac{1}{3}\overrightarrow{\not{D}}\epsilon\,, \quad \delta\overline{\psi} = \frac{1}{3}\overline{\epsilon}\overleftarrow{\not{D}} \\
\delta A &= -\frac{i}{2}\left(\overline{\psi}\not{e}\epsilon + \overline{\epsilon}\not{e}\psi\right) \\
\delta\omega^a &= -\overline{\psi}\left(e^a + \epsilon^a{}_{bc}e^b\gamma^c\right)\epsilon - \overline{\epsilon}\left(e^a - \epsilon^a{}_{bc}e^b\gamma^c\right)\psi\,, \\
\delta e^a &= 0
\end{aligned} \tag{3.48}$$

where $\epsilon$ satisfies the no-spin-3/2 condition, $[\delta^\mu_\nu - (1/3)\gamma_\nu\gamma^\mu]D_\mu\epsilon = 0$. This condition can be fulfilled provided the spacetime and the connection fields admit a Killing spinor of a certain kind [15]. This is the case for the vacuum: AdS or Minkowski space without fermions or electromagnetic fields. This background is a full-BPS state preserving full supersymmetry, but there are configurations preserving 1/2 or 1/4 of SUSY, just like in $(2+1)$ SUGRA [14, 68]. A bosonic vacuum $\psi = 0$ remains invariant under (3.48) provided $\not{D}\epsilon = 0$, which is also a requirement that the background admits a Killing spinor.

Unconventional supersymmetries can also be constructed in higher dimensions based on a gauge superalgebra containing $\mathfrak{so}(2n,2)$ or $\mathfrak{so}(2n-1,1)$ as a proper subalgebra. In odd dimensions $D = 2n+1 \geq 5$, a similar CS construction can be set up, while for $D = 2n \geq 4$, since the CS forms are not defined, the construction requires a metric and the action can be of a Yang-Mills type, as we shall see in detail in Chapter 5. In both cases, the fermionic part of the connection can be construed as a composite of a vielbein and a spin-1/2 Dirac field [15]. For all $D \geq 4$, it can be expected that, as in the three-dimensional case discussed here, the vielbein would not contribute to the dynamic contents unless it possesses an independent kinetic term of its own; the effective gauge symmetry would correspond to the bosonic part of the superalgebra, and supersymmetry would be reduced to a rigid invariance conditioned by the existence of globally defined Killing spinors of the background. In other words, supersymmetry would be at most an approximate feature in some vacuum spacetime geometries, and the main footprint of its presence in the theory would be in the field content, the type of couplings and the parameters in the action.

As we shall see in Chapter 9, the conduction properties of graphene [69–71] can be very well described by the $\pi$-electrons in the two sublattices of the honeycomb structure as massless fermions in the long-wavelength limit [40]. It was already conjectured that the system studied here could reproduce the behaviour of these $\pi$-electrons [12, 14], while the very strong $\sigma$-bond of the remaining available electrons of the carbon atoms keep the geometry of the graphene layer fixed. Therefore it is expected that in the low energy (long wavelength) regime, the dynamical contents are essentially in the fermion sector, as pointed out here. Nevertheless, note that we have introduced a torsional mass term, which is required in principle by hermiticity. Such construction not only leads to a symmetry breaking mechanism but, it also allows the massive fermion to trigger a backreaction into the background, provided we use the contorsion as an effective cosmological constant. This implies a constant curvature background as illustrated in [12]. Following that line, an idea to be experimentally explored is whether specific graphene layers (or graphene-like material) can be manufactured which admit Killing-spinors in order to measure some induced SUSY effects. This would provide low-energy graphene models to test high-energy physics theories, whose observable effects are beyond reach in current particle accelerators [31, 32].

Besides providing a rigorous tool for identifying the dynamical DOF, the Hamiltonian formalism could be the preliminary warm-up towards a quantization procedure [25, 26], eventually leading to a quantum theory of graphene. In the system described here, the only dynamical degrees of freedom are those of the Dirac fermion; the bosonic connections $A$ and $\omega$ are described by CS actions and therefore have no local DOF, while the dreibein is an artefact that can be gauged away. This means that the bosonic fields could not contribute to the QFT other than as classical external fields; their quantum excitations would be produced by nontrivial global holonomies of a topological nature. Such fields would not propagate and hence should not generate perturbative corrections. In particular, there should be no perturbative corrections generated by quantum fluctuations of the bosonic fields in graphene, the system should behave like a free electron field propagating in a curved classical background and, therefore, one can reasonably expect to be renormalizable.

# Chapter 4

# Internal $SU(2)$ group

In this section, we shall describe an extension of the group $OSP(2|2)$ in order to include a particular non-Abelian internal gauge symmetry, as $SU(2)$. As we will see, the procedure to construct such a theory is very similar to that followed in Chapter 2. The only difference is the addition, due to the transformation of the fermion under $SU(2)$ fundamental representation, of an extra internal index for such a fermion. We also present some interesting bosonic solutions, as $\mathrm{AdS}_3$ and BTZ black holes, and their Killing spinors.

## 4.1 Connection, Lagrangian and field equations

The minimal graded Lie algebra where the fields are in a linear representation, which includes $\mathfrak{su}(2)$ and $\mathfrak{so}(2,1)$ as subgroups, is $\mathfrak{usp}(2,1|2)$ [14]. We take a one-form connection, which is spanned by the $\mathfrak{su}(2)$ generators $\mathbb{T}_I$, the Lorentz generators $\mathbb{J}_a$, SUSY generators $\mathbb{Q}_i^\alpha$, $\overline{\mathbb{Q}}^i_\alpha$, and it is needed an extra generator $\mathbb{Z}$ (a central $U(1)$ extension) in order to close the algebra, as we shown in Appendix B.2. With the role of dreibein as connecting the tangent space and the base manifold, such a one-form connection can be written as

$$\mathbb{A} = A^I \mathbb{T}_I + \overline{\psi}^i \not{e} \mathbb{Q}_i + \overline{\mathbb{Q}}^i \not{e} \psi_i + \omega^a \mathbb{J}_a + b\mathbb{Z} \ , \tag{4.1}$$

where this time we omitted the spinorial indexes for the sake of simple notation. The index $I = 1,2,3$ is in the adjoint representation of $\mathfrak{su}(2)$ subalgebra, while the index $i = 1,2$ is in the fundamental one. We define the Dirac adjoint as

$$\overline{\psi}^i_\alpha = i\psi_j^{\dagger\beta} C_{\beta\alpha} \delta^{ji} \ . \tag{4.2}$$

Here

$$(\gamma_a)^\dagger = C\gamma_a C\,, \quad C^\dagger = -C\,, \quad C^2 = -1\,.$$

so that $C^\dagger C = 1$. In this case, the spinor need not be charged with respect to $\mathfrak{u}(1)$ because of $\mathbb{Z}$ is central. Part of the geometric structure and the details of the representation can be readily seen from the curvature two-form[1], $\mathcal{F} \equiv d\mathbb{A} + \mathbb{A}^2$,

$$\mathcal{F} = \mathcal{F}^a \mathbb{J}_a + \mathcal{F}^I \mathbb{T}_I + \overline{\mathbb{Q}}^i \mathcal{F}_i + \overline{\mathcal{F}}^i \mathbb{Q}_i + \mathcal{F}_{(b)}\mathbb{Z}\,, \tag{4.3}$$

whose components are given by

$$\mathcal{F}^a = R^a - \epsilon^a{}_{bc} e^b e^c \overline{\psi}\psi\,, \tag{4.4}$$

$$\mathcal{F}^I = F^I - i\epsilon_{abc} e^a e^b \overline{\psi}\gamma^c \sigma^I \psi\,, \tag{4.5}$$

$$\mathcal{F}_i = D_i{}^j (\not{e}\psi_j)\,, \tag{4.6}$$

$$\overline{\mathcal{F}}^i = -(\overline{\psi}^j \not{e})\overleftarrow{D}_j{}^i\,, \tag{4.7}$$

$$\mathcal{F}_{(b)} = db - i\epsilon_{abc} e^a e^b \overline{\psi}\gamma^c \psi\,. \tag{4.8}$$

Here $R^a = d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b\omega^c$ is the Lorentz curvature 2-form, and $F^I = dA^I + \frac{1}{2}\epsilon^I{}_{JK}A^J A^K$ is the $su(2)$ curvature. We use $D$ for the exterior covariant derivative for an $so(1,2) \times su(2)$ connection. In particular, for the spin-1/2 fundamental representation,

$$\begin{aligned} \overrightarrow{D}_i{}^j &= \delta_i^j \overrightarrow{d} - \frac{i}{2} A^I (\sigma_I)_i{}^j + \frac{1}{2}\delta_i^j \omega^a \gamma_a \\ \overleftarrow{D}_i{}^j &= \delta_i^j \overleftarrow{d} + \frac{i}{2}(\sigma_I)_i{}^j A^I - \frac{1}{2}\delta_i^j \gamma_a \omega^a\,. \end{aligned} \tag{4.9}$$

In (4.9) we have used the full Lorentz connection (metric compatible) $\omega^a$. The Lorentz connection can be split uniquely as $\omega^a = \mathring{\omega}^a + \kappa^a$, where $\mathring{\omega}^a$ is the torsion-free connection ($de^a + \mathring{\omega}^a{}_b e^b = 0$) and $\kappa^a$ is the contorsion one-form, so that $T^a = \epsilon^a{}_{bc}\kappa^b e^c$.

As before, the Lagrangian for this theory can be written as a CS form,

$$L = \frac{\kappa}{2}\langle \mathbb{A}d\mathbb{A} + \frac{2}{3}\mathbb{A}^3 \rangle\,, \tag{4.10}$$

where $\langle \ldots \rangle$ is the invariant supertrace of $\mathfrak{su}(2)$ graded Lie algebra [14], (see Appendix B). This way, the Lagrangian can be written simply as

$$L = \frac{\kappa}{4}\left(A^I dA_I + \frac{1}{3}\epsilon_{IJK}A^I A^J A^K\right) + \frac{\kappa}{4}\left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right) \tag{4.11}$$

$$- \kappa\overline{\psi}\left(\gamma^\mu \overrightarrow{D}_\mu - \overleftarrow{D}_\mu \gamma^\mu + \frac{1}{2}\epsilon^a{}_{bc}T_a{}^{bc}\right)\psi|e|d^3x\,. \tag{4.12}$$

---

[1] We omit internal indexes and free spinorial indexes whenever it does not lead to ambiguities, so that $\overline{\psi}\psi \equiv \overline{\psi}^i\psi_i$ and $\overline{\psi}\sigma^I\psi \equiv \overline{\psi}^i(\sigma^I)_i{}^j\psi_j$.

The field equations are given by the zero-curvature conditions,

$$\mathcal{F}^a = 0\,, \quad \mathcal{F}^I = 0\,, \tag{4.13}$$

the Dirac equation and the resulting equation from the variation with respect to the local frame.
Varying with respect to $\overline{\psi}$ and dropping a boundary term $\partial_\mu(-\kappa\,|e|\delta\overline{\psi}\psi)$, we obtain the Dirac equation,

$$0 = 2[\gamma^\mu \mathring{D}_\mu + m]\psi + \epsilon_{abc}\mathring{\omega}^a{}_\mu E^{b\mu}\gamma^c\psi + |e|^{-1}\partial_\mu(|e|\gamma^\mu)\psi\,, \tag{4.14}$$

where the last two terms are required by hermiticity. In (4.14) we have defined the quantity,

$$m = \frac{1}{8}\epsilon^{abc}T_{abc}\,, \tag{4.15}$$

associated to a nonzero torsion of the spacetime.
Varying with respect to the vierbein $e^a_\mu$ gives $\delta(|e|\mathcal{L}_\psi) = |e|\delta e^a{}_\mu \tau_a{}^\mu + \kappa\,\partial_\mu[|e|\epsilon^{\mu\nu}{}_a \delta e^a{}_\nu \overline{\psi}\psi]$, where

$$\tau_a{}^\mu = \kappa E_{c\mu}\epsilon^{cbd}\left[\overline{\psi}\overleftarrow{D}_d\gamma_b\gamma_a\psi + \overline{\psi}\gamma_a\gamma_b\overrightarrow{D}_d\psi - \overline{\psi}\psi T_{adf}\epsilon^{cdf}\right]\,. \tag{4.16}$$

Therefore, the field equation

$$\tau_a{}^\mu = 0\,, \tag{4.17}$$

guarantees that the stress-energy tensor,

$$t^{\mu\nu} = -\frac{1}{4}\kappa E_{c\mu}E_{b\nu}\epsilon^{bcd}\left[\overline{\psi}\overleftarrow{D}_d\gamma_b\gamma_a\psi + \overline{\psi}\gamma_a\gamma_b\overrightarrow{D}_d\psi - \overline{\psi}\psi T_{abd}\right] + (\mu \leftrightarrow \nu)\,, \tag{4.18}$$

vanishes on-shell[2] as a consequence of Weyl invariance. Another way to read this is that the on-shell torsion acts as a source for the usual spinor contribution to the energy-momentum tensor.

## 4.2 Symmetries

Due to the way the dreibein is implemented, the Lagrangian (4.11) has the scale symmetry (2.14) and, because it is built from a CS form, it is immediately invariant under diffeomorphisms. A gauge transformation generated by a local, infinitesimal, $su(2,1|2)$-valued zero-form $G$,

$$G = \rho^I\mathbb{T}_I + \overline{\mathbb{Q}}^i\varepsilon_i - \overline{\varepsilon}^i\mathbb{Q}_i + \lambda^a\mathbb{J}_a + \lambda\mathbb{Z}\,, \tag{4.19}$$

[2]Here $t^{\mu\nu} \equiv (2/\sqrt{-g})\delta(\sqrt{-g}\mathcal{L}_\psi)/\delta g_{\mu\nu} = (1/2)\eta^{ab}(E_a{}^\mu\tau_b^\nu + E_a{}^\nu\tau_b^\mu)$, and we used a convention in which $s^{\mu\nu} = s^{(\mu\nu)}$ for symmetric tensors.

induces transformations on the component fields $\omega^a$, $A^I$, $e^a$, $\psi_i$ and $b$. While the transformation laws for $\omega^a$, $A^I$ and $b$ are straightforward to read off from the usual rule $\delta\mathbb{A} = d\mathbb{A} + [\mathbb{A}, G] \equiv DG$, extra care is required when handling $e^a$ and $\psi_i$, since this expression only determines the variation of the product $\not{e}\psi_i$. In order to see the form of the transformations on the fields $\psi_i$ and $e^a$, we follow the prescription given in Chapter 2, which basically ensures that the vielbein remains invariant under gauge and SUSY transformations, but rotates as a vector under the Lorentz subgroup. This is in line with the standard assumption that the metric is unaffected by internal gauge transformations like $U(1)$ and $SU(N)$.

### 4.2.1 Internal gauge symmetry.

The $u(1)$ transformations generated by $\lambda\mathbb{Z}$ affect only the $u(1)$ field $b$, which changes as $\delta b = d\lambda$. Under $su(2)$, the nonzero transformations are

$$\delta A^I = D\rho^I \,, \tag{4.20}$$

$$\delta\psi_i = \frac{i}{2}\rho^I (\sigma_I)_i{}^j \psi_j \,, \tag{4.21}$$

$$\delta\overline{\psi}^i = -\frac{i}{2}\overline{\psi}^j \rho^I (\sigma_I)_j{}^i \,, \tag{4.22}$$

where we have defined the $su(2)$ covariant derivative in the adjoint representation, $DS^I \equiv dS^I + \epsilon^{IJK} A_J S_K$. It is consistent to keep the same notation for the full $so(1,2) \times su(2)$ covariant derivative as long as it is used with the appropriate representation of the argument. In this realization there are no gauginos and the matter field $\psi_i$ transforms in the fundamental representation of $SU(2)$.

### 4.2.2 Lorentz symmetry.

Under Lorentz rotations, $e^a$ transforms as a vector while the metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$ is insensitive to the choice of local orthonormal basis in the tangent space. Consequently, one finds[3]

$$\delta\omega^a = D\lambda^a\,, \tag{4.23}$$

$$\delta e^a = -\epsilon^{abc}\lambda_b e_c\,, \tag{4.24}$$

$$\delta\psi_i = -\frac{1}{2}\lambda^a\gamma_a\psi_i\,, \tag{4.25}$$

$$\delta\overline{\psi}^i = \frac{1}{2}\lambda^a\overline{\psi}^i\gamma_a\,, \tag{4.26}$$

where $D\lambda^a \equiv d\lambda^a + \epsilon^{abc}\omega_b\lambda_c$ defines the Lorentz covariant derivative of $\lambda^a$. Note that the matter field $\psi_i$ transforms in the spin-1/2 representation automatically, and not in the adjoint of $so(1,2)$. Obviously, $A^I$ remains invariant under local Lorentz transformations.

### 4.2.3 Supersymmetric rotations.

Under SUSY transformations $\omega^a$, $A^I$ and $b$ change by

$$\delta\omega^a = e^a(\overline{\psi}\varepsilon + \overline{\varepsilon}\psi) - \epsilon^a{}_{bc}e^b(\overline{\psi}\gamma^c\varepsilon - \overline{\varepsilon}\gamma^c\psi)\;, \tag{4.27}$$

$$\delta A^I = -ie^a(\overline{\varepsilon}\sigma^I\gamma_a\psi + \overline{\psi}\sigma^I\gamma_a\varepsilon)\;, \tag{4.28}$$

$$\delta b = -ie^a(\overline{\varepsilon}\gamma_a\psi + \overline{\psi}\gamma_a\varepsilon)\;, \tag{4.29}$$

where the $su(2)$ indexes are traced over (and omitted). We used here the same prescription given in Section 2.3 to deal with the decomposition of the vierbein $e^a_\mu$ and the spinor $\psi$. This leads to the infinitesimal transformations,

$$\delta\psi_i = \frac{1}{3}\gamma^\mu(D_\mu\varepsilon)_i\;, \tag{4.30}$$

$$\delta\overline{\psi}^i = -\frac{1}{3}(\overline{\varepsilon}\overleftarrow{D}_\mu)^i\gamma^\mu\;, \tag{4.31}$$

$$\delta e^a = 0\;. \tag{4.32}$$

Here, as for the $OSP(2|2)$ case, $\varepsilon$ will be regarded as a global instead of a local parameter.

In the next subsection, we describe the classical solutions for these field equations that have vanishing vacuum energy but a nonzero $\Lambda_{\text{eff}}$.

[3] In this one-index notation, eq. (4.24) is equivalent to $\delta e^a = -\lambda^a{}_b e^b$, with $\lambda^{ab} = -\epsilon^{abc}\lambda_c$.

## 4.3 Vacuum solutions

The field equations (4.13) in the matter-free sector, $\psi = 0 = \overline{\psi}$, imply that spacetime is locally Lorentz flat ($R^{ab} = 0$) and $SU(2)$ flat ($F^I = 0$). The interesting point is that, however, this does not necessarily imply a trivial geometry or a trivial $SU(2)$ configuration. The field equations admit other nontrivial solutions depending on the topology and boundary conditions. Moreover, these equations do not completely determine the metric structure and there is a large family of nontrivial solutions that solve them, as discussed in this section.

#### Lorentz-flat geometry

As shown in [72], the most general (2+1)-geometry compatible with $R^{ab} = 0$ is a geometry of constant negative (Riemann) curvature, i.e., $\mathrm{AdS}_3$. Minkowski space is also allowed as a limiting case of vanishing cosmological constant. This can be seen as follows.

The Lorentz connection $\omega$ can be uniquely split into a torsion-free part and the *contorsion* as

$$\omega^a{}_b = \mathring{\omega}^a{}_b + \kappa^a{}_b, \tag{4.33}$$

The torsion-free condition can be solved for $\mathring{\omega}$ in terms of the vielbein and a symmetric affine connection (Christoffel symbol). The Lorentz curvature $R^{ab}$ splits into the torsion-free (Riemannian) curvature $\mathring{R}^{ab}$ and torsion-dependent terms,

$$R^{ab} = \mathring{R}^{ab} + \mathring{D}\kappa^{ab} + \kappa^a{}_c\kappa^{cb} \ , \tag{4.34}$$

where for $\mathring{D}(\ldots)$ we understand the covariant derivative with respect to the torsion-free connection $\mathring{\omega}^a{}_b$. Clearly the Lorentz-flat condition $R^{ab} = 0$ does not necessarily imply $\mathring{R}^{ab} = 0$. The Lorentz-flat condition (4.34), however, implies that the torsion-free connection $\mathring{\omega}^{ab}$ is not generically flat, but $\mathring{R}^{ab} = -\mathring{D}\kappa^{ab} - \kappa^a{}_c\kappa^{cb}$. From the field equation $\mathcal{F}^a = 0$, we have immediately $R^a_b e^b = 0$, which means $DT^a = 0$. In three dimensions, this equation has the solution [72]

$$T^a = \frac{\epsilon}{l}\epsilon^a{}_{bc}e^b e^c \ , \tag{4.35}$$

where $l > 0$ is an arbitrary integration constant with dimensions of length and $\epsilon = \pm 1$. The introduction of a dimensionful constant fixes the scale for the classical configuration, breaks Weyl invariance and gives us an effective mass term for the fermion

$$m = -\frac{3\epsilon}{2l} \, . \tag{4.36}$$

Thus, from relation (4.35), the contorsion can be written as

$$\kappa^a{}_b = -\frac{\epsilon}{l}\epsilon^a{}_{bc}e^c, \tag{4.37}$$

where we reserved the constant $l$ for the vacuum case and $l > 0$. It can be directly checked that $\mathring{D}\kappa^{ab} = 0$ and finally

$$\mathring{R}^{ab} = -\frac{1}{l^2}e^a e^b\,. \tag{4.38}$$

The torsion-free part of the Lorentz connection defines the Riemann tensor that accounts for the purely metric (torsion free) curvature,

$$\mathcal{R}^{\alpha\beta}{}_{\mu\nu} = E^\alpha{}_a E^\beta{}_b \mathring{R}^{ab}{}_{\mu\nu}. \tag{4.39}$$

Combining (4.38) and (4.39), the Riemann tensor for a Lorentz-flat connection is found to be [72]

$$\mathcal{R}^{\alpha\beta}{}_{\mu\nu} = -\frac{1}{l^2}\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right). \tag{4.40}$$

Therefore, even if the contribution to the vacuum energy from the fermion condensate were to vanish ($\overline{\psi}\psi = 0$), there is an effective cosmological constant $\Lambda_{\text{eff}} = -\frac{1}{l^2}$, where $l$ is an arbitrary integration constant. The solution with flat Riemann curvature can also be accommodated by taking $\epsilon = 0$ (or $l \to \infty$). Note that, while there is a sign ambiguity in the torsion ($\epsilon = \pm 1$), no such ambiguity exists for the curvature, which means that this result is not true for $\Lambda > 0$: de Sitter spacetime is not a Lorentz-flat geometry.

Considering that the symmetry used to define the model is a superextension of Lorentz symmetry, it is interesting that either flat or negative curvature spaces could emerge spontaneously. Positive curvature, however, is not allowed. We can compare this fact with the four-dimensional case in which de Sitter is not favoured by SUSY either [73].

Conversely, (4.38) implies that any simply connected patch of three-dimensional anti-de Sitter space can be endowed with a flat Lorentz connection, just like any patch of Minkowski space. This result can be seen as the Lorentzian version of Adams' theorem, which states that $S^3$ is parallelizable, i.e., it can be endowed with a globally defined flat $SO(3)$ connection [74, 75]. This theorem is only valid for $S^0, S^1, S^3$ and $S^7$, so it should not surprise us to have also a similar conclusion in $D = 7$ and in no other cases.

In the presence of matter the fermion condensate $\langle\overline{\psi}\psi\rangle$ relates to the curvature of space and the magnitude of the torsion by means of,

$$\mathring{R}^{ab} = (2\langle\overline{\psi}\psi\rangle - \frac{1}{l^2})e^a e^b\,, \tag{4.41}$$

and therefore, the effective cosmological constant is

$$\Lambda_{\text{eff}} = 2\left\langle \overline{\psi}\psi \right\rangle - \frac{1}{l^2}\,. \tag{4.42}$$

This implies that, in order to avoid the appearance of a tachyonic mass term, the following condition has to be satisfied,

$$\frac{1}{l^2} = 2\left\langle \overline{\psi}\psi \right\rangle - \Lambda_{\text{eff}} \geq 0\,, \tag{4.43}$$

or $\Lambda_{\text{eff}} \leq 2\left\langle \overline{\psi}\psi \right\rangle$. There are three different cases: First, if $\left\langle \overline{\psi}\psi \right\rangle < 0$ only AdS spaces are allowed. Second, if $\left\langle \overline{\psi}\psi \right\rangle = 0$, then spacetimes with $\Lambda_{\text{eff}} \leq 0$ are allowed. Finally, for $\left\langle \overline{\psi}\psi \right\rangle > 0$, $\Lambda_{\text{eff}}$ can take any values in the range $-\infty < \Lambda_{\text{eff}} \leq 2\left\langle \overline{\psi}\psi \right\rangle$ allowing for metrics that include flat, negative, and a small window of positive curvature spacetimes.

Summarizing, the general solution for the matter-free equations is a spacetime that is locally $AdS_3$ (or Minkowski), where the cosmological constant $\Lambda = -1/l^2$ is an arbitrary integration parameter. This family of geometries includes $AdS_3$ with or without identification, in particular, the $(2+1)$-black hole [76] and spinning point particles [77].

### 4.3.1 The $(2+1)$ black hole as a Lorentz-flat geometry

The $(2+1)$-black hole is locally $AdS_3$, and the local frame that corresponds to the rotating solution reads [78]

$$e^0 = f dt\,, \tag{4.44}$$

$$e^1 = f^{-1} dr\,, \tag{4.45}$$

$$e^2 = r\left(d\varphi + N^{\varphi} dt\right), \tag{4.46}$$

where

$$f(r) = \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{1/2}, \tag{4.47}$$

$$N^{\varphi} = -\frac{J}{2r^2}, \tag{4.48}$$

and $(M, J)$ are integration constants corresponding to the mass and angular momentum. The vanishing torsion condition, $de^a + \mathring{\omega}^a{}_b e^b = 0$, can be solved for the connection yielding

$$\mathring{\omega}^0{}_1 = \frac{r}{l^2} dt - \frac{J}{2r} d\varphi\,, \tag{4.49}$$

$$\mathring{\omega}^1{}_2 = -f d\varphi\,, \tag{4.50}$$

$$\mathring{\omega}^2{}_0 = -\frac{J}{2fr^2} dr\,. \tag{4.51}$$

The corresponding Riemannian two-form has constant, negative (or zero) curvature, $\mathring{R}^{ab} = -l^{-2} e^a e^b$. On the other hand, the full Lorentz connection $\omega^a{}_b$, including the contorsion (4.37), reads

$$\omega^0{}_1 = \left( \frac{r}{l} - \epsilon \frac{J}{2r} \right) \left[ \frac{1}{l} dt + \epsilon d\varphi \right] , \tag{4.52}$$

$$\omega^1{}_2 = -f \left[ \frac{\epsilon}{l} dt + d\varphi \right] , \tag{4.53}$$

$$\omega^0{}_2 = -\frac{1}{lf} \left( \frac{Jl}{2r^2} + \epsilon \right) dr , \tag{4.54}$$

and is explicitly checked to be flat, $R^{ab} = 0$.

Other black holes solutions in the presence of torsion have also been derived in detail under the Mielke-Baeckler model [79, 80], but we want merely to stress here the fact we can realize the BTZ solution as a Lorentz flat geometry irrespective of the model which it is implemented.

### 4.3.2 Flat $su(2)$ sector

In addition to being locally AdS$_3$, the vacuum solutions have a locally flat $SU(2)$ connection. This connection is locally pure gauge and therefore can be gauged away in any simply connected patch. But the possibility of gauging it away everywhere depends on the topology of the manifold.

Since $su(2)$ and $so(1,2)$ are locally isomorphic, and the corresponding generators $1/2\sigma_I$ and $1/2\gamma_a$ are the same up to factors of $\pm i$ [cf. eq. (B.6)], one can use the connection (4.52)–(4.54) to tailor the $su(2)$ field $A^I$ as

$$A^1 = -i \frac{\eta}{hs} \left(1 - \eta s V^\varphi\right) dr , \tag{4.55}$$

$$A^2 = -h \left[ \frac{\eta}{s} dt + d\varphi \right] , \tag{4.56}$$

$$A^3 = -i \frac{\eta r}{s} \left(1 + \eta s l V^\varphi\right) \left[ \frac{\eta}{s} dt + d\varphi \right] , \tag{4.57}$$

where $\eta = \pm 1$, $s$ is an arbitrary length scale (not necessarily equal to $l$), and

$$h(r) = \left( \frac{r^2}{s^2} - W + \frac{K^2}{4r^2} \right)^{1/2} , \tag{4.58}$$

$$V^\varphi = -\frac{K}{2r^2} . \tag{4.59}$$

The flat $su(2)$ solution (4.55)–(4.57) makes the asymptotic behavior of the field as

$$\overline{A}^1 = -i\frac{\eta}{r}dr \,, \tag{4.60}$$

$$\overline{A}^2 = -\frac{r}{s}\left[\frac{\eta}{s}dt + d\varphi\right] \,, \tag{4.61}$$

$$\overline{A}^3 = -i\frac{r}{s}\eta\left[\frac{\eta}{s}dt + d\varphi\right] \,. \tag{4.62}$$

Here $(W, K)$ are integration constants. This configuration for $A^I$ closely mimics the Lorentz connection $\omega^a{}_b$ [cf. eqs. (4.52)–(4.54)], but the field equations allow nonetheless for independent integration constants $(W, K, s)$. As in the Lorentz connection, there is a sign ambiguity $(\eta)$ in the solution for $F^I = 0$, but in this case it is not related to another structure because in $SU(2)$ there is no analogue for the local frame or the torsion. Additionally, the solution (4.55)–(4.57) allows another sign freedom that corresponds to the choice of sign in the square root to define $h$. There is no analogue of this sign freedom in the Lorentz connection, since it would amount to choosing a local basis in tangent space with the opposite handedness relative to the coordinate basis.

### 4.3.3 Conserved Charges

In order to define conserved charges in the presence of nontrivial boundary conditions it is necessary to add boundary terms to the action to make sure that the action remains stationary on the classical orbits. For trivial conditions, in which all the fields are fixed at the boundary and the space is asymptotically flat, it is often unnecessary to take these precautions, but for asymptotically AdS spacetimes boundary terms are often required. In particular, the CS terms in a non-compact manifold must be supplemented with a boundary term and appropriate boundary conditions that regularize the action [81]. Let us assume a three-dimensional manifold whose local geometry is of the form $\mathcal{M} = \mathbb{R}\times\Sigma$, where $\mathbb{R}$ represents the time direction and $\Sigma$ is the two-dimensional spatial section. As shown in [82–84], the regularized Chern-Simons action in three dimensions is given by the *transgression form* $\mathcal{T}$ defined by

$$\mathcal{T} = L_{\text{CS}}(\mathbb{A}) + B(\mathbb{A}, \overline{\mathbb{A}}) - L_{\text{CS}}(\overline{\mathbb{A}}) \,, \tag{4.63}$$

where $\overline{\mathbb{A}}$ corresponds to a fixed classical solution, that matches $\mathbb{A}$ at the boundary and has zero curvature $\mathcal{F}_{\overline{\mathbb{A}}} = 0$. The term $B$ lives on the boundary and is such that it makes the action gauge invariant. Varying $\mathcal{T}$ with respect to $\mathbb{A}$ (with $\delta\overline{\mathbb{A}} = 0$) yields

$$\delta\mathcal{T} = \kappa\left\langle \delta\mathbb{A}\mathcal{F}_{\mathbb{A}}\right\rangle - \frac{\kappa}{2}d\left\langle \mathbb{A}\delta\mathbb{A}\right\rangle + \delta B. \tag{4.64}$$

The first term on the right-hand side vanishes on-shell and therefore $B$ must be such that

$$\left[\delta B - \frac{\kappa}{2}\int_{\partial\mathcal{M}}\langle \mathbb{A}\delta\mathbb{A}\rangle\right]\Bigg|_{\text{on-shell}} = 0. \tag{4.65}$$

The boundary condition $\mathbb{A}\big|_{\partial\mathcal{M}} = \overline{\mathbb{A}}$ means that (4.65) is fulfilled if

$$B = -\frac{\kappa}{2}\int_{\partial\mathcal{M}}\left\langle \mathbb{A}\overline{\mathbb{A}}\right\rangle . \tag{4.66}$$

The variation of the transgression form around an arbitrary (infinitesimal) configuration $\delta\mathbb{A}$ will be given by

$$\delta\mathcal{T} = \kappa\langle\delta\mathbb{A}\mathcal{F}_{\mathbb{A}}\rangle + d\Theta. \tag{4.67}$$

where

$$\Theta = \frac{\kappa}{2}\langle\delta\mathbb{A}\left(\mathbb{A}-\overline{\mathbb{A}}\right)\rangle . \tag{4.68}$$

Under gauge transformations $\delta\mathbb{A} = D_{\mathbb{A}}G$, $\delta\overline{\mathbb{A}} = D_{\overline{\mathbb{A}}}G$, the transgression remains invariant off-shell, $\delta_{\text{gauge}}\mathcal{T} = 0$. Consequently, by Noether's theorem, there is a conserved current $*\mathcal{J} = -\Theta$ [84]. Demanding the reducibility condition at spatial infinity [85],

$$D\xi\big|_{\partial\Sigma} = 0\,, \tag{4.69}$$

the conserved charge can be found as [84]

$$Q[\xi] = \kappa\left\langle\left(\mathbb{A}-\overline{\mathbb{A}}\right)\xi\right\rangle . \tag{4.70}$$

This last expression will be used now to compute the global charges of nontrivial vacuum solutions.

The nontriviality of the configuration given in Sections 4.3.1 and 4.3.2 can be assessed by computing the conserved charges (4.70). For a generator $\xi = \alpha^a\mathbb{J}_a + \beta^I\mathbb{T}_I \in so(1,2)\oplus su(2)$ we explicitly have

$$Q[\xi] = \frac{\kappa}{4}\left[(\omega^a-\overline{\omega}^a)\alpha_a + (A^I-\overline{A}^I)\beta_I\right]. \tag{4.71}$$

This charge requires the definition of the asymptotic behavior of $\overline{\omega}^a$ and $\overline{A}^I$ [85], given by the leading order in $r$ for $r\to\infty$, where these connections approach those of the massless black hole and the uncharged $su(2)$ solution. Therefore

$$\omega^0 - \overline{\omega}^0 = -\frac{Ml}{2r}\left(\frac{\epsilon}{l}dt + d\varphi\right) + O\left(r^{-3}\right) , \tag{4.72}$$

$$\omega^1 - \overline{\omega}^1 = \frac{l^2}{2r^3}\left(\epsilon M + \frac{J}{l}\right)dr + O(r^{-5})\,, \tag{4.73}$$

$$\omega^2 - \overline{\omega}^2 = -\frac{J}{2r}\left(\frac{\epsilon}{l}dt + d\varphi\right) , \tag{4.74}$$

and

$$A^1 - \overline{A}^1 = -i\frac{s^2}{2r^3}\left(\eta W + \frac{K}{s}\right) dr + O(r^{-5}) \,, \tag{4.75}$$

$$A^2 - \overline{A}^2 = \frac{Ws}{2r}\left(\frac{\eta}{s}dt + d\varphi\right) + O\left(r^{-3}\right) \,, \tag{4.76}$$

$$A^3 - \overline{A}^3 = i\frac{K}{2r}\left(\frac{\eta}{s}dt + d\varphi\right) \,. \tag{4.77}$$

Now, the reducibility condition (4.69) for $\xi$ implies in the asymptotic region

$$d\alpha^a + \epsilon^a{}_{bc}\omega^b\alpha^c = 0 \,, \quad d\beta^I + \epsilon^I{}_{JK}A^J\beta^K = 0 \,. \tag{4.78}$$

The asymptotic solutions are

$$\alpha^0 = \epsilon c_1\left(r + \frac{\epsilon l J - l^2 M}{2r}\right) + O\left(r^{-2}\right) \,, \tag{4.79}$$

$$\alpha^1 = 0 \,, \tag{4.80}$$

$$\alpha^2 = c_1 r + O\left(r^{-1}\right) \,, \tag{4.81}$$

and

$$\beta^1 = 0 \,, \tag{4.82}$$

$$\beta^2 = \eta c_2\left(r + \frac{\eta s K - l^2 W}{2r}\right) + O\left(r^{-2}\right) \,, \tag{4.83}$$

$$\beta^3 = i\eta c_2 r + O\left(r^{-1}\right) \,. \tag{4.84}$$

with $c_1$ and $c_2$ some arbitrary constants. Finally, the Noether charge (4.71) is found to be

$$\begin{aligned} Q = \kappa\left(\frac{c_1}{8l}(\epsilon M l - J) + O\left(r^{-2}\right)\right)[\epsilon dt + l d\varphi] \\ + \kappa\left(\frac{c_2}{4s}(\eta W s - K) + O\left(r^{-2}\right)\right)[\eta dt + s d\varphi]. \end{aligned} \tag{4.85}$$

This charge must be integrated on a circle at spatial infinity of a time slice, to obtain the conserved quantities associated to the two symmetry groups,

$$\int_{S^1_\infty} Q = \frac{\pi\kappa}{2}(c_1 q_{SO(1,2)} + c_2 q_{SU(2)}) \,, \tag{4.86}$$

where

$$q_{SO(2,1)} = \epsilon M l - J \,, \tag{4.87}$$

$$q_{SU(2)} = \eta s W - K \,. \tag{4.88}$$

Each of the two symmetry groups have a single Casimir operator and this is reflected in the two charges produced by Noether's procedure. In order to see how these charges determine the configuration, let us consider the charge associated to the Lorentz group, which is determined by two continuous parameters ($M$,$J/l$) and one sign ($\epsilon$). For a fixed value of $q_{SO(2,1)}$, there are two sets of points in the ($M - J/l$) plane that correspond to it,

$$M = \pm\frac{1}{l}\left(J + q_{SO(2,1)}\right). \tag{4.89}$$

These are two straight lines of slope $\pm 1$ that intersect at the point $(Ml, J) = -(0, q_{SO(1,2)})$. As shown in Figure 4.1, these lines (dashed) correspond to all states for some negative value of $q_{SO(1,2)}$, which include black holes (upper wedge), point particles (lower wedge) and unphysical states (left and right wedges). Each of these lines intersects an extremal black hole, $M = |J|/l$, or an extremal spinning particle, $M = -|J|/l$ [77]. As will be shown in the next section, for $q_{SO(1,2)} \neq 0$ those extremal states admit-globally defined Killing spinors (BPS states).

A given value of the $SU(2)$ charge also corresponds to two lines in the $(W, K/y)$ plane, but in this case there is no geometric interpretation provided by the metric, which discriminates between black holes, point particles and unphysical states. In contrast with Poincaré or AdS gauge theories, here we have only one independent charge asociated to space-time or Lorentz gauge transformations [86–89]. It is important to clarify how the flat $SU(2)$ solution should be interpreted. The fact that the fundamental homotopy group of $SU(2)$ is trivial, $\pi_1(SU(2)) = 0$, tells us that that symmetry is necessarily broken in the solution of Section 4.3.2. This is the result of imposing specific asymptotic behavior (4.60)-(4.62) by demanding $D\beta^I = 0$ at spatial infinity. In this sense (4.88) is really an $SU(2)$-singlet charge, computed with respect to certain "orientation" of the $\beta^I$ parameter. The parameter $\beta^I$ is analogous to the Higgs-like field of the 't Hooft-Polyakov monopole solution [90–92]. Here we have not included the charge associated with the central generator $\mathbb{Z}$, which could be treated as a $U(1)$ charge in the usual sense and added to the other two charges trivially.

### 4.3.4 Killing Spinors

If a bosonic system has a classical solution, it is often sufficient to show that it admits globally defined Killing spinors in order to prove perturbative stability. The idea is to embed the theory into a supersymmetric one so that the supersymmetric action is stationary around the classical solution. Then, SUSY is typically enough to show that the classical solution is a local energy

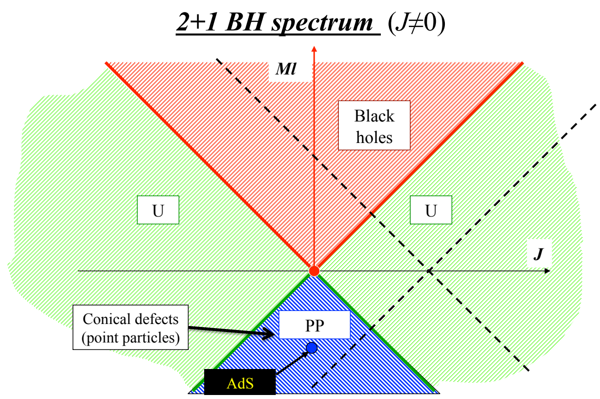


Figure 4.1: Mass-angular momentum phase diagram for three-dimensional solutions of gravity. The upper wedge (light blue, or darker grey in the printed version), $M > |J|/l$, corresponds to nonextremal black holes configurations. The lower wedge (light yellow in the electronic version), $M < -|J|/l$, corresponds to point particles. Left and right wedges, $|M| < |J|/l$, are unphysical configurations. At $M = -1$ we have anti-de Sitter spacetime (square). Solid lines correspond to $M \pm |J|/l = 0$. Dotted lines in the lower wedge correspond to $M \pm |J|/l = -1$. The dashed lines correspond to $-1 < M \pm |J|/l < 0$.

minimum and, therefore, perturbatively stable [93]. Using the covariant derivative (4.9), the Killing spinor equation $(D\psi)_i = 0$ can be written as[4]

$$d\psi + \frac{1}{2}\omega^a \gamma_a \psi - \frac{1}{2} A^I \sigma_I \psi = 0 \ , \tag{4.90}$$

where $\omega^a$ and $A^I$ are given by Equations (4.52)–(4.52) and (4.55)–(4.57), respectively.

### Solutions

The general solution for the Killing spinor equation is given by

$$\psi = U_X U_\gamma U_\sigma U_Y \ \psi_0, , \tag{4.91}$$

where $\psi_0$ is a constant spinor and

$$U_X = X\gamma_{-\epsilon} + \frac{1}{X}\gamma_\epsilon, \tag{4.92}$$

$$U_Y = Y\sigma_\eta + \frac{1}{Y}\sigma_{-\eta}, \tag{4.93}$$

$$U_\gamma = \exp\left[-\theta_{(\epsilon/l)}\gamma_0 \left(\left[-M + \frac{\epsilon J}{l}\right]\gamma_{-\epsilon} + \gamma_\epsilon\right)\right] \ , \tag{4.94}$$

$$U_\sigma = \exp\left[-i\theta_{(\eta/s)} \left(\left[-W + \frac{\eta K}{s}\right]\sigma_\eta + \sigma_{-\eta}\right)\sigma_2\right] \ . \tag{4.95}$$

In (4.92)–(4.95) we have defined

$$X = \left(f + \frac{r}{l} - \frac{\epsilon J}{2r}\right)^{1/2} \ , \tag{4.96}$$

$$Y = \left(h + \frac{r}{s} - \frac{\eta K}{2r}\right)^{1/2} \ , \tag{4.97}$$

$$\theta_{(v)} = \frac{1}{2}\left(vt + \varphi\right) \ , \tag{4.98}$$

$$\gamma_\epsilon = \frac{1}{2}\left(1 + \epsilon\gamma_1\right) , \quad \sigma_\eta = \frac{1}{2}\left(1 + \eta\sigma_1\right) . \tag{4.99}$$

Note that $U_\gamma$ and $U_\sigma$ depend only on $t$ and $\varphi$, while $U_X$ and $U_Y$ depend only on $r$ (see Appendix D for details).

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[4]Recalling that $\gamma$ and $\sigma$ belong to different spaces and therefore act on different indexes of $\psi$, we can safely omit all indexes and simply write $\gamma$ and $\sigma$ both acting on $\psi$ from the left.

**Periodicity Conditions**

Let us examine the periodicity of $U_\gamma$ and $U_\sigma$ under $\varphi \to \varphi + 2\pi$ for different values of $(M, J/l)$ and $(W, K/y)$. Under that rotation, the phases of $U_\gamma$ and $U_\sigma$,

$$S = -\gamma_0 \left( \left[ -M + \frac{\epsilon J}{l} \right] \gamma_{-\epsilon} + \gamma_\epsilon \right) , \tag{4.100}$$

$$Z = -i \left( \left[ -W + \frac{\eta K}{s} \right] \sigma_\eta + \sigma_{-\eta} \right) \sigma_2 \tag{4.101}$$

get multiplied by $2\pi$. There are two possibilities for periodicity to occur: i) if $S^2 = -1 = Z^2$, in which case the corresponding $U$s would be trigonometric functions of $\theta$, and ii) if $S^2 = 0 = Z^2$, in which case there is no $\varphi$-dependence at all. Direct computation yields

$$\begin{aligned} S^2 &= \left[ \gamma_0 \left( \left[ -M + \frac{\epsilon J}{l} \right] \gamma_{-\epsilon} + \gamma_\epsilon \right) \right]^2 \\ &= M - \frac{\epsilon J}{l}, \end{aligned} \tag{4.102}$$

$$\begin{aligned} Z^2 &= \left[ i \left( \left[ W - \frac{\eta K}{s} \right] \sigma_\eta - \sigma_{-\eta} \right) \sigma_2 \right]^2 \\ &= W - \frac{\eta K}{s}. \end{aligned} \tag{4.103}$$

In the $(M, J/l)$ plane one can distinguish three different cases: i) $M - \epsilon J/l = -1$, corresponding to two straight lines passing through the AdS point, $(-1, 0)$; ii) the generic extremal cases, $M = |J|/l \neq 0$; and iii) the zero mass extremal case, $M = 0 = |J|$. Three analogous cases can be distinguished in the $(W, K/s)$ plane simply replacing $(\epsilon, l, M, J)$ by $(\eta, s, W, K)$ which, together with the other three, produce nine combined cases. In each of these cases the number of globally-defined Killing spinors is different, as summarized in Table 4.1, also depicted in Figure 4.2 (see Appendix D for details).

In the case $M - \epsilon J/l = -1$, for each value of $\epsilon$ there are two well-defined solutions, the two basis for the constant spinor $\psi_0$, represented by the solid lines in Figure 4.2. In the second case, $M = |J|/l \neq 0$, there is only one well-defined solution corresponding to the basis spinor in the kernel of $\gamma_\epsilon$, represented by the two lines $M = \pm J/l$. We can also see that at the two green points there are three well-defined solutions: two for the value of $\epsilon$ such that $M - \epsilon J/l = -1$ and one for the value of $\epsilon$ such that $M - \epsilon J/l = 0$.

Since the $SO(1, 2)$ and $SU(2)$ symmetry groups are independent, each one with its own constants of integration, the total number of Killing spinors is

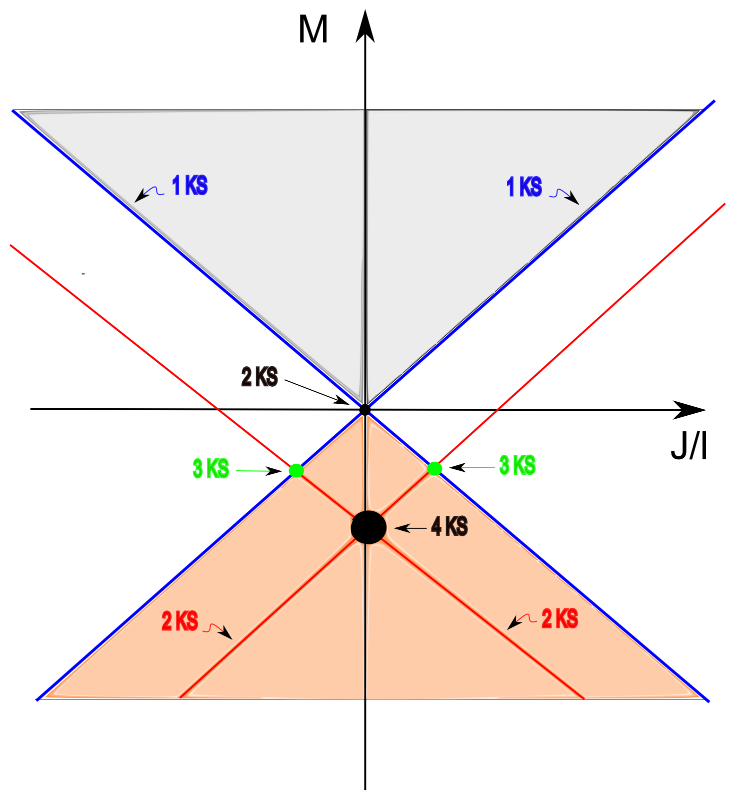



Figure 4.2: A generic configuration has no globally-defined Killing spinors (**KS**), but there may be up to 4 ks for special values of $M$ and $J$. Further explanation in the main text.

| $(W,K/s)$ \ $(M,J/l)$ | $(-1,0)$ | $(0,0)$ | $(\mathbb{R}^+,M)$ |
|---|---|---|---|
| $(-1,0)$ | 16 | 8 | 4 |
| $(0,0)$ | 8 | 4 | 2 |
| $(\mathbb{R}^+,W)$ | 4 | 2 | 1 |

Table 4.1: The number of Killing spinors for different values of $(M,J/l)$ and $(W,K/s)$ in the black hole region.

just the product of the number of well-defined solutions (4.91) of each sector. The number of complex components of $\psi_0$ is four, two from the spinor index and two from the internal index. The final number of Killing spinors for each case is given in Table 4.1. These could be considered as more general Killing-spinors solutions of the 'standard' BTZ, which were already found [68].

## 4.4 Discussion

We have considered the CS theory for the superalgebra $su(2,1|2)$. This algebra can be seen as the minimal supersymmetric extension of the algebra $so(1,2)\oplus su(2)$. This yields an action containing, in addition to the $SU(2)$ connection and $(2+1)$ gravity, a Dirac field minimally coupled to gravity and to the $SU(2)$ gauge field. As in [12], the cosmological constant and the mass of the fermion are related and determined by an integration constant, instead of being fundamental parameters in the action.

This system can be viewed as a three-dimensional toy model of a more "realistic" four-dimensional theory. However, the Dirac Lagrangian minimally coupled to the $SU(2)$ gauge field also seems appropriate to describe electrons in graphene in the long wavelength limit near the Dirac point, including the possibility of spin-spin coupling mediated by the $SU(2)$ gauge field. This interaction corresponds to assuming the freedom of choosing the spin quantization axis independently at each point in the graphene lattice, as done in the Jordan-Wigner transformation for the Hubbard model [94, 95]. Such an interaction might produce long-range correlations between electron pairs with antiparallel spins in a manner analogous to the Cooper pairs in the BCS theory, in which case, a superconducting phase could exist in graphene at low temperature.

The field equations in the matter-free case obtained by setting to zero the fermions, are those of a locally flat $SU(2)$ connection in a background of locally maximally symmetric three-dimensional spacetime, which includes

$AdS_3$, black holes, and point particles (conical singularities) in $AdS_3$, as well as their spinning counterparts. By exploiting the fact that for a particular choice of the torsion a locally $AdS_3$ geometry is Lorentz-flat, a globally nontrivial although locally flat $SU(2)$ connection is constructed mimicking the geometry of a $(2+1)$-black hole. The solution has a $SU(2)$ charge. However, as discussed in Section 4.3.3, this is not coloured but just an Abelian charge from the residual broken symmetry $SU(2) \to U(1)$. If the black hole is rotating, it is characterized by a combination of the parameters $(M, J)$ and $(W, K)$. For certain specific values of these parameters, the solutions admit globally defined Killing spinors, which means that the corresponding solutions are candidates for perturbatively stable ground state configurations with a number of unbroken rigid supersymmetries.

By a similar procedurewe we have seen in Chapter 3, in a generic SUSY extension of an internal non-Abelian gauge symmetry, the fermion excitations turn out to be the only contribution to the local DOF. In order to illustrate this, let us consider the split Lagrangian for the $SU(2)$ theory (4.11), which, up to a global factor, reads [14][5]

$$\begin{aligned} L_{SU(2)} &= \epsilon^{ij}\Big[-\eta_{ab}\dot{e}_i^a e_j^b \bar{\psi}_A \psi^A - \dot{\bar{\psi}}_A \gamma_{ij} \psi^A + \bar{\psi}_A \gamma_{ij} \dot{\psi}^A + \frac{1}{2}\eta_{ab}\dot{\omega}_i^a \omega_j^b + \frac{1}{2}\delta_{IJ}\dot{A}_i^I A_j^J\Big] \\ &\quad - e_t^a K_a + \omega_t^a J_a + A_t^I K_I \, , \end{aligned} \tag{4.104}$$

Here the indexes $A = 1, 2$ transform under the $2 \times 2$ vector representation of $SU(2)$ (Pauli matrices), while $I = 1, 2, 3$ refers to the adjoint representation (we follow the conventions of [14]). The primary constraints $(\varphi_a^i, \phi_a^i, \phi_I^i, \Omega^A, \bar{\Omega}_A)$ are defined in an analogous fashion to their $U(1)$ counterparts. If one omits the contraction in the $A$ index, i.e. $\bar{\psi}_A \psi^A = \bar{\psi}\psi$, the secondary constraints $K_a$ and $J_a$ adopt exactly the same form as (3.2,3.3) where the covariant derivatives are now gauged by $SO(2,1) \times SU(2)$. The remaining constraint reads

$$K_I = \epsilon^{ij}\delta_{IJ}(\frac{1}{2}F_{ij}^J - i\bar{\psi}\gamma_{ij}\sigma^J\psi) = \epsilon^{ij}\delta_{IJ}(\partial_i A_j^J + \frac{1}{2}\epsilon^J{}_{KL} A_i^K A_j^L - i\bar{\psi}\gamma_{ij}\sigma^J\psi) \, . \tag{4.105}$$

[5]The CS form also contains an abelian form $b$ associated to the central charge in $su(2,1|2)$. However, $b$ decouples from the action and therefore does not enter in the dynamical analysis.

Then, one can show that

$$\begin{aligned}
\tilde{J}_a &:= J_a + \epsilon_{ac}\varphi_b^j e_j^c + \frac{1}{2}(\bar{\Omega}\gamma_a\psi - \bar{\psi}\gamma_a\Omega) + D_j\phi_a^j\,, \\
\tilde{K}_I &:= K_I - \frac{i}{2}(\bar{\Omega}\sigma_I\psi - \bar{\psi}\sigma_I\Omega) + D_j\phi_I^j\,, \\
\Upsilon &:= -e_j^b\varphi_b^j + \bar{\Omega}\psi + \bar{\psi}\Omega\,, \\
\mathcal{H}_i &:= e_i^a K_a - e_i^a D_j\varphi_a^j + T_{ij}^a\varphi_a^j + \bar{\psi}\overleftarrow{D}_i\Omega + \bar{\Omega}\overrightarrow{D}_i\psi - \omega_i^a\tilde{J}_a - A_i^I\tilde{K}_I + \phi_I^j F_{ij}^I + \phi_a^j R_{ij}^a\,.
\end{aligned} \tag{4.106}$$

correspond to $F = 9$ first-class combinations generating $SO(2,1) \times SU(2) \times Weyl \times$ Diff transformations, respectively. In account of these and the remaining $S = 18 + 8n$ second class constraints, the original phase space of $N = 18 + 8n$ variables only contains

$$DOF_{SU(2)} = \frac{2N - 2F - S}{2} = 4n \tag{4.107}$$

degrees of freedom for a generic sector, exactly matching the double of the $U(1)$ case due to the doubling of the fermion fields. SUSY is again not realized as a first-class constraint, but is a rigid transformation for certain backgrounds. Such matters, together with the computation of the asymptotic charges, were already treated in Section 4.3.4.

# Chapter 5

# Unconventional Supersymmetry in $D = 4$

We would like to extend the model presented in previous chapters to $D = 4$, in order to obtain a more realistic fundamental theory, or at least an effective theory describing some particular behaviour of Nature. Following the same logic than these ones, the most naive choice for an invariant Lagrangian in even dimensions could be

$$P_{2n} = \langle \mathbb{F} \cdots \mathbb{F} \rangle \ , \tag{5.1}$$

where $\langle \cdots \rangle$ is a (super) trace in the Lie algebra, is an invariant polynomial 2n-form. However, this is a topological invariant and not a suitable Lagrangian. In fact, the Chern-Weil theorem asserts that any invariant polynomial of this form is necessarily closed, $dP_{2n} = 0$, and therefore, according to the Poincaré lemma, it is *locally* an exact form: $P_{2n} = dC_{2n-1}$ [96]. This means that its variations –under appropriate boundary conditions– identically vanish, or are just a boundary term, while the dynamics in the bulk remains arbitrary. Thus, in particular, there are no Lagrangians $L(\mathbb{F})$ constructed using only exterior products, invariant under the entire gauge group; the Euler-Lagrange equations for such "invariant Lagrangians" would have the trivial form $0 = 0$.

Therefore, in order to have dynamics in even dimensions, one must give up gauge invariance under the full gauge group. Instead, the form (5.1) must be constructed with a symmetric trace $\langle \cdots \rangle$ that is not invariant under the entire gauge symmetry group, but under a subgroup of it. This case is the only alternative in even dimensions and corresponds to the approach taken independently by Chamseddine and West [97], Mac Dowell and Mansouri [98], and by Townsend [99], to construct a four-dimensional (super-)gravity out of a superalgebra for the (super-)AdS symmetry. Those authors found that although the fields could be described by an $SO(3,2)$ ($AdS_4$) connection, the four-dimensional action could be at most invariant under the Lorentz group ($SO(3,1)$-invariant).

In all dimensions, YM Lagrangians can be constructed, provided the space-time is equipped with a metric structure with which the Hodge dual of $\mathbb{F}$ is defined. Thus, we tentatively define

$$L^{YM} = -\frac{1}{4} Str \left[\mathbb{F} \wedge^{\circledast} \mathbb{F}\right] \,, \tag{5.2}$$

where ${}^{\circledast}\mathbb{F}$ is the dual of $\mathbb{F}$. The metric structure required by this construction is provided by the vierbein form $e^a = e^a_\mu d^\mu$, where $\mu \in \{0, 1, 2, 3\}$.

## 5.1 The system

The simplest SUSY in $4D$ containing $U(1) \times SO(3,1)$ includes the (A)dS$_4$ generators $\mathbb{J}_a$ and $\mathbb{J}_{ab}$, the complex supercharge $\mathbb{Q}^\alpha$ in a spin 1/2 representation, and the $U(1)$ generator $\mathbb{K}$. This is the $\mathfrak{usp}(2,2|1)$ superalgebra, whose essential anticommutator is (see Appendix B.3 for details on this representation)

$$\{\mathbb{Q}^\alpha, \overline{\mathbb{Q}}_\beta\} = -i(\gamma^a)^\alpha_\beta \mathbb{J}_a + \frac{i}{2}(\gamma^{ab})^\alpha_\beta \mathbb{J}_{ab} - \delta^\alpha_\beta \mathbb{K} \,, \tag{5.3}$$

together with the trivial anticommutators $\{\overline{\mathbb{Q}}_\alpha, \overline{\mathbb{Q}}_\beta\} = 0 = \{\mathbb{Q}^\alpha, \mathbb{Q}^\beta\}$. An explicit $6 \times 6$ representation for the supercharges is

$$(\mathbb{Q}^\alpha)^A_B = -\frac{i}{s}(\delta^A_5 \delta^\alpha_B + C^{\alpha A}\delta^6_B), \quad (\overline{\mathbb{Q}}_\alpha)^A_B = \delta^A_\alpha \delta^5_B + \delta^A_6 C_{\alpha B} \,, \tag{5.4}$$

where $s^2 = -1$ corresponds to de Sitter, and $s^2 = 1$ to anti-de Sitter. Here $C_{\alpha\beta} = -C_{\beta\alpha}$ is the conjugation matrix, $C^{\alpha\beta}$ is its inverse[1]. In this representation, the $U(1)$ and AdS generators are [15]

$$(\mathbb{K})^A{}_B = i(\delta^A_5 \delta^5_B - \delta^A_6 \delta^6_B), \quad (\mathbb{J}_a)^A{}_B = \frac{1}{2}(\gamma_a)^\alpha{}_\beta \delta^A_\alpha \delta^\beta_B, \qquad (\mathbb{J}_{ab})^A{}_B = \frac{1}{2}(\gamma_{ab})^\alpha{}_\beta \delta^A_\alpha \delta^\beta_B \tag{5.5}$$

The connection can be written as

$$\mathbb{A} = A\mathbb{K} + \overline{\mathbb{Q}} \not{e} \psi + \overline{\psi} \not{e} \mathbb{Q} + f^a \mathbb{J}_a + \frac{1}{2}\omega^{ab}\mathbb{J}_{ab}, \tag{5.6}$$

[1]The indices $A, B = 1, ..., 6$ combine both spinor indices ($\alpha, \beta = 1, ..., 4$) and those of a two-dimensional representation ($r = 5, 6$) of $U(1)$, i.e., $A = (\alpha, r)$.

where $A = A_\mu dx^\mu$, $\not{e} = \gamma_a e^a_\mu dx^\mu$, $f^a = f^a_\mu dx^\mu$ and $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ are one-form fields (spinorial indices omitted). The curvature $\mathbb{F} = d\mathbb{A} + \mathbb{A}\mathbb{A}$ takes the form $\mathbb{F} = F_0\mathbb{K} + \overline{\mathbb{Q}}_\alpha \mathcal{F}^\alpha + \overline{\mathcal{F}}_\alpha \mathbb{Q}^\alpha + F^a \mathbb{J}_a + \frac{1}{2}F^{ab}\mathbb{J}_{ab}$, where

$$
\begin{aligned}
F_0 &= F - \overline{\psi}\not{e}\not{e}\psi\,, & (5.7)\\
\mathcal{F} &= \nabla(\not{e}\psi)\,, & (5.8)\\
\overline{\mathcal{F}} &= -(\overline{\psi}\not{e})\overleftarrow{\nabla}\,, & (5.9)\\
F^a &= Df^a - \frac{i}{s}\overline{\psi}\not{e}\gamma^a\not{e}\psi\,, & (5.10)\\
F^{ab} &= R^{ab} + s^2 f^a f^b + i\overline{\psi}\not{e}\gamma^{ab}\not{e}\psi\,, & (5.11)
\end{aligned}
$$

Here $F = dA$, $Df^a = df^a + \omega^a{}_b f^b$, and $R^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$. We have also used the notation $\not{f} = \gamma_a f^a$, and $\not{\omega} = \frac{1}{2}\gamma_{ab}\omega^{ab}$. The operators $\overrightarrow{\nabla} \equiv \left[\overrightarrow{d} - iA + \frac{s}{2}\not{f} + \frac{1}{2}\not{\omega}\right]$ and $\overleftarrow{\nabla} \equiv \left[\overleftarrow{d} + iA - \frac{s}{2}\not{f} - \frac{1}{2}\not{\omega}\right]$ are the full bosonic covariant derivatives in the $1/2-$spin representation.

Under SUSY transformation generated by $\Lambda = \overline{\mathbb{Q}}\epsilon - \overline{\epsilon}\mathbb{Q}$, the connecton $\mathbb{A}$ changes by $\delta\mathbb{A} = d\Lambda + [\mathbb{A}, \Lambda]$. Using the (anti-) commutation relations of the superalgebra, one finds

$$
\begin{aligned}
\delta A_\mu &= -\left(\overline{\epsilon}\gamma_\mu\psi + \overline{\psi}\gamma_\mu\epsilon\right) & (5.12)\\
\delta f^a &= -\frac{i}{s}\left(\overline{\epsilon}\gamma^a\not{e}\psi + \overline{\psi}\not{e}\gamma^a\epsilon\right) & (5.13)\\
\delta\omega^{ab} &= i\left(\overline{\epsilon}\gamma^{ab}\not{e}\psi + \overline{\psi}\not{e}\gamma^{ab}\epsilon\right) & (5.14)\\
\delta\left[\gamma_\mu\psi\right] &= \left[\partial_\mu - iA_\mu + \frac{s}{2}f^a_\mu\gamma_a + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\right]\epsilon \equiv \nabla_\mu\epsilon\,. & (5.15)
\end{aligned}
$$

As discussed above, using $\delta e^a = 0 = \delta\gamma_\mu$ in (5.15) implies $\delta\psi = \frac{1}{4}\gamma^\mu\nabla_\mu\epsilon$, and the consistency condition $\left[\delta^\mu_\nu - \frac{1}{4}\gamma_\nu\gamma^\mu\right]\nabla_\mu\epsilon = 0$ eliminates the spin-3/2 part.

### 5.1.1 Invariant Hodge trace

Starting from the connection (5.6), one can construct an action of the YM type. The Lagrangian is a four-form quadratic in curvature,

$$L = \kappa\langle\mathbb{F}^{\circledast}\mathbb{F}\rangle\,, \tag{5.16}$$

where ${}^{\circledast}\mathbb{F}$ stands for the dual of $\mathbb{F}$ with $({}^{\circledast})^2 = -1$ in the Lorentzian signature. Here we take duality as the Hodge dual $(*)$ in the spacetime, the $\gamma_5$-conjugate in spinor indices, and the dual in the AdS algebra, to wit,

$${}^{\circledast}\mathbb{F} = *F_0\mathbb{K} + (\overline{\mathbb{Q}})_\alpha(\gamma_5\mathcal{F})^\alpha + (\overline{\mathcal{F}})_\alpha(\gamma_5\mathbb{Q})^\alpha + \boldsymbol{\Upsilon}\left[F^a\mathbb{J}_a + \frac{1}{2}F^{ab}\mathbb{J}_{ab}\right]\,. \tag{5.17}$$

In the $6 \times 6$ representation, $(\Upsilon)^A{}_B = (\gamma_5)^\alpha{}_\beta \delta^\beta_B \delta^A_\alpha$, or

$$\Upsilon = \left[\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ \gamma_5 & 0 & 0 \\ & 0 & 0 \\ \hline 0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \end{array}\right] . \tag{5.18}$$

The three dualities square to minus the identity in their respective subspaces, $(*)^2 = (\gamma_5)^2 = (\Upsilon)^2 = -1$ [2]. Since $\Upsilon$ commutes with $\mathbb{K}$ and $\mathbb{J}_{ab}$, but not with $\mathbb{J}_a$ or $\mathbb{Q}^\alpha_i$, the resulting quadratic form (5.16) is invariant under $SO(3,1) \times U(1)$, the only remaining symmetry of the action out of the full AdS SUSY (5.3).

### 5.1.2 Four-dimensional Lagrangian

The nonvanishing supertraces, bilinear in the generators that appear in $L$, are

$$\langle \mathbb{K}\mathbb{K} \rangle = 2, \qquad \langle \mathbb{Q}^\alpha \overline{\mathbb{Q}}_\beta \rangle = 2i\delta^\beta_\alpha = -\langle \overline{\mathbb{Q}}_\alpha \mathbb{Q}^\beta \rangle, \qquad \langle \mathbb{J}_{ab} \Upsilon \mathbb{J}_{cd} \rangle = \epsilon_{abcd} \,, \tag{5.19}$$

and therefore,

$$\langle \mathbb{F}^{\circledast} \mathbb{F} \rangle = 2F_0 * F_0 + 4i\overline{\mathcal{F}}_\alpha (\gamma_5)^\alpha_\beta \mathcal{F}^\beta + \frac{1}{4}\epsilon_{abcd} F^{ab} F^{cd} \,. \tag{5.20}$$

From (5.8) and (5.10) it is clear that the covariant derivative acts on the components $\xi^\alpha_\mu \equiv \gamma_\mu \psi^\alpha$ which are in the kernel of the spin-3/2 projector, $P_\mu{}^\nu \gamma_\nu \psi = 0$. The second term of the r.h.s. of (5.20) contains only covariant derivatives in the spin-1/2 representation, so we can safely assume that no dynamical channels are available to switch on a spin-3/2 excitation.

The Lagrangian can also be expressed as

$$\begin{aligned} L = -\tfrac{1}{4}\langle \mathbb{F}^{\circledast}\mathbb{F} \rangle = \; & \mathcal{L}_{\rm EM} |e| d^4x + L_{\rm Grav}(\omega, f) \\ & + \tfrac{i}{2} s \overline{\psi} \left[ \overleftarrow{D} \gamma_5 \not{e} \not{f} \not{e} + \gamma_5 \not{e} \not{f} \not{e} \overrightarrow{D} \right] \psi + \tfrac{i}{2} s \overline{\psi} \left[ \gamma_5 (\not{T} \not{f} \not{e} - \not{e} \not{f} \not{T}) \right] \psi \\ & - \tfrac{i}{2} s^2 \overline{\psi} \gamma_5 \not{e} \not{f} \not{f} \not{e} \psi + 12 \left[ (\overline{\psi}\gamma_5\psi)^2 - (\overline{\psi}\psi)^2 \right] |e| d^4x \,, \end{aligned} \tag{5.21}$$

[2]This choice of the dual operator $\circledast$ ensures that it produces the right kinetic terms for the Maxwell filed, the gravitational action and the spinor.

where $\overrightarrow{D}\psi \equiv (\overrightarrow{d} - iA + \frac{1}{2}\not{\omega})\psi$, $\bar{\psi}\overleftarrow{D} \equiv \bar{\psi}(\overleftarrow{d} + iA - \frac{1}{2}\not{\omega})$, $\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, and $L_{\text{Grav}}(\omega, f) = -\frac{1}{16}\epsilon_{abcd}(R^{ab} + s^2 f^a f^b)(R^{cd} + s^2 f^c f^d)$.

The quartic fermionic expression is the Nambu–Jona-Lasinio (**NJL**) term, $g[(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma_5\psi)^2]$.

The field $f^a$ is undifferentiated and therefore its equation could –in principle– be algebraically solved and substituted back in the action. Since $f$ is a connection component, this means that the invariance of the theory under local AdS boosts is frozen, which is consistent with the fact that the action is not really invariant under local AdS boosts. The same is true about the vierbein $e^a$ in the first order formulation of four-dimensional gravity [100]: in that case, the torsion equation can be algebraically solved for the spin connection, underscoring the fact that 4D gravity has local $SO(3,1)$ invariance, and no $SO(3,2)$, $SO(4,1)$, or $ISO(3,1)$ local symmetry.

The tensor character of $f^a$ and $e^a$ is the same, and it was suggested in [99] that they should be proportional, $f^a = \mu e^a$, where $\mu$ is a constant with dimensions of (length)$^{-1}$. This choice eliminates parity-violating terms from the Lagrangian so that in the absence of parity changing interactions, this sector remains self-contained, but it might be of interest to see the consequences of relaxing this condition and to explore, in particular, whether this could lead to new phenomena in conflict with observations.

If one follows the proposal in [99] the Lagrangian becomes

$$\begin{aligned} L &= \mathcal{L}_{\text{EM}}|e|d^4x + L_{\text{Grav}}(\omega, e) \\ &-\tfrac{i}{2}s\mu\left[(\overline{\psi}\overleftarrow{D})\gamma^a\psi - \overline{\psi}\gamma^a(\overrightarrow{D}\psi)\right]\epsilon_{abcd}e^b e^c e^d + 2is\mu\overline{\psi}\gamma_5\gamma_a\psi(T_b e^b)e^a \\ &-\tfrac{i}{2}s^2\mu^2\overline{\psi}\psi\epsilon_{abcd}e^a e^b e^c e^d + 12\left[(\overline{\psi}\gamma_5\psi)^2 - (\overline{\psi}\psi)^2\right]|e|d^4x\,, \end{aligned} \tag{5.22}$$

in standard units, $\hbar = c = 1$, $\mu$ has units of mass. The spin-1/2 field with the right physical dimensions is $\psi_{physical} = \sqrt{6\mu}\psi$, where we have included a factor $\sqrt{6}$ for later convenience. Rewriting the Lagrangian in this convention, one obtains

$$L = [L_F + L_{EM}]\sqrt{|g|}d^4x + L_{\text{Grav}}(\omega, e), \tag{5.23}$$

where the fermionic Lagrangian is

$$L_F = -\frac{i}{2}s\left[\overline{\psi}(\overleftarrow{\not{D}} - \overrightarrow{\not{D}})\psi + 4\mu\overline{\psi}\psi\right] - ist^\mu\overline{\psi}\gamma_5\gamma_\mu\psi - \frac{1}{3\mu^2}\left[(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma_5\psi)^2\right]\,. \tag{5.24}$$

Here $\overrightarrow{\not{D}}\psi = (\overrightarrow{\not{\partial}} - i\not{A} + \frac{1}{2}\not{\omega})\psi$, and $\overline{\psi}\overleftarrow{\not{D}} = \overline{\psi}(\overleftarrow{\not{\partial}} + i\not{A} - \frac{1}{2}\not{\omega})$, are the covariant derivatives for the connection of the $SO(3,1)\times U(1)$ gauge group in the spinorial representation, and following [101], we defined $t^\mu \equiv -\frac{1}{3!}\varepsilon^{\mu\nu\rho\tau}e^a_\nu T_{a\rho\tau}|e|$.

The correct sign of Newton's constant in (5.23) is obtained for $s^2 = -1$, that is, for the de Sitter group only.

### 5.1.3 Field equations

Varying the action (5.22) with respect to the dynamical fields yields the following (we take the de Sitter signature):

$$\delta A_\nu : \quad \partial_\mu F^{\mu\nu} + i\overline{\psi}\gamma^\nu\psi = 0 \tag{5.25}$$

$$\delta \omega^{ab}{}_\mu : \quad \overline{\psi}\gamma_{ab}{}^c\psi E^\mu_c + 3\mu^2\left[E^\nu_a E^\lambda_b E^\mu_c + 2E^\mu_a E^\nu_b E^\lambda_c\right] T^c_{\nu\lambda} = 0 \tag{5.26}$$

$$\delta\overline{\psi}_\alpha : \quad -\overrightarrow{\not{D}}\psi + 2i\mu\psi + \gamma_5\gamma_\mu\psi t^\mu + \tfrac{2}{3\mu^2}\left[(\bar{\psi}\gamma_5\psi)\gamma_5 - (\bar{\psi}\psi)\right]\psi = 0 \tag{5.27}$$

$$\delta e^a : \quad \epsilon_{abcd}(R^{bc} - \mu^2 e^b e^c)e^d = \tau_a, \tag{5.28}$$

where $\tau_a$ is the stress-energy three-form, defined by $\delta\left(|e|[L_F + L_{EM}]\right) = \delta e^a \wedge \tau_a$. From the second equation it follows that $T^c_{\mu\nu}E^\nu_c = 0$, which means that torsion is determined by the local presence of fermions.

$$T^a_{\mu\nu} = \frac{-i}{3s\mu^2}\overline{\psi}\gamma^a{}_{bc}\psi e^b_\mu e^c_\nu\ . \tag{5.29}$$

Contracting the third equation with $\overline{\psi}_\alpha$ and its conjugate with $\psi^\alpha$, gives

$$\overline{\psi}_\alpha\frac{\delta L}{\delta\overline{\psi}_\alpha} - \frac{\delta L}{\delta\psi^\alpha}\psi^\alpha = \partial_\mu(is\sqrt{|g|}\overline{\psi}\gamma^\mu\psi)d^4x = 0\ , \tag{5.30}$$

which expresses the conservation of electric charge and coincides with the current conservation condition obtained from (5.25).

## 5.2 Discussion

The fermionic Lagrangian (5.23) describes an electrically charged spin-1/2 field minimally coupled to the spacetime background, plus non-minimal couplings that depend on the spacetime dimension. The coupling to torsion is not a new feature of this model but, as noted long ago by H. Weyl [55], it is present whenever the Dirac equation is written in a curved spacetime with torsion. The only new fermionic piece in the Lagrangian is the NJL term in the four-dimensional theory, which can be viewed as the main modification predicted by this model.[3]

[3] If instead of the $U(1)$ gauge group, one had considered $SU(2)$ or $SU(3)$, NJL term would have been of the form $C_{abcd}[(\overline{\psi}^a\psi^b)(\overline{\psi}^c\psi^d) - (\overline{\psi}^a\gamma_5\psi^b)(\overline{\psi}^c\gamma_5\psi^d)]$, where $C_{abcd}$ is an invariant tensor in the algebra.

Both the four-fermion NJL coupling and the Einstein-Hilbert gravitational action are perturbatively non-renormalizable. This strongly suggests that the whole system should be considered as a low energy, effective model and not a fully consistent quantum theory. However, the fact that the parameters of the theory are so tightly constrained that it is conceivable that the two evils may cancel each other. The exploration of this problem, however, lies well beyond the scope of this work.

The NJL term provides a mechanism for dynamical symmetry breaking that gives mass to the fermionic excitations in superconductivity and was originally proposed as a way to describe massive excitations in strong interactions [102–104]. The value of the fermion mass $m$ is produced through the gap equation for a cut-off $\mathcal{M}$,

$$\frac{m^2}{\mathcal{M}^2}\log\left[1+\frac{\mathcal{M}^2}{m^2}\right]=1-\frac{2\pi^2}{g\mathcal{M}^2}. \tag{5.31}$$

For $m=m_e\approx 0.5MeV$ and $\mathcal{M}=M_{Planck}=G^{-1/2}\approx 2.5\times 10^{22}m_e$, so that $m^2/\mathcal{M}^2\approx 10^{-45}$, the relation between the NJL coupling $g$ and the UV cut-off $\mathcal{M}$ must be extremely fine-tuned in the range $1<g\mathcal{M}^2/2\pi^2<1+10^{-43}$.

In four dimensions, the spacetime geometry is described by the Einstein-Hilbert action with cosmological constant $\Lambda=-3s^2\mu^2$, where $s^2=-1(resp.+1)$ for dS (resp. AdS) algebra [see, e.g., (B.10)]. The sign of the kinetic term in the gravitational action (5.22) is $-s^2$, which, in the standard convention, should be positive. Hence, both $\Lambda$ and $G$ should be positive. However, depending on the vacuum structure of the theory it might be worth considering the alternative where both $G$ and $\Lambda$ are negative, as in the case of the so-called topologically massive gravity in three dimensions [105, 106]. At any rate, the effective cosmological constant in the nontrivial vacuum should be given by $\Lambda_{Eff}=\Lambda-2i\mu s^2\langle\bar{\psi}\psi\rangle+g^2[\langle(\overline{\psi}\gamma_5\psi)^2\rangle-\langle(\overline{\psi}\psi)^2\rangle]$. It would be premature to claim something about the sign of $\Lambda_{Eff}$, especially in view of the fine tuning between $g$, $G$, $\Lambda$ and the cut-off $\mathcal{M}$.

The gravitational Lagrangian is a particular combination of the three Lovelock terms that occur in 4D that has the form of the Pfaffian of the (A)dS curvature. This combination can also be viewed as the gravitational analogue of Born-Infeld electrodynamics [107], and although the Gauss-Bonnet term has no effect on the field equations and hence is usually ignored, it can give a significant contribution to the global charges of the theory, and acts as a regulator that renders the charges well-defined and finite in the case of nontrivial asymptotics [108, 109]. It is, therefore, an interesting bonus of the model that the gravitational action is regularized by construction and no ad hoc counterterms are necessary to correctly define its thermodynamics.

Even as an effective low-energy model, a healthy theory should have a well-defined (stable) ground state, a vacuum around which it would make sense to expand perturbatively to study the quantum features of the theory (Killing spinors, BPS vacua). A vacuum without fermions (trivial vacuum, $\psi = 0$) would be invariant under SUSY provided $\delta\psi = \nabla\lambda = 0$, which means that $\lambda$ must be a covariantly constant (Killing) spinor. The number of linearly independent, globally defined solutions of this equation characterizes the residual supersymmetries of a particular background configurations, as was discussed in Section 4.3.4 for $D = 3$ bosonic solutions. Such backgrounds have been studied and a number of nontrivial BPS backgrounds are known [110, 111].

# Part II

# Massless Rarita-Schwinger Theory

# Chapter 6

# Free massless Rarita-Schwinger theory

In this Chapter, we shall discuss the theory of free 3/2-spin particle field. The action for describing this kind of particles was proposed by William Rarita and Julian Schwinger in 1941 [17], which in a modern notation can be written as (see next section for notational details)

$$S[\overline{\psi}, \psi] = \frac{i}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \left( \overline{\psi}_\mu \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma - im \overline{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right) \,, \tag{6.1}$$

where $m$ is proportional to a real particle mass. When the fermions $\overline{\psi}_{\mu\alpha}$ and $\psi^\alpha_\mu$ are coupled with a Maxwell field $A_\mu$ through minimal substitution $\partial_\mu \to D_\mu = \partial_\mu - igA_\mu$ (from now on, we will call it *gauged RS theory*), Velo and Zwanziger (**VZ**) shown that some inconsistences appear [20]. First, at classical level, the fermion wave fronts have superluminal modes, i.e., the velocity of such modes are strictly greater than speed of light $c$. Second, at quantum level, the anticommutation relation of the fermionic field operators are not positive definite. These inconsistences were late reinforced in $(2+1)$-dimensions by Hortaçsu [112] and in non-flat metrics by Deser and Waldron [21].

However, already in the work of VZ, we can see that the inconsistencies appearing in the gauged RS theory are more subtle for the $m = 0$ case, due to the fact that there are terms which become singular, as they contain the reciprocal of mass. With this in mind, and motivated by the fact that in the SM scheme the fermion mass leads to some problems[1], Stephen Adler

[1]This is why the spontaneous symmetry breaking is casting, through suitable potentials, in the SM rather than a fermion mass term. This spontaneous symmetry breaking could be generated either through couples to the Higgs boson or through the chiral symmetry breaking caused by particular fermion condensates.

recently reopened the discussion about the consistency of a classical and quantum massless RS field. In his works [22, 23], Adler claims that in the massless case there are no superluminal propagation modes and the quantum theory is consistent, as the anticommutators of the field fermionic operator are positive semidefinite. For some of these claims, it is used explicitly that, when $m = 0$, the massless RS has a *fermionic gauge symmetry*, i.e., the action is invariant under the transformation $\psi^\alpha_\mu \to \psi'^\alpha_\mu = \psi^\alpha_\mu + D_\mu \varepsilon^\alpha$, where $\varepsilon^\alpha$ is a fermionic 1/2-spin parameter.

Although it is relatively straightforward to show the free massless RS has this fermionic gauge invariance, the claim is more subtle for the gauged case, as we will see in Chapter 7. As this fermionic gauge invariance is important to avoid the inconsistencies of the massless case, it is relevant to make sure whether the symmetry actually exists for that case. As a warming-up exercise, we will apply the Dirac's Hamiltonian formalism [25] to the free massless RS case in order to compare later with the gauged case.

## 6.1 Classical action, symmetries and field equations

The functional action for the classical free massless RS theory (either for Majorana or Dirac fermions) is[2]

$$S[\overline{\psi}, \psi] = \frac{i}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \left( \overline{\psi}_\mu \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma - \partial_\mu \overline{\psi}_\nu \gamma_5 \gamma_\rho \psi_\sigma \right) \, , \tag{6.2}$$

where we added a boundary term (which does not alter the RS field equations) with respect to the action defined in [22] in order to write the action in a Hermitian way and take the conjugate convention $\overline{\psi}_\mu = i\psi^\dagger_\mu \gamma^0$. For Majorana fermions, the field $\psi_\mu$ must be anti-commuting, otherwise (6.2) is a null term without bulk field equations, as is also the case Dirac Lagrangian [113]. The field equations are obtained by varying $S[\overline{\psi}, \psi]$ with respect to $\overline{\psi}$ and $\psi$ , which are, respectively,

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \partial_\rho \psi_\sigma &= 0 \, , \\ \epsilon^{\mu\nu\rho\sigma} \partial_\mu \overline{\psi}_\nu \gamma_\rho &= 0 \, . \end{aligned} \tag{6.3}$$

[2]For notational simplicity, we are omitting the spinor indexes when there is no misunderstanding.

The zeroth component of (6.3) gives us[3]

$$\begin{aligned} \epsilon^{ijk}\partial_i\overline{\psi}_j\gamma_k &= 0 \ , \\ \epsilon^{ijk}\gamma_i\partial_j\psi_k &= 0 \ . \end{aligned} \tag{6.4}$$

With fermionic parameters $(\overline{\varepsilon}, \varepsilon)$ the RS fields gauge transformation is defined as

$$\begin{aligned} \overline{\psi}_\mu \to \overline{\psi}'_\mu &= \overline{\psi}_\mu + \partial_\mu\overline{\varepsilon} \ , \\ \psi_\mu \to \psi'_\mu &= \psi_\mu + \partial_\mu\varepsilon \ . \end{aligned} \tag{6.5}$$

Under the transformations (6.5), the action (6.2) changes according

$$\delta_G S[\overline{\psi}, \psi] = BT \ , \tag{6.6}$$

where for $BT$ we understand a boundary term.

The stress-energy tensor can be computed by varying with respect to $g_{\mu\nu}$ [114],

$$T^{\mu\nu}_{RS} = -\frac{i}{4}\epsilon^{\sigma\rho\lambda\tau}\left[\overline{\psi}_\sigma\gamma_5\left(\gamma^\nu\delta^\mu_\rho + \gamma^\mu\delta^\nu_\rho\right)\partial_\lambda\psi_\tau + \frac{1}{4}\partial_\alpha\left(\overline{\psi}_\sigma\gamma_5\gamma_\rho([\gamma^\alpha, \gamma^\mu]\delta^\nu_\lambda + [\gamma^\alpha, \gamma^\nu]\delta^\mu_\lambda)\right)\psi_\tau\right] . \tag{6.7}$$

which fulfills

$$\partial_\mu T^{\mu\nu}_{RS} = 0.$$

Using our particular representation of the Dirac matrices (see Appendix A.2 for the chosen representation and notation), we can write the left chiral component of the action (6.2) as (see Appendix E.1 for left and right-handed components decomposition)

$$\begin{aligned} S^L[\Psi^\dagger, \Psi] = \frac{1}{4}\int d^4x \Big( &-\vec{\Psi}^\dagger \cdot (\vec{\sigma} \times \dot{\vec{\Psi}}) + \dot{\vec{\Psi}}^\dagger \cdot (\vec{\sigma} \times \vec{\Psi}) + \vec{\Psi}^\dagger \cdot (\vec{\nabla} \times \vec{\Psi}) \\ &+ (\vec{\nabla} \times \vec{\Psi}^\dagger) \cdot \vec{\Psi} + \vec{\Psi}^\dagger \cdot (\vec{\sigma} \times \vec{\nabla}\Psi_0) - \vec{\nabla}\Psi_0^\dagger \cdot (\vec{\sigma} \times \vec{\Psi}) \\ &-\Psi_0^\dagger\vec{\sigma} \cdot (\vec{\nabla} \times \vec{\Psi}) - (\vec{\nabla} \times \vec{\Psi}^\dagger) \cdot \vec{\sigma}\Psi_0 \Big) \ , \end{aligned} \tag{6.8}$$

where, as usual, the dot implies time derivative and $\nabla^i = \partial^i = \partial_i$.

[3] We denote by the latin indexes $\{i, j, k, \ldots\}$ the spatial components and define $\epsilon^{ijk} \equiv -\epsilon^{0ijk}$.

## 6.2 Dirac's Hamiltonian formalism

### 6.2.1 Momenta, Poisson brackets and canonical Hamiltonian

The canonical momenta are given by[4]

$$\Xi^{\mu\alpha} = \frac{\partial^L \mathcal{L}}{\partial \dot{\Psi}^\dagger_{\mu\alpha}} \ , \ \Xi^{\dagger\mu}_\alpha = \frac{\partial^R \mathcal{L}}{\partial \dot{\Psi}^\alpha_\mu}, \tag{6.9}$$

where $\mathcal{L}$ is the Lagrangian density. The non-vanishing Poisson brackets at equal time between fields and momenta are (following convention of [26])

$$\begin{aligned} \left\{\Psi^\alpha_\mu(x), \Xi^{\dagger\nu}_\beta(y)\right\} &= \left\{\Xi^{\dagger\nu}_\beta(y), \Psi^\alpha_\mu(x)\right\} = \delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(x-y) \ , \\ \left\{\Psi^{\dagger\alpha}_\mu(x), \Xi^{\nu\beta}(y)\right\} &= \left\{\Xi^{\nu\beta}(y), \Psi^{\dagger\alpha}_\mu(x)\right\} = -\delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(x-y) \ . \end{aligned} \tag{6.10}$$

The time and spatial components of (6.9) are given by

$$\Xi^{0\alpha} = 0 \ , \ \vec{\Xi}^\alpha = \frac{1}{4}(\vec{\sigma})^\alpha{}_\beta \times \vec{\Psi}_\beta \ , \ \Xi^{\dagger 0}_\alpha = 0 \ , \ \vec{\Xi}^\dagger_\alpha = -\frac{1}{4}\vec{\Psi}^\dagger{}_\beta \times (\vec{\sigma})^\beta{}_\alpha \ .$$

We can write the canonical Hamiltonian as

$$\begin{aligned} H_0 &= \int d^3x \left(\Xi^{\dagger\mu}\dot{\Psi}_\mu + \overline{\Psi}^\dagger_\mu \Xi^\mu - \mathcal{L}\right) \\ &= \frac{1}{4}\int d^3x \Big(-\vec{\Psi}^\dagger \cdot (\vec{\nabla}\times\vec{\Psi}) - (\vec{\nabla}\times\vec{\Psi}^\dagger)\cdot\vec{\Psi} - \vec{\Psi}^\dagger\cdot(\vec{\sigma}\times\vec{\nabla}\Psi_0) \\ &\qquad + \vec{\nabla}\Psi^\dagger_0 \cdot(\vec{\sigma}\times\vec{\Psi}) + \Psi^\dagger_0 \vec{\sigma}\cdot(\vec{\nabla}\times\vec{\Psi}) + (\vec{\nabla}\times\vec{\Psi}^\dagger)\cdot\vec{\sigma}\Psi_0\Big) \ . \end{aligned} \tag{6.11}$$

### 6.2.2 Primary and secondary constraints

The primary constraints obtained from (6.9) are

$$\begin{aligned} \chi^{0\alpha} \equiv \Xi^{0\alpha} \approx 0 \quad &, \quad \vec{\chi}^\alpha \equiv \Xi^{0\alpha} - \frac{1}{4}(\vec{\sigma})^\alpha{}_\beta \times \vec{\Psi}_\beta \approx 0 \ , \\ \chi^{\dagger 0}_\alpha \equiv \Xi^{\dagger 0}_\alpha \approx 0 \quad &, \quad \vec{\chi}^\dagger_\alpha \equiv \vec{\Xi}^\dagger_\alpha + \frac{1}{4}\vec{\Psi}^\dagger \times \vec{\sigma} \approx 0 \ , \end{aligned} \tag{6.12}$$

whose only non-vanishing Poisson bracket is

$$\left\{\chi^{\dagger i}_\alpha(x), \chi^{j\beta}(y)\right\} = -\frac{1}{2}\epsilon^{ijk}(\sigma_k)^\beta{}_\alpha \delta^{(3)}(x-y) \ . \tag{6.13}$$

[4]In this subsection, we will write explicitly the spinorial indexes $\{\alpha, \beta, \ldots\}$.

Now, we can write the primary Hamiltonian as

$$H_T = H_0 + \int d^3x \left( \Lambda^\dagger_\mu \chi^\mu + \chi^{\dagger\mu} \Lambda_\mu \right) \ , \tag{6.14}$$

where $\Lambda^\dagger_\mu$ and $\Lambda_\mu$ are Lagrange multipliers.

We demand the primary constraints (6.12) to hold as the system evolves in time, i.e.,

$$\begin{aligned} \dot{\chi}^{0\alpha}(x) &= \left\{ \chi^{0\alpha}(x), H_T \right\} = -\frac{1}{2} (\vec{\sigma})^\alpha{}_\beta \cdot (\vec{\nabla} \times \vec{\Psi}^\beta) \approx 0 \ , \\ \dot{\chi}^{\dagger 0}_\alpha(x) &= \left\{ \chi^{\dagger 0}_\alpha(x), H_T \right\} = -\frac{1}{2} (\vec{\nabla} \times \vec{\Psi}^\dagger_\beta) \cdot (\vec{\sigma})^\beta{}_\alpha \approx 0 \ , \end{aligned}$$

where in the second equalities we used (6.10), (6.12) and (6.11) explicitly. This lead us to define the following secondary constraints

$$\begin{aligned} K^\alpha &\equiv \frac{1}{2} (\vec{\sigma})^\alpha{}_\beta \cdot (\vec{\nabla} \times \vec{\Psi}^\beta) \approx 0 \ , \\ K^\dagger_\alpha &\equiv \frac{1}{2} (\vec{\nabla} \times \vec{\Psi}^\dagger_\beta) \cdot (\vec{\sigma})^\beta{}_\alpha \approx 0 \ , \end{aligned} \tag{6.15}$$

whose non-vanishing Poisson brackets with the rest of constraints (6.12) are

$$\begin{aligned} \left\{ K^\alpha(x), \chi^{\dagger i}_\beta(y) \right\} &= \frac{1}{2} \epsilon^{ijk} (\sigma_j)^\alpha{}_\beta \partial^{(x)}_k \delta^{(3)}(x-y) \ , \\ \left\{ K^\dagger_\alpha(x), \chi^{i\beta}(y) \right\} &= -\frac{1}{2} \epsilon^{ijk} (\sigma_j)^\beta{}_\alpha \partial^{(x)}_k \delta^{(3)}(x-y) \end{aligned} \tag{6.16}$$

Preservation in time of the rest of primary constraints determines a possible set for the Lagrange multipliers $\Lambda^\dagger_i$ and $\Lambda_i$.

$$\begin{aligned} \dot{\chi}^{i\alpha}(x) &= \left\{ \chi^{i\alpha}(x), H_T \right\} = \frac{1}{2} \vec{\nabla} \times \vec{\Psi}^\alpha + \frac{1}{2} (\vec{\sigma})^\alpha{}_\beta \times \vec{\nabla} \Psi^\beta_0 - (\vec{\sigma})^\alpha{}_\beta \times \vec{\Lambda}^\beta \approx 0 \\ &\Longrightarrow \vec{\Lambda} \approx \vec{\nabla} \Psi_0 + i (\vec{\nabla} \times \vec{\Psi}) \ , \\ \dot{\chi}^{\dagger i}_\alpha(x) &= \left\{ \chi^{\dagger i}_\alpha(x), H_T \right\} = \frac{1}{2} \vec{\Psi}^\dagger_\alpha \times \vec{\nabla} - \frac{1}{2} \Psi^\dagger_{0\beta} \times \vec{\nabla} \times (\vec{\sigma})^\beta{}_\alpha + \vec{\Lambda}^\dagger_\beta \times (\vec{\sigma})^\beta{}_\alpha \approx 0 \\ &\Longrightarrow \vec{\Lambda}^\dagger \approx \vec{\nabla} \Psi^\dagger_0 - i (\vec{\nabla} \times \vec{\Psi}^\dagger) \ . \end{aligned}$$

where we used the properties (A.8-A.10). The other Lagrange multipliers $\Lambda^\dagger_0$ and $\Lambda_0$ are undetermined.

Finally, in other to see whether there is an extra constraint, we must check the secondary constraints $K^\dagger$ and $K$ are also preserved in time.

$$\begin{aligned} \dot{K}^\alpha(x) &= \{ K^\alpha(x), H_T \} = \int d^3y \left\{ K^\alpha(x), \chi^{\dagger i}_\beta(y) \right\} \Lambda^\beta_i = -\frac{1}{2} \epsilon^{ijk} (\sigma_j)^\alpha{}_\beta \partial_k \Lambda^\beta_i \\ &= -\frac{1}{2} ((\vec{\sigma})^\alpha{}_\beta \cdot (\vec{\nabla} \times \vec{\Lambda}^\beta). \end{aligned}$$

Replacing the value of $\vec{\Lambda}$ obtained above and using the fact that $\vec{\nabla}\times(\vec{\nabla}\Psi_0) = 0$, we get

$$\dot{K}^\alpha(x) \approx \frac{i}{2}(\vec{\sigma})^\alpha{}_\beta \cdot (\vec{\nabla}\times(\vec{\nabla}\times\vec{\Psi}^\beta)) = -\frac{i}{2}\vec{\nabla}\cdot((\vec{\sigma})^\alpha{}_\beta \times (\vec{\nabla}\times\vec{\Psi}^\beta))$$

Using $(\vec{\sigma})^\alpha{}_\beta K^\beta \approx 0$ and the property (A.8), we obtain $i(\vec{\sigma})^\alpha{}_\beta \times (\vec{\nabla}\times\vec{\Psi}^\beta) \approx \vec{\nabla}\times\vec{\Psi}^\alpha$, so

$$\dot{K}^\alpha(x) \approx \frac{1}{2}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\Psi}^\alpha) = 0 \,.$$

where in the last equality we assumed $\vec{\Psi}$ is smooth. Following the same steps, we can prove that

$$\dot{K}^\dagger_\alpha(x) \approx \frac{1}{2}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\Psi}^\alpha) = 0 \,.$$

This means there is no extra constraints, meaning we have the set of constraints: $\left\{\chi^{\dagger\mu}_\alpha, \chi^{\mu\alpha}, K^\dagger_\alpha, K^\alpha\right\}$.

### 6.2.3 First-class and second-class constraints

We already known from (6.12) that $\chi^0$ and $\chi^\dagger_0$ are first-class constraints. This is why the Lagrange multipliers $\Lambda^\dagger_0$ and $\Lambda_0$ are undetermined. The transformation associated to those constraints are arbitrary shifts on $\Psi^\dagger_0$ and $\Psi_0$, respectively.

As the action (6.1) has the symmetry (6.5), we can guess the following first-class constraints

$$\begin{aligned} \widetilde{K}^\alpha &\equiv \partial_\mu \chi^{\mu\alpha} - K^\alpha \\ \widetilde{K}^\dagger_\alpha &\equiv \partial_\mu \chi^{\dagger\mu}_\alpha - K^\dagger_\alpha \,. \end{aligned} \tag{6.17}$$

Indeed, using (6.13) and (6.16), it can be shown that (6.17) are first-class constraints and generates the gauge transformation (6.5), i.e., with arbitrary fermionic functions parameters $\varepsilon^\dagger$ and $\varepsilon$, at equal time, we have

$$\begin{aligned} \delta\vec{\Psi}^\dagger_\alpha(\vec{x}) &\equiv \left\{\int d^3y \varepsilon^\dagger_\beta(\vec{y})\widetilde{K}^\beta(\vec{y}), \vec{\Psi}^\dagger_\alpha(\vec{x})\right\} = \vec{\nabla}\varepsilon^\dagger_\alpha \,, \\ \delta\vec{\Psi}^\alpha(\vec{x}) &\equiv \left\{\int d^3y \widetilde{K}^\dagger_\beta(\vec{y})\varepsilon^\beta(\vec{y}), \vec{\Psi}^\alpha(\vec{x})\right\} = \vec{\nabla}\varepsilon^\alpha \,. \end{aligned} \tag{6.18}$$

### 6.2.4 Degrees of freedom counting

Once we classified the constraints in first-class $F$ and second-class $S$, the DOF are obtained following the formula already presented in (3.45) [26].

| Constraint kind | Number of this kind | Classification |
|---|---|---|
| $\chi^{0\alpha}$ | 2 | first-class |
| $\chi^{\dagger 0}_{\alpha}$ | 2 | first-class |
| $\widetilde{K}^{\alpha}$ | 2 | first-class |
| $\widetilde{K}^{\dagger}_{\alpha}$ | 2 | first-class |
| $\chi^{i\alpha}$ | 6 | second-class |
| $\chi^{\dagger i}_{\alpha}$ | 6 | second-class |

Table 6.1: Constraint classification for the free left-handed massless RS theory.

In this case $N = 2 \times 16 = 32$, $F = 8$ and $S = 20 - 8 = 12$ , as is shown in Table 6.1. Therefore, the formula (3.45) gives us $DOF = 2$. Of course, this is only when we take into account the left-handed theory (6.8). In the case of Majorana real spinors, the right-handed sector is determined by the left-handed one [115] and does not add new local DOF. In the complex Dirac spinor case, the right-handed sector brings in the same number of local DOF, leading to a total of $DOF = 4$. It is worth to nothing that these are the same results obtained in previous works [113, 116].

## 6.3 Faddeev-Jackiw method

We can apply for the free massless RS action (6.8) the Faddeev-Jackiw (**FJ**) method [27, 117]. Although this technique was already known and used some time before the paper of Ludvig Faddeed and Roman Jackiw for Lagrangians which are first order in the fields (see for instance [118], and references therein), we still shall call this *the Faddeev-Jackiw method*. This method has the advantage we do not need to introduce superfluous constraints, given us directly the local DOF without classifying such constraints in first and second class (if we are able to solve them, of course). For the case of massless RS theory, it elucidates which components of $\psi^{\alpha}_{\mu}$ we can choose as the local DOF of the system making use of the fact that the action (6.2) has fermionic gauge symmetry under the transformation (6.5).

Let us start with the Lagrangian density appearing in (6.8), which can be written as

$$\mathcal{L} = \frac{1}{4}\dot{\Psi}^{\dagger}_{i}\sigma_{j}\Psi^{k}\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi^{\dagger}_{i}\sigma_{j}\dot{\Psi}_{k} - \Psi^{\dagger}_{0}K - K^{\dagger}\Psi_{0} + \frac{1}{4}\Psi^{\dagger}_{i}\partial_{j}\Psi_{k}\epsilon^{ijk} + \frac{1}{4}\epsilon^{ijk}\partial_{i}\Psi^{\dagger}_{j}\Psi_{k} \; , \tag{6.19}$$

where $K^{\dagger}_{\alpha}$ and $K^{\alpha}$ are the constraints defined in (6.15). As we already known, these constraints can be improved to be first-class and generates the fermionic gauge transformation (6.5). Therefore, we have the freedom to change the

| Field | Number |
|---|---|
| $\Psi^{\alpha}_{3/2Ti}$ (with corresponding momentum $\Psi^{\dagger}_{3/2Ti\alpha}$) | 2 |

Table 6.2: Local DOF kept after solving the constraints for the free massless left-handed RS theory.

fermions as

$$\begin{aligned}\Psi^{\dagger}_{i\alpha} &= \Psi'^{\dagger}_{i\alpha} + \partial_i \zeta^{\dagger}_{\alpha} \,, \\ \Psi^{\alpha}_{i} &= \Psi'^{\alpha}_{i} + \partial_i \zeta^{\alpha} \,.\end{aligned}$$

We can fix the gauge choosing $\zeta^{\dagger}_{\alpha}$ and $\zeta^{\alpha}$ such that $\Psi'^{\dagger}_{i\alpha}(\sigma^i)^{\alpha}_{\beta} = 0$ and $(\sigma^i)^{\beta}_{\alpha}\Psi'^{\alpha}_{i} = 0$. This means (see Appendix (E.2))

$$\begin{aligned}\Psi'^{\dagger}_{1/2i\alpha} &= 0 \,, \\ \Psi'^{\alpha}_{1/2i} &= 0 \,.\end{aligned}$$

Even more, using (E.2), we can write the constraints as

$$\begin{aligned}K^{\alpha} &= -\frac{i}{2}\vec{\nabla}\cdot\vec{\Psi}^{\alpha}_{3/2} = -i\nabla^2\Theta^{\alpha} = 0, \\ K^{\dagger}_{\alpha} &= \frac{i}{2}\vec{\nabla}\cdot\vec{\Psi}^{\dagger}_{3/2\alpha} = -i\nabla^2\Theta^{\dagger}_{\alpha} = 0,\end{aligned}$$

where we decompose $\Psi^{\dagger}_{3/2i}$ and $\Psi_{3/2i}$ in their transversal (T) and longitudinal (L) components (see Appendix E.3). Then, the constraints $K^{\dagger}$ and $K$ can be seen as a set of differential equations (Laplace equations) for $\Theta^{\dagger}_{\alpha}$ and $\Theta^{\alpha}$, respectively. With suitable boundary condition (for instance, these functions decay sufficiently rapid at infinity) these equations can be solved ($\Theta^{\dagger} = \Theta = 0$, for the later boundary conditions). Taking all this into account and using again (E.2) for the kinetic term, we can write the Lagrangian density (6.19) as

$$\mathcal{L} = -\frac{i}{4}\dot{\vec{\Psi}}^{\dagger}_{3/2T}\cdot\vec{\Psi}_{3/2T}+\frac{i}{4}\vec{\Psi}^{\dagger}_{3/2T}\cdot\dot{\vec{\Psi}}_{3/2T}+\frac{1}{4}\Psi^{\dagger}_{3/2T}\cdot(\nabla\times\vec{\Psi}_{3/2T})+\frac{1}{4}(\nabla\times\vec{\Psi}^{\dagger}_{3/2T})\cdot\vec{\Psi}_{3/2T}, \tag{6.20}$$

indicating clearly $\Psi^{\alpha}_{3/2T}$ are the only degrees of freedom for the free massless RS theory, as shown in Table 6.2. . Therefore, $DOF = 2$ for the real Majorana spinor and, if we take into account the right-handed component for the complex Dirac spinor case, we have $DOF = 4$. It reassuring that this result coincides with the one obtained by using the Dirac's Hamiltonian formalism.

# Chapter 7

# Abelian gauged massless Rarita-Schwinger theory

As we have a more clear picture the local DOF of the free massless RS theory and how to apply both Dirac's Hamiltonian formalism and FJ method, we will move now to the gauged massless RS theory, starting from [22]. We shall apply the same methods than the free theory to elucidate if the fermionic gauge transformation is present in this case or not. As we mentioned before, this symmetry is crucial to overcoming some of the inconsistencies presented in [20], as the semidefinite positivity of the quantum field anticommutators. This is why once clarified the dynamical contents of the theory, we will compute the quantum fermion field anticommutators.

## 7.1 Classical action, symmetries and field equations

In the Abelian gauged case, the total action can be written as [22]

$$S[A,\overline{\psi},\psi] = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu\gamma_5\gamma_\nu\overrightarrow{D}_\rho\psi_\sigma + \overline{\psi}_\mu\overleftarrow{D}_\nu\gamma_5\gamma_\rho\psi_\sigma\right) \ , \tag{7.1}$$

where the anti-Hermitian convention for the Abelian covariant derivatives acting on $\overline{\psi}$ and $\psi$ are, respectively,

$$\begin{aligned} \overline{\psi}_\mu\overleftarrow{D}_\nu &= \partial_\nu\overline{\psi}_\mu + ig\overline{\psi}_\mu A_\nu \ , \\ \overrightarrow{D}_\mu\psi_\nu &= \partial_\mu\psi_\nu - igA_\mu\psi_\nu \ . \end{aligned} \tag{7.2}$$

The action (7.1) has the usual Abelian symmetry

$$\begin{aligned} A_\mu &\rightarrow A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \theta \ , \\ \psi^\dagger_{\mu\alpha} &\rightarrow \psi'^\dagger_{\mu\alpha} = e^{-i\theta}\psi^\dagger_{\mu\alpha} \ , \\ \psi^\alpha_\mu &\rightarrow \psi'^\alpha_\mu = e^{i\theta}\psi^\alpha_\mu \ , \end{aligned}$$

where $\theta$ is a local parameter and depends smoothly on the spacetime point $x$.

Using the two-component decomposition (E.1), we can write the left-chiral component action (7.1) as

$$\begin{aligned} S^L[A,\overline{\psi},\psi] &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\int d^4x \left[\dot{\Psi}^\dagger \sigma_j \Psi_k - \Psi^\dagger_i \sigma_j \dot{\Psi}_k\right] \\ &+\frac{1}{4}\int d^4x \left[\Psi^\dagger_i \sigma_j \overrightarrow{D}_k \Psi_0 + \Psi^\dagger_i \overleftarrow{D}_j \sigma_k \Psi_0 - \Psi^\dagger_0 \sigma_i \overrightarrow{D}_j \Psi_k - \Psi^\dagger_0 \overleftarrow{D}_i \sigma_j \Psi_k\right] \\ &+\frac{1}{4}\int d^4x \epsilon^{ijk}\left[\Psi^\dagger_i \overrightarrow{D}_j \Psi_k - \Psi^\dagger_i \overleftarrow{D}_j \Psi_k\right] \\ &+ \frac{ig}{2}\int d^4x \epsilon^{ijk} A_0 \Psi^\dagger_i \sigma_j \Psi_k \ . \end{aligned} \tag{7.3}$$

Taking into account the Bianchi identity and varying the action (7.3) with respect to $A_\mu$, $\Psi^\dagger_\mu$ and $\Psi_\mu$ gives us, respectively[1]

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \quad , \quad \partial_\nu F^{\mu\nu} = \frac{-ig}{2}\epsilon^{\mu\nu\rho\tau}\Psi^\dagger_\nu \sigma_\rho \Psi_\tau \ , \tag{7.4}$$

$$\epsilon^{\mu\nu\rho\tau}\sigma_\nu \overrightarrow{D}_\rho \Psi_\tau = 0 \quad , \quad \epsilon^{\mu\nu\rho\tau}\Psi^\dagger_\nu \overleftarrow{D}_\rho \sigma_\tau = 0 \ . \tag{7.5}$$

The equations (7.4) are the Maxwell equations with a RS fermion source and the (7.5) are the gauged version of (6.3).

## 7.2 Dirac's Hamiltonian formalism

### 7.2.1 Momenta, Poisson brackets and canonical Hamiltonian

The canonical momenta associated to the gauge field are

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \ , \tag{7.6a}$$

[1] As usual, we defined the cuadri-vector notation $\sigma_\mu = (\mathbb{I}, \vec{\sigma})$ and $\widetilde{\sigma}_\mu = (-\mathbb{I}, \vec{\sigma})$.

while the fermionic canonical momenta are given again by[2]

$$\begin{aligned} \Xi^{\mu\alpha} &= \frac{\partial^L \mathcal{L}}{\partial \dot{\Psi}^\dagger_{\mu\alpha}}, \\ \Xi^{\dagger\mu}_\alpha &= \frac{\partial^R \mathcal{L}}{\partial \dot{\Psi}^\alpha_\mu} . \end{aligned} \tag{7.6b}$$

The non-vanishing Poisson brackets, at equal time, between fields and momenta are (following the convention of [26])

$$\begin{aligned} \{A_\mu(x), \pi^\nu(y)\} &= -\{\pi^\nu(y), A_\mu(x)\} = \delta^\nu_\mu \delta^{(3)}(x-y) , \\ \left\{\Psi^\alpha_\mu(x), \Xi^{\dagger\nu}_\beta(y)\right\} &= \left\{\Xi^{\dagger\nu}_\beta(y), \Psi^\alpha_\mu(x)\right\} = \delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(x-y) , \\ \left\{\Psi^{\dagger\alpha}_\mu(x), \Xi^{\nu\beta}(y)\right\} &= \left\{\Xi^{\nu\beta}(y), \Psi^{\dagger\alpha}_\mu(x)\right\} = -\delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(x-y) . \end{aligned} \tag{7.7}$$

The time and spatial components of (7.6) are given by

$$\begin{aligned} \pi^0 = 0 \quad &, \quad \pi^i = F^{i0} , \\ \Xi^{0\alpha} = 0 \quad &, \quad \Xi^{i\alpha} = \frac{1}{4}\epsilon^{ijk}(\sigma_j)^\alpha{}_\beta \Psi^\beta_k , \\ \Xi^{\dagger 0}_\alpha = 0 \quad &, \quad \Xi^{\dagger i}_\alpha = -\frac{1}{4}\epsilon^{ijk}\Psi^\dagger_{j\beta}(\sigma_k)^\beta{}_\alpha . \end{aligned}$$

We can write the canonical Hamiltonian $H_0$ as

$$\begin{aligned} H_0 &= \int d^3x \left[\frac{1}{2}\pi^i\pi_i - A_0\left(\partial_i\pi^i + \frac{ig}{2}\Psi^\dagger_i\sigma_j\Psi_k\right) + \frac{1}{4}F_{ij}F^{ij}\right] \\ &\quad -\frac{1}{4}\int d^3x \epsilon^{ijk}\left[\Psi^\dagger_i\sigma_j\overrightarrow{D}_k\Psi_0 + \Psi^\dagger_i\overleftarrow{D}_j\sigma_k\Psi_0 - \Psi^\dagger_0\sigma_i\overrightarrow{D}_j\Psi_k \right. \\ &\quad \left. -\Psi^\dagger_0\overleftarrow{D}_i\sigma_j\Psi_k + \Psi^\dagger_i\overrightarrow{D}_j\Psi_k - \Psi^\dagger_i\overleftarrow{D}_j\Psi_k\right] . \end{aligned} \tag{7.8}$$

### 7.2.2 Primary and secondary constraints

The primary constraints are given by

$$\begin{aligned} \phi^0 &= \pi^0 \approx 0 , \\ \chi^{0\alpha} \equiv \Xi^{0\alpha} \approx 0 \quad &, \quad \chi^{i\alpha} \equiv \Xi^{i\alpha} - \frac{1}{4}\epsilon^{ijk}(\sigma_j)^\alpha{}_\beta\Psi^\beta_k \approx 0 , \\ \chi^{\dagger 0}_\alpha \equiv \Xi^{\dagger 0}_\alpha \approx 0 \quad &, \quad \chi^{\dagger i}_\alpha \equiv \Xi^{\dagger i}_\alpha + \frac{1}{4}\epsilon^{ijk}\Psi^\dagger_{j\beta}(\sigma_k)^\beta{}_\alpha \approx 0 , \end{aligned} \tag{7.9}$$

whose only non-vanishing Poisson bracket is

$$\left\{\chi^{\dagger i}_\alpha(x), \chi^{j\beta}(y)\right\} = -\frac{1}{2}\epsilon^{ijk}(\sigma_k)^\beta{}_\alpha\delta^{(3)}(x-y) . \tag{7.10}$$

[2] In this subsection, we will write explicitly the spinorial indexes $\{\alpha, \beta, \ldots\}$.

Now, we write the primary Hamiltonian

$$H_T = H_0 + \int d^3x \left(\Lambda^\dagger_\mu \chi^\mu + \chi^{\dagger\mu}\Lambda_\mu\right) \ ,$$

where $\Lambda^{\dagger\mu}$ and $\Lambda_\mu$ are Lagrange multipliers associated to the primary constraints, except for the constraint[3] $\pi^0$.

We demand the primary constraints (7.9) to hold as the system evolves in time, i.e.,

$$\begin{aligned}
\dot{\phi}^0(x) &= \left\{\phi^0(x), H_T\right\} = \partial_i\pi^i + \frac{ig}{2}\epsilon^{ijk}\Psi^\dagger_i\sigma_j\Psi_k \approx 0 \ , \\
\dot{\chi}^{0\alpha}(x) &= \left\{\chi^{0\alpha}(x), H_T\right\} = -\frac{1}{2}\epsilon^{ijk}(\sigma_i)^\alpha{}_\beta \overrightarrow{D}_j\Psi^\beta_k \approx 0 \ , \\
\dot{\chi}^{\dagger 0}_\alpha(x) &= \left\{\chi^{\dagger 0}_\alpha(x), H_T\right\} = -\frac{1}{2}\epsilon^{ijk}\Psi^\dagger_{j\beta}\overleftarrow{D}_i(\sigma_k)^\beta{}_\alpha \approx 0 \ ,
\end{aligned}$$

where in the second equalities we used (7.7), (7.9) and (7.8) explicitly. This lead us to define the following secondary constraints

$$\begin{aligned}
k &\equiv -\partial_i\pi^i - \frac{ig}{2}\epsilon^{ijk}\Psi^\dagger_i\sigma_j\Psi_k \approx 0 \ , \\
K^\alpha &\equiv \frac{1}{2}\epsilon^{ijk}(\sigma_i)^\alpha{}_\beta \overrightarrow{D}_j\Psi^\beta_k \approx 0 \ , \\
K^\dagger_\alpha &\equiv \frac{1}{2}\epsilon^{ijk}\Psi^\dagger_{j\beta}\overleftarrow{D}_i(\sigma_k)^\beta{}_\alpha \approx 0 \ ,
\end{aligned} \tag{7.11}$$

whose non-vanishing Poisson brackets with the rest of constraints (3.5) (and between them) are given in Appendix F.1.

Substituting (7.11) in (7.8), we get

$$\begin{aligned}
H_0 &= \int d^3x \left[A_0 k + \Psi^\dagger_0 K + K^\dagger\Psi_0 + \frac{1}{2}\pi^i\pi_i + \frac{1}{4}F_{ij}F^{ij}\right] \\
&\quad -\frac{1}{4}\int d^3x \epsilon^{ijk}\left[\Psi^\dagger_i\overrightarrow{D}_j\Psi_k - \Psi^\dagger_i\overleftarrow{D}_j\Psi_k\right] \ .
\end{aligned}$$

Preservation in time of the rest of primary constraints determines a possible set of Lagrange multipliers (see Appendix F.1 for computational details)

$$\begin{aligned}
\Lambda^\alpha_i &\approx igA_0\Psi^\alpha_i + \overrightarrow{D}_i\Psi^\alpha_0 + i\epsilon^{ijk}\overrightarrow{D}_j\Psi^\alpha_k \ , \\
\Lambda^\dagger_{i\alpha} &\approx -igA_0\Psi^\dagger_{i\alpha} + \Psi^\dagger_{0\alpha}\overleftarrow{D}_i - i\epsilon^{ijk}\Psi^\dagger_{k\alpha}\overleftarrow{D}_j \ .
\end{aligned}$$

where we used the properties (A.8-A.10).

[3] In Section 7.2.3 we present $\pi^0$ as a first-class constraint. Therefore, its Lagrange multiplier is undetermined.

Until now, everything was pretty like the free case, except the extra gauge field momenta and the covariant derivatives. However, demanding time preservation of the secondary constraints does not hold so easy as in the previous case. First of all, it can be proved with a little bit of algebra that the constraint $k$ does preserves in time, i.e.,

$$\dot{k}(x) = \{k(x), H_T\} \approx 0 \ .$$

But imposing preservation of $K^\alpha$ and $K^\dagger_\alpha$ gives us

$$\begin{aligned} \dot{K}^\alpha(x) &= \{K^\alpha(x), H_T\} \approx V^\alpha(x) \ , \\ \dot{K}^\dagger_\alpha(x) &= \left\{K^\dagger_\alpha(x), H_T\right\} \approx V^\dagger_\alpha(x) \ , \end{aligned}$$

where we defined

$$\begin{aligned} V^\alpha &= \frac{ig}{4}\epsilon^{\mu\nu\rho\tau}(\sigma_\mu)^\alpha{}_\beta F_{\nu\rho}\Psi^\beta_\tau \approx 0 \ , \\ V^\dagger_\alpha &= \frac{ig}{4}\epsilon^{\mu\nu\rho\tau}\Psi^\dagger_{\mu\beta}(\sigma_\nu)^\beta{}_\alpha F_{\rho\tau} \approx 0 \ . \end{aligned} \tag{7.12}$$

It is important to note that in spite we are using spacetime notation, due to definition of spatial gauge momenta $\pi^i = F^{i0}$, there are no velocities in definitions (7.12). So, in principle, it seems that (7.12) are new secondary (tertiary) constraints, as is claim in [22]. The non-vanishing Poisson brackets of the constraints (7.12) between them and the rest of primary and secondary constraints are given in Appendix F.1.

We can see from the first two brackets in (F.2) that $\chi^{\dagger 0}_\alpha$ and $\chi^{0\alpha}$ are not first-class constraints as in the free case. This means, the fields $\Psi^\dagger_0$ and $\Psi_0$ are not arbitrary anymore, but instead determined by the other canonical fields $\Psi^\dagger_i$ and $\Psi_i$, respectively. In order to see this fact explicitly, let us first define $\Sigma^i = \epsilon^{ijk}F_{jk}$, which is proportional to the magnetic field, in the same way that $\pi^i$ is proportional to the electric field. Now, we will prove a straightforward, but a useful result.

**Lemma 7.2.1.** *Let $\vec{A}$ be a spatial vector. Then $\vec{A} \neq \vec{0} \iff \vec{\sigma}\cdot\vec{A} \neq 0$.*

*Proof.* $(\Rightarrow)$

Using (A.6), we have

$$(\sigma\cdot\vec{A})(\sigma\cdot\vec{A}) = (\sigma^i\sigma^j)^\alpha{}_\beta A_i A_j = \vec{A}\cdot\vec{A}\delta^\alpha_\beta.$$

As $\vec{A} \neq 0$, then $\vec{A}\cdot\vec{A} \neq 0$ implying, due the above equality, $(\sigma\cdot\vec{A}) \neq 0$.

$(\Leftarrow)$

Suppose $\vec{A} = \vec{0}$. Then $\vec{\sigma}\cdot\vec{A} = 0$, but by hypothesis $\sigma\cdot\vec{A} \neq 0$, which is absurd. So, it is true $\vec{A} \neq \vec{0}$. □

Because of the Lemma 7.2.1, if the magnetic field is non-zero ($\vec{\Sigma}\cdot\vec{\Sigma} > 0$), then we can solve (7.12) for $\Psi_0$ and $\Psi_0^\dagger$ as function of the remaining dynamical fields. Indeed,

$$
\begin{aligned}
\Psi_0^\alpha &\approx \frac{1}{\vec{\Sigma}\cdot\vec{\Sigma}}\left[\vec{\Sigma}\cdot(\vec{\sigma})^\alpha_\beta(\vec{\Psi})^\beta\cdot\vec{\Sigma} + 2\vec{\Sigma}\cdot(\vec{\sigma})^\alpha_\beta\left[\vec{\pi}\cdot(\vec{\sigma}\times\vec{\Psi})^\beta\right]\right] , \qquad (7.13)\\
\vec{\Psi}^\dagger_{0\alpha} &\approx \frac{1}{\vec{\Sigma}\cdot\vec{\Sigma}}\left[\vec{\Sigma}\cdot(\vec{\Psi})^\dagger_\beta(\vec{\sigma})^\beta_\alpha\cdot\vec{\Sigma} - 2\left[(\vec{\Psi}^\dagger\times\vec{\sigma})_\beta\cdot\vec{\pi}\right](\vec{\sigma})^\beta_\alpha\cdot\vec{\Sigma}\right] .
\end{aligned}
$$

Demanding preservation in time of $V^\alpha$ and $V^\dagger_\alpha$ and using the brackets (F.2) allows us to solve the Lagrange multipliers $\Lambda_0$ and $\Lambda_0^\dagger$, respectively. Indeed,

$$
\begin{aligned}
\Lambda_0^\alpha &\approx \frac{\vec{\Sigma}\cdot(\vec{\sigma})^\alpha_\beta}{\vec{\Sigma}\cdot\vec{\Sigma}}\begin{pmatrix} 2[\vec{\nabla}\times\vec{\pi}\cdot\vec{\Psi}]^\beta - 2[\vec{\nabla}\times\vec{\pi}\cdot\vec{\sigma}\Psi_0]^\beta + \vec{\Sigma}\cdot\vec{\Lambda}^\alpha + 2[\vec{\pi}\cdot\vec{\sigma}\times\vec{\Lambda}]^\beta + \\ \left[\vec{\nabla}\times\vec{\Sigma} - ig\left(\vec{\Psi}^\dagger\times\vec{\sigma}\Psi_0 + \Psi_0^\dagger\vec{\sigma}\times\vec{\Psi} - \vec{\Psi}^\dagger\times\vec{\Psi}\right)\right]\cdot(\vec{\sigma}\times\vec{\Psi})^\beta \end{pmatrix} \qquad (7.14)\\
\Lambda^\dagger_{0\alpha} &\approx \begin{pmatrix} 2[\vec{\Psi}^\dagger\cdot\vec{\nabla}\times\vec{\pi}]_\beta - 2[\Psi_0^\dagger\vec{\sigma}\cdot\vec{\nabla}\times\vec{\pi}]_\beta + \vec{\Lambda}^\dagger_\beta\cdot\vec{\Sigma} - 2[\vec{\Lambda}^\dagger\times\vec{\sigma}\cdot\vec{\pi}]_\beta + \\ (\vec{\Psi}^\dagger\times\vec{\sigma})_\beta\cdot\left[-\vec{\nabla}\times\vec{\Sigma} + ig\left(\vec{\Psi}_0^\dagger\vec{\sigma}\times\vec{\Psi} + \vec{\Psi}^\dagger\times\vec{\sigma}\Psi_0 - \vec{\Psi}^\dagger\times\vec{\Psi}\right)\right] \end{pmatrix}\frac{(\vec{\sigma})^\beta_\alpha\cdot\vec{\Sigma}}{\vec{\Sigma}\cdot\vec{\Sigma}} .
\end{aligned}
$$

It can be proved as a crosscheck, as $\dot{\Psi}^\dagger_{0\alpha} = \{\Psi^\dagger_{0\alpha}, H_T\} = \Lambda^\dagger_{0\alpha}$ and $\dot{\Psi}_0^\alpha = \{\Psi_0^\alpha, H_T\} = \Lambda_0$, the set of equations (7.14) is the time derivative of (7.13), where the time derivatives of canonical variables were substituted by their field equations from (7.5). As a byproduct of these computations, we checked there is no more constraints for the gauged massless RS field theory. We have then the following set of $(2+24n)$ constraints[4]: $\left\{\phi^0, k, \chi^{\dagger\mu}_\alpha, \chi^{\mu\alpha}, K^\dagger_\alpha, K^\alpha, V^\dagger_\alpha, V^\alpha\right\}$.

### 7.2.3 First-class and second-class constraints

Strictly speaking, taking into account the above considerations, there is no first-class constraint associated with the fermionic gauge transformations unlike the free theory. It can be proved that the two first-class constraints are $\phi^0$ and the combination

$$
\widetilde{k} = k - ig\left(\vec{\Psi}^\dagger\cdot\vec{\chi} - \vec{\chi}^\dagger\cdot\vec{\Psi}\right) .
$$

This last improved version of $k$ constraint generates the usual infinitesimal Abelian gauge transformations, i.e.,

$$
\begin{aligned}
\delta_\theta A_i = \left\{A_i(x), \int d^3y\widetilde{k}(y)\theta(y)\right\} &= \frac{1}{g}\partial_i\theta(x) ,\\
\delta_\theta\Psi^\dagger_{i\alpha} = \left\{\Psi^\dagger_{i\alpha}(x), \int d^3y\widetilde{k}(y)\theta(y)\right\} &= -ig\theta(x)\Psi^\dagger_{i\alpha}(x) ,\\
\delta_\theta\Psi_i^\alpha = \left\{\Psi_i^\alpha(x), \int d^3y\widetilde{k}(y)\theta(y)\right\} &= ig\theta(x)\Psi_i^\alpha(x) .
\end{aligned}
$$

[4]Remember that $n = 1$ corresponds to real Majorana spinors, while $n = 2$ to complex Dirac spinors.

| Constraint kind | Number of this kind | Classification |
|---|---|---|
| $\phi^0$ | 1 | first-class |
| $\widetilde{k}$ | 1 | first-class |
| $\chi^{0\alpha}$ | 2 | second-class |
| $\chi^{\dagger 0}_{\alpha}$ | 2 | second-class |
| $\chi^{i\alpha}$ | 6 | second-class |
| $\chi^{\dagger i}_{\alpha}$ | 6 | second-class |
| $K^{\alpha}$ | 2 | second-class |
| $K^{\dagger}_{\alpha}$ | 2 | second-class |
| $V^{\dagger}_{\alpha}$ | 2 | second-class |
| $V^{\alpha}$ | 2 | second-class |

Table 7.1: Constraint classification for the massless left-handed RS theory coupled to a Maxwell field.

The remaining set of $24n$ constraints $\left\{\chi^{\dagger}_{\mu\alpha}, \chi^{\alpha}_{\mu}, K^{\dagger}_{\alpha}, K^{\alpha}, V^{\dagger}_{\alpha}, V^{\alpha}\right\}$ are second class, as we can see for the brackets computed in Appendix F.1.

### 7.2.4 Degrees of freedom counting

Applying again the formula (3.45), we have for this case $N = 2 \times (16 + 4) = 32 + 8$, $F = 2$, and $S = 24$, according to Table 7.1. Therefore, the number of degrees of freedom is now $DOF = 4 + 2 = 6$ for the real Majorana spinor. Taking into account also the right-handed fermion component for the complex Dirac spinor, we gave $DOF = 10$.

The main conclusion of this section is that the gauged massless RS theory is not invariant under fermionic gauge transformation (6.5). From the point of view of the formalism, the reason of that is the presence of tertiary constraints (7.12), which make the $\chi^{\dagger 0}, \chi^{0}, K^{\dagger}, K$ second-class constraints (and not first-class as the free case). So, there is a discontinuity on the DOF when one turns on the gauge coupling constant $g$.

## 7.3 Faddeev-Jackiw method

As in the free massless RS theory in Section 6.3, we can apply the FJ method for the gauged case. In such a case, the Lagrangian density appearing in

(7.3) can be written as

$$\begin{aligned}\mathcal{L} &= \dot{A}_i\pi^i + \frac{1}{4}\dot{\Psi}_i^\dagger\sigma_j\Psi^k\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi_i^\dagger\sigma_j\dot{\Psi}_k - A_0 k - \Psi_0^\dagger K - K^\dagger\Psi_0 \\ &- \frac{1}{2}\pi_i\pi^i - \frac{1}{4}F_{ij}F^{ij} + \frac{1}{4}\Psi_i^\dagger\overrightarrow{D}_j\Psi_k\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi_i^\dagger\overleftarrow{D}_j\Psi_k \,, \end{aligned} \tag{7.15}$$

where the constraints $k$, $K_\alpha^\dagger$ and $K^\dagger$ were defined in (7.11).

We first proceed by solving the constraint $k$. It will be useful to split the $A_i$ adn $\pi^i$ fields in their transversal (T) and longitudinal (L) components, i.e.,

$$\begin{aligned} A_i = A_{Ti} + A_{Li} \quad &, \quad \partial^i A_{Ti} = 0 \ , \ A_{Li} = \partial_i A \ , \\ \pi_i = \pi_{Ti} + \pi_{Li} \quad &, \quad \partial^i \pi_{Ti} = 0 \ , \ \pi_{Li} = \partial_i \pi \ , \end{aligned} \tag{7.16}$$

where $A$ and $\pi$ are scalars which define locally the longitudinal components of $A_i$ and $\pi^i$, respectively. The constraint $k$ can be read as

$$\partial_i\pi_L^i = -\frac{ig}{2}\epsilon^{ijk}\Psi_i^\dagger\sigma_j\Psi_k \Longrightarrow \nabla^2\pi = -\rho \,, \tag{7.17}$$

where we defined $\rho = \frac{ig}{2}\epsilon^{ijk}\Psi_i^\dagger\sigma_j\Psi_k$. This means we can solve $\pi$ from the Poisson equation (7.17) with suitable boundary conditions. Let us write symbolically[5] $\pi = -(\nabla^2)^{-1}\rho$, meaning $\pi$ is a solution of equation (7.17).

Now, we can observe that

$$\dot{A}_i\pi^i = \dot{A}_{Ti}\pi_T^i + \partial_i\dot{A}\partial^i\pi + \dot{A}_{Ti}\partial^i\pi + \partial_i\dot{A}\pi_T^i = \dot{A}_{Ti}\pi_T^i + \dot{A}\rho \,,$$

where in the second equality we integrated by parts and used (7.17). Similarly,

$$-\frac{1}{2}\pi_i\pi^i = -\frac{1}{2}\pi_{Ti}\pi_T^i + \frac{1}{2}\rho(\nabla^2)^{-1}\rho \,.$$

We can observe also that

$$F_{ij} = \partial_i A_j - \partial_j A_i = \partial_i A_{Tj} - \partial_j A_{Ti} = F_{Tij} \,,$$

where in the second equality we considered $A_i$ as a smooth function on the coordinates. Therefore, the Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \dot{A}_{Ti}\pi_T^i + \dot{A}\rho + \frac{1}{4}\dot{\Psi}_i^\dagger\sigma_j\Psi^k\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi_i^\dagger\sigma_j\dot{\Psi}_k - \Psi_0^\dagger K - K^\dagger\Psi_0 \\ &- \frac{1}{2}\pi_{Ti}\pi_T^i + \frac{1}{2}\rho(\nabla^2)^{-1}\rho - \frac{1}{4}F_{Tij}F_T^{ij} + \frac{1}{4}\Psi_i^\dagger\overrightarrow{D}_j\Psi_k\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi_i^\dagger\overleftarrow{D}_j\Psi_k \,. \end{aligned}$$

[5] Being more rigourous, one solution of the Poisson equation (7.17) is $\pi(x) = -\int d^3y G(x,y)\rho(y)$, where $G(x,y) = -\frac{1}{4\pi|x-y|}$ is the Laplacian Green function, which fulfils $\nabla^2 G(x,y) = \delta^{(3)}(x-y)$ [119, 120].

| Field | Number |
|---|---|
| $A_T$ (with corresponding momentum $\pi_T^i$) | 2 |
| $\Psi^\alpha_{3/2Ti}$ (with corresponding momentum $\Psi^\dagger_{3/2Ti\alpha}$) | 2 |
| $\Psi^\alpha_{1/2i}$ (with corresponding momentum $\Psi^\dagger_{1/2i\alpha}$) | 2 |

Table 7.2: Degrees of freedom left after solving the constraints for the massless left-handed RS theory coupled to a Maxwell field.

We can transform the fermions in the following way,

$$\begin{aligned} \Psi^\dagger_{\mu\alpha} &\rightarrow \Psi'^\dagger_{\mu\alpha} = \Psi^\dagger_{\mu\alpha} e^{-iA} , \\ \Psi^\alpha_\mu &\rightarrow \Psi'^\alpha_\mu = e^{iA}\Psi^\alpha_\mu , \end{aligned}$$

writing the lagrangian density as

$$\begin{aligned} \mathcal{L} &= \dot{A}_{Ti}\pi^i_T + \frac{1}{4}\dot{\Psi}'^\dagger_i\sigma_j\Psi'^k\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi'^\dagger_i\sigma_j\dot{\Psi}'_k - \Psi'^\dagger_0 K' - K'^\dagger\Psi'_0 \\ &- \frac{1}{2}\pi_{Ti}\pi^i_T + \frac{1}{2}\rho(\nabla^2)^{-1}\rho - \frac{1}{4}F_{Tij}F^{ij}_T + \frac{1}{4}\Psi'^\dagger_i\overrightarrow{D}_{Tj}\Psi'_k\epsilon^{ijk} - \frac{1}{4}\epsilon^{ijk}\Psi'^\dagger_i\overleftarrow{D}_{Tj}\Psi'_k , \end{aligned}$$

where $D_{iT}$ is the covariant derivative considering only the transversal component of the gauge field $A_i$.

We solve now the fermionic constraints $K'^\dagger_\alpha$ and $K'^\alpha$, respectively

$$\begin{aligned} &\frac{i}{2}\nabla^2\Theta^\alpha + \frac{g}{2}A_{Ti}\partial^i\Theta^\alpha + \frac{g}{2}A_{Ti}\Psi'^\alpha_{3/2Ti} - i\overrightarrow{D}_i\Psi'^\alpha_{1/2i} = 0 , \\ &-\frac{i}{2}\nabla^2\Theta^\dagger_\alpha + \frac{g}{2}A_{Ti}\partial^i\Theta^\dagger_\alpha + \frac{g}{2}\Psi'^\dagger_{3/2Ti\alpha}A_{Ti} - i\Psi'^\dagger_{1/2i\alpha}\overleftarrow{D}_i = 0 , \end{aligned} \tag{7.18}$$

where we decompose the fermions in 3/2−spin and 1/2−spin components the former split in its transversal, $\partial^i\Psi_{3/2Ti} = 0$ and longitudinal part, $\Psi_{3/2Li} = \partial^i\Theta$ , (see Appendix E). With suitable boundary conditions, we can solve the above differential equations for $\Theta^\alpha$ and $\Theta^\dagger_\alpha$, respectively.

At the end, the only degrees of freedom of the gauged massless RS are in Table 7.2. . Therefore, $DOF = 2 + 4 = 6$ for the real Majorana spinor, as the Dirac's Hamiltonian formalism. If we take into account the right-handed components for the complex Dirac spinor, we have $DOF = 10$. Of course, instead of solving $\Theta^\dagger$ and $\Theta$ as a function of $\Psi^\dagger_{1/2i}$ and $\Psi_{i1/2}$ in (7.18), we can solve the last ones as function of the formers, as the number of variables coincides each other. In this case, the local DOF are both the transversal and longitudinal 3/2-spin particle components.

## 7.4 Wave fronts and quantization in the presence of an external magnetic field

Once the Dirac's constraint classification is done, we are ready to the quantization of gauged RS field through the construction of Dirac bracket [26, 121]. For the sake of simplicity, we will deal with an external time-independent gauge field which is coupled with the RS fields through the covariant derivatives (7.2). In this way, we can realize there is an external magnetic field $\Sigma^i = \epsilon^{ijk}F_{jk}$, as defined above. The fermionic constraints are as (7.9) and (7.11), and the total Hamiltonian reads

$$H_T = \int d^3x \left[ -\frac{ig}{2}A_0\epsilon^{ijk}\Psi_i^\dagger\sigma_j\Psi_k + \Psi_0^\dagger K + K^\dagger\Psi_0 + \frac{1}{4}\Psi_i^\dagger\left(\overleftarrow{D}_j - \overrightarrow{D}_j\right)\Psi_k + \chi_\mu^\dagger\Lambda^\mu + \Lambda_\mu^\dagger\chi^\mu\right] ,$$

where $\Lambda_\mu$ and $\Lambda_\mu^\dagger$ are Lagrange multipliers and $K_\alpha^\dagger$, $K^\alpha$ as defined in (7.11).

For this case, we can verify preservation in time of secondary constraints $K^\alpha$ and $K_\alpha^\dagger$ implies the tertiary constraints

$$V^\alpha \equiv -\frac{i}{4}g\left(\not{\Sigma}^\alpha{}_\beta\Psi_0^\beta - \Sigma^i\Psi_i^\alpha - 2\epsilon^{ijk}(\sigma_i)^\alpha{}_\beta\psi_k^\beta\partial_jA_0\right) \approx 0 , \tag{7.19}$$
$$V_\alpha^\dagger \equiv \frac{i}{4}g\left(\Psi_{0\beta}^\dagger\not{\Sigma}^\beta{}_\alpha - \Sigma^i\Psi_{i\alpha}^\dagger + 2\epsilon^{ijk}\psi_{k\beta}^\dagger(\sigma_k)^\alpha{}_\beta\partial_jA_0\right) \approx 0 ,$$

where we introduced the notation $\not{\Sigma}^\alpha{}_\beta \equiv \Sigma^i(\sigma_i)^\beta{}_\alpha$.

Preservation in time of constraints (7.19), determines the Lagrange multipliers $\Lambda_0^\alpha$ and $\Lambda_{0\alpha}^\dagger$, respectively, while preservation of $K^\alpha$ and $K_\alpha^\dagger$ determines $\Lambda_i^\alpha$ and $\Lambda_{i\alpha}^\dagger$, respectively. Therefore, there are no extra constraints. We have already shown all the constraint are second class.

In order to compute the Dirac brackets, we will proceed by steps (see for instance Exercise 1.12 of [26]).

1. Consider the second-class constraints $\chi_\alpha^{\dagger i}$ and $\chi_\alpha^i$. We have then the following partial constraints matrix[6]

$$\begin{pmatrix} & \chi_\beta^{\dagger j}(y) & \chi^{j\beta}(y) \\ \chi_\alpha^{\dagger i}(x) & 0 & -\frac{1}{2}\epsilon^{ijk}(\sigma_k)^\alpha{}_\beta \\ \chi^{i\alpha}(x) & \frac{1}{2}\epsilon^{ijk}(\sigma_k)^\alpha{}_\beta & 0 \end{pmatrix}\delta^{(3)}(x-y) = M(x,y) ,$$

whose inverse is

$$\begin{pmatrix} & \chi_\beta^{\dagger j}(y) & \chi^{j\beta}(y) \\ \chi_\alpha^{\dagger i}(x) & 0 & -i(\sigma_i\sigma_j)^\alpha{}_\beta \\ \chi^{i\alpha}(x) & i(\sigma_i\sigma_j)^\beta{}_\alpha & 0 \end{pmatrix}\delta^{(3)}(x-y) = M^{-1}(x,y) .$$

[6]The following description is just mnemonic and cannot be realized as a formal mathematical equality.

We can immediately compute the bracket at level 1, i.e.,

$$\begin{aligned}\left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^* &= \left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\} - \int d^3w d^3z \left\{\Psi^\dagger_{i\alpha},\chi^{k\gamma}(w)\right\}\left(M^{-1}\right)^{\tau}_{jk\gamma}(w,z)\left\{\chi^{\dagger l}_\tau(z),\Psi^\beta_j(y)\right\} \\ &= -i(\sigma_i\sigma_j)^\beta{}_\alpha\delta^{(3)}(x-y)\ .\end{aligned}$$

Other important brackets which are modified at this level are

$$\begin{aligned}\left\{K^\dagger_\alpha(x),K^\beta(y)\right\}^* &= -\frac{i}{4}g\not\Sigma^\beta{}_\alpha\delta^{(3)}(x-y)\ , \\ \left\{V^\dagger_\alpha(x),V^\beta(y)\right\}^* &= \frac{g}{2}\Sigma^i\left(\partial_iA_0 - i\frac{g}{8}\Sigma_i\right)\delta^\beta_\alpha\delta^{(3)}(x-y)\ .\end{aligned}$$

2. Consider the second-class constraints $\chi^{\dagger 0}_\alpha$, $\chi^0_\alpha$, $V^\dagger_\alpha$ and $V^\alpha$. We have then the following partial constraints matrix

$$\begin{pmatrix} & \chi^{\dagger 0}_\beta(y) & V^\dagger_\beta(y) & \chi^{0\beta}(y) & V^\beta(y) \\ \chi^{\dagger 0}_\alpha(x) & 0 & 0 & 0 & -\frac{i}{4}g\not\Sigma^\beta{}_\alpha \\ V^\dagger_\alpha(x) & 0 & 0 & -\frac{i}{4}g\not\Sigma^\beta{}_\alpha & \frac{g}{2}\Sigma^i\left(\partial_iA_0 - i\frac{g}{8}\Sigma_i\right)\delta^\beta_\alpha \\ \chi^{0\alpha}(x) & 0 & -\frac{i}{4}g\not\Sigma^\alpha{}_\beta & 0 & 0 \\ V^\alpha(x) & -\frac{i}{4}g\not\Sigma^\alpha{}_\beta & \frac{g}{2}\Sigma^i\left(\partial_iA_0 - i\frac{g}{8}\Sigma_i\right)\delta^\alpha_\beta & 0 & 0\end{pmatrix}\delta^{(3)}(x-y) = (N)(x,y)\ ,$$

whose inverse is

$$\begin{pmatrix} & \chi^{\dagger 0}_\beta(y) & V^\dagger_\beta(y) & \chi^{0\beta}(y) & V^\beta(y) \\ \chi^{\dagger 0}_\alpha(x) & 0 & 0 & \frac{2}{g}\frac{\delta^\alpha_\beta}{\Sigma^i\partial_iA_0 - \frac{ig}{8}\vec\Sigma\cdot\vec\Sigma} & \frac{4i}{g}\not\Sigma^\alpha{}_\beta \\ V^\dagger_\alpha(x) & 0 & 0 & \frac{4i}{g}\not\Sigma^\alpha{}_\beta & 0 \\ \chi^{0\alpha}(x) & \frac{2}{g}\frac{\delta^\beta_\alpha}{\Sigma^i\partial_iA_0 - \frac{ig}{8}\vec\Sigma\cdot\vec\Sigma} & \frac{4i}{g}\not\Sigma^\beta{}_\alpha & 0 & 0 \\ V^\alpha(x) & \frac{4i}{g}\not\Sigma^\beta{}_\alpha & 0 & 0 & 0\end{pmatrix}\delta^{(3)}(x-y) = (N^{-1})(x,y)\ .$$

We can see the modified bracket at level 2 of $\Psi$ with $\Psi^\dagger$ and $K^\dagger$ with $K$ are unchanged, i.e.,

$$\begin{aligned}\left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^{**} &= \left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^* - \int d^3w d^3z \left\{\Psi^\dagger_{i\alpha},V^\gamma(w)\right\}\left(N^{-1}\right)^{\tau}_{\gamma}(w,z)\left\{V^\dagger_\tau(z),\Psi^\beta_j(y)\right\} \\ &= -i(\sigma_i\sigma_j)^\beta{}_\alpha\delta^{(3)}(x-y)\ , \\ \left\{K^\dagger_\alpha(x),K^\beta(y)\right\}^{**} &= -\frac{i}{4}g\not\Sigma^\beta{}_\alpha\delta^{(3)}(x-y)\ .\end{aligned}$$

Other modified brackets are, for instance,

$$\begin{aligned}\left\{\Psi^\dagger_{0\alpha}(x),\Psi^\beta_i(y)\right\}^{**} &= -\frac{i}{\vec\Sigma\cdot\vec\Sigma}\left(\not\Sigma\sigma_i\not\Sigma\right)^\beta_\alpha\delta^{(3)}(x-y)\ , \\ \left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_0(y)\right\}^{**} &= -\frac{i}{\vec\Sigma\cdot\vec\Sigma}\left(\not\Sigma\sigma_i\not\Sigma\right)^\beta_\alpha\delta^{(3)}(x-y)\ .\end{aligned}$$

3. Consider the second-class constraints $K^\dagger_\alpha$ and $K^\alpha$. We have then the following partial constraints matrix

$$\begin{pmatrix} & K^\dagger_\beta(y) & K^\beta(y) \\ K^\dagger_\alpha(x) & 0 & -\frac{i}{4}g\Sigma^\beta{}_\alpha \\ K^\alpha(x) & -\frac{i}{4}g\Sigma^\alpha{}_\beta & 0 \end{pmatrix} \delta^{(3)}(x-y) = R(x,y) \ ,$$

whose inverse is

$$\begin{pmatrix} & K^\dagger_\beta(y) & K^\beta(y) \\ K^\dagger_\alpha(x) & 0 & -\frac{4i}{g\vec{\Sigma}\cdot\vec{\Sigma}}\Sigma^\alpha{}_\beta \\ K^\alpha(x) & -\frac{4i}{g\vec{\Sigma}\cdot\vec{\Sigma}}\Sigma^\beta{}_\alpha & 0 \end{pmatrix} \delta^{(3)}(x-y) = R^{-1}(x,y) \ .$$

Finally, we compute the bracket at level 3 (the Dirac bracket), i.e.,

$$\begin{aligned} \left\{\Psi^\dagger_{i\alpha}(x), \Psi^\beta_j(y)\right\}^{***} &= \left\{\Psi^\dagger_{i\alpha}(x), \Psi^\beta_j(y)\right\}^{**} - \int d^3w d^3z \left\{\Psi^\dagger_{i\alpha}, K^\gamma(w)\right\}^{**} (R^{-1})^\tau_\gamma(w,z) \left\{K^\dagger_\tau(z), \Psi^\beta_j(y)\right\}^{**} \\ &= -i(\sigma_i\sigma_j)^\beta{}_\alpha \delta^{(3)}(x-y) + \frac{4i}{g}\int d^3z \overrightarrow{D}^{(z)}_i \left[\delta^{(3)}(x-z)\right] \frac{\Sigma^\beta{}_\alpha}{\vec{\Sigma}\cdot\vec{\Sigma}} \left[\delta^{(3)}(z-y)\right] \overleftarrow{D}^{(z)}_j \\ &\equiv \left\{\Psi^\dagger_{i\alpha}(x), \Psi^\beta_j(y)\right\}_D \ , \end{aligned} \tag{7.20}$$

which is the same as obtained in [23].

As there are no first-class constraints in this case, the extended Hamiltonian for the gauged RS theory coincides with the total Hamiltonian, where the second-class constraints are strongly imposed to be zero, and is

$$H_E = \int d^3x \left[ -\frac{ig}{2} A_0 \epsilon^{ijk} \Psi_i \sigma_j \Psi_k + \frac{1}{4}\epsilon^{ijk}\Psi^\dagger_i \left[\overleftarrow{D}_j - \overrightarrow{D}_j\right] \Psi_k \right] \ . \tag{7.21}$$

Now, we can obtain the dynamics of the canonical fields $\Psi^\dagger_{i\alpha}$ and $\Psi^\alpha_i$ (as we saw in Subsection 7.2.2, the fields $\Psi^\dagger_{0\alpha}$ and $\Psi^\alpha_0$ do have dynamics, but they are completely determined by $\Psi^\dagger_{i\alpha}$ and $\Psi^\alpha_i$) with their Dirac brackets with respect to $H_E$,

$$\begin{aligned} \dot{\Psi}^\dagger_{i\alpha} &= \left\{\Psi^\dagger_{i\alpha}, H_E\right\}_D \ , \\ \dot{\Psi}^\alpha_i &= \{\Psi^\alpha_i, H_E\}_D \ , \\ K^\dagger_\alpha &= K^\alpha = 0 \ . \end{aligned}$$

Note that the equality on the constraints are implemented strongly as the Dirac brackets preserves the constraint surface in the phase space [26]. It can be verified that one obtains the same field equations (7.5) for the canonical fields, i.e.,

$$
\begin{aligned}
\dot{\Psi}^\dagger_{i\alpha} &= -ig\Psi^\dagger_i A_0 + i\epsilon_i{}^{jk}\Psi^\dagger_{k\alpha}\overleftarrow{D}_k + \left[\frac{\Psi^\dagger_{j\beta}\Sigma^j \not{\Sigma}^\beta{}_\alpha - 2\epsilon^{jkl}\Psi^\dagger_{j\gamma}(\sigma_l)^\gamma{}_\beta \partial_k A_0 \not{\Sigma}^\beta{}_\alpha}{\vec{\Sigma}\cdot\vec{\Sigma}}\right]\overleftarrow{D}_i \ , \\
\dot{\Psi}^\alpha_i &= igA_0\Psi^\alpha_i + i\epsilon_i{}^{jk}\overrightarrow{D}_j\Psi^\alpha_k + \overrightarrow{D}_i\left[\frac{\not{\Sigma}^\alpha{}_\beta\Sigma^j\Psi^\beta_j + 2\epsilon^{jkl}\not{\Sigma}^\alpha{}_\beta(\sigma_j)^\beta{}_\gamma\partial_k A_0\Psi^\gamma_l}{\vec{\Sigma}\cdot\vec{\Sigma}}\right] \ , \\
K^\dagger_\alpha &= K^\alpha = 0 \ . 
\end{aligned}
\tag{7.22}
$$

As a crosscheck, we can verify that the quantities between straight brackets in the first two lines in (7.22) are $\Psi^\dagger_0$ and $\Psi_0$ by solving $V^\dagger_\alpha$ and $V^\alpha$ from (7.19), respectively. We can see in (7.22), even in the case of limiting $g \to 0$,

$$
\dot{\Psi}^\alpha_i \sim i\epsilon_i{}^{jk}\partial_j\Psi^\alpha_k + \partial_i\left[\frac{\not{\Sigma}^\alpha{}_\beta\Sigma^j\Psi^\beta_j + 2\epsilon^{jkl}\not{\Sigma}^\alpha{}_\beta(\sigma_j)^\beta{}_\gamma\partial_k A_0\Psi^\gamma_l}{\vec{\Sigma}\cdot\vec{\Sigma}}\right]
$$

we cannot avoid the second term, which contains information about the external gauge field. This is the Adler's argument about retained memory of gauge fields at zero amplitude limit (see Section VIII.A. of [22]).

This dynamical analysis can be used to prove there are no superluminal modes in the gauged massless RS theory. Let us consider the equation of wave fronts in the neighborhood of a spacetime point $x^*$, and we choose $A_0 = 0$. Writing (7.22) at first order,

$$
\dot{\Psi}^\alpha_i \sim i\epsilon_i{}^{jk}\partial_j\Psi^\alpha_k + \partial_i\left[\frac{\not{\Sigma}^\alpha{}_{\beta *}\Sigma^j_*\Psi^\beta_j}{\vec{\Sigma}_*\cdot\vec{\Sigma}_*}\right] + \Delta^i\left[\Psi_i, x_*, x\right] \tag{7.23}
$$

where $\Sigma_* = \Delta(x_*)$ and for $\Delta[\ldots]$ we symbolically denote terms with no first derivatives of its arguments. The term containing $\Delta$ can be avoided [22], as in

$$
\lim_{\delta\to 0}\int_{-\delta}^{\delta} d\ell\,\dot{\Psi}^\alpha_i ,
$$

discontinuities across wave fronts contribute only through the first derivative terms, but when the external field $\vec{\Sigma}$ is smooth, the $\Delta[\Psi_i, x_*, x]$ term makes a vanishing contribution. Therefore, dropping this term in (7.23) and multiplying by $\Sigma^2_*$, we get the following condition

$$
F^\alpha_i \equiv \Sigma^2_*\dot{\Psi}^\alpha_i - \Sigma^j_*\not{\Sigma}^\alpha{}_{\beta *}\partial_i\Psi^\beta_j - i\Sigma^2_*\epsilon_i{}^{jk}\partial_j\Psi^\alpha_k = 0 \ . \tag{7.24}
$$

Besides this, we have to take into account the secondary constraint (7.11), which, with the same arguments given before, leads to the linear condition

$$
G^\alpha \equiv \epsilon^{ijk}(\sigma_i)^\alpha{}_\beta\partial_j\Psi^\beta_k = 0 \ . \tag{7.25}
$$

Observing the two conditions (7.24) and (7.25) are linear equations with constant coefficients, the general solutions are plane waves. Without lost of generality, we can assume the vector wave $\vec{k}$ is opposite to the $x_3$-axis, i.e., $\vec{k} = -k\hat{x}_3$. So, we take the ansatz

$$\Psi_i^\alpha(t,z) = C_i^\alpha e^{i\Omega t + ikz} \ , \tag{7.26}$$

where $\Omega$, $k$, and $C_i^\alpha$ are constants non-zero constants. Therefore, the six conditions (7.24) become

$$\begin{aligned} \Omega C_1^\alpha &= -ikC_2^\alpha \ , \\ \Omega C_2^\alpha &= ikC_1^\alpha \ , \\ \vec{\Sigma}_* \cdot \vec{\Sigma}_* \Omega C_3^\alpha &= k\Sigma_*^j \not{\Sigma}^\alpha{}_{\beta *} C_j^\beta \ , \end{aligned} \tag{7.27}$$

while the two conditions (7.25) can be read as

$$(\sigma_2)^\alpha{}_\beta C_1^\beta = (\sigma_1)^\alpha{}_\beta C_2^\beta \ .$$

Writing explicitly the Pauli matrices and splitting the constants as column spinor $C_i = \begin{pmatrix} C_i^\uparrow \\ C_i^\downarrow \end{pmatrix}$, we get

$$\begin{aligned} -iC_1^\downarrow &= C_2^\downarrow \ , \\ iC_1^\uparrow &= C_2^\uparrow \ . \end{aligned} \tag{7.28}$$

The first two lines of (7.27) together with (7.28), have solutions

$$\begin{aligned} C_1^\uparrow = C \quad &, \quad C_2^\uparrow = iC \ , \ \Omega = k \ , \ , \\ C_1^\downarrow = C \quad &, \quad C_2^\downarrow = -iC \ , \ \Omega = -k \ , \ , \end{aligned} \tag{7.29}$$

meaning that the transversal modes are exactly luminal, as $|\Omega/k| = 1$. For the longitudinal modes, the third line in (7.27), has non-trivial solution only if

$$\begin{vmatrix} \vec{\Sigma} \cdot \vec{\Sigma}\Omega - k(\Sigma^3)^2 & k\Omega^3\,(\Omega^1 - i\Omega^2) \\ k\Omega^3\,(\Omega^1 + i\Omega^2) & \Sigma_* \cdot \Sigma_* \Omega - k(\Sigma^3)^2 \end{vmatrix} = 0 \ ,$$

where we assume the magentic field is evaluated in the spacetime point $x^*$. Solving the wave front velocity $V = |\Omega/k|$ from the last equation, we obtain

$$\left|\frac{\Omega}{k}\right| = \frac{(\Sigma^3)^2}{(\Sigma^1)^2 + (\Sigma^2)^2 + (\Sigma^3)^2} \leq 1 \ , \tag{7.30}$$

meaning the longitudinal modes are subluminal.

In order to quantize the theory, as this system is a second-class one, we have to promote the canonical variables as operators acting on the Fock space, whose quantum anticommutators $[,]_+$ are the Dirac brackets times $i\hbar$,

$$\left[\Psi_{i\alpha}^\dagger(x), \Psi_j^\beta(y)\right]_+ = i\hbar \left\{\Psi_{i\alpha}^\dagger(x), \Psi_j^\beta(y)\right\}_D \ . \tag{7.31}$$

By virtue of (7.20), it is verified the canonical anticommutator (7.31) is not positive definite.

## 7.5 Discussion

The main conclusion emerged from the detailed dynamical analysis for the massless gauged RS theory is that, unlike to the free case, there is no fermionic gauge invariance due to the fact there is no first-class constraint associated with it. This implies there is a discontinuity on the DOF when it is turned on the coupling constant $g$, according to our counting of first and second-class constraints in Chapters 6 and 7. Therefore, for the gauged case, we cannot use the fermionic gauge symmetry to restrict the quantum anticommutators to belong to the definite positive sector, as suggested in [23]. However, from the classical side, there are no superliminial modes for massless RS field, even in the presence of an external magnetic field, as is claimed in [22].

As also noted in [22], we can observe directly from (7.22) that perturbation theory is not well defined as we cannot decouple the free term from the external magnetic field term, due to the fact that there is no $g$-proportionality in the last one. In order words, the asymptotic states for the gauged massless RS theory are not well defined. Therefore, some *no-go* theorems about the quantum consistency of RS fields, which make use of this fact, should be revisited for the massless case.

We can wonder if an extension of the massless gauges RS which restore the gauge symmetry, as proposed in [22], could have a not ill-defined quantum physics. This extension consists in the addition of a spin-1/2 field to the Lagrangian, which it is coupled with both, the electromagnetic and the RS fields. The dynamical contents, including the local DOF counting, of such a model will be discussed in Chapter 8. Even it is true that one can always extend a second-class system, as the massless gauged RS theory, to become it a first-class one [26], perhaps this extended model could be used also to avoid anomalies in Grand Unified Theories [19].

# Chapter 8

# Extended gauged massless Rarita-Schwinger theory

We saw in Chapter 7 that, by a detailed study of their dynamical contents, the massless RS theory has not a fermionic gauge invariance. An extension of the massless RS to restore the gauge symmetry was proposed [22], by adding the 1/2-spin fermionic fields $\overline{\xi}_\alpha, \xi^\alpha$ as

$$
\begin{aligned}
S[A,\overline{\psi},\psi,\overline{\xi},\xi] &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu\gamma_5\gamma_\nu\overrightarrow{D}_\rho\psi_\sigma + \overline{\psi}_\mu\overleftarrow{D}_\mu\gamma_5\gamma_\rho\psi_\sigma\right) \\
&- \frac{g}{4}\int d^4x \epsilon^{\mu\nu\rho\sigma}\left(\overline{\xi}F_{\mu\nu}\gamma_5\gamma_\rho\psi_\sigma - \overline{\psi}_\mu\gamma_\nu\gamma_5F_{\rho\sigma}\xi\right) \qquad (8.1) \\
&+ \frac{g}{8}\int d^4x \epsilon^{\mu\nu\rho\sigma}\left(\overline{\xi}F_{\mu\nu}\gamma_5\gamma_\rho\overrightarrow{D}_\sigma\xi - \overline{\xi}\overleftarrow{D}_\mu\gamma_\nu\gamma_5F_{\rho\sigma}\xi\right) .
\end{aligned}
$$

The first observation we can make is that the new fermions $\overline{\xi}, \xi$ are not, in principle, auxiliary fields as they have dynamics. This fact can be seen in the last line of the action (8.1). The second observation, is that the action (8.1) has the following (infinitesimal) fermionic gauge symmetry

$$
\delta A_\mu = 0\ , \delta\overline{\psi}_\mu = \overline{\zeta}\overleftarrow{D}_\mu\ , \delta\psi_\mu = \overrightarrow{D}_\mu\zeta\ , \delta\overline{\xi} = \overline{\zeta}\ , \delta\xi = \zeta\ . \qquad (8.2)
$$

This statement can be shown immediately if we rewrite the Lagrangian density appearing in (8.1) in a more compact form,

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu - \overline{\xi}\overleftarrow{D}_\mu\right)\overleftarrow{D}_\nu\gamma_5\gamma_\rho\left(\psi_\sigma - \overrightarrow{D}_\sigma\xi\right) + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu - \overline{\xi}\overleftarrow{D}_\mu\right)\gamma_5\gamma_\nu\overrightarrow{D}_\rho\left(\psi_\sigma - \overrightarrow{D}_\sigma\xi\right) ,
$$

where we integrated by parts, dropped the boundary term and used the identity $\epsilon^{\mu\nu\rho\sigma}\overrightarrow{D}_\mu\overrightarrow{D}_\nu\xi = -\frac{ig}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\xi$ (and similarly for $\xi^\dagger$). The action (8.1) has also the standard gauge symmetry, which can be written in an infinitesimal form as

$$
\delta A_\mu = \frac{1}{g}\partial_\mu\alpha\ , \delta\overline{\psi}_\mu = -i\alpha\overline{\psi}_\mu\ , \delta\psi_\mu = i\alpha\psi_\mu\ , \delta\overline{\xi} = -i\alpha\overline{\xi}\ , \delta\xi = i\alpha\xi\ . \qquad (8.3)
$$

The transformations (8.2) and (8.3) are independent each other and can be done separately.

As in the other sections, it is better to work with the left-handed decomposition shown in Appendix E.1. In such a case, the action (8.1) can be read as

$$\begin{aligned} S[A,\overline{\psi},\psi,\overline{\xi},\xi] &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\int d^4x \epsilon^{\mu\nu\rho\tau}\left(\Psi^\dagger_\mu \sigma_\nu \overrightarrow{D}_\rho \Psi_\tau + \Psi^\dagger_\mu \overleftarrow{D}_\mu \sigma_\rho \psi_\tau\right) \\ &+ \frac{ig}{4}\int d^4x \epsilon^{\mu\nu\rho\tau} F_{\mu\nu}\left(\xi^\dagger \sigma_\rho \psi_\tau + \psi^\dagger_\rho \sigma_\tau \xi - \frac{1}{2}\xi^\dagger \sigma_\rho \overrightarrow{D}_\tau \xi - \frac{1}{2}\xi^\dagger \overleftarrow{D}_\rho \sigma_\tau \xi\right) . \end{aligned} \quad (8.4)$$

The field equations obtained from varying (8.1) with respect to $A_\mu$, $\Psi^\dagger_\mu$, $\Psi_\mu$, $\overline{\xi}$ and $\xi$ are, respectively,

$$\partial_\nu F^{\mu\nu} + \frac{ig}{2}\epsilon^{\mu\nu\rho\sigma}\left[\Psi^\dagger_\nu \sigma_\rho \Psi_\tau + \partial_\nu\left(\xi^\dagger \sigma_\rho \Psi_\rho + \Psi^\dagger_\rho \sigma_\tau \xi - \frac{1}{2}\xi^\dagger \sigma_\rho \overrightarrow{D}_\tau \xi - \frac{1}{2}\xi^\dagger \overleftarrow{D}_\rho \sigma_\tau \xi\right)\right] = 0 ,$$

$$\begin{aligned} \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_\nu\left[\overrightarrow{D}_\rho \Psi_\tau + \frac{ig}{2}F_{\rho\tau}\xi\right] &= 0 , \\ -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left[\Psi^\dagger_\nu \overleftarrow{D}_\rho + \frac{ig}{2}\xi^\dagger F_{\rho\tau}\right]\sigma_\tau &= 0 , \\ \frac{ig}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\sigma_\rho\left[\Psi_\tau - \overrightarrow{D}_\tau \xi\right] &= 0 , \\ \frac{ig}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\left[\Psi^\dagger_\rho - \xi^\dagger \overleftarrow{D}_\rho\right]\sigma_\tau &= 0 . \end{aligned} \quad (8.5)$$

We observe the second and third equations are consequence of the fourth and last equations, respectively. This fact is a consequence of the fermionic gauge symmetry which translates in a redundance of the dynamical content of the action (8.4).

## 8.1 Dirac's Hamiltonian formalism

### 8.1.1 Momenta, Poisson brackets and canonical Hamiltonian

The canonical momenta associated with the gauge field $A_\mu$ and the fermions $\Psi^\dagger_\mu, \Psi_\mu, \xi^\dagger, \xi$ are, respectively,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} , \Xi^{\dagger\mu}_\alpha = \frac{\partial^R \mathcal{L}}{\partial \dot{\Psi}^\alpha_\mu} , \Xi^{\mu\alpha} = \frac{\partial^L \mathcal{L}}{\partial \dot{\Psi}^\dagger_{\mu\alpha}} , p^\dagger_\alpha = \frac{\partial^R \mathcal{L}}{\partial \dot{\xi}^\alpha} , p^\alpha = \frac{\partial^L \mathcal{L}}{\partial \dot{\xi}^\dagger_\alpha} . \quad (8.6)$$

The non-vanishing Poisson brackets, at equal time, between fields and momenta are (following convention of [26])

$$\begin{aligned}
\{A_\mu(x), \pi^\nu(y)\} &= -\{\pi^\nu(y), A_\mu(x)\} = \delta^\nu_\mu \delta^{(3)}(x-y) \;, \\
\left\{\Psi^{\dagger\alpha}_\mu(x), \Xi^{\nu\beta}(y)\right\} &= \left\{\Xi^{\nu\beta}(y), \Psi^{\dagger\alpha}_\mu(x)\right\} = -\delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(x-y) \;, \\
\left\{\Psi^\alpha_\mu(x), \Xi^{\dagger\nu}_\beta(y)\right\} &= \left\{\Xi^{\dagger\nu}_\beta(y), \Psi^\alpha_\mu(x)\right\} = \delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(x-y) \;, \\
\left\{\xi^\dagger_\alpha(x), p^\beta(y)\right\} &= \left\{p^\nu_\beta(y), \xi^\dagger_\alpha(x)\right\} = \delta^\beta_\alpha \delta^{(3)}(x-y) \;, \\
\left\{\xi^\alpha(x), p^\dagger_\beta(y)\right\} &= \left\{p^\dagger_\beta(y), \xi^\alpha(x)\right\} = \delta^\alpha_\beta \delta^{(3)}(x-y) \;.
\end{aligned} \tag{8.7}$$

The time and spatial components of (7.6) are given by

$$\begin{aligned}
\pi^0 = 0 \quad &, \quad \pi^i = F^{i0} - \frac{ig}{2}\epsilon^{ijk}\left[\xi^\dagger\sigma_j\Psi_k + \Psi^\dagger_j\sigma_k\xi - \frac{1}{2}\xi^\dagger\sigma_j\overrightarrow{D}_k\xi - \frac{1}{2}\xi^\dagger\overleftarrow{D}_j\sigma_k\xi\right] \;, \\
\Xi^{\dagger 0}_\alpha = 0 \quad &, \quad \vec{\Xi}^\dagger_\alpha = -\frac{1}{4}\vec{\Psi}^\dagger{}_\beta \times (\vec{\sigma})^\beta{}_\alpha . \\
\Xi^{0\alpha} = 0 \quad &, \quad \vec{\Xi}^\alpha = \frac{1}{4}(\vec{\sigma})^\alpha{}_\beta \times \vec{\Psi}_\beta \;, \\
p^\dagger_\alpha = -\frac{ig}{8}\epsilon^{ijk}F_{ij}\xi^\dagger_\beta(\sigma_k)^\beta_\alpha \quad &, \quad p^\alpha = \frac{ig}{8}\epsilon^{ijk}F_{ij}(\sigma_k)^\alpha_\beta\xi^\beta \;.
\end{aligned}$$

At this point, it is convenient to introduce some definitions:

$$\begin{aligned}
\theta^\dagger_{i\alpha} &\equiv \Psi^\dagger_{i\alpha} - \frac{1}{2}\xi^\dagger_\alpha\overleftarrow{D}_i \;, \\
\theta^\alpha_i &\equiv \Psi^\alpha_i - \frac{1}{2}\overrightarrow{D}_i\xi^\alpha \;,
\end{aligned} \tag{8.8}$$

$$\odot^i \equiv \epsilon^{ijk}\left(\xi^\dagger\sigma_j\theta_k + \theta^\dagger_j\sigma_k\xi\right) \;,$$

$$® \equiv \epsilon^{ijk}\left(-\Psi^\dagger_i\overrightarrow{D}_j\Psi_k + \Psi^\dagger_i\overleftarrow{D}_j\Psi_k + igF_{ij}\left[\xi^\dagger\theta_k - \theta^\dagger_k\xi\right]\right) \;. \tag{8.9}$$

With (8.8), can write the canonical Hamiltonian $H_0$ as

$$\begin{aligned}
H_0 &= \int d^3x \left[A_0\left(-\partial_i\pi^i - \frac{ig}{2}\Psi^\dagger_i\sigma_j\Psi_k + \frac{g^2}{4}F_{ij}\xi^\dagger\sigma_k\xi\right)\right] \qquad (8.10) \\
&+ \int d^3x \left[\Psi^\dagger_0\epsilon^{ijk}\left(\frac{1}{2}\sigma_i\overrightarrow{D}_j\Psi_k + \frac{ig}{4}F_{ij}\sigma_k\xi\right) + \left(-\frac{1}{2}\Psi^\dagger_i\overleftarrow{D}_j\sigma_k - \frac{ig}{4}F_{ij}\xi^\dagger\sigma_k\right)\epsilon^{ijk}\Psi_0\right] \\
&+ \int d^3x \left[\frac{1}{2}\left(\pi^i + \frac{ig}{2}\odot^i\right)\left(\pi_i + \frac{ig}{2}\odot_i\right) + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{4}®\right] .
\end{aligned}$$

### 8.1.2 Primary and secondary constraints

The primary constraints are given by

$$\begin{aligned} \phi^0 &\equiv \pi^0 \approx 0 \, , \\ \chi^{0\alpha} \equiv \Xi^{0\alpha} \approx 0 \quad &, \quad \vec{\chi}^{\alpha} \equiv \Xi^{0\alpha} - \frac{1}{4}(\vec{\sigma})^{\alpha}{}_{\beta} \times \vec{\Psi}_{\beta} \approx 0 \, , \\ \vec{\chi}^{\dagger}_{\alpha} \equiv \vec{\Xi}^{\dagger}_{\alpha} + \frac{1}{4}\vec{\Psi}^{\dagger} \times \vec{\sigma} \approx 0 \quad &, \quad \chi^{\dagger 0}_{\alpha} \equiv \Xi^{\dagger 0}_{\alpha} \approx 0 \, , \\ \varphi^{\dagger}_{\alpha} \equiv p^{\dagger}_{\alpha} + \frac{ig}{8}\epsilon^{ijk}F_{ij}\xi^{\dagger}_{\beta}(\sigma_k)^{\beta}_{\alpha} \quad &, \quad \varphi^{\alpha} \equiv p^{\alpha} - \frac{ig}{8}\epsilon^{ijk}F_{ij}(\sigma_k)^{\alpha}_{\beta}\xi^{\beta} \, . \end{aligned} \tag{8.11}$$

whose only non-vanishing Poisson brackets are

$$\begin{aligned} \left\{\chi^{\dagger i}_{\alpha}(x), \chi^{j\beta}(y)\right\} &= -\frac{1}{2}\epsilon^{ijk}(\sigma_k)^{\beta}{}_{\alpha}\delta^{(3)}(x-y) \, , \\ \left\{\varphi^{\dagger i}_{\alpha}(x), \varphi^{\beta}(y)\right\} &= -\frac{ig}{4}\epsilon^{ijk}F_{ij}(\sigma_k)^{\beta}{}_{\alpha}\delta^{(3)}(x-y) \, . \end{aligned} \tag{8.12}$$

Now, we write the primary Hamiltonian

$$H_T = H_0 + \int d^3x \left(\Lambda^{\dagger}_{\mu}\chi^{\mu} + \chi^{\dagger\mu}\Lambda_{\mu} + \lambda^{\dagger}\varphi + \varphi^{\dagger}\lambda\right) \, ,$$

where $\Lambda^{\dagger\mu}, \Lambda_{\mu}, \lambda^{\dagger}, \lambda$ are Lagrange multipliers associated to the primary constraints, except for the constraint[1] $\pi^0$.

We demand the primary constraints (8.11) to hold as the system evolves in time, i.e.,

$$\begin{aligned} \dot{\phi}^0(x) &= \left\{\phi^0(x), H_T\right\} = \partial_i\pi^i + \frac{ig}{2}\epsilon^{ijk}\Psi^{\dagger}_i\sigma_j\Psi_k - \frac{g^2}{4}F_{ij}\xi^{\dagger}\sigma_k\xi \approx 0 \, , \\ \dot{\chi}^{\dagger 0}_{\alpha}(x) &= \left\{\chi^{\dagger 0}_{\alpha}(x), H_T\right\} = \frac{1}{2}\epsilon^{ijk}\left(\Psi^{\dagger}_{i\beta}\overleftarrow{D}_j + \frac{ig}{2}F_{ij}\xi^{\dagger}_{\beta}\right)(\sigma_k)^{\beta}{}_{\alpha} \approx 0 \, , \\ \dot{\chi}^{0\alpha}(x) &= \left\{\chi^{0\alpha}(x), H_T\right\} = -\frac{1}{2}\epsilon^{ijk}(\sigma_i)^{\alpha}{}_{\beta}\left(\overrightarrow{D}_j\Psi^{\beta}_k + \frac{ig}{2}F_{ij}\xi^{\beta}\right) \approx 0 \, , \end{aligned}$$

where in the second equalities we used (8.7), (8.11) and (8.10) explicitly. This lead us to define the following secondary constraints

$$\begin{aligned} k &\equiv -\partial_i\pi^i - \frac{ig}{2}\epsilon^{ijk}\left(\Psi^{\dagger}_i\sigma_j\Psi_k + \frac{ig}{2}F_{ij}\xi^{\dagger}\sigma_k\xi\right) \approx 0 \, , \\ K^{\dagger}_{\alpha} &\equiv -\frac{1}{2}\epsilon^{ijk}\left(\Psi^{\dagger}_{i\beta}\overleftarrow{D}_j + \frac{ig}{2}F_{ij}\xi^{\dagger}_{\beta}\right)(\sigma_k)^{\beta}{}_{\alpha} \approx 0 \, , \\ K^{\alpha} &\equiv \frac{1}{2}\epsilon^{ijk}(\sigma_i)^{\alpha}{}_{\beta}\left(\overrightarrow{D}_j\Psi^{\beta}_k + \frac{ig}{2}F_{ij}\xi^{\beta}\right) \approx 0 \, . \end{aligned} \tag{8.13}$$

[1] In Section 7.2.3 we present $\pi^0$ ($= \phi^0$) as a first-class constraint. Therefore, its Lagrange multiplier is undetermined.

The non-vanishing Poisson brackets of (8.13) with the rest of constraints (8.11), and between them, are shown in Appendix F.2.

Substituting (8.13) in (8.10), and using the notation $E^i \equiv \pi_i + \frac{ig}{2}\odot_i = F^{0i}$ ($\vec{E}$ is proportional to the electric field) and $\Sigma^i \equiv \epsilon^{ijk}F_{jk}$ ($\vec{\Sigma}$ is proportional to the magnetic field), we get

$$H_0 = \int d^3x \left[A_0 k + \Psi_0^\dagger K + K^\dagger \Psi_0 + \frac{1}{2}E^i E_i + \frac{1}{8}\Sigma^i \Sigma_i + \frac{1}{4}Ⓡ\right] .$$

Preservation in time of the rest of primary constraints determines a possible set of Lagrange multipliers (see Appendix F.2 for computational details),

$$\begin{aligned}
\Lambda^\dagger_{i\alpha} &\approx -igA_0\Psi^\dagger_{i\alpha} + \Psi^\dagger_{0\alpha}\overleftarrow{D}_i + i\epsilon^{ijk}\Psi^\dagger_{j\alpha}\overleftarrow{D}_k + ig\left[E^i + \frac{i}{2}\Sigma^i\right] ,\\
\dot{\varphi}^\alpha(x) &= \{\varphi^\alpha(x), H_T\}\\
\Lambda^\alpha_i &\approx igA_0\Psi^\alpha_i + \overrightarrow{D}_i\Psi^\alpha_0 + i\epsilon^{ijk}\overrightarrow{D}_j\Psi^\alpha_k + ig\left[E^i - \frac{i}{2}\Sigma^i\right] ,\\
\lambda^\dagger_\alpha &\approx -igA_0\xi^\dagger_\alpha + \Psi^\dagger_{0\alpha}\\
&+ \frac{1}{\vec{\Sigma}\cdot\vec{\Sigma}}\left[-2(\vec{E}\times\vec{\Sigma})\cdot\vec{\vartheta}^\dagger + 2i(\vec{E}\cdot\vec{\Sigma})(\vec{\vartheta}^\dagger\cdot\vec{\sigma}) - 2i(\vec{\Sigma}\cdot\vec{\vartheta}^\dagger)(\vec{\sigma}\cdot\vec{E}) - (\vec{\Sigma}\cdot\vec{\vartheta}^\dagger)(\vec{\sigma}\cdot\vec{\Sigma})\right]_\alpha ,\\
\lambda^\alpha &\approx igA_0\xi^\alpha + \Psi^\alpha_0\\
&+ \frac{1}{\vec{\Sigma}\cdot\vec{\Sigma}}\left[2(\vec{E}\times\vec{\Sigma})\cdot\vec{\vartheta} + 2i(\vec{E}\cdot\vec{\Sigma})(\vec{\sigma}\cdot\vec{\vartheta}) - 2i(\vec{E}\cdot\vec{\sigma})(\vec{\Sigma}\cdot\vec{\vartheta}) + (\vec{\Sigma}\cdot\vec{\sigma})(\vec{\Sigma}\cdot\vec{\vartheta})\right]^\alpha .
\end{aligned}$$

One can prove also that with these Lagrange multipliers the preservation in time of the secondary constraints $k, K^\dagger, K$ does not requirers extra constraints. Therefore, we completed the set of constraints, and in the next subsection, we will classify them according to be first-class or second-class ones.

### 8.1.3 First-class and second-class constraints

We already known from (8.11) that $\phi^0, \chi^\dagger_0, \chi^{0\alpha}$ are first-class constraints. As the action (8.1) has the symmetries (8.3), (8.2), we can guess the following first-class constraints

$$\begin{aligned}
\widetilde{k} &\equiv \frac{1}{g}k + i\chi^{i\dagger}\Psi_i - i\Psi^\dagger_i\chi^i + i\varphi^i\xi - i\xi^\dagger\varphi , && (8.14)\\
\widetilde{K}^\alpha &\equiv K^\alpha - \partial_\mu\chi^{\mu\alpha} + \varphi^\alpha\\
\widetilde{K}^\dagger_\alpha &\equiv K^\dagger_\alpha - \partial_\mu\chi^{\dagger\mu}_\alpha + \varphi^\dagger_\alpha . && (8.15)
\end{aligned}$$

Indeed, using (6.13) and (6.16), it can be shown that (6.17) are first-class constraints and generates the gauge transformations (8.3), (8.2), respectively.

| Constraint kind | Number of this kind | Classification |
|---|---|---|
| $\phi^0$ | 1 | first-class |
| $\widetilde{k}$ | 1 | first-class |
| $\chi^{0\alpha}$ | 2 | first-class |
| $\chi^{\dagger 0}_{\alpha}$ | 2 | first-class |
| $\widetilde{K}^{\alpha}$ | 2 | first-class |
| $\widetilde{K}^{\dagger}_{\alpha}$ | 2 | first-class |
| $\chi^{i\alpha}$ | 6 | second-class |
| $\chi^{\dagger i}_{\alpha}$ | 6 | second-class |
| $\varphi^{\dagger}_{\alpha}$ | $2^*$ | second-class |
| $\varphi^{\alpha}$ | $2^*$ | second-class |

Table 8.1: Constraint classification for the extended massless left-handed RS theory coupled to a Maxwell field.

### 8.1.4 Degrees of freedom counting

Applying again the formula (3.45), we have for this case[2] $N = 2 \times (4 + 16 + 4^*) = 4 + 32 + 16^*$, $F = (2 + 8)$, and $S = (12 + 4^*)$, according to Table 8.1. Therefore, the number of degrees of freedom is now $DOF = (2 + 2 + 2^*)$ for the real Majorana spinor. Taking into account also the right-handed fermion component for the complex Dirac spinor case, we gave $DOF = (2 + 4 + 4^*)$. We can see in this extended case we restore the local fermionic DOF of the free case (see Section 3.2).

## 8.2 Faddeev-Jackiw method

Following the same steps and definitions as in Sections 6.3 and 7.3, we can apply the FJ method to this extended massless RS. Starting with the Lagrangian density appearing in (8.4), we can write it as

$$\begin{aligned}\mathcal{L} &= \dot{A}_i\pi^i + \frac{1}{4}\epsilon^{ijk}\left(\dot{\Psi}^{\dagger}_i\sigma_j\Psi^k - \Psi^{\dagger}_i\sigma_j\dot{\Psi}_k\right) + \frac{ig}{8}\epsilon^{ijk}F_{ij}\left(\dot{\xi}^{\dagger}\sigma_k\xi - \xi^{\dagger}\sigma_k\dot{\xi}\right)\\ &- A_0k - \Psi^{\dagger}_0K - K^{\dagger}\Psi_0 - \frac{1}{2}(\pi_i + \frac{ig}{2}\odot_i)(\pi^i + \frac{ig}{2}\odot^i) - \frac{1}{4}F_{ij}F^{ij} - \frac{1}{4}®\,,\end{aligned}$$

where the quantities $\odot^i$, ® and the constraints $k$, $K^{\dagger}_{\alpha}$, $K^{\dagger}$ were defined in Sections 8.1.1 and 8.1.2, respectively.

[2] The asterisk $^*$ in the right of the number is just distinguish the local DOF coming from the extra fields $\xi^{\dagger}$ , $\xi$ with respect to the original fields $\Psi^{\dagger}_i$ , $\Psi_i$.

Splitting $A_i$ and $\pi^i$ in transversal and longitudinal components, defining the transformations

$$\begin{aligned} \Psi^\dagger_{\mu\alpha} \to \Psi'^\dagger_{\mu\alpha} = \Psi^\dagger_{\mu\alpha} e^{-iA} \quad &, \quad \Psi^\alpha_\mu \to \Psi'^\alpha_\mu = e^{iA}\Psi^\alpha_\mu \ , \\ \xi^\dagger_\alpha \to \xi'^\dagger_\alpha = \xi^\dagger_\alpha e^{-iA} \quad &, \quad \xi^\alpha \to \xi'^\alpha = e^{iA}\xi^\alpha \ , \end{aligned}$$

and solving for $k = 0$, as in Section 7.3, we have

$$\begin{aligned} \mathcal{L} &= \dot{A}_{Ti}\pi^i_T + \frac{1}{4}\epsilon^{ijk}\left(\dot{\Psi'}^\dagger_i \sigma_j \Psi'^k - \Psi'^\dagger_i \sigma_j \dot{\Psi'}_k\right) + \frac{ig}{8}\epsilon^{ijk}F_{ij}\left(\dot{\xi}'^\dagger \sigma_k \xi' - \xi'^\dagger \sigma_k \dot{\xi}'\right) \\ &- \Psi'^\dagger_0 K' - K'^\dagger \Psi'_0 - \frac{1}{2}(\pi_{Ti} + \frac{ig}{2}\odot_i)(\pi^i_T + \frac{ig}{2}\odot^i) - \frac{1}{4}F_{Tij}F^{ij}_T - \frac{1}{4}\circledR \ , \end{aligned}$$

where in this case $\odot^i$ and $\circledR$ contain primed fields and transversal covariant derivatives $D_T$.

As we already known this extended massless RS has the fermionic gauge invariance (8.2), we can induce a second field transformation

$$\begin{aligned} \Psi'^\dagger_{\mu\alpha} \to \Psi''^\dagger_{\mu\alpha} = \Psi'^\dagger_{\mu\alpha} + \partial_\mu \zeta^\dagger_\alpha \quad &, \quad \Psi'^\alpha_\mu \to \Psi''^\alpha_\mu = \Psi'^\alpha_\mu + \partial_\mu \zeta^\alpha \ , \\ \xi'^\dagger_\alpha \to \xi''^\dagger_\alpha = \xi'^\dagger_\alpha + \zeta^\dagger_\alpha \quad &, \quad \xi'^\alpha \to \xi''^\alpha = \xi'^\alpha + \zeta^\alpha \ , \end{aligned}$$

where, as in Section (6.3), we choose the fermionic parameters $\zeta^\dagger$ and $\zeta$ such that

$$\begin{aligned} \Psi''^\dagger_{i\alpha}(\sigma^i)^\alpha{}_\beta = 0 &\Longrightarrow \Psi''^\dagger_{1/2i\alpha} = 0 \ , \\ (\sigma^i)^\beta{}_\alpha \Psi''^\alpha_i = 0 &\Longrightarrow \Psi''^\alpha_{1/2i} = 0 \ . \end{aligned}$$

Now we left with only the $3/2-$spin components of $\Psi^\dagger$ and $\Psi$, we can split them in transversal and longitudinal components, then we can solve the fermionic constraints,

$$\begin{aligned} K''^\alpha &= \frac{i}{2}\nabla^2\Theta^\alpha + \frac{g}{2}A_{Ti}\partial^i\Theta^\alpha + \frac{g}{2}A_{Ti}\Psi'^\alpha_{3/2Ti} + \frac{ig}{2}\Sigma_k(\sigma^k)^\alpha_\beta \xi''^\beta = 0 \ , \\ K''^\dagger_\alpha &= -\frac{i}{2}\nabla^2\Theta^\dagger_\alpha + \frac{g}{2}A_{Ti}\partial^i\Theta^\dagger_\alpha + \frac{g}{2}\Psi'^\dagger_{3/2Ti\alpha}A_{Ti} - \frac{ig}{2}\xi''_\beta \Sigma_k(\sigma^k)^\beta_\alpha = 0 \ , \end{aligned}$$

in a frame where the magnetic field is not zero ($\vec{\Sigma}\cdot\vec{\Sigma} \neq 0$). Therefore, solving the fermionic constraints can be seen as solving the above differential equations for $\Theta^\dagger$ and $\Theta$ with suitable boundary conditions.

At the end, the only degrees of freedom left are those shown in Table 8.2 . Therefore, for the extended left-handed massless RS theory (real Majorana spinor), we have $dof = (2 + 2 + 2^*)$, as the Dirac's Hamiltonian formalism. If we take into account the right-handed components for the complex Dirac spinor case, we have $dof = 2 + 4 + 4^*$.

| Field | Number |
|---|---|
| $A_T$ (with corresponding momentum $\pi^i_T$) | 2 |
| $\Psi^\alpha_{3/2Ti}$ (with corresponding momentum $\Psi^\dagger_{3/2Ti\alpha}$) | 2 |
| $\xi^\alpha$ (with corresponding momentum $\xi^\dagger_\alpha$) | 2* |

Table 8.2: Degrees of freedom left after solving the constraints for the extended massless left-handed RS theory.

## 8.3 Quantization in the presence of an external magnetic field

The total Hamiltonian for the extended gauged massless RS with an external magnetic field (we choose $A_0 = 0$) can be written as

$$\begin{aligned} H_T &= \int d^3x \left[ \frac{1}{4} \Psi^\dagger_i \left( \overleftarrow{D}_j - \overrightarrow{D}_j \right) \Psi_k + \frac{ig}{4} \Sigma^i \left( \xi^\dagger \Psi_i - \Psi^\dagger_i \xi \right) + \frac{ig}{8} \Sigma^i \xi^\dagger \left( \overleftarrow{D}_i - \overrightarrow{D}_i \right) \xi \right. \\ &\quad \left. + \Psi^\dagger_0 K + K^\dagger \Psi_0 + \chi^\dagger_\mu \Lambda^\mu + \Lambda^\dagger_\mu \chi^\mu + \lambda^\dagger \varphi + \varphi^\dagger \lambda \right] , \end{aligned} \tag{8.16}$$

where the constraints are defined as (8.11) and (8.13). Regarding the quantization of the theory, we cannot apply directly the aforementioned proceeding as the system has first-class constraints. In order to transform the system in a second-class one, we need to add suitable equal number gauge-fixing condition (GFC) as first-class constraints [121]. These conditions must be well-defined in the sense that if $\Phi_A$ are first-class constraints and $\Gamma_A$ GFCs, then $\{\Gamma_A(x), \int d^3y \Phi_B(y) \varepsilon^B(y)\} = 0$ implies $\varepsilon^A = 0$. With the addition of these GFCs the system of constraints become second-class, and therefore, we can apply the same procedure than the last section.

### 8.3.1 $\xi^\dagger_\alpha = \xi^\alpha = 0$ gauge

The easiest way to implement the GFCs is to choose $\Psi^\alpha_0 = \Psi^\dagger_{0\alpha} = 0$ and $\xi^\alpha = \xi^\dagger_\alpha = 0$, i.e.,

$$\begin{aligned} \Omega^\alpha &= \xi^\alpha \approx 0 , \\ \Omega^\dagger_\alpha &= \xi^\dagger_\alpha \approx 0 , \\ \Upsilon^\alpha &= \Psi^\alpha_0 \approx 0 , \\ \Upsilon^\dagger_\alpha &= \Psi^\dagger_{0\alpha} \approx 0 . \end{aligned} \tag{8.17}$$

The extended Dirac bracket can be obtained in steps as the previous case.

1. Consider the second-class constraints $\chi^{\dagger i}_\alpha$, $\chi^i_\alpha$, $\varphi^\dagger_\alpha$ and $\varphi^\alpha$. We have then the following partial constraints matrix

$$\begin{pmatrix} & \chi^{\dagger j}_\beta(y) & \varphi^\dagger_\alpha(y) & \chi^{j\beta}(y) & \varphi^\beta(y) \\ \chi^{\dagger i}_\alpha(x) & 0 & 0 & -\frac{1}{2}\epsilon^{ijk}(\sigma_k)^\beta{}_\alpha & 0 \\ \varphi^\dagger_\alpha(x) & 0 & 0 & 0 & \frac{ig}{4}\not{\Sigma}^\beta{}_\alpha \\ \chi^{i\alpha}(x) & \frac{1}{2}\epsilon^{ijk}(\sigma_k)^\alpha{}_\beta & 0 & 0 & 0 \\ \varphi^\alpha(x) & 0 & \frac{ig}{4}\not{\Sigma}^\alpha{}_\beta & 0 & 0 \end{pmatrix}\delta^{(3)}(x-y) = M(x,y) ,$$

whose inverse is

$$\begin{pmatrix} & \chi^{\dagger j}_\beta(y) & \varphi^\dagger_\beta(y) & \chi^{j\beta}(y) & \varphi^\beta(y) \\ \chi^{\dagger i}_\alpha(x) & 0 & 0 & -i(\sigma_i\sigma_j)^\alpha{}_\beta & 0 \\ \varphi^\dagger_\alpha(x) & 0 & 0 & 0 & \frac{4i}{g}\frac{\not{\Sigma}^\alpha{}_\beta}{\vec{\Sigma}\cdot\vec{\Sigma}} \\ \chi^{i\alpha}(x) & i(\sigma_i\sigma_j)^\beta{}_\alpha & 0 & 0 & 0 \\ \varphi^\alpha(x) & 0 & \frac{4i}{g}\frac{\not{\Sigma}^\beta{}_\alpha}{\vec{\Sigma}\cdot\vec{\Sigma}} & 0 & 0 \end{pmatrix}\delta^{(3)}(x-y) = M^{-1}(x,y) .$$

We can immediately compute the bracket at level 1, i.e.,

$$\begin{aligned} \left\{\Psi^\dagger_{i\alpha}(x), \Psi^\beta_j(y)\right\}^* &= \left\{\Psi^\dagger_{i\alpha}(x), \Psi^\beta_j(y)\right\} - \int d^3w d^3z \left\{\Psi^\dagger_{i\alpha}, \chi^{k\gamma}(w)\right\}\left(M^{-1}\right)^\tau_{jk\gamma}(w,z)\left\{\chi^{\dagger l}_\tau(z), \Psi^\beta_j(y)\right\} \\ &= -i(\sigma_i\sigma_j)^\beta{}_\alpha\delta^{(3)}(x-y) , \\ \left\{\xi^\dagger_\alpha(x), \xi^\beta(y)\right\}^* &= \left\{\xi^\dagger_\alpha(x), \xi^\beta(y)\right\} - \int d^3w d^3z \left\{\xi^\dagger_\alpha(x), \varphi^\gamma(w)\right\}\left(M^{-1}\right)^\tau_\gamma(w,z)\left\{\varphi^\dagger_\tau(z), \xi^\beta(y)\right\} \\ &= \frac{4i}{g}\frac{\not{\Sigma}^\beta{}_\alpha}{\vec{\Sigma}\cdot\vec{\Sigma}}\delta^{(3)}(x-y) . \end{aligned}$$

Other important brackets which are modified at this level are

$$\begin{aligned} \left\{K^\dagger_\alpha(x), K^\beta(y)\right\}^* &= 0 , \\ \left\{K^\dagger_\alpha(x), \Omega^\beta(y)\right\}^* &= \delta^\beta_\alpha\delta^{(3)}(x-y) , \\ \left\{\Omega^\dagger_\beta(x), K^\beta(y)\right\}^* &= -\delta^\beta_\alpha\delta^{(3)}(x-y) , \\ \left\{\Omega^\dagger_\beta(x), \Omega^\beta(y)\right\}^* &= \frac{4i}{g}\frac{\not{\Sigma}^\beta{}_\alpha}{\vec{\Sigma}\cdot\vec{\Sigma}}\delta^{(3)}(x-y) . \end{aligned}$$

2. Consider the first-class constraints $K^\dagger_\alpha$, $K^\alpha$ and the GFCs $\Omega^\dagger_\alpha$ and $\Omega^\alpha$. We have then the following partial constraints matrix

$$\begin{pmatrix} & K^\dagger_\beta(y) & \Omega^\dagger_\beta(y) & K^\beta(y) & \Omega^\beta(y) \\ K^\dagger_\alpha(x) & 0 & 0 & 0 & \delta^\beta_\alpha \\ \Omega^\dagger_\alpha(x) & 0 & 0 & -\delta^\beta_\alpha & \frac{ig}{4}\not{\Sigma}^\beta{}_\alpha \\ K^\alpha(x) & 0 & -\delta^\alpha_\beta & 0 & 0 \\ \Omega^\alpha(x) & \delta^\alpha_\beta & \frac{ig}{4}\not{\Sigma}^\beta{}_\alpha & 0 & 0 \end{pmatrix}\delta^{(3)}(x-y) = G(x,y) ,$$

whose inverse is

$$\begin{pmatrix} & K^\dagger_\beta(y) & \Omega^\dagger_\beta(y) & K^\beta(y) & \Omega^\beta(y) \\ K^\dagger_\alpha(x) & 0 & 0 & \frac{4i}{g}\frac{\Sigma^\alpha{}_\beta}{\vec\Sigma\cdot\vec\Sigma} & \delta^\alpha_\beta \\ \Omega^\dagger_\alpha(x) & 0 & 0 & -\delta^\alpha_\beta & 0 \\ K^\alpha(x) & \frac{4i}{g}\frac{\Sigma^\beta{}_\alpha}{\vec\Sigma\cdot\vec\Sigma} & -\delta^\beta_\alpha & 0 & 0 \\ \Omega^\alpha(x) & \delta^\beta_\alpha & 0 & 0 & 0 \end{pmatrix}\delta^{(3)}(x-y) = G^{-1}(x,y)\ .$$

We can immediately compute the bracket at level 2, i.e.,

$$\begin{aligned}\left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^{**} &= \left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^{*} - \int d^3w d^3z \left\{\Psi^\dagger_{i\alpha},K^\gamma(w)\right\}^{*}\left(G^{-1}\right)^\tau_\gamma(w,z)\left\{K^\dagger_\tau(z),\Psi^\beta_j(y)\right\}^{*} \\ &= -i(\sigma_i\sigma_j)^\beta{}_\alpha\delta^{(3)}(x-y) + \frac{4i}{g}\int d^{(3)}z\left[\delta^{(3)}(z-y)\right]\overleftarrow{D}_j\frac{4i}{g}\frac{\Sigma^\tau{}_\gamma}{\vec\Sigma\cdot\vec\Sigma}\overrightarrow{D}_i\left[\delta^{(3)}(x-z)\right]\ .\end{aligned}$$

3. Finally, consider the first-class constraints $\chi^{\dagger 0}_\alpha$, $\chi^{0\alpha}$ and the GFCs $\Upsilon^\dagger_\alpha$ and $\Upsilon^\alpha$, which give us the final partial constraint matrix

$$\begin{pmatrix} & \chi^{\dagger 0}_\beta(y) & \Upsilon^\dagger_\beta(y) & \chi^{0\beta}(y) & \Upsilon^\beta(y) \\ \chi^{\dagger 0}_\alpha(x) & 0 & 0 & 0 & \delta^\beta_\alpha \\ \Upsilon^\dagger_\alpha(x) & 0 & 0 & -\delta^\beta_\alpha & 0 \\ \chi^{0\alpha}(x) & 0 & -\delta^\alpha_\beta & 0 & 0 \\ \Upsilon^\alpha(x) & \delta^\alpha_\beta & 0 & 0 & 0 \end{pmatrix}\delta^{(3)}(x-y) = H(x,y)\ ,$$

whose inverse is

$$\begin{pmatrix} & \chi^{\dagger 0}_\beta(y) & \Upsilon^\dagger_\beta(y) & \chi^{0\beta}(y) & \Upsilon^\beta(y) \\ \chi^{\dagger 0}_\alpha(x) & 0 & 0 & 0 & \delta^\alpha_\beta \\ \Upsilon^\dagger_\alpha(x) & 0 & 0 & -\delta^\alpha_\beta & 0 \\ \chi^{0\alpha}(x) & 0 & -\delta^\beta_\alpha & 0 & 0 \\ \Upsilon^\alpha(x) & \delta^\beta_\alpha & 0 & 0 & 0 \end{pmatrix}\delta^{(3)}(x-y) = H^{-1}(x,y)\ .$$

We can immediately compute the bracket at level 3, i.e.,

$$\begin{aligned}\left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^{***} &= \left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}^{**} \\ &= -i(\sigma_i\sigma_j)^\beta{}_\alpha\delta^{(3)}(x-y) + \frac{4i}{g}\int d^{(3)}z\left[\delta^{(3)}(z-y)\right]\overleftarrow{D}_j\frac{\Sigma^\tau{}_\gamma}{\vec\Sigma\cdot\vec\Sigma}\overrightarrow{D}_i\left[\delta^{(3)}(x-z)\right] \\ &\equiv \left\{\Psi^\dagger_{i\alpha}(x),\Psi^\beta_j(y)\right\}_D\ , \end{aligned} \tag{8.18}$$

which is exactly the same as the Dirac bracket for the standard gauged case (7.20). With the GFCs $\Upsilon^\dagger_\alpha$ and $\Upsilon^\alpha$, we just kill the fields $\Psi^\dagger_{0\alpha}$ and $\Psi^\alpha_0$, respectively.

The extended Hamiltonian is

$$H_E = \frac{1}{4}\int d^3x \epsilon^{ijk}\Psi_i^\dagger \left[\overleftarrow{D}_j - \overrightarrow{D}_j\right]\Psi_k \,. \tag{8.19}$$

We obtain the dynamics of the canonical fields with the modified Poisson bracket $\{\ldots,\ldots\}_{D^*}$, i.e.,

$$\begin{aligned} \dot{\Psi}_{i\alpha}^\dagger &= \left\{\Psi_{i\alpha}^\dagger, H_E\right\}_D \,, \\ \dot{\Psi}_i^\alpha &= \{\Psi_i^\alpha, H_E\}_D \,, \\ K_\alpha^\dagger &= K^\alpha = 0 \,. \end{aligned}$$

Note that the equality on the constraints are implemented strongly as the Dirac brackets preserves the constraint surface in the phase space [26]. It can be verified that one obtains the same field equations (7.22) for the canonical fields, except now $\Psi_{0\alpha}^\dagger = \Psi_0^\alpha = 0$.

Of course we can take a different set of GFCs than (8.17). For instance, to get exactly the same field equations than (7.5), we can choose the following set,

$$\begin{aligned} \Omega^\alpha &= \xi^\alpha \approx 0 \,, \\ \Omega_\alpha^\dagger &= \xi_\alpha^\dagger \approx 0 \,, \\ V^\alpha &= -\frac{ig}{4}\left[\not{\Sigma}^\alpha{}_\beta \Psi_0^\beta - \Sigma^i \Psi_i^\alpha\right] \approx 0 \,, \\ V_\alpha^\dagger &= \frac{ig}{4}\left[\Psi_{0\beta}^\dagger \not{\Sigma}^\beta{}_\alpha - \Sigma^i \Psi_{i\alpha}^\dagger\right] \approx 0 \,. \end{aligned}$$

Note that $V^\alpha$ and $V_\alpha^\dagger$ are exactly the tertiary constraints (7.12) for the case of an external magnetic field. The difference is that in this case is implemented as a GFC form *outside*, where in the other case it is a constraint coming from the Hamiltonian formalism when we require the time preservation of the secondary constraints $K^\alpha$ and $K_\alpha^\dagger$, respectively.

### 8.3.2 Extended covariant radiation gauge

As the $\xi$ has dimension of a spin-1/2 field, even another GFCs could be to add a multiple of $\not{\partial}\xi$ to $\overrightarrow{D}^i\Psi_i$ to form an extended gauge covariant radiation gauge fixing. This GFCs leads to particularly simple formulas

$$\begin{aligned} \omega^\alpha &\equiv \frac{1}{2}\overrightarrow{D}^i\Psi_i^\alpha + \frac{g}{4}\not{\Sigma}^\alpha{}_\beta\xi^\beta \approx 0 \,, \\ \omega_\alpha^\dagger &\equiv \frac{1}{2}\Psi_{i\alpha}^\dagger\overleftarrow{D}^i + \frac{g}{4}\xi_\beta^\dagger\not{\Sigma}^\beta{}_\alpha \approx 0 \,, \\ \Upsilon^\alpha &= \Psi_0^\alpha \approx 0 \,, \\ \Upsilon_\alpha^\dagger &= \Psi_{0\alpha}^\dagger \approx 0 \,. \end{aligned} \tag{8.20}$$

We note the last two GFCs only rule out $\Psi_0^\dagger$ and $\Psi_0$ of the game. The other observation is that

$$(\sigma_i)^\alpha{}_\beta \overrightarrow{D}^i (\sigma_j)^\beta{}_\gamma \Psi_i^\gamma = \overrightarrow{D}^i \Psi_i^\alpha + i\epsilon^{ijk}(\sigma_i)^\alpha{}_\beta \overrightarrow{D}_j \Psi_k^\beta = \omega^\alpha + 2iK^\alpha \approx 0$$

So, if $\vec{\sigma}\cdot\vec{D}$ is invertible, this means $\vec{\sigma}\cdot\vec{\Psi} \approx 0$, and analogously for $\Psi^\dagger$. Therefore, according to E.2, we are selecting the 3/2-spin component of $\Psi$. The third observation is that by gauge transformation (8.2), the first two GFC in (8.20) transform as

$$\begin{aligned} \omega'^\alpha &= \omega^\alpha + (\sigma^i\sigma^j)^\alpha{}_\beta \overrightarrow{D}_i \overrightarrow{D}_j \zeta^\beta \,, \\ \omega'^\dagger_\alpha &= \omega^\dagger_\alpha + \zeta^\dagger_\beta \overleftarrow{D}_i \overleftarrow{D}_j (\sigma^i\sigma^j)^\beta{}_\alpha \,, \end{aligned}$$

meaning that, if we insist in the invertibility of $\vec{\sigma}\cdot\vec{D}$, then this gauge condition is attainable for any configuration space sector. In this case, the non-vanishing brackets are[3]

$$\left(\begin{array}{ccccc} & K^\dagger_\beta(y) & \omega^\dagger_\beta(y) & K^\beta(y) & \omega^\beta(y) \\ K^\dagger_\alpha(x) & 0 & 0 & 0 & \frac{1}{2}\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(y)\beta}_\alpha \delta^{(3)}(x-y) \\ \omega^\dagger_\alpha(x) & 0 & 0 & -\frac{1}{2}\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(y)\beta}_\alpha \delta^{(3)}(x-y) & \frac{ig}{4}\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(y)\beta}_\alpha \delta^{(3)}(x-y) \\ K^\alpha(x) & 0 & -\frac{1}{2}\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(y)\alpha}_\beta \delta^{(3)}(x-y) & 0 & 0 \\ \omega^\alpha(x) & \frac{1}{2}\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(y)\alpha}_\beta \delta^{(3)}(x-y) & \frac{ig}{4}\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(y)\alpha}_\beta \delta^{(3)}(x-y) & 0 & 0 \end{array}\right) = N(x,y) \,,$$

where we introduced the notation $\overrightarrow{\not{D}}{}^\alpha_\beta \equiv (\sigma^i)^\alpha{}_\beta \overrightarrow{D}_i$. Assuming, as we said before, that $\vec{\sigma}\cdot\vec{D}$ is invertible, we define the following distribution

$$\int d^3z \mathcal{Z}^\alpha_\gamma(x,z)\overrightarrow{\not{D}}\overrightarrow{\not{D}}{}^{(z)\gamma}_\beta \delta^{(3)}(z-y) = \delta^\alpha_\beta \delta^{(3)}(x-y) = \int d^3z \overrightarrow{\not{D}}{}^{(z)\alpha}_\gamma \delta^{(3)}(z-y)\mathcal{Z}^\gamma_\beta(x,z)\overleftarrow{\not{D}} \,. \tag{8.21}$$

Roughly speaking, $\mathcal{Z}^\alpha_\beta$ is the inverse of the operator $\overrightarrow{\not{D}}\overrightarrow{\not{D}}$. We can now invert the matrix $N(x,y)$, obtaining

$$\left(\begin{array}{ccccc} & K^\dagger_\beta(y) & \Omega^\dagger_\beta(y) & K^\beta(y) & \Omega^\beta(y) \\ K^\dagger_\alpha(x) & 0 & 0 & i\mathcal{Z}^\alpha_\beta(x,y) & 2\mathcal{Z}^\alpha_\beta(x,y) \\ \Omega^\dagger_\alpha(x) & 0 & 0 & -2\mathcal{Z}^\alpha_\beta(x,y) & 0 \\ K^\alpha(x) & i\mathcal{Z}^\beta_\alpha(x,y) & -2\mathcal{Z}^\beta_\alpha(x,y) & 0 & 0 \\ \Omega^\alpha(x) & 2\mathcal{Z}^\beta_\alpha(x,y) & 0 & 0 & 0 \end{array}\right) = N^{-1}(x,y) \,.$$

[3]We proceed in the same lines as the Section 8.3.1, in the sense that we first construct level 1 brackets $\{\ldots,\ldots\}^*$ by solving the constraints $\chi^\dagger$, $\chi$, $\varphi^\dagger$ and $\varphi$. Then we move to level 2 brackets $\{\ldots,\ldots\}^{**}$, by solving the rest of the constraints.

Now we are ready to obtain the level 2 bracket,

$$
\begin{aligned}
\left\{\Psi^{\dagger}_{i\alpha}(x),\Psi^{\beta}_{j}(y)\right\}^{**} &= -i(\sigma_i\sigma_j)^{\beta}{}_{\alpha}\delta^{(3)}(x-y) - i\overrightarrow{D}^{(y)}_{j}\mathcal{Z}^{\beta}_{\alpha}(x,y)\overleftarrow{D}^{(x)}_{i} \\
&\quad +\epsilon_{j}{}^{kl}(\sigma_k)^{\gamma}{}_{\alpha}\overrightarrow{D}^{(y)}_{l}\mathcal{Z}^{\beta}_{\gamma}(x,y)\overleftarrow{D}^{(x)}_{i} + \overrightarrow{D}^{(y)}_{j}\mathcal{Z}^{\beta}_{\gamma}(x,y)\epsilon_{i}{}^{kl}\overleftarrow{D}^{(x)}_{k}(\sigma_l)^{\gamma}{}_{\alpha}\,, \\
\left\{\xi^{\dagger}_{\alpha}(x),\Psi^{\beta}_{i}(y)\right\}^{**} &= -2i\overrightarrow{D}^{(y)}_{i}\mathcal{Z}^{\beta}_{\alpha}(x,y) - \epsilon_{i}{}^{jk}(\sigma_j)^{\beta}{}_{\gamma}\overrightarrow{D}^{(y)}_{k}\mathcal{Z}^{\gamma}_{\alpha}(x,y) \\
\left\{\Psi^{\dagger}_{i\alpha}(x),\xi^{\beta}(y)\right\}^{**} &= -2i\mathcal{Z}^{\beta}_{\alpha}(x,y)\overleftarrow{D}^{(y)}_{i} + \mathcal{Z}^{\gamma}_{\alpha}(x,y)\overleftarrow{D}^{(y)}_{j}\epsilon_{i}{}^{jk}(\sigma_k)^{\beta}{}_{\gamma} \\
\left\{\xi^{\dagger}_{\alpha}(x),\xi^{\beta}(y)\right\}^{**} &= \frac{4i}{g}\frac{\not{\Sigma}^{\beta}{}_{\alpha}}{\vec{\Sigma}\cdot\vec{\Sigma}}\delta^{(3)}(x-y) - 3i\mathcal{Z}^{\beta}_{\alpha}(x,y)\,. \qquad (8.22)
\end{aligned}
$$

As the effect of taking into account the $\Upsilon^{\dagger}$ and $\Upsilon$ is to read off the variables $\Psi^{\dagger}_{0}$ and $\Psi_0$, respectively, the brackets (8.22) are the Dirac brackets.

We can see from (8.22) that once one goes to the quantum anticommutator relations (multiplying by $i\hbar$ the Dirac brackets) the last term for sure is singular when the magnetic field $\Sigma$ goes to zero.

# Part III

# Strained Graphene

# Chapter 9

# Electronic properties of Graphene

Besides the promising technological and industrial applications of their electronic properties, graphene has the potential to be used as a table-top to testing fundamental physics. A remarkable example is the possibility to deform the graphene sheet to mimic different spacetime horizons [31, 32]. This is due to the effective Dirac massless description of the hopping electrons in the flat case along with the possibility of the graphene sheet to acquire different background shapes, as for instance Beltrami pseudo-sphere [36]. This Chapter is devoted to the basic electronic properties of flat graphene without any kind of deformations (i.e, there is no curvature, no torsion). The intention is not to be exhaustive on this topic, as there is a lot of literature both from the theoretical and experimental point of view, but only to introduce basic properties and some notation. In the next two chapters, we will focus on the strained graphene and the electronic properties associated with it.

## 9.1 Honeycomb lattice

Graphene is an allotrope of carbon forming a one-atom-thick, hence it is the closest in Nature to a two-dimensional system. It is fair to say that was first theoretically speculated in [69, 70], and, decades later, experimentally found [71]. The honeycomb lattice of graphene is made of two intertwined triangular sub-lattices, which is called *honeycomb lattice*[1]. Three of carbon's

[1]We call the honeycomb structure a lattice even if strictly speaking the two triangular sub-lattice structure is not a lattice, as there is not a primitive translation vector basis associated to it [122].

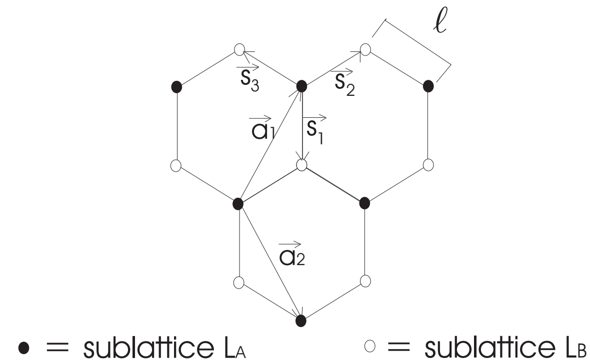



Figure 9.1: The honeycomb lattice of graphene, and its two triangular sublattices, with our choice of the basis vectors, $(\vec{a}_1, \vec{a}_2)$ and $(\vec{s}_1, \vec{s}_2, \vec{s}_3)$. This figure was taken from [31].

four electrons available to form covalent bonds are shared by the three nearest neighbours, forming so-called $\sigma$-bonds (associated to the atomic 2s-orbitals). The fourth electron also forms a covalent bond, which is called $\pi$-bond (associated to the atomic 2p-orbitals), but only with one of their neighbours. Because the $\pi$ bonds are 'weaker' than the $\sigma$ ones, these $\pi$ electrons can hop more easily. It turns out that in the vicinity of the points (in the momentum space) where the conductivity and valence bands touch, these electrons can be well described by Dirac equation in two spatial dimensions ($QED_{2+1}$) [40], as we shall recollect below. The *electronic properties* of graphene are due to the electrons belonging to the $\pi$ orbitals.

In the honeycomb lattice, there are two inequivalent sites per unit cell, which are symbolled as black and white dots in Figure 9.1. As the vectors $\{\vec{a}_1, \vec{a}_2\}$ are not enough to reach all black and white points, an extra set of vectors $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ are needed. The last set of vectors describes the position of the three near neighbors for each atom. The vectors of both sets are bi-dimensional $\vec{r} = (x, y)$, and we choose here the following convention for them (see Figure 9.1)

$$
\begin{aligned}
\vec{s}_1 &= \ell(0,-1) \quad , \quad \vec{s}_2 = \frac{\ell}{2}(\sqrt{3},1) \ , \ \vec{s}_3 = \frac{\ell}{2}(-\sqrt{3},1) \ , \\
\vec{a}_1 &= \frac{\ell}{2}(\sqrt{3},3) \quad , \quad \vec{a}_1 = \frac{\ell}{2}(\sqrt{3},-3) \ , \qquad (9.1)
\end{aligned}
$$

where $\ell \simeq 1.42$Å is the carbon-carbon near neighbors distance [40], that we call *lattice length*[2]. From the three near-neighbors vectors $\{\vec{s}_i\}$ we can

[2]As usual, the *lattice spacing* is the length of the basis vectors $|\vec{a}_i| = \sqrt{3}\ell \sim 2.46$Å.

construct the six next-to-near neighbor (**NNN**) vectors as

$$\begin{aligned} \vec{t}_1 = \vec{s}_2 - \vec{s}_3 \quad &, \quad \vec{t}_2 = \vec{s}_2 - \vec{s}_1 \ , \ \vec{t}_3 = \vec{s}_3 - \vec{s}_1 \ , \\ \vec{t}_4 = -\vec{t}_1 \quad &, \quad \vec{t}_5 = -\vec{t}_2 \ , \ \vec{t}_6 = -\vec{t}_3 \ . \end{aligned} \tag{9.2}$$

We will use these vectors to derive the dispersion relation of the $\pi$ electrons, up to NNN-order corrections in Section 9.2.

## 9.2 Massless Dirac structure of $\pi$ electrons

Let us briefly recapitulate the very well known facts happening at the low-energy limit of the physics of the $\pi$ electrons in graphene, in the half-filling regime. The tight-binding (**TB**) Hamiltonian is[3](using natural units $\hbar = 1$)

$$\begin{aligned} H \ = \ & -\sum_{\vec{r} \in L_A} \left[ \eta \sum_{i=1}^{i=3} \left( a^\dagger(\vec{r}) b(\vec{r} + \vec{s}_i) + b^\dagger(\vec{r} + \vec{s}_i) a(\vec{r}) \right) \right. \\ & \left. + \eta' \sum_{i=1}^{i=6} \left( a^\dagger(\vec{r}) a(\vec{r} + \vec{t}_i) + b^\dagger(\vec{r}) b(\vec{r} + \vec{t}_i) \right) \right] \ , \end{aligned} \tag{9.3}$$

where $\eta$ is the nearest-neighbor hopping energy which is approximately 2.8 eV, $\eta'$ is NNN hopping energy[4] (hopping in the same lattice), and $a, a^\dagger (b, b^\dagger)$ are the anticommuting annihilation and creation operators for the planar electrons in the sub-lattice $L_A(L_B)$ of the honeycomb lattice realized by the $\sigma$-bonds (see Figure 9.1).

If we make a Fourier transformation to momenta space $\vec{k} = (k_x, k_y)$ of the annihilation and creation operators,

$$a(\vec{r}) = \sum_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \ , \ b(\vec{r}) = \sum_{\vec{k}} b_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \ , \tag{9.4}$$

then

$$H = -\sum_{\vec{k}} \left[ \eta \sum_{i=1}^{i=3} \left( a^\dagger_{\vec{k}} b_{\vec{k}} e^{i\vec{k}\cdot\vec{s}_i} + b^\dagger_{\vec{k}} a_{\vec{k}} e^{-i\vec{k}\cdot\vec{s}_i} \right) + \eta' \sum_{i=1}^{i=6} \left( a^\dagger_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{t}_i} + b^\dagger_{\vec{k}} b_{\vec{k}} e^{-i\vec{k}\cdot\vec{t}_i} \right) \right] \ .$$

[3] We are not considering here the spinorial nature of the $\pi$ electrons as its contribution to TB Hamiltonian is weaker than the hopping energy.

[4] The value of $\eta'$ is still not well determined, but it is though to be around $0.02\eta \leq \eta' \leq 0.2\eta$ [40].

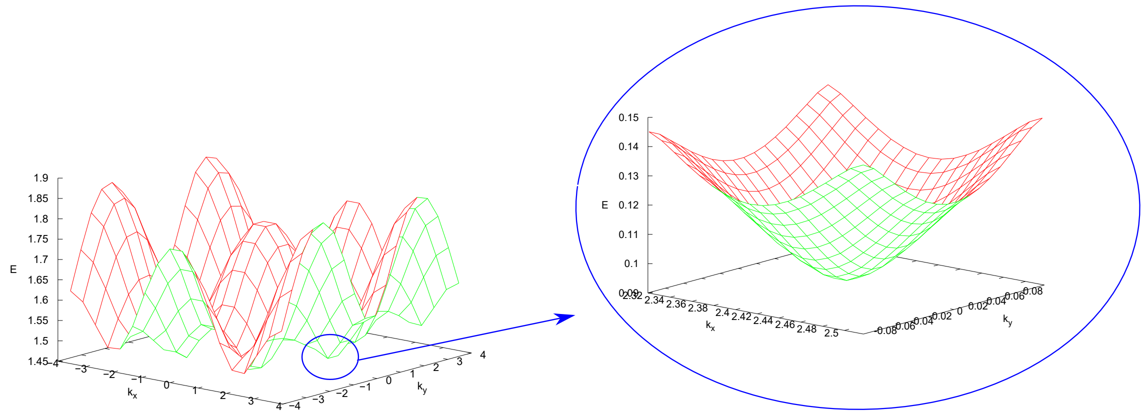



Figure 9.2: (a) The dispersion relation $E(\vec{k})$ for the $\pi$ electrons in graphene, setting $\eta\ell = 1$. We only take into account the near neighbors contribution in (9.3). (b) A zoom near the Dirac point $K_{D+}$ showing the linear approximation works well in the low energies regime.

Using our conventions (9.1) and (9.2), we define the functions

$$f_1(\vec{k}) \equiv -\eta \sum_{i=1}^{i=3} e^{i\vec{k}\cdot\vec{s}_i} = -\eta e^{i\ell k_y}\left[1 + 2e^{\frac{3i}{2}\ell k_y}\cos\left(\frac{\sqrt{3}}{2}\ell k_x\right)\right] , \quad (9.5)$$

$$f_2(\vec{k}) \equiv -\eta' \sum_{i=1}^{i=6} e^{i\vec{k}\cdot\vec{t}_i} = 2\eta'\left(\cos\left(\sqrt{3}\ell k_x\right) + 4\cos\left(\frac{\sqrt{3}}{2}\ell k_x\right)\cos\left(\frac{\sqrt{3}}{2}\ell k_y\right)\right) ,$$

leading to

$$H = -\sum_{\vec{k}}\left[f_1(\vec{k})a^\dagger_{\vec{k}}b_{\vec{k}} + f_1^*(\vec{k})b^\dagger_{\vec{k}}a_{\vec{k}} + f_2(\vec{k})a^\dagger_{\vec{k}}a_{\vec{k}} + f_2(\vec{k})b^\dagger_{\vec{k}}b_{\vec{k}}\right] .$$

For the case we consider only near neighbors contribution ($\eta' = 0$), the conduction and valence bands touch each other in the first Brillouin zone (if they do) when $|f_1(\vec{K})| = 0$. For the case of $\pi$ electrons in graphene they touch at $K_{D\pm} = (\pm\frac{4\pi}{3\sqrt{3}\ell}, 0)$, as we can check from (9.5). Actually, there are six of that points, but the only two shown above are inequivalent under lattice discrete symmetry. These points are called *Dirac points*. A sketch for the dispersion relation $E(\vec{k}) = |f(\vec{k})|$, for $\eta\ell = 1$, is shown in Figure 9.2 (a). Now, let us analyze the energy behaviour of the $\pi$ electrons around the Dirac points $K_{D\pm}$. In order to do that, we expand $f_1(\vec{k})$ as $\vec{k}_\pm = \vec{K}_{D\pm} + \vec{p}$, where it is assumed $|p| \ll |K_D|$,

$$\begin{aligned} f_{1+}(\vec{p}) \equiv f(\vec{k}_+) &= v_F\,(p_x + ip_y) \ , \\ f_{1-}(\vec{p}) \equiv f(\vec{k}_-) &= -v_F\,(p_x - ip_y) \ , \end{aligned}$$

where $v_F \equiv \frac{3}{2}\eta\ell \sim c/300$ is the *Fermi velocity.* We can see from this that the dispersion relation for the $\pi$ electrons around the fermi point is

$$|E_\pm(\vec{p})| = v_F|\vec{p}| \;, \tag{9.6}$$

which is the dispersion relation for a massless particle (see Figure 9.2 (b)).

Defining $a_\pm \equiv a(\vec{k}_\pm)$ and $b_\pm \equiv b(\vec{k}_\pm)$, the Hamiltonian (9.3), up to near neighbors correction (avoiding $f_2$), can be written as

$$\begin{aligned}
H|_{k_\pm} &= \sum_{\vec{p}} \left[f_+ a_+^\dagger b_+ + f_- a_-^\dagger b_- + f_+^* b_+^\dagger a_+ + f_-^* b_-^\dagger a_-\right] \\
&= v_F \sum_{\vec{p}} \left[a_+^\dagger \left(p_x + ip_y\right) b_+ - a_-^\dagger \left(p_x - ip_y\right) b_- + b_+^\dagger \left(p_x - ip_y\right) a_+ - b_-^\dagger \left(p_x + ip_y\right) a_-\right] \\
&= v_F \sum_{\vec{p}} \left[\begin{pmatrix} b_+^\dagger & a_+^\dagger \end{pmatrix} \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} \begin{pmatrix} b_+ \\ a_+ \end{pmatrix} - \begin{pmatrix} b_-^\dagger & a_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} \begin{pmatrix} b_- \\ a_- \end{pmatrix}\right] .
\end{aligned}$$

Arranging the creation (annihilation) operators as a column (row) vector $\psi_\pm = \begin{pmatrix} b_\pm \\ a_\pm \end{pmatrix}$ ; $\psi_\pm^\dagger = \begin{pmatrix} b_\pm^\dagger & a_\pm^\dagger \end{pmatrix}$, then

$$H = v_F \sum_{\vec{p}} \left[\psi_+^\dagger \vec{\sigma} \cdot \vec{p} \psi_+ - \psi_-^\dagger \vec{\sigma}^* \cdot \vec{p} \psi_-\right] \;, \tag{9.7}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2)$ and $\vec{\sigma}^* = (\sigma_1, -\sigma_2)$, being $\sigma_i$ the Pauli matrices. The Hamiltonian (9.7) is the known Dirac massless pseudoparticles description for the $\pi$ electrons. This description works well for energies of order of the hopping near neighbors energy $\eta$, as shown in Figure 9.2 (b). Also, the velocity of the $\pi$ electrons in this massless description is $v_F$ instead of speed of light $c$, so they are three hundred times slower than photons. However, this still a very high mobility for the conductor materials, being this one of the features that make graphene appealing for electronic devices applications.

We must stress here that the spinor nature of $\pi$ electrons emerged entirely from the lattice honeycomb structure of graphene, as we explicitly ignore the intrinsic spinorial nature of the electrons[5].

The two spinors $\psi_+$ and $\psi_-$ are connected by inversion of the full momentum $\vec{K}_{D+} + \vec{p} \longrightarrow -\vec{K}_{D+} - \vec{p} = \vec{K}_{D-} - \vec{p}$. This is consistent with the following picture: if near a given Dirac point, say $\vec{K}_{D+}$, the physics is described by the spinor $\psi_+$, then the physics for the upside-down inside the membrane is described by the spinor $\psi_-$ with opposite momentum [123]. If nothing mixes the two sides of the graphene sheet, as is the case for no curvature and no torsion on it, the physics of $\pi$ electrons at low energy are described by the

[5]We could insist in taking into account the intrinsic spinorial nature of the electrons with the possibility of giving rise interesting spin(intrinsic)-spin(lattice) interactions [14].

spinors $\psi_+$ and $\psi_-$, belonging each one in their own world. However, as we shall see later, this has implications when we deal with strained graphene, as the two Dirac points react differently under same strain tensor field.

Finally, we move back to the configuration space, which is equivalent to make the usual substitution $\vec{p} \to -i\vec{\nabla}$,

$$H = -iv_F \int d^2x \left[\psi_+^\dagger \vec{\sigma} \cdot \vec{\nabla}\psi_+ - \psi_-^\dagger \vec{\sigma}^* \cdot \vec{\nabla}\psi_-\right] \, , \tag{9.8}$$

where sums turned into integrals because continuum limit were assumed.

The next two chapters will be devoted to studying how the Hamiltonian (9.8) is modified when the graphene sheet is in-plane strained.

# Chapter 10

# Top-down approach: Spin-connection and Weyl gauge field for strained graphene

As is we learned in Chapter 9, the physics of the long wavelength (low energy) $\pi$ electrons in graphene, can be efficiently encoded within the Dirac massless two-dimensional Hamiltonian (9.8). In the approach followed in this Chapter, time will be included to make the formalism fully relativistic, although with the speed of light $c$ traded for the Fermi velocity $v_F$ (see, e.g., [31, 32]), hence the formalism becomes $(2+1)$-dimensional where the action is

$$S = i\hbar v_F \int d^3q \bar{\psi}\gamma^a \partial_a \psi \ , \tag{10.1}$$

here $q^a = (t, x, y)$ are the flat spacetime coordinates, $\gamma^a$ are the usual Dirac matrices in three dimensions, and we expand around only one of the two Dirac points.

Following the conventions in [124], the metric[1]

$$g_{\mu\nu}(q) = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & -g_{ij}(x,y) \end{pmatrix} \tag{10.2}$$

can also describe strain. Hence, we shall use the customary Dirac action in that curvilinear background. As also pointed out in [124], we recall

[1] In this Chapter, we changed the signature of the $(2+1)$-dimensional metric to follow the conventions in the literature of the subject.

that although the system is $(2+1)$ dimensional, the Riemann tensor $R^{\lambda}{}_{\mu\nu\rho}$, $\lambda, ... = \underline{0}, \underline{1}, \underline{2}$, has only one independent component, proportional to the Gaussian curvature $\mathcal{K}$. Furthermore, surfaces of zero or constant $\mathcal{K}$, make the metric (10.2) flat or conformally flat, respectively, and both cases can be treated at once within a formalism that uses

$$g_{\mu\nu}(Q) = e^{2\Sigma(Q)}\eta_{\mu\nu} \ , \tag{10.3}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$, and the information about the metric being flat or not is encoded in the conformal factor $\Sigma$. The coordinates where the metric can be explicitly written in a conformally flat fashion,

$$Q^{\mu} \equiv (T, X, Y) \ , \tag{10.4}$$

and are, in general, different from the original coordinates $q^{\mu}$. Therefore, the natural candidate action to describe strained graphene is (for a while we shall use $\hbar = 1 = v_F$)

$$S = i \int d^3Q \sqrt{g} \bar{\psi} E^{\mu}_{a} \gamma^{a} (\partial_{\mu} + \Omega_{\mu}) \psi \ , \tag{10.5}$$

where $\Omega_{\mu} = \frac{1}{2}\omega_{\mu}^{\ ab} J_{ab}$, with $J_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, the Lorentz generators, $E^{\mu}_{a}$ is the inverse of the three dimensional vielbein $e^{a}_{\mu}$ (the dreibein), $\omega_{\mu}^{\ ab}$ is the spin connection, and being in $(2+1)$ dimensions, we can write the spin connection in the one-index notation as in Chapters 2 and 4. Here, $a$ is the non-Abelian index of the local Lorentz group SO(2,1), and $\mu$ is the vector index on the spacetime base manifold.

The metric (10.2) can always be written in more suitable coordinates $\tilde{q}^{\mu} = (t, \tilde{x}, \tilde{y})$, where $t$ is the same time for both coordinate systems, and $(\tilde{x}, \tilde{y})$ are the *spatial isothermal* coordinates of the surface

$$g_{\mu\nu}(\tilde{q}) = \frac{\partial q^{\lambda}}{\partial \tilde{q}^{\mu}} \frac{\partial q^{\kappa}}{\partial \tilde{q}^{\nu}} g_{\lambda\kappa}(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e^{2\sigma(\tilde{x},\tilde{y})} & 0 \\ 0 & 0 & -e^{2\sigma(\tilde{x},\tilde{y})} \end{pmatrix} . \tag{10.6}$$

The single scalar function $\sigma$ identifies the surface/graphene membrane. Other isothermal coordinates can be found, say $\tilde{\tilde{x}}, \tilde{\tilde{y}}$, but the function identifying the surface is always the same[2]:

$$\sigma(\tilde{\tilde{x}}(\tilde{x}, \tilde{y}), \tilde{\tilde{y}}(\tilde{x}, \tilde{y})) = \sigma(\tilde{x}, \tilde{y}) \ . \tag{10.7}$$

[2] As an example of formula (10.7), the metric of a sphere of radius $r = 1$ could be written as $ds^2 = e^{2\sigma}\left(d\tilde{x}^2 + d\tilde{y}^2\right)$ with conformal factor $\sigma(\tilde{x}, \tilde{y}) = \ln(1/\cosh\tilde{y})$ being $\tilde{x} \in [0, 2\pi]$ and $\tilde{y} \in (-\infty, +\infty)$. On the other hand, using the coordinates $\tilde{\tilde{y}} = \begin{cases} \arcsin(1/\cosh\tilde{y}), & \text{if } \tilde{y} \geq 0 \\ \pi - \arcsin(1/\cosh\tilde{y}), & \text{if } \tilde{y} < 0 \end{cases}$, and $\tilde{\tilde{x}} = \tilde{x}$, $\sigma(\tilde{\tilde{x}}, \tilde{\tilde{y}}) = \ln\sin\tilde{\tilde{y}}$, with $\tilde{\tilde{y}} \in (0, \pi)$, we recover the standard metric of the sphere $ds^2 = \sin^2\tilde{\tilde{y}} d\tilde{\tilde{x}}^2 + d\tilde{\tilde{y}}^2$.

## 10.1 Connecting space and spacetime conformal factors

When the surface described by $\sigma$ has zero or constant curvature, the two metrics (10.3) and (10.6) both describe the same spacetime, although with different coordinates,

$$g_{\mu\nu}(Q) = e^{2\Sigma(Q)}\eta_{\mu\nu} = \frac{\partial\tilde{q}^{\lambda}}{\partial Q^{\mu}}\frac{\partial\tilde{q}^{\kappa}}{\partial Q^{\nu}}g_{\lambda\kappa}(\tilde{q}) \; ;$$

hence, their Gaussian curvature cannot differ. Due to the simple structure of the metric in the two coordinate frames, it is easy to compute the Gaussian curvature

$$\begin{aligned} \mathcal{K} &= -e^{-2\sigma}[\nabla^2\sigma] \\ &= e^{-2\Sigma}\left[2\Box\Sigma + (\partial_a\Sigma)(\partial^a\Sigma)\right] \end{aligned}$$

where $\nabla^2 = \partial^2_{\tilde{x}} + \partial^2_{\tilde{y}}$, and $\Box = \partial^2_T - \partial^2_X - \partial^2_Y$. Clearly, the two conformal factors are related, $\Sigma(\sigma)$: if we know $Q^\mu(\tilde{q})$, we can write $\Sigma(\tilde{x},\tilde{y}) = \Sigma(T(\tilde{x},\tilde{y}), X(\tilde{x},\tilde{y}), Y(\tilde{x},\tilde{y}))$, and then knowing $\sigma(\tilde{x},\tilde{y})$, we obtain $\Sigma(\sigma)$. Nonetheless, we have the general equations that $\Sigma$ has to satisfy for the two cases, for $\mathcal{K} = 0$,

$$\Box\Sigma = -\frac{1}{2}(\partial_a\Sigma)(\partial^a\Sigma) \; , \tag{10.8}$$

which corresponds to $\sigma$ harmonic functions (i.e., solutions of $\nabla^2\sigma = 0$), and for $\mathcal{K}$ =constant

$$\Box\Sigma = -\frac{1}{2}(\partial_a\Sigma)(\partial^a\Sigma) + \frac{1}{2}\mathcal{K}e^{2\Sigma} \; , \tag{10.9}$$

which corresponds to $\sigma$ Liouville functions (i.e., solutions of $\nabla^2\sigma = -\mathcal{K}e^{2\sigma}$).

Let us focus on the flat case which, since it corresponds to pure strained graphene, is the one of interest here. In this case, besides the obvious constant solution of (10.8), $\Sigma_{flat} = C$, we also have

$$\Sigma_{flat} = -\ln(T^2 - X^2 - Y^2) + C \; . \tag{10.10}$$

The constant $C$ could be set to zero, but we shall keep it, to later compare with the curved case. For the conformal factor (10.8), the metric (10.3) reads

$$g_{\mu\nu}(Q) = e^{2\Sigma_{flat}}\eta_{\mu\nu} = \frac{e^{2C}}{(T^2 - X^2 - Y^2)^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; . \tag{10.11}$$

Hence, nothing constrains the norm of vectors just as for the Minkowski case

$$\|Q\|^2 = g_{\mu\nu}Q^\mu Q^\nu = \frac{e^{2C}}{\eta_{\mu\nu}Q^\mu Q^\nu} \tag{10.12}$$

that means that $Q^\mu$ truly describes a flat three-dimensional spacetime. In order to see explicitly that (10.11) is in fact the Minkowski spacetime, take the following change of coordinates:

$$\begin{aligned} t &= \frac{e^C T}{T^2 - X^2 - Y^2}, \\ x &= \frac{e^C X}{T^2 - X^2 - Y^2}, \\ y &= \frac{e^C Y}{T^2 - X^2 - Y^2}. \end{aligned} \tag{10.13}$$

(10.14)

On these coordinates the line element is $ds^2 = dt^2 - dx^2 - dy^2$, showing that the singularities appearing in the metric (10.11) are simply coordinates singularities due to our choice of nonstandard coordinates[3]. As a result of that, when we use $\Sigma_{flat}$ in the spin connection of (10.5)

$$\omega_{\mu ab} = \delta^c_\mu (\eta_{ca}\delta^\nu_b - \eta_{cb}\delta^\nu_a)\Sigma_\nu \ , \tag{10.15}$$

we can safely use $\delta^\mu_a$ as a proper dreibein, because it indeed connects the tangent space with a flat manifold. Here $\Sigma_\mu = \partial_\mu \Sigma$, and $\Sigma_a = \partial_a \Sigma$, and we used the result that in three dimensions $\gamma^a J_{ab} = \gamma_b$.

### 10.1.1 Zero curvature: no physical effects of strain through the spin-connection

The action (10.5) for the metric (10.3) is

$$S = i \int d^3Q \ e^{2\Sigma} \ \bar{\psi}\gamma^a(\partial_a + \Sigma_a)\psi \ , \tag{10.16}$$

and when the Dirac field is properly transformed

$$\psi = e^{-\Sigma(Q)}\psi' \ , \tag{10.17}$$

where $\psi'$ refers to the Minkowskian flat spacetime $\eta_{\mu\nu}$ [see (10.3)], the action (10.5) is simply

$$S = i \int d^3Q \bar{\psi}'\gamma^a\partial_a\psi' \ , \tag{10.18}$$

which refers to the background metric $\eta_{\mu\nu}$, which in turn is the unstrained situation. Strain is gone altogether. It has no physical effects.

[3] To see whether one has a true or a coordinate singularity is, in general, not an easy task. On this, see, e.g., [125].

Another way to see this, is to stop at the action (10.16), i.e., *before* implementing the transformation of the spinor as in (10.17), and consider the gauge field $\Sigma_a$. This gauge field is itself a pure derivative, hence it cannot produce any physical effect through its field strength

$$F_{ab} = \partial_a \Sigma_b - \partial_b \Sigma_a = (\partial_a \partial_b - \partial_b \partial_a)\Sigma = 0 \ , \qquad (10.19)$$

as can be seen also by explicitly computing $\vec{B}^{\Sigma_{flat}} \equiv \vec{\nabla} \times \vec{\Sigma}_{flat}$ and $\vec{E}^{\Sigma_{flat}} \equiv -\vec{\nabla}\Sigma^0 + \partial_T \vec{\Sigma}_{flat}$ with $\Sigma_{flat}$ in (10.10). The result is zero $\vec{B}^{\Sigma_{flat}} = 0 = \vec{E}^{\Sigma_{flat}}$. Therefore, from here, we see that the very well-known pseudomagnetic field (and, for what matters, even a putative pseudoelectric field) induced by pure strain, cannot be accounted for by the spin-connection/Weyl pure-gauge field.

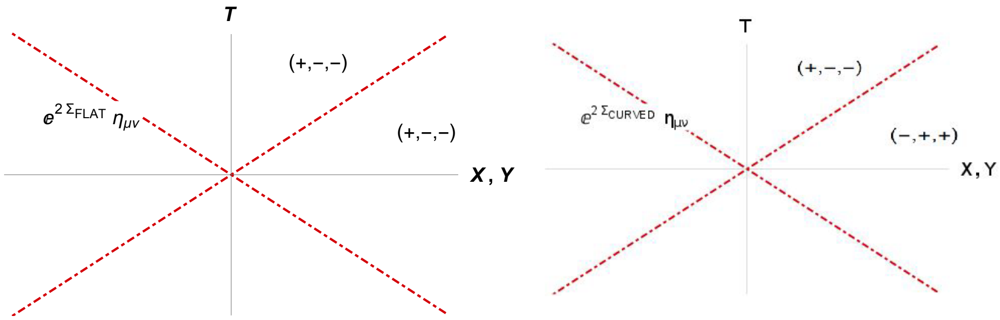
Let us now comment on $\Sigma_a$. As seen, this is a pure gauge field associated to the local Weyl transformations (10.3) and (10.17). Indeed, the Weyl field transforms as

$$W_\mu \to W_\mu - \partial_\mu \Sigma \ .$$

The reason why we do not have here the full Weyl gauge field, $W_\mu$, but only its pure gauge part, is due to the local Weyl invariance of (10.5), see [124]. Like any other Weyl field, $\Sigma_a$ is an Abelian gauge field. Abelian gauge fields are those routinely used in the graphene literature, [40] and [43]. On the other hand, $\Sigma_a$ carries information on the non-Abelian structure of the local Lorentz group that is encoded in the spin connection. This information is in the tangent space index "$a$" of $\Sigma_a$; see (10.15) and discussion below it. Indeed, (local) Weyl transformations, in general, rephrase spacetime scaling as an internal transformation. Non-Abelian gauge fields have also appeared in various discussions on the gauge field approach to strained graphene [44, 126].

The above-mentioned properties are common to the full Weyl gauge field $W_\mu$; hence, they hold also for theories that do not have local Weyl invariance. The extra property of $\Sigma_a$ is that it is $\partial_a \Sigma$; i.e. the true degree of freedom is just one scalar, $\Sigma$, the one related to the two-dimensional spatial phonon, $\sigma$, of the graphene membrane.

These facts are fully transparent in the coordinates $Q^\mu$ in (10.4). When the coordinates are different, the three aspects of this gauge field—(i) scalar nature, (ii) abelian field, and (iii) non-Abelian Lorentz structure—get mixed together, and they may appear, in the standard coordinates/frames used in graphene, as originating from different gauge fields, as we show in Section 10.2.




(a) (b)

Figure 10.1: (a) For $\Sigma_{flat}$ the spacetime is essentially the same as Minkowski. Modulo an inessential singularity at the light cone, the causal structures are identical; hence, the spacetime is three-dimensional and flat all the way. (b) In the case corresponding to $\Sigma_{curved}$, we see that there is a change of signature crossing the light cone, that signals a behavior similar to a black hole horizon. In this latter case, the true spacetime is one dimension smaller, due to the constraint (10.1.2) that coordinates need to satisfy when curvature is present.

### 10.1.2 Nonzero curvature: the classical manifestation of the quantum Weyl anomaly

Apparently, all seems clear: when $\Sigma = \Sigma_{flat}$, which should describe pure strain, no physical effects can be described by the QFT in the curved spacetime approach. Nonetheless, we also saw that when the Gaussian curvature of the membrane is constant, a similar procedure could be applied; hence, this seems to lead to conclude that also in that case as well, there is no physical effect. However, as we shall now show, this is not the case.

When one solves (10.9), e.g., for negative curvature, $\mathcal{K} = -r^{-2}$, one obtains

$$\Sigma_{curved} = -\frac{1}{2}\ln(T^2 - X^2 - Y^2) + \ln r \,, \tag{10.20}$$

and evidently the associated $\vec{B}^{\Sigma_{curved}}$ and $\vec{E}^{\Sigma_{curved}}$ are zero. Nonetheless, this time the metric (10.3) reads

$$g_{\mu\nu}(Q) = e^{2\Sigma_{curved}}\eta_{\mu\nu} = \frac{r^2}{T^2 - X^2 - Y^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \,, \tag{10.21}$$

and the light-cone becomes essentially singular due to the change of signature, typical of horizons in black hole spacetimes. Hence, the light-cone is a proper Killing horizon, as shown already when considering Unruh effects reproduced on graphene; see, e.g., [31, 32]. Let us now see what happens to the length of a position vector,

$$\|Q\|^2 = g_{\mu\nu} Q^\mu Q^\nu = r^2 > 0 \ .$$

That is a dramatic difference with the previous case (10.12) (amusingly, the difference comes about because of the different multiplicative constant factors: $-1$ for the flat vs $-1/2$ for the curved case). In the curved case the coordinates obey a constraint; hence, the effective theory lives in one dimension less, with the coordinates that we call $P^\alpha$, $\alpha = \underline{0}, \underline{1}$. The behavior of the flat and curved case is depicted in Figure 10.1. All of this makes us conclude that the action that corresponds to the curved case (in the negative curvature case) is

$$S^{(2)}_{eff} \approx i \int d^2 P \sqrt{g^{(2)}} \bar{\chi} \gamma^\alpha (\partial_\alpha + \Omega_\alpha) \chi \ ,$$

where everything has been dimensionally reduced to the two dimensions (including the fermions writing in this reduced case as $\chi$) and, in particular, the metric is the induced curved metric:

$$g^{(2)}_{\alpha\beta}(P) = \eta_{ab} \frac{\partial Q^a}{\partial P^\alpha} \frac{\partial Q^b}{\partial P^\beta} \ .$$

As well known, this two-dimensional theory has a quantum Weyl anomaly. Hence, interestingly, through the classical constraints we have a manifestation of a quantum Weyl anomaly.

The Weyl (trace) anomaly is known to be in one-to-one correspondence with the Hawking radiation [127]. This is an interesting road to pursue an alternative computation of the Hawking phenomenon on graphene as the one presented in other works [31, 32].

## 10.2 Equivalence of the static Hamiltonian and the fully relativistic approaches

To make contact with the literature focused on the phenomenology of graphene (see, e.g., [41]) we need first to move from the action in the $Q^\mu$ coordinates to the Hamiltonian in the $\tilde{q}^\mu$ coordinates, keeping the curvature radius $r$ finite. For $\Sigma = -\frac{1}{2}\ln(T^2 - X^2 - Y^2) + \ln r = -t/r$, the $Q^\mu$ coordinates are

$$\begin{aligned} T &= e^{t/r} \sqrt{e^{2\sigma(\tilde{y})} + r^2} && (10.22) \\ X &= e^{t/r} e^{\sigma(\tilde{y})} \cos \tilde{x} && (10.23) \\ Y &= e^{t/r} e^{\sigma(\tilde{y})} \sin \tilde{x} && (10.24) \end{aligned}$$

To write the Hamiltonian in the $\tilde{q}$ coordinates, we have to consider the action (10.16), Legendre-transform it

$$\mathcal{A} = \int dt d\tilde{x} d\tilde{y} \mathcal{L}(\tilde{q}) = \int dt d\tilde{x} d\tilde{y} \left[ \left\| \frac{\partial Q}{\partial \tilde{q}} \right\| (\pi_\psi \partial_T \psi)(\tilde{q}) - \mathcal{H} \right],$$

and follow the steps presented in the Appendix G. This gives, for $H = \int d\tilde{x} d\tilde{y} \mathcal{H}(\tilde{q})$,

$$H = -i\hbar \int d\tilde{x} d\tilde{y}\, \psi^\dagger \left( \tau^i \left[ v_F(\sigma(\tilde{y})) \right]_i^{\ \tilde{j}} \partial_{\tilde{j}} + v_F \phi + v_F \tau^i A_i \right) \psi \qquad (10.25)$$

where $\tau^i$ are the Pauli matrices, we have re-introduced $\hbar$ and $v_F$, and

$$\phi = \frac{1}{r}\, e^{2\sigma} \sigma_{\tilde{y}} \qquad (10.26)$$

$$A_1 = -\frac{1}{r} \frac{e^{3\sigma}\ \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}} \cos\tilde{x}\ ,\ A_2 = -\frac{1}{r} \frac{e^{3\sigma}\ \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}} \sin\tilde{x}\ , \qquad (10.27)$$

$$\left[ v_F(\sigma(\tilde{y})) \right]_i^{\ \tilde{j}} = v_F \begin{pmatrix} v_{1\tilde{x}} & v_{1\tilde{y}} \\ v_{2\tilde{x}} & v_{2\tilde{y}} \end{pmatrix}, \qquad (10.28)$$

with

$$v_{1\tilde{x}} = \frac{r\ e^{\sigma}\ \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}} \sin\tilde{x} \qquad (10.29)$$

$$v_{1\tilde{y}} = -\frac{e^{\sigma}}{r} \sqrt{e^{2\sigma} + r^2} \cos\tilde{x} \qquad (10.30)$$

$$v_{2\tilde{x}} = -\frac{r\ e^{\sigma}\ \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}} \cos\tilde{x} \qquad (10.31)$$

$$v_{2\tilde{y}} = -\frac{e^{\sigma}}{r} \sqrt{e^{2\sigma} + r^2} \sin\tilde{x}\ . \qquad (10.32)$$

The computations here were carried on for the $\Sigma$ in (10.20), for which we can present the coordinates (10.22)-(10.24); hence, the expressions for $\phi$, $A_i$ and $[v_F]_i^{\ \tilde{j}}$ depend of that choice. Nonetheless, even though the detailed expression of those quantities change for $\Sigma$ in (10.10), the structure of the Hamiltonian (10.25) remains the same for the case of interest of pure strain.

We see here that, through this top-down method, we were able to reproduce all the terms except one that customarily appears in the literature of strained graphene; see, e.g., [41]. The latter is the one gauge field that gives unambiguous physical effects, and that couples to the spinors with an imaginary factor (an instance that, on its own right, is an indication that such field cannot be a Weyl field, see, e.g., [124]). In Chapter 11 we shall extensively comment on this field. Before moving to that, let us get one step closer to the language usually adopted in the graphene literature.

In fact, the former expressions are written in the language of conformal factors $\sigma$ for the membrane and in isothermal coordinates whereas the usual approach employs strain tensors $u_{ij} = 1/2(\partial_i u_j + \partial_j u_i)$ (where, as customary, the $u_i$ measures the departure from the unstrained position), and Cartesian coordinates.

Although we started off with a fully relativistic formalism, due to the structure of the metric (10.2), everything of the previous expressions necessarily depends only on the spatial coordinates. Henceforth, we can focus on the spatial metric only and make the customary ansatz that, in Cartesian coordinates $(x, y)$,

$$g_{ij}(x, y) \simeq \delta_{ij} + 2u_{ij} \ . \tag{10.33}$$

On the other hand, this metric is related through a coordinate change to the one in (10.6)

$$g_{ij}(q) = \frac{\partial \tilde{q}^k}{\partial q^i} \frac{\partial \tilde{q}^l}{\partial q^j} \, \delta_{kl} e^{2\sigma(\tilde{q})} = L_{ij} \, e^{2\sigma(\tilde{q}(q))} \ . \tag{10.34}$$

This needs to be considered in its infinitesimal form, i.e., $\tilde{q}^i(q) \simeq q^i + \tilde{u}^i$, so that $\partial \tilde{q}^i / \partial q^j \simeq \delta^i_j + \partial_j \tilde{u}^i$, which gives, at first order, $L_{ij} \simeq \delta_{ij} + 2\tilde{u}_{ij}$; hence,

$$g_{ij}(x, y) \simeq (\delta_{ij} + 2\tilde{u}_{ij}) \, (1 + 2\sigma) \ . \tag{10.35}$$

Comparing the two expressions (10.33) and (10.35), we obtain the wanted link between conformal factor, Cartesian strain tensor, and isothermal strain tensor

$$u_{ij} \equiv \sigma \delta_{ij} + \tilde{u}_{ij} \ .$$

# Chapter 11

# Bottom-up approach: emergence of a honeycomb structure gauge field

We learned in Chapter 10 that in pure in-plane strained graphene sheet, we cannot obtain a gauge field which has physical significance from top-down approach (fully relativistic with Weyl invariance taking into account), as the curvature produced by it is zero, see (10.19). So, in this Chapter, we will take the bottom-up approach: starting from strained Hamiltonian we will see what we could learn to apply for sensible QG scenarios. As a warming-up exercise, we start off with homogeneous strain (the strain tensor is constant on the graphene sheet). After that, we discuss the more interesting case of inhomogeneous strain, which give rise to measurable results as the celebrated *pseudo-magnetic* field of around 300 T [38].

## 11.1 Homogeneous strain

Let us now take a homogeneously deformed graphene membrane. In this case the strain tensor $u_{ij}$ does not depend on the coordinates $\vec{x}$. Allowing an independent variation of the hopping energy of the three nearest-neighbours $t_n$ due to variation of the vector basis $\vec{s}_i' = \vec{s}_i + \vec{\Delta}_i = (\mathbb{1} + \overleftrightarrow{\epsilon})\vec{s}_i$, where $I$ is the identity matrix and $\epsilon_{ij} = \partial_i u_j$ (see Figure 11.1), we get to first order

$$t_i = \eta(1 - \beta\delta_i), \tag{11.1}$$

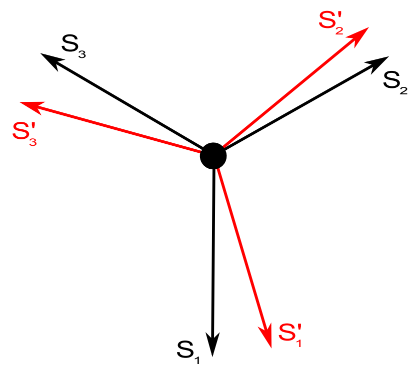



Figure 11.1: Near-neighbor vectors transformation $\vec{s}'_i = (\mathbb{1} + \overleftrightarrow{\epsilon})\vec{s}_i$ under homogeneous strain.

with $\beta = \left|\frac{\partial \log \eta}{\partial \log \ell}\right|$ the *electron Grüneisen parameter*, and $\delta_i = \frac{1}{\ell^2}\vec{s}_i \cdot \vec{\Delta}_i$. The TB Hamiltonian, taking into account this displacements, is

$$H = -\sum_{\vec{r}\in L_A}\sum_{i=1}^{i=3}\left(a^\dagger(\vec{r})t_i b(\vec{r}+\vec{s}'_i) + b^\dagger(\vec{r}+\vec{s}'_i)t_i a(\vec{r})\right). \tag{11.2}$$

Making a Fourier transformation and with analogous replacements as those leading to (9.7), we obtain

$$H = \sum_{\vec{k}}\left(h(\vec{k})a^\dagger(\vec{k})b(\vec{k}) + c.c.\right), \tag{11.3}$$

where

$$h(\vec{k}) = -\sum_{i=1}^{i=3} t_i e^{i\vec{k}\cdot\vec{s}'_i} = -\sum_{i=1}^{i=3} t_i e^{i\vec{k}'\cdot\vec{s}_i},$$

with $\vec{k}' = (\mathbb{1} + \overleftrightarrow{\epsilon})\vec{k}$. It can be checked the function $h(\vec{k})$ is the generalization of the near neighbor function energy $f_1(\vec{k})$ in (9.5), as they coincide when $\vec{\Delta}_i = 0$ (unstrained graphene case). We can note already here that the effect of strain is twofold:

1. it changes the hopping energy ($\beta$-dependent term in the literature);
2. it changes the reciprocal lattice in momentum space ($\beta$-independent term).

Now we proceed in similar to the lines in Chapter 9. Expanding $h(\vec{k})$ to first order in the displacements we get

$$h(\vec{k}) = f(\vec{k}') + \frac{\eta}{\ell^2}\sum_{i=1}^{i=3}\left(\vec{s}_i \cdot \vec{\Delta}_i\right)e^{i\vec{k}'\cdot\vec{s}_i}.$$

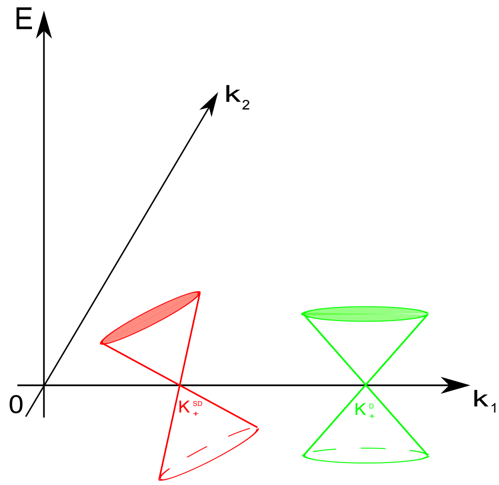



Figure 11.2: The twofold effect of the strain: shifting of the Dirac points $\vec{w_\pm}$ and changing the slope of the Fermi cone.

Making the expansion around the above Dirac points $\vec{k} = \vec{K'^D}_\pm + \vec{p}$, gives us a correction respect to the unstrained graphene

$$H = v_F \sum_{\vec{p}} \left( \psi_+^\dagger \vec{\sigma} \cdot \left( \vec{p'} + \vec{A} \right) \psi_+ - \psi_-^\dagger \vec{\sigma}^* \cdot \left( \vec{p'} - \vec{A} \right) \psi_- \right),$$

where $A^j = -\frac{\beta}{2\ell} \epsilon^{jp} K^{pmn} u_{mn}$ is the usual pseudo-gauge field interpretation of the in-plane strain effect in the graphene sheet [39]. The third rank anisotropic tensor $K^{jmn}$ will be soon very important (see Appendix H for the definition of this tensor). Going back to the configuration space, we end up with the result

$$H = -iv_F \int d^2x \left( \psi_+^\dagger \vec{\sigma} \cdot \left( \vec{\nabla} + i\vec{A} \right) \psi_+ - \psi_-^\dagger \vec{\sigma}^* \cdot \left( \vec{\nabla} - i\vec{A} \right) \psi_- \right) . \quad (11.4)$$

We can see here that the effect of strain does not break time-reversal symmetry with respect to the two Dirac points. We also see space-dependent Fermi velocity through $\vec{p'}$ vector, i.e., strain changes the slope of the Fermi cone and shifts the Dirac points [45] (see Figure 11.2). Indeed, the zeros of $h(\vec{k})$ in (11.3) are now localized in the new points $\vec{k_\pm^{SD}} = \vec{k_\pm^D} + \vec{w_\pm}$, where $w_{\pm 1} = \mp A_1$ and $w_{\pm 2} = \mp A_2$. The expansion around this shifted Dirac points $\vec{k_\pm} = \vec{k_\pm^{SD}} + \vec{p}$ is what gives us again the Dirac Hamiltonian for a massless (Weyl) spinor

$$H = v_F \sum_{\vec{p}} \left[ \left( b_+'^\dagger, a_+'^\dagger \right) \vec{\sigma} \cdot \vec{p} \begin{pmatrix} b_+' \\ a_+' \end{pmatrix} - \left( b_-'^\dagger, a_-'^\dagger \right) \vec{\sigma}^* \cdot \vec{p} \begin{pmatrix} b_-' \\ a_-' \end{pmatrix} \right], \quad (11.5)$$

but now $a'$ and $b'$ are taking into account that we are expanding respect to the true Dirac points $\vec{k}^{SD}_{\pm}$. This means that $a'(\vec{x})$ is the annihilation operator on $L_A$ but taking into account that we are now expanding around $\vec{k}^{SD}_{\pm}$ and not around $\vec{k}^{D}_{\pm}$. The same for the operator $b'(\vec{x})$, related to $L_B$. In the configuration space, the Hamiltonian (11.5) takes the form

$$H = -iv_F \int d^2x \left( \psi'^{\dagger}_{+} \vec{\sigma} \cdot \vec{\nabla} \psi'_{+} - \psi'^{\dagger}_{-} \vec{\sigma}^{*} \cdot \vec{\nabla} \psi'_{-} \right) \tag{11.6}$$

We can obtain the Hamiltonian (11.4) if we define that the fermion transformation

$$\psi'_{\pm} = e^{-i\vec{w_{\pm}}\cdot\vec{x}} \psi_{\pm}.$$

We could interpret this transformation, in the homogeneous strain regime, as a phase transformation. At the end, we can relate the unstained fermion $\psi_0$ with the strained one $\psi_{\pm}$ via

$$\psi_{\pm} = e^{i\vec{w_{\pm}}\cdot\vec{x}} \psi_{0\pm}. \tag{11.7}$$

## 11.2 Inhomogeneous strain

The above analysis works very well for the homogeneous strain: we shift the Dirac points in the Hamiltonian formulation which could be done either adding the term $i\vec{A}$, or transforming the fermion as (11.7). We can wonder what happens now in an inhomogeneous strain. From a general point of view, this is a very complicated issue and probably is not analytically solvable. Due to this, we shall make a reasonable approximation: the variation of the strain is very small on the scale of the lattice length $\ell$. In other words, the Fourier modes of the strain tensor $u_{ij}$ are much smaller than the original Dirac points $|\vec{k}^{D}_{\pm}|$. The modification of the Hamiltonian (11.4) in the presence of an inhomogeneous strain is computed in [43]. In the fermion response interpretation, we can write the formal solution $\psi$ of the of the strained graphene Hamiltonian following the Dirac's prescription, as [128, 129]

$$\psi(\vec{x}) = e^{i\int \vec{w}_{\pm}(\vec{x}')\cdot d\vec{x}'} \psi_0(\vec{x}) = e^{\mp i\int \vec{A}(\vec{x}')\cdot d\vec{x}'} \psi_0(\vec{x}), \tag{11.8}$$

where $\psi_0$ is the unstrained solution[1]. We can select a point $\vec{x}_0$ as a reference strain, i.e., a point where the strain effect $\vec{w}(\vec{x}_0)$ could be gauged away. The phase acquired in (11.8) could be interpreted as the circulation from the zero strain region $\vec{x}_0$ to the strained one $\vec{x}$. We stress the fact that

[1]We can see from (11.8) that each Dirac point is charged with opposite sign as a consequence of time-reversal symmetry.

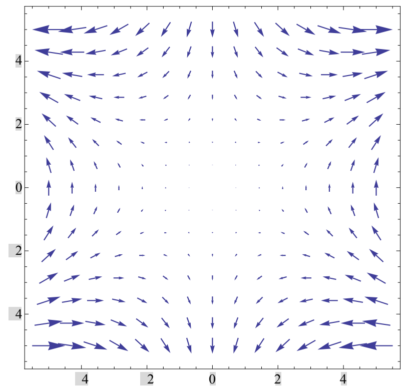



Figure 11.3: Strained graphene with a deformation vector defined as $\vec{u} = (2xy, x^2 - y^2)u_0/L$.

the solution (11.8) is *formal*: once the curl of $\vec{w}$ is nonzero, the integral in (11.8) is path-dependent (is a non-reversible process). Realizing that the effect of the inhomogeneous strain is to shift the momenta of the Fourier modes in a space-dependent way, we can interpret the situation as the $\psi$ fermion is in presence of an *effective* gauge field $\vec{A}$, which could give us a nonzero effective magnetic field if the curl of $\vec{A}$ is nonzero, giving rise to the characteristic Landau levels. This gauge field, frequently called in the literature *pseudo-gauge field*, is the one we announced in the through the Part III of this Thesis. Clearly, it could not have guessed from the QFT in curved space description.

We can follow the trail of the non-trivial behaviour of $\vec{A}$ as the result of the contraction of the strain tensor $u_{ij}$ with the third rank tensor $K^{ijk}$. This tensor is not isotropic[2] and its presence is due to the triad $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ specific to the structure of graphene, which is built from two sub lattices [39]. The non-triviality resulting from this contraction could be seen in a simple example [131]. Consider the deformation vector $\vec{u} = (2xy, x^2 - y^2)u_0/L$, where $u_0$ is the maximum value of the strain and $L$ is the length of the graphene sample, as represented in Figure 11.3. This is a non-singular vector field, in the sense that its curl and divergence is zero, so it is an irrotational vector field without any source or sink.

However, the vector field $\vec{A}$ resulting from the contraction of the corresponding strain tensor with $K^{ijk}$ is a clockwise rotational vector field with constant curl, as we can see in Figure 11.4. This means that, not only the displacement vector $\vec{u}$ matters, but also the *orientation* of this vector with respect to the near-neighbour vectors triad $\vec{s}_i$. So, the $K^{ijk}$ carries some memory of the

[2]In fact, the only isotropic tensors of rank three are proportional to the Levi-Civita antisymmetric tensor $\epsilon_{ijk}$, which is zero in two dimensions [130].

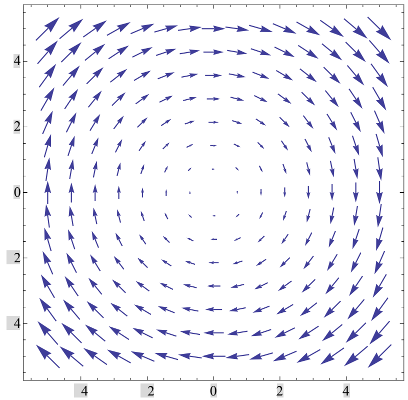



Figure 11.4: Pseudo-gauge field produced using the deformation vector $\vec{u} = (2xy, x^2 - y^2)u_0/L$.

honeycomb structure, even in the low-energy (long-wavelength) limit. To see more concretely that this is the case, suppose that we have two unstrained graphene sheet samples. The honeycomb orientation of both samples are such that one of them, let us say the sample 1, has the vector $\vec{s}_1$ of the triad parallel to the $y$-axis, as in the last Section and the sample 2 has vector $\vec{s}_1 = \ell(-1, 0)$ parallel to the $x$-axis, see Figure 11.5. The orientation of the triads is to both the same. A direct computation shows that the $K^{ijk}$ tensor are different for the sample 1 and sample 2. Now let us apply to both samples the same strain deformation vector shown in Figure 11.3. For the sample 1, the pseudo-gauge vector field gives us a constant pseudo-magnetic field $B = \frac{\beta}{\ell}u_0$, while for the sample 2 the pseudo-magnetic field is zero. So, even if in the long wavelength limit both samples look the same, the pseudo-magnetic field remembers the honeycomb orientation. A detailed study of the so-called "memory tensors" in the graphene honeycomb and Kagomé lattices is found in [132]. Once the curl of $\vec{w}$ is not zero, equation (11.8) is not single-valued, as we can see if we take a loop around the origin. As we said above, we are assuming that the variation of the strain tensor $u_{ij}$ is very small so, following the lines of [133, 134], we can envisage a process which we can extract a physical meaning of the solution (11.8), even if it is not single-valued. Consider a small planar box on the graphene sheet (but very large compared with the lattice length $\ell$) situated at $\vec{R}$ (see Figure 11.6). In the case of unstrained graphene ($\vec{A} = 0$), the solutions for the $\pi$ electrons have the form[3] $\psi_0^\dagger(\vec{r} - \vec{R}) = \psi\dagger_0|0>$. Now, the idea is to strain the graphene sheet in such a way that in the box the strain is homogeneous (constant shift $\vec{w}$ of the Dirac point as in the previous Section) and the associated magnetic field is almost zero in that region. The solution of the $\pi$ electrons in the box

[3] Because we are in the second quantization formalism, $\psi$ are operators and the wave-packets are these operators applied to the vacuum.

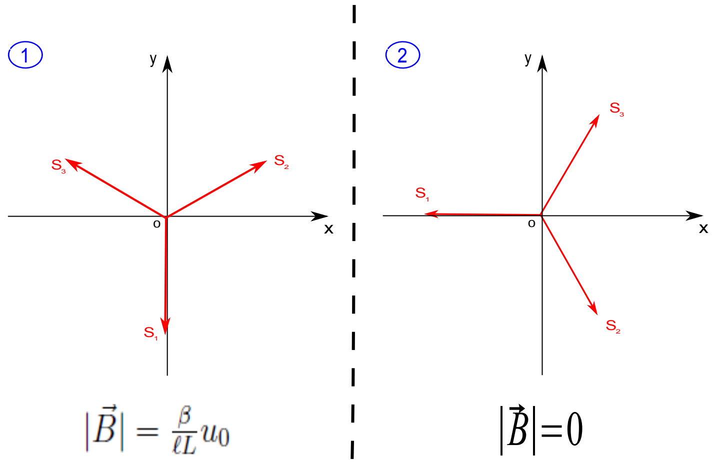



Figure 11.5: Two samples of graphene unstrained sheets. Sample 2 is clockwise rotated respect to sample 1 by $\pi/2$ radians.

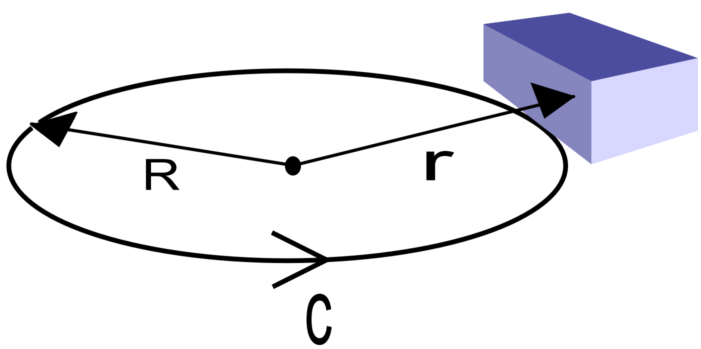



Figure 11.6: Loop $C$ of the box (in blue) situated at $\vec{R}$. All the fermions inside the box could be described as $\psi(\vec{r} - \vec{R})$.

could be written using the Dirac's prescription (11.8)

$$\psi(\vec{r}-\vec{R}) = e^{-i\int_{\vec{R}}^{\vec{r}} \vec{A}(\vec{r'})\cdot d\vec{r'}}\psi_0(\vec{r}-\vec{R}). \tag{11.9}$$

We can see that $\psi$ in (11.9) is single-valued in $\vec{r}$ and $\vec{R}$ locally. Now let the box be transported around a loop $C$. After completion of the loop, there will be a geometrical phase change that can be computed using Berry's formula [96]

$$\gamma(C) = i\oint_C \psi^\dagger(\vec{r}-\vec{R})\nabla_{\vec{R}}\psi(\vec{r}-\vec{R})\cdot d\vec{R} \tag{11.10}$$

Taking into account that the $\psi$ are normalized, we end up with the result

$$\gamma(C) = -\oint_C A(\vec{R})\cdot d\vec{R} = -\Phi, \tag{11.11}$$

where $\Phi$ is the magnetic flux enclosed in the loop.

## 11.3 Discussion

We conclude that when only strain is present, the only gauge field that has unambiguous physical effects is the one just discussed in this Chapter. The structure we saw there, is reminiscent of the gauge field arising in the Aharonov-Bohm effect, although strictly speaking, in order to have this effect, we need a magnetic flux line crossing the loop $C$ and a zero magnetic field outside the line. This is also reflected by the fact that the flux obtained in (11.11) is not constant and depends on the area enclosed by the loop. However, it could be possible to imagine a strain vector concentrated in some region of the graphene and assumed to be zero outside this region. Then, this procedure, recover the Aharonov-Bohm result. The Aharonov-Bohm effect is not new in the graphene literature and some examples of strain vectors were proposed in order to see this effect with particular procedures, see for instance [135, 136].

This gauge field could not have guessed from a top-down approach based on the standard QFT in curved spacetime. In fact, we saw here that if we define standard classical functions on graphene membrane the only fields emerging are the metric and the spin connection, both which do not reproduce the behaviour of this pseudogauge field. This $U(1)$ field can be put in correspondence with quantum field theoretical structures, such as the quantum anomalies. To see it, one just needs to realize that the origin of such gauge field is entirely quantum mechanical, and related to the fact that a "translation" in configuration space $T:\vec{x}\to\vec{x}+\vec{u}$, that is the straining of the graphene membrane, is necessarily associated with a "translation"[4] in momentum space $B : \vec{k} \to \vec{k} + \vec{v}$. Hence, in the first quantization language of the wave functions, those operations are carried on by quantum operators $\mathcal{U} : T \times B \to \mathbb{C}$ that, for the very meaning of quantum mechanics (that is the Heisenberg uncertainty principle) need to obey

$$\mathcal{U}(T_1, B_1)\mathcal{U}(T_2, B_2) = e^{(\vec{u}_1\vec{v}_2 - \vec{u}_2\vec{v}_1)}\mathcal{U}(T_2, B_2)\mathcal{U}(T_1, B_1) \,, \tag{11.12}$$

that are recognized, in the second quantization language, as instances of the nontriviality of the quantum field theoretical vacuum, and in turn of the quantum anomaly [138].

Therefore, our simple table-top laboratory, can help explore yet another arena of fundamental physics, that is the deep meaning of the quantum anomaly[5].

On the other hand, this pseudogauge field, for the use of graphene we have in mind, is intriguing also because, as explained in the paper, it is a memory of the lattice structure (that is the physics of the wavelengths comparable to $\ell$, the lattice spacing) in the continuum limit (that is the physics of the large wavelengths). That means it is a relic at a low energy of the high-energy behaviour of the system. This is yet another reason why we cannot reproduce this field from a top-down approach because in such approach the isotropy of the graphene membrane is tacitly assumed.

Effects of this kind would be of paramount importance to bring high energy theoretical constructions under the control of experiments. One example that comes to mind is the standard model extension of [140–142], where tensorial fields that are relics of the Lorentz invariant high-energy string theory combine with the fields of the Standard Model (SM) and their derivatives within Lorentz violating terms that have the form

$$T^{(k)}{}_{\mu\ldots\nu}(\text{SM fields and derivatives})^{\mu\ldots\nu} \,, \tag{11.13}$$

where $T^{(k)} \sim \ell^k_{\rm Planck}$. In our "graphene universe" $\ell_{\rm Planck} \approx \ell$.

To explore these scenarios, and to address the full variety of possible gauge fields in graphene, namely those arising beyond the pure strain limitation, would be a good future project.

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[4] A better name for this operation is "Galilean boost", that is why we use "B" for it. On this note see, e.g., [137].

[5] In our view, the very existence of a quantum anomaly is a sign of lack of understanding of how nature works at the most fundamental level, or, in other words, it is a sign of our ignorance of what is a proper quantum Noether theorem [139].

# Appendix A

# General definitions and useful Dirac matrices properties

## A.1 $D = 3$

Through this work we extensively use the Clifford algebra in $D = 3$. Some basic properties and definitions are[1]

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\,, \quad \gamma_{ab} \equiv \gamma_{[a}\gamma_{b]} = \frac{1}{2}[\gamma_a, \gamma_b]\,, \quad \gamma_{ab} = \epsilon_{abc}\gamma^c\,,$$
$$\frac{1}{2}[\gamma_{ab}, \gamma_c] = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b\,, \quad \gamma_{abc} = \epsilon_{abc} = \frac{1}{2}\{\gamma_{ab}, \gamma_c\} = \gamma_{[a|}\gamma_b\gamma_{|c]}$$

Let $\psi$ be a two-component Dirac spinor with odd Grassman parity . We define its Dirac conjugate by

$$\overline{\psi} = i\psi^\dagger\gamma_0. \tag{A.1}$$

or explicitly as $\bar{\psi}_\beta = i\psi^{\alpha *}(\gamma_0)_{\alpha\beta}$, $\alpha, \beta = 1, 2$. With this prescription, we have the conjugacy properties

$$\begin{aligned}(\overline{\chi}\psi)^* &= \overline{\psi}\chi, \\ (\overline{\chi}\gamma_a\psi)^* &= -(\overline{\psi}\gamma_a\chi). \\ \overline{(\gamma_a\psi)} &= -\overline{\psi}\gamma_a\end{aligned} \tag{A.2}$$

[1] We adopt the convention $\epsilon_{012} = -\epsilon^{012} = 1$ and the definition $T_{[a_1\ldots a_p]} = \frac{1}{p!}\delta^{b_1\ldots b_p}_{a_1\ldots a_p}T_{b_1\ldots b_p}$, where the generalized Kronecker delta can be written as the following determinant $\delta^{b_1\ldots b_p}_{a_1\ldots a_p} = \begin{vmatrix} \delta^{a_1}_{b_1} & \ldots & \delta^{a_1}_{b_p} \\ \vdots & \ddots & \vdots \\ \delta^{a_p}_{b_1} & \ldots & \delta^{a_p}_{b_p}\end{vmatrix}$. In the coordinate basis, we define $\epsilon_{ij} \equiv \epsilon_{tij}$.

## A.2 $D = 4$

The $\gamma$-matrices in Minkowskian $D = 4$ with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, are in a $4 \times 4$ spinorial-representation of the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$. The indices of the tangent space $a, b$ take the values 0,1,2 and 3.

From this properties, a number of useful representation independent identities follow,

1. (Three $\gamma$'s product property)

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c - \eta_{ac}\gamma_b + \eta_{bc}\gamma_a - \epsilon_{abcd}\gamma_5\gamma^d \ .$$

2. (Three $\gamma$'s $+\gamma_5$ product property)

$$\gamma_5\gamma_a\gamma_b\gamma_c = \gamma_5[\eta_{ab}\gamma_c - \eta_{ac}\gamma_b + \eta_{bc}\gamma_a] + \epsilon_{abcd}\gamma^d \ ,$$

3. (Four $\gamma$'s $+\gamma_5$ product property)

$$\begin{aligned}\gamma_5\gamma_a\gamma_b\gamma_c\gamma_d =& \epsilon_{abcd}\mathbb{1} + \gamma_5[\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}] \\ &+ \gamma_5[\eta_{ab}\gamma_{cd} - \eta_{ac}\gamma_{bd} + \eta_{ad}\gamma_{bc} + \eta_{bc}\gamma_{ad} - \eta_{bd}\gamma_{ac} + \eta_{cd}\gamma_{ab}] \ ,\end{aligned}$$

In the representation chosen here, the Dirac matrices are written in terms of the sigma Pauli matrices and $2 \times 2$ identity [22]

$$\begin{aligned}\gamma_0 &= -\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \gamma_i &= \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix},\end{aligned} \tag{A.3}$$

with also the $\gamma_5$ matrices are defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

We take the following convention for the Levi-Civita skew-symmetric tensor

$$\epsilon_{0123} = -\epsilon^{0123} = 1, \tag{A.5}$$

and, for the spatial Levi-Civita, the following orientation $\epsilon_{0ijk} = \epsilon_{ijk}$. In this case, we can take advantage of the Pauli matrices properties

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \ , \tag{A.6}$$

and vectorial product notation to write

$$\begin{aligned} \left(\vec{A}\times\vec{B}\right)^i &= \epsilon^{ijk}A_jB_k, \\ \vec{A}\cdot(\vec{B}\times\vec{C}) &= \epsilon^{ijk}A_iB_jC_k, \end{aligned} \tag{A.7}$$

for $\vec{A}$, $\vec{B}$ and $\vec{C}$ arbitrary spatial vectors.

Some useful properties, could be derived from the Pauli sigma matrices algebra, which are

$$\vec{\sigma}\times(\vec{\sigma}\times\vec{A}) = -2\vec{A}+i\vec{\sigma}\times\vec{A}\,, \tag{A.8}$$

$$(\vec{A}\times\vec{\sigma})\times\vec{\sigma} = -2\vec{A}+i\vec{A}\times\vec{\sigma}\,, \tag{A.9}$$

$$\vec{\sigma}(\vec{\sigma}\cdot\vec{A}) = \vec{A}-i\vec{\sigma}\times\vec{A}\,. \tag{A.10}$$

# Appendix B

# Graded Lie Algebra Representations

## B.1 Representation of the $\mathfrak{osp}(2,2)$ superalgebra

Let $z^A = (\varepsilon^\alpha; x^i)$ a vector constructed from odd Grassmann numbers $\varepsilon^\alpha$, $\alpha \in \{1,\dots,M\}$, and $x^i$, $i \in \{1,\dots,N\}$ real numbers. Defining the matrix

$$G_{AB} = \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & \delta_{ij} \end{pmatrix} ,$$

where $C_{\alpha\beta}$ is the charge conjugation matrix. The orthosymplectic group $OSp(N|M)$ is the group which leaves invariant the quadratic form [143]

$$z^A G_{AB} z^B = \varepsilon^\alpha C_{\alpha\beta} \varepsilon^\beta + x^i x_i \ . \tag{B.1}$$

We can see immediately from (B.1) that the $OSp(M|N)$ group contains the direct product of the simplectic group $Sp(M)$ and the special ortogonal group $SO(N)$, i.e., $Sp(M) \otimes SO(N) \subset OSp(M|N)$. This group has $\frac{M(M+1)+N(N-1)}{2}$ bosonic generators and $M \times N$ fermionic generators.

In Chapter 2, we deal with a particular case of the orthosymplectic group, i.e., the graded Lie algebra $\mathfrak{osp}(2|2)$. A particular representation for such a algebra in $3 \times 3$ matrices for the bosonic generators is [12]

$$\mathbb{K} = \begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & \frac{i}{2} & 0 \\ 0 & 0 & i \end{pmatrix} , \ \mathbb{J}_a = \begin{pmatrix} \multicolumn{2}{c}{\frac{1}{2}\gamma_a} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{B.2}$$

while for the fermionic generators,

$$\begin{aligned}\mathbb{Q}^1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathbb{Q}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \overline{\mathbb{Q}}_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \overline{\mathbb{Q}}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.3}$$

We can verify directly from (B.2-B.3) that

$$\begin{aligned}[\mathbb{J}_a, \mathbb{J}_b] &= \epsilon_{abc}\mathbb{J}^c, \quad [\mathbb{J}_a, \mathbb{Q}^\alpha] = -\frac{1}{2}(\gamma_a)^\alpha_\beta \mathbb{Q}^\beta, \quad \left[\mathbb{J}_a, \overline{\mathbb{Q}}_\alpha\right] = \frac{1}{2}(\gamma_a)^\beta_\alpha \overline{\mathbb{Q}}_\beta, \\ [\mathbb{Z}, \mathbb{Q}^\alpha] &= \frac{i}{2}\mathbb{Q}^\alpha, \quad \left[\mathbb{Z}, \overline{\mathbb{Q}}_\alpha\right] = -\frac{i}{2}\overline{\mathbb{Q}}_\alpha, \quad \left\{\mathbb{Q}^\alpha, \overline{\mathbb{Q}}_\beta\right\} = (\gamma^a)^\alpha_\beta \mathbb{J}_a - i\delta^\alpha_\beta \mathbb{Z},\end{aligned}$$

which is exactly (2.3). As shown in (B.2-B.3), these generators have vanishing supertrace. However, the non-vanishing quadratic supertraces are

$$\langle \mathbb{J}_a \mathbb{J}_b \rangle = \frac{1}{2}\eta_{ab}, \quad \langle \mathbb{Z}\mathbb{Z} \rangle = \frac{1}{2}, \quad \langle \overline{\mathbb{Q}}_\alpha \mathbb{Q}^\beta \rangle = \delta^\beta_\alpha, \quad \langle \mathbb{Q}^\alpha \overline{\mathbb{Q}}_\beta \rangle = -\delta^\alpha_\beta\ . \tag{B.4}$$

The covariant derivative $D_\mu$ induced by (2.3) appears naturally in (2.5). Acting on a Lorentz vector $\Sigma_a$ and 1/2-spinors ($\psi^\alpha$ and $\overline{\psi}_\alpha$) this reads

$$\begin{aligned} D_\mu \Sigma_a &= \partial_\mu \Sigma_a + \epsilon_{ab}{}^c \omega^b_\mu \Sigma_c, \\ \overrightarrow{D}_\mu \psi^\alpha &= \partial_\mu \psi^\alpha - \frac{i}{2}A_\mu \psi^\alpha + \frac{1}{2}\omega^a_\mu (\gamma_a)^\alpha_\beta \psi^\beta, \\ \overline{\psi}_\alpha \overleftarrow{D}_\mu &= \partial_\mu \overline{\psi}_\alpha + \frac{i}{2}A_\mu \overline{\psi}_\alpha - \frac{1}{2}\overline{\psi}_\beta (\gamma_a)^\beta_\alpha \omega^a_\mu = \overline{(\overrightarrow{D}_\mu \psi)}_\alpha. \end{aligned} \tag{B.5}$$

## B.2 Representation of the $\mathfrak{usp}(2,1|2)$ superalgebra

The following matrices provide a natural representation for the $\mathfrak{usp}(2,1|2)$ superalgebra,

$$\mathbb{J}_a = \left[\begin{array}{c|c} \frac{1}{2}(\gamma_a)^\alpha{}_\beta & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{2\times 2} \end{array}\right], \quad \mathbb{T}_I = \left[\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & -\frac{i}{2}(u\sigma_I u^\dagger)^i{}_j \end{array}\right], \tag{B.6}$$

where $\gamma_a$, $a = 0, 1, 2$, are Dirac matrices, with $\alpha, \beta = 1, 2$, and $\sigma_I$, $I = 1, 2, 3$, are Pauli matrices, with $i, j = 1, 2$. A metric to raise and lower latin indexes is given by $[u^{ij}] = i\sigma_2$. A generic supermatrix $\mathbb{M}$ has the following index structure:

$$\mathbb{M} = \left[\begin{array}{c|c} M^\alpha{}_\beta & M^\alpha{}_j \\ \hline M^i{}_\beta & M^i{}_j \end{array}\right].$$

In terms of components, (B.6) are given by

$$(\mathbb{J}_a)^A{}_B = \frac{1}{2}(\gamma_a)^A{}_B\,, \quad (\mathbb{T}_I)^A{}_B = -\frac{i}{2}u^{Ai}(\sigma_I)_i{}^j u_{jB}\,, \tag{B.7}$$

A direct calculation shows that[1]

$$[\mathbb{J}_a, \mathbb{J}_b] = \epsilon_{ab}{}^c \mathbb{J}_c\,,$$
$$[\mathbb{T}_I, \mathbb{T}_J] = \epsilon_{IJ}{}^K \mathbb{T}_K\,,$$

and $[\mathbb{J}_a, \mathbb{T}_I] = 0$. The fermionic generators

$$(\mathbb{Q}_i^\alpha)^A{}_B = \delta_i^A \delta_B^\alpha\,, \quad (\overline{\mathbb{Q}}_\alpha^i)^A{}_B = \delta_\alpha^A \delta_B^i\,, \tag{B.8}$$

are defined so that

$$(\overline{\psi}\mathbb{Q})^A{}_B = \left[\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 2} \\ \hline \overline{\psi}^A_B & 0_{2\times 2}\end{array}\right], \quad (\overline{\mathbb{Q}}\psi)^A{}_B = \left[\begin{array}{c|c} 0_{2\times 2} & \psi^A_B \\ \hline 0_{2\times 2} & 0_{2\times 2}\end{array}\right].$$

Direct computation gives

$$\{\mathbb{Q}_i^\alpha, \mathbb{Q}_j^\beta\} = 0, \quad \{\overline{\mathbb{Q}}_\alpha^i, \overline{\mathbb{Q}}_\beta^j\} = 0,$$

and

$$[\{Q_i^\alpha, \overline{Q}_j^\beta\}]^A{}_C = \delta_i^j \delta_\beta^A \delta_C^\alpha + \delta_\beta^\alpha \delta_i^A \delta_C^j\,, \tag{B.9}$$

The completeness relations for Dirac and Pauli matrices can be used to recast this as

$$\{\mathbb{Q}_i^\alpha, \overline{\mathbb{Q}}_\beta^j\} = \delta_i^j (\gamma^a)^\alpha{}_\beta \mathbb{J}_a - i\delta_\beta^\alpha (\sigma^I)_i{}^j \mathbb{T}_I - i\delta_i^j \delta_\beta^\alpha \mathbb{Z}\,,$$

where $\mathbb{Z}$ is a new bosonic generator represented by a diagonal matrix with vanishing supertrace,

$$\mathbb{Z}^A{}_B = \frac{i}{2}(\delta_\alpha^A \delta_B^\alpha + \delta_i^A \delta_B^i)\,.$$

This generator is a central charge that commutes with all generators in the superalgebra. The only non-vanishing commutators are

$$[\mathbb{J}_a, \mathbb{Q}_i^\alpha] = -\frac{1}{2}(\gamma_a)^\alpha{}_\beta \mathbb{Q}_i^\beta\,, \quad [\mathbb{J}_a, \overline{\mathbb{Q}}_\alpha^i] = \frac{1}{2}(\gamma_a)^\beta{}_\alpha \overline{\mathbb{Q}}_\beta^i\,,$$
$$[\mathbb{T}_I, \mathbb{Q}_i^\alpha] = \frac{i}{2}(\sigma_I)_i{}^j \mathbb{Q}_j^\alpha\,, \quad [\mathbb{T}_I, \overline{\mathbb{Q}}_\alpha^i] = -\frac{i}{2}(\sigma_I)_j{}^i \overline{\mathbb{Q}}_\alpha^j\,.$$

[1]Flat Lorentz and $SU(2)$ indexes in the adjoint representations are lowered and raised using the Lorentzian and Euclidean metrics $\eta_{ab}$ and $\delta^{IJ}$, respectively.

This completes the algebra. It can be directly checked that each of these generators have vanishing supertrace, while the supertrace of quadratic terms are

$$\begin{aligned}\langle \mathbb{J}_a\, \mathbb{J}_b\rangle &= \frac{1}{2}\eta_{ab}\ ,\\ \langle \mathbb{T}_I\, \mathbb{T}_J\rangle &= \frac{1}{2}\delta_{IJ}\ ,\\ \langle \mathbb{Q}_i^\alpha\, \overline{\mathbb{Q}}_\beta^j\rangle &= -\delta_\beta^\alpha\delta_i^j\ .\end{aligned}$$

## B.3 Representation of the $\mathfrak{usp}(2,2|1)$ superalgebra

The generators $\mathbb{J}_a$ and $\mathbb{J}_{ab}$ form the $4D$ algebra

$$[\mathbb{J}_a,\mathbb{J}_b] = s^2\mathbb{J}_{ab}\,,\quad [\mathbb{J}_a,\mathbb{J}_{bc}] = \eta_{ab}\mathbb{J}_c - \eta_{ac}\mathbb{J}_b\,, \tag{B.10}$$
$$[\mathbb{J}_{ab},\mathbb{J}_{cd}] = \eta_{ad}\mathbb{J}_{bc} - \eta_{ac}\mathbb{J}_{bd} + \eta_{bc}\mathbb{J}_{ad} - \eta_{bd}\mathbb{J}_{ac}\,, \tag{B.11}$$

which corresponds to anti-de Sitter ($so(3,2)$) for $s=1$ and to de Sitter ($so(4,1)$) for $s=i$. The supercharge $\mathbb{Q}$ belongs to a spin 1/2 representation, that is

$$[\mathbb{J}_a,\mathbb{Q}^\alpha] = -\tfrac{s}{2}(\gamma_a)^\alpha{}_\beta\mathbb{Q}^\beta,\quad [\mathbb{J}_a,\overline{\mathbb{Q}}_\alpha] = \tfrac{s}{2}\overline{\mathbb{Q}}_\beta(\gamma_a)^\beta{}_\alpha, \tag{B.12}$$
$$[\mathbb{J}_{ab},\mathbb{Q}^\alpha] = -\tfrac{1}{2}(\gamma_{ab})^\alpha{}_\beta\mathbb{Q}^\beta,\quad [\mathbb{J}_{ab},\overline{\mathbb{Q}}_\alpha] = \tfrac{1}{2}\overline{\mathbb{Q}}_\beta(\gamma_{ab})^\beta{}_\alpha. \tag{B.13}$$

Since $\mathbb{Q}$ is complex, it has the following commutators with the $U(1)$ generator

$$[\mathbb{K},\mathbb{Q}^\alpha] = i\mathbb{Q}^\alpha\,,\qquad [\mathbb{K},\overline{\mathbb{Q}}_\alpha] = -i\overline{\mathbb{Q}}_\alpha\ . \tag{B.14}$$

The algebra is completed by the anticommutators of supercharges,

$$\{\mathbb{Q}^\alpha,\overline{\mathbb{Q}}_\beta\} = -\frac{i}{s}(\gamma^a)^\alpha_\beta\mathbb{J}_a + \frac{i}{2}(\gamma^{ab})^\alpha_\beta\mathbb{J}_{ab} - \delta^\alpha_\beta\mathbb{K}\ , \tag{B.15}$$

together with the trivial anticommutators $\{\overline{\mathbb{Q}}_\alpha, \overline{\mathbb{Q}}_\beta\} = 0 = \{\mathbb{Q}^\alpha, \mathbb{Q}^\beta\}$. An explicit $6 \times 6$ representation for the supercharges is the following

$$(\mathbb{Q}^\alpha)^A_{\ B} = -i \left[ \begin{array}{c|cc} & 0 & \\ & 0 & \\ 0_{4\times 4} & 0 & C^{\alpha A} \\ & 0 & \\ \hline \delta^\alpha_B & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \end{array} \right] = -i(\delta^A_5 \delta^\alpha_B + C^{\alpha A}\delta^6_B) \ , \quad \text{(B.16)}$$

$$(\overline{\mathbb{Q}}_\alpha)^A_{\ B} = \left[ \begin{array}{c|cc} & & 0 \\ & & 0 \\ 0_{4\times 4} & \delta^A_\alpha & 0 \\ & & 0 \\ \hline 0\,0\,0\,0 & 0 & 0 \\ C_{\alpha B} & 0 & 0 \end{array} \right] = \delta^A_\alpha \delta^5_B + \delta^A_6 C_{\alpha B} \ , \qquad \text{(B.17)}$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugation matrix, and $C^{\alpha\beta}$ is its inverse. In this representation, the $U(1)$ and (A)dS generators are

$$(\mathbb{K})^A{}_B = \left[ \begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ 0_{4\times 4} & 0 & 0 \\ & 0 & 0 \\ \hline 0\,0\,0\,0 & i & 0 \\ 0\,0\,0\,0 & 0 & -i \end{array} \right] = i(\delta^A_5 \delta^5_B - \delta^A_6 \delta^6_B) \ , \qquad \text{(B.18)}$$

$$(\mathbb{J}_a)^A{}_B = \left[ \begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ \frac{s}{2}\gamma_a & 0 & 0 \\ & 0 & 0 \\ \hline 0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \end{array} \right] = \frac{s}{2}(\gamma_a)^\alpha{}_\beta \delta^A_\alpha \delta^\beta_B \ , \qquad \text{(B.19)}$$

$$(\mathbb{J}_{ab})^A{}_B = \left[ \begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ \frac{1}{2}\gamma_{ab} & 0 & 0 \\ & 0 & 0 \\ \hline 0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0 & 0 & 0 \end{array} \right] = \frac{1}{2}(\gamma_{ab})^\alpha{}_\beta \delta^A_\alpha \delta^\beta_B \ . \qquad \text{(B.20)}$$

# Appendix C

# Dynamical details on unconventional SUSY in $D = 3$

## C.1 Momenta, Constraints and Poisson brackets

For the starting action (3.1), the canonical momenta associated to the dynamical fields are given by

$$\pi^i \approx \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{1}{2}\epsilon^{ij} A_j, \quad \pi^i_a \approx \frac{\partial \mathcal{L}}{\partial \dot{\omega}^a_i} = \frac{1}{2}\epsilon^{ij}\eta_{ab}\omega^b_j, \quad p^i_a \approx \frac{\partial \mathcal{L}}{\partial \dot{e}^a_i} = -2\epsilon^{ij}\eta_{ab} e^b_j \overline{\psi}\psi$$
$$\chi^\alpha \approx \frac{\partial^L \mathcal{L}}{\partial \dot{\overline{\psi}}_\alpha} = -\epsilon^{ij}(\gamma_{ij})^\alpha{}_\beta \psi^\beta \quad , \quad \overline{\chi}_\alpha \approx \frac{\partial^R \mathcal{L}}{\partial \dot{\psi}^\alpha} = \epsilon^{ij}\overline{\psi}_\beta (\gamma_{ij})^\beta{}_\alpha . \tag{C.1}$$

The non-vanishing Poisson brackets between the fields and their respective momenta are defined as in [26] [1]

$$\begin{aligned} \left\{A_i, \pi^j\right\} &= -\left\{\pi^j, A_i\right\} = \delta^j_i, \quad \left\{\omega^a_i, \pi^j_b\right\} = -\left\{\pi^j_b, \omega^a_i\right\} = \left\{e^a_i, p^j_b\right\} = -\left\{p^j_b, e^a_i\right\} = \delta^j_i \delta^a_b, \\ \left\{\psi^\alpha, \overline{\chi}_\beta\right\} &= \left\{\overline{\chi}_\beta, \psi^\alpha\right\} = \delta^\alpha_\beta, \quad \left\{\overline{\psi}_\alpha, \chi^\beta\right\} = \left\{\chi^\beta, \overline{\psi}_\alpha\right\} = -\delta^\beta_\alpha . \end{aligned} \tag{C.2}$$

It is worth to note the relative sign between the two brackets on the last line: This choice is consistent with $\overline{\psi}_\alpha = i\psi^{\beta *}(\gamma_0)_{\beta\alpha}$ and $\overline{\chi}_\alpha = i\chi^{\beta *}(\gamma_0)_{\beta\alpha}$. Now, the primary constraints (3.5) satisfy

$$\begin{aligned} \{\varphi^i_a, \varphi^j_b\} = 4\epsilon^{ij}\eta_{ab}\bar{\psi}\psi, \quad \{\Omega, \varphi^j_b\} &= 2\epsilon^{ij} e^a_i \gamma_a \gamma_b \psi, \quad \{\bar{\Omega}, \varphi^j_b\} = 2\epsilon^{ij} e^a_i \bar{\psi}\gamma_b\gamma_a, \\ \{\bar{\Omega}, \Omega\} = 2\epsilon^{ij}\gamma_{ij}, \quad \{\phi^i, \phi^j\} &= -\epsilon^{ij}, \quad \{\phi^i_a, \phi^j_b\} = -\epsilon^{ij}\eta_{ab} . \end{aligned} \tag{C.3}$$

[1] Hereafter we will omit the $\delta^2(x-y)$ factors when computing the brackets. Spinor indexes may also be omitted for simplicity.

Using (C.3) and the definitions (3.7) one can show that the generators $\tilde{J}_a$, $\tilde{K}$ and $\Upsilon$ satisfy the following:

$$\begin{aligned}
\{\tilde{J}_a, \phi^i_b\} &= \epsilon_{ab}{}^c \phi^i_c, \quad \{\tilde{J}_a, \Omega\} = -\frac{1}{2}\gamma_a \Omega, \quad \{\tilde{J}_a, \bar{\Omega}\} = \frac{1}{2}\bar{\Omega}\gamma_a, \quad \{\tilde{J}_a, \varphi^i_b\} = \epsilon_{ab}{}^c \varphi^i_c, \\
\{\tilde{J}_a, J_b\} &= \epsilon_{ab}{}^c J_c, \quad \{\tilde{J}_a, K_b\} = \epsilon_{ab}{}^c K_c, \quad \{\tilde{K}, \Omega\} = \frac{i}{2}\Omega, \quad \{\tilde{K}, \bar{\Omega}\} = -\frac{i}{2}\bar{\Omega}, \qquad \text{(C.4)} \\
\{\Upsilon, \varphi^j_a\} &= -\varphi^j_a, \quad \{\Upsilon, K_a\} = -K_a, \quad \{\Upsilon, \Omega\} = \Omega, \quad \{\Upsilon, \bar{\Omega}\} = \bar{\Omega},
\end{aligned}$$

where the remaining brackets with constraints (3.2)-(3.5) vanish strongly. We then conclude that the constraints $\tilde{J}_a$, $\tilde{K}$,$\Upsilon$, together with the generator $\mathcal{H}_i$, are first-class.

The consistency of the primary constraints (3.5) with respect to the extended Hamiltonian (3.6) yields the following set of equations

$$\begin{aligned}
0 = \left\{\phi^i, H_T\right\} &= \epsilon^{ij}\left(\partial_j A_t + 2ie^a_t e^b_i \overline{\psi}\gamma_{ab}\psi - \lambda_j\right), \\
0 = \left\{\phi^i_a, H_T\right\} &= \epsilon^{ij}\left(\eta_{ab} D_j \omega^b_t + 2\epsilon_{abc} e^b_t e^c_j \overline{\psi}\psi - \eta_{ab}\Lambda^b_j\right), \\
0 = \left\{\varphi^i_a, H_T\right\} &= 2\epsilon^{ij}\left(\epsilon_{abc}\omega^b_t e^c_j \overline{\psi}\psi - iA_t e^b_j \overline{\psi}\gamma_{ab}\psi - 2\eta_{ab}\partial_j\left(e^b_t \overline{\psi}\psi\right) - \epsilon_{abc}\omega^b_j e^c_t \overline{\psi}\psi \right. \qquad \text{(C.5)} \\
&\quad \left. + e^b_t(\overline{\psi}\overleftarrow{D}_j \gamma_a\gamma_b\psi + \overline{\psi}\gamma_b\gamma_a \overrightarrow{D}_j\psi) + 2\eta_{ab}\lambda^b_j \overline{\psi}\psi + e^b_j\left(\overline{\Lambda}\gamma_b\gamma_a\psi + \overline{\psi}\gamma_a\gamma_b\Lambda\right)\right), \\
0 = \left\{\Omega, H_T\right\} &= -\epsilon^{ij}\left(iA_t e^a_i e^b_j \gamma_{ab}\psi + \epsilon_{abc}\omega^a_t e^b_i e^c_j \psi + 2\eta_{ab} e^a_t T^b_{ij}\psi - 2e^a_t e^b_i \gamma_a\gamma_b D_j\psi \right. \\
&\quad \left. + 2D_j\left(e^a_t e^b_i \gamma_b\gamma_a\psi\right) + 2\lambda^a_i e^b_j \gamma_b\gamma_a\psi - 2e^a_i e^b_j \gamma_{ab}\Lambda\right), \\
0 = \left\{\overline{\Omega}, H_T\right\} &= \epsilon^{ij}\left(iA_t e^a_i e^b_j \overline{\psi}\gamma_{ab} - \epsilon_{abc}\omega^a_t e^b_i e^c_j \overline{\psi} - 2\eta_{ab} e^a_t T^b_{ij}\overline{\psi} + 2e^a_t e^b_i (\overline{\psi}\overleftarrow{D}_j)\gamma_b\gamma_a \right. \\
&\quad \left. -2\left(e^a_t e^b_i \overline{\psi}\gamma_a\gamma_b\right)\overleftarrow{D}_j - 2\lambda^a_i e^b_j \overline{\psi}\gamma_a\gamma_b - 2e^a_i e^b_j \overline{\Lambda}\gamma_{ab}\right).
\end{aligned}$$

This system of $(14+4n)$ equations determines up to an equal number of Lagrange multipliers, leaving seven free parameters. This means that in a generic sector (maximum rank), there are $S = 14+4n$ second-class and $F = 7$ first-class constraints. Also, if one choose $(e^a_t, \omega^a_t, A_t)$ as the free parameters, the consistency of the secondary constraints $K_a$, $J_a$ and $K$ can be readily shown to follow. In Appendix C.2 we exhibit a solution for (C.5).

## C.2 Solving the consistency equations

Let us now choose tensors $\zeta_a$ and $T^a{}_{bc} = T^a{}_{[bc]}$, depending on the dynamical fields, such that

$$\begin{aligned}
T^a{}_{bc} e^b_i e^c_j &= T^a_{ij} \qquad \text{(C.6)} \\
&= D_i e^a_j - D_j e^a_i, \\
e^a_i \zeta_a &= D_i \psi. \qquad \text{(C.7)}
\end{aligned}$$

Equation (C.6) relates the nine Lorentz covariant components $T^a_{\ bc}$ to the three field dependent quantities on the RHS. Similarly, equation (C.7) expresses the vector-spinor $\zeta_a$ as a function of two components on the RHS. This means there are six real and one spinorial indeterminate components respectively[2], which will be fixed by the consistency equations. Now, let us take the Lagrange multipliers in equation (3.6) as

$$\begin{aligned}
\lambda^b_j &= -ve^b_j - \epsilon^b_{\ cd}\omega^c_t e^d_j + D_j e^b_t + T^b_{\ ac}e^a_t e^c_j \,, \\
\Lambda &= v\psi + \frac{i}{2}A_t\psi - \frac{1}{2}\omega^c_t\gamma_c\psi + e^a_t\zeta_a \,, \\
\bar\Lambda &= v\bar\psi - \frac{i}{2}A_t\bar\psi + \frac{1}{2}\omega^c_t\bar\psi\gamma_c + \bar\zeta_a e^a_t \,, \\
\Lambda^b_j &= D_j\omega^b_t + 2e^c_t e^a_j \epsilon^b_{\ ca}\bar\psi\psi \,, \\
\lambda_j &= \partial_j A_t + 2ie^a_t e^b_j \epsilon_{abc}\bar\psi\gamma^c\psi \,.
\end{aligned} \tag{C.8}$$

After inserting (C.8) into the consistency conditions (C.5), these reduce to

$$0 = e^b_t e^c_j(\eta_{ad}T^d_{\ cb}\bar\psi\psi - \bar\psi\gamma_a\gamma_{[c}\zeta_{b]} - \bar\zeta_{[b}\gamma_{c]}\gamma_a\psi) \,, \tag{C.9}$$

$$0 = |e|\epsilon^{abc}(\gamma_{ab}\zeta_c - \frac{1}{2}\gamma_a\gamma_d T^d_{\ bc}\psi) \,, \tag{C.10}$$

together with the conjugate of the last equation. For an arbitrary dreibein, equation (C.10) can be used to fix the remaining free component of $\zeta_a$ as a function of the dynamical fields and $T^a_{\ bc}$. On the other hand, using the constraint $K_a \approx 0$, one can show that (C.9) correspond to six independent equations for an equal number of free components in $T^a_{\ bc}$, once $\zeta_a$ is replaced.

Note now that the parameter $v$ does not show up in (C.9,C.10), this indicates that the complete set of equations is not independent. In fact, one can readily check that the following shift

$$T^a_{\ bc} \to T^a_{\ bc} + 2\beta\delta^a_{[b}E^t_{c]} \quad , \qquad \zeta_c \to \zeta_c - \beta E^t_c\psi \,, \tag{C.11}$$

leaves (C.6,C.7) and (C.9,C.10) invariant. This is related to the Weyl invariance, shifting the multiplier $v \to v - \beta$ in (C.8). We thus have the following picture: If the three components $e^a_t$ remain arbitrary, then one can solve the $(14 + 4n)$ multipliers as in (C.8), but this leaves a degeneracy in $v$ to be fixed afterwards. Otherwise one may restrict one of the components $e^a_t$

[2] For $|e| \neq 0$, one can put for instance $\zeta_a = E^i_a D_i\psi + E^t_a\xi$ and $T^a_{\ bc} = E^i_b E^j_c T^a_{ij} + E^t_{[b}E^i_{c]}\xi^a_i$ for arbitrary $\xi$ and $\xi^a_i$.

while leaving the scaling parameter $v$ completely free, as we explain below. In view of the counting argument of Section 3.2, we expect in general that one combination among the eight parameters $(A_t, \omega_t^a, e_t^a, v)$ will be found fixed in a generic sector, so that the number of functionally independent first-class constraints is reduced to $F = 7$. Note that the degeneracy in $v$ also suggests there could be certain configurations of the dynamical fields such that (C.9,C.10) have no solution: The consistency equations would lead to secondary constraints in this sectors.

The above reasoning is illustrated with the spin-1/2 sector described in Section 3.1.2. In that case one chooses the gauge $v = 0$ a priori, and then proceed to count the DOF considering the residual symmetries. This gauge fixing is equivalent to choose the solution

$$T^a{}_{bc} = 2\alpha\epsilon^a{}_{bc}\,, \tag{C.12}$$

$$\zeta_a = \gamma_a\psi\,, \tag{C.13}$$

for (C.6,C.7) and (C.9,C.10), provided (3.19,3.18). Inserting this into the multipliers (C.8) and then into the total Hamiltonian (3.6), one directly gets the form (3.27). Note that this Hamiltonian preserves the gauge, and possesses only seven free parameters $(A_t, \omega_t^a, e_t^a)$ corresponding to the generators of residual symmetries.

On the other hand, in a generic sector one can always use the 'degenerate gauge' (3.13) for counting purposes. Using $K_a \approx 0$, this election is readily seen to close the consistencies (C.9,C.10) and puts the total Hamiltonian in the form (3.12). However, by doing so one needs to assume there is in fact a solution for $\zeta_a$ and $T^a{}_{bc}$, in order to extend the sector for non-degenerate choices with $|e| \neq 0$. Thus, in any generic sector, a realization of the first-class constraints can be easily obtained by means of the degenerate gauge, leaving also $F = 7$ free parameters $(A_t, \omega_t^a, \xi^i, v)$.

# Appendix D

# Killing spinors

In this Appendix we present a detailed computation of the Killing spinors mentioned in Table 4.1. The radial, time, and angle components of this equation read

$$
\begin{aligned}
0 =& \partial_r \psi + \frac{1}{2}\left(\frac{\epsilon}{l} - N^{\varphi}\right) f^{-1}\gamma_1\psi \\
& - \frac{1}{2}\left(\frac{\eta}{s} - V^{\varphi}\right) h^{-1}\sigma_1\psi \,, \qquad \text{(D.1)}\\
0 =& \partial_t \psi + \frac{\epsilon}{2l}\left(f\gamma_0 + r\left(\frac{\epsilon}{l} + N^{\varphi}\right)\gamma_2\right)\psi \\
& + \frac{\eta}{2s}\left(ih\sigma_2 - r\left(\frac{\eta}{s} + V^{\varphi}\right)\sigma_3\right)\psi \,, \qquad \text{(D.2)}\\
0 =& \partial_\varphi \psi + \frac{1}{2}\left(f\gamma_0 + r\left(\frac{\epsilon}{l} + N^{\varphi}\right)\gamma_2\right)\psi \\
& + \frac{1}{2}\left(ih\sigma_2 - r\left(\frac{\eta}{s} + V^{\varphi}\right)\sigma_3\right)\psi \,. \qquad \text{(D.3)}
\end{aligned}
$$

With $X$ and $Y$ defined in (4.96,4.97), (D.1) becomes

$$
\partial_r \psi = \frac{d}{dr}\left(-\epsilon\gamma_1 \ln X\right)\psi + \frac{d}{dr}\left(\eta\sigma_1 \ln Y\right)\psi, \qquad \text{(D.4)}
$$

and the solution of (D.4) can be written as

$$
\psi = U_X U_Y \xi \,. \qquad \text{(D.5)}
$$

Here $\xi$ is an $r$-independent spinor with $U_X$ and $U_Y$ defined in (4.92) and (4.93). Replacing (D.5) in (D.2) and (D.3), and using the properties of these projectors [1] leads to

$$\begin{aligned}
0 =& \partial_t \xi + \frac{\epsilon}{2l}\gamma_0 \left[\left(-M + \frac{\epsilon J}{l}\right)\gamma_{-\epsilon} + \gamma_\epsilon\right]\xi \\
&+ \frac{i\eta}{2s}\sigma_2 \left[\left(-W + \frac{\eta K}{s}\right)\sigma_\eta - \sigma_{-\eta}\right]\xi\,, \\
0 =& \partial_\varphi \xi + \frac{1}{2}\gamma_0 \left[\left(-M + \frac{\epsilon J}{l}\right)\gamma_{-\epsilon} + \gamma_\epsilon\right]\xi \\
&+ \frac{i}{2}\sigma_2 \left[\left(W - \frac{\eta K}{s}\right)\sigma_\eta - \sigma_{-\eta}\right]\xi\,,
\end{aligned}$$

whose solution is given by (4.91). Next, we present those solutions with well defined periodicity conditions for different values of $(Ml, J)$ and $(Wl, K)$. As the $SO(1,2)$ and $SU(2)$ sector are decoupled, we will consider in detail only the cases where $M = W$ and $|J|\,l = |K|\,s$. The remaining cases in Table 4.1 can be obtained from these in a straightforward way.

## D.1 Case $M = W = -1$; $J = K = 0$

In this case, the functions $X$ and $Y$ take the form

$$\begin{aligned}
X &= \left(\frac{r}{l} + n\right)^{1/2} \quad , \quad n = \left(\frac{r^2}{l^2} + 1\right)^{1/2} \\
Y &= \left(\frac{r}{s} + \tilde{n}\right)^{1/2} \quad , \quad \tilde{n} = \left(\frac{r^2}{s^2} + 1\right)^{1/2} ,
\end{aligned}$$

and (4.91) reduces to

$$\begin{aligned}
\psi =& \left[\left(\frac{n+1}{2}\right)^{1/2} - \epsilon\left(\frac{n-1}{2}\right)^{1/2}\gamma_1\right] \\
&\times \left(\cos\theta_{(\epsilon/l)} - \gamma_0 \sin\theta_{(\epsilon/l)}\right) \\
&\times \left[\left(\frac{\tilde{n}+1}{2}\right)^{1/2} + \eta\left(\frac{\tilde{n}-1}{2}\right)^{1/2}\sigma_1\right] \\
&\times \left(\cos\theta_{(\eta/s)} - i\sigma_2 \sin\theta_{(\eta/s)}\right)\psi_0\,. \qquad \text{(D.6)}
\end{aligned}$$

As $\psi_0$ has the form

$$\psi_0 = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} ,$$

[1] These projectors satisfy (i) $\gamma_\pm^2 = \gamma_\pm$, (ii) $\gamma_\pm\gamma_\mp = 0$, (iii) $\gamma_+ + \gamma_- = \mathbb{1}$, (iv) $\gamma_{0,2}\gamma_\pm = \gamma_\mp\gamma_{0,2}$, (v) $\gamma_1\gamma_\pm = \pm\gamma_1$, and similarly for $\sigma_\pm$.

with $a, b, c, d$ arbitrary real numbers, it can be spanned in a four dimensional basis. Therefore, there are four spinors for each value of $\epsilon$ and $\eta$ leading to a total of sixteen Killing spinors.

## D.2 Case $M = J = W = K = 0$

In this case $U_\gamma = \exp\left[-\frac{1}{2}\theta_{(\epsilon/l)}\left(\gamma_0 + \epsilon\gamma_2\right)\right]$ and $U_\sigma = \exp\left[\frac{i}{2}\theta_{(\eta/s)}\left(\sigma_2 - i\eta\sigma_3\right)\right]$. Since $(\gamma_0 + \epsilon\gamma_2)$ is nilpotent, we can write $U_\gamma = \mathbb{1} - \frac{1}{2}\theta_{(\epsilon/l)}\left(\gamma_0 + \epsilon\gamma_2\right)$, and similarly for $U_\sigma$. Hence, in order to get rid of the linear dependence of $\psi$ in $\theta_{(\epsilon/l)}$ and $\theta_{(\eta/s)}$, $\psi_0$ must be in the kernel of $(\gamma_0 + \epsilon\gamma_2)$ and $(\sigma_2 - i\eta\sigma_3)$, i.e.,

$$\left(\gamma_0 + \epsilon\gamma_2\right)\psi_0 = 0 = \psi_0\left(\sigma_2 - i\eta\sigma_3\right) ,$$

which is satisfied provided $\psi_0$ is one of the eigenvector of $\gamma_1$ and $\sigma_1$ depending on $\epsilon$ and $\eta$. Hence $\psi_0$ can have the form

$$\psi_0^{(\epsilon,\eta)} = \begin{pmatrix} 1 \\ -\epsilon \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \eta \end{pmatrix} ,$$

As in this case $X = \left(\frac{2r}{l}\right)^{1/2}$ and $Y = \left(\frac{2r}{s}\right)^{1/2}$, we obtain

$$\psi = \frac{2r}{\sqrt{ls}}\psi_0^{(\epsilon,\eta)} . \tag{D.7}$$

Therefore, in this case there are four Killing spinors, one for each value of $\epsilon$ and $\eta$.

## D.3 Case $M, W > 0$; $M = |J|/l$, $W = |K|/y$

Let us consider the first the option $M = J/l$, $W = K/y$. Then, (4.102) and (4.103) take the form

$$\left[\gamma_0\left(\left(-M + \frac{\epsilon J}{l}\right)\gamma_{-\epsilon} + \gamma_\epsilon\right)\right]^2 = M\left(1 - \epsilon\right) ,$$
$$\left[-i\sigma_2\left(\left(W - \frac{\eta K}{s}\right)\sigma_\eta + \sigma_{-\eta}\right)\right]^2 = W\left(1 - \eta\right) .$$

Nilpotency is achieved in this case for $\epsilon = \eta = 1$ leading to $\psi = U_X^{(+)}U_\gamma^{(+)}U_\sigma^{(+)}U_Y^{(+)}\psi_0$, where $U_X^{(+)} = U_X|_{\epsilon=+1}$ and $U_Y^{(+)} = U_Y|_{\eta=+1}$. Since $\theta_{(1/l)}\gamma_0\gamma_+$ and $\theta_{(1/s)}\sigma_-\sigma_2$ are nilpotent,

$$U_\gamma^{(+)} = 1 - \theta_{(1/l)}\gamma_0\gamma_+, \text{ and } U_\sigma^{(+)} = 1 + i\theta_{(1/s)}\sigma_-\sigma_2.$$

Hence, $\psi_0$ must be in the kernel of $\gamma_+$ and $\sigma_-$,

$$\gamma_+\psi_0 = 0 = \psi_0\sigma_- \,,$$

which is satisfied by

$$\psi_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

As in this case $X = \sqrt{\frac{2r}{l} - \frac{Ml}{r}}, Y = \sqrt{\frac{2r}{s} - \frac{Ws}{r}}$, we finally arrive to

$$\psi = \sqrt{\left(\frac{2r}{l} - \frac{Ml}{r}\right)\left(\frac{2r}{s} - \frac{Ws}{r}\right)}\psi_0 \,. \tag{D.8}$$

Therefore, in this case there is only one Killing spinor. A similar analysis can be done for all the possible particular cases of $|J| = Ml$ and $|K| = Ws$ leading essentially to the same result. Hence, the extreme case has always only one well-defined Killing spinor.

# Appendix E

# Spinor decompositions

## E.1 Two-component form decomposition

For the sake of simplicity, and because of the chiral symmetry in the massless case in Chapter 6, it is better to work with the two-component spinors (left chiral-component) instead of the four-component formalism. The two-component four vector spinors $\Psi_\mu$ and $\Psi^\dagger_\mu$ can be defined as the eigenvectors of the left-projection operator $P_L = \frac{1}{2}(1+\gamma_5)$, i.e.,

$$\begin{aligned} P_L \psi_\mu &= \begin{pmatrix} \Psi_\mu \\ 0 \end{pmatrix} \\ \psi^\dagger_\mu P_L &= \begin{pmatrix} \Psi^\dagger_\mu & 0 \end{pmatrix} . \end{aligned}$$

## E.2 Spin projector operators

It is useful to decompose the left-handed spinor $\Psi^\alpha_i$ into the 3/2-spin and 1/2- spin components. This can be done using the *spin projector operators* introduced by Frondsal [144, 145]. One can define the following operators

$$\begin{aligned} (P^i_{3/2j})^\alpha{}_\beta &= \left( \delta^i_j \delta^\alpha_\beta - \frac{1}{3}(\sigma^i \sigma_j)^\alpha{}_\beta \right) , \\ (P^i_{1/2j})^\alpha{}_\beta &= \frac{1}{3}(\sigma^i \sigma_j)^\alpha{}_\beta . \end{aligned} \tag{E.1}$$

We can verify by inspection that

$$\begin{aligned}
(P^i_{3/2j})^\alpha{}_\beta (P^j_{3/2k})^\beta{}_\gamma &= (P^i_{3/2k})^\alpha{}_\gamma \ , \\
(P^i_{1/2j})^\alpha{}_\beta (P^j_{1/2k})^\beta{}_\gamma &= (P^i_{1/2k})^\alpha{}_\gamma \ , \\
(P^i_{3/2j})^\alpha{}_\beta (P^j_{1/2k})^\beta{}_\gamma &= (P^i_{1/2j})^\alpha{}_\beta (P^j_{3/2k})^\beta{}_\gamma = 0 \text{ (Orthogonality)} \ , \\
(P^i_{3/2j})^\alpha{}_\beta + (P^i_{1/2j})^\alpha{}_\beta &= \delta^i_j \delta^\alpha_\beta \text{ (Completeness)} \ , \\
(P^i_{3/2j})^\alpha{}_\beta (\sigma^j)^\beta{}_\gamma &= (\sigma_i)^\alpha{}_\beta (P^i_{3/2j})^\beta{}_\alpha = 0 \ .
\end{aligned}$$

These properties show that (E.1) are indeed projectors operators. We can observe that, as projectors, they have eigenvalues 0 or 1. Indeed, suppose a projector $P$ has a non-null eigenvector $v$ with eigenvalue $\lambda$, then using $P^2 = P$, we have $\lambda^2 v = P^2(v) = P(v) = \lambda v$, but, as $v \neq \vec{0}$, is necessarily true that $\lambda^2 = \lambda$, which proves the above claim. Therefore, the trace of each projector tell us the dimension number they span. We have then

$$\begin{aligned}
(P^i_{3/2i})^\alpha{}_\alpha &= 4 \ , \\
(P^i_{1/2i})^\alpha{}_\alpha &= 2 \ .
\end{aligned}$$

This means indeed $(P_{3/2})$ projects over a vectorial space of dimension four (3/2−spin field) and $(P_{1/2})$ over a vectorial space of dimension two (1/2−spin field). With the completeness property, we can decompose the left-handed fermions as

$$\begin{aligned}
\Psi^\dagger_{i\alpha} &= \Psi^\dagger_{j\beta} \delta^j_i \delta^\beta_\alpha = \Psi^\dagger_{j\beta} \left[ (P^j_{3/2i})^\beta{}_\alpha + (P^j_{1/2i})^\beta{}_\alpha \right] = \Psi^\dagger_{3/2i\alpha} + \Psi^\dagger_{1/2i\alpha} \ , \\
\Psi^\alpha_i &= \delta^i_j \delta^\alpha_\beta \Psi^\beta_j = \left[ (P^j_{3/2i})^\alpha{}_\beta + (P^j_{1/2i})^\alpha{}_\beta \right] \Psi^\beta_j = \Psi^\alpha_{3/2i} + \Psi^\alpha_{1/2i} \ .
\end{aligned}$$

We can split also some operators into spinor projectors components. For instance,

$$\epsilon^{ijk} (\sigma_j)^\alpha{}_\beta = -i (P^{ik}_{3/2})^\alpha{}_\beta + 2i (P^{ik}_{1/2})^\alpha{}_\beta \ . \tag{E.2}$$

## E.3 Transversal and longitudinal decomposition

Besides the left-handed–right-handed and 3/2−spin–1/2−spin decompositions, we can distinguish the transversal and longitudinal part of a 3/2−spin as

$$\begin{aligned}
\Psi^\dagger_{i\alpha} &= \Psi^\dagger_{3/2Ti\alpha} + \Psi^\dagger_{3/2Li\alpha} \ , \\
\Psi^\alpha_i &= \Psi^\alpha_{3/2Ti} + \Psi^\alpha_{3/2Li} \ ,
\end{aligned}$$

where $\partial^i \Psi^\dagger_{3/2Ti\alpha} = \partial^i \Psi^\alpha_{3/2Ti} = 0$ and, locally $\Psi^\dagger_{3/2Li\alpha} = \partial_i \Theta^\dagger_\alpha$ and $\Psi^\alpha_{3/2Li} = \partial_i \Theta^\alpha$.

# Appendix F

# Constraint brackets of the massless Rarita-Schwinger theory

## F.1 Gauged Massless Rarita-Schwinger theory

In this section of the Appendix, we will write in more detail the constraint Poisson brackets for the gauged RS theory given in Section (7.2.2). Demanding preservation in time of the primary constraints $\phi^0$, $\chi^{\dagger 0}_\alpha$, $\chi^{0\alpha}$ (7.9), it is obtained the secondary constraints (7.11). The non-vanishing Poisson brackets between them are,

$$\begin{aligned}
\left\{k(x), \chi^{i\alpha}(y)\right\} &= -\frac{ig}{2}\epsilon^{ijk}(\sigma_j)^\alpha{}_\beta \Psi^\beta_k(x)\delta^{(3)}(x-y) \, , \\
\left\{k(x), \chi^{\dagger i}_\alpha(y)\right\} &= -\frac{ig}{2}\epsilon^{ijk}\Psi^\dagger_{j\beta}(x)(\sigma_k)^\beta{}_\alpha \delta^{(3)}(x-y) \, , \\
\left\{K^\alpha(x), \chi^{\dagger i}_\beta(y)\right\} &= \frac{1}{2}\epsilon^{ijk}(\sigma_j)^\alpha{}_\beta \overrightarrow{D}^{(x)}_k \delta^{(3)}(x-y) \, , \\
\left\{K^\dagger_\alpha(x), \chi^{i\beta}(y)\right\} &= \frac{1}{2}\epsilon^{ijk}\delta^{(3)}(x-y)\overleftarrow{D}^{(x)}_j (\sigma_k)^\beta{}_\alpha \, , \\
\left\{k(x), K^\alpha(y)\right\} &= \frac{ig}{2}\epsilon^{ijk}(\sigma_i)^\alpha{}_\beta \Psi^\beta_j(y)\partial^{(x)}_k \delta^{(3)}(x-y) \, , \\
\left\{k(x), K^\dagger_\alpha(y)\right\} &= \frac{ig}{2}\epsilon^{ijk}\Psi^\dagger_i(y)(\sigma_j)^\beta{}_\alpha \partial^{(x)}_k \delta^{(3)}(x-y) \, .
\end{aligned} \tag{F.1}$$

Preservation in time of the rest of primary constraints in (7.9), determines the Lagrange multipliers $\Lambda^{\dagger}_{i\alpha}$ and $\Lambda_i^{\alpha}$ as,

$$
\begin{aligned}
\dot{\chi}^{i\alpha}(x) &= \left\{\chi^{i\alpha}(x), H_T\right\} = \frac{1}{2}\epsilon^{ijk}\left(\overrightarrow{D}_j\Psi_k^{\alpha} + (\sigma_j)^{\alpha}{}_{\beta}(\overrightarrow{D}_k\Psi_0^{\beta} + \frac{ig}{2}A_0\Psi_k^{\beta} - \Lambda_k^{\beta})\right) \approx 0 \\
&\Longrightarrow \Lambda_i^{\alpha} \approx igA_0\Psi_i^{\alpha} + \overrightarrow{D}_i\Psi_0^{\alpha} + i\epsilon^{ijk}\overrightarrow{D}_j\Psi_k^{\alpha}\ , \\
\dot{\chi}^{\dagger i}_{\alpha}(x) &= \left\{\chi^{\dagger i}_{\alpha}(x), H_T\right\} = \frac{1}{2}\epsilon^{ijk}\left(\Psi^{\dagger}_{k\alpha}\overleftarrow{D}_j + (\Psi^{\dagger}_{0\beta}\overleftarrow{D}_k + \frac{ig}{2}A_0\Psi^{\dagger}_{j\beta} + \Lambda^{\dagger}_{j\beta})(\sigma_j)^{\beta}{}_{\alpha}\right) \approx 0 \\
&\Longrightarrow \Lambda^{\dagger}_{i\alpha} \approx -igA_0\Psi^{\dagger}_{i\alpha} + \Psi^{\dagger}_{0\alpha}\overleftarrow{D}_i - i\epsilon^{ijk}\Psi^{\dagger}_{k\alpha}\overleftarrow{D}_j\ .
\end{aligned}
$$

In fact, as the action (7.1) is of first order, it is linear in time derivative. Therefore, the Lagrange multipliers $\Lambda^{\dagger}_{i\alpha}$ and $\Lambda_i^{\alpha}$ are $\dot{\Psi}^{\dagger}_{i\alpha}$ and $\dot{\Psi}_i^{\alpha}$, respectively. We now descent to the next level of the Dirac procedure demanding preservation in time of the secondary constraint, obtaining the tertiary constraints

(7.12), whose non-vanishing Poisson brackets are

$$
\begin{aligned}
\left\{V^\alpha(x), \chi^{\dagger 0}_\beta(y)\right\} &= \frac{ig}{4}\epsilon^{ijk}(\sigma_i)^\alpha_\beta F_{jk}\delta^{(3)}(x-y)\ , \\
\left\{V^\dagger_\alpha(x), \chi^{0\beta}(y)\right\} &= \frac{ig}{4}\epsilon^{ijk}F_{ij}(\sigma_k)^\beta_\alpha\delta^{(3)}(x-y)\ , \\
\left\{V^\alpha(x), \chi^{\dagger i}_\beta(y)\right\} &= -\frac{ig}{4}\epsilon^{ijk}F_{jk}\delta^\alpha_\beta\delta^{(3)}(x-y) - 2\epsilon^{ijk}(\sigma_j)^\beta_\alpha\pi_k\delta^{(3)}(x-y)\ , \\
\left\{V^\dagger_\alpha(x), \chi^{i\beta}(y)\right\} &= -\frac{ig}{4}\epsilon^{ijk}F_{jk}\delta^\beta_\alpha\delta^{(3)}(x-y) - 2\epsilon^{ijk}\pi_j(\sigma_k)^\beta_\alpha\delta^{(3)}(x-y)\ , \\
\left\{V^\alpha(x), k(y)\right\} &= \frac{ig}{2}\epsilon^{ijk}\Psi^\alpha_k(x)\partial^{(y)}_i\partial^{(x)}_j\delta^{(3)}(x-y) + 2\epsilon^{ijk}(\sigma_i)^\alpha_\beta\Psi^\beta_0\partial^{(x)}_j\partial^{(y)}_k\delta^{(3)}(x-y)\ , \\
\left\{V^\dagger_\alpha(x), k(y)\right\} &= \frac{ig}{2}\epsilon^{ijk}\Psi^\dagger_{i\alpha}(x)\partial^{(x)}_j\partial^{(y)}_k\delta^{(3)}(x-y) + 2\epsilon^{ijk}\Psi^\dagger_{0\beta}(x)(\sigma_k)^\beta_\alpha\partial^{(y)}_i\partial^{(x)}_j\delta^{(3)}(x-y)\ , \\
\left\{V^\alpha(x), K^\beta(y)\right\} &= -\frac{g^2}{4}\left(\vec\sigma\times\vec\Psi\right)^\alpha\cdot\left(\vec\sigma\times\vec\Psi\right)^\beta\delta^{(3)}(x-y)\ , \\
\left\{V^\dagger_\alpha(x), K^\beta(y)\right\} &= -\frac{g^2}{4}\left(\vec\Psi^\dagger\times\vec\sigma\right)_\alpha\cdot\left(\vec\sigma\times\vec\Psi\right)^\beta\delta^{(3)}(x-y)\ , \\
\left\{V^\alpha(x), K^\dagger_\beta(y)\right\} &= -\frac{g^2}{4}\left(\vec\sigma\times\vec\Psi\right)^\alpha\cdot\left(\vec\Psi^\dagger\times\vec\sigma\right)_\beta\delta^{(3)}(x-y)\ , \qquad \text{(F.2)} \\
\left\{V^\dagger_\alpha(x), K^\dagger_\beta(y)\right\} &= -\frac{g^2}{4}\left(\vec\Psi^\dagger\times\vec\sigma\right)_\alpha\cdot\left(\vec\Psi^\dagger\times\vec\sigma\right)_\beta\delta^{(3)}(x-y)\ , \\
\left\{V^\alpha(x), V^\beta(y)\right\} &= -\frac{g^2}{4}(\vec\sigma\times\vec\Psi(x))^\alpha\cdot\left(\vec\nabla^{(y)}\delta^{(3)}(x-y)\times(\vec\sigma\Psi_0(y)+\vec\Psi(y))\right)^\beta \\
&\quad +\frac{g^2}{4}\left(\vec\nabla^{(x)}\delta^{(3)}(x-y)\times(\vec\sigma\Psi_0(x)+\vec\Psi(x))\right)^\alpha\cdot(\vec\sigma\times\vec\Psi(y))^\beta \\
\left\{V^\alpha(x), V^\dagger_\beta(y)\right\} &= -\frac{g^2}{4}\left(\vec\nabla^{(x)}\delta^{(3)}(x-y)\times(\vec\sigma\Psi_0(x)-\vec\Psi(x))\right)^\alpha\cdot(\vec\Psi^\dagger(y)\times\vec\sigma)_\beta \\
&\quad -\frac{g^2}{4}(\vec\sigma\times\vec\Psi(x))^\alpha\cdot\left(\vec\nabla^{(y)}\delta^{(3)}(x-y)\times(\Psi^\dagger_0(y)\vec\sigma-\vec\Psi^\dagger(y))\right)_\beta \\
\left\{V^\dagger_\alpha(x), V^\dagger_\beta(y)\right\} &= \frac{g^2}{4}\left(\vec\nabla^{(x)}\delta^{(3)}(x-y)\times(\Psi^\dagger_0(x)\vec\sigma-\vec\Psi^\dagger(x))\right)_\alpha\cdot(\vec\Psi^\dagger(y)\times\vec\sigma)_\beta \\
&\quad -\frac{g^2}{4}(\vec\Psi^\dagger(x)\times\vec\sigma)_\alpha\cdot\left(\vec\nabla^{(y)}\delta^{(3)}(x-y)\times(\Psi^\dagger_0(y)\vec\sigma-\vec\Psi^\dagger(y))\right)_\beta
\end{aligned}
$$

Demanding preservation of the tertiary constraints (7.12),

$$
\begin{aligned}
\dot{V}^{\alpha}(x) &= -2[\vec{\nabla}\times\vec{\pi}\cdot\vec{\Psi}]^{\alpha}+2[\vec{\nabla}\times\vec{\pi}\cdot\vec{\sigma}\Psi_0]^{\alpha}-\vec{\Sigma}\cdot\vec{\Lambda}^{\alpha}-2[\vec{\pi}\cdot\vec{\sigma}\times\vec{\Lambda}]^{\alpha}+(\vec{\Sigma}\cdot\vec{\sigma}\Lambda_0)^{\alpha} \\
&\quad \left[-\vec{\nabla}\times\vec{\Sigma}+ig\left(\vec{\Psi}^{\dagger}\times\vec{\sigma}\Psi_0+\Psi_0^{\dagger}\vec{\sigma}\times\vec{\Psi}-\vec{\Psi}^{\dagger}\times\vec{\Psi}\right)\right]\cdot(\vec{\sigma}\times\vec{\Psi})^{\alpha}\approx 0 \\
&\quad \Longrightarrow \Lambda_0^{\alpha}\approx\frac{\vec{\Sigma}\cdot(\vec{\sigma})^{\alpha}_{\beta}}{\vec{\Sigma}\cdot\vec{\Sigma}}\left(\begin{array}{c} 2[\vec{\nabla}\times\vec{\pi}\cdot\vec{\Psi}]^{\beta}-2[\vec{\nabla}\times\vec{\pi}\cdot\vec{\sigma}\Psi_0]^{\beta}+\vec{\Sigma}\cdot\vec{\Lambda}^{\alpha}+2[\vec{\pi}\cdot\vec{\sigma}\times\vec{\Lambda}]^{\beta}+ \\ \left[\vec{\nabla}\times\vec{\Sigma}-ig\left(\vec{\Psi}^{\dagger}\times\vec{\sigma}\Psi_0+\Psi_0^{\dagger}\vec{\sigma}\times\vec{\Psi}-\vec{\Psi}^{\dagger}\times\vec{\Psi}\right)\right]\cdot(\vec{\sigma}\times\vec{\Psi})^{\beta}\end{array}\right) \\
\dot{V}^{\dagger}_{\alpha}(x) &= 2[\vec{\Psi}^{\dagger}\cdot\vec{\nabla}\times\vec{\pi}]_{\alpha}-2[\Psi_0^{\dagger}\vec{\sigma}\cdot\vec{\nabla}\times\vec{\pi}]_{\alpha}+\vec{\Lambda}^{\dagger}_{\alpha}\cdot\vec{\Sigma}-2[\vec{\Lambda}^{\dagger}\times\vec{\sigma}\cdot\vec{\pi}]_{\alpha}-(\Lambda_0^{\dagger}\vec{\sigma})_{\alpha}\cdot\vec{\Sigma} \\
&\quad (\vec{\Psi}^{\dagger}\times\vec{\sigma})_{\alpha}\cdot\left[-\vec{\nabla}\times\vec{\Sigma}+ig\left(\vec{\Psi}_0^{\dagger}\vec{\sigma}\times\vec{\Psi}+\vec{\Psi}^{\dagger}\times\vec{\sigma}\Psi_0-\vec{\Psi}^{\dagger}\times\vec{\Psi}\right)\right]\approx 0 \\
&\quad \Longrightarrow \Lambda^{\dagger}_{0\alpha}\approx\left(\begin{array}{c} 2[\vec{\Psi}^{\dagger}\cdot\vec{\nabla}\times\vec{\pi}]_{\beta}-2[\Psi_0^{\dagger}\vec{\sigma}\cdot\vec{\nabla}\times\vec{\pi}]_{\beta}+\vec{\Lambda}^{\dagger}_{\beta}\cdot\vec{\Sigma}-2[\vec{\Lambda}^{\dagger}\times\vec{\sigma}\cdot\vec{\pi}]_{\beta}+ \\ (\vec{\Psi}^{\dagger}\times\vec{\sigma})_{\beta}\cdot\left[-\vec{\nabla}\times\vec{\Sigma}+ig\left(\vec{\Psi}_0^{\dagger}\vec{\sigma}\times\vec{\Psi}+\vec{\Psi}^{\dagger}\times\vec{\sigma}\Psi_0-\vec{\Psi}^{\dagger}\times\vec{\Psi}\right)\right]\end{array}\right)\frac{(\vec{\sigma})^{\beta}_{\alpha}\cdot\vec{\Sigma}}{\vec{\Sigma}\cdot\vec{\Sigma}} .
\end{aligned}
$$

We can check that no extra constraints are produced, meaning the algorithm to find constraints is over at this level [26, 121].

## F.2 Extended gauged Rarita-Schwinger theory

Now, we show the constraints and their Poisson brackets regarding to the extended gauged RS theory presented in Chapter 8. Demanding preservation in time of the primary constraints $\phi^0$, $\chi^{\dagger 0}_{\alpha}$, $\chi^{0\alpha}$ (8.11), it is obtained the secondary constraints (8.13). The non-vanishing Poisson brackets between them are,

$$
\begin{aligned}
\left\{k(x),\chi^{i\alpha}(y)\right\} &= -\frac{ig}{2}\epsilon^{ijk}(\sigma_j)^{\alpha}{}_{\beta}\Psi^{\beta}_k(x)\delta^{(3)}(x-y)\,, \\
\left\{k(x),\chi^{\dagger i}_{\alpha}(y)\right\} &= -\frac{ig}{2}\epsilon^{ijk}\Psi^{\dagger}_{j\beta}(x)(\sigma_k)^{\beta}{}_{\alpha}\delta^{(3)}(x-y)\,, \\
\left\{k(x),\varphi^{\alpha}(y)\right\} &= \frac{ig}{4}\epsilon^{ijk}(\sigma_k)_{\beta}{}^{\alpha}\xi^{\beta}(x)\partial^{(x)}_i\partial^{(y)}_j\delta^{(3)}(x-y)+\frac{g^2}{4}\epsilon^{ijk}F_{ij}(\sigma_k)_{\beta}{}^{\alpha}\xi^{\beta}(x)\delta^{(3)}(x-y)\,, \\
\left\{k(x),\varphi^{\dagger}_{\alpha}(y)\right\} &= -\frac{ig}{4}\epsilon^{ijk}\xi^{\dagger}_{\beta}(x)(\sigma_k)^{\beta}{}_{\alpha}\partial^{(x)}_i\partial^{(y)}_j\delta^{(3)}(x-y)+\frac{g^2}{4}\epsilon^{ijk}F_{ij}\xi^{\dagger}_{\beta}(x)(\sigma_k)^{\beta}{}_{\alpha}\delta^{(3)}(x-y)\,, \\
\left\{k(x),K^{\alpha}(y)\right\} &= -\frac{ig}{2}\epsilon^{ijk}(\sigma_i)^{\alpha}{}_{\beta}\Psi^{\beta}_k(y)\partial^{(x)}_j\delta^{(3)}(x-y)-\frac{ig}{2}\epsilon^{ijk}(\sigma_i)^{\alpha}{}_{\beta}\xi^{\beta}(y)\partial^{(x)}_j\partial^{(y)}_k\delta^{(3)}(x-y)\,, \\
\left\{k(x),K^{\dagger}_{\alpha}(y)\right\} &= \frac{ig}{2}\epsilon^{ijk}\Psi^{\dagger}_i(y)(\sigma_j)^{\beta}{}_{\alpha}\partial^{(x)}_k\delta^{(3)}(x-y)+\frac{ig}{2}\epsilon^{ijk}\xi^{\dagger}_{\beta}(y)(\sigma_i)^{\beta}{}_{\alpha}\partial^{(x)}_j\partial^{(y)}_k\delta^{(3)}(x-y)\,, \\
\left\{K^{\alpha}(x),\chi^{\dagger i}_{\beta}(y)\right\} &= \frac{1}{2}\epsilon^{ijk}(\sigma_j)^{\alpha}{}_{\beta}\overrightarrow{D}^{(x)}_k\delta^{(3)}(x-y)\,, \qquad\qquad (F.3) \\
\left\{K^{\dagger}_{\alpha}(x),\chi^{i\beta}(y)\right\} &= \frac{1}{2}\epsilon^{ijk}\delta^{(3)}(x-y)\overleftarrow{D}^{(x)}_j(\sigma_k)^{\beta}{}_{\alpha}\,, \\
\left\{K^{\alpha}(x),\varphi^{\dagger}_{\beta}(y)\right\} &= \frac{ig}{4}\epsilon^{ijk}(\sigma_i)^{\alpha}{}_{\beta}F_{jk}(x)\delta^{(3)}(x-y)\,, \\
\left\{K^{\dagger}_{\alpha}(x),\varphi^{\beta}(y)\right\} &= \frac{ig}{4}\epsilon^{ijk}F_{ij}(x)(\sigma_k)^{\beta}{}_{\alpha}\delta^{(3)}(x-y)\,.
\end{aligned}
$$

Preservation in time of the rest of primary constraints $\chi^{\dagger i}, \chi^i, \varphi^\dagger, \varphi$ determines a possible set of Lagrange multipliers.

$$\begin{aligned}
\dot{\chi}^{i\alpha}(x) &= \{\chi^{i\alpha}(x), H_T\} \Longrightarrow \Lambda_i^\alpha \approx igA_0\Psi_i^\alpha + \overrightarrow{D}_i\Psi_0^\alpha + i\epsilon^{ijk}\overrightarrow{D}_j\Psi_k^\alpha + ig\left[E^i - \frac{i}{2}\Sigma^i\right] , \\
\dot{\chi}_\alpha^{\dagger i}(x) &= \left\{\chi_\alpha^{\dagger i}(x), H_T\right\} \Longrightarrow \Lambda_{i\alpha}^\dagger \approx -igA_0\Psi_{i\alpha}^\dagger + \Psi_{0\alpha}^\dagger\overleftarrow{D}_i + i\epsilon^{ijk}\Psi_{j\alpha}^\dagger\overleftarrow{D}_k + ig\left[E^i + \frac{i}{2}\Sigma^i\right] , \\
\dot{\varphi}^\alpha(x) &= \{\varphi^\alpha(x), H_T\} \\
&\Longrightarrow \lambda^\alpha \approx igA_0\xi^\alpha + \Psi_0^\alpha + \frac{1}{\vec{\Sigma}\cdot\vec{\Sigma}}\left[2(\vec{E}\times\vec{\Sigma})\cdot\vec{\vartheta} + 2i(\vec{E}\cdot\vec{\Sigma})(\vec{\sigma}\cdot\vec{\vartheta}) - 2i(\vec{E}\cdot\vec{\sigma})(\vec{\Sigma}\cdot\vec{\vartheta}) + (\vec{\Sigma}\cdot\vec{\sigma})(\vec{\Sigma}\cdot\vec{\vartheta})\right]^\alpha , \\
\dot{\varphi}_\alpha^\dagger(x) &= \left\{\varphi_\alpha^\dagger(x), H_T\right\} \\
&\Longrightarrow \lambda_\alpha^\dagger \approx -igA_0\xi_\alpha^\dagger + \Psi_{0\alpha}^\dagger + \frac{1}{\vec{\Sigma}\cdot\vec{\Sigma}}\left[-2(\vec{E}\times\vec{\Sigma})\cdot\vec{\vartheta}^\dagger + 2i(\vec{E}\cdot\vec{\Sigma})(\vec{\vartheta}^\dagger\cdot\vec{\sigma}) - 2i(\vec{\Sigma}\cdot\vec{\vartheta}^\dagger)(\vec{\sigma}\cdot\vec{E}) - (\vec{\Sigma}\cdot\vec{\vartheta}^\dagger)(\vec{\sigma}\cdot\vec{\Sigma})\right]_\alpha ,
\end{aligned}$$

where we used the properties (A.8-A.10). We defined also, to solve the last two Lagrange multipliers $\lambda^\dagger, \lambda$, the quantities

$$\begin{aligned}
\vartheta_i^\alpha &= \Psi_i^\alpha - \overrightarrow{D}_i\xi^\alpha , \\
\vartheta_{i\alpha}^\dagger &= \Psi_{i\alpha}^\dagger - \xi_\alpha^\dagger\overleftarrow{D}_i ,
\end{aligned}$$

assuming there is a frame where $\vec{\Sigma} \neq 0$, in order to apply Lemma 7.2.1.

# Appendix G

# From $(2+1)$-dimensional Lagrangian to static two-dimensional Hamiltonian of $\pi$ electrons

To be specific, let us focus on the case on negative constant Gaussian curvature, although similar formulae must hold the pure strain/fully flat case. The coordinates are

$$\begin{aligned} T &= e^{t/r}\sqrt{e^{2\sigma(\tilde{y})}+r^2} \\ X &= e^{t/r}e^{\sigma(\tilde{y})}\cos\tilde{x} \\ Y &= e^{t/r}e^{\sigma(\tilde{y})}\sin\tilde{x} \end{aligned}$$

and

$$\Sigma = -\frac{1}{2}\ln(T^2-X^2-Y^2)+\ln r = -t/r\ .$$

To write the Hamiltonian in the $\tilde{q}$ coordinates, we have to consider the action (10.16)

$$S = i\int dT dX dY e^{2\Sigma}\left[\bar{\psi}\gamma^0(\partial_T+\Sigma_T)\psi+\bar{\psi}\gamma^1(\partial_X+\Sigma_X)\psi+\bar{\psi}\gamma^2(\partial_Y+\Sigma_Y)\psi\right]\ .$$

We have to write this in the $\tilde{q}^\mu$ coordinates. We need the Jacobian

$$\|\frac{\partial Q^\mu}{\partial\tilde{q}^\nu}\| = -e^{3t/r}\,\frac{r\ e^{2\sigma(\tilde{y})}\ \sigma_{\tilde{y}}(\tilde{y})}{\sqrt{e^{2\sigma(\tilde{y})}+r^2}} \qquad \text{(G.1)}$$

the three terms

$$
\begin{aligned}
\Sigma_T &= -\frac{T}{T^2 - X^2 - Y^2} = -\frac{e^{-t/r}}{r^2}\sqrt{e^{2\sigma(\tilde{y})} + r^2} \\
\Sigma_X &= \frac{X}{T^2 - X^2 - Y^2} = \frac{e^{-t/r}}{r^2} e^{\sigma(\tilde{y})} \cos\tilde{x} \\
\Sigma_Y &= \frac{Y}{T^2 - X^2 - Y^2} = \frac{e^{-t/r}}{r^2} e^{\sigma(\tilde{y})} \sin\tilde{x}
\end{aligned}
$$

and to re-express the derivatives, e.g., $\partial_X = \tilde{x}_X \partial_{\tilde{x}} + \tilde{y}_X \partial_{\tilde{y}}$, etc., where, as usual, $\sigma_{\tilde{y}}(\tilde{y}) = \partial_{\tilde{y}}\sigma(\tilde{y})$, $\tilde{x}_X = \partial\tilde{x}/\partial X$, etc. Then we use

$$
S = \int dt d\tilde{x} d\tilde{y} \mathcal{L}(\tilde{q}) = \int dt d\tilde{x} d\tilde{y} \left[ \left\| \frac{\partial Q}{\partial \tilde{q}} \right\| (\pi_\psi \partial_T \psi)(\tilde{q}) - \mathcal{H} \right],
$$

from which we can read off the Hamiltonian $H = \int d\tilde{x} d\tilde{y} \mathcal{H}(\tilde{q})$. The final expression is

$$
\begin{aligned}
H &= -i\hbar v_F \int d\tilde{x} d\tilde{y} \left( -\frac{r\, e^{2\sigma}\, \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}} \right) \left[ \psi^\dagger \left( -\frac{\sqrt{e^{2\sigma} + r^2}}{r^2} \right) \psi \right. \\
&+ \psi^\dagger \gamma^0 \gamma^1 \Big( -e^{-\sigma} \sin\tilde{x}\, \partial_{\tilde{x}} + \frac{e^{-\sigma}}{\sigma_{\tilde{y}}} \frac{e^{2\sigma} + r^2}{r^2} \cos\tilde{x}\, \partial_{\tilde{y}} + \frac{e^{\sigma}}{r^2} \cos\tilde{x} \Big) \psi \\
&+ \left. \psi^\dagger \gamma^0 \gamma^2 \Big( e^{-\sigma} \cos\tilde{x}\, \partial_{\tilde{x}} + \frac{e^{-\sigma}}{\sigma_{\tilde{y}}} \frac{e^{2\sigma} + r^2}{r^2} \sin\tilde{x}\, \partial_{\tilde{y}} + \frac{e^{\sigma}}{r^2} \sin\tilde{x} \Big) \psi \right]
\end{aligned}
$$

where we have reintroduced $\hbar$ and the Fermi velocity $v_F$. Note that nothing depends on $t$, as it must be.

The formula above for $H$ gives a field of the type $\psi^\dagger\psi$ in (G.2)

$$
\phi = \frac{1}{r}\, e^{2\sigma} \sigma_{\tilde{y}} \,,
$$

the $A_1$ and $A_2$ fields in the non-derivative terms of (G.2) and (G.2) (here we identify $A_1$ and $A_2$ according the Pauli matrices $\tau^1$ and $\tau^2$)

$$
A_1 = -\frac{1}{r}\, \frac{e^{3\sigma}\, \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}}\, \cos\tilde{x} \,, \; A_2 = -\frac{1}{r}\, \frac{e^{3\sigma}\, \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma} + r^2}}\, \sin\tilde{x} \,,
$$

and the space-dependent, inhomogeneous Fermi velocity tensor

$$
\left[ v_F(\sigma(\tilde{y})) \right]_i^{\;\tilde{j}} = v_F \begin{pmatrix} v_{1\tilde{x}} & v_{1\tilde{y}} \\ v_{2\tilde{x}} & v_{2\tilde{y}} \end{pmatrix}
$$

with

$$
\begin{aligned}
v_{1\tilde{x}} &= \frac{r\ e^{\sigma}\ \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma}+r^2}}\ \sin\tilde{x} \\
v_{1\tilde{y}} &= -\frac{e^{\sigma}}{r}\ \sqrt{e^{2\sigma}+r^2}\ \cos\tilde{x} \\
v_{2\tilde{x}} &= -\frac{r\ e^{\sigma}\ \sigma_{\tilde{y}}}{\sqrt{e^{2\sigma}+r^2}}\ \cos\tilde{x} \\
v_{2\tilde{y}} &= -\frac{e^{\sigma}}{r}\ \sqrt{e^{2\sigma}+r^2}\ \sin\tilde{x}
\end{aligned}
$$

Therefore, obtaining the expression (10.25) we have in the main text.

# Appendix H

# Some details of the tight-binding computations

We start with the effective function of the hopping energy $t_i$ with respect to intercarbon distance [45],

$$t_i = -\eta \exp\left[-\beta(\frac{\mid \vec{s_i'} \mid}{\ell} - 1)\right], \tag{H.1}$$

where $\eta \simeq 2.8$ eV is the equilibrium hopping energy, $\vec{s_i'}$ (with $i = 1, 2, 3$) are the variation of the basis vectors $\vec{s_i}$, and $\beta = \left|\frac{\partial \log \eta}{\partial \log \ell}\right|$ is the Grüneisen parameter. In order to deal with inhomogeneous strain, we consider the expansion of (H.1) up to the first derivative of the strain tensor $u_{ij}(\vec{x})$, i.e.,

$$t_i = \eta\left[1 - \frac{\beta}{\ell^2}(s_i)^m u_{mn}(s_i)^n - \frac{\beta}{2\ell^2}(s_i)^k(s_i)^m(s_i)^n\partial_k u_{mn}\right], \tag{H.2}$$

where with some abuse of notation, $(s_i)^m$ stands for the $m$ component of the vector $\vec{s_i}$, the indices $k, m, n$ are contracted (dummy indices) and $\partial_k \equiv \frac{\partial}{\partial x^k}$. The tight-binding Hamiltonian, in the second quantization formalism[1], could be written as

$$H = -\sum_{\vec{x}\in L_A}\sum_{i=1}^{i=3}\left(a^\dagger(\vec{x})t_i b(\vec{x}+\vec{s_i}) + c.c.\right),$$

[1]If we use the second quantization formalism, some subtitles appear with respect to the vacuum where the creation and annihilation operators act. This subtitles will not be considerer here because the system is simple enough but, in the case of curved graphene, extra care must be taken. This is because the presence of defects on the honeycomb makes the vacuum nontrivial [31].

where $L_a$ stands for the sublattice $A$ (see Figure 9.1) and, in order to have a Hermitian Hamiltonian, we added the complex conjugate of the first term. Using the following expressions for the Fourier transforms

$$a(\vec{x}) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} a_{\vec{k}} \quad , \qquad b(\vec{x}) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} b_{\vec{k}} \quad , \qquad u^m(\vec{x}) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} u^m_{\vec{k}} \quad ,$$

we can write the Hamiltonian in the Fourier space as

$$H = -\eta \sum_{\vec{k},\vec{q}} \sum_{i=1}^{i=3} \left( b^\dagger_{\vec{k}-\vec{q}}, a^\dagger_{\vec{k}} \right) \begin{pmatrix} 0 & e^{-i(\vec{k}-\vec{q})\cdot\vec{s}_i} T^\dagger_{i,\vec{q}} \\ e^{i(\vec{k}-\vec{q})\cdot\vec{s}_i} T_{i,\vec{q}} & 0 \end{pmatrix} \begin{pmatrix} b_{\vec{k}-\vec{q}} \\ a_{\vec{k}} \end{pmatrix}, \tag{H.3}$$

where we used the symmetry property of $u_{ij}$ and defined $T_{i,\vec{q}}$ as

$$T_{i,\vec{q}} = \delta(\vec{q}) + i(s_i)^m u_{m,\vec{q}} (s_i)^n - (s_i)^j (s_i)^m (s_i)^n q_j q_m u_{n,\vec{q}}.$$

Now, we expand the Hamiltonian around one Dirac point $K_\pm$. For the sake of simplicity, we expand around $\vec{K}_+ = \left(\frac{4\pi}{3\sqrt{3}\ell}, 0\right)$, such that $\vec{k} = \vec{K}_+ + \vec{p}$. We can work a little bit more on the matrix content of (H.3) as

$$\begin{pmatrix} 0 & T^\dagger_{i,\vec{q}} e^{-i(\vec{K}_+-\vec{q})\cdot\vec{s}_i} \\ T_{i,\vec{q}} e^{i(\vec{K}_+-\vec{q})\cdot\vec{s}_i} & 0 \end{pmatrix} = \frac{i}{\ell} \vec{\sigma}\cdot\vec{s}_i \sigma_3 \left(\mathbb{I} + i\sigma_3(\vec{p}-\vec{q})\right) \begin{pmatrix} T^\dagger_{i,\vec{q}} & 0 \\ 0 & T_{i,\vec{q}} \end{pmatrix},$$

where in the last equality we made use of the identity[2] [43]

$$\begin{pmatrix} 0 & e^{-i(\vec{K}_+-\vec{q})\cdot\vec{s}_i} \\ e^{i(\vec{K}_+-\vec{q})\cdot\vec{s}_i} & 0 \end{pmatrix} = \frac{i}{\ell} \vec{\sigma} \cdot \vec{s}_i \sigma_3.$$

At this point, it will be useful to show the following identities [43]

$$\sum_{i=1}^{i=3} (s_i)^m = 0,$$

$$\frac{1}{\ell^2} \sum_{i=1}^{i=3} (s_i)^m (s_i)^n = \frac{3}{2} \delta^{mn},$$

$$\frac{1}{\ell^3} \sum_{i=1}^{i=3} (s_i)^j (s_i)^m (s_i)^n = -\frac{3}{4} K^{jmn},$$

[2] In the case of expanding around the other inequivalent Dirac point $\vec{K}_- = \left(-\frac{4\pi}{3\sqrt{3}\ell}, 0\right)$, the formula is

$$\begin{pmatrix} 0 & e^{-i(\vec{K}_+-\vec{q})\cdot\vec{s}_i} \\ e^{i(\vec{K}_+-\vec{q})\cdot\vec{s}_i} & 0 \end{pmatrix} = \frac{-i}{\ell} \vec{\sigma}^* \cdot \vec{s}_i \sigma_3$$

.

$$\frac{1}{\ell^4}\sum_{i=1}^{i=3}(s_i)^j(s_i)^m(s_i)^n(s_i)^r = \frac{3}{8}\left(\delta^{jm}\delta^{nr} + \delta^{jn}\delta^{mr} + \delta^{jr}\delta^{mn}\right),$$

The tensor $K^{jmn}$, defined in the third identity, is an invariant under the discrete $C_3$ rotations. This tensor is very important in the discussion about the anisotropy of the strain-induced gauge field. For our choice of basis vectors $\{\vec{s_i}\}$, its only nonzero components are $K^{222} = -K^{112} = -K^{121} = -K^{112} = 1$. We also note that the other three tensors are all isotropic. Making use of all this, doing some standard algebra and going back to the configurations space via the anti-Fourier transforms of $a$, $b$ and $u^m$, we end up with the following Hamiltonian for the inhomogeneous strain

$$H = -i\sum_{\vec{x}}\psi_+^\dagger\sigma^j\left(v_{jm}\partial_m + iv_F A^j - v_F\Gamma^j\right)\psi_+,$$

where $v_{jm} = v_F\left(\delta_{jm} - \frac{\beta}{4}(u_{nn}\delta_{jm} + 2u_{jm})\right)$ is the celebrated space-dependent Fermi velocity [43], $A^j = \frac{\beta}{2\ell}\epsilon^{jp}K^{pmn}u_{mn}$ is the pseudogauge field, and $\Gamma^j = \frac{\beta}{4}\left(\partial_m u_{jm} + \frac{1}{2}\partial_j u_{mm}\right)$ is a connection-like coefficient. The Fermi velocity in the unstrained graphene is $v_F = \frac{3}{2}\eta\ell$.

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