The Synergies Between Data Envelopment Analysis and Multi-Criteria Decision Aid: Case of the PROMETHEE Method

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Thèse présentée en vue de l’obtention du grade de Docteur en Sciences de l’Ingénieur sous la direction du Professeur Yves De Smet

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July 4, 2017
Acknowledgements

There are many people whom I should acknowledge for all hardship they endured in helping me during my research period to make this thesis possible.

First of all, I owe my gratitude to my family, specially my mother, who always has stood by me and pulled me by her love through in tough tasks. It is hard to thank my sister and her family enough for their care and spiritual support through all these years and to thank my brother for his distance support. I think they already know what they mean to me and how appreciated I am for their ongoing encouragements as I work towards my academic dreams.

It is difficult for me to sum up an acknowledge of my supervisor, Prof. Dr. Yves De Smet in giving me the opportunity to work on this thesis. He was patient, helpful and supportive. Although he was strict with me, his rigor and strictness help me so much to improve the level of my research. It was an honor and a great pleasure for me to work with and learn from him.

I would like to thank the members of my jury for their interest in my work: Marc Pirlot, Stijn Vansummeren, Hugues Bersini, and Johan Springael. I specially thank Marc Pirlot and Stijn Vansummeren for the annual meetings we had to discuss my progress.

I wish to appreciate my colleagues who I have met in CoDE-SMG unit for their support, their encouragements, discussions, remarks, comments and help during the course of this thesis: Karim Lidouh, Anh Vu Doan, Stefan Eppe, Dimitri Van Assche, Renaud Sarrazin, Jean-Philippe Hubinont, Jean Rosenfeld, and Issam Banamar. I would like to extend my appreciation to professors Bertrand Mareschal, Elke Hermans, Yongjun Shen, and Martin J. Geiger, and academic colleagues I have encountered Quantin Hayez, Seddigheh Babaei, and Hatice Calik who supported me to do better my thesis.

In particular, I would like to thank Adel Hatami-Marbini for his precious comments and helps in Data Envelopment Analysis. I also thank Nabi Nabiollahi for his special help in rechecking some of my MATLAB codes.

I am totally grateful to have friends who supported me from all over the world: Narjes Javadi Mottaghi who we shared our moments even from distance, Mahtab Pesaran who her friendship means a lot to me, and Younes Faghihi who supported me spiritually during my hard periods. Moreover, I would like to appreciate Danial Khatib for all funny times that we shared and all helping in computer dilemmas, Anita Golzar who encouraged me for passing the difficulties, Solmaz Javanbakhti who we laughed a lot with each other, and Bruno Zavatara who we experienced beautiful moments with each other.

Of course, I could not mention all of the friends and family members who gave me the chance to feel them beside me to go on whenever I needed it. These six years have been a long journey, but I learned and experienced things I would not have otherwise.
Abstract

For a little less than twenty years, researchers have worked on integrating Data Envelopment Analysis (DEA) and Multi-Criteria Decision Aid (MCDA). Several contributions have been done by integrating DEA with different MCDA methods to bring this field to what it is today. After studying the course of Multi-Criteria Data Envelopment Analysis (MCDEA) integration through numerous works, the future of such an attempt can be questionable. For this aim, the PROMETHEE method in MCDA has been integrated with DEA. To the best of our knowledge, this synergy has been done for the first time in this thesis.

Two synergies have been conducted: Using PROMETHEE in DEA and vice versa. The first contribution applies PROMETHEE in DEA to develop a new weight restricted DEA model. This new model has two main characteristics: more discrimination power between efficient units and engaging a priori information of decision makers in DEA. The second contribution uses both DEA and PROMETHEE to propose a new ranking technique. DEA is employed to generate a pairwise comparison matrix to be used in PROMETHEE for the purpose of ranking alternatives. The last contribution uses DEA in PROMETHEE. It presents a new algorithm to propose weights in the context of the PROMETHEE II method based on DEA. Furthermore, these two methods can be used in parallel. Comparing the results obtained from DEA and PROMETHEE in evaluating the performance of units enriches the analysis of decision-making problem by confirming the robustness of answers.

The purpose of this integration is to provide some tools to help decision makers in the process of evaluating the performance of alternatives and analyzing the multicriteria decision-making problems.
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List of Abbreviations

ARWU = Academic Ranking of World Universities
AHP = Analytical Hierarchy Process
AR = Assurance Regions
BCC = Banker, Charnes, Cooper
BFS = Basic Feasible Solution
CCA = Canonical Correlation Analysis
CCR = Charnes, Cooper, Rhodes
CI = Consistency Index
CRS = Constant Return to Scale
CSW = Common Set of Weights
CV = Coefficient of Variation
D = Dual
DEA = Data Envelopment Analysis
DM = Decision Maker
DMU = Decision Making Unit
DRS = Decreasing Returns to Scale
ELECTRE = ELimination Et Choix Traduisant la REalité/ ELimination and Choice Translating Reality
FDH = Free Disposal Hull
FMS = Flexible Manufacturing System
GAIA = Geometrical Analysis for Interactive Aid
GDEA = General Data Envelopment Analysis
GDP = Gross Domestic Product
GM = Geometric Mean
H & M = Hennes & Mauritz
HDI = Human Development Index
IMDb = Internet Movie Database
IMP = IMProved
I-O = Input-Oriented
IRS = Increasing Returns to Scale
IWR = Input Weight Restriction
LP = Linear Programming
MACBETH = Measuring Attractiveness by a Categorical Based Evaluation Technique
MAUT = Multi-Attribute Utility Theory
MCDA = Multiple Criteria Decision Aid/ Multicriteria Decision Aid
MCDEA = Multi-Criteria Data Envelopment Analysis
MPG = Miles Per Gallon
MS = Management Science
NLP = Non-Linear Programming
O-O = Output-Oriented
OR = Operations Research
OWR = Output Weight Restriction
P = Primal
PCA = Principal Component Analysis
PIIWCCR = PROMETHEE II Weight restricted CCR model
PPS = Production Possibility Set
PROMETHEE = Preference Ranking Organization METHod for Enrichment Evaluation
PTE = Pure Technical Efficient
RI = Random Index
RTS = Returns to Scale
SBM = Slack Base Measurement
SE = Scale Efficient
SEI = Sustainable Energy Index
TE = Technical Efficient
TSP = Travelling Salesman Problem
VEA = Vertex Enumeration Algorithm
VRS = Variable Return to Scale
WCCR = Weight restricted CCR
List of Notations

Chapter 1

- The optimal solution is: \( x^* = \text{argmax} \{ f(x) | x \in \mathcal{A} \} \);
- An input vector \( X_j = (x_{1j}; x_{2j}; \ldots; x_{ij}; \ldots; x_{mj})^T \); \( x_{ij} \) is the quantity of input \( i \) used by \( DMU_j \);
- An output vector \( Y_j = (y_{1j}; y_{2j}; \ldots; y_{rj}; \ldots; y_{sj})^T \); \( y_{rj} \) is the quantity of output \( r \) obtained by \( DMU_j \); where \( j = 1, 2, \ldots, n \), \( i = 1, 2, \ldots, m \) and \( r = 1, 2, \ldots, s \);
- \( DMU_j \) is the unit \( j \);
- Virtual input = \( \sum_{i=1}^{m} v_i^* X_j, j = 1, \ldots, n \);
- Virtual output = \( \sum_{r=1}^{s} u_r^* Y_j, j = 1, \ldots, n \);

where \( v^* = (v_1^*, \ldots, v_m^*) \): non-negative input optimal weight vector and \( u^* = (u_1^*, \ldots, u_s^*) \): non-negative output optimal weight vector;

- \( \geq \) in \( X_j \geq 0 \) shows a non-negative input vector; \( \exists j: x_{ij} \neq 0 \) means at least one of the input vector’s elements is not zero;
- \( \geq \) in \( Y_j \geq 0 \) shows a non-negative input vector; \( \exists j: y_{rj} \neq 0 \) means at least one of the output vector’s elements is not zero;
- \( (X, Y) \) Activity;
- \( X' \geq X \) and \( Y' \leq Y \): any activity with input no less than \( X \) in any component and with output no greater than \( Y \) in any component;

In this thesis, vectors are compared in their magnitude.

- Convex set: \( (X, Y) \in PPS, (X', Y') \in PPS \) and \( \lambda \in (0, 1) \Rightarrow \lambda (X, Y) + (1 - \lambda) (X', Y') \in PPS \);
- \( PPS_{CRS} = \{(X, Y) | X \geq \sum_{j=1}^{n} X_j \lambda_j, Y \leq \sum_{j=1}^{n} Y_j \lambda_j, \lambda_j \geq 0 \} \);
- \( PPS_{VRS} = \{(X, Y) | X \geq \sum_{j=1}^{n} X_j \lambda_j, Y \leq \sum_{j=1}^{n} Y_j \lambda_j, \lambda_j = 1, \lambda_j \geq 0 \} \);
- \( \theta \) is the efficiency score of \( DMU_j \): the optimal solution of primal LP (CCR, BCC and Additive I-O multiplier models);
- \( Z \) is the efficiency score of \( DMU_j \): the optimal solution of dual LP (CCR, BCC and Additive I-O envelopment models);
- \( u_r \) is the weight of output \( r \);
- \( v_i \) is the weight of input \( i \);
\( (Z^*, \lambda^*) \) is an optimal solution for the CCR I-O envelopment model;

\( (\phi^*, \hat{\lambda}^*) \) is optimal for the corresponding CCR O-O model;

\( \lambda_j \) the dual variable;

\( u_o \) the dual variable of BCC model;

\( s_i^- \) is the shortage quantity of input \( i \) in \( DMU_j \);

\( s_r^+ \) are and extra quantity of output \( r \) in \( DMU_j \);

\( \varepsilon > 0 \), where \( \varepsilon \) is a non-Archimedean element smaller than any positive real number;

\( Q \) is the efficiency score of the O-O multiplier BCC model;

\( (\bar{x}_o, \bar{y}_o) \) The projected point on the efficient frontier in CCR and BCC models.

Chapter 2

\( F = \{f_1, ..., f_k, ..., f_q\} \) Family of criteria;

\( \mathcal{A} = \{a_1, ..., a_j, ..., a_n\} \) Set of alternatives;

\( f_k(a) \) The evaluation of action \( a \) (alternative \( a \)) according to criterion \( j \);

\( aPb \ a \) is preferred to \( b \);

\( aIa \ a \) is indifferent to \( b \);

\( aRb \ a \) is incomparable to \( b \);

\( a_j \rightarrow P a_i: P \) is “asymmetric”;

\( a_i \rightarrow R a_i: R \) is irreflexive;

\( S = (P \cup I) \) an outranking relation;

\( aDb \leftrightarrow f_k(a) \geq f_k(b): a, be\mathcal{A}, k = \{1,2, ..., q\}: a \) dominates \( b \);

\( \exists k \in \{1,2, ..., q\}: f_k(a) > f_k(b): a, be\mathcal{A}; a \) is efficient in comparison with \( b \);

\( U(x) \) is the utility function;

\( \nu_{n,q} \) : The cells of the matrix contain estimates of the performance of each alternative on each of the criteria provided by an expert or various experts in MAUT;

\( U(a) > U(b) \leftrightarrow a > b \ (a \ is \ preferred \ to \ b) \);

\( U(a) = U(b) \leftrightarrow a = b \ (a \ is \ indifference \ to \ b) \);

\( U_k \) is the utility function of criterion \( k \): \( k = \{1,2, ..., q\} \);
- $C_{ik}; i, k = 1, 2, \ldots, q$ expresses the relative importance of the criterion $i$ over the criterion $k$ in AHP;
- $C$ the weight matrix in AHP;
- $CI$ consistency index;
- $c(a_iS_{a_j})$ Concordance index in ELECTRE;
- $d(a_iS_{a_j})$ Discordance index;
- $v$ discordance threshold;
- $\forall a_j \in A \setminus A'$ the solutions, which are not belongs to $A$;
- $s_1$ and $s_2$ concordance thresholds;
- $v_k(f_k(a_i))$ veto threshold;
- $S(a_i, a_j)$ the credibility degree;
- $q_k$ and $p_k$ are indifference and preference thresholds;
- $d_k(a_i, a_j) = f_k(a_i) - f_k(a_j)$, the differences between each pair of alternatives on each criterion in PROMETHEE;
- $P_k(a_i, a_j)$ predefined preference function;
- $P_k(d_k(a, b))$ Function of preference of one action over another;
- $\pi_k(a_i, a_j)$ unicriterion preference degrees;
- $\pi(a_i, a_j)$ outranking degree;
- $\phi(a) = \phi^+(a) - \phi^-(a)$, The net outranking flow is defined as the difference between the positive flow and the negative one;
- $\phi_k(a_j)$ the unicriterion net flow score of alternative $a_j$;
- $(S^+, I^+)$ and $(S^-, I^-)$ be the complete pre-orders obtained from the positive and negative flows;
- $(P^I, I^I, R^I)$ correspond to the preference, indifference and incomparability of each pair of alternatives in PROMETHEE I;
- $(P^{II}, I^{II})$ are the preference and indifference relations between each pair of alternatives in PROMETHEE II;
- $\phi$ matrix of unicriterion net flow scores;
- $a_j$ is the coordinate vector of each alternative;
• $e_k$ is an axis for each criterion;
• $w$ is the weight vector;
• $\pi$ the decision stick;
• $\delta$ is the amount of information preserved by GAIA plane;
• $\Delta(a_i, a_j) = \emptyset(a_i) - \emptyset(a_j)$;
• $\Delta'(a_i, a_j) = \emptyset'(a_i) - \emptyset'(a_j)$;
• $\Delta_k(a_i, a_j) = \emptyset_k(a_i) - \emptyset_k(a_j)$;
• $\alpha$ and $\beta$ the factor of rearranging weights;
• $\alpha_k^-$ the lower bound of $\alpha$;
• $\Omega^- = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A}, \text{s.t. } \Delta(a_i, a_j) \Delta_k(a_i, a_j) < 0\}$;
• $\alpha_k^+$ the upper bound of $\alpha$;
• $\Omega^+ = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A}, \text{s.t. } \Delta(a_i, a_j) \Delta_k(a_i, a_j) > \Delta^2(a_i, a_j)\}$;
• $\Omega^0 = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A}, \text{s.t. } \Delta(a_i, a_j) = 0 \text{ and } \Delta_k \neq 0\}$.

Chapter 3
• $\geq$ and $>$ mean $\geq$ and $>$;
• $\Delta$ Objective function of GDEA model;
• $d_j$ is the maximum of deviation between weighted investigated DMU and other DMUs.
• $Y^*_r$ is the output of the ideal point;
• $s_j$ is the relative distance to the ideal point.

Chapter 4
• $(\alpha_i, \beta_i, Y_r, \delta_i, \eta_i, \theta_r, k_i, \rho_i, \sigma_r, \tau_r, \varphi_i)$ are constants which imposed to weights in different weight restricted DEA model (5-2);
• $V = \{v | Cv \geq 0\}$ convex cone for the inputs weights in Cone-Ratio DEA model: intersection form;
• $U = \{u | Du \geq 0\}$ convex cone for the outputs weights in Cone-Ratio DEA model: intersection form;
• $R^+_m$ the non-negative real numbers: domain of inputs weights;
• $R^+_s$ the non-negative real numbers: domain of outputs weights;
• $\mathcal{V}$ convex cone for the inputs weights in Cone-Ratio DEA model: sum form;
• $\mathcal{U}$ convex cone for the outputs weights in Cone-Ratio DEA model: sum form;
• $w_i^*$ the central weights in the vector $W$
• $\phi_i(a_j) = 1$ : the dummy input added to unicriterion net flow score matrix ;
• $v_l$ associated weight to dummy input;
• $\alpha_k, \beta_k$ and $\lambda_j$ are the dual variables of the model PIIWCCR;
• $\phi_k$ the output vector of dual form of PIIWCCR;
• $\lambda, A$ and $B$ are the vectors of dual variables of PIIWCCR;
• $W^+$ and $W^-$ are also lower and upper bounds vectors;
• $(Z^*, C_0^*, \lambda^*, s^{++}, s^{--}, A^*, B^*)$ an optimal solution of dual PIIWCCR ;
• $E_0$ reference set ;
• $(\tilde{\phi}_{to}, \tilde{\phi}_{ko})$ the improved activity in dual of PIIWCCR ;
• $P_{imp}$ : the primal form according to the improved activity;
• $D_{imp}$ : the dual form according to the improved activity;
• $(Z_{oimp} = 1, \lambda_{imp}^* = \lambda^*, s_{imp}^{--}, s_{imp}^{++}, A_{imp}^*, B_{imp}^* = 0)$ the optimal solution of dual problem according to improved activity;
• $(v_{imp}^*, w_{imp}^*)$ the optimal solution of primal problem according to improved activity.

Chapter 5

• $v_{ik}^*$ and $u_{rk}^*$ : the optimal weights of inputs and outputs resulted by running CCR;
• $E_{kj}$ : the related score of $DMU_j$, using weights of $DMU_k$ ;
• $E_k$: the average cross efficiency scores;
• $M_k$: maverick index;
• $\rho_o^*$ : the benchmark score of efficient $DMU_o$ ;
• $Z_j$ and $W_j$ : input and output composites, respectively ;
• $r_{ZW}$ : the coefficient correlation between composite input and output;
• $S_{XX}, S_{YY}$: the matrices of the sums of squares of the input and output variables, respectively;
• $S_{XY}$: the matrix of the sums of products of the input and output variables;
- $T_j$: DEA scores of DMUs of canonical correlation analysis technique;
- $a_{jk}$: the evaluation of unit $j$ over unit $k$ in the pairwise comparison matrix in AHP;
- $v_q$: the weights of the criteria in the Jablonsky’s model;
- $w_{ij}$: the preference indices of efficient DMUs;
- $E_{AB}$ and $E_{BA}$: the cross efficiency scores of each pair of $A$ and $B$;
- $E_{ij}^*$: the cross efficiency score of unit $i$ in comparison with unit $j$;
- $E_i^*$ and $E_i^{i+k}$: the efficiency score of unit $i$ and $i + k$, respectively, in comparison between $n$ units with a single input and a single output;
- $A_i = \frac{x_i}{y_j}$;
- $E_{i,i+k}^*$, $E_{i+k,i}^*$: the efficiency scores of $DMU_i$ and $DMU_{i+k}$, respectively;
- $E''_i$, $E''_{i+k}$: the efficiency scores of $DMU_i$ and $DMU_{i+k}$ after adding $\alpha$ to their outputs, respectively;
- $\emptyset^*(a_i), \emptyset^*(a_{i+k})$: the net flow scores of $DMU_i$ and $DMU_{i+k}$ after adding $\alpha$ to the outputs in DEA model.

**Chapter 6**

- $w_k^-$ and $w_k^+$: weight intervals within which the values are likely to vary, determined by DM;
- $\Delta_{ij} = \emptyset_k(a_i) - \emptyset_k(a_j)$;
- $P$ is a convex polyhedron;
- $R^q$ $q$-dimentional space of polyhedron;
- $v$ is a vertex of polyhedron $P$;
- $B$ is the set of basic points;
- $N$ is the set of co-basic points;
- $M = c(n, 2)$ is the number of constraints resulted by a super-efficient ranking;
- $M' = c(n, 2) - c(N(eficiency = 1), 2)$ is the number of constraints resulted by a CCR ranking;
- $w_{kj}$ is the weight of criterion $k$ in the constraint $j$;
- $GM$ the geometric mean in weight matrix.
General Introduction

The process of decision-making has attracted the attention of many philosophers since early times. Great philosophers like Aristotle, Plato, Thomas Aquinas, etc. discussed the capacity of humans to decide, and claimed that thought is what distinguishes humans from animals (Figueira, J. et al., 2005).

Nowadays, evaluation performance and multicriteria decision-making have become quite popular. They rely on the assessment of “alternatives” or “decision making units” according to multiple indicators that have been previously identified by Decision Makers (DMs) as being relevant and important in the considered context. Famous websites like the Academic Ranking of World Universities (ARWU) (http://www.arwu.org), the Gross Domestic Product (GDP) ranking of countries (http://data.worldbank.org), the Internet Movie Database (IMDb) (http://www.imdb.com), and the Trends Gazelles ranking of companies (http://www.trends.be) show the growing interests on this topic. On the one hand, these simple and user-friendly interfaces often hide the implemented methods, the underlying assumptions and the detailed computation steps. On the other hand, a lot of researchers have developed and studied models to evaluate, choose, rank, sort, cluster, etc. a set of alternatives. Within this huge Operational Research (OR) literature, we will consider two main chapters: Data Envelopment Analysis (DEA) and, in Multi-Criteria Decision Aid (MCDA), the PROMETHEE methods. The topic of this PhD thesis will be to investigate the added value of the potential integration between these two research areas.

DEA, as introduced by Charnes et al. (1978), is a performance assessment methodology to measure the relative efficiency of a set of homogeneous Decision Making Units (DMUs). These are assumed to consume a single or multiple inputs to produce a single or multiple outputs without pre-determined weights (see Chapter 1).

MCDA is a discipline developed to assist Decision Makers (DMs) to choose, rank, sort, etc. a set of alternatives in the presence of multiple conflicting criteria. Preference modeling (for instance the weight elicitation problem) is at the core of this discipline. During the last forty years, different approaches (MAUT, AHP, Outranking methods, etc.) have been proposed to address these problems (see Chapter 2).

Researchers have already started to study the potential synergies between DEA and MCDA. For instance, Belton and Vickers (1993) discussed on advantages and limitations of DEA and MCDA methods. Belton and Stewart (1999) studied the possibility of analogy among them to benefit from one another and cover the deficiencies of each other. Since then, for over twenty years, researchers have worked on the integration of DEA with different methods in MCDA (see Chapter 3). To the best of our knowledge, this work is the first to consider the integration with the PROMETHEE methodology.
At the beginning of this work, two main observations were made:

1. One of the main features of DEA models is the freedom of inputs and outputs’ weights (i.e. these values are automatically determined in order to compute the best possible efficiency score for each DMU). However, this distinctive feature has also been considered as one of the main drawbacks (Allen et al., 1997). Under the total weights flexibility assumption, a DEA model:
   - can assign very low or very high value to the weights of some given inputs and/or outputs, which is not desirable (or even unrealistic);
   - cannot integrate a priori information of DMs about weights;
   - leads to assess several DMUs as being efficient without being able to further discriminate them.

In the literature, one of the solutions to overcome this problem is to restrict the freedom of weights assumption. One way for generating weight restrictions in DEA is to rely on MCDA (Allen et al., 1997). In this perspective, different MCDA methods, such as AHP and MACBETH, have already been integrated DEA (Shang and Sueyoshi, 1995; Zhu, 1996; Premachandra, 2001; Entani et al., 2004; Junior, 2008; Pakkar; 2014-2016). In this work, these questions will be addressed by using the PROMETHEE method.

2. One common point of most MCDA methodologies is the important role played by the DM in the preference elicitation phase. In complex multicriteria problems, when there is not enough a priori available information, the determination of the criteria importance may be a difficult task for DMs. As already stressed, in DEA, weights are determined automatically in order to implement the best efficiency score for each DMU. This thesis will investigate if a similar approach can be easily integrated with PROMETHEE in order to ease the preference elicitation.

This thesis is structured in two main parts. The first part (which covers Chapters 1, 2, and 3) is dedicated to a literature review and an introduction to the main notions related to DEA and MCDA (with a special focus on PROMETHEE) as well as a comparison between DEA and MCDA. Additionally, two MCDEA models are introduced to show how a DEA model or an extension of a DEA model can be used to solve a multicriteria problem.

The second part of this PhD thesis is constituted by three main original contributions:

**Contribution 1** (Chapter 4)

- In the first contribution, a summary of some weight restricted DEA models is presented. Then, an integrated Multi Criteria Data Envelopment Analysis (MCDEA) model is introduced, which can be applied to increase the discrimination power of DEA. To achieve this goal, the stability intervals are used based on PROMETHEE II as weight constraints in DEA. Furthermore, the unicriterion net flow scores matrix is used instead of the initial evaluation matrix. By doing so, preferential information is already integrated in the DEA process. By construction, the best
results are compatible with the PROMETHEE II ranking. Besides, additional sensitivity analysis is done. Some comparisons with the outputs of other decision-making techniques are provided based on three examples.

Contribution 2: (Chapter 5)

- In the second contribution, a new two-step algorithm is presented to completely rank units according to multiple inputs and outputs. In the first step, DEA is applied between each pair of DMUs independently to generate a pairwise comparison matrix. In the second step, the obtained matrix is exploited by means of PROMETHEE. A complete ranking of units is achieved based on the net flow scores of PROMETHEE. The compatibility between the resulting ranking of DEA and DEA-PROMETHEE methods is demonstrated while there exist just one input and one output. It is also discussed the monotonicity property of the method. Additionally, DEA-PROMETHEE is compared with an integrated DEA-AHP approach on a numerical example.

Contribution 3: (Chapter 6)

- In the third contribution, it is aimed to investigate how DEA can be used to propose weights in the context of the PROMETHEE II method. More precisely, one suggests an extension of the so-called “decision maker brain” used in the GAIA plane (also known as PROMETHEE VI) but on the basis of DEA. The aim is to try enriching the understanding of the problem by inducing an initial ranking through GAIA plane. The underlying idea is based on the computation of weights in PROMETHEE (GAIA brain), which are compatible with the DEA analysis. This chapter is ended with a numerical example.

Let mention that a fourth contribution is also presented in Appendix 2 but it will not be discussed in details in this work.

Contribution 4: (Appendix 2)

- In the fourth contribution, it is aimed to use DEA and MCDA in parallel to evaluate the overall performance and ranking of a sample of 55 drivers, aged 70 and older, by using the concept of composite indicators and PROMETHEE. The data set is extracted from an assessment battery and a fixed-based driving simulator. To achieve this objective, drivers completed tests of an assessment battery of psychological and physical aspects as well as knowledge of road signs. Moreover, they took part in a driving simulator test in which difficult scenarios for older drivers were included. In this case study, a DEA model is applied to calculate the optimal performance index score for each driver. It is also applied PROMETHEE II to enrich the analysis of this problem by considering preferential information from DMs using both the raw and the normalized data. Applying GAIA plane and Spider web, as two graphic tools in PROMETHEE can help DMs to analyze drivers’ profiles, give complementary advices to them and deepen the results. The results of this study show that the best and the worst drivers identified by the two methods are similar. Respectively, this complementary approach leads us to global improvement of the analysis and suggest older drivers more insight in characterizing their driving performances. These observations
point out the interest of using PROMETHEE II and DEA. The significant correlation between these results confirms the robustness of answers.

Finally, we conclude this work by summarizing the main findings of the PhD thesis and by identifying some perspectives for future research. To conclude, let us point out that the results of this PhD thesis has led to several publications and presentations in international conferences:

**Publications**

**Journal papers**


**Peer-reviewed conference proceeding**


**Presenting in conferences**


Part 1: Operations Research
Chapter One

Review of literature I: Data Envelopment Analysis (DEA)

Abstract
In this introductory chapter, the general idea and basic models of Data Envelopment Analysis are concisely presented: CCR, BCC and Additive models. This chapter is finalized by pointing out the strengths and limitations of Data Envelopment Analysis, which to our point of view, motivate applying the multicriteria methods in decision-making.

1.1. Overview
The structure of nowadays societies requires the use of existing and available sources in the optimal way. To achieve this goal, many efforts and researches have been done. The problem of measuring efficiency of a production unit constitutes one of these efforts, which are important in different sectors such as business, industry, economy, and management (Avkiran and Rowlands, 2006). The efficiency is measured when the best possible economic effects (outputs) are obtained with the minimum use of the available resources (inputs). In a simple case where units have just a single input and a single output, efficiency is defined as the ratio of output to input. In real case problems, organizational units have multiple inputs and outputs. Data Envelopment Analysis (DEA) is a decision making tool based on Linear Programming (LP) for measuring the relative efficiency of a set of comparable units according to multiple inputs and multiple outputs. DEA identifies the relative efficiency and inefficiency of units. Furthermore, it identifies the sources and level of inefficiency for each of the inputs and outputs.

There are a few important issues that should be considered before one applies a DEA model: defining the units to be analyzed (Decision Making Units: DMUs), and the performance measures used to analyze DMUs (inputs and outputs).

DMUs are units under evaluation such as bank branches, hospitals, universities, information systems, products, cities, countries, governments, airlines, airports, etc.
Once DMUs are identified, a set of performance measures should be determined to characterize the DMUs. Such measures that can reflect the performance or efficiency of DMUs are classified as inputs and outputs. The production process for each DMU is to use a set of inputs to generate a set of outputs. Each DMU has a varying level of inputs and yields a varying level of outputs. DEA inputs are those one usually tries to minimize like costs, No. of personnel, materials used, etc. and DEA outputs are those one usually tries to maximize such as profit, revenue, products, etc.

Let suppose that a bank branch is considered as an excellent bank with outstanding profit performance. Here, to evaluate multiple performance measures of the branch, some questions may arise:

- Does the branch use its resources efficiently?
- Could the branch reduce its operating costs and increase its profitability?
- How should be integrated the multiple measures?

To integrate multiple measures, weights on the measures should be considered.

- Are there available the weights on multiple measures?
- Is there the trade-offs among the multiple measures?

Occasionally, there is no a priori information available on weights or there exists just a partial information. Further, sometimes there is no trade-off among the multiple measures. For example, what is a trade-off between the number of times clients visit the bank branch and the number of employees work on that branch?

DEA is able dealing with multiple performance measures (inputs and outputs) in a single integrated model. To measure the efficiency performance of the bank branch, DEA includes any necessary measures related to the characterization of banking performance. Some measures that can be identified in the problem of comparing the operational efficiency of bank branches are as follows: Inputs (No. of employees, Operating costs, Branch size, Computer terminals, etc.) and outputs (Total value of accounts, No. of transactions, Credit transactions, Deposit transactions, etc.).

DEA identifies a best practice frontier for comparing different bank branches. According to the set of available inputs and outputs, DEA aims to determine which of the branches are efficient, and shows specific inefficiencies of the other branches. Furthermore, it provides specific targets for improvement of branches that did not act efficiently. Let consider evaluating performance of 8 bank branches with a single input and a single output. When the data is plotted, DEA shows where the best practice is.
Example 1.1 (Bank branches) - A bank wishes to compare the operational efficiency of its 8 branches. Table 1.1 shows for each branch the total value of accounts (output), expressed in hundreds of thousands of euros, and the number of staff employed (inputs), with the corresponding efficiency values calculated based on the ratio of output to input. The Figure 1.2 shows for each branch the number of employees on the horizontal axis and the value of accounts on the vertical axis. The slope of the line connecting each point to the origin represents the efficiency value associated with the corresponding branch. The line with the maximum slope represents the efficient frontier for all branches being analyzed (passed from E and G). The branches that lie on this line correspond to efficient units, while evidently, the branches that are below the efficient frontier are inefficient units. The area between the efficient frontier and the positive horizontal semi-axis is called the Production Possibility Set (PPS) [PPS is explained in Section 1.2.2.1].

<table>
<thead>
<tr>
<th>Bank branches</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Staff size (input) N.</td>
<td>4</td>
<td>1.5</td>
<td>3</td>
<td>3.5</td>
<td>5</td>
<td>5</td>
<td>3.5</td>
<td>4</td>
</tr>
<tr>
<td>Account value (output) €100,000</td>
<td>2</td>
<td>1</td>
<td>2.5</td>
<td>2.5</td>
<td>5</td>
<td>3</td>
<td>3.5</td>
<td>1.5</td>
</tr>
<tr>
<td>Staff/Account (efficiency)</td>
<td>0.5</td>
<td>0.667</td>
<td>0.833</td>
<td>0.714</td>
<td>1</td>
<td>0.600</td>
<td>1</td>
<td>0.375</td>
</tr>
</tbody>
</table>

Table 1.1 Input, output, and efficiency values for the bank branches

In this case, bank branches E and G show the best performance with efficiency scores equal to 1 [The efficiency in DEA models is discussed later in Section 1.2.2.2].

In the presence of multiple inputs and multiple outputs, plotting data to identify the best practice efficient frontier is not an easy task. DEA applies a system of LP to integrate the multiple measures. For this reason, DEA determines a system of weights such that each DMU can attain its best possible efficiency score.
To be summarized, DEA gives:

- The efficient frontier consisting of the best performing DMUs based upon the current data set;
- Efficiency Score for each DMU;
- Efficiency reference set: peer groups;
- Target for the inefficient DMUs;

Information on how much inputs can be decreased (Input-Orientation) and/or outputs can be increased (Output-Orientation) to make an inefficient unit efficient (improving productivity and efficiency of inefficient DMUs). This process can be considered as a guide to “what to do” for the managers. In the Figure 1.2, in an Input-Orientation system, the target of unit F is G, and in an Output-Orientation, the target of unit F is E on the efficient frontier [input and output orientation is defined in Definition 1.4].

1.1.1 Introduction

Data Envelopment Analysis (DEA) is becoming a growingly accepted Operations Research (OR) tool in management sectors. DEA is a data oriented method to evaluate a set of entities called DMUs, since it benefits from a certain decisional autonomy. It is a mathematical programming method to measure the efficiency of DMUs by converting multiple inputs into multiple outputs. To achieve the optimal goal in DEA, the essential decisions can be taken by seeing the efficiency or inefficiency of DMUs. The approach that is applied in DEA to make a frontier by enveloping all the observed input/output measures of DMUs is why it was first afforded this name (Cooper et al., 2005). This frontier, later, is used to evaluate the performances of all DMUs.

The definition of DMU is flexible for any such unit and gives room to a wide range of descriptions. In recent years, great variety of DEA’s applications in different sectors and activities have been seen to evaluate the performance of different DMUs such as bank branches (Ebrahimnejad et al., 2014; Zervopoulos et al., 2016), bank clients (Cheng et al., 2007), health care sectors: hospitals, doctors (Flokou et al., 2010), educational institutions (Salerno, 2006), manufacturing units, e.g. factories and power plants (Ahn et al., 1988), benchmarking (Bogetoft and Otto, 2011), administrative management (Avkiran and Rowlands, 2006), fast food restaurants (Choi et al., 2007), electric utility sectors (Vaninsky, 2008), bus routes (Sheth et al., 2007), travel agencies (Köksal and Aksu, 2007), sports: rating of fitness centers to evaluate effectiveness of players in different sports (Cooper et al., 2008), oil companies branches (Thompson, 1991), insurance companies branches (Shiuh-Nan and Tong-Liang, 2008) and regional subdivisions (Meimand et al., 2002). These applications extend to evaluate the performances of cities, regions and countries with many different kinds of inputs such as different types of costs, and outputs like quality-of-life and revenues (Hashimoto et al., 2009).

DEA in its classic form is not able to consider managers preferences in evaluation performance of different units. Moreover, in some specific areas, it is unable to portray the problem properly,
when several conflicting objectives needs to be considered simultaneously instead of one single objective (Belton and Stewart, 1999; Aouni and Laflamme, 2014). Thus, in the situations where priorities and managerial preferences should be taken into account or/and there exist several conflicting criteria to be evaluated simultaneously, multicriteria approaches may be used as useful tools to characterize the problem.

This chapter, first, briefly talks about the background of DEA and summarizes some concepts and definitions such as Production Possibility Set (PPS) and efficiency. Then, it presents some classical DEA models like CCR, BCC and Additive. The chapter is continued by fleetingly expressing the Returns to Scale (RTS) principles. It finally discusses some strengths and limitations of DEA, which leads to plead in favor of using a multicriteria method to support Decision Makers (DMs).

1.2. Generalities, definitions and concepts of DEA

In this section, some important definitions in DEA is presented.

DEA is a method that has been designed to compare and evaluate the relative efficiency of a number of DMUs (Cooper et al., 2004, 2005, 2007 and 2011). Several factors make DEA a relatively worthy tool for the evaluation of efficiency. Firstly, DEA is a non-statistical and an extreme-point method. It compares each unit with only the best unit(s) without using any special statistical distribution, e.g. the normal distribution. In contrast, some statistical approaches, e.g. regression analysis, have a central tendency and evaluate units relative to an average unit (Cooper et al., 2004, 2005, 2007 and 2011). Secondly, DEA is a non-parametric method in OR for the estimation of production frontier. It does not require any basic assumption of a functional form relating inputs to outputs. DEA builds its own frontier, using the set of inputs and outputs of different DMUs. Consecutively, the issue of specifying inaccurate frontier is not considerable (Cooper et al., 2004, 2005, 2007 and 2011). Thirdly, DEA takes into account the set of all multiple inputs and multiple outputs simultaneously to quantify efficiency, while parametric approaches, e.g. stochastic frontier analysis, in the process of measuring efficiency, consume all inputs to generate an output (Jarzebowski, 2013). Finally, DEA works well in small sample size [as a rule of thumb, the sample size should be at least three times more than the sum of the number of inputs and outputs] (Nunamaker, 1985; Franchon, 2003; Cooper et al., 2005). While, computing stochastic errors in the parametric approaches is possible for the sample size more than 100 observations (Aigner et al., 1977).

DEA as an input-output non-parametric method employs some form of LP to determine an efficient frontier, which is composed of a set of DMUs, characterized with efficiency scores equal to 1. The relative efficiency of other inefficient observations is estimated based on their deviations from the efficient frontier.

The strength of DEA is that the relative efficiency can be measured with few assumptions: identifying DMUs, inputs/outputs factors, and a proper model. As explained above, on the one hand, it works with a small number of DMUs (Nunamaker, 1985; Franchon, 2003; Cooper et al.,
Thus, it can be appropriate in small-scale problems such as selecting the best candidate for a job, the best candidate for a loan, the best firm and so forth. On the other hand, when the number of DMUs is small in comparison with the number of inputs and outputs, the number of efficient DMUs that lie on the efficient frontier, is increased. This has led to the construction of tools that can force DEA to distinguish more among its efficient units when it is necessary (Adler et al., 2002). Multicriteria approach can be such a useful tool that may help to increase the discrimination power of DEA, by putting preferential information, such as weight restrictions, in the structure of DEA.

The next sub-section presents briefly the background of DEA. Additionally, it shows the growth of DEA literature through two figures.

1.2.1. Background of DEA

The early work of Farrell (1957) that extended the concept of “productivity” to the more general concept of “efficiency” flashed the mind of Charnes, Cooper and Rhodes (CCR) to initiate DEA in 1978 (Cooper et al., 2004). The original CCR model (Charnes et al., 1978) was applicable only to models globally characterized by a constant scale (Banker et al., 2004). In what turned out to be a major advance, Banker, Charnes, and Cooper (BCC) in 1984 extended the CCR model to improve DEA to a variable scale (Banker et al., 2004) [Constant Returns to Scale (CRS) and Variable returns to Scale (VRS) are explained later in the Section 1.4]. In the following years, methodological contributions from a large number of researchers collected into a significant volume of literature around the CCR–BCC models and other general approaches of DEA. The fast growth of DEA, as an acceptable method of efficiency analysis, can be understood from different bibliography studies. Seiford (1997) listed no fewer than 472 published articles and accepted Ph.D. dissertations in this regard. In a more recent bibliography, Emrouznejad et al. (2008) includes 3,183 items from 2,152 different authors (Figure 1.3). Liu et al. (2013) adopted the database of ISI web of science as the source for their study and a total of 4936 papers were listed in their work (Figure 1.4). Indeed, at the present moment, an internet search in ScienceDirect homepage for DEA produces 205,000 entries (retrieved from http://www.sciencedirect.com, April 2017)!
Figure 1.4 Growth curve of DEA literature (Liu et al., 2013)

The curve line in the middle of the Figure 1.4 shows the direct estimate from the growth curve analysis. The highlighted area’s margins include the 90% assurance regions (Liu et al., 2013).

Further, parallel development of programs and software (Lingo, KONSI DEA Analysis, Open source DEA, DEA Frontier software, DEA zone, PIM-DEA …) for solving the DEA problems made it considerably easier to use DEA in practical applications. Some websites of DEA software are listed in bibliography.

In what follows, some principles and definitions are considered to be able understanding better the concept of DEA.

1.2.2. Basic principles and definitions in DEA

This part presents some principles and definitions used in DEA such as size of the problem, “Production Possibility Set”, “input and output orientations”, and “efficiency”.

It is assumed that each DMU is characterized by an input vector and an output vector. It may be supposed there are \( n \) DMUs to be rated, where each DMU uses \( m \) inputs to produce \( s \) outputs relative to \( n - 1 \) other DMUs. \( DMU_j \) has an input vector \( X_j = (x_{1j}; x_{2j}; \ldots; x_{ij}; \ldots; x_{mj})^T \) and an output vector \( Y_j = (y_{1j}; y_{2j}; \ldots; y_{rj}; \ldots; y_{sj})^T \), where \( j = 1, 2, \ldots, n \), \( i = 1, 2, \ldots, m \) and \( r = 1, 2, \ldots, s \). \( x_{ij} \) and \( y_{rj} \) are the quantity of input \( i \) used by \( DMU_j \) and the quantity of output \( r \) acquired by \( DMU_j \), respectively. Hence, \( n \) blocks of system have an input matrix \( X \) and an output matrix \( Y \). By convention, the efficiency of a DMU is estimated as a fraction between 0 and 1 (the mathematical models are shown in Section 1.3).

Basic principles in DEA

In the classical DEA models, there are some basic principles as follows (Cooper et al., 2004, 2005, 2007 and 2011):
1- **Problem size**: Generally, it is better to keep the following relation \( n \geq 3(m + s) \), as an empirical principle, between the combined number of inputs \((m)\) and outputs \((s)\) with the number of units \((n)\) (Nunamaker, 1985; Franchon, 2003; Cooper et al., 2005). When the number of DMUs is less than the combined number of inputs and outputs \([n < 3(m + s)]\), a large number of DMUs will be identified as efficient (with efficiency scores equal to one). Consecutively, the efficiency discrimination power of DEA is problematic due to an inadequate number of freedom degrees (Cooper et al., 2005). Banker et al. (1989) suggested a rough rule of thumb in envelopment models. They proposed that the sample size should satisfy: \( n \geq \max\{m, s, 3(m, s)\}\).

A way to increase the discrimination power of DEA is applying the weight restricted DEA models such as the Assurance Regions (AR) methods (Thompson et al., 1986) and the Cone-Ratio (Charnes et al., 1990). In this thesis, a multicriteria method (PROMETHEE) is applied to enrich the evaluation of DMUs in DEA (Bagherikahvarin and De Smet, 2016) [please see Chapter 4].

Indeed, the selection of inputs and outputs is a crucial matter in DEA. Cooper et al. (2005) suggested a selection process. They proposed to first choose a small group of inputs and outputs and then progressively increase these values to discern the effects of the new adding factors. Morita and Avkiran (2009) and Luo et al. (2012) proposed statistical approaches to choose inputs and outputs factors.

2- **Semi-positivity**: It is assumed the non-negativity of input and output vectors with at least one positive element: \( X_j \geq 0 \) such that \( \exists j: x_{ij} \neq 0 \) and \( Y_j \geq 0 \) such that \( \exists j: y_{rf} \neq 0 \) where \( j = 1,2,...,n, i = 1,2,...,m \) and \( r = 1,2,...,s \).

**Main definitions**

In this part, the Production Possibility Set (PPS) is introduced. The standard DEA models and their feasible solutions can be produced on it. Furthermore, the definition of efficiency is presented.

**1.2.2.1. The Production Possibility Set (PPS)**

Each pair of such semi-positive input \( X \in \mathbb{R}^m \) and output \( Y \in \mathbb{R}^s \) is called an “activity” \((X, Y)\) \([m\) and \(s\) identify the number of dimensions required to express inputs and outputs, correspondingly\]. The “Production Possibility Set” (PPS) is the set of feasible activities. The following properties are assumed for PPS (Cooper et al., 2005):

1- The observed activity \((X, Y)\) belongs to PPS.
2- If any activity \((X, Y)\) belongs to PPS, then the activity \((\alpha X, \alpha Y)\) belongs also to PPS for any positive scalar \(\alpha\) [Constant Returns to Scale (CRS) assumption].
3- For any activity \((X, Y)\) in PPS, a semi-positive feasible activity \((X', Y')\) with \(X' \geq X\) and \(Y' \leq Y\) is included in PPS.
4- Any semi positive linear combination of activities in PPS belongs to PPS (Convex set):
\( (X,Y) \in PPS, (X',Y') \in PPS \) and \( \lambda \in [0,1] \) \( \Rightarrow \)

\[ \lambda(X,Y) + (1-\lambda)(X',Y') \in PPS \] (1.1)

Putting the data sets in vectors \( X_j = (x_{ij})^T \) and \( Y_j = (y_{rj})^T \), we can define the PPS satisfying 1 through 4 by:

\[ PPS_{CRS} = \{ (X,Y) \mid X \geq \sum_{j=1}^{n} X_j \lambda_j, Y \leq \sum_{j=1}^{n} Y_j \lambda_j, \lambda_j \geq 0 \} \] (1.2)

where \( \lambda_j \) is a semi-positive vector in \( R^n \).

Figure 1.2 in Example 1.1 illustrates a classic PPS in two dimensions [with single input \( (i = 1) \) and single output \( (r = 1) \)], which follows the Constant Returns to Scale (CRS) condition (Banker et al., 2004). The PPS, which ensures the condition of a Variable Returns to Scale (VRS) (Banker et al., 2004) is as follows:

\[ PPS_{VRS} = \{ (X,Y) \mid X \geq \sum_{j=1}^{n} X_j \lambda_j, Y \leq \sum_{j=1}^{n} Y_j \lambda_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \} \] (1.3)

The convexity condition \( \sum_{j=1}^{n} \lambda_j = 1 \) is added to this set. Figure 1.4 in Example 1.2 shows a standard PPS in two dimensions (with two inputs and single output) that reflects the VRS condition (Banker R. D. et al., 1984).

**Definition 1.1. Relative Efficiency** – “A DMU is to be evaluated as fully (100%) efficient based on the available empirical information if and only if the performances of other DMUs do not show that some of its inputs or outputs can be improved without worsening some of its other inputs or outputs” (Cooper et al., 2004).

DEA determines the best possible relative efficiency of a given DMU by assigning weights automatically to the inputs and outputs of that unit. Thus, it does not require assigning priori values for these factors. This characteristic of DEA may help DM assigning priorities of criteria in multicriteria decision-making problems. In the absence of enough priority information between criteria, DEA can be used to initiate a weight vector (Bagherikahvarin and De Smet, 2016). This contribution can be seen in Chapter 6.

According to the definition of efficiency, a characterization of being “inefficient” holds only if an evaluated DMU (output to input relation) does not lie on the efficient frontier in the PPS.

Now, Example 1.1 that is considered the efficiency evaluation of bank branches is more clear.

As mentioned in the introduction, it is interesting to note the difference between statistical approaches such as regression line and DEA. The regression line evaluates each unit’s performance relative to an average unit, whereas DEA do this evaluation by comparing DMUs with the efficient frontier (Cooper et al., 2004, 2005, 2007 and 2011). The green dotted line in Figure 1.2 represents the regression line.
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Definition 1.2. General efficiency dominance - Let us suppose that $X_h = (x_{1h}; x_{2h}; \ldots; x_{ih}; \ldots; x_{mh})^T$ and $Y_h = (y_{1h}; y_{2h}; \ldots; y_{rh}; \ldots; y_{sh})^T$ are input and output vectors of $DMU_h$, respectively, where $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, s$. The input and output vectors of $DMU_k$ are determined similarly. Let $(X_h, Y_h)$ and $(X_k, Y_k) \in PPS$. It can be said that $DMU_k$ “dominates” $DMU_h$ if and only if $X_k^T \leq X_h^T$ and $Y_k^T \geq Y_h^T$ with respect to PPS. A strict inequality is established at least for one component in the input and/or the output vector (Cooper et al., 2004, 2005, 2007 and 2011).

In Figure 1.2, all DMUs (bank branches) are dominated by $DMU_E$ and $DMU_G$. For example, $DMU_D$ is dominated by $DMU_E$ since $DMU_D$ has less amount of account value and number of staff than $DMU_E$, or it is dominated by $DMU_G$ because however, it has the same number of staff but its account value is less than $DMU_G$. $DMU_A$ and $DMU_H$ lie further to “efficient frontier” than other bank branches; evidently, they have less efficiency scores than other branches.

A DMU in PPS is not dominated if and only if there is no other point in PPS, which satisfies the dominance definition. This leads to the efficiency of a DMU. Let us stress that one finds similar notions in Multiple-Criteria Decision Aid (MCDA).

“Efficiency” is the main specification of the DEA concept. In the next sub-section, it is discussed.

1.2.2.2. Efficiency

Definition 1.3. Efficiency - $DMU_o$ is efficient with respect to PPS if and only if there is no $DMU_j \in PPS$ for $j = 1, 2, \ldots, n$ and $j \neq o$, such that $DMU_o$ is dominated by $DMU_j$ (Cooper et al., 2004, 2005, 2007 and 2011).

Let $N = \{1, 2, \ldots, n\}$ denotes the set of units being compared by evaluating their efficiencies. If each unit produces a single output ($y_j$) using a single input ($x_j$) only, the “efficiency” of the $j^{th}$ unit is defined as follows:

$$\theta_j = \frac{Output}{Input} = \frac{y_j}{x_j}$$

(1.4)

If the unit generates multiple outputs using various inputs, the efficiency of $DMU_j$ is defined as the ratio between the sum of the weighted outputs to the sum of the weighted inputs. In the absence of input or output vector, a dummy input or output vector is added to the data set. The efficiency of $DMU_j$ is defined as follows:

$$\theta_j = \frac{u_1y_{1j} + u_2y_{2j} + \cdots + u_sy_{sj}}{v_1x_{1j} + v_2x_{2j} + \cdots + v_my_{mj}} = \frac{\sum_{r=1}^{s} u_r y_{rij}}{\sum_{i=1}^{m} v_i x_{ij}}$$

(1.5)

where $v = (v_1, v_2, \ldots, v_m)$ and $u = (u_1, u_2, \ldots, u_s)$: $i = 1, \ldots, m$ and $u_r: r = 1, \ldots, s$ are the associated non-negative weights of input $i$ and output $r$, respectively (these weights will be discussed later in this section).
Several assumptions should be respected before using the weighted sum, such as compensability, additivity and determining weights. Compensability between input and output factors (adding input/output to one another) clearly means: profiles that show different performances might result in having same scores (e.g. a high performance on a given factor can have less impact when there is a low performance on another one). Additivity of input/output factors assumes that there is no interaction between the factors taken into consideration (i.e. all inputs and outputs are chosen weights independently to reach the maximum efficiency) which is rarely true in practice. In the presence of multiple inputs and multiple outputs, the efficiency of $DMU_j$ depends strongly on the system of introduced weights. One main drawback of the weighted summation of outputs to inputs is that for different set of weights, the efficiency value may undergo different values. Thus, it may become a difficult task to fix a unified system of weights, which can be shared and accepted for all the evaluated units. In DEA, the LP structure evaluates units through a weight system that is the best for each DMU individually. It allows the efficiency value of each DMU to be maximized. The purpose of DEA is to identify the efficient units, which lie on the “efficient frontier” and which ones do not lie on it (Cooper et al., 2004). Further, to control the value of generated weight factors, weight restricted DEA models were initiated (Thompson et al., 1986). This thesis, in Chapter 4, presents a model to control DEA’s weights based on a multicriteria approach (Bagherikahvarin and De Smet, 2016\textsuperscript{a}).

**Efficient frontier**

The relationship between the inputs used and the outputs produced is the sense of “efficient frontier” (it is also known as “production function”). It determines a line which connects the origin to the efficient units (efficiency scores equal to 1). The efficient frontier is shown in different figures in this Chapter (Figures 1.2, 1.5, 1.6, 1.7, and 1.8).

There are two possible definitions of efficiency depending on the purpose of the evaluation to improve the performance of inefficient units. It identifies for each input level, the output level that can be achieved in the condition of efficiency. Similarly, it identifies for each output level the minimum level of input to be taken to become efficient.

**Definition 1.4. Input and Output Orientation (I-O and O-O)** - One might be pointed out how to improve the input features of a unit to become efficient while keeping the same output level (possible reduction of inputs). These models are Input-Oriented (I-O). Reversely, in an Output-Oriented (O-O) model, the purpose is to increase the outputs level of a unit to become efficient, without changing its inputs proportions (Cooper et al., 2004, 2005, 2007 and 2011).

In the Example 1.1, the I-O and the O-O are illustrated for unit $F$. As it can be seen $F$ needs to increase its account value (output) to a point between $E$ and $G$ to lie on the efficient frontier and become efficient (O-O). It can also decrease the number of its staff (input) to a point close to $G$ to become efficient (I-O). Further, $F$ can simultaneously increase its output and decrease its input (with a combination of input and output) to become efficient. The last can be done via Additive model, which is explained in Section 1.3.3 (Charnes, et al., 1985).
When a unit produces a single output by using two inputs, the efficient frontier supposes the shape shown in Figure 1.5. To have a unified frontier in this case, it can be normalized by dividing inputs by the only output. Example 1.2 considers this type of problems. Indeed, this single output can be considered as a dummy output equal to 1.

**Example 1.2** (Cooper et al., 2004) - Table 1.2 shows the data set of a problem to evaluate the efficiency scores of 7 bank branches according to 2 inputs (No. of employees and Operating cost) and 1 output (Total account value). Figure 1.5 illustrates this problem. Branches C, D, E, and F are efficient with efficiency scores equal to 1. The efficiency value of an inefficient branch is evaluated by the length of the segment connecting the unit to the efficient frontier along the line passing through the origin O. The efficiency value of DMUA is θA = \( \overline{OQ} / \overline{OA} \), where \( \overline{OQ} \) and \( \overline{OA} \) represent the lengths of segments OQ and OA, respectively.

The branch A may be made efficient by a displacement along segment OA that moves it onto the efficient frontier. Such displacement is identical to progressively decreasing the quantity of both inputs while keeping unchanged the quantity of output (I-O). The PPS is defined as the region delimited by the efficient frontier where the observed units being compared are found.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input 1 (employees)</td>
<td>4</td>
<td>7</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>Input 2 (operation cost)</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>Output (total account)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

| Efficiency  | 0.8571 | 0.6315 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.6667 |

Table 1.2 Efficiency evaluation of 7 bank branches with 2 inputs and 1 output (Cooper et al., 2004)

![Figure 1.5 Two inputs and single output (Cooper et al., 2004)](image-url)
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The DMUs on the frontier are also called “non-dominated” or “Pareto optimal” units.

The set of weights and the efficiency scores can be computed by running CCR-I-O model (1.8) [please see Example 1.3].

In the next section, the classical DEA models are presented.

1.3. The classical models in DEA

This section considers some classical DEA models as follows: CCR, BCC and Additive models.

1.3.1. CCR model

Charnes, Cooper and Rhodes (CCR) first initiated the standard DEA model in 1978. CCR is built on the notion of efficiency as defined in (1.5): the ratio of sum of the weighted multiple outputs to sum of the weighted multiple inputs. It computes the relative efficiency of \( DMU_o \) in comparison to other DMUs. In mathematical programming, to find the related multipliers (weights), this function should be maximized as an objective function of the DMU under evaluation:

\[
Max \theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \tag{1.6}
\]

Relation (1.6) without any other additional constraints is unbounded; thus a set of normalizing constraints for every DMU (including \( DMU_o \)) must be added and put it equal or less than 1:

\[
Max \theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}
\]

s. t.

\[
\frac{\sum_{r=1}^{s} u_r y_{ri}}{\sum_{i=1}^{m} v_i x_{ii}} \leq 1, j = 1, ..., n
\]

\[
u_r, v_i \geq 0, r = 1, ..., s, i = 1, ..., m
\]

After presentation of CCR model (Charnes et al., 1978), an amendment has been issued by a proven example that revealed an inefficient DMU might be evaluated efficient (Cooper et al., 2004). This phenomenon is happened when the CCR optimization model ignores some weights to give the maximum efficiency score to a DMU. Therefore, the constraints \( u_r, v_i \geq \varepsilon : r = 1, ..., s, i = 1, ..., m \) may be replaced the non-negative weights, where \( \varepsilon > 0 \), is a non-Archimedean element smaller than any positive real number. It prevents the unit from assigning a null weight to an input or output (Arnold et al., 1998; Cooper et al., 2004).

Model (1.7) can be transformed to a LP by putting the sum of weighted input of \( DMU_o \) equal to 1. This condition leads to an alternative optimization problem, the CCR-I-O model, where the objective function consists of the maximization of the sum of the weighted outputs in \( DMU_o \).
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Max $\theta_o = \sum_{r=1}^{s} u_r y_{ro}$

s.t.

$\sum_{i=1}^{m} v_i x_{io} = 1$  \hspace{1cm} (1.8)

$\sum_{r=1}^{s} u_r y_{rij} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, \ldots, n$

$u_r, v_i \geq 0, r = 1, \ldots, s, i = 1, \ldots, m$

Definition 1.5. CCR-efficiency (Cooper et al., 2005) - Let $\theta^*$ be the optimum value of the objective function corresponding to the optimal solutions $(\nu^*, u^*)$ of CCR model (1.8). $DMU_j, j = 1, \ldots, n$, is said to be efficient if $\theta^* = 1$ and if there exists at least one optimal solution $(\nu^*, u^*)$ such that $\nu^* > 0$ and $u^* > 0$; otherwise $DMU_j$ is CCR-inefficient.

The inefficiency of CCR means that either $\theta^* < 1$ or $\theta^* = 1$ and at least there is some zero weights in the $(\nu^*, u^*)$ for every optimal solution of (1.8). By repeating a similar optimization problem for each of the $n$ individual DMUs, $n$ vectors of weights are obtained. The system chooses weights with full flexibility, which would be a certain advantage. However, this advantage might be seen as a disadvantage in some points such as the impossibility of contributing the prior information of managers or ignoring some inputs and/or outputs (generating zero or near zero weights). For this reason, the weight restricted DEA models were initiated by Thompson et al. (1986). In Chapter 4 one presents a new weight restricted DEA model based on the PROMETHEE method (Bagherikahvarin and De Smet, 2016a).

Example 1.3 Table 1.3 shows the result of running CCR-I-O model (1.8) on the data set of Table 1.2 (Cooper, et al., 2004) [All DEA problems are solved through LP in MATLAB].

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_1$</td>
<td>0.1428</td>
<td>0.0526</td>
<td>0.0637</td>
<td>0.1306</td>
<td>0.4918</td>
<td>0</td>
<td>0.3333</td>
</tr>
<tr>
<td>$\nu_2$</td>
<td>0.1428</td>
<td>0.2105</td>
<td>0.4902</td>
<td>0.2387</td>
<td>0.0041</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$u$</td>
<td>0.8571</td>
<td>0.6315</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.6667</td>
</tr>
</tbody>
</table>

Table 1.3 The result of running CCR model on data set of Table 1.2

According to Table 1.3 and as explained in Example 1.2, bank branches $C, D, E,$ and $F$ are CCR-efficient by Definition 1.5 and they lie on the efficient frontier. Branch $F$ is not fully efficient; however, it looks like an efficient DMU with efficiency score equal to 1 but it has a zero variable $(\nu_{1F}^* = 0)$; thus, it has an excess in its input 1. Unit $F$ is called a “weak efficient” unit. The preliminary relation between the number of DMUs with inputs and outputs is not satisfied: $7 < 3(2 + 1) = 9$. Thus, the problem of discrimination between DMUs occurs. Further, the full flexibility of weights in DEA gives a zero weight in branch $F$. Implementing some weight restrictions may increase the discrimination between DMUs besides eliminating the effect of zero weights.
According to optimal weight vectors, the relative importance of input and output factors can be computed.

**Definition 1.6. Virtual inputs and Virtual outputs** - Assume that there is a set of \( n \) DMUs with an input vector \( X_j \) and an output vector \( Y_j \), where \( j = 1, 2, \ldots, n \). \( v^* = (v_1^*, \ldots, v_i^*, \ldots, v_n^*) \), \( i = 1, 2, \ldots, m \) and \( u^* = (u_1^*, \ldots, u_r^*, \ldots, u_s^*) \), \( r = 1, 2, \ldots, s \) are non-negative input and output optimal weight vectors, consecutively. These weights are the result of running DEA primal models [formulations number (1.8) and (1.16)]. The product of the input’s vector and its corresponding optimal weight gives the virtual input. The virtual outputs are obtained similarly. As a convention, while the sum of the virtual outputs is equal to the efficiency of a DMU under evaluation, the virtual inputs always sum up to the maximum efficiency score (i.e. 1). A virtual input or output provides information about the relative importance of each given factor (Cooper et al., 2004, 2005, 2007 and 2011).

Virtual input = \( \sum_{i=1}^{m} v_i^* X_j, j = 1, \ldots, n \) (1.9)

Virtual output = \( \sum_{r=1}^{s} u_r^* Y_j, j = 1, \ldots, n \) (1.10)

To characterize the dual form of model (1.8), the objective function should be minimized. The equality \( \sum_{i=1}^{m} v_i x_{io} = 1 \) corresponds to \( Z \) (the dual variable). The constraint \( \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \) determines the sign of \( \lambda_j \). Finally, the non-negativity of \( u_r \) and \( v_i \) indicates the sign of the main constraints in the body of (1.11) as follows:

\[ \begin{align*}
\text{Min } & Z \\
\text{s. t. } & \sum_{j=1}^{n} \lambda_j x_{ij} - Z x_{io} \leq 0, i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro} \geq 0, r = 1, \ldots, s \\
& \lambda_j \geq 0, j = 1, \ldots, n 
\end{align*} \] (1.11)

Both problems (1.8) and (1.11) can be used to compute the efficiency scores. Each feasible solution of the dual problem (minimization) provides an upper bound to the solution of the primal problem (maximization). A solution of (1.11) always exists, since we can set \( Z^* = 1 \) and \( \lambda_0^* = 1 \) and \( \lambda_j^* = 0 \) for all \( j \neq 0 \). This solution confirms the optimum solution (the efficiency score) of \( DMU_j \) is equal to or less than 1 (\( Z^* \leq 1 \)). An inefficient unit (\( DMU_0 \)) can move to the efficient frontier by compensating its shortage equal to \( Z^* x_{io} \).

For each feasible solution \( (Z, \lambda) \) of problem (1.11), the slack variables \( s_{i}^-, i = 1, \ldots, m \) and \( s_{r}^+, r = 1, \ldots, s \), can be defined as follows:

\[ \begin{align*}
& s_{i}^- = Z x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij}, i = 1, \ldots, m \\
& s_{r}^+ = \sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro}, r = 1, \ldots, s
\end{align*} \] (1.12, 1.13)
where $s_i^-$ and $s_r^+$ are the excess quantity of input $i$ and shortage quantity of output $r$ in $DMU_o$, respectively.

To find the value of $s_i^-$ and $s_r^+$ (the input excess and the output shortage), following problem is solved:

$$
\begin{align*}
\text{Min } Z &= \varepsilon (\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} - Z x_{io} \leq 0, i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rf} - y_{ro} \geq 0, r = 1, \ldots, s \\
& \lambda_j s_i^-, s_r^+ \geq 0, j = 1, \ldots, n
\end{align*}
$$

(1.14)

As mentioned earlier, $\varepsilon$ is a non-Archimedean element, smaller than any other positive real number. The problem (1.14) should be solved in two phases. In the first phase, the optimal objective solution ($Z^*$) of the problem (1.11) is computed, which, by the virtue of the duality theorem, can be equal to the optimal objective value ($\theta^*$) of the (1.8). In the second phase, this optimal value ($Z^*$) is integrated to the following problem to find the slack variables:

$$
\begin{align*}
\text{Max } \omega &= \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{s.t.} & s_i^- = Z^* x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij}, i = 1, \ldots, m \\
& s_r^+ = \sum_{j=1}^{n} \lambda_j y_{rf} - y_{ro}, r = 1, \ldots, s
\end{align*}
$$

(1.15)

**Theorem 1.1** $DMU_o$ is “fully-efficient” if and only if $Z^* = 1$ and the optimum values of the slack variables are equal to zero: $s_i^--= 0, i = 1, \ldots, m$ and $s_r^+ = 0, r = 1, \ldots, s$ (Cooper et al. 2005).

When $Z^* = 1$, but some slack variables are non-zero, $DMU_o$ is “weak-efficient”. $DMU_F$ in Example 1.3 is weak efficient. Consecutively, a DMU with $Z^* < 1$ is an inefficient DMU.

**Theorem 1.2** Let $(Z^*, \lambda^*)$ be an optimal solution for the CCR-I-O envelopment model (1.11). Then $(1/Z^*, \lambda^*/Z^*) = (\phi^*, \hat{\lambda}^*)$ is optimal for the corresponding CCR-O-O model. Similarly if $(\phi^*, \hat{\lambda}^*)$ is an optimal solution for the O-O model, then $(1/\phi^*, \hat{\lambda}^*/\phi^*) = (Z^*, \lambda^*)$ is optimal for the I-O model. Likewise, if in (1.8), CCR-I-O multiplier model, $(v^*, u^*)$ is the optimal solution, then $(\tilde{v}^*, \tilde{u}^*) = (v^*/\theta^*, u^*/\theta^*)$ (Cooper et al., 2004, 2005, 2007 and 2011).

**Example 1.4** Table 1.4 shows the result of running CCR-O-O model (refers to O-O models in Table 1.4) on data set of Table 1.2 (Cooper, et al., 2004). Additionally, it compares these results with the CCR-I-O model in Table 1.3.

Considering $DMUA$, the CCR-I-O model gives $\theta^* = Z^* = 0.8571$, which yields the values of $\lambda^*_k = (0, 0, 0, 0.7143, 0.2857, 0, 0)$ in envelopment form and $(v^*, u^*) = (0.1428, 0.1428, 0.8571)$ in multiplier form. Thus, from (1.8) $\theta^* = 1 \times 0.8571/(4 \times 0.1428 + 3 \times 0.1428) = 0.8571$. 

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Alternatively, solving the CCR-O-O models confirms the principles of Theorem 1.2:
\[
\left(1/Z^*, \lambda^*/Z^* \right) = \left(\phi^*, \tilde{\lambda}^* \right) \rightarrow \left(\frac{1}{0.8571}, \frac{0}{0.8571}, \frac{0.7143}{0.8571}, \frac{0.2857}{0.8571}\right) = (1.1667, 0, 0.8333, 0.3333).
\]

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1^*)</td>
<td>0.1428</td>
<td>0.0526</td>
<td>0.0637</td>
<td>0.1306</td>
<td>0.4918</td>
<td>0</td>
<td>0.3333</td>
</tr>
<tr>
<td>(v_2^*)</td>
<td>0.1428</td>
<td>0.2105</td>
<td>0.4902</td>
<td>0.2387</td>
<td>0.0041</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(u^*)</td>
<td>0.8571</td>
<td>0.6315</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.6667</td>
</tr>
<tr>
<td>(\lambda_{Z}^*)</td>
<td>0</td>
<td>0.1053</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(\lambda_{D}^*)</td>
<td>0.7143</td>
<td>0.8947</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\lambda_{E}^*)</td>
<td>0.2857</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

| CCR-I-O \((\theta^* = Z^*)\) | 0.8571 | 0.6315 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.6667 |
| \(v_1^*\) | 0.1667 | 0.0833 | 0.0833 | 0.0833 | 0.1667 | 0 | 0.5000 |
| \(v_2^*\) | 0.1667 | 0.3333 | 0.3333 | 0.3333 | 0.1667 | 1 | 0 |
| \(u^*\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(\lambda_{Z}^*\) | 0 | 0.1667 | 1 | 0 | 0 | 1 | 0 |
| \(\lambda_{D}^*\) | 0.8333 | 1.4167 | 0 | 1 | 0 | 0 | 0 |
| \(\lambda_{E}^*\) | 0.3333 | 0 | 0 | 0 | 1 | 0 | 1.500 |

| CCR-O-O \((Q^* = \phi^*)\) | 1.1667 | 1.5835 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.4999 |

Table 1.4 Optimal solution values for the CCR-I-O and CCR-O-O models on the data set of Table 1.2

Knox Lovell and Pastor (1999) demonstrated that a CCR-I-O model without outputs is meaningless. Since, in such a case, all DMUs are evaluated as inefficient with the worst efficiency score equal to zero. Therefore, a dummy output should be added to the problem. A CCR-O-O model without inputs is also meaningless. The absence of input causes efficiency scores equal to \(+\infty\). This means that all units are assigned the worst possible efficiency score. Hence, a dummy input is added to make this problem solvable. They also proved that a CCR model with a single constant input or output accords with the corresponding BCC model (BCC model is explained in the Section 1.3.2). Consecutively, the efficiency scores of both models are equal. This can be seen in Chapter 3, Example 3.1. The details in DEA models without inputs or without outputs are available in (Knox Lovell and Pastor, 1999).

Several applications of CCR model can be seen in literature such as technology forecasting for wireless communication (Anderson et al., 2008), identification of the new business area (Seol et al., 2011) and the selection of stock portfolio (Isaias et al., 2015).

Table 1.5 represents different CCR models.
Input-Oriented

<table>
<thead>
<tr>
<th>Envelopment model</th>
<th>Multiplier model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \min Z - \varepsilon (\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) )</td>
<td>( \max \theta = \sum_{r=1}^{s} u_r y_{ro} )</td>
</tr>
<tr>
<td>s.t. ( \sum_{j=1}^{n} x_{ij} \lambda_j + s_i^- = x_{i0}, \ i = 1,2, \ldots, m )</td>
<td>s.t. ( \sum_{i=1}^{m} v_i x_{i0} = 1, i = 1,2, \ldots, m )</td>
</tr>
<tr>
<td>( \sum_{j=1}^{n} y_{rj} \lambda_j - s_r^+ = y_{ro}, \ r = 1,2,\ldots, s )</td>
<td>( \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \ r = 1,2,\ldots, s )</td>
</tr>
<tr>
<td>( \lambda_j, s_i^-, s_r^+ \geq 0, \ j = 1,2,\ldots, n )</td>
<td>( u_r, v_i \geq \varepsilon &gt; 0 )</td>
</tr>
</tbody>
</table>

Output-Oriented

<table>
<thead>
<tr>
<th>Envelopment model</th>
<th>Multiplier model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max \phi + \varepsilon (\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) )</td>
<td>( \min Q = \sum_{i=1}^{m} \bar{v}<em>i x</em>{i0} )</td>
</tr>
<tr>
<td>s.t. ( \sum_{j=1}^{n} x_{ij} \lambda_j + \tilde{s}<em>i^- = x</em>{i0}, \ i = 1,2, \ldots, m )</td>
<td>s.t. ( \sum_{r=1}^{s} \bar{u}<em>r y</em>{ro} = 1, r = 1,2, \ldots, s )</td>
</tr>
<tr>
<td>( \sum_{j=1}^{n} y_{rj} \lambda_j - \phi y_{ro}, \ r = 1,2,\ldots, s )</td>
<td>( \sum_{i=1}^{m} \bar{v}<em>i x</em>{ij} - \sum_{r=1}^{s} \bar{u}<em>r y</em>{rj} \geq 0, \ i = 1,2, \ldots, m )</td>
</tr>
<tr>
<td>( \lambda_j, \tilde{s}_i^-, s_r^+ \geq 0, \ j = 1,2,\ldots, n )</td>
<td>( \bar{u}_r, \bar{v}_i \geq \varepsilon &gt; 0 )</td>
</tr>
</tbody>
</table>

Table 1.5 Different CCR models (Cooper et al., 2004, 2005, 2007 and 2011)

Not every DMU under the constant scale production (CCR/CRS) can show the relative efficiency. The size of the DMU is not static, different period maybe in a state of increasing or decreasing. Therefore, Banker et al. (1984) added a convexity assumption \( (\sum_{j=1}^{n} \lambda_j = 1) \) on the CCR model to show the change scale reward. The new model is called the BCC model. This extra assumption introduces an additional variable into the multiplier model. As it can be seen in Section 1.4, this extra variable makes it possible to effect RTS evaluations (increasing, constant, and decreasing). Therefore, the BCC model is referred to as a VRS model and distinguished from the CCR model, which is referred to as a CRS. This assumption ensures that the region specified by a set of points forms a convex set. This convexity condition means that each composite unit is a convex combination of its reference units.

1.3.2. BCC model

This section introduces BCC model with a simple illustration. Figure 1.6 exhibits four DMUs \( A(2,1), B(3,4), C(5,6), \) and \( D(4,3) \), each with one input and one output.
The dotted line is the efficient frontier of the CCR model. The bold blue broken line $ABC$ is the frontier of the BCC model. The PPS is the area consisting of the frontier together with observed activities or possible activities (with an excess of input and/or a shortfall in output compared with the frontiers). The only CCR-efficient unit is $B$; however, $A$, $B$ and $C$ are BCC-efficient with efficiency scores equal to 1.

The BCC-efficiency of $P$ is estimated as $PR/PD = 2.67/4 \simeq 0.67$, while its CCR-efficiency is smaller with the value $PQ/PD = 2/4 = 0.5$.

Actually, the CCR-efficiency score is equal to or less than BCC-efficiency score in an I-O multiplier model (Cooper et al., 2004). Reversely, in an O-O form, BCC-efficiency score is not more than the optimal objective value of the CCR model, since BCC model (1.17) imposes one additional constraint $\sum_{j=1}^{n} \lambda_j = 1$, so its feasible region is a subset of CCR feasible region (Cooper et al. 2007).

In the BCC-O-O model, efficiency of $D$ is $ST/DT = 5/3 \simeq 1.67$. This means that $D$ can be efficient by increasing its output to $1.67 \times 3 \equiv 5$. In the CCR-I-O model, efficiency of $D$ is attainable by decreasing its input to 2.

Banker, Charnes and Cooper (1984) published the BCC model whose PPS is defined as the presented set in (1.3). The BCC model differs from the CCR model only in its convexity condition. This condition demonstrates possible ways of DMUs combination.

The BCC-I-O multiplier model evaluates the efficiency of $DMU_0$ by solving the following LP:
\[ \text{Max } \theta = \sum_{r=1}^{s} u_r y_{ro} - u_o \]
\[ \text{s.t.} \]
\[ \sum_{i=1}^{m} v_i x_{io} = 1 \]
\[ \sum_{r=1}^{s} u_r y_{rf} - \sum_{i=1}^{m} v_i x_{ij} - u_o \leq 0, j = 1, \ldots, n \]
\[ u_r, v_i \geq 0, r = 1, \ldots, s, i = 1, \ldots, m, u_o \text{ is free in sign} \]

where \( \theta \) and \( u_o \) are scalars. \( u_o \) is “free in sign” and may be positive, negative or zero. The optimal value of this variable may be used to identify Returns to Scale (RTS).

The dual envelopment form of (1.16) is as follows:

\[ \text{Min } Z, \]
\[ \text{s.t.} \]
\[ \sum_{j=1}^{n} \lambda_j x_{jy} - Z x_{io} + s^* = 0, i = 1, \ldots, m \]
\[ \sum_{j=1}^{n} \lambda_j y_{rf} - y_{ro} - s^{**} = 0, r = 1, \ldots, s \]
\[ \sum_{j=1}^{n} \lambda_j = 1 \]
\[ \lambda_j \geq 0, j = 1, \ldots, n \]

This dual problem (1.17) is solved using a two-phase procedure similar to the CCR case (1.14) and (1.15). In the first phase, \( Z \) is minimized and in the second phase, the sum of the input excesses and output shortfalls is maximized, keeping \( Z = Z^* \) (the optimal objective value in the first phase).

An optimal solution of (1.17) is \((Z^*, \lambda^*, s^{*-}, s^{**})\), where \( s^{*-} \) and \( s^{**} \) represent the maximal input excesses and output shortfalls, respectively.

**Definition 1.7. BCC-Efficiency** - If an optimal solution \((Z^*, \lambda^*, s^{*-}, s^{**})\) for (1.17) satisfies \( Z^* = 1 \) and it has no slack variables \((s^{*-}, s^{**} = 0)\), then \( DMU_o \) is called “BCC-efficient”; otherwise it is BCC-inefficient. If \( Z^* = 1 \), but there still is some non-zero slacks, \( DMU_o \) is weak-efficient (Cooper et al., 2004, 2005, 2007 and 2011).

The BCC-O-O multiplier model, correspondent to (1.16), is as follows:

\[ \text{Min } Q = \sum_{i=1}^{m} \bar{v}_i x_{io} - v_o \]
\[ \text{s.t.} \]
\[ \sum_{r=1}^{s} \bar{u}_r y_{ro} = 1, r = 1, 2, \ldots, s \]
\[ \sum_{i=1}^{m} \bar{v}_i x_{ij} - \sum_{r=1}^{s} \bar{u}_r y_{rf} - v_o \geq 0, i = 1, 2, \ldots, m \]
\[ \bar{u}_r, \bar{v}_i \geq 0, v_o \text{ is free in sign} \]

The dual of (1.18) [BCC-O-O envelopment model] is as follows:
Chapter 1 – Data Envelopment Analysis

\[\begin{align*}
\text{Max } \phi \\
\text{s. t.} \\
\sum_{j=1}^{n} x_{ij} \hat{\lambda}_j &\leq x_{io}, \ i = 1, 2, \ldots, m \quad (1.19) \\
\sum_{j=1}^{n} y_{rj} \hat{\lambda}_j &\geq \phi y_{r0}, \ r = 1, 2, \ldots, s \\
\sum_{j=1}^{n} \lambda_j & = 1 \\
\hat{\lambda}_j, s_i^-, s_r^+ &\geq 0, j = 1, 2, \ldots, n
\end{align*}\]

**Example 1.5** Table 1.6 presents the results of running BCC-I-O models (1.16) and (1.17) on the data set of Table 1.2.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_1^*)</td>
<td>0.1429</td>
<td>0.0526</td>
<td>0.0833</td>
<td>0.0833</td>
<td>0.1667</td>
<td>0</td>
<td>0.3333</td>
</tr>
<tr>
<td>(v_2^*)</td>
<td>0.1429</td>
<td>0.2105</td>
<td>0.3333</td>
<td>0.3333</td>
<td>0.1667</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(u^*)</td>
<td>0.8571</td>
<td>0.6316</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.6667</td>
</tr>
<tr>
<td>(\lambda_c^*)</td>
<td>0</td>
<td>0.1053</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>I-O</td>
<td>(\lambda_d^*)</td>
<td>0.7143</td>
<td>0.8947</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\lambda_f^*)</td>
<td>0.2857</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(s_i^-)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(s_2^-)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.6667</td>
</tr>
<tr>
<td>(s_i^+)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 1.6 Optimal solution values for the BCC-I-O models on the data set of Table 1.2**

In this example, the BCC efficiency scores are equal to CCR ones. Some values of weights and slack variables are different from CCR. The BCC-I-O model gives the unique optimal solution \(\theta^* = Z^* = 0.8571\) for branch A, which yields the values of \(\lambda^*_A = (0, 0, 0, 0.7143, 0.2857, 0, 0)\) in envelopment form and \((\nu^*, u^*) = (0.1429, 0.1429, 0.8571)\) in multiplier form. Obviously, \(\theta^* = 1 \times 0.8571/(4 \times 0.1429 + 3 \times 0.1429) = 0.8571\).

According to Tables 1.2 and 1.6, branch A needs to reduce its input two to the amount used by branch D (from 3 to 2) to be efficient. Repeating this problem for other DMUs, reveals branches C, D, E, and F are efficient.

Applying (1.17) for branch F gives \(Z_F^* = 1\) indicating this unit is on the efficient frontier. However, maximizing the slack variables in (1.17) for branch F reveals a non-zero slack: \((s_1^+, s_2^-, s_3^+)^* = (2, 0, 0)\). This solution shows that branch F can reduce its first input by 2 units to meet \(DMU_F\) on the efficient frontier with totally zero slacks. Therefore, branch F according to Definition 1.7 is weakly efficient.
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Knox Lovell and Pastor (1999) considered the behavior of a BCC model without inputs or without outputs. A BCC-I-O (BCC-O-O) model with a single constant output (input) is correspondent to a BCC-I-O (BCC-O-O) model without outputs (inputs). The proof is available in (Knox Lovell and Pastor, 1999).

The BCC model is a reference model in several applications such as measuring the efficiency of facility layout in manufacturing systems (Toloo and Nalchigar, 2009), evaluating the efficiency of European health services (Kleine et al., 2014), and measuring the efficiency and productivity growth of new-technology based firms (Grilo and Santos, 2015).

Integrating both I-O and O-O models in just one model gives the Additive model.

1.3.3. Additive model

Charnes and his colleagues (1985) combined both I-O and O-O forms of BCC model in a single model, called “Additive model”. Indeed, the “Additive model” considers the outputs shortfall \( s^- \) and the inputs excess \( s^+ \) simultaneously:

\[
\begin{align*}
\text{Max } \theta &= \sum_{i=1}^{m} s^-_i + \sum_{r=1}^{s} s^+_r \\
\text{s.t. } &\sum_{j=1}^{n} \lambda_j x_{ij} + s^-_i = x_{io}, i = 1, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} - s^+_r = y_{ro}, r = 1, \ldots, s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j, s^-_i, s^+_r \geq 0, r = 1, \ldots, s, i = 1, \ldots, m
\end{align*}
\]  

(1.20)

The dual problem of (1.20) is expressed as follows:

\[
\begin{align*}
\text{Min } Z &= \sum_{i=1}^{m} v_i x_{io} - \sum_{r=1}^{s} u_r y_{ro} - u_o \\
\text{s.t. } &\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} - u_o \geq 0, j = 1, \ldots, n \\
&v_i, u_r \geq 0, r = 1, \ldots, s, i = 1, \ldots, m, u_o \text{ is free in sign}
\end{align*}
\]  

(1.21)

**Definition 1.8. Additive efficiency** - When the optimal solution of (1.20) is \((\lambda^*, s^-^*, s^+^*)\), \(DMU_o\) is Additive-efficient if and only if \( s^-^* \) and \( s^+^* = 0 \).

In the Additive model, the efficiency score does not have a scalar value. The slacks in the objective function of (1.20) are maximized in order to identify the inefficiencies. Normally, any non-zero slacks cause a variation in the input or output proportions. The main constraints in (1.20) gives the same PPS as BCC model. The omission of \( \sum_{j=1}^{n} \lambda_j = 1 \) from (1.20) gives an Additive model based on CCR.

Figure 1.7 illustrates four DMUs \( A, B, C, \) and \( D \), each with one input and one output. \( DMU_D \) can be replaced by the arrows \( s^- \) and \( s^+ \). The maximal value of \( s^- + s^+ \) is attained at \( B \), which is most distant from \( D \).
**Theorem 1.3** $DMU_o$ is Additive-efficient if and only if it is BCC-efficient.

A proof of this theorem can be found in Ahn et al. (1988).

**Example 1.6** Table 1.7 compares the result of the BCC model (1.17) and the Additive model (1.20) using the data set of Table 1.2.

![Figure 1.7 Additive model (Cooper et al., 2004)](image)

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^{*-}$</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$s_2^{*-}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$s_1^{*+}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 1.7 The slack variables of Additive model based on data set of Table 1.2**

Table 1.7 shows the full BCC and Additive efficiency of bank branches $C$, $D$, and $E$ (Theorem 1.3). However, the BCC efficiency score of $F$ is equal to 1, but its input excess ($s_1^{*-F} = 2$) causes the weak efficiency in both models. The other DMUs are all inefficient (they neither have efficiency equal to 1 nor slacks equal to zero).

The detailed information in other DEA models such as Slack Base Measurement (SBM) (Tone, 2002) and Free Disposal Hull (FDH) (Deprins et al., 1984) is available in (Cooper et al., 2004, 2005, 2007 and 2011; Zhu, 2015). Some weight restricted DEA models and various algorithms for complete ranking of units in DEA are introduced in the Chapters 4 and 5, respectively.

As presented, CCR is a CRS model and BCC is a VRS model. In what follows, the RTS characteristics of DEA models is just shown through a theorem.
1.4. Returns to Scale (RTS) characterizations

In this section, the aim is to just briefly review the subject of Returns to Scale (RTS) [increasing, decreasing, or constant] in the BCC-I-O multiplier model (1.16) (Banker et al., 1984). Banker (1984) expansively described RTS in CCR model in his work.

In this regard, Banker and Thrall (1992) proved following theorem:

**Theorem 1.4** When $DMU_o$ is efficient, the optimal value of variable $u_o$, i.e. $u_o^*$ in the corresponding formulation of BCC model (1.16), identifies the characterization of RTS through following conditions:

1- “Increasing Returns to Scale” (IRS) holds for the point $(\bar{x}_o, \bar{y}_o)$ if and only if $u_o^* < 0$ for all optimal solutions;

2- “Constant Returns to Scale” (CRS) holds for the point $(\bar{x}_o, \bar{y}_o)$ if and only if $u_o^* = 0$ for at least one optimal solution;

3- “Decreasing Returns to Scale” (DRS) holds for the point $(\bar{x}_o, \bar{y}_o)$ if and only if $u_o^* > 0$ for all optimal solutions.

The point $(\bar{x}_o, \bar{y}_o)$ is supposed to be the coordinates of a point on the efficient frontier. Nevertheless, the efficiency status of the point has no need to be concerned since efficiency can always be achieved. Let suppose that $DMU_o$ is an inefficient BCC point via (1.16). By applying the dual form (1.17), and finding the optimal value of $DMU_o (Z^*, \lambda^*, s^{-*}, s^{+-})$, the inefficient point can be projected on the efficient frontier as follows:

\[ \bar{x}_o = Z^x x_{lo} - s_i^{-*} = \sum_{j=1}^{n} \lambda_j x_{ij}, i = 1, \ldots, m \]  
(1.22)

\[ \bar{y}_o = y_{ro} + s_r^{+-} = \sum_{j=1}^{n} \lambda_j y_{rj}, r = 1, \ldots, s \]  
(1.23)

Ramanathan (2003) explained CRS means that a unit is able to scale the inputs and the outputs linearly without increasing or decreasing efficiency; while VRS is defined as the ability of that unit to catch up, given limitations such as different constraints that may cause the unit not to be operating at optimal scale.

Figure 1.8, basically, illustrates RTS characteristics, based on 6 DMUs, in the case of 1 input and 1 output. The line connects DMUs B and C that pass from origin is the CCR efficient frontier. These two DMUs are located between two different RTS and presents CRS. The segment ABCD composes the BCC efficient frontier, which shows DRS, CRS, and IRS. $DMU_E$ can be projected on the CCR efficient frontier [by (1.22) and (1.23)], thus this DMU shows CRS (Banker, 1984). On the line segment AB, to the left of B, IRS occurs, where on the line segment CD, to the right of C, DRS occurs. Applying (1.22) and (1.23), to project $DMU_E$ on the efficient frontier, gives point $E'$ on the efficient frontier, therefore $DMU_E$ exhibits IRS in a BCC-I-O model, whereas this projection under BCC-O-O model, gives point $E''$ that shows DRS behaviour. Hence, the concept of RTS is affected by orientation of the model [the BCC-I-O and BCC-O-O models yield different projection points on the VRS frontier] (Seiford and Zhu, 1999a).
Further details, theorems and their proofs can be found in (Banker et al., 1984; Tone, 1996; Seiford and Zhu, 1999; Banker et al., 2004).

This chapter is finalized by addressing some of the main strengths and limitations of the DEA method.

1.5. Some strengths and limitations in DEA

Some of the main features of DEA that make it a solid tool to measure the efficiency are listed below (Cooper et al., 2004, 2005, 2007 and 2011; Zhu 2002 and 2015):

- DEA is able to handle multiple inputs and multiple outputs that can be measured in diverse units. For example, in measuring banks efficiencies, in the presence of two inputs (capital and number of employees) and one output (profit), there are two different units: capital is measured in money units while employees are measured in number units.

The input-output data table of DEA is similar to evaluation table in some MCDA methods such as PROMETHEE.

- DEA measures the efficiency of units based on their numerical value, and not by applying the priori idea of people in importance of factors. DEA maximizes the objective function (profit) using available data.

This characteristic of DEA may help solve multicriteria problems in evaluating performance of units by means of less interactions with DMs.

- DEA is a non-parametric method and it does not require any basic assumption of a functional form relating inputs to outputs. DEA builds its own functional form, using the set of
inputs and outputs of different DMUs. However, it should be noted that the self-built functional form of DEA is a kind of weighted sum.

- DEA focuses on the best-practice frontier instead of central tendencies (Charnes et al., 1994). Each unit is compared to an efficient unit or several efficient units on the efficient frontier to distinguish inefficient units.

- In DEA, each inefficient DMU can be projected on the efficient frontier; thus, its improvements to be efficient can be identified.

- In DEA, there is only the non-negativity restriction on the multipliers (weights). This flexibility of weights helps DMUs to achieve the best possible ratings in their self-evaluation.

The same features that make DEA a good tool in measuring efficiency can also cause some problems. The DEA’s limitations should be considered when choosing whether to use DEA. Some of these limitations are as follows:

- DEA needs solving $n$ different LP for a set of $N = \{1, \ldots, n\}$ DMUs. Therefore, problems with many DMUs leads to intensive computations.

- DEA is an extreme point technique (the efficiency frontier is composed by the DMUs with the best performance). Therefore, outliers due to measurement errors may affect DEA results significantly if these errors falsely results in observations determining the frontier. Since that will then affect the efficiency scores of all observations for which the observation on the frontier forms the basis (reference DMU), causing their inefficiency enhanced.

- DEA is a non-parametric technique, which makes statistical hypothesis tests difficult. For example, estimating confidence intervals, within which DEA’s efficiencies are calculated, may not be possible. The confidence interval can be important in robustness and sensitivity analysis of decision-making problems.

- DEA is generally designed to measure efficiency of units in the presence of one/more inputs and one/more outputs. However, in some real applications, there is only inputs or outputs. This problem can be solved by adding a dummy output or input. Example 1.2 shows a sample with two inputs and without outputs. This example was solved by adding a dummy output (a vector with all entities equal to 1).

- DEA shows how well a DMU acts in comparison to its peers (set of efficient units on the efficient frontier). This indicates the relative efficiency that converges very slowly to absolute efficiency. It means a DMU is not compared with a defined maximum index. This may cause the possibility of having non-complete ranking that gives several efficient units on the efficient frontier with efficiency scores equal to 1.

- Further, one of the main limitations of DEA is the difficulty of applying flexible weights. The classical DEA models such as CCR and BCC are originally unbounded models (except the non-negativity constraint on weights). Through these models, a DMU can put a very high weight
value on one or some inputs and/or outputs while it assigns zero or very small weight values on other inputs and/or outputs. This full flexibility characteristic of DEA may cause undesirable consequences, since it may be difficult to justify the efficiency of each unit. In two main situations the additional control on weights is needed (Charnes et al., 1994):

1- When the number of inputs and outputs is relatively large compared with the number of DMUs, many DMUs are considered as efficient. Consequently, discrimination power of DEA is decreased. Thus, applying bounds on weights can help to discriminate stronger between DMUs.

2- The classical DEA models ignore additional information such as preferences in computing efficiency, which can be in contradictions with expert opinions.

1.6. Conclusion

In this chapter, an introduction of DEA methodology in measuring the efficiency scores of DMUs is presented. The chapter is started with the most general definitions and concepts of DEA: the Production Possibility Set and efficiency concept in this set. Then, introducing the top basic models helped to deepen in this topic. Further, noting on Returns to Scales clarifies some attributes in the introduced DEA models. As a final point, some strengths and limitations of DEA are identified.

The DEA strengths leads us to apply it on some multicriteria methods with the aim of helping DMs in some complex situations, where there is no clear preferences between criteria weights. Bagherikahvarin and De Smet (2016b) suggested an algorithm to compute weights in PROMETHEE, which are compatible with the DEA analysis. This contribution is available in Chapter 6.

One of the main limitations of DEA is the deficiency of discrimination power between efficient DMUs. As discussed in CCR and BCC models, DEA divide units into efficient and inefficient DMUs. There is no distinction between efficient units. Thus, several approaches are developed to increase the discrimination power of DEA, such as cross efficiency, super efficiency, benchmarking, multivariate statistics, and DEA-MCDA integrated techniques (these techniques are explained in Chapter 5: Section 5.2). In this thesis, a multicriteria method (PROMETHEE II) is integrated with DEA to develop a new ranking algorithm, which decreases the number of ties in DEA results. The PROMETHEE weight stability intervals are applied as the weight restrictions in DEA to reduce the space of solution; hence decreasing the number of efficient units to increase the discriminatory power of DEA. Furthermore, the net flow scores of PROMETHEE is used as the output vector in DEA. This matrix includes preferential information of the decision-making problem (Bagherikahvarin, M. and De Smet, Y., 2016a). In some extensions of DEA problem such as weight restricted DEA models, the increased number of restrictions may cause some computational difficulties and/or infeasibility. Bagherikahvarin and De Smet (2016a) proposed a weight restricted DEA model based on absolute weight restrictions that has less constraints than models built on Assurance Regions (AR).
In another contribution of this thesis, a two-step algorithm is proposed to obtain a complete ranking in DEA (Bagherikahvarin, M., 2016). In the first step, DEA is applied between each pair of units to generate a pairwise comparison matrix; in the second step, the PROMETHEE II is used to aggregate final scores from the generated pairwise comparison matrix and give a complete ranking.

In the next chapter, it is focused on summarizing the multicriteria methodologies and their principles.
Chapter Two

Review of literature II: Multi-Criteria Decision Aid (MCDA)

Abstract
This chapter presents a summary of the multicriteria paradigm. The discussion covers some main concepts and characteristics in the field of Multi-Criteria Decision Aid (MCDA) as well as an introduction to some classical methods. This chapter is completed by noting generally some strengths and limitations of multicriteria methods, which leads us to integrate it with some helping tools such as Data Envelopment Analysis (DEA) to support Decision Makers (DMs).

2.1. Introduction
MCDA is a sub-discipline of Operations Research (OR). It offers a set of methods, which focus on several objectives at the same time. MCDA methods are based on a decision aid activity that exists to support the DM in taking better decisions (Roy, 1993). The distinctiveness feature of MCDA compared to other decision aiding techniques is that it takes into consideration several criteria simultaneously to help the DM.

This chapter first briefly introduces the basics of unicriterion and multicriteria paradigms in order to justify the choice to use such a paradigm. Then, it presents the decision aid context in multicriteria situations. At that point, the principal definitions and different types of MCDA problems are concisely presented besides introducing some techniques such as: MAUT (Keeney and Raiffa, 1976), AHP (Saaty, 1980), MACBETH (Bana e Costa et al., 2005), ELECTRE (Roy, 1968), and PROMETHEE (Brans, 1982). The focus of this chapter is more on the PROMETHEE and its graphical tool, GAIA method, since it is the main multicriteria method used in this thesis. For more details in the MCDA definitions and methods, one can refer to (Vincke, 1992; Bana e Costa, 1993; Bouyssou, 1993; Roy, 1993 and 1996; De Smet and Vincke, 2002; Figueira et al., 2005). This chapter is closed by noting some general features of MCDA methods that motivate us to use integrated approaches.
2.1.1. The unicriterion Paradigm

A unicriterion optimization problem can be formulated as follows:

\[
\max f(x) : x \in \mathcal{A}
\]  \hspace{1cm} (2.1)

where \( f \) is the objective function optimizing (without loss of generality to maximize the function) the solutions denoted by \( x \). The objective function \( f \) is also called the criterion on which the potential solutions are optimized. \( \mathcal{A} \) is the set of feasible solutions. The optimal solution is:

\[
x' = \arg \max \{ f(x) | x \in \mathcal{A} \}
\]  \hspace{1cm} (2.2)

Vincke (1992) defined criterion as follows:

**Definition 2.1. Criterion** – “A criterion is defined as a function \( f \), defined on \( \mathcal{A} \), taking its values in a totally ordered set, and representing the DM’s preferences according to some points of view.”

In these problems, the set of alternatives \( \mathcal{A} \) has been defined implicitly (i.e. by formulating constraints) or explicitly (i.e. enumeration). Some typical examples of unicriterion paradigm are Linear and Non-Linear Programming (LP or NLP). Data Envelopment Analysis (DEA) (Cooper et al., 2005) and Travelling Salesman Problem (TSP) (Applegate et al., 2006) are two particular LPs.

An LP can be formulated as follows:

\[
\max c^t x \\
\text{s.t.} \\
Ax \leq b \\
x \geq 0
\]  \hspace{1cm} (2.3)

where \( b \) is the right hand-side vector and \( c \) is the vector of coefficients, \( \mathcal{A} \) is the matrix of coefficients and \( x \) is the vector of variables (continuous, Boolean or integer variables) to be determined. The simplex algorithm (Dantzig, 1951) and the interior point method (Karmarkar, 1984) are two well-structured methods to solve an LP problem.

**Example 2.1** A typical unicriterion example is DEA. Chapter 1 explained briefly the DEA methodology and some of its classical models. Measuring the efficiency of bank branches is such a case in DEA (Ebrahimnejad et al., 2014 and Zervopoulos et al., 2016). Example 1.1 considers the efficiency of each bank’s branch by maximizing the ratio of the staff size (input) to the value of accounts (output).

The unicriterion (mono-objective) problems are mathematically well-stated. Nevertheless, they take only an objective into consideration. The next sub-section presents in brief some reasons of moving from unicriterion to multicriteria paradigm in solving decision-making problems with several criteria/objectives.
2.1.2. From the unicriterion paradigm to the multicriteria paradigm

The models that belong to unicriterion paradigm optimize only a single criterion (objective) (e.g. maximizing profit, efficiency or power vs. minimizing costs, delays or risk). This single criterion synthesizes all the features of the problem. Some other criteria can be satisfied by considering them as constraints. It should be noted that several real world decision-making problems could not be considered as a unicriterion problem. They usually involve several conflicting criteria concurrently. For example, the construction of a new car factory would involve several issues, at the same time. These issues could be very different in nature, such as economic (cost, profit), social (mobility, society's consumption), environmental (pollution, noise), etc. This simple example shows that even if the unicriterion paradigm is a common used approach, since there is no achievable optimum solution in these situations, it cannot be applied when several criteria have to be simultaneously taken into account. Additionally, a multicriteria model is closer to the real life problems; thus, it is a richer problem in its structure to take supportable decisions for future (Aouni and Laflamme, 2014).

Some features of unicriterion optimization problems are categorized into three following groups (Scharlig, 1985):

1- **Stable set of actions**: During an optimization process, former than analysis, the set of options is assumed to be known. This set does not change through the decision process. While, in some situations, this set is likely to change by introducing new alternatives or removing some existing ones.

2- **Elite actions**: Each elite action is supposed to perfectly reveal all the features of the problem; i.e. it performs well enough on all existing criteria.

3- **Transitivity**: The transitivity of DM’s preferences is needed to rank options from the worst to the best in order to find an optimal solution.

These features led researchers towards multicriteria paradigm.

2.1.3. The multicriteria paradigm

A multicriteria problem can be formulated as:

$$
\max \{ f_1(x), f_2(x), \ldots, f_q(x) \mid x \in \mathcal{A} \}
$$

(2.4)

where $\mathcal{A}$ is a set of potential actions (alternatives), $x$ is a solution of $\mathcal{A}$ and $f_k(\cdot): \mathcal{A} \rightarrow R^+, k \in \{1,2,\ldots,q\}$ build a set of $q$ evaluated criteria: $F = \{f_1, f_2, \ldots, f_q\}$, which without loss of generality needs to be maximized (for each criterion $k$, map each potential action to a real positive value). Multicriteria decision problems are mathematically ill-defined, since, in these problems, most of the time, there is no optimal solution [like optimal solution of LP in (2.3)] that is better than any other solution when considering all criteria, simultaneously (Vincke, 1992).

Multicriteria paradigm follows two main views: it involves finding a set of Pareto optimal solutions (Definitions 2.5 and 2.6 define the “Pareto dominance” and “Pareto efficient” solutions,
respectively) with respect to a set of criteria, and it takes into account the choice of alternatives, criteria and preferences of DMs. In this order, the analyst helps DMs in the process of decision-making by gathering information about different features of the problem such as the importance of criteria (weights); then, put them on a proper model in order to find the compromise solutions that best satisfy them.

There exists a possibility of conflicts between the criteria being considered, such that satisfying one criterion may dissatisfy other ones (e.g. car road policies in reducing motorized transport has a positive environmental effect, but it is likely to have a harmful effect in terms of social equity, at least in the short term). Multicriteria methods are used to accept such inherent conflicts, by revealing and analyzing them in order to find ways to lessen the existing conflicts.

In some cases, incomparability and incommensurability occur in complex decision-making problems. Multicriteria approaches are in general able to take them into account when deriving a choice or ranking structure. Incomparability does not necessarily lead to a situation where no decision-making is possible (refer to Figure 2.2). It means that additional approaches such as iterations and back stepping in the decision process have to be used. This suggests that the analysis of the decision problem does not allow identifying a global preference or indifference relation between two or more items (Roy, 1996). Incommensurability does not mean the impossibility of comparison but it means the absence of a common unit of measurement. Sunstein (1997) defined incommensurability as follows: “The incommensurability occurs when the options cannot be ranged along a single metric without doing violence to our considered judgments about how these goods are best characterized.”

In some situations, a unicriterion model such as DEA needs to consider DMs priorities in its structure to be able to discriminate among actions. The unicriterion models may also be used as a tool in multicriteria problems in order to help DMs. For example, a DEA model can be applied to propose a weight vector where there is not enough distinct preferences between criteria (Bagherikahvarin and De Smet, 2016b).

During the last four decades, some general approaches were used to integrate these two paradigms (Tone, 1996; Yu et al. 1996; Nakayama et al., 2002; Kleine, 2004; Ksala and Aksub, 2006; Zhao et al., 2006; Emrouznejad et al., 2008; Jahanshahloo et al., 2009; Bagherikahvarin and De Smet, 2016a and 2016b; Bagherikahvarin, 2016). In the following chapters, some of these integrated approaches are presented.

Facilitating decision-making needs some tools that are designated for this reason. In this regard, the next sub-section introduces decision aid concept.

### 2.1.4. Decision aid

Scientists from different fields considered the philosophy behind decision-making and the acceptance or rejection of consequences of the decision for centuries. For example, the science of probability appeared with the aim of determining some decision aid tools to explain the life likelihoods and human behaviour using mathematical concepts (Roy, 1993). Through a scientific
analysis, the unknown behind human thoughts and decision-making could be understood. This analysis needs some convenient tools, which can be named as “decision aid” tools. It takes shape under the name of Management Sciences (MS) and Operations Research (OR).

Bernard Roy (1993) defined “decision aid” as follows:

**Definition 2.2.** “The activity of one who, in ways we call scientific, helps to obtain elements of answers to questions asked by actors involved in a decision-making process, elements helping to clarify this decision in order to provide actors with the most favourable conditions possible for that type of behaviour, which will increase coherence between the evolution of the process on the one hand and the goals and/or systems of values within which these actors operate on the other”.

Actually, decision-making is a complex activity composed of several conflicting criteria, which together leads us taking a decision. Multi-Criteria Decision Aid (MCDA) is a method emerged as a decision aid tool encountering the complexity of decision-making. Next sub-section considers MCDA and its process.

### 2.1.5. What is Multi-Criteria Decision Aid?

As mentioned above, the main objective of MCDA is to provide a decision aid tool in complicated problems encountered several factors simultaneously, regard to the DMs priorities and knowledge on problem. Accordingly, the optimal solution (in the traditional optimization sense) leaves room for the compromise solutions faced with multiple conflicting criteria.

The survey in decision-making problems has a long history. It dates back to the 18th century, when De Condorcet (1785) published the context of “the plurality of voices” in an essay. Nevertheless, it was after the Second World War that the science behind reasoning decisions built the name of Management Sciences (MS) with two main courses: Operations Research (OR) and decision aid. The mathematical modelling of decision-making problems in OR were highlighted in the work of Von Neumann and Morgenstern (1944) by introducing “game theory”, Samuelson (1948) by entering the “consumption theory” in economics, Arrow (1951) by suggesting an alternative way in the “choice theory”, and Luce and Raiffa (1957) by opening discussion in psycho mathematical facets of individual decisions. Simon (1960) suggested decomposing the decision process in three main steps: 1- Collecting information about the problem (Intelligence), 2- Defining the criteria and creating possible solutions (Design) and 3- Estimating the solutions and choosing the best answer (Choice). In the late 60s, the Simon’s decision process was improved to MCDA notions and expressions. Goal Programming, initiated in 1955 (Charnes et al., 1955), was among the first extension of LP problems in connection with MCDA principals. The first international Multi-Criteria Decision Making (MCDM) conference was held in 1972. From then on, an impressive progress was seen in different domains of MCDA. Several multicriteria approaches were developed and built MCDA as we know today (Keeney and Raiffa, 1976 and 1993; Saaty, 1980; Brans, 1982; Roy, 1991 and 1996). For a review on the early history of MCDM, it is referred to Koksalan et al. (2013).
Multi-Criteria Decision Aid/ Multi-Criteria Decision Analysis (MCDA) and Multi-Criteria Decision Making (MCDM) are different terminologies of this discipline. MCDA is developed in Europe and it refers to the European school (French school) of this thought while MCDM is developed in the United States of America, known as American school. This distinction is not only in the name but also in the beneath philosophy of how to help DMs to make better decisions. MCDA investigates discrete methods and outranking relations to understand the preference model behind the DM’s decisions. These methods include for instance graphical tools to represent the data. MCDM method heavily relies on precise knowledge and value judgements and the goal is reaching an optimal decision using utility/value functions and multi-objective optimization (Roy, 1996).

Unlike many techniques, which let model find solutions, when the problem is given, MCDA leads to problem identification and structuring. It provides the theoretical background to build a model (Belton and Stewart, 2002).

The MCDA process is shown in Figure 2.1, inspired by Belton and Stewart (2002).

![Figure 2.1. MCDA process inspired by Belton and Stewart (2002)](image-url)
Chapter 2 – Multi-Criteria Decision Aid

The first step in MCDA is identifying and structuring problem by detecting important factors such as alternatives, criteria, preference values and goals. In the next step, the key identified issues and values may help DMs to proceed building a model. Then, DMs may reflect their ideas on the model in a way that supports a more detailed and precise evaluation of problem to move forward. In the final step, the model is used to synthesize gathered information to give possible choices. Robustness and sensitivity analyses in the process of problem solving can be the extra steps in identifying or adding new alternatives. Indeed, the main goal of MCDA is to develop a plan to be implemented in the future or/and helping DMs in structuring and solving problems that may change over time (Belton and Stewart, 2002).

The context of the next two examples gives a more precise idea of what a multicriteria problem is.

Example 2.2 Choosing a location for a nuclear power plant - In making the decision of whether to build a nuclear plant or not, choosing a proper location is one of the most important issues. After identifying possible locations, the associated criteria should be recognized such as proximity of plant to other facilities (distance from cities, water drinking resources, public transportsations, green spaces, the user factories, etc.), environmental impact (pollution), costs (construction, maintenance, equipment, material, etc.), revenue, etc. In a complex critical issue like this problem (facing several conflicting criteria) when there are multiple stakeholders who are deeply affected from the consequences, the decisions cannot be made intuitively. Structuring the issue like a multicriteria problem, gives us the opportunity of ranking different locations according to different criteria.

Example 2.3 Ranking Wallonia municipalities- Assessing the level of well-being in different municipalities of Wallonia in Belgium is a typical issue to give a life quality index for those who live there. One establishes a ranking of municipalities based on criteria such as health-care, accommodation, education and training, employment, income and purchasing power, mobility, quality of life and environment, proximity to shopping centers, security of life and environment, administrative institutions, the situation of marital and family, and the revenue of municipality. The data set of this example (Charlier et al., 2014) is used by Bagherikahvarin and De Smet (2016) to test the validity of a ranking approach.

The following section explains the main concepts and definitions in the process of MCDA.

2.2. Main Concepts in MCDA

Structuring a decision problem is a core task in multicriteria activities. As explained, it requires identifying and understanding the key elements and definitions. This section defines some main concepts in a multicriteria problem such as “alternatives” and the way to evaluate and compare them, “criteria”, the important factors for DMs, different type of MCDA problems and the preference modelling.
2.2.1. Alternatives and criteria

Two fundamental concepts in MCDA are alternatives and criteria.

**What is an alternative (action) in MCDA?**

Philippe Vincke (1992) denoted the set of “actions” by \( \mathcal{A} \) and defined it in MCDA as follows:

**Definition 2.3. Actions (alternatives) - “The set of objects, decisions, candidates, projects ... to be explored during the decision procedure is called the set of actions”**.

He also explained how this set, before ongoing with the decision process, may be defined (Vincke, 1992):

- By implicitly defining the properties, which characterizes its elements, when it is infinite or finite but too large for a possible enumeration. Choosing the price of an article in an interval \( (\mathcal{A}) \) of the real line would be an example of this kind of sets.

- By explicitly listing all members of this set when it is defined finite and small enough for a possible enumeration. A location to be chosen among 20 options \( (\mathcal{A}) \) for constructing a school, nuclear power plant, laboratory, etc. would be an example of this case.

In MCDA, the word “action” in the set \( \mathcal{A} \) does not refer to an activity but to any type of results or solutions (Laaribi, 2000). The set \( \mathcal{A} \) can be stable and it is not changed during the decision process. Nevertheless, in some decision situations, some other alternatives should be added to the current set; thus, this set can be also evolutive and change during this period. Furthermore, \( \mathcal{A} \) can be either “globalized”, meaning any action in \( \mathcal{A} \) excludes any others, or “fragmented”, meaning combining actions of \( \mathcal{A} \) generates possible solution of decision problem (Vincke, 1992). The term “alternative” can be seen in many problems instead of “action”.

The alternatives in a decision problem are compared according to different factors. These factors are called “criteria”.

**What is a criterion in MCDA?**

“Criterion” means “a standard for judging” in the dictionary form (Belton and Stewart, 2002). It can also be considered as a scale to evaluate and compare potential alternatives according to a well-defined point of view (Roy, 2005). Different MCDA methods often use different terms for “criterion”: goal, objective or attribute.

Goals can be defined as levels of ambition or priority values, while in this regard, objective is generally something measurable. For example, in the problem of choosing a location for a nuclear power plant, one criterion may be the footprint of CO\(_2\) in environment. Then, a goal would be a certain limit of pollution with CO\(_2\), whereas the objective is minimizing this footprint. The way that goal and objective can be achievable defines an “attribute” for criterion. An “attribute” provides the means of evaluating the levels of achievement for a criterion (or objective/goal). It usually shows the performance level or properties of an alternative. Therefore, an “attribute”
should be defined on a proper scale with the main issue of decision problem under evaluation. The descriptions of three main scales are as follows (Roy, 2005):

a) “Quantitative scales”: These types of scales also called “cardinal” or “ratio” scales. They assign a clear and meaningful quantity for each attribute by a number. Zero shows the absence of quantity.

b) “Qualitative scales”: These are also called “ordinal” scales, when the level of difference between two attributes does not have a measurable value (such as verbal comparisons of two criteria). In the Example 2.3, the criterion of quality of life is measured qualitatively in five levels: very bad, bad, medium, good and very good.

c) “Interval scales”: These scales have measurements where the difference between values is meaningful. In other words, equal increments in value on a partial value function should represent equal trade-offs with other criteria. For example, the difference between a 100 degrees F and 80 degrees F is the same difference as between 20 degrees F and 0 degrees F.

The criteria and their attributes are the main concerns of the DM in the process of structuring a decision-making problem; thus, their scale of measures should be meaningful for DMs. It needs to note that sometimes, a criterion as defined in Definition 2.1 is very general or unclear. Therefore, choosing a “consistent family of criteria”, and the scale measures of criteria and their attributes are a core task in decision-making problems. In this regard, there are various techniques to define more precisely a criterion. For example, after choosing the criteria, depending on the size of the problem, a hierarchy of criteria can be built (finding sub-criteria, which show criteria more clear in details). In Example 2.2, choosing a location for nuclear power plant, cost as a criterion divided into four or more sub-criteria: cost of construction, maintenance, equipment, material, etc. Keeney (1992) proposed some approaches to direct the search and the thinking about values as the main driving force in identifying the true criteria. Roy (1985 and 1996) defined the “consistent family of criteria” as follows:

Definition 2.4. The consistent family of criteria – A set $F$ of criteria, which has following three properties, is consistent (Roy, 1985 and 1996):

1- **Exhaustive**: If $f_k(a_i) = f_k(a_j), \forall k$, then there is no preference between $a_i$ and $a_j$.

Exhaustiveness of evaluation means that it should provide a complete evaluation of set $\mathcal{A}$ and guaranty that all aspects of the problem is considered.

2- **Cohesive**: If $f_k(a_i) = f_k(a_j)$, and $\forall l \neq k: f_l(a_i) > f_l(a_j)$ $a_i$ is preferred to $a_j$ for $f_l$; then, $a_i$ is preferred to $a_j$.

Cohesiveness means that each criterion in $\mathcal{A}$ has a specific direction of preferences: minimized or maximized, contributing to the overall performance of problem.

3- **Non-redundant**: This property implies the exclusion of unnecessary criteria from $F$. 
Non-redundancy means that suppression of a criterion from $F$ lead to a set of criteria satisfying two first properties.

These three properties formalize selection of criteria.

In the next chapter, the criteria are compared with inputs and outputs of a specific kind of optimization problem: DEA.

The evaluation table of a multicriteria problem is defined according to its alternatives and criteria. Next sub-section considers different types of MCDA problems: choice, sorting, ranking, description, design and portfolio.

2.2.2. The MCDA problem

Mostly, the purpose of a MCDA method is to provide answers for a specific multicriteria problem. This purpose can be: choosing an alternative from $\mathcal{A}$, allocating different alternatives to different clusters, evaluating the performance of all alternatives and rank them, etc.

As described in the Section 2.2.1, modelling decision problems can be divided into two main classes based on the related type of actions and criteria (Vincke, 1992). The first class of problems includes those where the set of actions are implicitly defined (i.e. using constraints). The number of actions is infinite or uncountable and it needs to be built during the decision-making process. The problems in this class are generally called multi-objective problems (Evans and Steuer, 1973). The second class involves those, where actions are explicitly defined. The set $\mathcal{A}$ is finite. This class of problems are commonly named as multi-attribute or multicriteria evaluation problems. This thesis studies the second class of problems where the set $\mathcal{A}$ is finite.

Roy (1976) defined the four main types of decision-making problems in the second class as follows:

- The “choice problem” ($\alpha$ problem): The process of choosing one action or a subset of possible actions from $\mathcal{A}$ with respect to the family of criteria. Choosing one or several proper locations for building a nuclear plant would be an example of $\alpha$ problem. Another example would be choosing an appropriate plan for water resource management (Yilmaz and Yurdusev, 2011).

- The “sorting problem” ($\beta$ problem): The process of dividing $\mathcal{A}$ into subsets when actions must be sorted into pre-determined ordered (or not-ordered) categories. One $\beta$ problem would be sorting information in different categories in Gmail.

- The “ranking problem” ($\gamma$ problem): The process of complete or partial pre-order of actions in the set $\mathcal{A}$ increasingly or decreasingly. It is used to rate actions by comparing them to other possible solutions. Academic ranking of world universities (retrieved from http://www.arwu.org) is an example of $\gamma$ problem.

- The “description problem” ($\delta$ problem): The process of describing actions and their criteria (attributes) in a formalized manner to help the DM in evaluating the actions, criteria and their related preferences, which can lead to a solution of the decision problem. The GAIA
visualization tool that is explained later in PROMETHEE part of this thesis is one of the appropriate descriptive tools.

Belton and Stewart (2002) have added two more types:

- The “design problem”: The process of searching, identifying, or creating new actions to face the goals and ambitions recognized through the MCDA process. Keeney (1992) called the design problem “value focused thinking.”

- The “portfolio problem”: The process of choosing actions from a large set of possibilities, considering not only the features of the individual actions but also the way they interact (the positive or negative synergies).

Nominal (ordinal) classification and clustering are among some other types of problems that can be solved by using MCDA (Doumpos and Zopounidis, 2002).

Table 2.1 represents the evaluation table of a multicriteria problem.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>$f_1(\cdot)$</th>
<th>$f_2(\cdot)$</th>
<th>...</th>
<th>$f_q(\cdot)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$f_1(a_1)$</td>
<td>$f_2(a_1)$</td>
<td>...</td>
<td>$f_q(a_1)$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$f_1(a_2)$</td>
<td>$f_2(a_2)$</td>
<td>...</td>
<td>$f_q(a_2)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_n$</td>
<td>$f_1(a_n)$</td>
<td>$f_2(a_n)$</td>
<td>...</td>
<td>$f_q(a_n)$</td>
</tr>
</tbody>
</table>

Table 2.1 Evaluation table (Brans et al. 1984)

The only information that can be extracted from the evaluation table is the “Pareto dominance” relation. This relation is defined in the following sub-section.

2.2.3. Dominance and Efficiency

**Definition 2.5. Dominance** (Vinceke, 1992) - Let $D$ indicate the “Pareto Dominance” relation:

$$a_i D a_j \iff f_k(a_i) \geq f_k(a_j); a_i, a_j \in A, \forall k \in \{1, 2, ..., q\}$$

and $\exists k \in \{1, 2, ..., q\}: f_k(a_i) > f_k(a_j)$ \hspace{1cm} (2.5)

It means that alternative $a_i$ “dominates” alternative $a_j$, if $a_i$ is better than or at least as good as $a_j$ on all the considered criteria. At least one of the inequalities holds strictly.

The definition of dominancy makes it possible to filter alternatives in order to keep only the non-dominated solutions. It can help reducing the number of alternatives under evaluation when it is too high.

We already defined efficiency in a specific unicriterion optimization method, DEA (see please Definition 1.3 in Chapter 1). Efficiency in multicriteria method is defined as follows:
Definition 2.6. Efficiency (Vincke, 1992) - Alternative \( a_i \) in \( \mathcal{A} \) is said to be “efficient” if:

\[
\exists a_j \in \mathcal{A}: a_j \succ a_i
\]

(2.6)

Thus, efficiency is the state of an alternative, which is not dominated.

The set of “efficient” solutions is often called a “Pareto optimal frontier” or “non-dominated” solutions.

Definition 2.7. Ideal point - The ideal point, in \( \mathbb{R}^n \) is the point whose coordinates are \((Z_1^*, \ldots, Z_q^*)\) where \( Z^*_k = \max_{x \in \mathcal{A}} f_k(x) \) and \( k = 1, \ldots, q \). The alternative, which is the best according to criterion \( k \) is denoted by \( \tilde{a}^k = Z^*_k \). There may be several ideal points (Vincke, 1992).

In the next chapter: Section 3.2, the difference between efficiency and Pareto optimality is explained in DEA and MCDA. Pareto optimality in MCDA is understood in criteria space while in DEA, it is identified in the space of output to input ratios. This is the reason for having different positions of alternatives (DMUs) in these spaces (Abraham et al., 2005; Opricovic and Tzeng, 2008).

At this point, since the cardinality of the efficient set may be too high, the dominance relation is not enough to solve a multicriteria problem. Logically, this seeks more clear preference relations.

2.2.4. Preference modelling

As mentioned above, some multicriteria decision-making problems cannot be solved with regard to the dominance relation. Therefore, DM should be able to give more precise preference information on the relationships. This requires the parametrization of a particular mathematical model in a formal way to represent her/his preferences. Hence, before presenting various MCDA methods, the main concepts in preference modelling are introduced.

The pairwise comparisons of alternatives in the unicriterion optimization models give only two relationships:

1- The preference relationship \((a_i \succ a_j)\): \( a_i \) is better than \( a_j \).

2- The indifference relationship \((a_i \sim a_j)\): \( a_i \) is not better than \( a_j \) and \( a_j \) is not better than \( a_i \).

The interested reader can refer to Roy (1976, 1991 and 1996) for more details of these relations.

Similarly, some MCDA methods like multi-attribute utility functions, aggregate all criteria into a unique value; thus the multicriteria problem is converted to a unicriterion optimization problem. In this framework, the preference and indifference relations are supposed to be transitive, but also these assumptions have been criticized. Luce (1956) considered the non-transitivity of indifference relation in an example. Let consider 40 cups of coffee that all contain an equal amount of coffee but cup 2 contains slightly less sugar than cup 3, cup 3 slightly less sugar than cup 4 and so on. If a person were to taste all cups from 1 to 40, she/he would hardly be able to discriminate the cups, since the difference in taste between successive cups is too small. Nevertheless, comparing cup 1

Considering a problem with several criteria by modelling the DM’s preferences, leads to three binary relations result from the pairwise comparison of $a_i$ and $a_j$, $i, j = 1, ..., n$, in $\mathcal{A}$ (Vincke, 1992; Roy, 1996):

1. The preference relationship $(a_i Pa_j)$: $a_i$ is preferred to $a_j$. It shows the presence of clear reasons that validate preference of $a_i$.

2. The indifference relationship $(a_i I a_j)$: $a_i$ is not better than $a_j$ and $a_j$ is not better than $a_i$. It shows the presence of clear reasons that validate the equivalence of both actions.

3. The incomparability $(a_i Ra_j)$: $a_i$ is not comparable with $a_j$. It shows the absence of clear reasons that validates any of the relations $P$ and $I$.

The set of $\{P, I, R\}$ as a binary relation satisfies the following properties for $(a_i, a_j) \in \mathcal{A}$:

\[
\begin{align*}
    a_i Pa_j & \Rightarrow a_j \neg I a_i: P \text{ is “asymmetric”;} \\
    a_i I a_i & : I \text{ is “reflexive”;} \\
    a_i I a_j & = a_j I a_i \text{ and } a_i Ra_j = a_j Ra_i: I \text{ and } R \text{ are “symmetric”;} \\
    a_i \neg R a_i & : R \text{ is “irreflexive”}.
\end{align*}
\]

**Definition 2.8.** If the set of three relations $\{P, I, R\}$ satisfies the mentioned properties and if for any two elements in $\mathcal{A}$: $(a_i, a_j) \in \mathcal{A}$, one and only one of the properties of $a_i Pa_j, a_j Pa_i, a_i I a_j, a_i Ra_j$ is true. This set forms a preference structure on $\mathcal{A}$ (Vincke, 1992).

The incomparability between two actions is the distinctive feature one can recognize in the French school of multicriteria problems. This relation is stated when neither $P$, nor $I$ can be identified as a relation between two actions. It may happen due to the insufficiency of information, incompatible profiles or uncertainty.

Figure 2.2 illustrates the different preference structures between four alternatives when $\mathcal{A} = \{a, b, c, d\}$. The alternatives $b$ and $c$ are incomparable in this figure. They are such that each one of them can be better than the other. This means that without any additional information, it is impossible to certainly identify which one is better. Indeed, we can see that alternative $a$ is better than/or at least as good as any of the three other alternatives. Alternative $d$ is indifferent to alternative $c$, but never better than the others. More explanations and details on preference relations can be seen in (Tsoukiàs and Vincke, 1997).

Another binary relation that can enrich the preference structure is the relation $Q$. It shows a “weak preference” relation $(a_i Q a_j)$ vs. $P$ as a strict preference relation. It means that, the DM knows the preference relation is asymmetric $(a_j \neg P a_i)$, however there does not exist a clear
distinction between preference and indifference relations. In other words, the difference level of preference between two actions is so low.

\begin{equation}
\forall a_i, a_j \in A \left\{ \begin{array}{l}
a_i Pa_j \iff a_i S a_j, a_j \sim a_i \\
a_i I a_j \iff a_i S a_j, a_j S a_i \\
a_i R a_j \iff a_i \sim a_j, a_j \sim a_i
\end{array} \right.
\end{equation}

Figure 2.2. Preference structure between four alternatives

The outranking relation $a_i S a_j$, when $S = (P \cup I)$, stands for $a_i$ is at least as good as $a_j$. A through result of this relation is as follows:

The preference relationships can be strengthened by putting a degree of intensity by DM. For instance, $P$ may express a strict preference between each pair of alternatives by a valued preference relation. This valued relation often takes place in interval $[0,1]$. It is later explained in the PROMETHEE families (Brans, 1982): Section 2.3.

The next section presents some multicriteria methods.

### 2.3. Some MCDA methods

The final step of preference modelling is to provide means of stating the relationship between alternatives by considering all criteria. This step is done by aggregating the DM’s preferences, through a method, over the considered criteria. In this part, some multicriteria methods are briefly introduced: MAUT (Keeney and Raiffa, 1976), AHP (Saaty, 1980), MACBETH (Bana e Costa et al. 2005), ELECTRE (Roy, 1968) and PROMETHEE (Brans, 1982). More details about the mentioned and some other methods can be found in (Vincke, 1992; Figueira et al., 2005).

#### Multi-Attribute Utility Theory

The Multi-Attribute Utility Theory (MAUT) is a structured method in MCDM that is designated to handle the trade-offs among multiple objectives. It was first introduced by Fishburn (1970) and developed by Keeney and Raiffa (Keeney and Raiffa, 1976 and 1993). One of the first applications of MAUT involved in a study of finding a location for a new airport in Mexico City in the early
1970s (Keeney and Raiffa, 1976). The considered criteria were cost, capacity, access time to the airport, safety, social disruption and noise pollution.

The MAUT belongs to the family of aggregation methods. It aims to substitute the initial multicriteria problem \( \max \{ f_1(x), f_2(x), \ldots, f_q(x) \mid x \in \mathcal{A} \} \) to the following unicriterion problem:

\[
\max \{ U(x) \mid x \in \mathcal{A} \}
\]  

(2.9)

where \( U(x) \) is the utility function and aggregates all the criteria to a single criterion:

\[
U(x) = U[f_1(x), f_2(x), \ldots, f_q(x)]
\]  

(2.10)

Data in MAUT is given by a matrix in which each row corresponds to an action and each column represents a criterion. The cells of the matrix contain estimates of the performance of each alternative on each of the criterion provided by an expert or various experts \( (v_{nq}) \). Table 2.2 shows the MAUT matrix with a finite set of actions. Detailed information in finite and infinite set \( \mathcal{A} \) can be studied in (Keeney and Raiffa, 1976 and 1993).

<table>
<thead>
<tr>
<th></th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>\cdots</th>
<th>( f_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( v_{11} )</td>
<td>( v_{12} )</td>
<td>\cdots</td>
<td>( v_{1q} )</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( v_{21} )</td>
<td></td>
<td>\cdots</td>
<td>( v_{2q} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td></td>
<td>\cdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( a_n )</td>
<td>( v_{n1} )</td>
<td></td>
<td>\cdots</td>
<td>( v_{nq} )</td>
</tr>
</tbody>
</table>

Table 2.2. MAUT matrix

Generally, the utility function is a non-linear function defined such that between two alternatives \( a_i, a_j \in \mathcal{A} \):

\[
U(a_i) > U(a_j) \iff a_i > a_j \quad (a_i \text{ is preferred to } a_j)
\]  

(2.11)

\[
U(a_i) = U(a_j) \iff a_i = a_j \quad (a_i \text{ is indifference to } a_j)
\]  

(2.12)

Different models exist to build function \( U \). The additive model (Huber, 1974) is one of the simplest models:

\[
U(a_i) = \sum_{k=1}^{q} U_k(f_k(a_i))
\]  

(2.13)

where \( U_k \) is the utility function of criterion \( k: k = \{1, 2, \ldots, q\} \). It is a strictly increasing real function. The aim is to return values with a same scale, let the criteria be compared and combined with different units of measurement. Through this utility function, it is possible to compute an aggregated score for each option and rank them for the sake of choosing the best ones.
A core subject in MAUT technique is the determination of the marginal utility functions $U_k$. There are two ways to construct these functions (Fishburn, 1970 and Vincke, 1992): 1- *Direct methods*: $U_k$ is built based on the information provided by experts. In this approach, an interactive questioning procedure is used to elicit the model’s parameters. 2- *Indirect methods*: $U_k$ is built based on the global judgements made by DM on the actions and it does not need the intensive participation of experts, which is a time-consuming process.

Finally, after determination of $U_k$, the MAUT ensures two main steps (Chen and Klein, 1997):

1- **Aggregating**: $U(a_i)$ gives a global value for each alternative;

2- **Ranking or Sorting**: the computed $U(a_i)$ in the first step is used to rank alternatives or sort them in the predefined clusters.

Different types of utility functions, applications and developments of MAUT can be seen in (Keeney and Raiffa, 1993; Dyer, 2005).

**Analytical Hierarchy Process**

The Analytical Hierarchy Process (AHP) (Saaty, 1980) is a widely used tool in multicriteria problems (Saaty, 1988, 1990, 2003 and 2005). A literature review of this method can be found in (Ho, 2008). AHP was developed to facilitate decision-making process when one is faced with a combination of conflicting quantitative and qualitative criteria.

First, the AHP users decompose the problem into a hierarchy of more comprehended sub-problems (alternatives and criteria); then, each can be analysed separately. Once the hierarchy is built, the pairwise comparison matrix is created based on DM or group of DMs judgement among each pair, using the verbal scale (equally, moderately, strongly, very strongly preferred, extremely preferred). This verbal scale is linear type and rates by numerical order (1, 3, 5, 7, 9). Even numbers in this order are intermediate values for the scale, e.g. 2 shows equally to moderately prefer one item to another. Different scales for comparing each pair can be seen in Ishizaka and Lusti (2006). However, this verbal comparison is one of the distinctive features of AHP, but the method does not accept any verbal scale for each pairwise comparison, which does not belong to the predefined scale. Hence, the evaluation table (Table 2.1) is not the input data of AHP but the DM’s preference matrix. A large number of different techniques have been proposed to compute the priority vector such as eigenvector, geometric mean, least square, and normalized column. Ishizaka and Lusti (2006) compared several methods to derive priorities in AHP. In this section, the normalized right-hand side eigenvector of this matrix is applied to compute the final alternatives scores and the importance of each criterion [please see the equation (2.18)]. Here, it is limited calculating criteria weights. A similar technique is used to evaluate alternatives scores.

The following matrix represents the result of pairwise comparisons between criteria:
where $c_{ik}$: $i, k = \{1, 2, \ldots, q\}$ expresses the relative importance of the criterion $i$ over the criterion $k$, based on the verbal scale. As mentioned above, a 9-point rating system translates the verbal judgements of DM/DMs to real values. In an ideal case, each element of this matrix needs to respect the consistency condition as follows:

$$C_{kl} = \frac{1}{c_{ik}} \forall i, k \in \{1, 2, \ldots, q\}$$  \hspace{1cm} (2.15)

A pairwise comparison matrix is held to be consistent if the following property is satisfied:

$$C_{ik} = C_{ij} \times C_{jk} \forall i, j, k \in \{1, 2, \ldots, q\}$$  \hspace{1cm} (2.16)

It means that the weights are consistent if they are transitive; e.g., if element $a$ is 3 times better than element $b$ and $b$ is 5 times better than $c$, the relation (2.16) gives $a$ is 15 times better than $c$. In the presence of full consistency, the weight’s matrix $C$ (2.14) can be presented as follows:

$$C = \begin{pmatrix}
1 & c_{12} & \cdots & c_{1k} & \cdots & c_{1q} \\
\frac{1}{c_{12}} & 1 & \cdots & c_{2k} & \cdots & c_{2q} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{1}{c_{1k}} & \frac{1}{c_{2k}} & \cdots & 1 & \cdots & c_{iq} \\
\frac{1}{c_{1q}} & \frac{1}{c_{2q}} & \cdots & c_{iq} & \cdots & 1 \\
\end{pmatrix}$$

(2.14)

where $w_1, w_2, \ldots, w_q$ are the criteria weights. In case of inconsistency, the eigenvector of matrix $C$ gives the relative importance of criteria (weights of each criterion):

$$CW = qW$$  \hspace{1cm} (2.18)

The vector $W$ is the right hand side eigenvector. $W = (w_1, w_2, \ldots, w_q)$ is normalized to sum of its elements to make a unique weight vector.

In the real situations, for matrices involving human value judgements, holding the condition $C_{ik} = C_{ij} \times C_{jk}$ represents too much consistency, which is undesirable [in the mentioned example under relation (2.16), $a$ is not necessarily 15 times better than $c$]. Thus it does not hold, as human judgements can be inconsistent to a greater or lesser degree. Accordingly, the Consistency Index (CI) (Saaty, 1990) is defined as follows to measure the consistency level:

$$CI = \frac{\lambda_{\text{max}} - q}{q - 1}$$  \hspace{1cm} (2.19)
where $\lambda_{max}$ is the highest value in the eigenvector of matrix $C$ and $q$ is the matrix size. If $\lambda_{max} = q$, then $CI = 0$ and the pairwise comparison matrix is a perfect consistent matrix. The difference between $\lambda_{max}$ and $q$ indicates the inconsistency of the judgements.

$CI$ is then compared to a set of predefined Random Index ($RI$). The $RI$ is achieved by randomly generating matrix and taking the average $CI$ value. The “Consistency Ratio” $CR$ is the rescaled version of $CI$ and is defined as:

$$CR = \frac{CI}{RI}$$

When the value of $CR$ is smaller or equal to 0.1, the level of inconsistency is judged to be acceptable. Otherwise, it is rejected and the DM has to revise judgements. $CR = 0.1$ means that the judgments are 10% as inconsistent as if they had been given randomly.

However, a certain level of inconsistency may still be accepted (Saaty, 1990), but one disadvantageous of AHP is the problem of inconsistency. Encountering this issue, while the size of decision problem is large, takes time to redo the comparisons.

The pairwise comparison matrix between alternatives is also obtained by the same process as $C$. The DM is asked to compare alternatives based on each criterion. Finally, ranking of alternatives is obtained according to the global priorities between criteria.

Another important criticism in AHP is its sensibility to rank reversal. Due to the nature of pairwise comparisons to give final ranking, adding an alternative(s) at the end of the process can cause a reverse or change in the ranking (Belton and Gear, 1983; Balali et al. 2014). The forth Chapter, Section 4.4.2 compares and summarizes the features of AHP and another MCDA methodology, PROMETHEE, in several facts, including rank reversal.

**Definition 2.9. Rank reversal** (Belton and Gear, 1983) – A rank reversal in multicriteria decision-making problems is a change in the final rank order of alternatives, when for example, the method of ranking, the set of alternatives, and/or criteria change.

**MACBETH**

Measuring Attractiveness by a Categorical Based Evaluation Technique (MACBETH) was designated by Bana e Costa (Bana e Costa and Vansnick, 1999; Bana e Costa et al., 2005). The MACBETH approach assumes that the DM provides both ordinal (comparison of alternatives) and cardinal (intensities of preferences) information. Literally, it facilitates the path from ordinal to cardinal preference modelling (Bana e Costa et al., 2005).

The first step, similar to AHP, is to build the hierarchy structure of the problem. It is an interactive method, which uses semantic judgements about differences in attractiveness of several options with respect to each of the criteria. MACBETH aims helping DM to compute the relative attractiveness of each option (Bana e Costa and Vansnick, 1999). It is based initially on iterative questioning that compares each pair of options, demanding only a qualitative preference judgement to build a matrix. The pairwise comparisons are based on predefined 7-scale semantic categories:
indifference (1), very weak (2), weak (3), moderate (4), strong (5), very strong (6) and extreme (7). The weighting of the criteria is done with a similar questioning process. MACBETH method allows checking the consistency of matrix. Nevertheless, the method tests the compatibility of the information collected with the cardinal information obtained by pairwise comparison based on 7-scale. When an incompatibility is noticed, the judgement matrix is inconsistent and judgements should be revised (Bana e Costa et al., 2005). Unlike AHP, which accepts a certain level of inconsistency (Saaty, 1990), MACBETH does not accept any level of it. By means of an LP, the qualitative judgements of DMs are translated to values on an interval scale (interval scale is defined in Section 2.2). Finally, MACBETH computes the overall scores for each option with an additive aggregation model (ranking).

MACBETH has been extensively applied in various evaluation contexts, specifically strategic plan development, resource allocation, performance evaluation for employees, etc. (Bana e Costa et al., 2005). MACBETH, similar to AHP, is an understandable approach for users and the pairwise comparisons are easy to make (Dyer and Forman, 1992). Although, as a pairwise comparison approach, based on semantic judgements of DMs, it has been criticized for requesting a prior information preparation. It intensely depends on the number of criteria involved and consequently the number of comparisons to be done (Ferreira et al., 2011a).

Next sub-section considers two outranking methods: ELECTRE and PROMETHEE.

2.3.1. Outranking methods

Outranking methods, unlike AHP and MACBETH, are not based on an underlying value function. The result of outranking approaches is an outranking relation on the set of alternatives [and not a value for each alternative] (Belton and Stewart, 2002). These models are used in discrete problems; naturally, they stand on pairwise comparisons of actions but need less input such as preference and judgements of DMs between criteria and alternatives.

ELECTRE (Roy, 1974) and PROMETHEE (Brans, 1984) are two families of outranking methods in MCDA French school. Incomparability of items, which is the key concept in outranking methods, is one of the main differences with methods in MCDM American School such as AHP (Climaco, 1997).

The ELECTRE families (Roy, 1974; Figueira et al. 2005) are first presented briefly and then in more details PROMETHEE (Brans, 1984) is considered.

The ELECTRE methods

The ELECTRE (ELimination Et Choix Traduisant la REalité) or (ELimination and Choice Translating REality) was first proposed by (Benayoun et al., 1966; Roy, 1974). From then on, it was developed (Roy, 1991; Figueira et al., 2005). Different ELECTRE methods are: ELECTRE I, ELECTRE II, ELECTRE III, ELECTRE IV, ELECTRE IS, and ELECTRE TRI (Vincke, 1992; Figueira et al., 2005). These methods can be applied to choose, rank and sort items. The principle
of the original ELECTRE is to evaluate the outranking relation of two alternatives from two different aspects: the concordance and the discordance indices (Junhua and Jiuping, 2006). In this part, it is only explained the basics of ELECTRE. More details can be found in (Vincke, 1992; Figueira et al., 2005).

1- ELECTRE I

ELECTRE I is in the group of “choice problem” (α problem) method. The outranking relation $a_i S a_j$ (2.8) is held under two conditions:

1- The concordant coalition should be strong enough to support $a_i S a_j$, and
2- The discordance should not be existed against $a_i S a_j$.

In this regard, the “concordance” and “discordance” indices are defined.

The “concordance index” is the sum of the weights associated to the criteria and represents a measure of statement $a_i S a_j$:

$$c(a_i S a_j) = \sum_{k; f_k(a_j) \geq f_k(a_i)} w_k,$$

$$i, j = 1, ..., n, k \in \{1, 2, ..., q\}$$

where $w_k$ is the assigned weight of each criterion and $\sum_{k=1}^q w_k = 1, w_k \geq 0$.

This index validates $a_i S a_j$, a sufficient majority of criteria in favour of this relation should occur. The “concordance index” must be equal to or greater than a predefined concordance threshold $s [c(a_i S a_j) \geq s]$. The threshold $s$ is within the range of $[0.5, 1 − \min(w_k)]$. Clearly, the concordance index lies between 0 and 1.

The “discordance index” does not validate the relation $a_i S a_j$ if it validates $a_j S a_i$ on at least one criterion. It is defined as follows:

$$d(a_i S a_j) = \max_{(k; f_k(a_j) < f_k(a_i))} \{f_k(a_j) − f_k(a_i)\}$$

The discordance index has no power when it is less than a predefined discordance threshold $v (d(a_i S a_j) < v)$. The threshold $v$ is defined alongside the outranking relation $S$.

Both of these indices should be computed for every pair of actions in $\mathcal{A}$.

In this step, a subset of alternatives $A'$ in $\mathcal{A}$ ($A' \subset \mathcal{A}$) defined as the “kernel”, which within the outranking relation in (2.8) is satisfied. The kernel holds these two conditions: 1- the solutions, which are not member of this set, are outranked by at least one alternative in $A'$, and 2- $A'$ is the set of the compromise solutions that its members do not outrank each other (the members are incomparable):

$$\begin{cases} 1 − \forall a_j \in \mathcal{A} \setminus A', \exists a_i \in A': a_i S a_j \\ 2 − \forall a_i, a_j \in A', a_i \overline{S} a_j \end{cases}$$

(2.23)
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The kernel exists and is unique when the outranking relation \( S \) does not contain circuit in its graph (Vincke, 1992).

2- ELECTRE II

ELECTRE II was the first method in ELECTRE family designed to deal with \( \gamma \) problem (ranking). In this approach, two outranking relations (strong and weak) must be built. These relations are built due to two predefined concordance thresholds \( s_1 \) and \( s_2 \), while \( s_1, s_2 \in [0.5, 1 - \min(w_k)] \) and \( s_1 > s_2 \). The strong outranking relation \( S^1 \) and a weak outranking relation \( S^2 \) are defined as follows:

\[
a_i S^l a_j \iff \begin{cases} 
    c(a_i S a_j) \geq s_l, \forall l = 1, 2 \\
    \sum_{k: f_k(a_i) \geq f_k(a_j)} w_k > \sum_{k: f_k(a_i) < f_k(a_j)} w_k, k \in \{1, 2, \ldots, q\} 
  \end{cases}
\] (2.24)

Alternative \( a_i \) outranks \( a_j \) when \( c(a_i S a_j) \geq c(a_j S a_i) \). The discordance index can also induce two relations by setting two discordance thresholds.

To ranking alternatives, first, a subset \( B \) is determined from \( S^1 \) such that members of \( B \) are not strongly outranked by any others. In the second step, a subset \( Z^1 \) is determined from \( B \) and \( S^2 \) such that the alternatives belong to it are not weakly outranked by any alternatives from \( B \). \( Z^1 \) contains the best class of alternatives. Then, \( Z^1 \) is removed and the process of building and removing subsets is repeated until obtaining a complete pre-order from the best to the worst alternatives. The same process can be done to generate another complete pre-order but this time from the worst alternatives to the best ones.

3- ELECTRE III

ELECTRE III was proposed to improve ELECTRE II to deal with inaccuracy, uncertainty, imperfection or ill-determination of ordinal data (Figueira et al., 2005). It presents “pseudo-criteria”, which is based on definition of two thresholds: indifference \( (q_k) \) and preference \( (p_k) \).

Definition 2.10. Pseudo-Criterion (Roy, 1996) - A “pseudo-criterion” is a function \( f_k \) associated with two thresholds \( q_k \) and \( p_k \), satisfying the following conditions: for all ordered pair of actions \((a_i, a_j) \in A \times A\) such that: 1- \( f_k(a_i) \geq f_k(a_j) \) and 2- \( f_k(a_i) + p_k(f_k(a_i)) \) and \( f_k(a_i) + q_k(f_k(a_j)) \) are non-decreasing monotone functions of \( f_k(a_i) \), such that \( p_k(f_k(a_j)) \geq q_k(f_k(a_j)), \) for all \( a_j \in A \).

In ELECTRE III, the “credibility degree” \([S(a_i, a_j)]\) of outranking relation between each pair of actions is defined to show \( a_i \) outperforms \( a_j \). To determine \([S(a_i, a_j)]\), the concordance and discordance indices should be defined. The concordance index is computed as follows:

\[
c(a_i S a_j) = \sum_{k=1}^{q} w_k c_k(a_i S a_j), i, j = 1, \ldots, n, k \in \{1, 2, \ldots, q\} \] (2.25)
where \(w_k\) is the assigned weight of each criterion and \(\sum_{k=1}^{q} w_k = 1, w_k \geq 0\), then:

\[
c_k(a_iS_a_j) = \begin{cases} 
0 & \text{if } f_k(a_i) + p_k(f_k(a_i)) \leq f_k(a_j) \\
\text{otherwise} & \text{if } \frac{f_k(a_j) - f_k(a_i) - q_k(f_k(a_i))}{p_k(f_k(a_i)) - q_k(f_k(a_i))} \\
1 & \text{if } f_k(a_i) + q_k(f_k(a_i)) > f_k(a_j)
\end{cases} \quad (2.26)
\]

where \(q_k\) and \(p_k\) are indifference and preference thresholds, respectively, which given by DM.

To compute the discordance index, the “veto threshold” \([v_k(f_k(a_i))]\) should be computed. This threshold is at least equal to \(p_k\) for each criterion \(k\) such that \(a_i\) cannot outrank \(a_j\) if \(f_k(a_j) \geq f_k(a_i) + v_k(f_k(a_i))\).

**Definition 2.11. Veto threshold** (Figueira et al., 2005) - The “veto threshold” gives the possibility to the criterion \(f_k\) to impose its veto power, which means that \(f_k(a_j)\) is so much better than \(f_k(a_i)\). Thus, relation \(a_iS_a_j\) is not possible.

The discordance index is as follows:

\[
d_k(a_iS_a_j) = \begin{cases} 
0 & \text{if } f_k(a_i) + p_k(f_k(a_i)) \geq f_k(a_j) \\
\text{otherwise} & \text{if } \frac{f_k(a_j) - f_k(a_i) - p_k(f_k(a_i))}{v_k(f_k(a_i)) - p_k(f_k(a_i))} \\
1 & \text{if } f_k(a_i) + v_k(f_k(a_i)) < f_k(a_j)
\end{cases} \quad (2.27)
\]

Finally, the credibility degree (outranking degree) is defined as follows:

\[
S(a_i, a_j) = \begin{cases} 
c(a_iS_a_j) & \text{if } d_k(a_iS_a_j) \leq c(a_iS_a_j) \quad \forall k \in \{1, 2, \ldots, q\} \\
c(a_iS_a_j) \prod_{k \in F : d_k(a_iS_a_j) > c(a_iS_a_j)} \frac{1 - d_k(a_iS_a_j)}{1 - c(a_iS_a_j)} & \text{otherwise}
\end{cases} \quad (2.28)
\]

where \(F\) is the set of criteria when \(d_k(a_i, a_j) > c(a_i, a_j)\). As it can be seen in (2.28), in the absence of discordant index, the outranking degree is equal to concordance index; if not, the outranking degree is changed proportionally with concordance and discordance index.

In this step, a threshold \(s(\lambda)\) should be defined, where \(\lambda = \max_{a_i, a_j \in \mathcal{A}, i \neq j} S(a_i, a_j)\). The outranking degrees with value equal to or greater than \(\lambda - s(\lambda)\) are considered. Next step is inducing the rank of alternatives. For this aim, a “qualification index” should be determined for each alternative \(a_i \in \mathcal{A}\). This index shows the difference between the number of alternatives, which
outranks the alternative $a_i$ and be outranked by alternative $a_j$. The “highest qualification index” of alternatives builds a set that called the first distillate $D_1$. The process is repeated for $D_1$ when it has more than one alternative. In the presence of just one alternative, this process is done in $\mathcal{A} \setminus D_1$ to build $D_2$. The procedure is repeated until $D_1$ becomes empty from alternatives and before starting with the set of $\mathcal{A} \setminus D_1$. The same procedure is done for $D_1 \setminus D_2$ when $D_2$ has one alternative or in $D_2$ when it has more than one alternative. Continuing this process induces the first complete pre-order. A second pre-order can be made by considering the set of the lowest qualification indices.

Ranking of alternatives based upon a score assigned to an alternative is one of the characteristic of MCDA methods. The ELECTRE methods are not appropriate to compute scores of each alternative in final ranking. When the scores should be computed, the PROMETHEE outranking method can be applied (Brans and Mareschal, 2005). The following part studies the PROMETHEE method.

The PROMETHEE families

PROMETHEE (Preference Ranking Organisation METHod for Enrichment Evaluation), which belongs to the family of the outranking methods (Vincke, 1992), has been initiated by Brans in (1982), and developed by Brans (1984) and Brans and Vincke (1985). It is one of the known and widely applied outranking methods (Behzadian et al., 2010) because of its simplicity in computational process and an important number of applications in different fields such as banking, education, industrial location, manpower planning, water resource management, investment, medicine, chemistry, health care, tourism, ethics in OR and dynamic management. It has also user-friendly software such as PROMCALC (Brans and Mareschal, 1994), DECISION LAB 2000 (Mareschal, 2000), Visual PROMETHEE (Mareschal, 2011), D-Sight (Hayez et al., 2012), and Smart Picker (Ishizaka and Nemery, 2013). In this section, the main steps of PROMETHEE method are explained briefly. More details for interested reader can be found in (Brans et al., 1984; Brans and Vincke, 1985; Brans and Mareschal, 2002; De Smet and Lidouh, 2013).

The PROMETHEE families based on pairwise comparisons of actions on each criterion according to DM’s preferences, resulting in scores on each criterion. Finally, these scores are aggregated into a global score, which leads to a partial pre-order (PROMETHEE I) or complete pre-order (PROMETHEE II). It can be applied to any discrete set of actions $\mathcal{A}$, evaluated over a set of $q$ criteria $\mathcal{F} = \{f_1(a_j), ..., f_k(a_j), ..., f_q(a_j)\}$, where $f_k(a_j)$ is the evaluation of action $a_j$ on criterion $k$.

In the first step, the differences between each pair of alternatives on each criterion are computed:

$$d_k(a_i, a_j) = f_k(a_i) - f_k(a_j),$$

$\forall a_i, a_j \in \mathcal{A}, \forall k \in \{1, 2, ..., q\}$ (2.29)
When $d_k(a_i, a_j)$ is enough small that DM can neglect it, $a_i$ and $a_j$ are considered indifferent for the criterion $k$. With the higher value of this difference, a preference function $P_k(a_i, a_j)$ should be defined to enrich the poor dominance (binary) relation. The framework of PROMETHEE methods is based on the definition of these preference functions to aggregate the related information on each criterion. Therefore, one has to associate a generalized criterion $\{f_k(a_j), P_k(a_i, a_j)\}$ to each criterion. $P_k(a_i, a_j)$ provides the preference strength of action $a_i$ over $a_j$. This is characterized by the function $P_k(d_k(a_i, a_j))$. The higher the value of $d_k$, the stronger the preference of $a_i$ over $a_j$ on criterion $f_k$. It is assumed to be a non-decreasing function of $d_k(a_i, a_j)$:

$$P_k: R \rightarrow [0, 1]: d_k(a_i, a_j) \mapsto P_k(d_k(a_i, a_j))$$

(2.30)

This preference degree takes values between 0 and 1. The value of 0 means that the difference between the two alternatives is not significant $[d_k(a_i, a_j) < 0 \rightarrow P_k(d_k(a_i, a_j)) = 0]$ and the value of 1 means that the difference is strong enough to make DM strictly prefer one alternative to other one.

The method provides the DM with a set of predefined preference functions for each of which at most two parameters have to be defined, indifference $q_k$ and preference $p_k$ thresholds. There exist six main different types of preference functions as presented in Figure 2.3 (Brans and Vincke, 1985; Brans and Mareschal, 2002).

An example of these functions can be the fifth type, linear function (Figure 2.3). The preference function has two thresholds for each criterion $k$, indifference $q_k$ and preference $p_k$ thresholds. When the difference between scores of actions $a_i$ and $a_j$ in criterion $k$, $d_k(a_i, a_j)$, is less than $q_k$, $a_i$ and $a_j$ are considered indifferently on this criterion. If $d_k$ is greater than $p_k$, $a_i$ is strictly preferred to $a_j$. In the distance between these two thresholds, preference function linearly increases. This function is suited for quantitative criteria. Additionally, it covers other types of preference functions.

In the third step, PROMETHEE aggregates the preference degrees to compute a score for each alternative. For this aim, the concept of preference function is used to transform the differences into uncriterion preference degrees, thus:

$$\pi_k(a_i, a_j) = P_k[d_k(a_i, a_j)],$$

$$\forall a_i, a_j \in A, \forall k \in \{1, 2, ..., q\}$$

(2.31)

The computation of outranking (preference) degree of action $a_i$ over $a_j$ is as follows:

$$\pi(a_i, a_j) = \sum_{k=1}^{q} \pi_k(a_i, a_j) \cdot w_k,$$

$$\forall a_i, a_j \in A, \forall k \in \{1, 2, ..., q\}$$

(2.32)
where $w_k$ is the normalized positive weight associated to each criterion $k$. This degree varies between 0 and 1. For each preference index, following relations are considered:

$$\pi(a_i, a_j) \geq 0, \quad \pi(a_i, a_j) + \pi(a_j, a_i) \leq 1,$$

$$\forall a_i, a_j \in A$$  (2.33)

In step four, when each alternative $a_j$ is compared with $(n - 1)$ other alternatives in $A$, the positive and negative “outranking flows” are defined as follows:

$$\phi^+(a_j) = \frac{1}{n-1} \sum_{x \in A} \pi(a_j, x)$$

$$\phi^-(a_j) = \frac{1}{n-1} \sum_{x, a \in A} \pi(x, a)$$  (2.34)

The positive outranking flows, $\phi^+(a_j)$, expresses how $a_j$ outranks other alternatives and the negative outranking flows, $\phi^-(a_j)$, expresses how $a_j$ is outranked by other alternatives. Further, $\phi^+$ and $\phi^-$ $\in [0, 1]$.

1- PROMETHEE I

Let $(S^+, I^+)$ and $(S^-, I^-)$ be the two pre-orders obtained from the positive and negative flows:

$$(a_i S^+ a_j \leftrightarrow \phi^+(a_i) > \phi^+(a_j))$$

$$(a_i I^+ a_j \leftrightarrow \phi^+(a_i) = \phi^+(a_j))$$

(2.35)

$$(a_i S^- a_j \leftrightarrow \phi^-(a_i) < \phi^-(a_j))$$

$$(a_i I^- a_j \leftrightarrow \phi^-(a_i) = \phi^-(a_j))$$

It shows that the higher the positive flow and the lower the negative flow of an alternative, the better the alternative. The “PROMETHEE I” allows obtaining a partial ranking of the alternatives by taking the intersection of these two pre-orders in (2.35):

$$\begin{cases} a_i S^+ a_j \text{ and } a_i S^- a_j \\ a_i P^+ a_j \text{ and } a_i I^- a_j \\ a_i I^+ a_j \text{ and } a_i S^- a_j \\ a_i I^+ a_j \text{ and } a_i I^- a_j \\ a_i R^+ a_j \text{ else} \end{cases}$$  (2.36)

When $(P^+, I^+, R^+)$ corresponds to the preference, indifference and incomparability of each pair of alternatives in PROMETHEE I, respectively.

2- PROMETHEE II

A complete pre-order, called PROMETHEE II, can be obtained on the basis of the net flow score as follows:
Table 2.3 Preference functions (Brans and Mareschal, 2002)
\[ \phi(a_j) = \phi^+(a_j) - \phi^-(a_j), \forall a_j \in \mathcal{A} \]  \hspace{1cm} (2.37)

Following relations can be defined:

\[ \begin{align*}
& a_i P^{II} a_j \iff \phi(a_i) > \phi(a_j) \\
& a_i I^{II} a_j \iff \phi(a_i) = \phi(a_j)
\end{align*} \]  \hspace{1cm} (2.38)

where \((P^{II}, I^{II})\) are the preference and indifference relations between each pair of alternatives, consecutively. The complete ranking in PROMETHEE II is obtained, since it does not give room to incomparability between alternatives. These relations express that the higher the net flow score, the better the alternative. The net flow scores can also be computed as follows:

\[ \phi(a_j) = \sum_{k=1}^q w_k \phi_k(a_j) = \frac{1}{n-1} \sum_{j=1}^n \left[ \pi(a_i, a_j) - \pi(a_j, a_i) \right] \]  \hspace{1cm} (2.39)

where \(\phi_k(a_j)\) is a score computed per criterion for each alternative and is called the unicriterion net flow score of alternative \(a_j\); it can be figured as follows:

\[ \phi_k(a_j) = \frac{1}{n-1} \sum_{x \in \mathcal{A}} \{ P_k(a_j, x) - P_k(x, a_j) \} \]

\(\forall a_j \in \mathcal{A}, \forall k \in \{1, 2, ..., q\}\)  \hspace{1cm} (2.40)

The Values of \(\phi_k(a_j)\) already integrate intra-criterion parameters and all values lie within the same range \((-1 \leq \phi_k(a_j) \leq 1\).

Indeed, the unicriterion net flow scores matrix is a matrix that includes the preference information of the decision problem.

Mareschal and Brans (Mareschal and Brans, 1988) have proposed a complementary geometrical tool that helps the DM/user to understand better the results presented by the PROMETHEE rankings. This tool called GAIA (Geometrical Analysis for Interactive Aid), is based on a Principal Component Analysis (PCA) of the unicriterion net flow scores. It is one of the differences of PROMETHEE with other multicriteria methods.

3- The GAIA plane

The GAIA plane displays a visual descriptive analysis when there are several criteria (more than 3 criteria). It aims to present as much information as possible from the multicriteria decision-making problem. The GAIA plane generates a two-dimensional view of the multicriteria problem, using a Principal Component Analysis (PCA). If matrix \(\phi = (\phi_k(a_j)), \forall a_j \in \mathcal{A}, \forall k \in \{1, 2, ..., q\}\) consists of all unicriterion net flows of the decision problem \([\phi_k(a_j)\) is defined in Equation (2.40)], a PCA is then used on the corresponding variance-covariance matrix \(C\):

\[ nC = \phi'\phi \]  \hspace{1cm} (2.41)
The eigenvectors and eigenvalues of $\mathcal{C}$ characterize the GAIA plane. The eigenvectors are orthogonal. They point to a direction of alternatives dispersion’s positioning. The eigenvalues present this dispersion. Finally, in order to show the two-dimensional plane in the criteria space, two vectors $u$ and $v$ are selected, as the eigenvectors associated to the two highest eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $\mathcal{C}$. All the components of the problem are projected on the plane defined by them:

$$\alpha_j = ((\varnothing_1(a_j), (\varnothing_2(a_j), \ldots, (\varnothing_q(a_j)), \forall a_j \in \mathcal{A}) \quad (2.42)$$

$$e_k = (0, \ldots, 0, 1, 0, \ldots, 0), k \epsilon \{1, 2, \ldots, q\} \quad (2.43)$$

$$w = (w_1, w_2, \ldots, w_k, \ldots, w_q) \quad (2.44)$$

where $\alpha_j$ is the coordinate vector of each alternative, $e_k$ is an axis for each criterion and $w$ is the weight vector. The whole projection of all these components called the GAIA plane (an illustration can be seen in Figure 2.4).

The quality of the projection can be computed as follows:

$$\delta = \frac{\lambda_1 + \lambda_2}{\sum_{k=1}^{q} \lambda_k} \quad (2.45)$$

Delta ($\delta$) is the amount of information preserved by GAIA plane and gives us a confidence level for the results. In Figure 2.4, 87% of information is kept from projection of a problem with 10 alternatives and 5 criteria on the GAIA plane. Since this value is high enough, thus, the projections seem confidential. Nevertheless, it should not be forgot that this projection causes some missing information.

Each element on the GAIA plane has a specific interpretation according to the decision problem:

- **Relative positions of alternatives ($a$)**

The location of an alternative on the GAIA plane presents us its features. This location can be considered related to other alternatives, to the criteria and to the decision stick as follows:

  - **Related to other alternatives**: Group of alternatives, projected close to each other on the GAIA plane, are considered as alternatives with similar profiles. In Figure 2.4, alternatives $a_3$ and $a_5$, which are close to each other, seem to have similar profiles. Nevertheless, actions $a_1$, $a_6$ and $a_9$ are located far from each other; thus, they have very different characteristics.

  - **Related to the criteria**: The alternatives, which are located in the direction of each criterion, act better on that specific criterion and point out the strongest and weakest features of a solution. In Figure 2.4, alternatives $a_9$ and $a_{10}$ are oriented towards $C_4$, while behave poorly on $C_1$. Action $a_2$ has a very strong profile in terms of $C_2$ and $C_3$. Action $a_5$ is good enough on $C_5$ but not enough good on all other criteria. Alternative $a_4$ has approximately weak profile on all criteria.
Figure 2.4. GAIA plane of a problem with 10 alternatives and 5 criteria

- **Related to the decision stick (π):** The projection of alternatives on the decision stick (π) may be the representation of PROMETHEE ranking. The further the action on it, the better the rank of action. However, the ranking inferred from this projection may be different from final ranking (due to the loss of information), but it still can be a helpful tool when more specific information is not available. The ranking obtained from the plane of Figure 2.4 is \( a_2, a_6, a_{10}, a_4, a_9, a_8, a_7, a_5, a_3, \) and \( a_1 \).

- **Positions of the criteria (C)**

  The direction of each criterion’s axe on the plane indicates which criteria are compatible with each other and which ones are in conflict. The conflicting criteria point to opposite or very different orientations. It would mean that finding the alternatives, which act similar on conflicting criteria is a difficult task. The compatible criteria means it is simple finding alternatives that act well on them simultaneously or finding some alternatives have bad evaluations on these criteria. For example, in Figure 2.4, alternative \( a_2 \), which is good on both \( C_2 \) and \( C_3 \), acts absolutely poor on \( C_5 \).

  Further, the size of the criteria axes measures its discrimination power in a problem. A short criterion does not differentiate enough between actions as the other criteria do and vice versa. Since the GAIA plane has been chosen to capture the maximum distinction of the alternatives, the criteria that do not do this favour will likely end up being orthogonal to the plane. In Figure 2.4, criterion \( C_5 \) has a short length related to other criteria. Thus, it can be said that the value of \( C_5 \) measured for each action of this problem does not discriminate the actions as the other criteria do.
Position of the decision stick ($\pi$)

Projecting the normalized vector of $w = (w_1, w_2, ..., w_k, ..., w_q)$, $w/\parallel w \parallel$, on the GAIA plane characterizes the decision stick. Thus, it reflects the importance that DM gives to each criterion. In Figure 2.4, the decision stick points slightly more towards $C_2$ and $C_3$ than the other criteria (DM gave more importance to these criteria). Indeed, changing the weights modifies the direction of the decision stick.

A further use of the GAIA plane relies on the sensitivity analysis and robustness. The DM may investigate the impact of weight variations in the final ranking when she/he hesitates to assign precise weight values to the different criteria. On the one hand, the DM may explicitly change the values, one by one, and control their impacts. This can be done with the so-called “walking weights” tool. On the other hand, she/he may want to determine weight intervals within which the values are likely to vary. In these intervals, the rank of alternatives does not change (typically, the rank of the first alternative is reserved). This information will determine a hypercube of possible values in the k-dimensional space of unicriterion net flow scores. The projection of this hypercube on the GAIA plane will permit to recognize an area where the decision stick will belong. This area is called “GAIA brain” or brain of the DM. If this brain does not contain the origin, the decision problem is an easy problem to solve (Brans and Mareschal, 2002).

4- The weight stability intervals

The purpose of the weight stability intervals technique (Mareschal, 1988) is to preserve the preference ranking of a subset of alternatives within the intervals of criteria weights. Its main strength is the automated generation of intervals limits that evaluates the robustness of PROMETHEE II outputs (typically maintaining the rank of the first alternative).

In a complete pre-order, the structure of $(P, I)$ between a pair of actions $(a_i, a_j)$ is already defined in (2.38). In this context, we want to investigate the potential modification in the $(P, I)$ structure when the weights are changed. Let us state $w'_l = (1 + \beta)w_l$ (with $\beta \geq -1$) for criterion $f_l$. To maintain the normalization of modified weights, all the other weights should be rearranged as $w'_k = \alpha w_k$ when $k \neq l$ and $k = \{1, ..., l, ..., q\}$. The parameters $\alpha$ and $\beta$ are related as follows:

$$\alpha = \frac{1-(1+\beta)w_l}{1-w_l}$$ (2.46)

Constraints on $\alpha$ and $\beta$ guarantee the non-negativity of the modified weights:

$$-1 \leq \beta \leq \frac{1-w_l}{w_l}$$ (2.47)

$$0 \leq \alpha \leq \frac{1}{1-w_l}$$ (2.48)

As a consequence, $w_l$ can be increased or decreased from its initial value according to the values of $\alpha$ and $\beta$. Let us denote $(P', I')$ is the complete pre-order resulted by the modified weight $(w'_l)$. It can be noted:
\[ \Delta(a_i, a_j) = \phi(a_i) - \phi(a_j) \]  
\[ \Delta'(a_i, a_j) = \phi'(a_i) - \phi'(a_j) \]  
\[ \Delta_l(a_i, a_j) = \phi_l(a_i) - \phi_l(a_j) \]

It is easy to show that:

\[ \Delta'(a_i, a_j) = \alpha \Delta(a_i, a_j) + (1 - \alpha) \Delta_l(a_i, a_j) \]  

The comparison between \((P, I)\) and \((P', I')\), restricted on the pair \((a_i, a_j)\), can lead to three different cases:

- **a.** \( a_i \) \( P a_j, a_j \) \( P' a_i \): the preference is inverted;
- **b.** \( a_i \) \( P a_j, a_i \) \( l' a_j \): the preference is changed to indifference and
- **c.** \( a_i \) \( l a_j, a_i \) \( P' a_j \): the indifference is changed to preference.

The first case can also be written like this:

\[ \Delta(a_i, a_j) \Delta'(a_i, a_j) < 0 \]  

In other words, if one wants to hold the relative position between two pairs of actions, the following constraint have to be satisfied:

\[ \Delta(a_i, a_j) \Delta'(a_i, a_j) > 0 \quad \text{s. t.} \quad \Delta(a_i, a_j) \neq 0 \]  

By replacing \( \Delta' \) from (2.52), relation (2.54) is changed to:

\[ \Delta(a_i, a_j) [\alpha \Delta(a_i, a_j) + (1 - \alpha) \Delta_l(a_i, a_j)] > 0 \]  

Thus:

\[ \alpha \Delta(a_i, a_j) \Delta_l(a_i, a_j) - \Delta^2(a_i, a_j) \]  

\[ < \Delta(a_i, a_j) \Delta_l(a_i, a_j) \]  

Stability condition in (2.54) implies when \( a_i \) and \( a_j \) are indifferent, \( \Delta(a_i, a_j) = 0 \) and this condition is hold if and only if \( \Delta'(a_i, a_j) = 0 \). Further, relation (2.55) shows two conditions to reach stability: (a) \( \Delta'(a_i, a_j) = 0 \) or (b) \( \alpha = 1 \). In condition (a) \( a_i \) and \( a_j \) remain indifferent (\( \alpha \) has no bound). In condition (b) indifference should be changed to preference to let \( w_k \) modify to \( w'_k \). Finally, the relation (2.56) is the condition which builds the \( \alpha \)'s constraints. This ensures the stability for pair of actions \((a_i, a_j)\) in three different situations:

1. When \( f_l \) is in disagreement with ranking of \( a_i \) and \( a_j \leftrightarrow \Delta(a_i, a_j) \Delta_l(a_i, a_j) < 0 \), the inequality (2.56) gives the lower bound of \( \alpha (\alpha_l^-) \):

\[ \frac{\Delta(a_i, a_j) \Delta_l(a_i, a_j)}{\Delta(a_i, a_j) \Delta_l(a_i, a_j) - \Delta^2(a_i, a_j)} < \alpha \]  

\[ 2.57 \]
This bound has to be satisfied for all pairs of actions that fall in this category. Therefore, let define $\Omega^-$ as follows:

$$\Omega^- = \left\{ (a_i, a_j) \in \mathcal{A} \times \mathcal{A}, s.t. \right\} 
\Delta(a_i, a_j) \Delta_l(a_i, a_j) < 0 \right\}$$ (2.58)

The lower bound of $\alpha$ is:

$$\alpha_l^- = \max_{(a_i, a_j) \in \Omega^-} \frac{\Delta(a_i, a_j) \Delta_l(a_i, a_j)}{\Delta(a_i, a_j) \Delta_l(a_i, a_j) - \Delta^2(a_i, a_j)}$$ (2.59)

2. When $f_l$ is strongly in agreement with ranking of $a_i$ and $a_j \leftrightarrow \Delta(a_i, a_j) \Delta_l(a_i, a_j) > \Delta^2(a_i, a_j)$, (2.56) gives the upper bound of $\alpha$ ($\alpha_l^+$). In this situation it corresponds to:

$$\frac{\Delta(a_i, a_j) \Delta_l(a_i, a_j)}{\Delta(a_i, a_j) \Delta_l(a_i, a_j) - \Delta^2(a_i, a_j)} > \alpha$$ (2.60)

As before, this bound has to be satisfied for all pairs of actions that fall in this category. Therefore, let us define $\Omega^+$ as follows:

$$\Omega^+ = \left\{ (a_i, a_j) \in \mathcal{A} \times \mathcal{A}, s.t. \right\} 
\Delta(a_i, a_j) \Delta_l(a_i, a_j) > \Delta^2(a_i, a_j) \right\}$$ (2.61)

Then, the upper bound of $\alpha$ is:

$$\alpha_l^+ = \min_{(a_i, a_j) \in \Omega^+} \frac{\Delta(a_i, a_j) \Delta_l(a_i, a_j)}{\Delta(a_i, a_j) \Delta_l(a_i, a_j) - \Delta^2(a_i, a_j)}$$ (2.62)

3. Finally when $0 \leq \Delta(a_i, a_j) \Delta_l(a_i, a_j) \leq \Delta^2(a_i, a_j)$, then inversion between $a_i$ and $a_j$ is not possible and $\Omega^0$ is:

$$\Omega^0 = \left\{ (a_i, a_j) \in \mathcal{A} \times \mathcal{A}, s.t. \right\} 
\Delta(a_i, a_j) = 0 \text{ and } \Delta_l \neq 0 \right\}$$ (2.63)

According to (2.56) this situation shows that there exist two states for the set $\Omega^0$:

(I) If $\Omega^0 \neq \emptyset$, then value of $\alpha$ is fixed in 1. It means no change of weights is allowed (no inversions between preferences and only changing from indifference to preference can happen).

(II) If $\Omega^0 = \emptyset$ then $\alpha_l^- < \alpha < \alpha_l^+$.

The final step is determining the weight stability intervals for criterion $f_l$. From (2.46), $\beta$ is:

$$\frac{1-w_l}{w_l} (1-\alpha) \text{ whereas its stability area is: } \beta_l^- \leq \beta \leq \beta_l^+.$$ When $W_l^- = (1 + \beta_l^-)w_l$ and $W_l^+ = (1 + \beta_l^+)w_l$, replacing $\beta$ gives the weight stability interval for criterion $f_l$ as follows:

$$WSI_l = (W_l^-, W_l^+)$$

$$= (1 - (1 - w_l)\alpha_l^+, 1 - (1 - w_l)\alpha_l^-)$$ (2.64)
These intervals should be determined for all criteria one by one. The weight stability intervals are consist of the values that the weight of one criterion can take without changing the results given by the initial set of weights, all other weights being kept constant. In Chapter 4 these intervals are used for all criteria as \((W_k^-, W_k^+)\): \(k = 1, \ldots, q\).

In the following example, the concepts of PROMETHEE II in choosing preference parameters, weight stability intervals and GAIA plane is studied.

Example 2.4 Ranking medium-sized company in Brussels (The data set is retrieved from http://www.trends.be) - In this example, the focus is on the ranking of medium-sized companies in Brussels built on PROMETHEE II. This is based on the growth value of each criterion during 4 consecutive years (criteria are introduced below). A rank is assigned to each criterion for each company during the years 2008 to 2012. The resulted ranking by “Gazelles” is obtained by adding the rank of each company on each criterion. Thus, all criteria should be minimized.

Table A1 in Appendix 1 presents the related data of 75 companies according to 6 criteria, as well as their ranking results from PROMETHEE II and Gazelles. Revenue (turnover), cash flow and employees are three main criteria that are considered in two groups: Absolute growth and Relative growth. To be included in the category of medium-sized companies, their revenue should be between 1 and 10 million Euros in the starting year. In this study, the company has to be registered for at least five years, presented a positive cash flow and engaged at least 20 persons in the last year. More information about data can be found in following website: www.trends.be.

In this example, the ranking orders in Gazelles and PROMETHEE II are compared. The preference functions are all chosen linear type. In the process of choosing thresholds, approximately always the indifference and preference thresholds are fixed to a quarter and the third quarter of the difference of the greatest and the smallest value of alternatives in each criterion. Weights are equally distributed. Indeed, the criteria engaged in this problem should be maximized in nature, but as explained above, the criteria are minimized, since the data set in Gazelles is based on the rank of each criterion: less it is, better performs in the final ranking. Tables 2.3 and 2.4 display PROMETHEE II parameters and the corresponding stability intervals in level 1 \((r = 1)\).

<table>
<thead>
<tr>
<th>Criteria</th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
<th>(C_4)</th>
<th>(C_5)</th>
<th>(C_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min/Max</td>
<td>Min</td>
<td>Min</td>
<td>Min</td>
<td>Min</td>
<td>Min</td>
<td>Min</td>
</tr>
<tr>
<td>Type</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
</tr>
<tr>
<td>Thresholds</td>
<td>(q = 39.25, p = 117.75)</td>
<td>(q = 43.5, p = 130.5)</td>
<td>(q = 54.75, p = 164.25)</td>
<td>(q = 54, p = 162)</td>
<td>(q = 38.25, p = 114.75)</td>
<td>(q = 38.5, p = 115.5)</td>
</tr>
<tr>
<td>Weights</td>
<td>0.167</td>
<td>0.167</td>
<td>0.167</td>
<td>0.167</td>
<td>0.167</td>
<td>0.167</td>
</tr>
</tbody>
</table>

Table 2.3 PROMETHEE parameters (Gazelles)

According to data in Table A1 in Appendix 1, the company “Hennes & Mauritz Logistic” has rank 1, in both methods. This pleads in favor of a robust conclusion. The first two companies in Gazelles and PROMETHEE II are the same (second company is Pull & Bear). The DM can determine the best company by analysing ranking order of both methods in this table. The Kendal’s tau rank correlation between ranking results of PROMETHEE II and Gazelles is 0.896 that indicates
a high correlation. This correlation confirms the fact that the chosen parameters for PROMETHEE II are proper enough to have approximately compatible ranking with the Gazelles one. However, at the level of stability equal to 1, generated intervals (Table 2.4) are not tight enough, but they confirm the stable place of the first company with changing weights within them.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Min weight</th>
<th>Value</th>
<th>Max weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>0</td>
<td>0.167</td>
<td>1</td>
</tr>
<tr>
<td>C₂</td>
<td>0</td>
<td>0.167</td>
<td>1</td>
</tr>
<tr>
<td>C₃</td>
<td>0</td>
<td>0.167</td>
<td>0.889</td>
</tr>
<tr>
<td>C₄</td>
<td>0</td>
<td>0.167</td>
<td>0.677</td>
</tr>
<tr>
<td>C₅</td>
<td>0</td>
<td>0.167</td>
<td>1</td>
</tr>
<tr>
<td>C₆</td>
<td>0</td>
<td>0.167</td>
<td>0.978</td>
</tr>
</tbody>
</table>

Table 2.4 Weight Stability Intervals of PROMETHEE II in \( r = 1 \) (Gazelles)

The ranking order of three first companies, when \( r \) is fixed at level 3, does not change. As expected, Hennes & Mauritz Logistic always stays at the first rank. The GAIA plane in Figure 2.5 also shows alternative 1 (H&M logistic) is the best company. Further, as expected, the criteria in the same group have similar profiles. The value of delta is 73.6%; hence, the amount of reserved information is reasonable. Enlarging Figure 2.5 demonstrates the group of alternatives that act alike.
The next section summarizes some of the limitations and strengths of MCDA problems.

### 2.4. Some strengths and limitations in MCDA

Mostly, there exist some common weak and strong features between MCDA methods, but each method has its own particular advantages and disadvantages. Table 2.6 summarizes some strengths and weaknesses of MAUT, AHP, ELCETRE and PROMETHEE methods that are considered earlier in this chapter (De Keyser and Peeters, 1996; Linkov et al., 2006; Velasquez and Hester, 2013). A general disadvantage of multicriteria problems is as follows:

- A multicriteria problem, in fact, is an ill-defined mathematical problem, since there is no optimal and objective solution for all criteria. It means there is generally an action, which outperforms one other for some criteria but simultaneously it is outperformed by another action for some other criteria. So that some pairs of actions remain incomparable. Thus, some trades-off must be done among the different points of view to determine an acceptable solution for the decision-making problem. Additionally, a ranking problem result in an objective solution only if all the criteria considered yield the same ranking, which is clearly exceptional (Vincze, 1992; Kujawski, 2003).

Choosing the compromise solutions in multicriteria problems when the decision aid tools are applied is dependent on the DM’s personality. This characteristic is not a weakness; however, it
can be relatively awkward for the researchers who are used to solving problems independent to DMs prior information (Vincke, 1992).

<table>
<thead>
<tr>
<th>Methods</th>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
</table>
| MAUT     | • Easier to compare alternatives whose overall scores are expressed as single numbers;  
          • Theoretically sound – based on utilitarian philosophy;  
          • Can incorporate preferences. (Keeney and Raiffa, 1976) | • Criteria weights obtained through less rigorous stakeholder surveys may not accurately reflect stakeholders true preferences (Linkov et al., 2006a);  
          • Preferences need to be precise (Velasquez and Hester, 2013). |
| AHP      | • Surveying pairwise comparisons is easy to implement;  
          • Verbal comparisons;  
          • Hierarchy structure can easily adjust to fit large sized problems;  
          • Scalable. (Saaty, 1988, 1990, 2003 and 2005) | • The weights obtained from pairwise comparison are strongly criticized for not reflecting people’s true preferences (Linkov et al., 2006a);  
          • The interdependence of actions and criteria may cause problems (Velasquez and Hester, 2013);  
          • Inconsistency (Saaty, 1980);  
          • Rank reversal (Belton and Gear, 1983). |
| ELECTRE  | • Does not require the reduction of all criteria to a single unit;  
          • Can discard some alternatives from evaluation and prepare the limited set of alternatives to be used by another MCDA method: saving time. (Roy, 1974; Figueira et al. 2005) | • Its process and outcome can be difficult to express;  
          • Does not provide a score for each alternative;  
          • Rank reversal. (Linkov et al., 2006a; Velasquez and Hester, 2013) |
| PROMETHEE | • Easy to use (Behzadian et al., 2010);  
            • Software;  
            • Strong graphical tool (Mareschal and Brans, 1988). | • Does not provide a clear method by which to assign weights;  
            • Works on ratio scales (De Keyser and Peeters, 1996);  
            • Not taking the discordance into account on constructing the outranking relations (De Keyser and Peeters, 1996);  
            • Rank reversal (De Keyser and Peeters, 1996). |

Table 2.6 Strengths and weaknesses of MAUT, AHP and outranking methods

Besides this feature of MCDA, of course, there are several positive points in multicriteria problems. Generally, they are all well-structured methods according to decision-making dedicated to real life problems. These problems structured based on the idea and preference information of DMs and managers.
2.5. Conclusion

This chapter concisely gives an introduction in the unicriterion and multicriteria paradigms. Then, it summarizes MCDA problem and the basic definitions and concepts behalf of it. It also briefly presents some of the multicriteria approaches: MAUT, AHP, MACBETH and ELECTRE, and explains more PROMETHEE and GAIA plane. Furthermore, it introduces the notion of weight stability intervals in PROMETHEE. The chapter is finalized with generally mentioning some limitations and advantages of multicriteria methods.

From a practical point of view, it is essential to reflect on the diverse situations in which MCDA may be applied. The MCDA framework leads to use it alongside some specific type of unicriterion optimization problems. In this thesis, PROMETHEE outranking method is applied in integration with DEA with the aim of increasing the discrimination power between units. DEA, as explained in the first chapter, is a type of LP to evaluate efficiency of DMUs. Classical DEA divides units into two groups: 1) efficient and 2) inefficient, while there is no discrimination between efficient units. One of the main approaches to encounter this problem is to limit the freedom of weights in DEA. Some of these models emerged beneath the name of “weight restricted DEA” models. Here, for the first time the weight stability intervals of PROMETHEE is applied to restraint multipliers in DEA. Furthermore, the unicriterion net flow scores matrix of PROMETHEE II is used instead of outputs of DEA. Accordingly, the discrimination power between units is increased (Bagherikahvarin and De Smet, 2016a). Chapter 4 presents this contribution.

In another work, through a two-phase algorithm, a complete ranking of DMUs in DEA is generated by using PROMETHEE (Bagherikahvarin, 2016). In the first phase, DEA is applied to generate a pairwise comparison matrix. Then, in the second phase, PROMETHEE II is used to gather information from this matrix and give the final score of each option. Chapter 5 presents this contribution.

In a particular situation, e.g. where there is not available enough preference information, DEA can support PROMETHEE in recognizing compromise solutions. Initiatively, it is suggested an extension of the so-called “decision maker brain” used in the GAIA plane. Built on this conception, a new integrated approach is proposed to give a first practical idea about importance of the criteria. Then, DM feels free to refine these values. The underlying idea is based on the computation of weights in PROMETHEE (GAIA brain) that are compatible with the DEA analysis. Further, depending on the position of this brain with respect to the alternatives, one may determine those that could eventually become good compromise solutions and those that will never be considered as good candidates. Through this contribution, it is tried to help DMs in determining some preference options in problems that are more complex or the ones where there are not clear preferences between criteria. This contribution is studied in Chapter 6.
The next chapter aims to compare MCDA and DEA problems in certain points to consider the possibility of their integration. Moreover, two existed DEA-MCDA integrated approaches are presented.
Chapter Three

Multi-Criteria Decision Aid - Data Envelopment Analysis synergies

Abstract

In this chapter, a general comparison between DEA and MCDA is presented. Additionally, two MCDEA integrated models are briefly discussed.

3.1. Introduction

Data Envelopment Analysis (DEA) and Multi-Criteria Decision Aid (MCDA) are two important research areas in Operations Research (OR) and Management Sciences (MS). DEA and MCDA, each one has their share of responsibility to the extent that they can help Decision Makers (DMs) to analyze and solve the decision-making problems.

There is an analogy between the problems tackled by DEA and the ranking problems in multicriteria analysis (Roy, 1985). The DEA and the MCDA problems, indeed, are similar, by interpreting the Decision Making Units (DMUs) as a set of candidates to be evaluated, and the input/output factors as criteria to be aggregated in the best manner. This parallel has led some researchers to begin studying the interactions between DEA and different approaches of MCDA (Belton and Vickers, 1993; Stewart, 1996; Belton and Stewart, 1999 and 2002; Bouyssou, 1999; Nakayama et al., 2002). Some questions regarding the possibility of considering DEA as a MCDA tool or MCDA as a supportive tool in DEA have been raised and criticisms have also been made in past researches (Bouyssou, 1999).

DEA, mainly, is a tool for screening and controlling the performance of several similar units, whereas MCDA is a tool for choosing, ranking, and sorting purpose (different types of MCDA problem can be seen in Chapter 2: Section 2.2.2). Besides, DEA can be seen as a “retrospective” method; since, it typically looks at background and existing record of DMUs evaluation performance, while MCDA is considered as a “prospective” method (Belton and Stewart, 1999). Nevertheless, some researchers reject these grouping labels [this is explained in Section 3.2] (Cook et al., 1992; Doyle and Green, 1993; Shang and Sueyoshi, 1995; Papagapiou et al., 1997; Sarrico
et al., 1997; Takamura and Tone, 2003). Despite the differences among DEA and MCDA (some differences are considered in the Section 3.2), which put them between competing approaches, there is growing attention in incorporating value judgments into DEA to increase its control power in monitoring of units. Accordingly, they can also be seen as complementary methods (Allen et al., 1997).

This chapter first attempts to generally compare DEA and MCDA. The connections, similarities and differences between these two methods are briefly discussed in several viewpoints such as the structure of both methods in subjectivity and objectivity, retrospective and prospective sense, model structuring, different options and their factors, weights of factors, and efficiency and Pareto optimality. Furthermore, two Multi-Criteria Data Envelopment Analysis (MCDEA) existing methods are concisely presented.

3.2. A general comparison between DEA and MCDA approaches (similarities and differences)

Different MCDA methods have been used for aiding DMs to make their final decisions in multicriteria contexts. DEA can be used as a useful tool for screening alternatives within MCDA. In a decision space, where discrete multi-dimensional alternatives exist, DEA can be linked with some MCDA methods (Amin et al., 2006) [this link is explained in the current section]. Therefore, DEA and MCDA would be considered as complementary methods; nevertheless, there are some differences between them (Belton and Stewart, 1999 and 2002):

- **Objectivity vs. Subjectivity**

MCDA lays stress on taking into account the value judgments throughout the decision making process. This process is about the specification of alternatives, criteria, the weight elicitation for the criteria, which reflects the importance of each factor, and other preference information provided by DMs. Thus, MCDA methods are dependent on subjective judgements of DMs (Belton and Stewart, 1999). This fact confirms that different DMs may provide a different priori information; hence, different results can be achieved through MCDA, whereas via classical DEA models, without engaging a priory information, a single result can be obtained.

Although MCDA relies on the subjective preference data, its approach is different in outranking methods such as PROMETHEE (Belton and Stewart, 1999). As explained in Chapter 2, PROMETHEE focuses on the degree of differences between evaluation of each pair of alternatives on each criterion that is an objective action. Then, preference functions are used for relating preferences to objectively measured performances.

The guiding instructions of DEA in identifying factors to be implemented in the problem analysis is based on statistical approaches such as correlation analysis and classification of inputs and outputs rather than value judgements (Belton and Stewart, 1999; Morita and Avkiran, 2009; Luo et al., 2012). Indeed, the rationality assumption in DEA models tries to find the best possible efficiency for each unit. Therefore, the factors’ weights are determined by model and not by the
DMs (Belton and Stewart, 1999). However, choosing model, DMUs, inputs, and outputs in DEA is a subjective action.

Besides, in several applications of decision-making problems, DEA is unable to handle qualitative scales without help of significant subjective judgements to be able to transfer such data to quantitative scale (Papagapiou et al., 1997). Furthermore, the act of monitoring DMUs in DEA, in the presence of weight freedom, is not enough a discriminant action; i.e. DEA is not able to discriminate completely between different efficient units. The concept of subjective value judgements was entered into DEA structure to incorporate the preference information, as well, increase the discrimination power of DEA (Allen et al., 1997).

- **Retrospective vs. Prospective**

The terminologies “retrospective” in DEA and “prospective” in MCDA were first used by Belton and Stewart (1999). However, retrospectivity and prospectivity concepts were from one of the founder of DEA, William Cooper, in a meeting in Jerusalem (1985). He referred to the role of “ex-post” evaluation in DEA for controlling and monitoring purposes while MCDA is a family of methods based on “ex-ante” evaluation for planning and choosing acts. He emphasized that the different use of these two techniques should not obscure their similarities in mathematical formulations.

As pointed out in introduction, a common use of DEA is established on applying historical data (using the historical data has the concept of subjectivity in its core) like personal experiences, data sets existing in organizations or articles. These data are used in evaluating performance of similar units, such as educational institutions (Salerno, 2006), bank branches (Paradi et al., 2011), health care centers (Flokou et al., 2010), and electric utility sectors (Vaninsky, 2008). The focus of DEA is identifying the units who perform well or badly and consecutively, based on the performance of former, setting targets for the latter (Belton and Stewart, 1999).

In many cases, MCDA does not have the opportunistic viewpoint of DEA: they do not use the existing data of evaluated units to make a choice for future. Therefore, a considerable attempt is needed to elicit different information such as inter-criteria importance. This can be applied to the choice of a new location for an electrical center, localization of hypermarkets (Brans and Mareschal, 2002), or selecting the right candidate for a job. These choices are based on decisions that open new opportunities in future, which have already no consideration to existing history or just have a partially attention to available data.

This fundamental difference between DEA and MCDA approaches cannot make an obstacle in their integration for the purpose of evaluating units. The retrospective models (like DEA) evaluate units based on the historical data to improve their future performance. However, DMs should use their experience to elicitate preferences in MCDA problems. There is a growing number of applications that use DEA in multicriteria problems and vice versa. For example, Islei et al. (1991) used a MCDA model to monitor and control (monitoring and controlling is the DEA’s task) the progressing process in research and development of ICI pharmaceutical units. The results of
their model provide a cohesive analysis and reporting system that helps decision making and improves interactions among a team of research managers. Sarrico et al. (1997) used DEA in a multicriteria decision-making problem to measure the performance of different educational units in the UK. They helped potential students choose their preferred future university. Gouveial et al. (2008) developed an extended Additive DEA model, based on an MCDA method, to overcome the shortcomings of DEA in discriminating between DMUs. They inverted input and output factors into utility functions that are aggregated using a weighted sum. For this reason, an additive model of MAUT method is applied. Consecutively, the new DEA model lets each DMU choose the weights associated with these utility functions while minimizing the difference of utility to the best DMU. These utility functions act like constraints on weights and eventually, increase the discrimination power of DEA. Hatefi and Torabi (2010) created a common weight MCDA-DEA model to construct two important composite indicators: Sustainable Energy Index (SEI) and Human Development Index (HDI). Wang (2015) constructed a composite indicator of the SEI for 109 countries, applying preference information to limit weights in DEA. Nevertheless, the existing overlap between DEA and MCDA would be meaningful exploring what each approach can give as an extra benefit to other or what each approach can learn from the other.

- Model structuring

The structures of DEA and MCDA approaches were already explained in the 1st and the 2nd Chapters. Structuring a model is an initial phase in characterizing a decision-making problem. It can determine whether DEA, MCDA or other way of analysis, is an applicable method to solve the problem. In structuring phase, the actions and attributes should be defined carefully to determine the notions of problem. As mentioned earlier in current section, the guidelines of DEA in identifying attributes (input/output factors) are based more on statistical approaches while MCDA relies on value judgements to choose these factors. The DEA models are built based mostly on a comparison between inputs and outputs like costs as inputs and profits as outputs. Several multicriteria problems are built on a similar structure with two general factors: minimized (costs) and maximized (profit) criteria. However, in an MCDA decision-making problem, depending upon the context of the problem, there are some other possible combinations such as low quality (min) vs high quality (max) products, short-term (min) vs long-term (max) effects and number of employee (min) vs amount of production (max) (Belton and Stewart, 1999 and 2002). Aggregating different inputs and outputs into an efficiency measure leads this argument to a bridge between DEA and MCDA [efficiency and Pareto optimality is discussed later in this section] (Belton and Vickers, 1993).

In some real-world decision-making problems, while several conflicting input and output factors should be considered, applying a DEA model can be problematic. In such situations, DEA may be unable in the choice problem. Indeed MCDA methods, as explained in the 2nd Chapter, are structured to face to the problems with several conflicting criteria, simultaneously. Therefore, looking to the structure of DEA as a multicriteria problem can be helpful to evaluate units. MCDA methods are able to choose among various options with involving preferences to the structure of
problem. In this regard, putting prior information in the structure of a DEA model is the milestone of relationship between DEA and MCDA (Belton and Vickers, 1993; Belton and Stewart, 1999 and 2002; Madlener et al., 2006).

- **Alternatives/DMUs and Criteria**

Decision Making Units (DMUs) in DEA are playing the role of decision alternatives in MCDA. DEA has a capacity to evaluate large data sets. As already stressed, the users are advised to keep the following relation between the number of inputs and outputs with DMUs: \( n \geq 3(m + s) \) [more explanations are available in Section 1.2.3] (Cooper et al., 2005). The fewer the number of DMUs in comparison with the number of inputs and outputs, the more efficient the units. This is an undesirable characteristic of DEA. This problem has led several researchers to work on it for developing some models such as inclusion of more qualitative descriptors or weight restricted DEA models (Allen et al., 1997) to improve the discrimination power of DEA. This subject is discussed later in Chapter 4. The consequence of these working groups was increasing the subjectivity in DEA concepts.

In contrast, several MCDA methods work on limited size set of alternatives. Actually, defining or generating alternatives is an important step in MCDA. Alternatives may not be well defined in the first step and further consideration is needed (as explained in Chapter 2: Section 2.2.1, the set of alternatives can be stable or evolutive). It is likely using a tool to initially screen a longlist of possible alternatives to be evaluated. DEA can be such a tool to help DMs in the phase of screening alternatives and identifying the efficient alternatives among a longlist of alternatives.

Generally, inputs and outputs in DEA can be seen as criteria in MCDA. This parallel gives the sense that inputs/outputs (criteria) are issues of concern in choosing one or more potential actions. Most of the criteria with a behavior like inputs and outputs variables must be minimized and maximized, respectively (Stewart, 1996). Multiple inputs contribute negatively and multiple outputs take part positively to the overall performance evaluation. For example, in the problem of locating a hypermarket, if there are three criteria (cost, availability of customers and parking access), cost is an input factor in a DEA view of issue and it must be minimized in multicriteria models. Similarly, availability and parking access that are considered as outputs must be maximized in an MCDA method. However, exceptions are possible. Some factors are taken into account as outputs in performance evaluation of a decision-making problem, while they should be minimized in MCDA and vice versa. For example, in the evaluation of water resource planning, each plan results in a certain amount of pollution. Therefore, in DEA viewpoint, it is taken into consideration as an output, while in MCDA structure, it should be minimized as a harmful effect of plans. The risk resulted by investment in bourse is another type of these exceptions. While it is an output in DEA concept, it should be minimized in multicriteria problems.

In DEA, the introduced weights of inputs and outputs are independent. DEA system gives the best possible performance to the DMU under evaluation by assigning weights to the inputs and outputs of that DMU. In MCDA additive model, criteria should be preference independent- so that scores can be assigned on each criterion without having to know the scores on any of the other
criteria. If not, the multicriteria problem should be restructured or the criteria should be redefined (Goodwin and Wright, 1998).

Belton and Stewart (1999) remarked an advantage of looking at criteria in MCDA as inputs and outputs in DEA context. The performance evaluation of any alternative is possible by taking into consideration only relative weights within the input criteria and within the output criteria independently.

- **Inter-criteria relations (Weights)**

Basic DEA models do not generate a common set of weights; i.e. a DEA model chooses the best possible weights for the input and output factors of each DMU. Whereas MCDA models provide a common set of weights for criteria determined by subjective judgements of DMs (Parkan and Wu, 2000).

As explained in Chapter 1, the DEA is established on an LP model for inputs/outputs weights determination. However, this gives the best possible light for each unit but the freedom of weights causes some inefficient units to be assessed as efficient on the basis of only one input and/or one output. This problem has led researchers to restrict weights freedom, which enter an element of subjectivity in DEA (some DEA weight restricted models and related critics are discussed in Chapter 4).

Further, several MCDA methods look for some approaches to explicitly elicit the importance of each criterion; i.e. nevertheless, the DMs play important role in the determining of weights values, but, some mathematical approaches can be helpful also [e.g. AHP proposes the eigenvector method to generate the importance of criteria from the pairwise comparison matrix (Saaty, 1980)]. Additionally, the degree of subjectivity in MCDA may avert DMs to specify final valid weights; thus, uncertainties of DMs express the necessity of some sensitivity or robustness analysis. This analysis is possible through some multicriteria methodologies such as PROMETHEE, facilitated by some software like D-Sight (such an analysis is explained in Chapter 2, Section 2.3.1: weight stability intervals).

- **Efficiency and Pareto optimality**

The core concept in DEA and MCDA is “Pareto optimality”; nevertheless, there is a difference in their concepts. The Pareto optimality in MCDA is defined in criteria space; and, this concept in DEA is determined in the space of outputs to inputs ratios (this ratio represents efficiency score of each DMU). This is the reason of different positions of alternatives (DMUs) in these spaces (Opricovic and Tzeng, 2008).

Definition 1.3 in Chapter 1 and Definition 2.6 in Chapter 2 define efficiency in DEA and MCDA, respectively. The set of “efficient” solutions is often called “Pareto optimal” solutions. The “Pareto optimal” solutions are the solutions that are not dominated by other solutions in both DEA and MCDA. The definition of dominancy (non-dominated solutions) in DEA and MCDA can be seen in Definitions 1.2 and 2.5.
Each “Pareto optimal” solution in MCDA represents a different compromise among criteria (Abraham et al., 2005). The “Pareto optimal” frontier in MCDA is the image of the set of Pareto optimal solutions in the space of criteria. The shape of this frontier indicates the nature of trade-offs between different criteria (changing in the importance of criteria). Each criterion can be maximized or minimized. The frontier presents the set of non-dominated solutions [compromise solutions] (Abraham et al., 2005; Opricovic and Tzeng, 2008). There is no solution better than a non-dominated solution according to all criteria (Goel et al., 2007). The “Pareto optimal” frontier in DEA is identified by the outputs to inputs ratios of efficient DMUs. Any DMU that performs the best on one specific ratio of an output to an input is called efficient by DEA.

An efficient unit in DEA is a non-dominated solution in the space of output/input ratios. The set of efficient units determined by DEA has no relationship with non-dominated solutions within MCDA. A non-dominated solution in MCDA could be an inefficient or an efficient unit in DEA (Opricovic and Tzeng, 2008). Commonly, the efficient units by DEA are highly ranked by MCDA methods, and the worst units by DEA are given low rankings by MCDA (Goel et al., 2007; Opricovic and Tzeng, 2008); nevertheless, there is no proved theory on this order. In Example 3.1, it can be seen that unit 1 has the first rank in both DEA and MCDA methods.

Kao (2010) solved a multicriteria decision-making problem by looking at its structure in DEA concept. He considered alternatives as DMUs and criteria as inputs and outputs. Hence, the DEA technique is applied to identify the non-dominated alternatives. This problem is briefly presented in Section 3.3.

- **Usefulness:**

DEA evaluates the performance of DMUs. It identifies the efficient units and determines the possible improvements for inefficient units (this improvements for inefficient units shows how a DMU needs to decrease its inputs or increase its outputs in order to become efficient). This is an issue of less interest in multicriteria problems. MCDA methods rank alternatives in the presence of simultaneous conflicting criteria.

MCDA is more concerned with understanding the structure of decision-making problems based on interactions with DMs. DEA can be seen as a useful tool to solve multicriteria problems, when the method increases its contacts with DMs (Belton and Vickers, 1993; Stewart, 1996). As discussed several times in current chapter, DM can play important role in a DEA problem by restricting the weights freedom. Effectively, the weight restricted DEA models can be one possible interaction, linking DEA and MCDA. Next chapter summarizes some important weight restricted DEA models. Furthermore, a new weight restricted DEA model is discussed (Bagherikahvarin and De Smet, 2016a).

Before concluding this part, it would be interesting to note a potential concern with respect to the DEA and MCDA methods. Both of these problems in computing the score of each unit, take into account the information about all other units in the set \( \mathcal{A} \). Therefore, the introduction of a new unit or the elimination of an existing one can change the relative position of existing units (ranking
of units). This introduces “rank reversal” problem, which is a light of future research in considering the common behaviour of DEA and MCDA.

In the next example, the multicriteria problem presented in Example 2.4 (Chapter 2) is structured like a DEA model. The evaluation table in PROMETHEE II, including alternatives and criteria, is considered as the evaluation table of DEA with DMUs and inputs. As explained in Example 2.4, the criteria have the nature of outputs but considered as inputs, since the data set in Gazelles (retrieved from http://www.trends.be) is based on the rank of each criterion. Further, in PROMETHEE the DM determines the preference information; while in DEA, it does not require to provide a priori information related to the importance of factors.

**Example 3.1** Example 2.4 referred to use of PROMETHEE method to rank medium-sized companies in Brussels (retrieved from http://www.trends.be). In current example, the CCR and BCC classical DEA models [(1.8) and (1.16)] are used to evaluate the performance of these companies, based on the related dataset from Table A1 in Appendix 1. Furthermore, the DEA and PROMETHEE results are compared.

Tables 3.1 presents the efficiency scores of CCR and BCC models. This problem is a full input case, thus a dummy output with value of 1 for all the DMUs is added to the problem to solve the CCR model. As explained in the 1st Chapter, a CCR-I-O model with a constant single output corresponds with its BCC model (Knox Lovell and Pastor, 1999). Thus, the number of equal efficient DMUs in both CCR and BCC is identical and equal to 7 (the efficiency scores are also equal). The results show that DMUs 1, 2, 6, 14, 15, 37, and 38 are the best efficient units in both mentioned DEA models. The bold numbers in Table 3.1 represent the efficiency scores equal to 1. Table 3.2 presents the ranking results of Gazelles, PROMETHEE II, SECCR and SEBCC. The SECCR and SEBCC models are the super-efficient rankings of CCR and BCC, respectively. Super efficiency is introduced in Chapter 5: Section 5.2.2. Further, the PROMETHEE II parameters can be checked in Chapter 2: Table 2.3. The efficient units number 1 and 2 are the two first ranks of PROMETHEE and Gazelles; however, some other efficient units of DEA models are not among the high ranked units of PROMETHEE and Gazelles, and vice versa. Let point out that the company “Hennes & Mauritz Logistic” has rank 1, in all methods. This pleads in favour of a robust conclusion. The DM can determine the best company by analysing rank order of different methods in this table.

Table 3.3 presents the Kendall’s Tau’s rank correlation between the rank results of different methods.

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**Table 3.3 Kendall’s Tau rank correlation (Gazelles)**
The correlation coefficient between PROMETHEE II and other methods can vary by changing the preference parameters of PROMETHEE II such as weights, preference functions and thresholds.

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Table 3.1 The efficiency scores of DEA models

This example is also considered in the next chapter for the proposed model of Bagherikahvarin and De Smet (2016a). They suggested using the weight stability intervals of PROMETHEE in a DEA model to increase the discrimination power of DEA.
Chapter 3 – Multi-Criteria Decision Aid - Data Envelopment Analysis synergies

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Table 3.2 Ranking results of mediums sized companies in Brussels

In spite of the differences between DEA and MCDA discussed in the current section, DEA could be considered as a pre-process method in MCDA. It can provide a substantial screening of alternatives for multicriteria problems by choosing the efficient units. Therefore, the number of units to be evaluated is decreased. Then, a MCDA method is applied in the resized problem to choose between options. Since, DEA determines the efficient DMUs without any information of the relative importance of inputs and outputs, it could be a useful means in MCDA, particularly, when a DM is not able to express a preference at the beginning of system designing or planning. However, DEA cannot replace MCDA in finding the compromise solutions. In this regard, a new ranking approach, based on DEA and PROMETHEE II is suggested in Chapter 5. Moreover, an algorithm based on DEA to enrich the understanding of the multicriteria problem is considered
in Chapter 6. Besides, MCDA could be seen as a helpful tool to increase the discrimination power of DEA by determining the weight restrictions in DEA. This problem is studied in Chapter 4.

This chapter provides arguments to show that DEA and MCDA can be considered as complementary methods. If a final solution of a decision-making problem is determined by seeing efficiencies (or inefficiencies) of alternatives, the idea of DEA can be applied to MCDA problems. Then, it should be noted that the efficiency ratio (output to input ratio) in DEA can also depend on value judgments or can be concerned as a value judgement (Nakayama et al., 2002). In applying DEA to a wide range of multicriteria decision-making problems, the ratio of efficiency is not enough to be considered as value judgements. Therefore, to make a link between DEA and MCDA, a model is needed, which can apply the value judgments in its structure (Nakayama et al., 2002). In what follows, two Multi-Criteria Data Envelopment analysis (MCDEA) models are presented. These models are just two examples among many existing models that can be used as a tool to solve multicriteria decision-making problems.

3.3. Two MCDEA models that can help Decision Makers in multicriteria problems

In this section, two MCDEA models are briefly presented: A General Data Envelopment Analysis (GDEA) model (Nakayama et al., 2002) and a common weight DEA model (Kao, 2010).

- **Data Envelopment Analysis in Multicriteria Decision Making** (Nakayama et al., 2002)

Nakayama et al. (2002) introduced a GDEA model (an extension of a DEA model) to evaluate the efficiency of units through several DEA models such as CCR and BCC in a unified way. The main idea of GDEA is to structure a model to evaluate the performance of DMUs, integrating somehow value judgments of DMs.

In MCDA, it may exist several Pareto efficient solutions (a Pareto efficient solution is not essentially a unique solution). Indeed, to take a meaningful decision, one should be able to determine a Pareto efficient solution among several existing solutions. For this aim, a model that can reflect value judgements of DMs is needed. The GDEA model of Nakayama et al. (2002) is such a model. Generally, the viewpoint of DEA can be applied to solve multicriteria decision-making problems, if DM chooses a final solution by evaluating the efficiency of alternatives instead of choosing the compromise solutions (Yu et al., 1996; Nakayama et al., 2002; Kleine, 2004; Jahanshahloo et al. 2009). Furthermore, it should be noted that the ratio of output to input (constrained to value of 1) can be defined as a value judgement in DEA.

The key idea in the GDEA model of Nakayama et al. (2002) is to provide a partial or full ranking between alternatives by varying one parameter ($\alpha$). It evaluates the efficiency of units in some basic DEA models, as follows:
Chapter 3 – Multi-Criteria Decision Aid - Data Envelopment Analysis synergies

(GDEA)

\[
\begin{align*}
\text{Max } & \Delta \\
\text{subject to } & \Delta \leq \tilde{d}_j + \alpha [\sum_{r=1}^{s} u_r(y_{ro} - y_{rf}) + \\
& \sum_{i=1}^{m} v_i(-x_{io} + x_{ij})], \quad j = 1, \ldots, n \\
\sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i &= 1 \\
u_r, v_i &\geq 0, \ r = 1, \ldots, s, i = 1, \ldots, m
\end{align*}
\]

where \( \tilde{d}_j = \max_{r=1,\ldots,m} \{ u_r(y_{ro} - y_{rf}), v_i(-x_{io} + x_{ij}) \}, j = 1, \ldots, n \) is the maximum of deviation between weighted unit/alternative/DMU under evaluation and other units; \( \alpha \) is a positive parameter that can be varied; \( u_r, r = 1, \ldots, s \), is the weight of outputs and \( v_i, i = 1, \ldots, m \), is the weight of inputs. \( y_{rf} \) is the output \( r \) and \( x_{ij} \) is the value of input \( i \) in \( DMU_j \), respectively. Constraint (3.1) shows that the efficiency score cannot be more than the value of deviation between the weighted unit under evaluation and other units. Indeed, when \( j=0 \), the right-hand side of equation (3.1) is zero. Therefore, the optimal value is not greater than zero (in the best case the optimal value is equal to zero). In Example 3.2, it can be seen that the efficient units have efficiency scores equal to 0. Equation (3.2) presents the weights normalization constraint; and the constraints in (3.3) explains the non-negative weights can be replaced by positive weights, where \( \varepsilon > 0 \) is a non-Archimedean element smaller than any positive real number.

Contrary to the classical DEA models, the GDEA model can integrate value judgments of DMs by varying the parameter \( \alpha \).

It should be noted that the presented GDEA model is structured by employing augmented Tchebyshev scalarizing function (the concept of deviations \( \tilde{d}_j \)). The definition and details of this function is available in Sawaragi et al. (1985).

The following definition and theorems consider efficiency of a unit through the above GDEA model (Nakayama et al., 2002).

**Definition 3.1. Efficiency in GDEA** \((\alpha - efficiency)\) - For a giving positive number \( \alpha \) by DM, \( DMU_o \) is defined to be \( \alpha - efficient \) if and only if the optimal value of the GDEA problem is equal to zero. Otherwise, \( DMU_o \) is said to be \( \alpha - inefficient \).

Two following theorems show that the parameter \( \alpha \) determines relationships between dominance structure in GDEA model and two basic DEA models: CCR and BCC.

**Theorem 3.1** \( DMU_o \) is “CCR-efficient” if and only if \( DMU_o \) is \( \alpha - efficient \) for sufficiently large positive \( \alpha \), in problem \( GDEA' \). The problem \( GDEA' \) is obtained by adding constraint \( \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} = 0 \) to the problem GDEA.
Theorem 3.2 DMU<sub>o</sub> is “BCC-efficient” if and only if DMU<sub>o</sub> is α - efficient for sufficiently large positive number α in GDEA.

Varying the parameter α in the GDEA' and GDEA models provides different dominance structure (ranking) between alternatives. These structures can be consistent with the dominance structures resulted by CCR (1.8) and BCC (1.16) models. The value of this parameter is given by DM.

For the proof of these two theorems, it can be referred to (Yun et al., 2004).

Example 3.2 (Nakayama et al., 2002) - Suppose that there are six DMUs, which consume one input to produce one output (Table 3.4). DMUs and input/output are assumed to be alternatives and criteria in MCDA, respectively.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td>2</td>
<td>3</td>
<td>4.5</td>
<td>4</td>
<td>6</td>
<td>5.5</td>
</tr>
<tr>
<td>Output</td>
<td>1</td>
<td>3</td>
<td>3.5</td>
<td>2</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.4 The dataset of 6 DMUs with 1 input and 1 output (Nakayama et al., 2002)

Table 3.5 presents the result of α - efficiency in the basic DEA models, GDEA and GDEA'. In the DEA models (CCR and BCC), it can be seen that a DMU is efficient if the optimal value is equal to 1. The α - efficiency is obtained by changing the value of α in both GDEA and GDEA'. When α-efficiency= 0, the unit is efficient. The DM put α = 10 in both models. DMU<sub>B</sub> is efficient in all four models. DMU<sub>A</sub> and DMU<sub>E</sub> are just BCC-efficient and α - efficient in GDEA.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCR model</td>
<td>0.5</td>
<td>1.00</td>
<td>0.78</td>
<td>0.50</td>
<td>0.83</td>
<td>0.73</td>
</tr>
<tr>
<td>BCC model</td>
<td>1.00</td>
<td>1.00</td>
<td>0.83</td>
<td>0.63</td>
<td>1.00</td>
<td>0.75</td>
</tr>
<tr>
<td>α=10 (GDEA')</td>
<td>-9.33</td>
<td>0.00</td>
<td>-3.25</td>
<td>-11.33</td>
<td>-0.73</td>
<td>-3.74</td>
</tr>
<tr>
<td>α=10 (GDEA)</td>
<td>0.00</td>
<td>0.00</td>
<td>-2.10</td>
<td>-11.0</td>
<td>0.00</td>
<td>-3.35</td>
</tr>
</tbody>
</table>

Table 3.5 Results of classical DEA models, GDEA and GDEA'

The Figures 3.1 and 3.2 represent, consecutively, the efficient frontier generated by GDEA' and GDEA when α = 10. The equivalence of GDEA' with CCR and GDEA with BCC efficient frontiers is evident in these figures.

The GDEA model is useful for evaluating the efficiency of complex management systems in business, industry and social problems when the computational complexity is high (Nakayama et al., 2002).
Hereafter, another MCDEA model is introduced to evaluate performance and rank alternatives in multicriteria problems based on Common Set of Weights (CSW) in DEA (Kao, 2010).

**A common-weight DEA method for ranking alternatives in MCDA** (Kao, 2010)

One of the main characteristics of DEA is to choose the weight of inputs and outputs automatically by model. It may be an advantage in calculating the efficiency scores of DMUs. As noted in Chapters 1 and 3, this flexibility can also make troubles in ranking DMUs. Further, weights used in computing the efficiency scores, are varied from unit to unit. In this regard, many researchers tried to find a CSW in DEA, based on compromise solutions in MCDA method (Kao and Hung, 2005; Kao, 2010; Chunheng and Wenbin, 2015).
Kao (2010) structured a compromise programming technique to determine weights of criteria (a CSW model). He took the difference between the performance of an alternative and the ideal point as the distance criterion. The final ranking is based on the aggregate performance calculated from the set of weights. Kao (2010) suggested a measure of relative distance. Interesting point of the relative distance measures is that the rankings obtained based on the distance to the ideal and those obtained based on the distance to the anti-ideal are the same. An ideal point as defined in Chapter 2 (Definition 2.7) is a virtual point that can consume the least inputs to produce the most outputs, while an anti-ideal point is a virtual point that uses the most inputs only to generate the least outputs. An ideal point can act as a benchmark to be compared with other points. Naturally, the aggregate performance of an ideal alternative is better than any other alternative, no matter what weights are applied to individual criteria. Therefore, the aggregate performance of the \( j \)th alternative relative to the ideal unit is computed as \( P_j = \sum_{r=1}^{s} w_r Y_{rj}/ \sum_{r=1}^{s} w_r Y_r^* < 1 \) (since the ideal unit has an aggregate performance equal to 1 to normalize the weights; \( \sum_{r=1}^{s} w_r Y_r^* = 1 \)), where \( Y_{rj} \) denotes the value of the \( r \)th criterion (output), \( r = 1, 2, ..., s \), in the \( j \)th alternative; \( j = 1, ..., n \) and \( w_r \) is the associated weight to the \( r \)th criterion. Then, \( P^* = \sum_{r=1}^{s} w_r Y_r^* / \sum_{r=1}^{s} w_r Y_r^* = 1 \) represents a relative performance of the ideal alternative. Consequently, \( \sum_{r=1}^{s} w_r Y_{rj} \) becomes the relative performance of the \( j \)th alternative. The relative distance between the \( j \)th alternative and the ideal point, in terms of the aggregate performance, is defined by \( s_j = 1 - P_j \) (\( s_j \) is always positive because every unit is dominated by ideal unit). Since \( s_j \) represents the relative position of alternative \( j \) from the origin to its projection on the frontier, it is a relative distance measurement (Kao, 2010).

Kao (2010) considered a multicriteria problem as a DEA problem without input. Then, he proposed the following model to minimize the total squared difference \( s_j \):

\[
\begin{align*}
\text{Min } & \sum_{j=1}^{n} s_j^2 \\
\text{s.t.} & \\
\sum_{r=1}^{s} w_r Y_{rj} + s_j &= 1, j = 1, ..., n \tag{3.4} \\
\sum_{r=1}^{s} w_r Y_r^* &= 1 \\
w_r & \geq \varepsilon, \quad r = 1, 2, ..., s
\end{align*}
\]

This model generates a CSW \((w_r^*)\). These optimal weights are then used to compute the relative performance of the \( j \)th alternative. Thus, the relative distance to the ideal point is computed as:

\[
s_j^* = \sum_{r=1}^{s} w_r^* Y_r - \sum_{r=1}^{s} w_r Y_{rj} \\
= \sum_{r=1}^{s} w_r^* (Y_r - Y_{rj}) \tag{3.5}
\]

The value of \( s_j^* \) is the base of ranking.
Similarly, the relative distance to the anti-ideal can be computed.

**Example 3.3** Kao (2010) considered a data set with five DMUs and only two outputs. Table 3.6 presents the related data set. Furthermore, the ranking results of relative distance to ideal and anti-ideal points is compared with the result of a DEA model. DEA model is a CCR model without inputs; a dummy input is added to solve it.

<table>
<thead>
<tr>
<th>Alts./DMUs</th>
<th>Y₁</th>
<th>Y₂</th>
<th>DEA</th>
<th>Relative Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Efficiency</td>
<td>Rank</td>
<td>Ideal</td>
<td>Rank</td>
</tr>
<tr>
<td>A</td>
<td>2</td>
<td>4</td>
<td>0.8</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>4</td>
<td>0.9</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
<td>2</td>
<td>0.998</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 3.6 DEA and relative distance comparisons in an example (Kao, 2010)**

Figure 3.3 illustrates the data set of Example 3.3 (five alternatives with just two outputs \( Y₁ \) and \( Y₂ \)). Kao computed weights via model (3.4): \( w₁^* = 11/90 \) and \( w₂^* = 7/90 \). Then replaced them in (3.5) to compute scores. Alternatives \( B \) and \( D \) are recognized as the efficient DMUs/alternatives (lied on the efficient frontier). Alternative \( E \) located in the vertical line (extended down form \( D \)), is weakly efficient; therefore, it is ranked next to \( B \) and \( D \). Points \( A \) and \( C \) are dominated alternatives (non-efficient). Line segments SBDT characterizes the efficient frontier. The ratios of \( \frac{OA}{OA'} \) and \( \frac{OC}{OC'} \) is the measure of \( A \) and \( C \)’s efficiency that are equal with the obtained scores from (3.5) \( [A' \) and \( C' \) are the projections on the efficient frontier]. \( C \) performs better than \( A \) while it is closer to the efficient frontier; further, its efficiency score is more than \( A \).

Kao (2010) gave a geometrical interpretation in Figure 3.3 from the relative performance compared with the ideal point, as follows:

A frontier \( UW \) is defined passed from the ideal point. Each unit is compared with its projection on this frontier, to compute its aggregate performance. For example, the performance measure of \( D \) is \( OD/OD^* \) (\( D^* \) is the projection of \( D \) on the ideal frontier). The value of \( OD/OD^* \) is the relative distance to the anti-ideal while \( DD^*/OD^* \) is the relative distance to the ideal point. Suppose that \( U'W' \) is parallel to the frontier \( UW \) passing through \( D \). This line is represented by \( w₁^*Y₁ + w₂^*Y₂ = \left(\frac{11}{90}\right)Y₁ + \left(\frac{7}{90}\right)Y₂ = 1 - s_D = \frac{76}{90} = 0.85 \). Assume that the line connecting the origin and the ideal point intersects line \( U'W' \) at \( D^o \). It can be shown that \( OD/OD^* = OD^o/OI \). Since the length of \( OI \) has been rescaled to 1, \( OD/OD^* = OD^o \) (\( w₁^*Y₁D + w₂^*Y₂D = 0.85 \) is the aggregate performance of \( D \)). Thus, comparing an alternative with the ideal point is equivalent to compare it with its projection on the ideal frontier. Line \( U''W'' \), another parallel line with frontier \( UW \), passes from the non-efficient unit \( A \). Similarly, one can draw a parallel line with equation \( \left(\frac{11}{90}\right)Y₁ + \left(\frac{7}{90}\right)Y₂ = \).
1 - s_A = \frac{50}{90} = 0.56. It attributes the character of point A. The length of OA^o is the aggregate performance of alternative A, with a value of w_1Y_{1A} + w_2Y_{2A} = 0.56. For the general case of n alternatives, n parallel hyperplanes are constructed; each has a particular distance to the efficient ideal frontier. The best alternative is the one that has the shortest distance to the ideal frontier.

Figure 3.3 The Geometrical representation of the relative performance compared with the ideal point I (Kao, 2010)

In Table 3.6, as expected, the ranking resulting from the relative distance to the ideal and the anti-ideal are the same. Ranking order of alternatives C and E in DEA (the 4th and 3rd places, respectively) is reverse of ranking obtained by relative distances (C 3rd and E 4th). The detailed calculation of relative distance can be seen in (Kao, 2010).

One more time, Kao’s technique shows that the idea of DEA can be used in MCDA problems. Through the proposed model, the weights do not have to be chosen by DMs and it is the model that generates a CSW.

Generally, the objective of MCDEA methods is structuring a model to solve a multicriteria decision-making problem. DEA and MCDA are supportive methods. However, DEA cannot be replaced by MCDA, but it can be used as a tool to help DMs. Through DEA, it is possible to evaluate the performance of alternatives instead of choosing the compromise solutions. Furthermore, the value judgements of DMs can be reflected in DEA models. This would make possible to evaluate more precisely the alternatives. Besides, looking at the structure of a DEA
problem as a MCDA problem allows ranking DMUs. Hence, it can be understood that DEA and MCDA, both can be helpful for each other.

In following chapters, three different MCDEA algorithms are presented. Each one of these algorithms tries to solve one of the difficulties of DEA and/or MCDA that are mentioned in Chapters 1 and 2.

### 3.4. Conclusion

Through this chapter, a brief comparison between MCDA and DEA methodologies is done from the perspective of subjectivity/objectivity, retrospective/prospective, model structuring and the components of each structure (alternatives/DMUs and criteria/inputs and outputs), weights, efficiency and Pareto optimality (solutions) and their usefulness. Despite these differences in background and philosophy, each approach can learn from the other. Additionally, two MCDEA models are presented: GDEA and CSW. Both models treated a multicriteria problem by means of a DEA structure.

The 2nd part of this thesis presents the three main contributions. Each one of these contributions is a MCDEA technique in the form of DEA-PROMETHEE integration. The proposed integrated algorithms try to help DMs in solving multicriteria decision-making problems.

As mentioned in the 1st Chapter, within the DEA context, problem of discrimination between efficient and inefficient DMUs often arises, mainly if there is a relatively large number of input and output factors with respect to units to be evaluated. In the next chapter (Chapter 4), first, some important weight restriction techniques in DEA are introduced. Then, in this regard, a new DEA-PROMETHEE integrated model is presented, which increases the discrimination power of DEA by bounding freedom of weights based on the weight stability intervals of PROMETHEE (Bagherikahvarin and De Smet, 2016a).

One issue in the DEA structure is that it is not able to fully rank DMUs. In Chapter 5, an integrated DEA-PROMETHEE algorithm is discussed, which gives a complete rank of units (Bagherikahvarin, 2016).

One main fact in MCDA method is determining a priori information such as weights to structure model. In some complicated situations where prioritization of criteria is not a simple task, DEA can be applied to help DMs and give the initial view on importance of criteria. The last contribution of current thesis (Chapter 6) introduces an algorithm to compute weights in PROMETHEE that are compatible with the DEA analysis (Bagherikahvarin and De Smet, 2016b).
Part 2: The Contributions
Chapter Four

A ranking method based on DEA and PROMETHEE II


Abstract

In this chapter, the motivation for integrating weight restrictions in DEA are briefly presented. Moreover, some of the most important weight restricted DEA models are summarized. The focus of this chapter is on the generating weight restrictions in DEA models by using MCDA methods. In this regard, the PROMETHEE and AHP methods are compared to discuss the pros and cons of using each one to generate bounds of weights in DEA. Finally, a new weight restricted DEA model based on PROMETHEE II is investigated.

4.1. Introduction

The issue of engaging value judgements and preferences very often arises in both theoretical thoughts and practical applications (real life problems) of different extensions of DEA. It is one of the reasons that motivates researchers to integrate DEA and MCDA in order to put the preferential information resulted by a MCDA problem into a DEA model. Robinson et al. (1976) suggested that it is important to recognize and decide which types of opinions are to be used in evaluation process: “value judgements” and/or “preferences”. “Value judgements” show whether something is good or bad in some respects. Thus, the judgements can be used as a critical opinion based on an assessment of merit or against a standard of comparison, e.g. item X is good. Whereas “preference” is defined as an opinion that explicitly relates to an individual(s) “liking” or “disliking” based on experience, e.g. I like item X (Robinson et al., 1976). Weight restrictions in DEA can be seen as a particular case of value judgements and/or preferences. When there are enough reasons to determine and put some priori information into DEA model, it is called value judgements. When the chosen information to put into model is depends on the feelings of DMs, it is called preferences. Liu et al. (2006b) explained that value judgements in DEA are closely associated with preferences of DMs; thus, both terms can be used as the same concepts in the performance evaluation of DMUs. In this thesis, both terms are used. Allen et al. (1997) expressed value judgements as: “the concept of value judgments, albeit frequently discussed, is lacking a
formal definition in the context of DEA, and therefore value judgments are considered as logical constructs, incorporated within an efficiency assessment study, reflecting the Decision Makers’ (DMs) preferences in the process of assessing efficiency”. The aim of incorporating value judgments into DEA method is to indicate the importance of using prior preferences or information in efficiency evaluation. Different approaches can be applied to incorporate the prior information in DEA like weight restrictions.

This chapter is structured and developed as follows: The second section highlights the motivation for incorporating weight restrictions in DEA. In the third section, main types of weight restrictions are reviewed. The fourth section introduces some weight restricted DEA models based on MCDA methods. In this section, a new ranking model is presented based on DEA and PROMETHEE II. The dual problem of this ranking model and some of its properties are also considered.

4.2. Motivation for incorporating weight restrictions in DEA

As explained in the first part of this thesis, a classical DEA model assigns weights to inputs and outputs to compute the relative efficiency of a DMU such that the ratio of its weighted outputs to weighted inputs is maximized. There are only two constraints, which restrict the weights: the efficiency of DMUs should not exceed one and the weights should not be negative (this condition can be changed to non-zero weights). Hence, DEA in its purest form allows almost total flexibility in the selection of weights. The argument with respect to weight flexibility is that the efficiency of different DMUs under evaluation is computed using different sets of weights to let each DMU achieves its maximum efficiency rating feasible for its inputs and outputs. This total flexibility is important in distinguishing of inefficient DMUs. These DMUs lie under Pareto frontier and are outperformed by efficient units even though using their own favorable possible set of weights.

However, this total weight flexibility has led to some criticisms in application. The most important drawbacks are as follows:

1- The total flexibility in weights may allow a classical DEA model assigns very low or even zero values to some unfavorable input(s) and/or output(s) of a DMU in order to exclude these input(s) and/or output(s) from the evaluation process and become efficient (regardless of unfavorable input/output factors). In an extreme case, a DMU may appear efficient compared to the equivalent ratios of other units, for just a single ratio of an output to an input, while ignoring other inputs and outputs, which may also be important. Consequently, the classical DEA models with unbounded weights (the DEA models, which have only the non-negativity bounds on weights) may generate some unrealistic results. In this regard, a data set from Wong and Beasely (1990) is considered bellow.

**Example 4.1** Wong and Beasely (1990) evaluated the relative efficiency of 7 departments of a university according to 3 inputs (I): Number of Academic Staff in each department (NAS), Salary of Academic Staff (SAS) and, Salary of Support Staff (SSS), and 3 outputs (O): Number of
Undergraduate Students (NUS), Number of Postgraduate Students (NPS), and Number of Published Research papers (NPR)]. Table 4.1 illustrates the corresponding data set.

<table>
<thead>
<tr>
<th>Departments</th>
<th>I/O</th>
<th>I₁ (NAS)</th>
<th>I₂ (SAS)</th>
<th>I₃ (SSS)</th>
<th>O₁ (NUS)</th>
<th>O₂ (NPS)</th>
<th>O₃ (NPR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₁</td>
<td></td>
<td>12</td>
<td>400</td>
<td>20</td>
<td>60</td>
<td>35</td>
<td>17</td>
</tr>
<tr>
<td>D₂</td>
<td></td>
<td>19</td>
<td>750</td>
<td>70</td>
<td>139</td>
<td>41</td>
<td>40</td>
</tr>
<tr>
<td>D₃</td>
<td></td>
<td>42</td>
<td>1500</td>
<td>70</td>
<td>225</td>
<td>68</td>
<td>75</td>
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<td>D₄</td>
<td></td>
<td>15</td>
<td>600</td>
<td>100</td>
<td>90</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>D₅</td>
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<td></td>
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<td>730</td>
<td>50</td>
<td>132</td>
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</tr>
<tr>
<td>D₇</td>
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<td>2350</td>
<td>600</td>
<td>305</td>
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<td>97</td>
</tr>
</tbody>
</table>

Table 4.1 Data set of 7 departments (Wong and Beasley, 1990)

A classic CCR model [model (1.8)] is run on this data set. The resulted data in Table 4.2 shows that just department 4 is inefficient while it ignores the salary of academic and support staff as inputs and number of published papers as output. All other departments are evaluated as efficient while some of their inputs and outputs are ignored. Therefore, for a DM who thinks for example the number of published paper or the salary of academic staff is an important factor to evaluate the efficiency of each department, this result is not acceptable. Wong and Beasley (1990) created a virtual weight restricted DEA model (this model is presented in Section 4.3.2) based on the use of proportions to control weight flexibility. They set the lower and upper bounds for each weight factor. Consecutively, the number of efficient DMUs reduces to 4 departments while most of the weight factors are considered in this assessment (not equal to zero).

<table>
<thead>
<tr>
<th>Departments</th>
<th>Efficiency</th>
<th>v₁</th>
<th>v₂</th>
<th>v₃</th>
<th>u₁</th>
<th>u₂</th>
<th>u₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₁</td>
<td>1</td>
<td>0</td>
<td>0.0285</td>
<td>0</td>
<td>0</td>
<td>0.0025</td>
<td>0</td>
</tr>
<tr>
<td>D₂</td>
<td>1</td>
<td>0.006</td>
<td>0.0039</td>
<td>0</td>
<td>0</td>
<td>0.0013</td>
<td>0</td>
</tr>
<tr>
<td>D₃</td>
<td>1</td>
<td>0.004</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0004</td>
<td>0.0063</td>
</tr>
<tr>
<td>D₄</td>
<td>0.82</td>
<td>0.009</td>
<td>0</td>
<td>0</td>
<td>0.0641</td>
<td>0.000062</td>
<td>0</td>
</tr>
<tr>
<td>D₅</td>
<td>1</td>
<td>0.0016</td>
<td>0</td>
<td>0.0045</td>
<td>0.0218</td>
<td>0</td>
<td>0.000064</td>
</tr>
<tr>
<td>D₆</td>
<td>1</td>
<td>0.0039</td>
<td>0.108</td>
<td>0.0522</td>
<td>0</td>
<td>0</td>
<td>0.00015</td>
</tr>
<tr>
<td>D₇</td>
<td>1</td>
<td>0.00174</td>
<td>0.0048</td>
<td>0.2332</td>
<td>0</td>
<td>0</td>
<td>0.000069</td>
</tr>
</tbody>
</table>

Table 4.2 Efficiency scores and weight factors resulted from unbounded CCR model

2- The process of estimating DMUs efficiencies under complete weight flexibility allows the DMUs under evaluation assigning vastly different weights for the same factor. In some points, it can be inappropriate, since similar DMUs in DEA use similar inputs to produce the same kind of outputs. Of course, some degree of weight flexibility can also be desirable, since each DMU is characterized through its factors (inputs/outputs) and tend to reflect its specific circumstances
regard to these factors. Thus, one input or output may be more important in reflecting the feature of one DMU compared to another DMU (Pedraja et al., 1997).

3- One considerable problem in unbounded DEA models is that they do not allow to incorporate any prior information and preferences of DMs and managers into the analysis regarding the importance of inputs and/or outputs.

From the first application of DEA by Charnes et al. (1978) in evaluating the performance of the “Program follow through” in the U.S.A., a growing attention was concerned to control the full flexibility of weights. In this case, analyzing data shows numerous DMUs evaluated as efficient, which ignores value judgements and managerial preferences on the weights. In this regard and during the last 3 decades, researchers developed several weight restricted DEA models to avoid drawbacks of total weight flexibility and to incorporate the value judgements into them (Allen et al., 1997; Thanassoulis et al., 2004). Within these models, they tried to set upper and/or lower bounds within which factor weights are allowed to vary. Weight restrictions possibly reduce the region of feasible solutions for the weights. Therefore, it reduces the efficiency of the DMUs and the number of efficient DMUs. Consequently, it increases the discrimination power of DEA between DMUs.

Below, some purposes are listed for which weight restriction(s) could be used in DEA models (Allen et al., 1997):

- To increase the discrimination power of DEA when the best DMU(s) should be chosen instead of a group of efficient units.

The problem of weak discriminating power of the classical DEA models (CCR and BCC) between DMUs, when they have full weight flexibility, often occurs once the number of DMUs under evaluation is small, compared to the total number of outputs and inputs. Since, in this situation, each DMU, to obtain its efficiency score, mostly focuses on a specific input-output combination, which is not directly comparable with the same combination in other DMUs. Therefore, many DMUs are able to achieve their maximum efficiency scores and are identified as efficient, which is not desirable (Thanassoulis et al., 2004). First, Thompson et al. (1986) encounters this problem when they tried to locate the high-energy physics sites in Texas, U.S.A., by the means of DEA. Five sites out of six were evaluated as efficient due to the CCR model (1.8) while incorporating Assurance Regions (ARs) as the constraints on free weights, based on expert opinions, reduced this number to just one site (the ARs are presented in the Section 4.3.1). In the Example 4.1, data in Table 4.2 shows that six departments between seven departments are considered efficient. Incorporating value judgements as constraints decreased this number to four efficient departments. As already stressed in an empirical principle (please see Chapter 1: Section 1.2.3), it is better that the number of DMUs be equal or greater than the sum of the inputs and outputs: \( n \geq 3(m + s) \) (Cooper et al., 2005).
A new DEA-based ranking algorithm is introduced in this chapter (Section 4.4.4). The weight stability intervals of PROMETHEE are used as weight restrictions to increase the discrimination power of DEA (Bagherikahvarin and De Smet, 2016a).

- To incorporate prior views on the values of individual input and output factors in DEA. As explained in the 3rd drawback of weight flexibility, by assigning bounds to weights, the DM can express her/his opinion about the relative importance of the factors. Chilingerian and Sherman (1997) rated physician practice patterns (DMUs) by applying weight restrictions. These restrictions were determined by the health maintenance office director and helped to identify the efficient patterns whose were in line with the preferences of the director. In another study, Li et al (2008) evaluated the performance of participating nations in six summer Olympic Games through a DEA-AR model. The outputs of their model were the number of gold, silver, and bronze medals, and the inputs were GDP per capita (US dollars) and the population of each country. They adjusted the ARs only on the outputs and based on the importance of the ratio of the gold medal to silver and bronze medals. The results as can be seen in Li et al. (2008) shows an improvement in discrimination between participating nations.

- To incorporate prior views on efficient and inefficient DMUs to consider prior perceptions of management in a whole problem. Often management has prior insights (regarding to expert opinions) in performance of some DMUs. Weight restricted DEA models allow management to integrate these insights, about good or poor DMUs, into DEA analysis. Charnes et al. (1990) assessed the performance of large commercial banks in the U.S.A. by developing the Cone-Ratio model (Cone-Ratio model is explained in Section 4.3.3). They imposed management views on three preselected banks (which were recognized as very good banks). While CCR model identified some inefficient banks as efficient (for just a particular ratio of output to input, the DMU is considered efficient while the other factors were ignored), incorporating weight restrictions on preselected banks improved this evaluation.

- To ensure the incorporation of all inputs and outputs in the performance assessment of DMUs. Weight restricted DEA models ensure that likely all factors are considered in the analysis (please refer to Example 4.1 to see the weights in an unbounded DEA model).

- To make a relation between values of certain inputs with values of certain outputs. In 1995, DEA was proposed to assess the efficiency of perinatal care units in the U.K. The district health authorities discussed on using a standard survival rate named as “survival rate of babies at risk” for the aim of this assessment and recognizing environmental impacts on mortality. This rate was a key factor in measuring the quality of medical outcomes in perinatal care units. However, considering it as an input or output in a DEA model was problematic. The survival rate disturbed the assumption of CRS that was likely for the other inputs and outputs factors. Since, the factors,
which affected the number of survivals and the number of babies at risk, might be different. Further, babies at risk was estimated by a different approach from survivals. Thanassoulis et al. (1995) addressed this problem by treating the "babies at risk" as an input and the "number of survivals" as an output. A DEA model, with total weight flexibility, relates the ratio of survivals to babies at risk to all other output to input ratios that is not desirable in this problem. Therefore, they developed a weight restricted DEA model (ARII) considered several set of constraints on weights to evaluate this efficiency. One of the most discussed constraints was considering the weight on the "babies at risk" equal to the weight on the "number of survivals". The logic behind equating the two weights was that these two factors together well-defined the “survival rate of babies at risk”. The information on the preferences between factors was collected from expert opinions. One can refer to Thanassoulis et al. (1995) for details of this problem.

In the next section, some important weight restricted DEA models are summarized.

### 4.3. Different techniques for imposing weight restrictions in DEA

In this section, some techniques are briefly introduced, used to impose weight restrictions into CCR-I-O multiplier DEA model. Allen et al. (1997) classified these techniques in three major groups:

a) Direct restrictions on weights (Absolute weight restrictions and Assurance Regions);

b) Restricting weight flexibility by restricting the weighted inputs and outputs (Virtual weight restrictions DEA model and Contingent restrictions on weights);

c) Adjusting the observed input-output levels to capture value judgements (Cone-Ratio).

It should be noted that in this thesis, the weight restrictions are imposed to all inputs and outputs of all DMUs and not to a specific inputs and/or outputs.

#### 4.3.1. Direct restrictions on weight

Additional constraints involving the weights are included in the existing DEA models as direct restrictions on the weights \([r_1-r_7\) in (4.1)].

Allen et al. (1997) presented following multiplier model to impose direct restrictions on the CCR model [presented in the first chapter (1.8)]:

\[
\begin{align*}
\text{Max } \theta &= \sum_{r=1}^{s} u_r y_{ro} \\
\text{s.t.} \\
\sum_{i=1}^{m} v_i x_{io} &= 1 \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} &\leq 0, j = 1, ..., n \\
\alpha_i &\leq v_i \leq \beta_i \text{ (r1)} \\
\gamma_r &\leq u_r \leq \delta_r \text{ (r2)} \\
\eta_i v_i + \eta_{i+1} v_{i+1} &\leq v_{i+2} \text{ (r3)}
\end{align*}
\]
\[
\theta_r u_r + \theta_{r+1} u_{r+1} \leq u_{r+2} \quad (r_4)
\]
\[
k_i \leq v_i/v_{i+1} \leq \rho_i \quad (r_5)
\]
\[
\sigma_r \leq u_r/u_{r+1} \leq \tau_r \quad (r_6)
\]
\[
\varphi_i v_i \leq u_r \quad (r_7)
\]
\[
\zeta_i \leq v_i x_{ij} / \sum_{i=1}^{m} v_i x_{ij} \leq \chi_i \quad (r_8)
\]
\[
\psi_r \leq u_r y_{rj} / \sum_{r=1}^{s} u_r y_{rj} \leq \omega_r \quad (r_9)
\]

where \(v_i \) and \(u_r \) are the weights of \(i^{th}\) input and \(r^{th}\) output, respectively. The parameters \((\alpha_i, \beta_i, \gamma_r, \delta_i, \eta_i, \theta_r, k_i, \rho_i, \sigma_r, \tau_r, \varphi_i, \zeta_i, \chi_i, \psi_r, \omega_r)\) are constants, which are specified by users/experts and reflect value judgments in relative importance of input/output factors.

Allen et al. (1997) divided restrictions \(r_1\) to \(r_7\) into three main categories of direct restrictions on CCR model:

1. **Absolute restrictions on weights:**
   - Input Weights Restrictions (IWR): \(\alpha_i \leq v_i \leq \beta_i \quad (r_1)\)
   - Output Weights Restrictions (OWR): \(\gamma_r \leq u_r \leq \delta_r \quad (r_2)\)

   This type of restrictions was first introduced by Dyson and Thanassoulis (1988) to evaluate the efficiency of university departments. Cook et al. (1991) used these restrictions to assess efficiency of highway maintenance patrols. The absolute constraints on weights impose upper and/or lower bounds on the input and/or output weights (Roll et al., 1991; Roll and Golany, 1993) in order to prevent the ignorance or over emphasized of inputs/outputs (Allen et al. 1997). Further, these bounds can reflect the preferences of DMs. Each factor may take the lower bound, the upper bound or both bounds, simultaneously, dependent upon experts’ opinions and problems contexts.

**Example 4.2** (Liu et al., 2006a) - The following table presents the results (outputs) of three students evaluated in mathematics, physics and chemistry.

<table>
<thead>
<tr>
<th>Student</th>
<th>Mathematics</th>
<th>Physics</th>
<th>Chemistry</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>71</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>70</td>
<td>70</td>
<td>70</td>
</tr>
<tr>
<td>C</td>
<td>50</td>
<td>50</td>
<td>99</td>
</tr>
</tbody>
</table>

Table 4.3 Results of 3 students (Liu et al., 2006a)

Table 4.4 shows the results of the simple CCR and absolute weight restricted CCR models.

This problem is a fully output case and is solved by adding a dummy input equal to vector 1. Running CCR model without any restrictions cause ignoring some outputs by assigning zero weights, such as physics and chemistry for student A and mathematics and physics for student C.
### Table 4.4 Results of CCR and absolute weight restricted CCR models

<table>
<thead>
<tr>
<th></th>
<th>CCR</th>
<th>Absolute weight restricted CCR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Score $u_1^<em>$ $u_2^</em>$ $u_3^*$</td>
<td>Score $u_1^<em>$ $u_2^</em>$ $u_3^*$</td>
</tr>
<tr>
<td>Student A</td>
<td>1 0.0141 0 0</td>
<td>0.3978 0.0053 0.0045 0.0045</td>
</tr>
<tr>
<td>Student B</td>
<td>1 0 0.0099 0.0044</td>
<td>1 0.0045 0.0045 0.0053</td>
</tr>
<tr>
<td>Student C</td>
<td>1 0 0 0.0101</td>
<td>0.9733 0.0045 0.0045 0.0053</td>
</tr>
</tbody>
</table>

Further, all DMUs seems to be efficient. It happens since each DMU has the complete freedom to select the weights, which are the most favorable for its assessment to achieve the maximum efficiency score. If the DM believes that, the students should have good progress in some subjects, so, the weight factor of each subject should not be allowed too low. In this problem, DMs added constraint $0.0045 \leq u_r$ for $r = 1, 2, 3$. The new result shows only student B as efficient DMU. It can be mentioned that the lower bounds less than 0.004 are affectless in the results. Moreover, the lower bounds more than 0.005 cause infeasibility. This represents the dependence of the problem to the chosen restrictions by DM, which can be considered as a weakness.

The absolute weight restrictions increase the discrimination power of DEA and decrease the number of dispersed weights; however, there are some limits in this type of weight restrictions. The interpretation of these bounds in the efficiency evaluation is a challenge, since the efficiency in DEA is defined based on the weights relative; i.e. there is a strong interdependency between the bounds on different weights. For example, setting an upper bound on one input weight implicitly set a lower bound on the virtual input of all other factors. Thus, it affects the weight values of remaining inputs. Another difficulty with absolute restrictions is the possibility of infeasibility (Podinovski and Athanassopoulos, 1998; Podinovski, 1999, 2001, 2004, 2005).

The key difficulty in applying these restrictions is the determination of their values. Roll and Golany (1993) proposed a general method to specify parameters for absolute weights restrictions when there is no priori information about the factors. This method is composed of four steps as follows:

1) Run an unbounded CCR model to generate a weight matrix;
2) Find the average of each factor weight $(v_i^*, u_r^*)$ for all DMUs;
3) Determine the ratio of the highest weight value to the lowest one and put it equal to $d$;
4) Add $\frac{2u_r^*}{1+d} \leq u_r \leq \frac{2d u_r^*}{1+d}$ and $\frac{2v_i^*}{1+d} \leq v_i \leq \frac{2d v_i^*}{1+d}$ as constraints to the output and input weights in a CCR model. The logic behind using these bounds is that they cause the lower bounds to take a value smaller than 1 and the upper bounds to take a value greater than 1.

It should be noted that these bounds do not promise the feasibility of model. Besides, these bounds do not reflect prior information on factors, which is one of the main purposes of weight restrictions in DEA. Though, it guaranties that the weights do not take zero or extreme values. Furthermore, these weight restrictions cause increasing the discriminant between efficient DMUs.
In the case of existing partial priori information about factors, the determined bounds can be redefined or modified by attention to this information. The details of the proposed approach and the related example can be found in (Roll and Golany, 1993).

In this chapter, a new absolute weight restricted DEA model is presented, based on the PROMETHEE method. In this new ranking model, the weight stability intervals of PROMETHEE are used as the weights’ bounds in DEA. The structure of model is enriched by preference information of DMs not only through weight stability intervals but also by means of using the unicriterion net flow scores matrix of PROMETHEE. An important issue in this model is that weight restrictions are generated automatically by PROMETHEE. Actually, this model increases the discrimination power of DEA. It also decreases the weight disperse dependent upon stability level (Bagherikahvarin and De Smet, 2016a).

2- Assurance Region type I:

\[(IWR) \eta_i v_i + \eta_{i+1} v_{i+1} \leq v_{i+2} (r_3), k_i \leq v_i / v_{i+1} \leq \rho_i (r_5)\]

\[(OWR) \theta_r u_r + \theta_{r+1} u_{r+1} \leq u_{r+2} (r_4), \sigma_r \leq u_r / u_{r+1} \leq \tau_r (r_6)\]

In Assurance Region type I (ARI) lower and upper limits are imposed on the ratios of inputs and/or outputs weights. ARI was introduced to accomplish two purposes: 1- Incorporating relative values of inputs/outputs and 2- Incorporating a priori information on marginal rates of substitution between inputs and/or outputs (Charnes et al., 1985). First, Thompson et al. (1986) used the ARI to improve the discrimination on selection between physic laboratory sites. Since then, such weight restrictions have been applied in various applications based on either some existing information from market (Thompson et al., 1990; Thompson et al. 1996a; Thompson et al. 1996b; Taylor et al., 1997) or expert opinions (Beasley, 1990; Zhu, 1996; Takamura and Tone, 2003) on the relative importance of the inputs and/or outputs. Restrictions types \(r_5\) and \(r_6\) were used most frequently, reflecting better the marginal rates of substitution (Allen et al., 1997). Even though, the lower bounds (\(k_i, \sigma_r\)) or the upper bounds (\(\rho_i, \tau_r\)) are often dropped. Imposing the ARI, whether generate a complete ranking or no, but it increases the discrimination power of DEA (Charnes et al., 1990; Thompson et al., 1990).

Assurance regions reduce the space of feasibility (the PPS). Accordingly, some efficient units become inefficient (Cooper et al., 2005). Thus, clearly, the possibility of infeasibility is increased. The sensitivity analysis and more details on ARs can be found in (Podinovski and Athanassopoulos, 1998; Podinovski, 1999, 2001, 2004 and 2005). Figure 4.1 shows the decrease in the size of PPS space after adding ARs to the problem, by replacing the efficient frontier. The segment \(E_1E_2\) is moved to \(E'_1E'_2\) (the efficient frontier from \(E_1E_2E_3E_4\) is moved to \(E'_1E'_2E'_3E'_4\)). Therefore, \(DMU_4\) is no more on the efficient frontier (it is changed to an inefficient DMU).

Obviously, the values of bounds in ARI are dependent on the scaling of the inputs and outputs; i.e. these bounds are sensitive to the units of measurement of the related factors (Allen et al., 1997). Concurrently, the ARI bounds value in the multiplier DEA model [refer to LP (1.8)] may need to
be clarified for managers. Thus, its dual form should be resolved to interpret the extra quantity of input and shortage quantity of output (slack variables). In the case of using restrictions on all inputs and outputs ratios (or most of them), the dual form can have heavy structure. Accordingly, giving clear managerial meanings may be difficult. The related discussions can be seen in Allen et al. (1997) and Thanassoulis et al. (2004).

For instance, in Example 4.2, adding constraint $u_1 \geq u_2 \geq u_3 \geq 0$ to the CCR multiplier model shows that the chemistry mark is more important than marks of physics and physics is more important than mathematics. Table 4.5 gives student A inefficient while there is no discrimination between two other DMUs. In this case, clearly, the absolute weight restrictions act better to discriminate between DMUs. Table 4.4 shows the result of absolute weight restrictions.

![Figure 4.1 Reducing the size of PPS after adding the ARs](image)

**Table 4.5 Results of Example 4.2 with ARI**

<table>
<thead>
<tr>
<th>Score</th>
<th>$u_1'$</th>
<th>$u_2'$</th>
<th>$u_3'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student A</td>
<td>0.3619</td>
<td>0.0048</td>
<td>0.0048</td>
</tr>
<tr>
<td>Student B</td>
<td>1</td>
<td>0.0026</td>
<td>0.0058</td>
</tr>
<tr>
<td>Student C</td>
<td>1</td>
<td>0</td>
<td>0.0067</td>
</tr>
</tbody>
</table>

3- **Assurance region type II:**

(IWR and OWR) $\varphi_t \mu_t \leq u_r$ ($r_7$)

Thompson et al. (1990) called relationships between the input and output weights “Assurance Regions type II” (ARII) [bounds set on the ratios of output weights to input weights ($u_r/\mu_t$)]. In ARII the relation between input and output weights (clearly the case of ARII) is required since efficiency measure in DEA is represented by the combination of inputs and outputs factors (Thanassoulis et al., 2004). ARII was introduced to accomplish two purposes: 1) Incorporating a
prior information about the relative importance of an output to an input (Thanassoulis et al., 1995), and 2) Analyzing the profit efficiency of DMUs (Thompson et al., 1996). In this case, the bounds are set according the existing information in the market.

The application of DEA models with ARIIs are not as prevalent in literature as DEA models with absolute weight restrictions and ARI (Thanassoulis et al., 2004). Thanassoulis et al. (1995) assessed the UK’s perinatal care units to recognize environmental impacts on mortality using ARII. As explained earlier in Section 4.2, they make a link among the weight of the number of survivals as output to the number of babies at risk as input. In the process of efficiency measurement, this link prevents from assigning very high or very low weights on survivals or babies at risk, respectively.

ARII may cause infeasible DEA model as well as two other direct weight restrictions (Podinovski and Athanassopoulos, 1998; Podinovski, 1999, 2001, 2004, 2005). Furthermore, it depends on the scaling of the inputs and outputs as it can be seen in ARI (Allen et al., 1997).

4.3.2. Restricting Weight Flexibility by Restricting the Weighted Inputs and Outputs

In this section, two different methods are briefly introduced, which impose constraints on the weighted inputs and outputs. The first approach, “virtual weight restriction”, was proposed by Wong and Beasley (1990) and the second one, “contingent weight restriction”, was developed by Pedraja et al. (1997). These models have not been applied widely in DEA literature (Allen et al., 1997).

1- Restrictions on Relative Importance of Factors (Virtual weight restrictions)

\[(IWR) \ z_i \leq v_i x_{ij} / \sum_{i=1}^{m} v_i x_{ij} \leq c_{i} \ (r8)\]

\[(OWR) \ z_r \leq u_r y_{rj} / \sum_{r=1}^{s} u_r y_{rj} \leq c_{r} \ (r9)\]

First, Wong and Beasley (1990) introduced virtual weight restrictions DEA model. The model imposes restrictions on the importance attached to a certain input(s) and/or output(s) of a DMU. This attached importance is the proportion of the total output/input devoted to that output/input. Thus, the importance attached to input \(i\) by \(DMU_j\) can be given as \(v_i x_{ij} / \sum_{i=1}^{m} v_i x_{ij}\) (similarly, the importance attached to output \(r\) by \(DMU_j\) can be given as \(u_r y_{rj} / \sum_{r=1}^{s} u_r y_{rj}\)). The virtual inputs and outputs are independent on the units of measurement.

The virtual input/output of a DMU as introduced in DEA (Chapter 1: Definition 1.6) shows the relative contribution of each input/output to its efficiency evaluation. Hence, it can be supportive in identifying strong and weak regions of DMU’s performance. The parameters (bounds) in \(r8\) and \(r9\) can be determined based on the expert ideas in the relative importance of the different inputs and outputs. Sarrico and Dyson (2004) transferred the proportional virtual weights restrictions of Wong and Beasley (1990) to non-proportional restrictions.

The virtual weight restrictions did not receive much attention in DEA (Allen et al., 1997), since: 1- The restrictions are implicit (i.e. restrictions are DMU specific), 2- Adding each
restriction in respect of all DMUs cause evaluating the efficiency of each DMU with $2n$ extra constraints ($n$ is the number of DMUs). This is computationally costly. Additionally, more constraints cause increasing the possibility of infeasibility in DEA (Estellitas Lins et al., 2007), and 3- These types of restrictions make DEA sensitive to input and output orientations; i.e. by shifting from an I-O to an O-O might produce different efficiency scores for DEA model (Thanassoulis et al., 2004).

2- Contingent restrictions on weights

\[(\text{OWR})c_{r'r'} \leq u_{ij}y_{ij}/u_{r'j}y_{r'j} \leq d_{r'r'} \ (r_{10})\]

Pedraja et al. (1997) proposed some type of weight restrictions taking into account the level of input and output factors in each DMU. This consideration confirms participation of those fain the DEA analysis, which have significant contribution to the total efficiency of each DMU.

Restriction $r_{10}$ was proposed for an output space, where $r, r' > 1$ and $j = 1, \ldots, n$. The value of $c_{r'r'}$ and $d_{r'r'}$ should be chosen by the expert analyst. Similar constraints can be applied in the input space. These types of restrictions work under condition that the proportion of total benefits attributed to an output (input) cannot exceed the level of that attributed to another output (input) by more than a certain multiplier (Pedraja et al., 1997).

The term of "contingent" was given to these types of weight restrictions, since the approach emphasizes that the pattern of weight determined depends on the levels of inputs and outputs chosen by the DMU. Because of this dependence, the DMU puts more weight on outputs whose levels are high (i.e. ones that it produces efficiently) and less weight on outputs whose levels are low (i.e. ones that increases its inefficiency).

4.3.3. Adjusting the Observed Input-Output Levels to Capture Value Judgements – The Artificial Data Sets Method

In the previous sections, two types of weight restrictions are discussed, which were imposed by adding additional constraints to the original DEA model. In this section, another approach is discussed to simulate weight restrictions that modifies the existing input-output data by multiplying to a vector (Golany, 1988; Charnes et al., 1990).

1- The Cone-Ratio Model

Charnes et al. (1990) introduced the Cone-Ratio model. Figure 4.2 presents geometrically a convex cone. Consider a company with six departments $A, B, \ldots, F$. Let each department employ two inputs to give a single service as output. Different preferences according to management ideas may exist. The management of the company would prefer departments to use more input 1 rather to input 2. Let also assume that management may decide to substitute one input by another input (e.g. if input 1 is machine hour and input 2 is labor hour, management may prefer to substitute the labors by machines). It should be noted that the change in one input affects another input; i.e. increasing input 1 causes a decrease in input 2 and vice versa. All departments produce the same amount of
service with the same quality. Figure 4.2 shows the plot of this example. Each department has its management preferences to represent the proportion in which it uses the two inputs. The dotted lines in this figure shows one possible convex cone in this problem. The convex cones linearly divide the management preferences based on a set of linear constraints; like ranges of substitution ratio between inputs. For example, the dotted line joining the origin and the department B shows all points that use both inputs with the same ratio as B (the same can be said for the line joining C and the origin). Therefore, the departments lying inside the “management preference type 1” cone have a ratio of input 1 to input 2 that lies between the corresponding ratios of departments B and C. It can be seen that however departments A and D are weakly efficient in a DEA model, but they are not any more inside the cone (management preference type 1). Obviously, the cone reduces the size of PPS, consecutively, it causes an increase in the discrimination power of DEA. Depends upon the context of the problem, the number of cones may be changed.

![Figure 4.2 Geometrical representation of a convex Cone](image)

To mathematically represent the cone ratio constraints, suppose that there are two inputs and \( \frac{v_1}{v_2} \) is the ratio of inputs weights. The admissible bounds of this ratio is commonly determined either by using the knowledge of experts on the market prices of inputs or first running the classical CCR model (1.8) and then selecting bounds depends on the set of optimal weights of the most efficient DMUs:

\[
c_1 \leq \frac{v_1}{v_2} \leq c_2
\]

(4.2)

where \( c_2 > c_1 \geq 0 \). Then (4.2) is changed to \( c_2 v_2 - v_1 \geq 0 \) and \( -c_1 v_2 + v_1 \geq 0 \) such that \( C = \begin{pmatrix} c_2 & -1 \\ -c_1 & 1 \end{pmatrix} \) and \( v = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \). The related convex cone for the input weights in \( m \) dimensional space is defined as follows:

\[
V = \{ v \in \mathbb{R}^m | Cv \geq 0 \}
\]

(4.3)
The set $V$ is a half-space form of polyhedral cone. $R^+_m$ is the space of non-negative input weights. Similarly, the convex cone of output weights is defined as $U = \{ueR^+_s | Du \geq 0\}$. Matrix $D$ is the equivalent of matrix $C$ for the output weights. $R^+_s$ is the space of non-negative output weights.

The convex cones of $V$ and $U$ are then be used to construct the following Cone-Ratio model based on CCR classical model (1.8):

\[
\begin{align*}
\text{Max } & u^T y_{ro} \\
\text{s.t. } & v^T x_{io} = 1 \\
& u^T y_{rf} - v^T x_{ij} \leq 0, j = 1, ..., n; \ i = 1, ..., m; r = 1, ..., s \ \\
& \forall eV < R^+_m, \ u \in U < R^+_s
\end{align*}
\] (4.4)

The sets $V$ and $U$ are the defined close convex cones in (4.3). This model produces the same efficiency scores as ARI [model (4.1)] when $r_5$ and $r_6$ are $V = \{v|cv \geq 0\}$ and $U = \{u|Du \geq 0\}$, respectively. Thus, ARI is a specific case of the Cone-Ratio model (Allen et al., 1997). When $V = R^+_m$ and $U = R^+_s$, model (4.4) is transformed to the CCR model (Charnes et al., 1978 and 1990). The way that cones $V$ and $U$ are constructed [as defined above in (4.3)] are called the “intersection form”. They are used as restrictions in a DEA models with ARs. Such cones can be defined with an alternative way for input weights as follows (Charnes et al., 1990):

\[
\begin{align*}
\bar{V} &= \text{cone}(\{a_1, ..., a_p\}) = \\
& \{v|v = \sum_{j=1}^{p} a_j a_j, a_j \in R^+_p, a_j \geq 0, j = 1, ..., p\}
\end{align*}
\] (4.5)

The cone $\bar{V}$ is generated by a finite set of vectors, where $a_1, ..., a_p$ are some vectors in $R^m$, which build matrix $A^T (m \times p)$ and $p$ is a positive integer. The cone $\bar{V}$ is generated by matrix $A^T$. The output cone $\bar{U}$ is made by matrix $B^T (s \times l)$ similar to $\bar{V}$ ($\bar{U} = \text{cone}(\{b_1, ..., b_l\}) = \{u|u = \sum_{j=1}^{l} \beta_j b_j, \beta_j \in R^+_l, \beta_j \geq 0, j = 1, ..., l\}$). This way is called “sum form”. In the small size matrices, the “intersection form” and “sum form” can be simply transformed to each other.

Multiplying the input and output vectors to matrices $A$ and $B$ of “sum form” transforms Cone-Ratio model in (4.4) into a CCR classic form:

\[
\begin{align*}
\text{Max } & \beta^T (BY_o) \\
\text{s.t. } & \alpha^T (AX_o) = 1 \\
& \boldsymbol{\beta}^T (BY) - \boldsymbol{\alpha}^T (AX) \leq 0, j = 1, ..., n; \ i = 1, ..., m; r = 1, ..., s \ \\
& \alpha, \beta \geq 0, \alpha \varepsilon _p^+, \beta \varepsilon _l^+
\end{align*}
\] (4.6)
where the matrices $A$ and $B$ are defined in relation to the matrices $C$ and $D$ above. These matrices are equivalent alternative forms, with $A^T = (C^T C)^{-1} C^T$ and $B^T = (D^T D)^{-1} D^T$ (Charnes et al., 1990; Brockett et al., 1997). Let $BY = Y'$ and $AX = X'$, thus it is clear that model (4.6) is equivalent with CCR model with transformed input vector ($X'$) and output vector ($Y'$). This transformation to CCR model allows the use of DEA software, which does not offer weight restrictions tools, to solve some kind of weight restricted DEA models. However, this advantage can also be seen as a disadvantage, since the transformed data must be transformed back to the original form that allows results to be interpreted. Another advantage of this model is that it allows applying zero or even negative observed data (Charnes et al., 1990; Allen et al., 1997; Thanassoulias et al., 2004).

The presented techniques in the current section, each one proposes a way to construct bounds on the weights of input and output factors. Quantifying these bounds is always a discussable subject for researchers in this domain. Value judgements and preferences of experts, managers, DMs, and available information can be used to quantify these bounds. In the next section, it is briefly presented that how some MCDA methods can be used to translate the value judgments into the bounds values.

4.4. Weight restricted DEA models based on MCDA methods

As explained in previous sections, the difficulty of interpreting weights in DEA, absence of DMs authority on the weights values, discrimination problem of DEA classical models, and some other issues have led researchers to develop some techniques restricting weight values in DEA. In Section 4.3, several techniques to impose weight restrictions into DEA models were introduced. One difficult task in incorporating these restrictions is how to quantify them. The value judgements that can be reflected by priori information of experts and DMs are commonly expressed imprecisely, such as “less interesting” or “more important”. These expressions have to be translated into precise mathematical relationship. MCDA methods, originally based on priori information, preferences of DMs and value judgements, can be used for such a translation and to propose weight restrictions into DEA structure. Nevertheless and as already discussed in Chapters 2 and 3, eliciting weight values is a vital subject for DMs in MCDA field. As well, development of weight restrictions techniques has led to new areas of DEA applications. One practical inferences of these applications is the use of weight restricted DEA models as a decision aid tool to analyze multicriteria decision-making problems, evaluate performance of units and rank them before eliciting weights (Cook et al., 1992; Allen et al., 1997). For this reason, weight restricted DEA models can be an area in which the DEA community learn from MCDA and interactive decision-making principles in MCDA. This collaboration between DEA and MCDA has highlighted problems in interpreting weight restrictions in DEA and eliciting weights in MCDA (Belton and Stewart, 2002).

In this section, some existing applications of MCDA methods in DEA are presented, such as applying MACBETH (Bana e Costa et al., 2005) and AHP (Saaty, 1980) to propose DEA weights
bounds. Further, AHP and PROMETHEE are compared in integrating with DEA models to evaluate their ability in proposing weight restrictions into DEA. Finally, a new model is presented that used PROMETHEE II as a tool, for the first time, to generate weight restrictions in DEA.

4.4.1. DEA-MACBETH integrated approach

In this sub-section, a weight restricted DEA model is concisely presented based on MACBETH methodology.

- **Multicriteria approach to DEA** (Junior, 2008)

Junior (2008) developed a multicriteria approach to DEA. He presented a weight restricted DEA model to evaluate decision alternatives without eliciting weights in a multicriteria problem. For this purpose, Junior (2008) proposed using MACBETH (Bana e Costa et al., 2005) as a multicriteria tool to determine the weights bounds to be applied in the virtual weight restrictions DEA model (Wong and Beasley, 1990).

Junior (2008) used the data set from Hokkanen and Salminen (1997) to evaluate the location of a solid waste management system in Oulu, Finland. He considered the maximizing criteria as outputs and the minimizing criteria as inputs. As explained in Chapter 2: Section 2.3, MACBETH methodology is started by asking the DM to verbally judge the difference of importance between each pair of alternatives on each criterion. These comparisons are made in a 7-point scale and make a judgment matrix that as the result generates the weights found by Hokkanen and Salminen (1997). The software M-MACBETH is employed to simulate the interviewing process and generate the lower and upper bounds of all inputs and outputs. The software outcomes (bounds) were normalized (divided by 100). Finally, these normalized weight restrictions were added to the virtual weight restricted DEA model [constraints $r_8$ and $r_9$ in (4.1)] (Wong and Beasley, 1990).

On the one hand, the DM did not choose weights directly, which can be a difficult task in the procedure of a multicriteria method. On the other hand, the result of proposed approach was improved; i.e. the discrimination power of DEA was increased. However, in some cases, the result of this method can be in contradiction with the original MACBETH ranking. To avoid this weakness, Junior (2008), added some additional constraints to the virtual weight restrictions. Unfortunately, as stated by Estellitas Lins et al. (2007), increasing the number of constraints can increase the likelihood of infeasibility.

PROMETHEE in contrast to MACBETH does not provide a guideline to generate weights of criteria. Nevertheless, PROMETHEE propose a technique to generate weight restrictions called weight stability intervals while MACBETH does not have a solid instructions and mathematical formulations to construct the bounds. This ability of PROMETHEE makes us using it as weight’s bound generator in DEA (Bagherikahvarin and De Smet, 2016a).

4.4.2. DEA-AHP integrated model applications

In this sub-section, four DEA-AHP integrated models briefly introduced to motivate using AHP as a tool to generate weight restrictions for DEA.
A unified framework for the selection of Flexible Manufacturing System (FMS) (Shang and Sueyoshi, 1995)

Shang and Sueyoshi (1995) addressed the problem of selecting the most appropriate Flexible Manufacturing System (FMS) by integrating DEA and AHP methods. A unified system was proposed to facilitate decision making in design and planning by a two-phase algorithm. The model contains the virtual weight restrictions in DEA (Wong and Beasley, 1990) that apply AHP (Saaty, 1980) to generate bounds of weights. In the first phase, AHP is used to make a pairwise comparison matrix between each pair of inputs and outputs. In the second phase, the results derived from the AHP analysis are served as a guideline for setting the upper and lower bounds for restrictions \( r_8 \) and \( r_9 \) in (4.1).

As explained in Chapter 2: Section 2.3, the eigenvector of constructed pairwise comparison matrix between inputs represents the weight vector \( W = (w_1, w_2, ..., w_m) \). This vector shows the importance of each input factor. The judgement matrix \( C \) will then be made based on this weight vector such that each component of it satisfies \( c_{ij} = w_i / w_j \), where \( i, j = 1, 2, ..., m \) (\( m \) is the number of inputs). This matrix has two principles: \( CW = mW \) and \( \sum_{i=1}^{m} w_i = 1 \). Shang and Sueyoshi (1995) substituted \( w_i \) by the ratio of the virtual inputs:

\[
W_i = \frac{v_i x_i}{\sum_{i=1}^{m} v_i x_i}
\]  

(4.7)

Since the normalization equation does not change:

\[
\sum_{i=1}^{m} W_i = \sum_{i=1}^{m} \left( \frac{v_i x_i}{\sum_{i=1}^{m} v_i x_i} \right) = \frac{\sum_{i=1}^{m} v_i x_i}{\sum_{i=1}^{m} v_i x_i} = 1
\]  

(4.8)

Equation (4.7) is also satisfied for \( W_{ij} \). Thus, the components of judgement matrix between inputs are made as follows:

\[
c_{ij} = \frac{w_i}{w_j} \cong \frac{v_i x_i}{\sum_{i=1}^{m} v_i x_i} / \frac{v_j x_j}{\sum_{i=1}^{m} v_j x_j} \equiv \frac{v_i x_i}{v_j x_j}
\]  

(4.9)

The relative weight \( w_i \), computed by AHP, may change its value, in accordance with the subjective judgments of DMs. Shang and Sueyoshi (1995) substituted the lower and the upper bounds of the virtual input \( (v_i x_i / \sum_{i=1}^{m} v_i x_i) \) with the smallest and the largest of such weight values \( (w_i) \). These constraints are then added to the CCR model similar to constraint \( (r_8) \) in model (4.1):

\[
\zeta_i \leq v_i x_{ij} / \sum_{i=1}^{m} v_i x_{ij} \leq \chi_i
\]

(4.8)

Compared with the classical DEA models, which gives zero importance to some inputs and outputs in the FMS problem, this integrated DEA-AHP weight restricted model removes zero weights. Furthermore, it increases the discrimination power of DEA by reducing the number of efficient manufacturing systems. Hence, the weight vectors measured by AHP can serve as a practical tool in generating the weight restrictions for a DEA model. The details of the problem, the related data set and results of this case study can be seen in Shang and Sueyoshi (1995).

Kong and Fu (2012) proposed an integrated DEA-AHP-AR model to construct a student-based performance evaluation model for business schools in Taiwan. In this model, AHP was employed
to generate weight restrictions to use in a CCR model. As explained above, the pairwise comparisons between inputs and outputs were made. The smallest and the largest values of the resulted weight vectors were later used as the ARs in a CCR model. The efficiency scores calculated by proposed model are more discriminant between units than the obtained scores by CCR model; i.e. the number of efficient units are seriously decreased. Further, the calculated weights are more reasonable than the weights calculated using the CCR; i.e. the zero weights are removed by restricting them.

Lee et al. (2012) suggested similar DEA-AHP-AR integrated method to evaluate the Photovoltaics firms in Taiwan.


Zhu (1996) employed a Cone-Ratio DEA model to evaluate the efficiency of the textile factories in the Nanjing, China, during the Chinese economic reforms. He used ARI and ARII models to specify inputs and outputs cones when $r_5$ and $r_6$ in (4.1) are equal to $V = \{v | Cv \geq 0\}$ and $U = \{u | Du \geq 0\}$ in model (4.4), respectively. Then, he applied AHP to determine the lower and upper bounds of ARs as explained above in the work of Shang and Sueyoshi (1995).

Meanwhile, most existing approaches involved ARs used cost-price market information to determine limits of weights, the Zhu’s approach (Zhu, 1996) integrated ARs in a Cone-Ratio model based on expert opinions (AHP) and not just market information. The problem was that Chinese government controlled all market prices and fixed it at certain level for relatively long period. Therefore, the real concept of cost-price could not be used in the Chinese economic reforms planning.

In this regard, Zhu (1996) used AHP to gather and evaluate value judgements between the textile factories. The output of AHP was then applied to set weight bounds in two Cone-Ratio-ARs models. The Cone-Ratio-ARI model was used to reflect the statement of governmental central planning economies, which gave more importance to the net industrial output. The Cone-Ratio-ARII model was also applied to reflect the concept of market economies and put its emphasis on the profits, taxes and revenues, in the process of evaluating Nanjing’s textile factories.

This integrated method effected the evaluation process and helped identifying the best corporations based on their flexibility and adoptability to real economic conditions. It refined DEA efficiency results to analyze the textile industrial behavior during Chinese economic reforms. Furthermore, while CCR and BCC analysis present 13 and 18 units efficient out of 35 corporations, respectively, the integrated model of Zhu (1996) gives just 4 efficient units. Obviously, the new model increased the discrimination power of DEA by taking into account the expert opinions in economic matters of textile industry.
DEA-AHP integrated model to Rank educational units (Premachandra, 2001)

Premachandra (2001) ranked educational units by applying AHP in a DEA classical model. He proposed incorporating some value judgements of the DMs on different factors of educational units, in order to obtain DEA results that are more reasonable. In this regard, he used the virtual inputs and outputs of each DMU to determine additional constraints for the unbounded DEA model.

In the AHP phase, according to Saaty’s 9-point scale, the pairwise comparison matrices are generated between each educational unit according to each input factor. The number of these matrices is equal to the total number of inputs. The priority vector corresponding to each matrix was gained by dividing each element of a column by its column total; then calculating the row averages. At this point, the DM has also to make a pairwise comparison matrix between inputs. The priority vector of this matrix is calculated by the same approach mentioned above. In the next step, the overall priorities for each DMU ($\alpha_k$) should be calculated with multiplying the priority of each input by the priority of each educational unit in each input. The overall priority is equal to the weighted averages of the corresponding elements or virtual inputs $\alpha_k = v_kx_{kj}/\sum_{i=1}^m v_ix_{ij}$, $k = 1, \ldots, m-1, j = 1, \ldots, n$. The value of $\alpha_k$ represents the importance that $DMU_j$ attaches to input $i$ in order to attain its maximum efficiency. Therefore, the following constraint is added to the unbounded DEA model:

$$v_kx_{kj} - \alpha_k \sum_{i=1}^m v_ix_{ij} = 0,$$

$$k = 1, \ldots, m-1, j = 1, \ldots, n$$

(4.10)

The same process is done to calculate the priority vector between outputs. Consecutively, a similar constraint is added to DEA model for output factors.

This new integrated DEA-AHP model involves several pairwise comparisons, which is time consuming; although, injecting preferences of DMs into the DEA model augments the discrimination power of DEA. Premachandra (2001) compared CCR and Cone-Ratio models with the new DEA-AHP model. CCR unbounded model gives 6 out of 7 educational units efficient while Cone-Ratio and the new DEA-AHP model provide 4 and 2 efficient units, respectively. Besides, the model allows DM to consider several conflicting criteria simultaneously. It should be noted that the high number of constraints in this problem increases the possibility of infeasibility (Estellitas Lins et al., 2007).

Interval DEA-AHP to evaluate units (Entani et al., 2004)

Entani et al. (2004) proposed a new method based on DEA and interval AHP. They used AHP to generate a pairwise comparison matrix between decision criteria. The obtained matrix can be generally inconsistent. Thus, it seems more reasonable to give criteria interval importance grades that shows inconsistency contained in the pairwise comparison matrix. The importance grade of each item is computed as following interval:

$$W_i = [L_{w_i}, U_{w_i}] = [w_i^c - d_i, w_i^c + d_i]$$

(4.11)
where $L_{w_i}$ and $U_{w_i}$ are the lower and upper bounds of the interval $W_i$, respectively. The value $w_i^C$ shows the center and $d_i$ is the radius.

The eigenvector of the pairwise comparison matrix gives the priority weight vector $w_i^C$ of each item. The vector of central weights should be normalized $\sum_{i=1}^{n} w_i^C = 1$.

Entani et al. (2004) approximated $a_{ij}$ as an interval ratio to hold relation (4.11) where $W_i$ and $W_j$ are the estimated interval importance grade and their relation ($\frac{W_i}{W_j}$) is defined as the maximum range of possible ratios:

$$ a_{ij} \in \left[ \frac{w_i^C - d_i}{w_j^C + d_j}, \frac{w_i^C + d_i}{w_j^C - d_j} \right], \quad i, j = 1, \ldots, n \quad (4.12) $$

Then interval regression analysis is applied to generate the radius. In this regard, the following problem is solved using $w_i^C$ to minimize the radiuses such that bounds in (4.11) hold for all $a_{ij}$:

$$ \begin{align*}
\min & \quad \lambda \\
\text{s.t.} & \quad \frac{w_i^C - d_i}{w_j^C + d_j} \leq a_{ij} \leq \frac{w_i^C + d_i}{w_j^C - d_j}, \quad i = 1, \ldots, n, \quad j = i + 1, \ldots, n \\
& \quad d_i \leq \lambda, \quad i = 1, \ldots, n 
\end{align*} \quad (4.13) $$

By solving this problem, $d_i$ is obtained to put in (4.11) and create the related intervals of each item. The created intervals were used in a DEA model with normalized data to evaluate units more efficiently according to expert opinions in AHP; consecutively, the discrimination power of DEA is strengthened. Entani et al. (2004) used normalized data to make inputs and outputs weights of DEA in accordance with the inputs and outputs importance grade in interval AHP. The DEA efficiency scores are the same with normalized and non-normalized data.

Pakkar (2014-2016) proposed several approaches based on integration of DEA and AHP for ratio analysis, where AHP is applied to create weight restrictions. The main objective, similar to other works, is increasing the discrimination power of DEA.

As it can be seen, several authors concentrated on the integration of DEA and AHP and applying AHP to quantify the value of constraints on weights. The resulted weight restricted DEA models, as expected, increase the discrimination power of DEA. However, there are some issues on applying AHP in the structure of DEA. One of these issues is the process of pairwise comparisons to generate weight restrictions that is time consuming. As far as it is concerned, PROMETHEE has not been yet applied for such an objective. In the next section, AHP is compared with PROMETHEE in several points. This tries to motivate using PROMETHEE to quantify weight restrictions in DEA.

### 4.4.3. AHP vs PROMETHEE

Several authors have compared AHP with PROMETHEE (Mahmoud and Garcia, 2000; Macharis et al., 2004; Turcksin et al., 2011; Balali et al., 2014; Mursanto and Halim, 2014) in several points.
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such as: the structuring problem, the paradigm, the judgement scale, the weight determination, the
handling of inconsistency, the evaluation elicitation phase, the rank reversal, the sensitivity
analysis, software packages, visualization problem and group decision making. Here, different
points of view of different authors are summarized in comparison between AHP and
PROMETHEE. Furthermore, the possible connections of each of the AHP and PROMETHEE
methods with DEA are reviewed:

1- Problem structuring

The most distinct characteristic of AHP is decomposing a decision problem into its sub-problems,
from general to detailed, to make the problem more understandable. Nevertheless, PROMETHEE
has somehow the ability of making a hierarchy between criteria but decomposing a problem is not
identified as a main character of it. When the number of criteria augments, it may become difficult
for the DM to evaluate the problem (Macharis et al., 2004).

Decomposing a problem is an advantage for AHP but this can make some difficulties in large
size problems. However, this step is important, since decomposing different structures may lead
to different ranking results [this can be a common problem in different MCDA methods with
decomposing structure] (Ishizaka and Nemery, 2013). Furthermore, a problem with so many sub-
criteria tends to receive more weights than when they are less decomposed (Saaty, 1990; Poyhonen
et al., 1997; Ishizaka and Nemery, 2013). In this situation, DM should try to rearrange these sub-
criteria in clusters (Saaty, 1990).

The initial problem structuring and evaluation table in PROMETHEE has similarity with some
kind of optimization models such as DEA. It divides the criteria into two groups (maximized and
minimized criteria) similar to DEA. The maximized and minimized criteria, generally, behave like
outputs and inputs in DEA (a more complete description of alternatives and criteria in DEA and
PROMETHEE is available in Chapter 3: Section 3.2). This similarity may help structuring a
multicriteria problem like a DEA problem to choose the best options when there is not enough
priori information on criteria. Or else, a DEA problem can be seen as a multicriteria decision
making problem. In this situation, the preference information of DMs can be integrated into the
DEA problem as weight restrictions.

2- The paradigm

The AHP method can be identified as a complete aggregation method in additive group (Roy and
Bouyssou, 1993) while PROMETHEE is a partial aggregation method by means of outranking
relations (Roy, 1991). Approximately, compensation, resulted by aggregation, between bad scores
in some criteria and good scores on some other criteria, can occur in both methods. By such
aggregation, AHP loses some detailed information on some criteria and sub-criteria. These trade-
offs are intensified in the presence of a large number of criteria (Saaty, 2003; Macharis et al.,
2004). PROMETHEE I has non-compensatory property that is a result of outranking feature
(Perny, 1998). In PROMETHEE I, the degradation of performance in a given criterion cannot be
compensated by good performance in others. Alternatively, the performance of actions on each criterion, good or bad, is kept. Consequently, this process prevents loss of information (Perny, 1998). It can be said that the possibility of incomparability in PROMETHEE causes less compensation problem rather than AHP. PROMETHEE II may have this problem of losing some detailed information, since PROMETHEE I is forced into a complete ranking of actions in PROMETHEE II (Macharis et al., 2004).

Regard to the compensation feature of PROMETHEE and AHP, one can prefer PROMETHEE to evaluate performance of actions and present a ranking order.

3- The inconsistency

The acceptance of limited inconsistency is an advantage in AHP, since it is manageable (Harker and Vargas, 1987). Measurement of consistency can be used to calculate consistency among DMs as well as consistency at different levels of hierarchy (Balali et al., 2014). However, it should be noted that the consistency ratio can allow contradictory judgements in a pairwise comparison matrix or can reject a reasonable matrix (Bana e Costa and Vansnick, 2008). Furthermore, re-judgement in a non-consistent matrix is a time consuming process. Too much consistency is also undesirable (Bana e Costa and Vansnick, 2008) [please see explanation in Chapter 2: Section 2.3]. However, inconsistency may exist in PROMETHEE pairwise comparison matrix, but the method does not provide a way to measure the value of consistency index.

4- The judgement Scales

The nature of verbal comparisons in AHP is interesting. It is user-friendly and evidently more understandable than numbers for the ones who use this model (as explained in Chapter 2: Section 2.3, the verbal comparisons are transformed to numerical measures by a 9-point scale). However, verbal comparisons may let some vagueness in non-trivial comparisons, which is not desirable (Ishizaka and Labib, 2009). Additionally, using different judgement scales to derive priorities, to convert them to numerical comparisons, may cause different results (Ishizaka and Labib, 2009). In this regard, Barzilai (2005) discussed another problem of AHP. In his opinion, the preferences cannot be denoted by ratio scales, since there is no absolute zero, like in temperature (real sense) or electrical tension. However, Saaty (1994) stated that if one wants to aggregate measurement (e.g. weighted sum), the ratio scales are the only possible measurement. Besides, there is a lack of sensitivity in comparing preferentially close elements. PROMETHEE does not have the possibility of verbal comparisons.

However, the classical DEA models do not consist of the value judgments of DMs, but as discussed earlier in this chapter, the weight restricted DEA models are based on value judgements and preference information of DMs and experts. In Section 4.4.2, different integrated DEA-AHP approaches were presented that used AHP to quantify weight restrictions in DEA models (Shang and Sueyoshi, 1995; Zhu, 1996; Premachandra, 2001; Entani et al., 2004; Kong and Fu, 2012; Pakkar, 2014-2016). In contrary with AHP, PROMETHEE provides just one way to determine these weight restrictions: “weight stability intervals”. It has clear mathematical formulations. As
already mentioned, in this thesis, PROMETHEE is applied to generate bounds of weights in DEA for the first time in the work of Bagherikahvarin and De Smet (2016).

5- The weight determination

In AHP, there are clear guidelines to determine weights, which is considered as an advantage. The relative priorities of weights are determined through a sequence of pairwise comparisons, using different methods like eigenvector, geometric mean, least square and normalized column (Ishizaka and Lusti, 2006). Evidently, on the one hand, the interpretation of obtained weights is not trivial (Belton, 1986). On the other hand, different DMs may ask different questions to create pairwise comparison matrices. Thus different results may occur, which is not desirable. Moreover, inconsistency \((C_R > 0.1)\) or too much consistency \((C_{ik} = C_{ij} \times C_{jk})\) may happen in pairwise comparison matrices that clearly is disagreeable (please see Chapter 2: Section: 2.3).

In comparison with AHP, one disadvantage in PROMETHEE method is the lack of specific guidelines to determine the priorities between weights (Mahmoud and Garcia, 2000; Macharis et al., 2004; Turcksin et al., 2011; Balali et al., 2014; Mursanto and Halim, 2014). DM should decide about priorities of criteria. Besides, determining the generalized criteria may be a difficult task for an inexpert user. On the contrary, PROMETHEE has several predefined tools to manage weights like weight stability intervals and walking weights, which can be more helpful in determining weight restrictions in DEA model rather than AHP (less amount of value judgements, no pairwise comparisons by DM, no inconsistency checking). It should be mentioned that in this regard, AHP has some tools, but the method does not provide a unified mathematical formulations for that.

6- The elicitation phase

In the elicitation phase, AHP obviously needs much more inputs than PROMETHEE. Decomposing problem and consecutively (high) number of pairwise comparisons \([n(n-1)/2]\) make high volume of interactions with DMs and experts. Further, the use of different numerical scale creates some limitations (Belton, 1986; Macharis et al., 2004). For example, in a 9-point scale, if item \(a\) is 3 times more important than item \(b\) and item \(b\) is 5 times more important than item \(c\), the AHP cannot cope this problem which indicates \(a\) is 15 times more important than \(c\). In such cases, Saaty (1988) proposed new clusters, and sub-clusters should be arranged, which allow comparing items in defined numerical scale. PROMETHEE needs much less inputs. It needs just eliciting the preference functions (their thresholds) and the weight evaluation of each criterion \((3q - 1\) or more).

7- The rank reversal problem

Both AHP and PROMETHEE methods suffer from rank reversal. A definition of rank reversal can be seen in Chapter 2: Section 2.3, Definition 2.9, but it does not have a cohesive definition in past literatures. In some cases, rank reversal problem is shifting the ranking of units by adding/removing a new alternative or a copy of existing alternative. In some other cases, rank reversal can happen with adding/removing a criterion. Belton and Gear (1983) and Barzilai et al.
Chapter 4: A ranking method based on DEA and PROMETHEE II

(1987) were among the first researchers considered rank reversal in AHP and De Keyser and Peeters (1996) studied it in PROMETHEE for the first time. Verly and De Smet (2013) focused on this problem in PROMETHEE I and II when a copy of an alternative is added. They proved that eliminating the worst unit does not reverse the ranking, whereas in AHP, this phenomenon may occur, since a new pairwise comparison matrix should be created. Furthermore, Verly and De Smet (2013) showed that rank reversal might only occur when the net flow scores of two units are close. They also proved that the remove of a non-discrimination criterion would never cause a rank reversal while this act causes rank reversal in AHP (Wijnmalen and Wedley, 2009). Pérez et al. (2006) was also pointed out that rank reversal in AHP could also be caused not only by deletion of a non-discrimination criterion but also by addition of such a criterion.

Some DEA models under certain conditions suffer from rank reversal, as well; e.g. Wang and Luo (2009) discussed on rank reversal in cross-efficiency DEA model when a non-discrimination criterion is deleted, which is a common case with AHP (Pérez et al., 2006).

As mentioned above, PROMETHEE does not show the rank reversal behavior in some cases such as eliminating the worst unit and a non-discriminating criterion, while AHP shows it. Accordingly, PROMETHEE may be a more useful tool to evaluate performance of DMUs in a DEA problem. This issue can be a light of future research.

8- Sensitivity analysis

Varying weights is an important issue in the sensitivity analysis of both methods. As explained several times, PROMETHEE method has special tool to vary weights in a secure limit (weight stability intervals) through a unified mathematical calculation. However, AHP has some tools for sensitivity analysis of weights, but the method, according to what we know so far, does not propose a unique way in this regard; further, sensitivity analysis has received less attention in the AHP literature (Ishizaka and Labib, 2009).

One more time, it is referred to this point that the weight stability intervals of PROMETHEE is used in a DEA model to limit the weights freedom in order to increasing the discrimination power of DEA (Bagherikahvarin and De Smet, 2016).

9- Software packages

Both methods have user-friendly software. PROMCALC (Brans and Mareschal, 1994), DECISION LAB 2000 (Mareschal, 2000), Visual PROMETHEE (Mareschal, 2011), D-Sight (Hayez et al., 2012), and Smart Picker (Ishizaka and Nemery, 2013) are PROMETHEE’s packages and Lens, Right Choice DSS, Criterium, EasyMind, WebAHP, and Expert Choice (Ishizaka and Labib, 2009) are AHP’s packages.

PROMETHEE software such as D-sight and Visual PROMETHEE have flexible tools to perform sensitivity analysis and visualization of problem.
In AHP, when new projects, alternatives or criteria are introduced to the problem, all the pairwise comparisons from the lowest level of the hierarchy should be re-entered into software (Expert Choice); whereas, in the PROMETHEE software (such as D-Sight or Decision Lab), this can be done without many changes. Evidently, this ability of PROMETHEE software is lied on the characteristic of the method.

However, DEA has several software such as Lingo, KONSI DEA Analysis, Open source DEA, DEA Frontier software, DEA zone and PIM-DEA, but looking to a DEA structure (evaluation table) as a multicriteria problem can provide the situation of using MCDA software to evaluate the performance of units in a DEA problem.

10- Visualization

To have a clear view of a decision-making problem, visualization is needed. Hence, visualization can be realized as a final step in a multicriteria problem. Both PROMETHEE and AHP methods have visualization tool; but existing the GAIA plane (Mareschal and Brans, 1988) in PROMETHEE makes this tool powerful (Macharis et al., 2004). The GAIA plane interprets the relative positions of alternatives, criteria, their similarities, conflicts to each other, and the position of decision stick. Further, through GAIA brain, where the decision stick belongs, one may determine the good compromise solutions and those that will never be considered as good candidates (the detailed information is available in Chapter 2: Section 2.3.1). However, some corresponding tools exist in the AHP method such as Treemaps (Asahi et al., 1995).

Based on the conception of GAIA brain, Bagherikahvarin and De Smet (2016b) proposed a new integrated DEA-PROMETHEE approach to enrich the understanding of the problem (this contribution is considered in Chapter 6).

11- Group decision making

Both AHP and PROMETHEE include some tools to help group decision making. For this reason, in the AHP, the geometric mean of the individual pairwise comparisons (Zahir, 1999) and in the PROMETHEE, the weighted sum of the individual net flows is calculated. Macharis et al. (1998) proposed a three-stage structure for group decision making in PROMETHEE.

DEA, classically, is an objective method to evaluate efficiency of DMUs. However, weight restricted DEA models are built on subjective preferences of a DM or a group of DMs. In this regard, depends upon the chosen MCDA method, which is integrated with DEA, the related approach of group decision making can be applied. This is not the issue of consideration in this thesis.

The comparison between AHP and PROMETHEE is made based on several points of view such as the problem structuring, the amount of interactions required between the analyst and the DMs, weight determination, sensitivity analysis, visualization and ease of understanding by experts and non-experts. As explained, in some points such as problem structuring, AHP is preferred to PROMETHEE and in some other points PROMETHEE can be the more interesting
method, some of which are sensitivity analysis, visualization and the amount of interactions with DMs. In some viewpoints including weight determination, a negative evaluation was given to both methods. With AHP, weights are obtained through the sequence of pairwise comparisons, but using this method may be troubled by the limit of 9-point scale and the amount of pairwise comparisons. With PROMETHEE, no guidelines are specified for weight determination. Besides, the advantage of AHP in decomposing a decision problem and building hierarchies of criteria may change to a disadvantage when there are many criteria; because it may become difficult for DMs to obtain clear view of the problem. Additionally, with an augmentation in the number of alternatives, the amount of interactions with the DM(s)/user(s) is increased. This may prevent the users from continuing the analysis and leading to some inconsistencies (Mahmoud and García, 2000; Macharis et al., 2004; Turcksin et al., 2011; Balali et al., 2014; Mursanto and Halim, 2014).

As explained above, both methods have some limitations and advantages. Macharis et al. (2004) summarized the mentioned characteristics of AHP and PROMETHEE in a table. Table 4.6 is an extraction of their table.

The signs are representing the following description: (--) the approach that was followed has important disadvantages; (-) the approach that was followed has disadvantages; (+) the approach that was followed has advantages; (+-) the approach may have both negative and positive aspects; (++) the approach that was followed has important advantages.

<table>
<thead>
<tr>
<th>Point of Views</th>
<th>AHP</th>
<th>PROMETHEE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem structuring</td>
<td>++</td>
<td>+</td>
</tr>
<tr>
<td>Paradigm</td>
<td>--</td>
<td>-</td>
</tr>
<tr>
<td>Judgement scale</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Weight determination</td>
<td>+-</td>
<td>-</td>
</tr>
<tr>
<td>Amount of evaluation</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Rank reversal</td>
<td>--</td>
<td>-</td>
</tr>
<tr>
<td>Software</td>
<td>++</td>
<td>++</td>
</tr>
<tr>
<td>Flexibility of Software</td>
<td>--</td>
<td>+</td>
</tr>
<tr>
<td>Visualization</td>
<td>+</td>
<td>++</td>
</tr>
</tbody>
</table>

Table 4.6 Summary of comparison between the AHP and the PROMETHEE, extracted from (Macharis et al., 2004)

Further, in each point, it was tried to consider the possibility of using DEA to solve multicriteria problems or/and in integration with/besides one of the AHP or PROMETHEE method. For example, in problem structuring, DEA is more compatible with evaluation table in PROMETHEE than AHP; since, maximized and minimized criteria in PROMETHEE can be directly seen as outputs and inputs in DEA. In sensitivity analysis, PROMETHEE has powerful tool named as “weight stability intervals”. This unique feature of PROMETHEE leads using it as weight restrictions to enhance the structure of DEA in discriminating among efficient DMUs (Bagherikahvarin and De Smet, 2016). However, AHP has some tools in this regard but as far as
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it is considered, it does not provide a clear unique mathematical formulation to generate weight restrictions. In visualization, Bagherikahvarin and De Smet (2016b) suggested a new approach in accordance with DEA analysis, based on GAIA brain conception, to enrich the understanding of problem and propose the first idea on weight vector. To the best of one’s knowledge, there is not a similar work in integration of AHP and DEA.

4.4.4. DEA-PROMETHEE integrated approach

Bagherikahvarin and De Smet (2016a) proposed a new DEA-PROMETHEE integrated model to increase the discrimination power of DEA.

The work proposes a weight restricted DEA model based on PROMETHEE II. The main objective of this model, as other weight restricted DEA models, is increasing the discrimination power of DEA. Besides, it allows avoiding undesirable results such as allocating very low or high weight values to some inputs or outputs. Nevertheless, the main distinctive characteristic of the new model is integrating the preferences of DMs, not only through weight restrictions, but also via the unicriteria net flow scores matrix as output of DEA model. This matrix already includes preferences of DM(s) and ensures that the compromise solution(s) identified in PROMETHEE II is characterized by an efficiency score equal to 1 in DEA.

Hypothesis

The following hypotheses are imposed on this approach:

1- It is assumed that the inputs and the outputs in a DEA problem are considered as criteria or attributes in multicriteria decision-making problem, with minimization of inputs and/or maximization of outputs as associated objectives (Doyle and Green, 1993). There are some exceptions; i.e. the minimized criteria are considered as outputs and vice versa. In this work, it is avoided to consider such criteria. For more explanations in this regard, one can refer to Chapter 3, Section 3.2.

2- It is assumed that the pre-determined criteria weights are chosen by DM due to a priori information. In the particular case: in the absence of distinct priorities between criteria, all weights are chosen equally.

3- It is assumed that the type of preference function is chosen linear type. This particular function is the most suited one for quantitative criteria (e.g. prices, costs, power, etc.). It involves an area of indifference and strict preference (it establishes a linear growth between the point of indifference, 0, and the point of strict preference, 1). Thus, it includes four other types of preference functions (Brans and Mareschal, 2002).

4- It is assumed that the outputs of PROMETHEE applied in DEA (weight stability intervals and unicriterion net flow scores matrix) transfer the DM idea into DEA.

5- Without loss of generality, it is assumed that the new weight restricted DEA model is feasible.
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**Problem definition**

As stated in this chapter, the main limitations of the classical DEA models can be summarized in three points:

- The weak discrimination power of DEA among efficient units;
- The lack of value judgements and preference information of DMs and managers in the structure of model;
- The undesirable weight values.

In order to address these limitations in this thesis, it is proposed a weight restricted DEA model based on PROMETHEE.

In what follows, the step of the designed algorithm can be studied.

**Algorithm steps**

The algorithm steps with respect to the hypothesis are as follows:

1- The PROMETHEE II is run within the data set in order to generate these results: a) net flow scores, b) unicriterion net flow scores, and c) the weight stability intervals.

These results are later used in the new DEA model.

2- The net flow scores of PROMETHEE II [Equation (2.39): \( \phi(a_j) = \sum_{k=1}^{q} w_k \phi_k(a_j) \)] are maximized as output in a CCR-I-O multiplier DEA model without input.

3- The weight stability intervals of PROMETHEE II are applied as weight restrictions in DEA.

More the level of stability intervals, more the discrimination power of DEA. Additionally, under certain stability level, the weights are less diffused (avoided from very low or very high weight values).

4- The results of the new algorithm are compared with other approaches such as PROMETHEE II and CCR.

The new model is called “PROMETHEE II Weight restricted CCR model” (PIIWCCR):

\[
E_o = \text{Max} \ [\phi(a_o) = \sum_{k=1}^{q} w_k \phi_k(a_o)]
\]

s.t.

\[
\sum_{k=1}^{q} w_k \phi_k(a_j) \leq 1; \ j = 1, 2, ..., n
\]

(4.14)

\[
W^- \leq w_k \leq W^+
\]

\[
w_k \geq 0, \forall k
\]
4.4.4.1. Advantages and limitations

The infeasibility did not occur in several case studies; nevertheless, one should pay attention that applying additional weight restrictions in DEA multiplier models may decrease the size of feasible solution (PS) and cause infeasibility. In this regard, researchers developed tools to resolve this problem (Allen et al., 1997; Podinovski and Athanassopoulos, 1998; Sarrico and Dyson, 2004; Podinovski, 1999-2005; Podinovski and Bouzdine-Chameeva, 2013 and 2015).

As mentioned in Section 4.3.1, imposing absolute restrictions on weights may cause infeasibility (Podinovski and Athanassopoulos, 1998; Podinovski, 1999-2005). Nevertheless, the lower number of constraints in such a model comparing with some models like virtual weight restrictions DEA model, decreases the likelihood of infeasibility (Estellitas Lins et al., 2007).

Moreover, it should be paid attention that transferring between input and output orientations in DEA models with absolute weight restrictions may give different efficiency scores. Therefore, the restrictions need to be set according to the model orientation used (Allen et al., 1997).

Another difficulty that may cause in some cases is the possibility of not having a complete ranking. Thus, the proposed weight restricted DEA model increases the discrimination power of DEA; however, it is probable to have more than one DMU with efficiency score equal to 1. Further, through weight restrictions, model does not provide any relations between specific inputs and outputs.

Besides these limitations, reducing the weight flexibility and using the unicriterion net flow scores of PROMETHEE as output of DEA in the proposed model lead to the following advantages:

1- Increase the discrimination power of DEA by reducing the number of DMUs that are characterized by an efficiency score equal to 1.
2- Integrate partly the preferential information of the DM in a DEA model. The unicator criterion net flow scores and the weigh stability intervals of PROMETHEE II are used as output and weight bounds value in an extended DEA model, respectively.

3- Decrease the role of DM by generating automatically the weight stability intervals, while determining bounds is the main difficulty in weight restricted DEA models (Allen, R. et al., 1997).

4- Have reasonable correlation with the outputs of some other DEA and MCDA methods. This point can be considered in Section 4.4.4.2 in Example 4.5.

5- In the case of necessity to have a complete ranking, the stability level of intervals can be increased. In the higher level of stability intervals, more complete ranking can be seen.

These advantages are studied in some examples in the following section. For detail information, please refer to Bagherikahvarin and De Smet (2016a).

Studying the dual form of a multiplier DEA model can be meaningful for people who needs understanding the DEA results. When the number of DMUs is more than the sum of the inputs and outputs, computationally solving a dual is easier, since it has less constraints \((m + s)\) than primal \((n)\). Further, in dual the solutions are characterized as inputs and outputs that corresponds to the original data while the weights obtained by solving primal represent the evaluations of inputs and outputs (Cooper et al., 2005). Next section contributes to the DEA literature by undertaking a sensitivity analysis in PIWCCR model presented in Section 4.4.4.

### 4.4.4.2. Some properties of PIWCCR model

In this section, a deeper look is taken at the PIWCCR model. The properties of its dual model is analyzed and studied. Some definitions in Chapter 1 (e.g. CCR-efficiency) are reconsidered in this section for the new model. Some new definitions regarding to the dual of PIWCCR are also considered. In this regard, three examples are presented.

**Duality analysis**

Model (4.14), after adding a dummy input to the unicator criterion net flow scores matrix, can be written as follows:

\[
(P)
\]

\[
\theta_o = \text{Max} [\emptyset(a_o) = \sum_{k=1}^{q} w_k \emptyset_k(a_o)]
\]

\[
s.t.
\]

\[
v_i \emptyset_i(a_o) = 1
\]

\[
\sum_{k=1}^{q} w_k \emptyset_k(a_j) - v_i \emptyset_i(a_j) \leq 0; j = 1, 2, ..., n
\]

\[
w_k \leq W^+_k, \forall k
\]

\[
-w_k \leq -W^-_k, \forall k
\]

\[
w_k, v_i \geq 0, \forall k
\]

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The dummy element $\phi_l(a_j) = 1$ considers as the input $l$ for $a_j$ – this is the same for all DMUs.

The dual of this model is as follows:

$$Z_o = \text{Min } (C_o + \sum_{k=1}^{q} \alpha_k W_k^{+} - \sum_{k=1}^{q} \beta_k W_k^{-})$$

s.t.

$$\sum_{j=1}^{n} \lambda_j \phi_l(a_j) \leq C_o \phi_l(a_o)$$

$$\sum_{j=1}^{n} \lambda_j \phi_k(a_j) + \sum_{k=1}^{q} \alpha_k - \sum_{k=1}^{q} \beta_k \geq \phi_k(a_o), \forall k$$

$$\alpha_k, \beta_k, \lambda_j \geq 0 \; \forall k, j$$

The dual problem $(D)$ seeks for an activity in PPS, which gives at least the output level $\phi_k(a_o)$ in $DMU_o$, in its all components, when reducing the input vector $\phi_l(a_l)$ radially to a value as small as possible. Since $\phi_l$ is a vector with all entities equal to 1; thus constraint (4.17) can be written $\lambda \leq C_o$.

**Property 4.1** The activity $(\phi_l\lambda, \phi_k\lambda)$ outperforms $(C_o\phi_l, \phi_k)$ when $Z_o^* < 1$ (Cooper et al., 2005).

The above property shows the presence of slack variables:
\[ s^- = C_o \varnothing_{lo} - \varnothing_{l} \lambda = C_o - \lambda \] (4.19)

\[ s^+ = \varnothing_{k} \lambda - \varnothing_{ko} + A - B \] (4.20)

while \( s^-, s^+ \geq 0 \) for any feasible solution \((Z_o, \lambda)\) in dual problem \((D)\). In Chapter 1, a two-phase LP problem \((1.14)\) was proposed to solve \((D)\) (Cooper et al., 2005).

The following definitions are inspired from (Cooper et al., 2005). The existing definitions are used to define the new model.

**Definition 4.1. Max slack solution** - An optimal solution of phase II of \((D)\) \((\lambda^*, s^+, s^-)\) is called the max-slack solution.

Assuming that both \((P)\) and \((D)\) problems have a finite positive optimal solution. Let an optimal solution of \((D)\) be \((Z_o^*, C_o^*, \lambda^*, s^+, s^-, A^*, B^*)\). Based on this solution PIWWCCR-efficiency is defined as follows:

**Definition 4.2. PIWWCCR-efficient** - The activity \((x_o, y_o) = (\varnothing_{lo}, \varnothing_{ko})\) associated with \(DMU_o\) is PIWWCCR-efficient if and only if an optimal solution satisfies following conditions: \(Z_o^* = 1, s^{++}, s^{--} = 0\), where \(s^{-} = C_o^* \varnothing_{lo} - \varnothing_{l} \lambda^* = C_o^* - \lambda^* \) and \(s^{++} = \varnothing_{k} \lambda^* - \varnothing_{ko} + A^* - B^*\).

**Definition 4.3. Reference set** - The reference set of an inefficient \(DMU_o\), based on the max-slack solution of dual problem \((D)\) is as follow:

\[ E_o = \{j|\lambda^*_j > 0\}, j \in \{1,\ldots, n\} \] (4.21)

where \(j \in E_o\) means the index \(j\) is included in the set \(E_o\).

**Definition 4.4. Improved activity** - The efficiency of an inefficient-PIWWCCR activity \((\varnothing_{lo}, \varnothing_{ko})\) of \(DMU_o\) can be improved to \((\widetilde{\varnothing}_{lo}, \widetilde{\varnothing}_{ko})\) in the reference set \(E_o\), if the input values decreased and similarly the output values increased by their excesses \((s^-)\) and shortfalls \((s^{++})\), respectively, as follows:

\[ \widetilde{\varnothing}_{lo} = C_o^* \varnothing_{lo} - s^{-} (= \varnothing_{l} \lambda^*) = C_o^* - s^{-} \] (4.22)

\[ \widetilde{\varnothing}_{ko} = \varnothing_{ko} + s^{++} - A^* + B^* (= \varnothing_{k} \lambda^*) \] (4.23)

Improved activity \((\widetilde{\varnothing}_{lo}, \widetilde{\varnothing}_{ko})\) projects \((\varnothing_{lo}, \varnothing_{ko})\) into the reference set \(E_o\). Any non-negative combination of activities in \(E_o\) is efficient (Cooper et al., 2005).

Projection \((4.23)\) that is a replace for \((1.23)\) allows for the extra variable vectors \(A^*\) and \(B^*\) of dual problem \((D)\) [envelopment model], as a result of the weight restrictions in \((P)\) [multiplier model].

**Theorem 4.1.** The improved activity \((\widetilde{\varnothing}_{lo}, \widetilde{\varnothing}_{ko})\) is PIWWCCR-efficient (Cooper et al., 2005).
Proof. Since the activity \((\overline{\phi}_{lo}, \overline{\phi}_{ko})\) belongs to PPS, one PIWCCR-efficiency solution can be obtained by solving:

\[
\begin{align*}
(D_{imp}) \\
Z_{oimp} &= \text{Min} \ (C_{oimp} + W^+ A_{imp} - W^- B_{imp}) \\
\text{s.t.} \\
C_{oimp} \overline{\phi}_{lo} - \phi_l \lambda_{imp} - s^-_{imp} &= 0 \quad (4.24) \\
\phi_k \lambda_{imp} + A_{imp} - B_{imp} - s^+_{imp} &= \overline{\phi}_{ko}, \forall k \\
\lambda_{imp} &\geq 0, s^-_{imp}, s^+_{imp}, A_{imp}, B_{imp} \geq 0
\end{align*}
\]

where index "imp" indicates the dependency to the improved activity. The problem \((D_{imp})\) has a feasible solution \((Z_{oimp} = 1, \lambda_{imp} = \lambda^*, s^-_{imp}, s^+_{imp}, A_{imp}, B_{imp} = 0)\). Therefore, \(0 \leq Z_{oimp}^* \leq 1\).

The dual of the \(D_{imp}\) refers to the \((P)\) as:

\[
(P_{imp}) \\
\theta_{oimp} = \text{Max} \ [\overline{\phi}_o = W_{imp} \overline{\phi}_{ko}]
\]

\[
\text{s.t.} \\
v_{imp} \overline{\phi}_{lo} = 1 \\
W_{imp} \overline{\phi}_k - v_{imp} \overline{\phi}_l \leq 0 \\
W_{imp} \leq W^+ \\
-W_{imp} \leq -W^- \\
v_{imp}, W_{imp} \geq 0
\]

where \(W_{imp}\) is the vector of output weights \(w_{kimp}, \forall k\) and \(v_{imp}\) is the vector of input weights.

**Definition 4.5. Complementary condition**- A complementary condition holds between any optimal solutions \((v^*, w^*)\) in \((P)\) and \((\lambda^*, s^{++}, s^{--})\) in \((D)\) as follows:

\[
v^* s^{--} = 0 \quad \text{and} \quad w^* s^{++} = 0 \quad (4.26)
\]

Relation (4.26) is known as the “complementary slackness” condition, which shows that if any element in \(v^*\) and \(u^*\) is positive then the equivalent element in \(s^{++}\) and \(s^{--}\) must be zero and conversely. Both factors can also be zero simultaneously. According to this condition, the following relation can be written:

\[
v^* s^{--} = w^* A^* = w^* s^{++} = w^* B^* = 0 \quad (4.27)
\]

Suppose that in the problem \((P_{imp})\) \(v^*_{imp} = v^*/Z^*_o\) and \(W^*_{imp} = w^*/Z^*_o\) when \((v^*, w^*)\) is an optimal solution of \((P)\) (please see Theorem 1.2, Chapter 1: Section 1.3.1). The complementary slackness conditions (4.27), when replacing \(\overline{\phi}_{lo}\) and \(\overline{\phi}_{ko}\) from (4.22) and (4.23) gives:
\[ v_{imp}^{*} \phi_{lo} = \frac{v^*}{z_o^*} (Z_o^* \phi_{lo} - s^{-}) = v^* \phi_{lo} = 1 \quad (4.28) \]

\[ W_{imp}^{*} \phi_{ko} = \frac{w^* (\phi_{ko} + s^++A^++B)}{z_o^*} = w^* \phi_{ko} = 1 \quad (4.29) \]

In (4.28), it should be paid attention that \( Z_o^* = C_o^* \) when \( \Lambda_{imp} = \Lambda_{imp}^* = 0 \). Therefore, the \( D_{imp} \) and \( P_{imp} \) optimal solutions are \( (Z_{0imp}^* = 1, \Lambda_{imp}^* = \Lambda_{imp}^*, s_{imp}^-, s_{imp}^+, A_{imp}^*, B_{imp}^* = 0) \) and \( (v_{imp}^*, w_{imp}^*) \), respectively.

Following example considers the PIIWCCR model (4.14) and the properties of its dual model (4.16).

**Example 4.3** Table 4.7 presents the data set for evaluating 12 hospitals according to 2 inputs [Number of doctors and nurses] and 2 outputs [Number of outpatients and inpatients: each in units of 100 persons per month] (Cooper et al., 2005).

<table>
<thead>
<tr>
<th>Hospitals</th>
<th>H₁</th>
<th>H₂</th>
<th>H₃</th>
<th>H₄</th>
<th>H₅</th>
<th>H₆</th>
<th>H₇</th>
<th>H₈</th>
<th>H₉</th>
<th>H₁₀</th>
<th>H₁₁</th>
<th>H₁₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>I₁: Doctor</td>
<td>20</td>
<td>19</td>
<td>25</td>
<td>27</td>
<td>22</td>
<td>55</td>
<td>33</td>
<td>31</td>
<td>30</td>
<td>50</td>
<td>53</td>
<td>38</td>
</tr>
<tr>
<td>I₂: Nurse</td>
<td>151</td>
<td>131</td>
<td>160</td>
<td>168</td>
<td>158</td>
<td>255</td>
<td>235</td>
<td>206</td>
<td>244</td>
<td>268</td>
<td>306</td>
<td>284</td>
</tr>
<tr>
<td>O₁: Outpatient</td>
<td>100</td>
<td>150</td>
<td>160</td>
<td>180</td>
<td>94</td>
<td>230</td>
<td>220</td>
<td>152</td>
<td>190</td>
<td>250</td>
<td>260</td>
<td>250</td>
</tr>
<tr>
<td>O₂: Inpatient</td>
<td>90</td>
<td>50</td>
<td>55</td>
<td>72</td>
<td>66</td>
<td>90</td>
<td>88</td>
<td>80</td>
<td>100</td>
<td>100</td>
<td>147</td>
<td>120</td>
</tr>
</tbody>
</table>

Table 4.7 The Hospital case (Cooper et al., 2005)

This data set is applied in the new model. Thus, first, the parameters of PROMETHEE II should be defined. Following table presents the related parameters:

<table>
<thead>
<tr>
<th>Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min/Max</td>
</tr>
<tr>
<td>Type</td>
</tr>
<tr>
<td>Thresholds</td>
</tr>
<tr>
<td>Weights</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Criteria</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min</td>
<td>Min</td>
<td>Min</td>
<td>Max</td>
<td>Max</td>
</tr>
<tr>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
</tr>
<tr>
<td>q=9</td>
<td>q=43.75</td>
<td>q=41.5</td>
<td>q=24.25</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td>0.30</td>
<td>0.2</td>
<td>0.15</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.8 PROMETHEE parameters (The hospital case)

The indifference and preference thresholds are fixed to the first and the third quartile of difference of maximum and minimum data in each criterion, consecutively. The weights are chosen by DM (\( w_1 = 35%, w_2 = 30%, w_3 = 20%, w_4 = 15% \)). As already stressed, the unicriterion net flow scores, \( \phi_k(a_j) \), are used as the output matrix in the DEA analysis. To avoid using negative values of \( \phi_k(a_j) \) in DEA problem, the unicriterion scores are transformed by using a linear transformation: \( [\phi_k(a_j) + 1] / 2 \).

Tables 4.9 and 4.10 represent the weight stability intervals and unicriterion scores, respectively.
Chapter 4: A ranking method based on DEA and PROMETHEE II

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Min weight</th>
<th>Value</th>
<th>Max weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>0.2072</td>
<td>0.2786</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>0.02164</td>
<td>0.0165</td>
<td>1</td>
</tr>
<tr>
<td>$C_3$</td>
<td>0.0547</td>
<td>0.0991</td>
<td>0.2671</td>
</tr>
<tr>
<td>$C_4$</td>
<td>0</td>
<td>0.0485</td>
<td>0.2240</td>
</tr>
</tbody>
</table>

Table 4.9 Weight Stability Intervals of PROMETHEE II in the level 1 (Hospital case)

<table>
<thead>
<tr>
<th>Hospitals</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>$H_5$</th>
<th>$H_6$</th>
<th>$H_7$</th>
<th>$H_8$</th>
<th>$H_9$</th>
<th>$H_{10}$</th>
<th>$H_{11}$</th>
<th>$H_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset(a_j)_{: Doc.}$</td>
<td>0.6767</td>
<td>0.6868</td>
<td>0.6414</td>
<td>0.6288</td>
<td>0.6591</td>
<td>0.1414</td>
<td>0.5530</td>
<td>0.5833</td>
<td>0.5959</td>
<td>0.2071</td>
<td>0.1641</td>
<td>0.4621</td>
</tr>
<tr>
<td>$\emptyset(a_j)_{: Nurs.}$</td>
<td>0.7126</td>
<td>0.7615</td>
<td>0.6854</td>
<td>0.6635</td>
<td>0.6917</td>
<td>0.3513</td>
<td>0.4191</td>
<td>0.5337</td>
<td>0.3883</td>
<td>0.3100</td>
<td>0.2161</td>
<td>0.2666</td>
</tr>
<tr>
<td>$\emptyset(a_j)_{: Outp.}$</td>
<td>0.2070</td>
<td>0.3743</td>
<td>0.4126</td>
<td>0.4885</td>
<td>0.1881</td>
<td>0.6522</td>
<td>0.6287</td>
<td>0.3819</td>
<td>0.5205</td>
<td>0.7062</td>
<td>0.7336</td>
<td>0.7062</td>
</tr>
<tr>
<td>$\emptyset(a_j)_{: Inp.}$</td>
<td>0.4887</td>
<td>0.3156</td>
<td>0.3484</td>
<td>0.4253</td>
<td>0.4084</td>
<td>0.4887</td>
<td>0.4813</td>
<td>0.4512</td>
<td>0.5349</td>
<td>0.5349</td>
<td>0.8611</td>
<td>0.6614</td>
</tr>
</tbody>
</table>

Table 4.10 The unicriterion score matrix $\emptyset_k(H_j)$ (The Hospital case)

Table 4.11 outlines the rank orders of CCR, PROMETHEE II (PII), and PIIWCCR models. The number of efficient DMUs in PIIWCCR is reduced to only one DMU (increasing the discrimination power of DEA). However, $H_2$ is the best hospital in three mentioned models, but the Kendal tau’s correlation between CCR and PIIWCCR ranking order is not adequately strong (0.333); whereas, as expected, the correlation between PII and PIIWCCR is high enough (0.800) [applying the unicriterion score matrix instead of evaluation table in DEA causes more compatibility between the results of PROMETHEE II and PIIWCCR]. It is worth to note that putting weight stability intervals of this data set in a classic weight restricted CCR (WCCR) model makes problem infeasible. The detail model of WCCR can be seen in Bagherikahvarin and De Smet (2016a). Therefore, according to duality theorem, its dual is unbounded.

Table 4.12 presents the results of sensitivity analysis of the problems $(P)$ and $(D)$. It shows the slack variables (all equal to zero) and consecutively the efficiency scores of 12 hospitals. Because of the weight restrictions in PIIWCCR, the optimal solutions (weights) of $(P)$: $(v^*_1, w^*_2, w^*_3, w^*_4)$ do not have very low (zero) or high value. Considering $H_2$ as the best DMU, yields $\theta^* = Z^* = 1$, when $\lambda_2^* = 1$ and $\lambda_j^* = 0 \forall j \neq 2$. As $H_2$ is the only efficient unit, it is the reference set for all other units. The excess and shortfall of inputs and outputs is $s^{*-}, s^{*+} = 0$, respectively. It confirms the full efficiency of this Hospital. The optimal solution of $H_5$ is $(Z^*, C^*, \lambda^*, s^{*-}, s^{*-}, A^*, B^*) = (0.959, 0.959, 0.959, 0, 0, 0), \lambda_j^* = 0 \forall j \neq 2$. From Equation (4.19): $s_j^{*-} = C_5^* \emptyset_{15} - C_5^* \lambda_2^* = C_5^* - \lambda_2^*$. While $s_j^{*-} = 0$, thus $C_5^* = \lambda_2^*$. In this case, the general term is $C_j^* = \lambda_2^*$. Further, the objective function of $(D)$ is $Z_j^* = C_5^* + W^+ A^* - W^- B^* = C_5^* = 0.959$. It also gives $Z_j^* = C_j^*$. Therefore, $Z_j^* = C_j^* = \lambda_2^*$. Relation (4.18) is $\emptyset_k \lambda + A - B = s^* = \emptyset_{ko}$. Clearly, for $DMU_j$ under evaluation $\lambda_2^* = \emptyset_k(H_j)/\emptyset_k(H_2)$. 

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<table>
<thead>
<tr>
<th>Hospitals</th>
<th>CCR rank</th>
<th>Scores</th>
<th>PII rank</th>
<th>PIIWCCR rank</th>
<th>Scores (θ*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>0.985</td>
</tr>
<tr>
<td>H2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>H3</td>
<td>8</td>
<td>0.883</td>
<td>4</td>
<td>4</td>
<td>0.933</td>
</tr>
<tr>
<td>H4</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>0.915</td>
</tr>
<tr>
<td>H5</td>
<td>12</td>
<td>0.763</td>
<td>5</td>
<td>3</td>
<td>0.959</td>
</tr>
<tr>
<td>H6</td>
<td>10</td>
<td>0.835</td>
<td>12</td>
<td>12</td>
<td>0.206</td>
</tr>
<tr>
<td>H7</td>
<td>7</td>
<td>0.902</td>
<td>6</td>
<td>8</td>
<td>0.805</td>
</tr>
<tr>
<td>H8</td>
<td>11</td>
<td>0.796</td>
<td>8</td>
<td>7</td>
<td>0.849</td>
</tr>
<tr>
<td>H9</td>
<td>4</td>
<td>0.960</td>
<td>7</td>
<td>6</td>
<td>0.867</td>
</tr>
<tr>
<td>H10</td>
<td>9</td>
<td>0.871</td>
<td>11</td>
<td>10</td>
<td>0.302</td>
</tr>
<tr>
<td>H11</td>
<td>6</td>
<td>0.955</td>
<td>10</td>
<td>11</td>
<td>0.239</td>
</tr>
<tr>
<td>H12</td>
<td>5</td>
<td>0.958</td>
<td>9</td>
<td>9</td>
<td>0.673</td>
</tr>
</tbody>
</table>

Table 4.11: Ranking order of different methods (Hospital case) [*: PROMETHEE II]

<table>
<thead>
<tr>
<th>Hospitals</th>
<th>( v_j^* )</th>
<th>( w_1^* )</th>
<th>( w_2^* )</th>
<th>( w_3^* )</th>
<th>( \lambda_2^* )</th>
<th>( s^* )</th>
<th>( s^{++} )</th>
<th>A'</th>
<th>B'</th>
<th>Z'</th>
<th>C'</th>
<th>Ref.</th>
<th>Imp. In.</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>1</td>
<td>0.962</td>
<td>0.326</td>
<td>0.058</td>
<td>0.214</td>
<td>0.985</td>
<td>0</td>
<td>0.985</td>
<td>0.985</td>
<td>H2</td>
<td>0.985</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H2</td>
<td>1</td>
<td>0.377</td>
<td>0.932</td>
<td>0.073</td>
<td>0.015</td>
<td>1</td>
<td>0</td>
<td>0.985</td>
<td>0.985</td>
<td>H2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H3</td>
<td>1</td>
<td>0.967</td>
<td>0.216</td>
<td>0.267</td>
<td>0.224</td>
<td>0.933</td>
<td>0</td>
<td>0</td>
<td>0.933</td>
<td>H2</td>
<td>0.933</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H4</td>
<td>1</td>
<td>0.967</td>
<td>0.216</td>
<td>0.267</td>
<td>0.224</td>
<td>0.915</td>
<td>0</td>
<td>0</td>
<td>0.915</td>
<td>H2</td>
<td>0.915</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H5</td>
<td>1</td>
<td>1</td>
<td>0.303</td>
<td>0.055</td>
<td>0.197</td>
<td>0.959</td>
<td>0</td>
<td>0</td>
<td>0.959</td>
<td>H2</td>
<td>0.959</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H6</td>
<td>1</td>
<td>0.207</td>
<td>0.902</td>
<td>0.267</td>
<td>0.224</td>
<td>0.206</td>
<td>0</td>
<td>0</td>
<td>0.206</td>
<td>H2</td>
<td>0.206</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H7</td>
<td>1</td>
<td>0.967</td>
<td>0.216</td>
<td>0.267</td>
<td>0.224</td>
<td>0.805</td>
<td>0</td>
<td>0</td>
<td>0.805</td>
<td>H2</td>
<td>0.805</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H8</td>
<td>1</td>
<td>0.967</td>
<td>0.216</td>
<td>0.267</td>
<td>0.224</td>
<td>0.849</td>
<td>0</td>
<td>0</td>
<td>0.849</td>
<td>H2</td>
<td>0.849</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H9</td>
<td>1</td>
<td>0.967</td>
<td>0.216</td>
<td>0.267</td>
<td>0.224</td>
<td>0.867</td>
<td>0</td>
<td>0</td>
<td>0.867</td>
<td>H2</td>
<td>0.867</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H10</td>
<td>1</td>
<td>0.207</td>
<td>0.902</td>
<td>0.267</td>
<td>0.224</td>
<td>0.302</td>
<td>0</td>
<td>0</td>
<td>0.302</td>
<td>H2</td>
<td>0.302</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H11</td>
<td>1</td>
<td>0.207</td>
<td>0.902</td>
<td>0.267</td>
<td>0.224</td>
<td>0.239</td>
<td>0</td>
<td>0</td>
<td>0.239</td>
<td>H2</td>
<td>0.239</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H12</td>
<td>1</td>
<td>0.967</td>
<td>0.216</td>
<td>0.267</td>
<td>0.224</td>
<td>0.673</td>
<td>0</td>
<td>0</td>
<td>0.673</td>
<td>H2</td>
<td>0.673</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.12: The results of sensitivity analysis (Hospital case)

The condition of “complementary slackness” is also held in this case, since \( v^* s'^{-} = w^* s'^{++} = w^* A^* = w^* B^* = 0 \).

An inefficient DMU can be made more efficient by projection on the efficient frontier. In this I-O model \((D)\), efficiency is increased through proportional decrease of inputs. However, it is necessary to discriminate between an efficient boundary unit and a boundary unit (Cooper et al., 2004). The last column of Table 4.12 shows the input improvement for putting each unit on the frontier. From equation (4.22), the improved input is \( \tilde{\Theta}_{lo} = C_0^* \Phi_{lo} - s'^{-}(= \Theta_1 \Lambda^*) = C_0^* - s'^{-} \). While \( s'^{-} = 0 \), \( \tilde{\Theta}_{lo} = C_0^* \). Finally, the following relation can be held between parameters of this example: \( Z_j^* = C_j^* = \tilde{\Theta}_{lj} = \lambda_2^* \).

The topic of sensitivity (stability or robustness) analysis has taken variety of forms in DEA analysis (Cooper et al., 2004, 2005, 2007 and 2011). One part of this literature is the variations in the data set. The next part presents an approach to check the stability of PIIWCCR efficiency.
evaluation (ranking) to variations in the data; i.e. how much the input and output vectors of DMUs are sensible to a fix rate of change. The focus is on the stability of classification of DMUs (it checks whether an efficient DMU remains efficient under a certain rate of data variation).

**Multiplier model approaches** (Thompson et al., 1996; Seiford and Zhu, 1998; Neralic, 2004)

Thompson et al. (1996) analyzed the multiplier DEA model. They showed that an efficient DMU under evaluation would remain efficient and its optimal values would remain valid for some range of variations in the evaluation table.

Actually, the sensitivity analysis in multiplier models studies the effect of percentage data changes on efficiency scores, which can simultaneously degrade efficient DMUs and upgrade inefficient DMUs.

The multiplier model approach can be exploited in problem (4) by defining a new vector $\mathbf{x} = (\mathbf{v}, \mathbf{w})$ and a function $h_j(M)$ as follows:

$$h_j(M) = \frac{f_j(M)}{y_j(M)} = \frac{\sum_{k=1}^{q} w_k \phi_k(a_j)}{v_l \phi_l(a_j)} \quad \forall j$$

(4.30)

Let $h_o(M)$ for $DMU_o$ under evaluation be better than the other DMUs:

$$h_o(M) = \text{Max } h_j(M) \geq h_j(M) \forall j \neq o$$

(4.31)

In the extreme efficient points, where $Z^* = 1$ in the (D) problem, for multiplier $M^* = (v^*, w^*)$:

$$h_o(M^*) > h_j(M^*) \forall j \neq o$$

(4.32)

which can be written:

$$h_o(M^*) = \frac{\sum_{k=1}^{q} w_k^* \phi_k(a_o)}{v_l^* \phi_l(a_o)} > 0$$

$$h_j(M^*) = \frac{\sum_{k=1}^{q} w_k \phi_k(a_j)}{v_l \phi_l(a_j)} \quad \forall j \neq o$$

(4.33)

This strict inequality will generally remain valid until some range of variations in evaluation table.

The vector $\phi_l(a_j), j = 1, ..., n$ is the dummy input equal to vector 1: $e = (1, ..., 1)$. Further, from $v_l \phi_l(a_o) = 1$ in (P), $v_l^* = 1$ is obtained. Therefore, the strict inequality (4.33) transformed to following form:

$$h_o(M^*) = \sum_{k=1}^{q} w_k^* \phi_k(a_o) > 0$$

$$\sum_{k=1}^{q} w_k \phi_k(a_j) = h_j(M^*) \forall j \neq o$$

(4.34)

Thompson et al. (1996) described a ranking principal as: “If $DMU_o$ is more efficient than any other $DMU_j$ relative to the vector $M^*$, then $DMU_o$ is said to be top ranked. Holding $M^*$ fixed, the data are then varied and $DMU_o$ is said to be top ranked if (4.34) continues to be held.”
For an extreme efficient $DMU_o$ (and all other efficient DMUs), the outputs are all decreased and the inputs are all increased by a same factor. In contrary, for inefficient DMUs, the outputs are all increased and the inputs are all decreased. Continuing in this manner, the value of $h_o(M^*)$ in the relation (4.34) is decreased and $h_j(M^*) \forall j \neq o$ is increased, over where a reversal occurs in this relation. In this point, the $DMU_o$ will be no more top ranked.

In the Example 4.3, $H_2$ is the best hospital. Applying the set of its optimized weights from Table 4.12 ($w_k^*, \forall k = 1, ..., 4$) in the formulation (4.34) gives $h_j(M^*) \forall j = 1, ..., 12$ as follows:

<table>
<thead>
<tr>
<th>DMUs</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>$H_5$</th>
<th>$H_6$</th>
<th>$H_7$</th>
<th>$H_8$</th>
<th>$H_9$</th>
<th>$H_{10}$</th>
<th>$H_{11}$</th>
<th>$H_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_j(M^*)$</td>
<td>0.941</td>
<td>1</td>
<td>0.915</td>
<td>0.897</td>
<td>0.912</td>
<td>0.435</td>
<td>0.652</td>
<td>0.751</td>
<td>0.632</td>
<td>0.426</td>
<td>0.329</td>
<td>0.484</td>
</tr>
</tbody>
</table>

Table 4.13 Efficiency variation in the hospital case

As it can be seen, $H_2$ still remains the top ranked unit [$h_2(M^*) = 1$]. The gaps between the top rank units in the Table 4.13 shows that some range of data variation may result again to the same top ranked unit. However, changing the top rank unit is also possible. In this regard, the sensitivity analysis process proceeds in step-wise of 5% increment in the input and 5% decrement in the outputs of $H_2$. This value touches inputs and outputs of other inefficient DMUs by decreasing 5% their inputs and increasing 5% their outputs.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
<th>$H_5$</th>
<th>$H_6$</th>
<th>$H_7$</th>
<th>$H_8$</th>
<th>$H_9$</th>
<th>$H_{10}$</th>
<th>$H_{11}$</th>
<th>$H_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_j(M^*)$</td>
<td>0.983</td>
<td>1</td>
<td>0.987</td>
<td>0.970</td>
<td>0.538</td>
<td>0.734</td>
<td>0.816</td>
<td>0.695</td>
<td>0.526</td>
<td>0.420</td>
<td>0.561</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.14 Efficiency variation after 5% data changes (the hospital case)

This data variation shows that $H_2$ is not any more top ranked hospital. Hospitals $H_1$ and $H_3$, which in the first round (Table 4.13), were the 2nd and the 3rd ranked, are now top ranked. In the next step, the optimized weights of $H_1$ and $H_3$ are applied to generate $h_j$. In each increment-decrement of data, some DMUs are replaced by the top ranked units. It should be noticed that the values of $h_j(M^*)$ for the efficient DMUs are decreased in each variation; reversely, this value are increased for inefficient DMUs.

The next part considers the variation of weights restrictions and consequently the effects of this variation in the final ranking of WCCR and PIIWCCR methods.

**Weight matrices:**

Imposing restrictions to DEA weights increases the discrimination power of DEA model; however, sometimes it does not decrease the number of zero weights significantly. It means depends to the factors, which have weight restrictions (or does not have) or depends to the bounds of restrictions, there is still possibility of existent zero weights.

Weight stability intervals of PROMETHEE II have the possibility of removing the zero weights in higher level of stability.
The PIIWCCR model (4.14) and a classic weight restricted DEA model (WCCR) are run, when the stability level is fixed in $r = 1$. The WCCR is constructed by adding weight stability intervals of PROMETHEE II to the CCR model (Bagherikahvarin and De Smet, 2016a). The generated weight matrices of each model are checked. If there are still some undesirable weight values, the length of weight restrictions is proposed to be reduced. This process can be done in two ways:

1) Increasing the level of stability ($r > 1$);

2) Adding and subtracting $\alpha\%$ to lower bounds and from upper bounds of weight stability intervals, respectively ($W_k^- + \alpha\%, W_k^+ - \alpha\%$).

The process can be improved by finding a turning point. In this regard, the level of stability/\(\alpha\) can be increased over where: 1) the infeasibility occurs, 2) the full ranking occurs, and 3) the weights become non-zero. Indeed, this turning point gives the maximum possible value of $r$ or $\alpha$.

**Example 4.4** Bagherikahvarin and De Smet (2016a) considered the data set of (Hokkanen and Salminen, 1997) to localize a waste management system in Oulu, Finland. They studied the condition of 22 locals according to 8 criteria and compared the results of several methods such as CCR, WCCR and PIIWCCR.

Table 4.15 presents the weight stability intervals in the first level resulted by the PROMETHEE analysis. Further, Table 4.16 compares the efficiency scores of CCR, WCCR and PIIWCCR. This table is extracted from Bagherikahvarin and De Smet (2016a). The bold numbers in Table 4.16 show the efficient units. In the classic CCR model the number of equal efficient DMUs (Efficiency=1) is 16 out of 22. This shows that CCR alone is not a good discriminator among DMUs. As expected, the number of equal efficient units in WCCR and PIIWCCR are reduced to 9 and 2, respectively. Though, incorporating weight stability intervals into the DEA classic model improves the DEA discrimination power by decreasing the number of efficient units. To have a complete ranking in the model PIIWCCR (4.14), the level of stability is augmented to $r = 2$. In this level, the number of efficient units is lessened to just 1 unit, number 10. This number in WCCR is reduced from 9 to only 5 units.

The weight matrices resulted by PIIWCCR when $r = 1, 2$ are available in Appendix 1: Tables A2 and A3. Comparison of data in these tables shows us, however the number of zero weights is reduced significantly in level 2 of stability intervals, but there are still some undesirable weight values. As explained above, to have a weight matrix with less dispersed and undesirable value, a value of $\alpha\%$ can be added and subtracted to lower bounds and from upper bounds of weight stability intervals in level 1, respectively. In this regard, variation of $\alpha$ provides us $\alpha = 5.6\%$ as a turning point ($\alpha > 5.6\%$ makes problem infeasible). By adding and subtracting this value to lower and from upper bounds in model (4.14), consecutively: ($W_k^- + 5.6\%, W_k^+ - 5.6\%$), the zero values were removed from weight matrix (Table A4 in Appendix 1 reveals this fact). The number of efficient units in WCCR is reduced to 3, after 5.6\% variation in the weight intervals. It is noticeable to mention that the problem remains feasible in the reduced weight restrictions.
Chapter 4: A ranking method based on DEA and PROMETHEE II

More detailed explanation from this example is available in Bagherikahvarin and De Smet (2016a).

Table 4.15 Weight Stability Intervals of PROMETHEE II in level 1 [localization of the waste management system] (Bagherikahvarin and De Smet, 2016a)

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Min weight</th>
<th>Value</th>
<th>Max weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>0</td>
<td>0.2786</td>
<td>1</td>
</tr>
<tr>
<td>C₂</td>
<td>0</td>
<td>0.0165</td>
<td>0.1840</td>
</tr>
<tr>
<td>C₃</td>
<td>0</td>
<td>0.0991</td>
<td>0.3947</td>
</tr>
<tr>
<td>C₄</td>
<td>0</td>
<td>0.0485</td>
<td>0.9800</td>
</tr>
<tr>
<td>C₅</td>
<td>0</td>
<td>0.0929</td>
<td>0.4336</td>
</tr>
<tr>
<td>C₆</td>
<td>0.0472</td>
<td>0.2683</td>
<td>1</td>
</tr>
<tr>
<td>C₇</td>
<td>0</td>
<td>0.0516</td>
<td>0.2000</td>
</tr>
<tr>
<td>C₈</td>
<td>0</td>
<td>0.1445</td>
<td>0.2877</td>
</tr>
</tbody>
</table>

Table 4.16 The efficiency scores of DEA models (localization of the solid waste management system in Oulu, Finland) extracted from Bagherikahvarin and De Smet (2016a)

<table>
<thead>
<tr>
<th>DMUs/Alts</th>
<th>CCR</th>
<th>WCCR</th>
<th>PIWCCR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.837</td>
<td>0.6833</td>
<td>0.558</td>
</tr>
<tr>
<td>2</td>
<td>0.871</td>
<td>0.691</td>
<td>0.358</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.904</td>
<td>0.283</td>
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<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0.983</td>
</tr>
<tr>
<td>5</td>
<td>0.991</td>
<td>0.985</td>
<td>0.727</td>
</tr>
<tr>
<td>6</td>
<td>0.985</td>
<td>0.986</td>
<td>0.384</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0.995</td>
<td>0.821</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
<td>0.462</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>11</td>
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<td>12</td>
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<td>0.425</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>1</td>
<td>0.993</td>
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<tr>
<td>14</td>
<td>1</td>
<td>0.967</td>
<td>0.552</td>
</tr>
<tr>
<td>15</td>
<td>0.977</td>
<td>0.9777</td>
<td>0.266</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>1</td>
<td>0.976</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>0.967</td>
<td>0.468</td>
</tr>
<tr>
<td>18</td>
<td>0.977</td>
<td>0.977</td>
<td>0.266</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>0.999</td>
<td>0.991</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>0.891</td>
<td>0.534</td>
</tr>
<tr>
<td>21</td>
<td>1</td>
<td>1</td>
<td>0.281</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
<td>0.999</td>
<td>0.281</td>
</tr>
</tbody>
</table>
Chapter 4: A ranking method based on DEA and PROMETHEE II

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Min weight</th>
<th>Value</th>
<th>Max weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>0.079</td>
<td>0.2786</td>
<td>0.9750</td>
</tr>
<tr>
<td>C₂</td>
<td>0</td>
<td>0.0165</td>
<td>0.1745</td>
</tr>
<tr>
<td>C₃</td>
<td>0.02</td>
<td>0.0991</td>
<td>0.3035</td>
</tr>
<tr>
<td>C₄</td>
<td>0.015</td>
<td>0.0485</td>
<td>0.9793</td>
</tr>
<tr>
<td>C₅</td>
<td>0</td>
<td>0.0929</td>
<td>0.4336</td>
</tr>
<tr>
<td>C₆</td>
<td>0.0877</td>
<td>0.2683</td>
<td>1</td>
</tr>
<tr>
<td>C₇</td>
<td>0</td>
<td>0.0516</td>
<td>0.2000</td>
</tr>
<tr>
<td>C₈</td>
<td>0.012</td>
<td>0.1445</td>
<td>0.2877</td>
</tr>
</tbody>
</table>

Table 4.17 Weight Stability Intervals of PROMETHEE II in level 2 (localization of the waste management system)

Next part explains briefly the coefficient of variation in efficiency scores with a numerical example from Bagherikahvarin and De Smet (2016).

### Ranking

The Coefficient of Variation (CV) is a measure of spread that describes the amount of variability relative to the mean. Because CV is unit-less, it can be used instead of the standard deviation to compare the spread of data sets that have different units or different means. It can be computed by dividing standard deviation to mean of data sets. More CV shows better performance of model in ranking result (Kong and Fu, 2012).

Table 4.18 gives the CV of efficiency scores in each ranking result (CCR, WCCR and PIIWCCR) in the problem of localization of the waste management system in Oulu (Bagherikahvarin and De Smet, 2016).

<table>
<thead>
<tr>
<th></th>
<th>WCCR</th>
<th>PIIWCCR</th>
<th>WCCR</th>
<th>PIIWCCR</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean score</td>
<td>0.9837</td>
<td>0.9556</td>
<td>0.6122</td>
<td>0.9331</td>
</tr>
<tr>
<td>Std. Score</td>
<td>0.0428</td>
<td>0.0919</td>
<td>0.2887</td>
<td>0.1086</td>
</tr>
<tr>
<td>C.V.</td>
<td>0.0435</td>
<td>0.0962</td>
<td>0.4716</td>
<td>0.1164</td>
</tr>
</tbody>
</table>

Table 4.18 The CV of efficiency scores in different rankings in the problem of localization of the waste management system

When \( r = 1 \) and after weight stability intervals variation, \( CV(CCR) < CV(WCCR) < CV(PIIWCCR) \); thus clearly not just for the more discriminatory power of PIIWCCR this ranking has better performance, but also for more CV.

### Efficiency Scores

By assigning constraints on weights, the region of search for those weights is reduced (Figure 4.1). So a DMU’s efficiency assessment cannot be increased, and it may possibly be reduced. Further, according to the dual problem \((\tilde{D})\), CCR efficiency is generally reduced in value, as expected from problems with extra variables. Thus, some efficient units become inefficient (Cooper et al., 2005).
Figure 4.4 presents the efficiency scores of the CCR, WCCR and PIIWCCR models (the related scores were shown in Table 4.16) in the problem of localization of the waste management system in Oulu (Bagherikahvarin and De Smet, 2016a). It is obvious that most of the efficiency scores, after adding constraints, are reduced in the value rather than CCR results. Obviously, the efficiency scores in PIIWCCR are much more reduced than WCCR model. For instance, some efficient DMUs in CCR and WCCR like DMUs 9, 12, 21, and 22 have just small efficiency values in PIIWCCR.

Degrees of Freedom

Another reason to add weight restrictions into the DEA models is the problem involving degrees of freedom, since the DEA has the orientation to relative efficiency. There is rough rule of thumb which can be a guidance in DEA models: \( n \geq \max\{m \times s, 3 \times (m + s)\} \), where \( n \), \( m \), and \( s \) are the number of DMUs, inputs, and outputs, respectively. In the lack of respect to the relation between the number of DMUs and input/output factors, the number of efficient DMUs is increased. In this regard, weight restrictions are used in DEA to reduce the number of efficient units.

In Example 4.4, the number of DMUs, inputs and outputs is \( n = 22 \), \( m = 5 \), and \( s = 3 \), respectively; so: \( 22 < \max\{15, 24\} = 24 \). The relation is not hold; thus, several DMUs have efficiency scores equal to 1 (16 out of 22 units become efficient in CCR model). By using weight stability intervals in PIIWCCR model, the number of efficient DMUs is reduced to only 2 units. Obviously, the discrimination power of DEA is increased.

Eliminating/adding a DMU

In Example 4.4, adding or eliminating the best and the worst DMUs, does not change the ranking results and efficiency scores of units in CCR, WCCR and PIIWCCR models.

Next part considers the effect of varying thresholds of PROMETHEE II in the final ranking result of Example 4.4.
**Variation of PROMETHEE thresholds (q, p)**

First, the PROMETHEE II rankings are compared in different q and p \((q, p + \alpha\%\)). In each iteration, the thresholds are varied 10\% (augmented 10\% of their original values). The related PROMETHEE table and values of thresholds are available in Bagherikahvarin and De Smet (2016a). Table 4.19 shows the results of Kendall’s Tau correlation between the ranking results of PROMETHEE II for different values of q and p in the problem of localization of a waste management system in Oulu (Hokkanen and Salminen, 1997).

<table>
<thead>
<tr>
<th>p, q + \alpha%</th>
<th>0%</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>35%</th>
<th>40%</th>
<th>45%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>1</td>
<td>0.982</td>
<td>0.974</td>
<td>0.956</td>
<td>0.974</td>
<td>0.974</td>
<td>0.965</td>
<td>0.957</td>
</tr>
<tr>
<td>10%</td>
<td>1</td>
<td>0.991</td>
<td>0.957</td>
<td>0.991</td>
<td>0.974</td>
<td>0.965</td>
<td>0.957</td>
<td>0.957</td>
</tr>
<tr>
<td>20%</td>
<td>1</td>
<td>0.950</td>
<td>0.983</td>
<td>0.965</td>
<td>0.957</td>
<td>0.948</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>1</td>
<td>0.950</td>
<td>0.965</td>
<td>0.974</td>
<td>0.983</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35%</td>
<td>1</td>
<td>0.983</td>
<td>0.974</td>
<td>0.965</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40%</td>
<td>1</td>
<td>0.991</td>
<td>0.983</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45%</td>
<td>1</td>
<td>0.991</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.19 Kendall’s Tau rank correlation coefficient of the ranking results of PROMETHEE II for different q and p

As the result indicates, the correlations between rankings in different values of q and p are significantly high. However, after each \(\alpha\%\) variation of q and p, the weight stability intervals are slightly changed, but the new ranking results of PIIWCCR are not expressively changed. In each variation, the new PIIWCCR, like the main ranking of PIIWCCR, gives DMUs 10 and 7 as the best units. This example shows that the PROMETHEE method can be resistant against the variation of its parameters (thresholds) in Example 4.4. Nevertheless, it does not a proof of PROMETHEE’s resistance against the variation of its thresholds.

**Example 4.5** This example is the case study of ranking medium-sized companies in Brussels that already studied in Chapter 2 and 3. Example 2.4 (Chapter 2) compared the ranking results of Gazelles (www.trends.be, 2014) with PROMETHEE II in the stability levels 1 and 3. Example 3.1 (Chapter 3) compared the ranking results of CCR and BCC models in DEA with the results of Example 2.4. This example compares previous results with the result of PIIWCCR (4.14) when \(r = 1\) and \(r = 3\).

Table 4.20 shows the ranking results of all mentioned models. The red numbers in this table shows the efficient units in DEA models (CCR, BCC and PIIWCCR). The number of efficient units from 7 units in CCR and BCC decreased to just 3 units in PIIWCCR. The results present DMUs 1, 14, and 37 as the best efficient units in all mentioned DEA models.

Table 4.21 gives the correlation between different ranking results. As discussed in Example 2.4, there is a good correlation value between PROMETHEE II and Gazelles. This confirms the fact that
the chosen parameters for PROMETHEE II leads to generate approximately compatible ranking with the initial ranking (Gazelles). The other correlations are significant, as well.

<table>
<thead>
<tr>
<th>PIIWCCR</th>
<th>PR.II</th>
<th>CCR</th>
<th>BCC</th>
<th>Gazelles</th>
</tr>
</thead>
<tbody>
<tr>
<td>PIIWCCR</td>
<td>1</td>
<td>0.576*</td>
<td>0.346</td>
<td>0.347</td>
</tr>
<tr>
<td>PR.II</td>
<td>1</td>
<td>0.459</td>
<td>0.460</td>
<td>0.896</td>
</tr>
<tr>
<td>CCR</td>
<td>1</td>
<td>0.995</td>
<td>0.463</td>
<td></td>
</tr>
<tr>
<td>BCC</td>
<td>1</td>
<td>0.464</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gazelles</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.21 Kendall’s Tau rank correlation values when $\tau = 1$ (Gazelles)

<table>
<thead>
<tr>
<th>Rank</th>
<th>GAZ.</th>
<th>PR.II</th>
<th>CCR</th>
<th>BCC</th>
<th>PIIW CCR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
<td>3</td>
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<td>14</td>
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<td>39</td>
<td>60</td>
<td>60</td>
<td>36</td>
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</table>

Table 4.20 Ranking results of medium-sized companies in Brussels (Gazelles)
At the level of stability equal to 1, generated intervals in Table 2.4 are not tight enough to have a clear effect in the new method. Thus, however the correlations between PIIWCCR and other models are significant but these values are not high enough. This leads us to another level of stability in PROMETHEE II to generate more restricted intervals. According to the stability intervals when $r = 3$ (Table 2.5), Table 4.22 presents the correlation of PIIWCCR ranking in this new stability level of intervals with ranking results of other models.

<table>
<thead>
<tr>
<th></th>
<th>PIIWCCR</th>
<th>PR.II</th>
<th>CCR</th>
<th>BCC</th>
<th>Gazelles</th>
</tr>
</thead>
<tbody>
<tr>
<td>PIIWCCR</td>
<td>1</td>
<td>0.750</td>
<td>0.435</td>
<td>0.436</td>
<td>0.744</td>
</tr>
<tr>
<td>PR.II</td>
<td>1</td>
<td>0.459</td>
<td>0.460</td>
<td>0.896</td>
<td></td>
</tr>
<tr>
<td>CCR</td>
<td>1</td>
<td>0.995</td>
<td>0.463</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BCC</td>
<td>1</td>
<td>0.464</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gazelles</td>
<td></td>
<td></td>
<td></td>
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<td>1</td>
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</tbody>
</table>

Table 4.22 Kendall’s Tau rank correlation values when $r = 3$ (Gazelles)

According to data in Table 4.22, the correlation of PROMETHEE II and PIIWCCR in $r = 3$ is increased significantly in comparison with the similar result in Table 4.21 (*).  

4.5. Conclusion

In this chapter, the introduction and literature review of the first contribution of this thesis is presented. The motivations for using weight restrictions in DEA models are discussed. Besides, some important weight restricted DEA models are summarized. Furthermore, AHP and PROMETHEE are compared in giving inspiration to integrate PROMETHEE with DEA.

The main objective of this contribution is increasing the discrimination power of DEA by proposing a new weight restricted DEA model based on PROMETHEE II. The new model, on the one hand, reduces the number of efficient DMUs and thus increases the discrimination power of DEA. On the other hand, it makes possible to partly integrate the preferential information of the DM in an extended DEA model. The weight stability intervals of PROMETHEE II play the role of weight restrictions in DEA. These intervals are generated automatically (depending on the acceptance of the PROMETHEE ranking and given a certain stability level). Further, the unicriterion net flow scores of PROMETHEE replace the output of DEA.

Moreover, the dual behaviour of the proposed model is studied. The properties and some definitions of the 1st Chapter are also reconsidered for the new model.

The new proposed model is called PIIWCCR. As far as it is considered, this is a first work in which one investigates the potential synergies between PROMETHEE and DEA.

In the next chapter, the second contribution of this thesis is presented, which considers a new ranking method in DEA. Through this contribution, a two-phase algorithm is suggested to fully rank units in DEA based on PROMETHEE (Bagherikahvarin, 2016).
Chapter Five

A DEA-PROMETHEE approach for complete ranking of units


Abstract

In this chapter, a new integrated approach based on DEA and PROMETHEE II is presented for complete ranking of units. The hypothesis, limitations and advantages of the model are also considered. In this order, some important techniques for complete ranking in DEA are briefly discussed: Cross efficiency, Super efficiency, Benchmarking, Multivariate statistics and DEA-MCDA integrated techniques.

5.1. Introduction

In the basic DEA models, the so-called efficient units have equal scores (equal to 1). Thus, the best-performing DMUs can be identified, but further distinction within classical DEA models is not possible. In order to overcome this limit and improve the discrimination power of DEA, researchers have developed methods to present a full ranking of DMUs. Adler and his colleagues (2002) and Hosseinzadeh Lotfi et al. (2013), classified the DEA ranking methods in six general groups: 1- Cross efficiency (Sexton et al., 1986), 2- Super efficiency (Andersen and Petersen, 1993), 3- Benchmarking (Torgersen et al., 1996), 4- Multivariate statistical techniques (Friedman and Sinuany-Stern, 1997 and 1998), 5- Slack-adjusted DEA models (Bardhan et al., 1996), and 6) DEA-MCDA integrated methods. The last group needs additional prior information from DMs (Thompson et al., 1986; Golany, 1988; Cook et al., 1990, 1992 and 1993; Shang et Sueyoshi, 1995; Sinuany-Stern et al., 2000; Ho and Oh, 2010; Jie, et al., 2010; Fulop and Markovits, 2012). However, some DEA-MCDA integrated methods exist that do not require any priori information (Adler et al., 2002); e.g. multi-objective model of (Li and Reeves, 1999). It should be mentioned that several researchers have proposed approaches in the literature, which cannot be easily categorized, into one or other groups. For instance, the work of Alirezaee and Afsharian (2007) is one of these approaches. They presented a Balance Index to completely rank DMUs. In modifying this model to give a more stable index, Guo and Wu (2013) proposed the maximal balance index. It can determine a unique ranking, using restrictions in DEA models. Wen
and Li (2009) introduced the application of Fuzzy Logic, which is one of the interesting paths for complete ranking of units. Alem et al. (2013) constructed a new fuzzy-DEA-AHP model to rank DMUs completely. In this chapter, five techniques of complete ranking in DEA are reviewed (Adler et al., 2002).

Some researchers have studied the connection between DEA and MCDA (Cook and Kress, 1990; Belton, 1992; Cook et al., 1992; Doyle and Green, 1993; Belton et Vickers, 1993; Stewart, 1994 and 1996). The general similarities, differences and integration between DEA and MCDA are considered in Chapter 3. Ranking is a common issue in both methods; therefore, several researchers paid attention to it. Shang et Sueyoshi (1995), Sinuany-Stern et al. (2000), Li and Ma (2008), Ho and Oh (2010), Jablonsky (2007 and 2012), and Fulop and Markovits (2012) proposed an DEA-AHP methodology for ranking DMUs. Shang et Sueyoshi (1995) ranked and selected Flexible Manufacturing Systems (FMS) by using the subjective AHP results in DEA (this model is presented in Chapter 4: Section 4.4.2). Sinuany-Stern et al. (2000) presented a two-stage DEA-AHP ranking model to remove the pitfalls of Shang and Sueyoshi (1995). In the first stage, DEA is used for each pair of units separately in order to generate a pairwise comparison matrix. This matrix is used to rank the units based on a single level AHP (this model is presented in this chapter: Section 5.2.5). Li and Ma (2008) developed an iterative method by integrating DEA, AHP and Gower plot techniques. Ho and Oh (2010) constructed a stock selection framework by integrating DEA and AHP. Fulop and Markovits (2012) used a variant of the CCR model to build a non-reciprocal pairwise comparison matrix and then use AHP to obtain ranking values of DMUs. Further, Rocio Guede and his colleagues (2012) attempted to analyze the innovation efficiency in Spain. They used ELECTRE Tri (Roy, 1991) to give a robustness analysis of the efficient activity branches to improve the DEA discrimination ability.

In the present contribution, PROMETHEE (Brans, 1982) is used to rank units. This method is based on pairwise comparisons of the actions. The new integrated DEA-PROMETHEE approach works in two phases. In the first phase, DEA is run between each pair of DMUs. In the second phase, the PROMETHEE II method is used to aggregate scores from the generated pairwise comparison matrix. In PROMETHEE, the DM subjectively chooses the preference functions, related thresholds and weights to generate the pairwise comparison matrix whilst the new integrated method relies on DEA to perform this task. The pros and cons of this approach are discussed in Section 5.3.1. Finally, while through the use of a classical DEA model, the efficient DMUs cannot be discriminated, the new model avoids this problem.

This chapter is organized as follows: In Section 5.2, five complete ranking techniques, including two DEA-AHP ranking techniques, are reviewed. This section is closed by an example that compares some of the introduced methods. In Section 5.3, the DEA-PROMETHEE integrated approach is explained. This algorithm is illustrated on a particular example. Furthermore, the possibility of compatibility between DEA and the new model is discussed as well as the monotonicity of the new model in the case of 1 input and 1 output.
Chapter 5: A DEA-PROMETHEE approach for complete ranking of units

5.2. Different techniques to complete ranking of units in DEA

As mentioned in the introduction, one of the main features of DEA is evaluating DMUs in two groups of efficient and inefficient units. In decision-making problems, which need distinguishing among efficient units and fully rank them to choose the best solution, it is needed to apply some techniques to do so. This section briefly introduces some important techniques used to fully rank units in DEA: Cross efficiency, Super efficiency, Benchmarking, Multivariate statistics in DEA and DEA-MCDA integrated techniques.

5.2.1. Cross efficiency ranking technique

The cross efficiency technique was first developed by Sexton et al. (1986), initiating the issue of complete ranking in DEA. While one of the critical points in DEA is the nature of self-evaluation, the cross efficiency was developed as a DEA extension with the idea being to use DEA to do peer-evaluation (Cook and Zhu, 2007). In the first step, this technique computes the efficiency score of each DMU through the CCR model (1.8) by separate LPs (n is the number of DMUs) to generate optimal weights of inputs and outputs, \( v^*_k \) and \( u^*_k \), respectively (self-evaluation). In the second step, it applies these weights to all peer DMUs to construct a cross efficiency matrix (peer-evaluation) with elements as follows:

\[
E_{kj} = \frac{\sum_{r=1}^{s} u^*_r y^*_r}{\sum_{l=1}^{m} v^*_l x^*_l} \tag{5.1}
\]

Element \( E_{kj} \) represents the related score of \( DMU_j \), using weights of \( DMU_k \) (when DEA runs for unit \( k \), it gives its weights). This is the reason why cross efficiency is regarded as a peer-evaluation while DEA is a self-evaluation. The elements of matrix \( E_{kj} \) are comprised between zero and one. The elements in the diagonal of matrix \( E_{kj} \), \( E_{kk} \), represent the standard efficiency scores of DMUs; thus, \( E_{kk} = 1 \) for efficient units and \( E_{kk} < 1 \) for inefficient ones [since model (1.8) is a CCR-I-O model, efficiency scores are not greater than 1].

The cross efficiency technique applies the result of matrix \( E_{kj} \) in order to rank DMUs. In this regard, the average of cross efficiency scores are used:

\[
\overline{E}_k = \frac{\sum_{j=1}^{n} E_{kj}}{n} \tag{5.2}
\]

The average \( \overline{E}_k \) is the score of \( DMU_k \). It is more representative than \( E_{kk} \) to present scores, since to calculate it, all elements of matrix \( E_{kj} \) are used, including the diagonal’s elements. It can be said that \( \overline{E}_k \) measures the total ratios over all the runs of all the DMUs (Adler, 2002). The maximum value of \( \overline{E}_k \) is 1 (i.e. all the units evaluate unit \( k \) as efficient, thus it is efficient).

Doyle and Green (1994) suggested some alternative ways instead of average scores such as computing the median, minimum, or variances of scores. Furthermore, they presented the “maverick index” in the cross efficiency technique, which is defined as follows:

\[
M_k = \frac{E_{kk} - e_k}{e_k}, \text{ where } e_k = \frac{1}{n-1} \sum_{j \neq k} E_{kj} \tag{5.3}
\]
Index $M_k$ is used to measure the deviation between diagonal’s elements, $E_{kk}$ (the self-evaluated scores) and the units peer scores. The higher value of $M_k$ gives a DMU a more maverick nature; i.e. the DMUs, which are evaluated as efficient under self-evaluation cannot be a member of reference set for inefficient DMUs (non peer-efficient). On the other hand, the low value of $M_k$ indicates DMUs that are often both self and peer efficient (Doyle and Green, 1994). Liang et al. (2008a,b) developed this method by working on the definition of non-uniqueness optimized weights in DEA. Further, Wang and Luo (2009) considered rank reversal problem in the process of cross efficiency evaluation when a DMU is added or removed. Lim and Zhu (2015) evaluates units by a DEA cross efficiency model under Variable Returns to Scale (VRS).

Some advantages and limitations of this technique are covered in Section 5.3.1.

5.2.2. Super efficiency ranking technique

Andersen and Petersen (1993) first proposed the super efficiency technique. This method was developed with the aim of removing some (all) ties in the process of computing efficiency scores in DEA. It ranks the efficient DMUs which have efficiency scores equal to 1 and presents a more complete final ranking to increase the discrimination power of DEA. It can be formulated as follows:

$$\text{Max } \theta = \sum_{r=1}^{s} u_r y_{ro}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, \ldots, n, j \neq o$$

$$u_r, v_i \geq 0, r = 1, \ldots, s, i = 1, \ldots, m$$

Model (5.4) is a “CCR-super-efficient” model. The model removes the $o^{th}$ constraint from the formulation to enable an extreme efficient unit $o$. The result could be an efficiency score greater than 1. It computes the distance between the evaluated unit $o$ and the “efficient frontier” that is evaluated without unit $o$ for $j = 1, \ldots, n, j \neq o$ (Seiford, 1996; Thrall, 1996; Seiford and Zhu, 1999b; Chen, 2005).

Figure 5.1 illustrates model (5.4), where there are 5 DMUs with 2 inputs and 1 output equal to 1 (dummy output). The line segments connecting DMUs $A$, $B$, $C$, and $D$ constitutes the efficient frontier; thus their efficiency scores are equal to 1. $DMU_E$ is inefficient. By using super efficiency model on $DMU_B$, it is excluded from the set of DMUs; therefore the new frontier is composed of DMUs $A$, $C$ and $D$. $B'$ is the projection of $B$ on the new frontier. The super efficiency score of $DMU_B$ under (5.4) is equal to $OB'/OB$, obviously, greater than 1. This implies that $DMU_B$ could increase both inputs and remain efficient. The inefficiency score of $DMU_E$ does not change by excluding from the reference set.
Chapter 5: A DEA-PROMETHEE approach for complete ranking of units

Figure 5.1 A CCR-I-O Super-efficient illustration (Chen and Du, 2015)

Model (5.5) is the dual model of (5.4). It measures the distance among Pareto frontier without unit \( o \) (DMU under evaluation) and the unit \( o \) itself. In Figure 5.1, \( BB' \) shows this distance.

\[
\begin{align*}
\text{Min} & \quad Z \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} - Z x_{io} \leq 0, i = 1, \ldots, m \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} - y_{ro} \geq 0, r = 1, \ldots, s \\
& \quad \lambda_j \geq 0, j = 1, \ldots, n
\end{align*}
\]  

(5.5)

Hadi-Vencheh and Esmaeilzadeh (2013) considered a new super-efficient model in the presence of negative data. Amirteimoori et al. (2014) initiated a super-efficient model with a common set of weights.

However, the super efficiency technique is easy to employ and understand, but there are some limitations as well. Some DMUs in the super-efficient model can attain an unusual excessive high score. Sueyoshi (1999) proposed adding bounds on the weights of model (5.4) to avoid this problem. Moreover, he computed an index to limit the super-efficient scores to a maximum value of 2.

Another limitation concerns the problem of infeasibility, which leads to a non-complete ranking. Thrall (1996) showed that the super-efficient CCR and BCC models may be infeasible; however, the super-efficient BCC model encounter this issue more than CCR. Seiford and Zhu
(1999b) discussed the infeasibility conditions of DEA super-efficient models. Mehrabian et al. (1999) proposed a modification in model (5.5) to avoid infeasibility.

5.2.3. Benchmarking ranking technique

Torgersen et al. (1996) suggested a two-phase complete ranking model by evaluating the importance of efficient DMUs as a benchmark for inefficient DMUs. In the first phase, they computed slack variables through the Additive model (1.20) [Chapter 1: Section 1.3.3]. In this phase, the set of efficient units $\mathcal{E}$ is identified as DMUs with slack values equal to zero. In the second phase, the following model is run for all DMUs:

$$
\frac{1}{\theta_o} = \text{Min } f_o \\
\text{s.t. } \\
\sum_{j \in E} \lambda_j x_{ij} + s^-_i = x_{io}, i = 1, \ldots, m \\
\sum_{j \in E} \lambda_j y_{rj} - s^+_r = f_o y_{ro}, r = 1, \ldots, s \\
\sum_{j \in E} \lambda_j = 1 \\
\lambda_j, s^-_i, s^+_r \geq 0, r = 1, \ldots, s, i = 1, \ldots, m
$$

(5.6)

Then, model (5.6) is used to calculate the individual reference weights as follows:

$$
\rho_o^r \equiv \frac{\sum_{j=1}^s \lambda_j (y_{rj}^o - y_{rj})}{y_r^o - y_r} \quad \forall o = 1, \ldots, E, r = 1, \ldots, s \\
\text{s.t. } \\
y_{rj}^o = \frac{y_{rj}}{\theta_j} + s^+_r
$$

(5.7)

The value of $\rho_o^r$ is the benchmark score of efficient $DMU_o$. It computes the ratio of the total aggregated possible increases in output $r$ over where $DMU_o$ acts as a member of the reference set. Then, an average value of $\rho_o$ can be calculated to rank all efficient units in set $E$. Furthermore, these individual reference weights consider DMUs, which have distinct importance to the industry (Torgersen et al., 1996). Torgersen et al. (1996) used their technique to rank unemployment offices in Norway. They compared their results with the output of super-efficient model (5.4) and showed different rankings, because of the outlier problem in super efficiency technique.

In a benchmarking technique, an efficient DMU is highly ranked based on the number of observations in the reference set. Thus, this technique is not robust enough in small size samples, since the number of observations is limited (Shuttleworth, 2005).

5.2.4. **Multivariate statistics in the DEA context**

An integration of DEA and statistical methods gives another technique to completely rank the units. The aim of this type of technique is closing the gap between DEA and classical statistical methods. As explained in Chapter 2, DEA is a technique that tends to frontier in comparison with regression analysis techniques, which have central tendency. Indeed, DEA designs the efficient frontier, which is based on the best observations. DEA is an optimization technique that considers each unit separately. However, regression is a parametric technique that sets a single function to the collected data. DEA assigns different weights to factors of different units. As discussed, this weight flexibility is one of the main features of the DEA. Although this flexibility can cause some difficulties in DEA (the discussion over DEA weight flexibility can be seen in Chapter 4).

Canonical correlation analysis (Sinuany-Stern and Friedman, 1998) is an extension of the regression analysis technique (Adler et al., 2002), which can be used to fully rank units. The objective is to find a unique common set of weights for the inputs and outputs of all units. In this regard, two input and output composites are constructed as linear combination of \( m \) inputs \( (Z_j) \) and \( s \) outputs \( (W_j) \):

\[
Z_j = V_1 x_{1j} + V_2 x_{2j} + \cdots + V_m x_{mj} \\
W_j = U_1 y_{1j} + U_2 y_{2j} + \cdots + U_r y_{rj}
\]  \( (5.8) \)

where canonical correlation analysis is used to determine input and output weight vectors, \( V' = (V_1, \ldots, V_m) \) and \( U' = (U_1, \ldots, U_r) \), correspondingly. Tatsuoka and Lohnes (1988) maximized the coefficient correlation between composite input and output \( (r_{ZW}) \) as follows:

\[
Max \ r_{ZW} = \frac{V'S_{xy}U}{\sqrt{(V'S_{xx}V)(U'S_{yy}U)}}
\]

\( S.t. \)

\[
V'S_{xy}V = 1 \\
U'S_{yy}U = 1
\]

where \( S_{xx}, S_{yy} \) and \( S_{xy} \) are the matrices of the sums of squares and products of the input and output variables, respectively. Further, in the context of DEA, \( U' \) and \( V' \) are non-negative. Friedman and Sinuany-Stern (1997) defined the ratio of output to input composites as scores of DMUs:

\[
T_j = \frac{w_j}{z_j}
\]  \( (5.10) \)

It should be noted that, unlike with DEA analysis that limits scores to a maximum of 1, the integrated method of canonical correlation analysis and DEA gives scores that are unbounded. Thus, here, it is the ranking of units that is important rather than scores values.
DEA and canonical correlation analysis are independent; nevertheless, the sum of weighted inputs and outputs in (5.8) are similar in both methods. Additionally, (5.10) is a similar ratio to compute scores in both techniques.

There are some other statistical methods that are integrated with DEA to rank units, such as linear discriminant analysis (Sinuany-Stern et al., 1994) and discriminant analysis of ratios (Sinuany-Stern and Friedman, 1998).

5.2.5. DEA-MCDA integrated techniques

Generally, DEA and MCDA were compared in Chapter 3. It was explained that in spite of differences between these methodologies, several researchers used them simultaneously to help DMs building a ranking algorithm. Two integrated MCDEA models were introduced as well (Nakayama et al., 2002 and Kao, 2010).

In this section, three MCDEA integrated techniques are reviewed to rank units completely. Li and Reeves (1999), Sinuany-Stern et al. (2000), and Jablonsky (2012) proposed such techniques.

- Multi-objective Linear Programming model (Li and Reeves, 1999)

Li and Reeves (1999) initiated a tri-objective LP model to limit the freedom of weights and consecutively, to increase the discrimination power of DEA. The constraints imposed in this model are the same as for the CCR model (1.8):

\[
\begin{align*}
\text{Min } s_o \\
\text{Min } \sum_{j=1}^{n} s_j \\
\text{Min } (\text{Max } s_j) \\
s.t. \\
\sum_{t=1}^{m} v_t x_{t0} = 1 \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{t=1}^{m} v_t x_{ij} + s_j = 0, j = 1, ..., n \\
u_r, v_i, s_j \geq 0, \forall i, r, j
\end{align*}
\]

where \( s_j \) represents the slack variables. When \( s_j \) is equal to zero, \( DMU_j \) is efficient and its deviation from the efficient frontier is null. The objective functions are represented as follows: the first one is similar to the classical CCR model; the second one is the minimum of the sum of deviations (slack variables) that optimizes the overall evaluation of all DMUs, from the view point of the DMU under evaluation, and the third objective is the minimax of deviations that is a measure of equity.

The method does not necessarily provide the complete ranking. Although, similarly to ARs and Cone-Ratio models, it bounds weights freedom, but it does not require additional information of DMs as other methods do.

Zhao et al. (2006) applied this tri-objective model to select dam location in China. Namorado et al. (2010) used it to evaluate the Brazilian privatized highways.
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- An DEA-AHP methodology for ranking DMUs (Sinuany-Stern et al., 2000)

Sinuany-Stern et al. (2000) introduced a ranking method in AHP (Saaty, 1980) context based on DEA. They suggested a two-step process. In the first step, a cross efficiency CCR model is used to compare each pair of DMUs. The results of these pairwise comparisons are held in a matrix. In the second step, the created matrix of the first step is applied by a single level AHP to provide ranking of units. If in pairwise comparisons of DMUs \( j \) and \( k \), \( E_{jk} \) shows the optimal efficiency score of unit \( j \) in comparison with unit \( k \) (\( E_{kj} \) is defined similarly to \( E_{jk} \)) and \( E_{jj} \) and \( E_{kk} \) are the diagonal elements of cross-evaluation matrix (this technique was explained in the present chapter: Section 5.2.1), each member of the pairwise comparison matrix is:

\[
A_{jk} = \frac{E_{jj} + E_{jk}}{E_{kk} + E_{kj}}, \quad a_{jj} = 1, j, k = 1, ..., n
\]  

where \( n \) is the number of DMUs, \( a_{jk} \) shows the evaluation of unit \( j \) over unit \( k \). \( a_{jj} = 1 \) reflects the diagonal elements of matrix. Obviously, similar to AHP, \( a_{kj} = 1/a_{jk} \).

In the original AHP method, the pairwise comparison matrix is created depending upon the subjective preference information of the DMs. However, the integrated DEA-AHP method of Sinuany-Stern et al. (2000) builds this matrix objectively, which is more favorable in the view point of DMs (it should be noted that there is always a degree of subjectivity in choosing inputs and outputs). This matrix covers one of the main drawbacks of AHP, when in evaluating units, the AHP method requires several pairwise comparisons. In this method, the multiple criteria are taken into consideration by DEA while the ranking is obtained by AHP.

An important limitation of this approach is the possibility of having many elements equal to 1 in the pairwise comparison matrix resulting from (5.12). This causes the pairs of alternatives to be evaluated equally. Consecutively, the final ranking is not complete that is not desirable (Sinuany-Stern et al., 2000). This problem leads us to implement another MCDA method (PROMETHEE) instead of AHP to generate a complete ranking (Bagherikahvarin, 2016). The next example shows this deficiency in the presented DEA-AHP ranking technique.

**Example 5.1 (Sinuany-Stern et al., 2000)** - Table 5.1 represents the data used by Sinuany-Stern and her colleagues (2000). The problem is characterized by 4 DMUs according to 2 inputs and 2 outputs.

<table>
<thead>
<tr>
<th>Input1</th>
<th>Input2</th>
<th>Output1</th>
<th>Output2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>50</td>
<td>55</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>130</td>
<td>60</td>
<td>12</td>
</tr>
<tr>
<td>C</td>
<td>68</td>
<td>96</td>
<td>45</td>
</tr>
<tr>
<td>D</td>
<td>45</td>
<td>30</td>
<td>35</td>
</tr>
</tbody>
</table>

Table 5.1. Database of Sinuany-Stern and her colleagues (2000)
Table 5.2 presents the pairwise comparison matrix in DEA-AHP method through formulation (5.12).

<table>
<thead>
<tr>
<th></th>
<th>a1</th>
<th>a2</th>
<th>a3</th>
<th>a4</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>a3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.8508</td>
</tr>
<tr>
<td>a4</td>
<td>1</td>
<td>1</td>
<td>1.1754</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.2 Generated pairwise comparison matrix by DEA to use in AHP

As it can be seen above, Table 5.2 shows a matrix with many elements equal to 1. Such a matrix is not appropriate to clearly discriminate alternatives in AHP, since 1 elements in the pairwise comparison matrix means that two DMUs are assumed to be similar. Consequently, a number of DMUs are likely to be considered having the same rank, meanwhile derived ranking weights from this matrix may be similar or have very close values.

Table 5.3 represents ranking results of CCR and DEA-AHP models.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>CCR Scores</th>
<th>DEA-AHP</th>
<th>DMUs rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0.4994</td>
<td>2-3</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0.4994</td>
<td>2-3</td>
</tr>
<tr>
<td>C</td>
<td>0.851</td>
<td>0.4800</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0.5204</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.3 Ranking results of DEA and DEA-AHP

The DMUs do not have a complete rank in DEA and DEA-AHP methods. In CCR, 3 units out of 4 have equal efficiency scores and in DEA-AHP, units A and B have the same scores. $DMU_A$ has the 2nd or 3rd rank in DEA-AHP, in common with $DMU_B$. In DEA, it is an efficient unit and has the same rank as B and $DMU_C$. In both rankings, is in the last place. In Section 5.3, a new MCDEA ranking algorithm, based on PROMETHEE II, is proposed, which gives a complete ranking under any condition.

- **An AHP approach for ranking efficient units in DEA models** (Jablonsky, 2012)

Jablonsky (2012) used AHP to discriminate among efficient DMUs. The structure of his AHP model has three hierarchy levels. The first level is the main objective (evaluation of efficient units), the second level includes inputs and outputs (criteria) of the problem and the last level encloses the DMUs identified as efficient through a DEA model. In the second level, Jablonsky (2012) used all possible ratios of outputs to inputs instead of using single input and output as criteria in AHP. The ratios are representations of the efficiency characteristic and are simply comparable. Figure 5.2 illustrates such a tri-level hierarchy.
As it can be seen in Figure 5.2, the second level contains criteria of the evaluation (i.e. specific efficiency measures). The weights of the criteria are shown by \( v_q \), where \( q = 1,2, \ldots, mr \) (\( m \) and \( r \) are the number of inputs and outputs, respectively). These weights can be obtained by pairwise comparisons of criteria in AHP, but Jablonsky (2012) proposed using geometric mean of appropriate input and output weights. On the last level, the preference indices of efficient DMUs are derived by pairwise comparisons of these DMUs with respect to the ratios of outputs to inputs in the second levels: \( w_{ij}, i = 1, \ldots, n, j = 1, \ldots, mr \). Finally, the sum of preference indices gives the global preference measures of all DMUs:

\[
 u(DMU_i) = \sum_{j=1}^{mr} w_{ij}, \; i = 1, \ldots, n
\]  

The obtained global preference measures give the ranking of efficient DMUs.

The augmented number of criteria in this technique, on the one hand, causes many comparisons. On the other hand, it may result in an inconsistent pairwise comparison matrix.

Jablonsky (2007) also evaluated the efficiency of pension funds in the Czech Republic, using the AHP with interval pairwise comparisons for evaluation and classification of efficient units.

In what follows, an example is considered, which compares some of the classical DEA models with the complete ranking techniques reviewed in this section. The data set is drawn from (Sexton et al., 1986).

**Example 5.2 (Sexton et al., 1986)** - The data set contains 6 nursing home (DMUs/alternatives) over 4 criteria (2 inputs and 2 outputs): staff hours per day (SHD), supplies per day (SPD), total Medicare and Medicaid reimbursed patient days (MRP), and total private patient days (TPP). Table 5.4 presents the data set. Table 5.5 shows the results of running models CCR [model (1.8)], BCC [model (1.16)], and Additive (Add) [model (1.20)]: the efficiency scores in CCR and BCC, and the slack variables in Additive models. Table 5.6 presents the results of cross efficiency (CE) [equations (5.1 and 5.2)], super efficiency (SE) [model (5.4)], Benchmarking (BM) [model (5.7)],
canonical correlation analysis (CCA) [model (5.10)], and tri-objective DEA-MCDA (3-O) [model (5.11)] models.

<table>
<thead>
<tr>
<th></th>
<th>SHD</th>
<th>SPD</th>
<th>MRP</th>
<th>TPP</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>150</td>
<td>0.2</td>
<td>14000</td>
<td>3500</td>
</tr>
<tr>
<td>B</td>
<td>400</td>
<td>0.7</td>
<td>14000</td>
<td>21000</td>
</tr>
<tr>
<td>C</td>
<td>320</td>
<td>1.2</td>
<td>42000</td>
<td>10500</td>
</tr>
<tr>
<td>D</td>
<td>520</td>
<td>2.0</td>
<td>28000</td>
<td>42000</td>
</tr>
<tr>
<td>E</td>
<td>350</td>
<td>1.2</td>
<td>19000</td>
<td>25000</td>
</tr>
<tr>
<td>F</td>
<td>320</td>
<td>0.7</td>
<td>14000</td>
<td>15000</td>
</tr>
</tbody>
</table>

Table 5.4 Database of Sexton et al. (1986)

<table>
<thead>
<tr>
<th>DMUs</th>
<th>CCR scores</th>
<th>BCC scores</th>
<th>Add (slacks sum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0.971</td>
<td>1</td>
<td>19.721</td>
</tr>
<tr>
<td>F</td>
<td>0.870</td>
<td>0.900</td>
<td>85.698</td>
</tr>
</tbody>
</table>

Table 5.5 Results of running classical DEA models

As expected from classical DEA models, Table 5.5 reveals the non-discriminatory character of DEA among efficient units. CCR and Additive models identifies 4 units out of 6 as efficient, while this number in BCC is 5. All three models introduce unit F as the worst unit (the last in the ranking). The next table compares the results of some complete ranking techniques in DEA.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>CE scores</th>
<th>SE scores</th>
<th>BM ranking</th>
<th>CCA ranking</th>
<th>3-O scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2.000</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0.955</td>
<td>1.412</td>
<td>2</td>
<td>2</td>
<td>0.953</td>
</tr>
<tr>
<td>C</td>
<td>0.886</td>
<td>1.400</td>
<td>4.5</td>
<td>3</td>
<td>0.883</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1.131</td>
<td>4.5</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>0.974</td>
<td>0.971</td>
<td>3</td>
<td>5</td>
<td>0.974</td>
</tr>
<tr>
<td>F</td>
<td>0.847</td>
<td>0.870</td>
<td>-</td>
<td>6</td>
<td>0.864</td>
</tr>
</tbody>
</table>

Table 5.6 Results of running some techniques of complete ranking in DEA

In this example, the cross efficiency technique increases the discrimination power of DEA. However, it does not give a complete ranking. Units A and D achieve efficiency scores equal to 1. The super-efficient model gives a complete ranking where A is the best unit. The inefficient units (E and F) have the same scores as in the CCR model. The benchmarking technique, based on the additive model with VRS, only gives the rankings of efficient DMUs (no evaluation for non-
efficient unit $F$). It shows $C$ and $D$ with the same ranking as the least efficient units. The canonical correlation analysis achieves a full ranking of units. The tri-objective LP model of Li and Reeves (1999), considers $A$ and $D$ as efficient, similarly to the cross efficiency technique. As it can be seen in Tables 5.5 and 5.6, units $A$ and $F$ are always the best and the worst in different ranking techniques, respectively. More explanations on this example can be found in Adler et al. (2002).

In the next section, a new ranking technique based on DEA and PROMETHEE II is discussed.

5.3. **DEA-PROMETHEE integrated ranking technique**

Bagherikahvarin (2016) developed a new DEA-PROMETHEE integrated technique to fully rank units in DEA.

As discussed, the original DEA models do not provide a full ranking; it possibly classifies units into two groups: efficient and inefficient. However, in practice a full ranking can be required for evaluating efficiency and comparing units in many applications such as bank branches, companies and universities. During the last two decades, there have been attempts to fully rank units in the context of DEA (Adler et al., 2002; Hosseinzadeh Lotfi et al., 2013). In this contribution, a two-step model is presented to rank units according to multiple inputs and outputs. In the first step, DEA is applied between each pair of DMUs independently to generate a pairwise comparison matrix. In the second step, the obtained matrix is exploited by means of PROMETHEE II to completely rank units. The compatibility and monotonicity properties, between the resulting rankings of DEA and the new DEA-PROMETHEE technique, are shown while just one input and one output exist.

The new technique aims to profit of both DEA and PROMETHEE to completely rank units in DEA: it plans to use DEA to mathematically generate the pairwise comparison matrix for PROMETHEE II without any subjective help from a DM; and to use PROMETHEE to gather information from a pairwise comparison matrix to fully discriminate among DMUs/alternatives.

**Hypothesis**

The following hypotheses are respected in this approach:

1- It is assumed that the multiple inputs and the multiple outputs respectively contributed negatively and positively to the overall evaluation of DMUs/alternatives through PROMETHEE II. This means that the inputs are considered as minimized criteria and the outputs are seen as maximized criteria.

2- It is assumed that in each pairwise comparison of $DMU_A$ and $DMU_B$, the CCR model [Problem (a)] is feasible;

3- It is assumed that in each pairwise comparison of $DMU_A$ and $DMU_B$, their cross efficiency scores are different: $E_{AB}^* \neq E_{BA}^*$.

4- It is assumed that $a_{ij}$ is the advantage (preference) of $a_i$ over $a_j$, in the pairwise comparison matrix. If $E_{ij}^*$ and $E_{ji}^*$ denote the efficiency scores of $a_i$ and $a_j$, respectively, $a_{ij}$ is
defined as $E^*_i - E^*_j$ when $E^*_i > E^*_j$. To be in accordance with the pairwise comparison matrix in PROMETHEE, it is also assumed that when $i = j$, $a_{ii} = 0$.

5- A CCR-I-O model (1.8) is run between $n$ units when there exist just a single input and a single output; it is assumed that if $DMU_i$ has a higher rank than $DMU_{i+k}$, for $i = 1,2,\ldots,n-1$ and $k = 1,\ldots,n-i$, then: $E^*_i > E^*_{i+k}$, while $E^*_i$ and $E^*_{i+k}$ are the efficiency scores of unit $i$ and unit $i+k$. This means that the assumption is based on a non-increasing ranking order of DMUs. This assumption is used to demonstrate some properties of the new algorithm in the presence of a single input and a single output. It can be seen later in the current section.

**Problem definition**

As stated in previous chapters, there are some limitations in DEA and PROMETHEE methods. One of the most important shortcomings of the classical DEA models is:

- The weak discrimination power of DEA among efficient units; i.e. the DEA classical models divide units into two groups: efficient and inefficient. There is no discrimination between efficient units.

In multicriteria decision-making problems, DMs should determine subjectively the preference information to evaluate units. In some situations, DMs may face some difficulties in determining a priori information; e.g. lack of enough preference information in complex multicriteria decision-making problems. In these situations, some tools are needed to help DMs. DEA can be such a tool (Thompson et al., 1986; Golany, 1988; Cook and Kress, 1990; Cook et al., 1990, 1992 and 1993; Belton, 1992; Doyle and Green, 1993; Belton et Vickers, 1993; Stewart, 1994 and 1996; Shang et Suyehoshi, 1995; Sinuany-Stern et al., 2000; Ho and Oh, 2010; Jie, et al., 2010; Fulop and Markovits, 2012). The non-subjective or less-subjective approaches are easier from the viewpoint of the DM, who does not have the stress of subjectively determining preferences and evaluating alternatives (Stewart, 1994; Sinuany-Stern et al., 2000). PROMETHEE as other MCDA methods may need such tools to help DMs.

While for creating the pairwise comparison matrix in PROMETHEE, the preference functions, related thresholds and weights should be determined by the DM, the new algorithm applies DEA to do so. In the present thesis, in order to address these issues, a new integrated DEA-PROMETHEE algorithm is proposed, which profits from both methods. The algorithm works within the framework of DEA. Mainly, it improves the DEA analysis beyond the mere classification of efficient and inefficient units to a full ranking, by incorporating some elements of PROMETHEE as further analysis. While in the original PROMETHEE method, the data in the pairwise comparison matrix results from subjective preferences of the DM; in the present model, this matrix is non-subjective. It applies DEA for all pairs of units, given the input and output factors of each. The new model takes into consideration the multiple criteria by means of DEA when the ranking is performed by PROMETHEE.
Algorithm steps

The steps of the algorithm concerning the hypothesis are as follows:

1- In the first step of this algorithm, the efficiency scores of each pair of unit $A$ and $B$ are computed through DEA: $E_{AB}^*$ and $E_{BA}^*$:

$$E_{AB} = \text{Max } \sum_{r=1}^{s} u_r y_{rA}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{iA} = 1,$$

$$\sum_{r=1}^{s} u_r y_{rA} + s_1 = 1,$$

$$\sum_{r=1}^{s} u_r y_{rB} - \sum_{i=1}^{m} v_i x_{iB} + s_2 = 0,$$

$$u_r, v_i \geq 0, \forall i = 1, 2, ..., m, r = 1, 2, ..., s$$

In the case of equality of $E_{AB}^*$ and $E_{BA}^*$ ($E_{AB}^* = E_{BA}^* = 1$), the cross efficiency technique [(5.1) and (5.2)] is applied to evaluate the pair. Based on these results, the pairwise comparison matrix between each pair of $i$ and $j$ is generated as follows for all $i \neq j$:

$$a_{ij} = \begin{cases} E_{ij}^* - E_{ji}^* & \text{if } E_{ij}^* > E_{ji}^* \\ 0 & \text{else} \end{cases} \quad (5.14)$$

As in PROMETHEE, the quantity $a_{ij}$ is interpreted as being the advantage (preference) of $A_i$ over $A_j$. Furthermore, for $i = j$: $a_{ii} = 0$. This gives a pairwise comparison matrix that is then exploited by means of the net flow scores.

In accordance with the PROMETHEE method, relation (2.33) in Chapter 2: $a_{ij} + a_{ji} \leq 1 \left[ \pi (a_i, a_j) + \pi (a_j, a_i) \leq 1 \right], a_{ij} \geq 0 \left[ \pi (a_i, a_j) \geq 0 \right]$ and $a_{ii} = 0 \left[ \pi (a_i, a_i) = 0 \right]$.

2- In the second step, net flow scores of the different units are computed from the new pairwise comparison matrix. Two cases may happen: either $E_{ij}^* > E_{ji}^*$ or the opposite. It is considered that $I_{ij} = \begin{cases} 1 & \text{if } E_{ij}^* > E_{ji}^* \\ 0 & \text{else} \end{cases}$. It is obvious that $I_{ij} = 1 - I_{ij}$.

Consequently, and according to Equation (2.39) in Chapter 2, the net flow scores $\phi (a_i)$ are calculated as follows:

$$\phi (a_i) = \frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{j \neq i} \left[ \pi (a_i, a_j) - \pi (a_j, a_i) \right]$$

$$= \frac{1}{n-1} \sum_{j=1}^{n} \left[ a_{ij} - a_{ji} \right]$$

$$= \frac{1}{n-1} \left( \sum_{j=1}^{n} (E_{ij}^* - E_{ji}^*) I_{ij} - \sum_{j=1}^{n} (E_{ji}^* - E_{ij}^*) I_{ji} \right)$$
\[
\phi(a_i) = \frac{1}{n-1} \sum_{j \neq i}^{n} [(E_{ij}^* - E_{ji}^*)I_{ij} - (E_{ji}^* - E_{ij}^*)I_{ij}]
\]

Therefore:

\[
\phi(a_i) = \frac{1}{n-1} \sum_{j \neq i}^{n} (E_{ij}^* - E_{ji}^*)
\]  

(5.15)

This section is continued with two examples. First, the result of the new proposed algorithm is compared with the output of Example 5.1. Then, the algorithm is illustrated using a new example.

**Example 5.3** Example 5.1 is done for the proposed algorithm. For this aim, each pair of actions is compared by means of Problem (\(a\)). Then, the cross efficiency is applied to compute the efficiency score of each unit. In this step, the elements of the pairwise comparison matrix is generated by means of (5.14). For example, when \(E_{12}^* = 1\) and \(E_{21}^* = 0.8197\), then \(a_{12} = 1 - 0.8197 = 0.1803\) and \(a_{21} = 0\). Table 5.7 presents this matrix.

<table>
<thead>
<tr>
<th></th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>0.1803</td>
<td>0.3571</td>
<td>0.1056</td>
<td></td>
</tr>
<tr>
<td>(a_2)</td>
<td>0</td>
<td>0.0412</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(a_3)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(a_4)</td>
<td>0</td>
<td>0.3585</td>
<td>0.1492</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.7: Generated pairwise comparison matrix by DEA to use in PROMETHEE II

Table 5.8 compares the results of DEA and DEA-AHP in Table 5.3 with the DEA-PROMETHEE II (DEA-PII) ranking. The DMUs do not have a complete rank in DEA and DEA-AHP techniques, while the new algorithm gives a complete ranking. \(DMU_a\) has the first rank in DEA-PII and it is efficient in CCR, but in DEA-AHP, \(DMU_B\) is the best unit. \(DMU_C\) has the last place in all mentioned rankings.

Indeed in PROMETHEE, the sum of all net flow scores is equal to zero. In this example, this sum is equal to zero as well. However, this is not a rule for the new ranking algorithm. Since, the pairwise comparison matrix is not provided by PROMETHEE. In the next example, it can be seen that this sum is not equal to zero.

<table>
<thead>
<tr>
<th>DMU</th>
<th>DEA-PII Scores</th>
<th>DMUs Rank</th>
<th>CCR Scores</th>
<th>DEA-AHP Scores</th>
<th>DMUs rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.2143</td>
<td>1</td>
<td>1</td>
<td>0.4994</td>
<td>2-3</td>
</tr>
<tr>
<td>B</td>
<td>-0.1658</td>
<td>3</td>
<td>1</td>
<td>0.4994</td>
<td>2-3</td>
</tr>
<tr>
<td>C</td>
<td>-0.1825</td>
<td>4</td>
<td>0.851</td>
<td>0.4800</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>0.1340</td>
<td>2</td>
<td>1</td>
<td>0.5204</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.8: Ranking results of DEA and DEA-AHP
Example 5.4 In this example, 10 top universities of the Academic Ranking of World Universities (ARWU), listed in the field of computer sciences, are evaluated according to 5 criteria. The alternatives are Stanford University (SU), Massachusetts Institute of Technology (MIT), University of California Berkeley (UCB), Princeton University (PU), Carnegie Mellon University (CMU), Cornell University (CU), University of Southern California (USC), The University of Texas at Austin (UTA), Harvard University (HU), and University of Toronto (UT). The criteria are as follows: Alumni (number of alumni from the institution winning Turing Awards since 1951), Awards (staff of an institution winning Turing Awards since 1961), HiCi (Highly Cited researchers), PUB (papers indexes in science citation index-expanded) and TOP (percentage of papers published in the top 20% journals on the field of computer science compared to the papers published in all journals of that subject field). The detailed information of this problem can be found in (http://www.arwu.org). The problem is constituted fully output. In this order, a dummy input is added to the evaluation table (Table 5.9).

<table>
<thead>
<tr>
<th>Universities</th>
<th>SU</th>
<th>MIT</th>
<th>UCB</th>
<th>PU</th>
<th>CMU</th>
<th>CU</th>
<th>USC</th>
<th>UTA</th>
<th>HU</th>
<th>UT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alumni</td>
<td>90.7</td>
<td>54.2</td>
<td>100</td>
<td>68.6</td>
<td>42</td>
<td>42</td>
<td>0</td>
<td>42</td>
<td>97</td>
<td>24.3</td>
</tr>
<tr>
<td>Awards</td>
<td>86.6</td>
<td>100</td>
<td>96.8</td>
<td>71.8</td>
<td>79.1</td>
<td>57.3</td>
<td>39.5</td>
<td>39.5</td>
<td>0</td>
<td>53</td>
</tr>
<tr>
<td>HiCi</td>
<td>100</td>
<td>89.2</td>
<td>42.9</td>
<td>60.6</td>
<td>55.3</td>
<td>55.3</td>
<td>65.5</td>
<td>55.3</td>
<td>42.9</td>
<td>49.5</td>
</tr>
<tr>
<td>PUB</td>
<td>80.9</td>
<td>87.8</td>
<td>76.7</td>
<td>63</td>
<td>85</td>
<td>57.3</td>
<td>68.4</td>
<td>70.4</td>
<td>65.5</td>
<td>71.1</td>
</tr>
<tr>
<td>TOP</td>
<td>97.9</td>
<td>89.3</td>
<td>86.1</td>
<td>94.7</td>
<td>75.4</td>
<td>85.5</td>
<td>86.6</td>
<td>77.2</td>
<td>93.7</td>
<td>78.3</td>
</tr>
</tbody>
</table>

Table 5.9 Evaluation table extracted from (www.arwu.org)

As explained in the first step of the algorithm, each pair of universities is compared by using the DEA approach, in Problem (α), to construct the required pairwise comparison matrix based on (5.14). In the case where Problem (α) gives two equal efficient units, the cross efficiency technique [(5.1) and (5.2)] is applied between each pair to compute \( E_{ij}^* \) and \( E_{ji}^* \). See Table 5.10 as a resulted matrix:

<table>
<thead>
<tr>
<th></th>
<th>SU</th>
<th>MIT</th>
<th>UCB</th>
<th>PU</th>
<th>CMU</th>
<th>CU</th>
<th>USC</th>
<th>UTA</th>
<th>HU</th>
<th>UT</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU</td>
<td>0</td>
<td>0.1038</td>
<td>0.1583</td>
<td>0.1645</td>
<td>0.2064</td>
<td>0.2811</td>
<td>0.4217</td>
<td>0.3093</td>
<td>0.3200</td>
<td>0.3519</td>
</tr>
<tr>
<td>MIT</td>
<td>0</td>
<td>0</td>
<td>0.0382</td>
<td>0.1289</td>
<td>0.1367</td>
<td>0.1772</td>
<td>0.2026</td>
<td>0.2409</td>
<td>0.1782</td>
<td>0.2543</td>
</tr>
<tr>
<td>UCB</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0644</td>
<td>0.0933</td>
<td>0.0971</td>
<td>0.1101</td>
<td>0.1186</td>
<td>0.0983</td>
</tr>
<tr>
<td>PU</td>
<td>0</td>
<td>0</td>
<td>0.0076</td>
<td>0</td>
<td>0.0640</td>
<td>0.1907</td>
<td>0.2669</td>
<td>0.1530</td>
<td>0.1692</td>
<td>0.1750</td>
</tr>
<tr>
<td>CMU</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0721</td>
<td>0.2670</td>
<td>0.0857</td>
<td>0.0941</td>
<td>0.1048</td>
</tr>
<tr>
<td>CU</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0540</td>
<td>0.0142</td>
<td>0.0586</td>
<td>0.0698</td>
</tr>
<tr>
<td>USC</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0078</td>
<td>0</td>
</tr>
<tr>
<td>UTA</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.1744</td>
</tr>
<tr>
<td>HU</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0240</td>
<td>0</td>
</tr>
<tr>
<td>UT</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2970</td>
<td>0</td>
<td>0</td>
<td>0.0329</td>
</tr>
</tbody>
</table>

Table 5.10 Generated pairwise comparison matrix by DEA
For example, in comparing \( a_{UCB} \) and \( a_{PU} \), \( E_{UCB,PU}^* = E_{PU,UCB}^* = 1 \). Hence, the cross efficiency technique is applied to discriminate between them. Finally, \( E_{UCB,PU}^* < E_{PU,UCB}^* \); thus, \( a_{UCB,PU} = 0 \) and \( a_{PU,UCB} = E_{PU,UCB}^* - E_{UCB,PU}^* = 1 - 0.9924 = 0.0076 \). In the second step of the algorithm, the net flow scores should be determined. \( \Phi(a_{UCB}) \) is computed as follows [based on (5.15)]:

\[
\Phi(a_{UCB}) = \frac{1}{10-1} \sum_{j=1}^{10} (E_{UCB,j}^* - E_{j,UCB}^*) = \frac{1}{9} (0 + 0 + 0 + 0 + 0.0644 + 0.0933 + 0.0971 + 0.1101 + 0.1186 + 0.0983 - 0.1583 - 0.0382 - 0 - 0.0076 - 0 - \cdots - 0) = \frac{0.3777}{9} = 0.0426.
\]

In Table 5.11, the scores and the ranking results of the CCR model (1.8) and the new method (DEA-PROMETHEE II) are provided.

<table>
<thead>
<tr>
<th>DEA-PII Scores</th>
<th>DMUs Ranking</th>
<th>CCR Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2574</td>
<td>SU</td>
<td>1</td>
</tr>
<tr>
<td>0.1392</td>
<td>MIT</td>
<td>1</td>
</tr>
<tr>
<td>0.0814</td>
<td>PU</td>
<td>0.9673</td>
</tr>
<tr>
<td>0.0426</td>
<td>UCB</td>
<td>1</td>
</tr>
<tr>
<td>-0.0169</td>
<td>CMU</td>
<td>0.9726</td>
</tr>
<tr>
<td>-0.0686</td>
<td>CU</td>
<td>0.8733</td>
</tr>
<tr>
<td>-0.0843</td>
<td>UTA</td>
<td>0.8299</td>
</tr>
<tr>
<td>-0.1062</td>
<td>HU</td>
<td>1</td>
</tr>
<tr>
<td>-0.1969</td>
<td>USC</td>
<td>0.8845</td>
</tr>
<tr>
<td>-0.8095</td>
<td>UT</td>
<td>0.8399</td>
</tr>
</tbody>
</table>

Table 5.11 Comparison between DEA and DEA-PROMETHEE II rankings

DEA gives 4 efficient units. In this example, the DEA and DEA-PROMETHEE rankings are not perfectly compatible. Nevertheless, 3 units out of the 4 efficient units in DEA are ranked in top places in the new model. \( a_{PU} \), which belongs to inefficient units in DEA, has a good rank in the DEA-PII. \( a_{HU} \) is an efficient unit that does not have good place in the new ranking (rank 8). The inefficient DMUs in DEA have close value inefficiency scores. \( a_{UT} \) in both methods has a low rank, but not the same rank. While DEA ranks 4 units out of 10 as efficient units, the DEA-PII provides a complete rank without engaging the preferential information of DM in choosing weights, preference functions, and thresholds.

Through this algorithm, Bagherikahvarin (2016) evaluated different locations for building hypermarkets in Belgium to provide a complete ranking of units.

In what follows, some properties of the proposed model (the compatibility and monotonicity between the ranking results of the DEA and DEA-PROMETHEE and rank reversal in new model) are discussed in the presence of just one input and one output.
Some properties of this model

To demonstrate some properties of the proposed algorithm, first, it is shown that, in comparison between \( n \) units with a single input and a single output, if \( E_i^* > E_{i+k}^* \), then \( E_{i,i+k}^* > E_{i+k,i}^* \) when \( i = 1, 2, \ldots, n-1 \) and \( k = 1, \ldots, n-i \).

The CCR model (1.8) is run between \( n \) units when there is just a single input and a single output. Let suppose that \( E_i^* > E_{i+k}^* \) for \( i = 1, 2, \ldots, n-1 \) and \( k = 1, \ldots, n-i \): Problem (a')

\[
E_i = \max u_i y_i \\
\text{s.t.} \\
v_i x_i = 1, \quad (5.16) \\
u_i y_i \leq 1, \quad (5.17) \\
u_i y_j - v_i x_j \leq 0, \forall j \neq i, j = 1, 2, \ldots, n \quad (5.18) \\
u_i, v_i \geq 0
\]

where \( x_i, y_i, v_i, \) and \( u_i \) are the single input, single output, and the weights associated to the single input and output, respectively. By solving Problem (a') the optimal solutions for \( DMU_i \) are \( v_i^* = 1/x_i \) [from (5.16)] and \( u_i^* = \min_j \left\{ \frac{x_j}{y_j} \right\} \) [from (5.17) and (5.18)] for \( j = 1, \ldots, n, j \neq i \). The hypothesis is \( E_i^* > E_{i+k}^* \) for \( i = 1, 2, \ldots, n-1 \) and \( k = 1, \ldots, n-i \). In other words \( E_1^* > E_2^* > \cdots > E_n^* \) and \( DMU_1 \) is the best unit (it has an efficiency score equal to 1: \( E_1^* = 1 \)). Consequently, the successive efficiency scores can be written as follows:

\[
\frac{y_1}{x_1} \min_j \left\{ \frac{x_j}{y_j} \right\} > \cdots > \frac{y_i}{x_i} \min_{j \neq i} \left\{ \frac{x_j}{y_j} \right\} > \cdots > \\
\frac{y_{i+k}}{x_{i+k}} \min_{j \neq i+k} \left\{ \frac{x_j}{y_j} \right\} > \cdots > \frac{y_n}{x_n} \min_{j \neq n} \left\{ \frac{x_j}{y_j} \right\} \quad (5.19)
\]

It is evident that:

\[
\begin{align*}
A_i \subset A_1 & \text{ since } A_i = A_1 \setminus \{i\} \\
A_{i+k} \subset A_1 & \text{ since } A_{i+k} = A_1 \setminus \{i+k\}
\end{align*} \quad (5.20)
\]

Thus:

\[
\min A_1 = \cdots = \min A_i = \cdots = \min A_{i+k} = \cdots = \min A_n = \frac{x_i}{y_i} \quad (5.21)
\]

Therefore, the efficiency scores of \( DMU_i \) and \( DMU_{i+k} \) (\( E_i^* \) and \( E_{i+k}^* \)) are as follows:

\[
E_i^* = u_i y_i = \frac{y_i}{x_i} \times \min A_i = \frac{y_{i+1}}{x_{i+1}} \quad (5.22)
\]
and

\[ E_{i+k}' = u_{i+k}^* y_{i+k} = \frac{y_{i+k}}{x_{i+k}} \times \min A_{i+k} = \frac{y_{i+k} x_1}{x_{i+k} y_1} \quad (5.23) \]

When \( E_i' > E_{i+k}' \) for \( i = 1, 2, \ldots, n - 1 \) and \( k = 1, \ldots, n - i \), it can be written

\[ \frac{y_i x_1}{x_i y_1} > \frac{y_{i+k} x_1}{x_{i+k} y_1} \]

thus:

\[ \frac{y_i}{x_i} > \frac{y_{i+k}}{x_{i+k}} \quad (5.24) \]

Running Problem \((a')\) between just two units, \( DMU_i \) and \( DMU_{i+k} \), gives:

\[ E_{i,i+k} = \max u_i y_i \]

\[ s.t. \]

\[ v_i x_i = 1 \]

\[ u_i y_i \leq 1 \]

\[ u_i y_{i+k} - v_i x_{i+k} \leq 0 \]

\[ u_i, v_i \geq 0 \]

Since from (5.24), the relation \( \frac{x_i}{y_i} < \frac{x_{i+k}}{y_{i+k}} \) is resulted, therefore, the efficiency scores of \( DMU_i \) and \( DMU_{i+k} \) \((E_{i,i+k}' \text{ and } E_{i+k,i}' \text{ in } (5.25))\) are:

\[ E_{i,i+k}' = u_i^* y_i = \min \left\{ \frac{1}{y_i} \frac{x_{i+k}}{y_{i+i+k}} \right\} \times y_i \]

\[ = \min \left\{ \frac{x_i}{y_i} \frac{x_{i+k}}{y_{i+k}} \right\} \times y_i = \frac{x_i}{y_i} \times \frac{y_i}{x_i} = 1 \quad (5.26) \]

\[ E_{i+k,i}' = u_{i+k}^* y_{i+k} = \min \left\{ \frac{1}{y_{i+k}} \frac{x_i}{x_{i+k}} \right\} \times y_{i+k} \]

\[ = \min \left\{ \frac{x_i}{y_i} \frac{x_{i+k}}{x_{i+k}} \right\} \times y_{i+k} = \frac{x_i}{y_i} \times \frac{y_{i+k}}{x_{i+k}} < 1 \quad (5.27) \]

So:

\[ \text{If } E_i' > E_{i+k}' \quad \forall i = 1, 2, \ldots, n - 1, k = 1, \ldots, n - i \]

\[ \text{i.e.: } E_{i,i+k}' > E_{i+k,i}' \quad \blacksquare \quad (5.28) \]

Now, let point out some properties of the proposed algorithm in the presence of a single input and a single output:

- **The compatibility between the ranking results of the DEA and DEA-PROMETHEE**

It is assumed that Problem \((a')\) is run. The aim is to show that if \( E_i' > E_{i+k}' \), \( i = 1, 2, \ldots, n - 1 \) and \( k = 1, \ldots, n - i \), then \( \phi(a_i) > \phi(a_{i+k}) \).

Due to formulation (5.15): \( \phi(a_i) = \frac{1}{n-1} \sum_{j \neq i}^n (E_{ij}' - E_{jj}') \) and \( \phi(a_{i+k}) = \frac{1}{n-1} \sum_{j \neq i}^n (E_{i+k,j}' - E_{j,i+k}') \). By (5.14), when \( E_{i,i+k}' > E_{i+k,i}' \) for \( i = 1, 2, \ldots, n - 1 \) and \( k = 1, \ldots, n - i \), then:
Chapter 5: A DEA-PROMETHEE approach for complete ranking of units

\[
\begin{align*}
& \begin{cases}
E_{ij}^* - E_{ji}^* = E_{i+k,j}^* - E_{j,i+k}^* = 0, & \forall j: 1 \leq j \leq i \\
E_{ij}^* - E_{ji}^* > 0, & \forall j: i + 1 \leq j \leq i + k \\
E_{ij}^* - E_{ji}^* > E_{i+k,j}^* - E_{j,i+k}^*, & \forall j: i + k + 1 \leq j \leq n
\end{cases} \\
& (5.29)
\end{align*}
\]

Equations (a) and (b) are evident. It is already shown in (5.28) that \(E_{i,i+k}^* > E_{i+k,i}^*\). Consecutively: \(E_{i,i+k+1}^* > E_{i+k+1,i}^*\), ... \(E_{i,i+n}^* > E_{n,i}^*\). To prove (c), based on (5.26) and (5.27) \(\forall j: i + k + 1 \leq j \leq n\) it can be written that:

\[
E_{ij}^* - E_{ji}^* = (E_{i,i+k+1}^* - E_{i+k+1,i}^*) + \cdots + (E_{i,n}^* - E_{n,i}^*)
\]

\[
= \left(1 - \frac{y_{i+k+1}x_{i+j}}{x_{i+k+1}y_{i+j}}\right) + \cdots + \left(1 - \frac{y_{n}x_{i+n}}{x_{n}y_{i+n}}\right)
\]

\[
(5.30)
\]

Relation (5.28) also claims that \(E_{i+k,i+k+1}^* > E_{i+k+1,i+k}^*\), ... \(E_{i+k,i+n}^* > E_{n,i+k}^*\):

\[
E_{i+k,j}^* - E_{j,i+k}^*
\]

\[
= (E_{i+k,i+k+1}^* - E_{i+k+1,i+k}^*) + \cdots + (E_{i+k,n}^* - E_{n,i+k}^*)
\]

\[
= \left(1 - \frac{y_{i+k+1}x_{i+j}}{x_{i+k+1}y_{i+j}}\right) + \cdots + \left(1 - \frac{y_{n}x_{i+n}}{x_{n}y_{i+n}}\right)
\]

\[
(5.31)
\]

From (5.24), the relation \(\frac{y_{i+j}}{x_{i+k}} > \frac{y_{i+k}}{x_{i+k}}\) is obtained, thus:

\[
\frac{y_{i+k}x_{i+k+1}}{x_{i+k+1}y_{i+k+1}} > \frac{y_{i+k}x_{i+k+1}}{x_{i+k}y_{i+k+1}} \Rightarrow \frac{x_{i+k+1}}{y_{i+k+1}} < \frac{x_{i+k}y_{i+k+1}}{y_{i+k}x_{i+k+1}} \Rightarrow 1 - \frac{x_{i+k}y_{i+k+1}}{y_{i+k}x_{i+k+1}} > 1 - \frac{x_{i+k}y_{i+k+1}}{y_{i+k}x_{i+k+1}}
\]

\[
(5.32)
\]

For other terms of (5.30) and (5.31), the same relations as (5.32) can be built. Obviously, \(\forall j: i + k + 1 \leq j \leq n\): \(E_{ij}^* - E_{ji}^* > E_{i+k,j}^* - E_{j,i+k}^*\).

Thus, \(\emptyset(a_i) > \emptyset(a_{i+k})\).\]

Actually, in this situation, this is no longer necessary using the cross efficiency since in each pairwise comparison \(E_{i,i+k}^* > E_{i+k,i}^*\).

Therefore, it can be observed that both methods are compatible in the presence of just one input and one output. On the other hand, in the presence of multiple inputs and outputs, the ranking results of DEA and DEA-PROMETHEE are not necessarily compatible. This was outlined in Example 5.3.

- **The monotonicity between the ranking results of the DEA and DEA-PROMETHEE**

A DEA ranking result is said to be monotone if by adding a constant to the output of all DMUs (input of all DMUs), in an I-O model (O-O), the ranking order of DMUs does not change. Consequently, the ranking in the new model is also monotone.
To show the monotonicity of the new model in the presence of just 1 input and 1 output, it is sufficient to show that the DEA model between \( n \) units is monotone. Problem \((a')\) is run while it is supposed that \( E^*_i > E^*_{i+k} \) for \( i = 1, 2, \ldots, n-1 \) and \( k = 1, \ldots, n-i \). From (5.24), the following relation can be written:

\[
\frac{y_i}{x_i} > \frac{y_{i+k}}{x_{i+k}} \Rightarrow \frac{y_i}{y_{i+k}} > \frac{x_i}{x_{i+k}} \tag{5.33}
\]

It is put \( y'_i = y_i + \alpha \) and \( y'_{i+k} = y_{i+k} + \alpha \) while \( \alpha \) is a constant. The aim is to show that \( E^{**}_i > E^{**}_{i+k} \) \((E^{**}_i \text{ and } E^{**}_{i+k} \text{ are efficiency scores of } DMU_i \text{ and } DMU_{i+k} \text{ after adding } \alpha \text{ to their outputs, respectively})\):

\[
\frac{y_i + \alpha}{y_{i+k} + \alpha} > \frac{y_i}{y_{i+k}} \Rightarrow \frac{y_i + \alpha}{x_i} > \frac{y_i + \alpha}{x_{i+k}} \tag{5.34}
\]

To show (5.34), a condition must be held. From (5.33) and (5.34), it can be deduced that:

\[
y_i y_{i+k} + \alpha y_{i+k} > y_{i+k} y_i + \alpha y_i \Rightarrow \alpha y_{i+k} > \alpha y_i \tag{5.35}
\]

Therefore, to have monotonicity in DEA ranking order, after adding \( \alpha \) to the outputs of the problem, the following condition should be satisfied:

\[
y_{i+k} > y_i \quad \forall \alpha > 0 \tag{5.36}
\]

According to condition (5.36) \( E^{**}_i > E^{**}_{i+k} \).

In (5.28), it is already shown that if \( E^*_i > E^*_{i+k} \), then \( E^{**}_{i,i+k} > E^{**}_{i,k+1} \) for \( i = 1, 2, \ldots, n-1 \) and \( k = 1, \ldots, n-i \). Consecutively, if \( E^{**}_i > E^{**}_{i+k} \), then \( E^{**}_{i,i+k} > E^{**}_{i,k+1} \); thus \( \emptyset'(a_i) > \emptyset'(a_{i+k}) \) and the new model is also monotone.■

Furthermore, the BCC-I-O model (1.17) displays “Translation Invariance” (Cooper et al., 2005 and 2011); therefore, it is monotone as well.

To show this characteristic of the BCC model, it is put \( y'_{rj} = y_{rj} + \alpha_r \) and \( y'_{ro} = y_{ro} + \alpha_r \) in the second constraint of (1.17) [the other parts of the model (1.17) are not be changed]:

\[
\sum_{j=1}^{n} (y'_{rj} - \alpha_r) \lambda_j - s_r^+ = y'_{ro} - \alpha_r \quad \text{for} \quad r = 1, \ldots, s
\]

The model (1.17) is changed as follows:

\[
\begin{align*}
M_i & \quad Z, \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s^- = Z x_{i0}, i = 1, \ldots, m \\
\sum_{j=1}^{n} \lambda_j (y'_{rj} - \alpha_r) - s^+ & = y'_{ro} - \alpha_r, r = 1, \ldots, s \\
\sum_{j=1}^{n} \lambda_j & = 1 \\
\lambda_j & \geq 0, \quad j = 1, \ldots, n
\end{align*}
\]

Constraint (5.37) is changed to:
The equality of (5.38) is the result of considering the convexity condition ($\sum_{j=1}^{n} \lambda_j = 1$). The optimal solution of both problems (1.17) and (5.37) is $(\lambda_j^*, s_r^+, s_i^*)$. Thus, this model is Translation Invariant and consecutively monotone. As it can be seen, the model is not further limited by a single input and a single output. In this thesis, BCC cross-efficient models are not discussed.

- **No rank reversal in the new DEA-PROMETHEE method by adding a copy of the worst unit**

In the PROMETHEE method, adding a copy of an existing alternative may result to rank reversal (Verly and De Smet 2013). The new algorithm however does not cause rank reversal by adding a copy of the worst alternative.

Let us, one more time, suppose that the primary ranking is $E_1^* > E_2^* > \cdots > E_n^*$, and $k = 1, ..., n - i$. It shows that $E_1^* > E_2^* > \cdots > E_n^*$. If a copy of the last unit is added to the existing units: $a_1, ..., a_n, a_{n+1}$ such that $a_n = a_{n+1}$ then $E_n^* = E_{n+1}^*$. Obviously, there is no rank reversal in Problem ($a'$). Respectively and with attention to (5.15), in the new pairwise comparison matrix, $\emptyset(a_n) = \emptyset(a_{n+1})$, and the ranking order between other alternatives does not change. Therefore, in the presence of just a single input and a single output, there is no rank reversal in the new ranking algorithm.

This chapter is closed by mentioning some limits and advantages of the proposed model.

### 5.3.1. Advantages and limitations

As mentioned above, the new ranking method that depends upon the context of DEA profits from the advantages of each method to cover the limits of other one. On the one hand, when the subjective determination of preference information may be a difficult task for the DM, PROMETHEE applies DEA to generate the pairwise comparison matrix. On the other hand, while DEA is not able to discriminate between efficient DMUs, it uses PROMETHEE to fully rank them. Furthermore, the new method is always feasible, since a DEA cross efficiency model based on CRS is always feasible (Lim and Zhu, 2015).

However, even though this method can fully discriminate between units, there are still several limitations. Using the CCR classical model to compare two units, mostly, gives two equal efficient units in comparison with the number of multiple inputs and multiple outputs. In order to cover this problem, the average cross efficiency measure is used to distinguish between two units. The cross efficiency technique has some advantages: proposing a complete order of units; eliminating unrealistic weight systems without requiring the elicitation of weight restrictions from value judgements of DMs; and differentiating between good and poor performers by average cross efficiency. Nevertheless, unfortunately, there are some limitations for this technique. One possible limitation is the non-uniqueness feature of cross efficiencies because of the DEA characteristic in generating weights. Multiple optimal solutions (weights) exist because one factor can be favorable to one DMU and not to another, or vice versa (Doyle and Green, 1994). Another problem that
arises according to a DEA feature lies on unrealistic weights for one unit by passively adopting the weights designed for other units. Therefore, the solutions of the cross efficiency method are not Pareto solutions. The third limitation is the losing association with the weights by averaging among the cross efficiency scores (Adler et al., 2002; Despotis, 2002). It means that this method cannot clearly provide the weights to help DM improving her/his performance.

Additionally, there is the possibility of incompatibility between the ranking of inefficient DMUs in a classical DEA model and the proposed algorithm. Alternatively, the efficient units in DEA may be ranked low in the new algorithm; e.g. in Example 5.4, when HU is evaluated as an efficient university by DEA, it receives a bad rank in the DEA-PROMETHEE algorithm.

Finally, one main feature of multicriteria problems is the possibility of engaging a priori information and preferences of DMs into the problem, which is not possible through this algorithm.

5.4. Conclusion

In this chapter, the introduction and literature review of the second contribution of this thesis is presented. First, some important ranking techniques are reviewed. The techniques provide a complete ranking in DEA (or nearly complete ranking). Then, three MCDEA models are presented, which increase the discrimination power of DEA. The chapter is continued by introducing a new ranking algorithm based on DEA and PROMETHEE II.

The main objective of this contribution is to fully rank units in the context of DEA by taking the benefits of both DEA and PROMETHEE to cover each other. The algorithm belongs to this contribution applies DEA to compare each pair of units and create the pairwise comparison matrix. Then, this matrix is used by means of PROMETHEE II to compute the final scores of each unit. While, DEA is a weak method to discriminate among efficient DMUs, this algorithm provides a full ranking. Moreover, in the situations where there is not enough a priori information about decision-making problem, DMs does not require determining the preference information for the PROMETHEE method to generate a pairwise comparison matrix. The algorithm is compared with a similar DEA-AHP algorithm through an example. While the new algorithm always gives a full ranking, the DEA-AHP approach does not guarantee a complete ranking.

Besides, some properties of this algorithm in the presence of a single input and a single output are discussed, such as compatibility and monotonicity between the resulting ranks of the DEA and DEA-PROMETHEE, and rank reversal in the new algorithm.

In the next chapter, the third contribution of this thesis is presented. The contribution presents a new algorithm based on DEA to propose weights in the context of the PROMETHEE II method (Bagherikahvarin and De Smet, 2016b).
Chapter Six

Determining new possible weight values: a procedure based on Data Envelopment Analysis


Abstract

In this chapter, a new algorithm is presented to propose weights in the context of the PROMETHEE II method, based on DEA. This algorithm can help enriching the understanding of multicriteria decision-making problem. The hypothesis, limitations and advantages of the algorithm are also considered.

6.1. Introduction

Nowadays, researchers in multicriteria decision-making problems are more and more interested in integrated approaches. As explained in previous chapters, the combined use of DEA and MCDA has already received a lot of attention.

In the context of MCDA, the notion of “optimal solution” (which is at the core of the traditional Operational Research models) disappears and leaves room for the concept of “compromise solution” which mainly relies on the preferences of DMs. Actually, there is a parallelism between the problems addressed by DEA and the decision-making problems in multicriteria analysis (Roy, 1985). Indeed, as several times explained, inputs and outputs in DEA may be viewed as attributes or criteria in MCDA. Moreover, DMUs might be seen as alternatives. Doyle and Green (1993) introduced DEA as an aid to MCDA methods. Stewart (1996) mentioned the connection of the two fields on the basis of the objective function. While in DEA, the efficiency frontier is determined by optimizing the weighted sum of outputs over the weighted sum of inputs, in MCDA, assessing and ranking alternatives are based on the conflicting set of criteria and subjective judgments. Similarly, Ishizaka and Nemery (2013) indicated the differences between DEA and MCDA based on their mechanism of comparing actions. Sarkis (2000) called DEA as a thoughtless approach for MCDA to evaluate alternatives.
objectively. Through others, this similarity has been clearly pointed out in the works of Belton and Vickers (1993) and has led to the creation of special interest groups to study the interactions between DEA and different methods of MCDA. In Chapter 3, DEA and MCDA are concisely compared and in Chapters 3, 4, and 5, some of these interactions were briefly explained; such as GDEA models (Yu et al., 1996; Nakayama et al., 2002; Kleine, 2004; Jahanshahloo et al., 2009), common weight DEA model (Kao, 2010), weight restricted DEA models based on MCDA (Shang and Sueyoshi, 1995; Zhu, 1996; Premachandra, 2001; Entani et al., 2004; Bagherikahvarin and De Smet, 2016) and some complete ranking techniques in DEA using MCDA models (Li and Reeves, 1999; Sinuany-Stern et al., 2000; Jablonsky, 2012; Bagherikahvarin, 2016).

One of the main differences between DEA and MCDA is the role played by DMs. Indeed, in a multicriteria framework, the preferences of DMs have to be determined cautiously. Among others, weight values have to be assessed and tested carefully whereas in DEA, it is the model itself, which generates the weights automatically.

In this chapter, it is investigated how a classical DEA model can be applied in the specific context of MCDA: the PROMETHEE II method. As far as it is considered, this is the first time one investigates the application of DEA in PROMETHEE to help DMs in computing weights. The originality of this contribution is using DEA to identify and illustrate possible weight values in PROMETHEE II that are simultaneously consistent with the induced DEA ranking. The underlying idea is to define a polyhedron of weight values (GAIA brain) that are compatible with the DEA analysis. This is characterized by a linear system. The Vertex Enumeration Algorithm (VEA) is used to enumerate possible vertices. Then, these points are projected on the GAIA plane and the associated convex hull is determined. Finally, a geometric mean is applied on the resulted weight matrix (vertices) to propose an initial weight vectors to DM.

The chapter is structured as follows: The second part describes the proposed approach, including the hypothesis, the problem definition, the algorithm steps and its limits and advantages. The algorithm is then applied on a small sample. In the third part, the Vertex Enumeration Algorithm is summarized. It is clarified through an example. This chapter is closed by an illustrative example.

6.2. Determining new possible weight values in PROMETHEE: a procedure based on DEA

As explained in Chapter 2 (Section 2.3.1), in the PROMETHEE method, a complementary visual tool, called GAIA (Mareschal and Brans, 1988) is often used in practice. The GAIA plane facilitates the decision making process. Indeed, several interpretations can be done according to the relative positions of the alternatives between themselves, in comparison with the different criteria, between the criteria and with the decision stick. Let us recall that the decision stick is the projection of the weights vector on the GAIA plane. For instance, it allows detecting groups of similar and/or incomparable alternatives, conflicting and/or redundant criteria, best compromise solutions, ranking of alternatives according to different criteria, etc. An advance
use of the GAIA plane relies on sensitivity analysis and robustness to enrich the understanding of the problem. Truly, choosing the exact weight values may be a difficult task for DM (DM may hesitate to assign precise weight values to the different criteria). Therefore, one may investigate the impact of weight variations in the final ranking. On the one hand, the DM may explicitly change the values, one by one, and control their impacts in the final ranking. This can be done with the so-called “walking weights” tool. On the other hand, she/he may want to define weight intervals within which the values are likely to vary:

\[ w_k^- \leq w_k \leq w_k^+, k = 1, \ldots, q \]  \hfill (6.1)

where \( w_k^- \) and \( w_k^+ \) are determined by DM. These intervals can also be verified by tolerating a percentage \( \alpha_k, k = 1, \ldots, q \) in this value:

\[ w_k \pm \alpha_k w_k, k = 1, \ldots, q \]  \hfill (6.2)

The value of \( \alpha_k \) is determined by DM(s). This information determines a hypercube of possible weight values in the k-dimensional space of unicriterion net flow scores; the projection of this hypercube on the GAIA plane permits to identify a region where the decision stick belongs. Depending on the position of this region with respect to the alternatives, one may determine those that could eventually become good compromise solutions and those that never considers as good candidates. This tool is referred to as the “GAIA brain” or “decision maker brain” (PROMETHEE VI) (Brans and Mareschal, 2002). Briefly, it allows assessing the impact of different weight values on the final ranking. In this context, the DM is supposed to provide weight intervals to be tested. The idea of the proposed algorithm is to suggest weights intervals for the PROMETHEE VI method that are compatible with the DEA analysis.

In this algorithm, it is analysed how the ranking induced by DEA may constraint the weights used in constructing an extension of the GAIA brain. This extension is not the same as the GAIA brain in PROMETHEE VI but it may provide DM the similar information about the space of freedom of the DM. It shows the importance of alternatives, which are lied in the same direction with the GAIA brain. More precisely, if the DEA analysis leads to state that the efficiency of alternative \( a_i \) is higher than alternative \( a_j \), weight values of the PROMETHEE II ranking are restricted such that \( a_i \) has a lower rank than \( a_j \) (a lower rank is assumed to be better than a higher rank). The weight values computed in this way are communicated to the DM as a starting point. Then, she/he remains free to modify these values in order to better represent the weights she/he wants to investigate. It should be noted that the aim of forcing alternatives in PROMETHEE to be in the same order with the DEA analysis is creating an extension of the GAIA brain, without knowledge in initial weights values. This may deepen the conception of the multicriteria problems and help DMs by inducing an initial ranking through GAIA plane.

**Hypothesis**

The following hypotheses are imposed in this approach:
Chapter 6 – Determining new possible weight values: a procedure based on Data Envelopment Analysis

1- It is assumed that the evaluation table, preference functions and parameters (indifference and preference thresholds and weights) are available. Preference functions are chosen linear type. As explained in Chapter 2, this type of preference function covers other types of preference functions. Nevertheless, the principles of this approach are not changed by changing the type of function.

2- It is assumed that the inputs and the outputs in a DEA model are identified as criteria in a multicriteria problem, with minimization of inputs and/or maximization of outputs as associated objectives (Doyle and Green, 1993).

3- It is assumed that the number of DMUs/alternatives to be evaluated are equal to or more than the number of inputs and outputs/criteria. As discussed the size of decision-making problems in DEA (Chapter 1: Section 1.2.3), it is better to keep the relation \( n \geq 3(m + s) \) between the combined number of inputs \( m \) and outputs \( s \) with the number of units \( n \) (Nunamaker, 1985; Franchon, 2003; Cooper et al., 2005). This increases the possibility of more discrimination between efficient units. However, the objective of present chapter is not discrimination between units through DEA. Hence, it is not necessary to respect mentioned relation. It is supposed to keep relation \( n \geq m + s \).

4- It is assumed that the net flow score of alternative \( a_i \) is better than \( a_j \), if, in DEA, DMU \( a_i \) has more efficiency score than \( a_j \). A description of this assumption can be seen in the 3rd step of the algorithm.

5- It is assumed that the DEA analysis of problem is feasible.

**Problem definition**

As discussed in earlier chapters, an important step in multicriteria decision-making problems is to discuss and determine a priori information. In complex problems, it is possible that DM faces some difficulties in choosing some parameters such as weight of criteria (Thompson et al., 1986; Golany, 1988; Cook and Kress, 1990; Cook et al., 1990, 1992 and 1993; Belton, 1992; Belton and Vickers, 1993; Doyle and Green, 1993; Stewart, 1994 and 1996; Shang et Sueyoshi, 1995; Sinuany-Stern et al., 2000; Ho and Oh, 2010; Jie, et al., 2010; Fulop and Markovits, 2012). Some methods like AHP has tool to generate weights. But, PROMETHEE suffers from:

- Absence of a clear technique to choose weight values for criteria.

In this chapter, concerning this issue in PROMETHEE, a new algorithm is designated. The algorithm is based on DEA to propose the initial weight vectors to DM. Then, she/he is free to refine the weight values.

**Algorithm steps**

The steps of the algorithm with reference to the hypothesis are as follows:
1- Criteria are partitioned into two groups: inputs (criteria to be minimized) and outputs (criteria to be maximized);

2- A DEA analysis is performed: DMUs are ranked according to their efficiency scores; whether they can be divided into efficient and non-efficient units or be ranked completely via the super-efficient Model (5.4);

3- The PROMETHEE unicriterion net flows, $\phi_k(a_j)$, are computed for all criteria and alternatives;

The ranking computed at step (2) is used to generate weight constraints on the PROMETHEE II ranking. More precisely, if $a_i$ has better efficiency score than that of $a_j$ in DEA [efficiency($a_i$) > efficiency($a_j$)] then the following linear constraint is defined in PROMETHEE II:

$$\sum_{k=1}^{q} w_k \cdot [\phi_k(a_i) - \phi_k(a_j)] > 0 \quad (6.3)$$

If $\phi_k(a_i) - \phi_k(a_j) = \Delta_{ij}$, the following linear system should be solved:

$$\begin{cases} \sum_{k=1}^{q} w_k \cdot \Delta_{ij} > 0 \\ \sum_{k=1}^{q} w_k = 1, w_k \geq 0 \end{cases} \quad (6.4)$$

The second equation in (6.4) shows that the sum of the nonnegative weights is normalized to 1.

4- These constraints define a polyhedron of PROMETHEE weights values that induce rankings which are compatible with the DEA analysis;

5- This polyhedron is projected on the GAIA plane;

6- A geometric mean is applied on the resulted weight matrix (vertices) of step 3 to propose an initial weight vectors to DM.

**Example 6.1** Let us illustrate the mentioned compatibility in step 4 with an artificial data set (in this case: $k = 3$ criteria and $n = 6$ alternatives). Table 6.1 summarizes the data associated to this instance. Two first criteria are taken as inputs and the last criterion is considered as output in DEA analysis. Figure 6.1 illustrates the potential Kendall’s correlation between the PROMETHEE II rankings and the DEA analysis for different values of $w_1$, $w_2$, and $w_3$ (since weights are normalized, then $1 - w_1 - w_2 = w_3$). In order to identify weight values that are compatible with the DEA analysis, 1000 instantiations (1000 normalized weight combinations) are randomly drawn from the hypercube $\sum_{k=1}^{3} w_k = 1$ and tested. Consecutively, each set of weights is used to generate a ranking in PROMETHEE II (1000 PROMETHEE II rankings are generated in these weights to compare with the DEA analysis). The compatible values help us to estimate the area where should be projected on the GAIA plane. The colour bar next to the figure displays the value of correlation regards to the colour. High correlation values correspond to the lighter part of the...
graph. It means that in lighter regions more weights induce the same ranking. Clearly, in this particular instance, a reasonable part of the weights domain is compatible with the DEA analysis.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min/Max</td>
<td>Min</td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td>Type</td>
<td>Linear</td>
<td>Linear</td>
<td>Linear</td>
</tr>
<tr>
<td>Thresholds</td>
<td>q=0, p=3</td>
<td>q=1, p=3</td>
<td>q=0, p=6</td>
</tr>
<tr>
<td>Weights</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Alts/DMUs</td>
<td>a₁</td>
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<td>1.7</td>
</tr>
<tr>
<td></td>
<td>a₂</td>
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<td>4.1</td>
</tr>
<tr>
<td></td>
<td>a₃</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>a₄</td>
<td>3</td>
<td>5.5</td>
</tr>
<tr>
<td></td>
<td>a₅</td>
<td>5</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>a₆</td>
<td>8</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 6.1 Illustrative example: evaluation table and preference parameters

Figure 6.1 Kendall’s Tau correlation between PROMETHEE II rankings and DEA ranking for an instance based on 3 criteria and 6 alternatives (n = 6, k = 3)

Each constraint (that imposes the compatibility) limits the area of the compatible domain.

The DEA ranking is a₅, a₆, a₂, a₁, a₄, a₃. Thus related constraints include: \( \sum_{k=1}^{3} w_k \cdot \Delta_{56} > 0 \), \( \sum_{k=1}^{3} w_k \cdot \Delta_{a5} > 0 \), \( \sum_{k=1}^{3} w_k \cdot \Delta_{a6} > 0 \). As it is clear, there exist 5 constraints and other constraints are out-spaced.
Conceptually, the features of the compatible weights domain can be represented as in Figure 6.2. The central white convex polytope is the result of the constraints satisfaction of all related constraints in the instance.

In this contribution, it is proposed to project this area on the GAIA plane as shown on Figure 6.3.

The green surrounded area, which is the result of the projection of central polytope in Figure 6.2 on the GAIA plane, is known as GAIA brain.

In this illustrative example, the $\delta$ value is equal to 90%. It means the amount of preserved information is highly significant. The dispersion of all criteria in different direction shows the presence of conflicts. Their lengths are long enough, thus all criteria are important in differentiation between alternatives. Alternatives $a_5$ and $a_6$ seem to be ranked before other alternatives since they are in the same direction with the GAIA brain. Alternative $a_5$ acts well on...
the C₁ and alternative a₆ is quite good on the C₂. Alternative a₃ acts good on the criteria C₃ but not on other criteria.

The reasonable size of the projected polyhedron in this plane indicates that few weight values are compatible with the DEA analysis. Moreover, it shows that all DEA compatible alternatives always lie in the right top part of the graph. This eventually leads to hesitations between a₄, a₅ and a₆ but not with other options. A detailed example is provided in the next section.

More formally, the intersection of constraints should be found by solving a system of linear equations. There exist different algorithms to solve such a system and find all its vertices. One of the interesting algorithms is the VEA (Avis and Fukuda, 1992) that is summarized in the next sub-section.

6.2.1. Vertex Enumeration Algorithm

A convex polyhedron is defined as: \( P = \{ x \in R^q: Ax \leq b \} \) such that \( A = (a_{ij}) \) is a given matrix, \( i = 1,2, ..., m \) and \( j = 1,2, ..., q \), and \( b \) is an \( m \) dimensional vector. Polyhedron \( P \) may be empty (\( P=\emptyset \)) or unbounded (e.g. \( P = \{ x \in R^q: x \geq 0 \} \)). A bounded polyhedron is a polytope. It is normally assumed a full dimensional situation for a polytope; i.e., there exists an interior point \( x \) that strictly satisfies all inequalities of \( P \). A vertex (\( v \)) of a polyhedron is a \( q \) dimensional point belongs to \( P \) if and only if it is a unique solution to a subset of \( q \) inequalities solved as equalities. (Avis and Fukuda, 1992). The vertex enumeration problem is to generate all vertices of a polytope \( P \).

Matheiss and Rubin (1980) and Dyer (1983) first investigated the problem of enumerating all vertices of a polyhedron. There are two classes of methods: the first class is the "double description" method of Motzkin et al. (1953), and the second class includes methods built on pivoting (Dyer, 1983). Avis and Fukuda (1992) presented a pivot-based algorithm to find \( v \) vertices of a polyhedron in \( R^q \) defined by a non-degenerate system of \( m \) inequalities. A vertex lying on more than \( q \) bounding hyperplanes in \( R^q \) is called degenerate. In the algorithm presented in this chapter, each vertex stays on exactly \( q \) hyperplanes. The related path to find each vertex is found by pivoting. It means interchanging one of the equations defining a vertex with one not currently used. Details about this method can be seen in (Avis and Fukuda, 1992). The basic idea is based on following steps:

1- Find any starting point as a vertex \( v \);
2- Create edges from \( v \) subsequently;
3- Move to any unsearched vertex adjacent to \( v \); If none back to previous step.

The definition of dictionaries is important in related computations. A new non-negative variable, for each inequality, is added to \( P \) to make a dictionary:

\[
x_{n+i} = b_i - \sum_{j=1}^{q} a_{ij}x_j, x_{n+i} \geq 0, i = 1,2, ..., m \quad (6.5)
\]
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The added variables are called slack variables and the original variables are decision variables. The variables on the left hand side of a dictionary are called basic: $B = \{ i: x_i \text{ is basic} \}$ and on the right hand side are called co-basic: $N = \{ j: x_j \text{ is co-basic} \}$. A pivot is interchanging an index between $B$ and $N$. This pivoting solves equations for the basic variables. By putting $x_j = 0$ for all $j \in N$, a basic solution of dictionary is obtained. A basic solution is feasible if all slack variables are non-negative. This Basic Feasible Solution (BFS) is a vertex. Thus, to find a starting point, a BFS should be found. The decision variables should be changed with left hand side variables by letting slacks to leave system (pivoting).

Next example simply presents a pivoting system to find the vertices of a polyhedron.

**Example 6.2** (Avis, 2010)- Polyhedron $P$ is as follows:

\[ x_1 - x_3 \leq 1 \]
\[ x_2 - x_3 \leq 1 \]
\[ -x_1 - x_3 \leq 1 \]
\[ -x_2 - x_3 \leq 1 \]
\[ x_3 \leq 0 \]

$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

An initial dictionary is:

\[ x_4 = 1 - x_1 + x_3 \geq 0 \]
\[ x_5 = 1 - x_2 + x_3 \geq 0 \]
\[ x_6 = 1 + x_1 + x_3 \geq 0 \]
\[ x_7 = 1 + x_2 + x_3 \geq 0 \]
\[ x_8 = -x_3 \geq 0 \]

In this problem, $B$ is \{4,5,6,7,8\}, $N$ is \{1,2,3\}, and the BFS, by setting $x_j = 0$ (the set $N$), is $x = (-1,-1,0,2,2,0,0,0)$ which shows the starting vertex $v = (-1,-1,0)$. Pivoting from $N = \{1,2,3\}$ to $N = \{1,3,7\}$ results:

\[
\begin{cases}
  x_4 = 1 - x_1 + x_3 \geq 0 \\
  x_5 = 1 - x_2 + x_3 \geq 0 \\
  x_6 = 1 + x_1 + x_3 \geq 0 \\
  x_7 = 1 + x_2 + x_3 \iff x_2 = -1 + x_7 - x_3 \geq 0 \\
  x_8 = -x_3 \geq 0 
\end{cases}
\]

It gives the vertex $v = (-1,1,0)$. Continuing pivoting between decision variables in the right hand side and the basic variables in left hand side generates all vertices of polyhedron $P$: $(0,0,-1), (-1,-1,0), (-1,1,0), (1,1,0)$, and $(1,-1,0)$.

There exist different software packages and programs to solve this problem. In this work, an extension of a MATLAB code is used (Kelder, 2005) to enumerate all vertices of a polyhedron while $m \geq q$ ($m$ constraints, $q$ variables). This program converts the polyhedron (convex polygon, polytope, etc.) defined by the system of inequalities $Ax \leq b$ and equalities into a list of vertices $v$. 

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The feasible region should have some finite extent in all dimensions. It also identifies the list of redundant constraints. Further, it finds the redundant vertices in dimensions higher than 2, detect their redundancies at up to 6 digits of precision and finally returns the unique vertices.

Another problem that may arise is the existence of unbounded constraints. Each inequality outlining the polyhedron \( P \) is bounded by a hyperplane to make a polytope. When these hyperplanes do not limit entirely the polyhedron (polyhedron \( P \) is unbounded), the program returns an error. In this case, to induce bounding and continue to solve the problem, bounds should be determined. This problem may be detected by defining large box constraints on the variables. For example, if \( A = [0 \ 1 \ 1 \ 0 \ 1 \ 1] \) and \( b = [1 \ 1 \ 1] \), the MATLAB code (Kelder, 2005) returns an error for unboundedness. \( A \) and \( b \) are defined as: \( A = [A; 0 \ -1 \ 0 \ 1 \ -1 \ 0 \ 1 \ 0] \) and \( b = [b; 2\ 1000; 2\ 1000] \). Actually, the bounding box of constraints on variables are determined within \([-1, 1000]\). Indeed, the constraints boxes are determined as large that do not modify tangibly the space as the definition of problem changes, but give bounds to variables. As a toy sample, if \( x \) is a tree’s height in meter, the box constraint \([-1, 1000]\) for \( x \) can be a reasonable choice to create boundedness, since all possible solutions for \( x \) are possible within this box.

The generated intersection points are the compatible weights between DEA analysis and PROMETHEE II ranking. In this step, these points are projected on the GAIA plane and the related convex hull is determined.

**Example 6.3** In order to illustrate the proposed method, the problem of ranking 12 hypermarkets in Belgium is considered according to five criteria. The data set is drawn from Brans and Mareschal (2002).

Table 6.2 performs the alternatives, the considered criteria, the evaluations as well as preference functions and parameters. In the first step, the criteria are partitioned into two groups. Criteria include: construction cost (\( C_1 \)) and competition (\( C_2 \)) [minimization] and population (\( C_3 \)), parking availability (\( C_4 \)), and network access (\( C_5 \)) [maximization]. The minimized and the maximized criteria are supposed to be the inputs and the outputs in DEA, respectively. As it has been explained earlier, in this work, the preference functions are limited to the linear type. The indifference (\( q \)) and preference (\( p \)) thresholds are determined by DMs according to their behavior. In the absence of clear distinction among the importance of the criteria, the weights are chosen equally; however, in this algorithm, the weights values are not required for computing the unicriterion net flow scores.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
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<tr>
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<td>Max</td>
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<td>( q=0, p=75 )</td>
<td>( q=0, p=225 )</td>
<td>( q=1, p=2 )</td>
</tr>
<tr>
<td>Weights</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
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<table>
<thead>
<tr>
<th>Alts/DMUs</th>
<th>Score</th>
<th>Super-efficient rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>21</td>
<td>a11</td>
</tr>
<tr>
<td>a2</td>
<td>21.3</td>
<td>a11</td>
</tr>
<tr>
<td>a3</td>
<td>8.2</td>
<td>a4</td>
</tr>
<tr>
<td>a4</td>
<td>6.6</td>
<td>a5</td>
</tr>
<tr>
<td>a5</td>
<td>4.9</td>
<td>a3</td>
</tr>
<tr>
<td>a6</td>
<td>21.3</td>
<td>a12</td>
</tr>
<tr>
<td>a7</td>
<td>17.9</td>
<td>a6</td>
</tr>
<tr>
<td>a8</td>
<td>17.3</td>
<td>a4</td>
</tr>
<tr>
<td>a9</td>
<td>14.2</td>
<td>a9</td>
</tr>
<tr>
<td>a10</td>
<td>10.4</td>
<td>0.987</td>
</tr>
<tr>
<td>a11</td>
<td>12.9</td>
<td>a12</td>
</tr>
<tr>
<td>a12</td>
<td>9.6</td>
<td>0.654</td>
</tr>
</tbody>
</table>

Table 6.2 Evaluation table and preference parameters, based on Brans and Mareschal (2002)

In the second step of the algorithm, the DEA-CCR model (1.8) is run in data set of Table 6.2 to analyse the multicriteria decision-making problem. Further, the super efficiency model (5.4) may be used to rank completely all units. Table 6.3 shows the DEA results:

<table>
<thead>
<tr>
<th>Classic rank</th>
<th>Score</th>
<th>Super-efficient rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>a11</td>
<td>1</td>
<td>a11</td>
</tr>
<tr>
<td>a12</td>
<td>1</td>
<td>a4</td>
</tr>
<tr>
<td>a5</td>
<td>1</td>
<td>a5</td>
</tr>
<tr>
<td>a6</td>
<td>1</td>
<td>a3</td>
</tr>
<tr>
<td>a4</td>
<td>1</td>
<td>a12</td>
</tr>
<tr>
<td>a3</td>
<td>1</td>
<td>a2</td>
</tr>
<tr>
<td>a2</td>
<td>1</td>
<td>a6</td>
</tr>
<tr>
<td>a9</td>
<td>1</td>
<td>a9</td>
</tr>
<tr>
<td>a7</td>
<td>0.987</td>
<td>a7</td>
</tr>
<tr>
<td>a8</td>
<td>0.884</td>
<td>a8</td>
</tr>
<tr>
<td>a1</td>
<td>0.865</td>
<td>a1</td>
</tr>
<tr>
<td>a10</td>
<td>0.654</td>
<td>a10</td>
</tr>
</tbody>
</table>

Table 6.3 DEA ranking

As result of DEA analysis in Table 6.3 presents, DMUs are divided into two groups: efficient and inefficient. Later in the next step, this rank order among different units is used to generate weight constraints on the PROMETHEE II ranking.

In step 3, the PROMETHEE unicriterion net flow scores \( \emptyset_k(a_j) \) (2.40) are computed for all criteria and alternatives. Table 6.4 presents the unicriterion net flow scores matrix.
Table 6.4 Unicriterion net flow scores matrix in Example 6.3

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1</td>
<td>-0.815</td>
<td>0.576</td>
<td>0.121</td>
<td>-0.701</td>
<td>-0.455</td>
</tr>
<tr>
<td>a2</td>
<td>-0.818</td>
<td>0.818</td>
<td>0.286</td>
<td>-0.656</td>
<td>-0.455</td>
</tr>
<tr>
<td>a3</td>
<td>0.596</td>
<td>0.333</td>
<td>-0.505</td>
<td>0.397</td>
<td>0.636</td>
</tr>
<tr>
<td>a4</td>
<td>0.808</td>
<td>0.576</td>
<td>-0.869</td>
<td>-0.045</td>
<td>-0.182</td>
</tr>
<tr>
<td>a5</td>
<td>0.949</td>
<td>0</td>
<td>-0.835</td>
<td>0.897</td>
<td>0.455</td>
</tr>
<tr>
<td>a6</td>
<td>-0.818</td>
<td>-0.606</td>
<td>1</td>
<td>0.36</td>
<td>-0.182</td>
</tr>
<tr>
<td>a7</td>
<td>-0.37</td>
<td>-0.606</td>
<td>0.495</td>
<td>0.13</td>
<td>-0.455</td>
</tr>
<tr>
<td>a8</td>
<td>-0.03</td>
<td>-0.818</td>
<td>0.553</td>
<td>-0.162</td>
<td>0.455</td>
</tr>
<tr>
<td>a9</td>
<td>0.364</td>
<td>0</td>
<td>-0.699</td>
<td>-0.216</td>
<td>0.455</td>
</tr>
<tr>
<td>a10</td>
<td>0.051</td>
<td>0.333</td>
<td>-0.139</td>
<td>0.147</td>
<td>0.636</td>
</tr>
<tr>
<td>a11</td>
<td>0.441</td>
<td>0</td>
<td>-0.224</td>
<td>0.848</td>
<td>-0.455</td>
</tr>
</tbody>
</table>

As explained in the third step of the algorithm, the ranking computed in step 2 (Table 6.3) generates weight constraints on the PROMETHEE II ranking. Generally, it can be said that the number of constraints in considering the complete ranking via super efficiency model (5.4) is:

\[ M = c(n, 2) \]  

The value of \( c(n, 2) \) represents the binomial coefficient. Thus, the number of possible resulted hyperplanes (constraints) is: \( M = c(12, 2) = 66 \). Between these constraints, the positive ones are considered in constructing the central polytope (GAIA brain). Furthermore, the normalization equation is also added to the system (\( \sum_{k=1}^{5} w_k = 1 \)).

In a classical DEA model (column 2 in Table 6.3), the number of constraints is computed as follows:

\[ M' = c(n, 2) - c(N(\text{efficiency} = 1), 2) \]  

When \( n \) is the number of alternatives/DMUs and \( N(\text{efficiency} = 1) \) is the number of equal efficient DMUs. Literally, the value of \( M' \) shows the number of absolute preferences between each pair of actions in DEA (not considering preferences between equal efficient units). Hence, \( M' = c(12, 2) - c(8, 2) = 38 \).

In this example, all possible constraints are studied: \( M + 1 = 67 \) (the case of super-efficient rank) to take into consideration all aspects of problem in generating GAIA brain. In applying (6.7), just the number of constraints is reduced. As a result, the size of GAIA brain is also reduced slightly. As mentioned, the priority in PROMETHEE II rankings should be compatible with the DEA analysis. For instance, in Table 6.3, the super-efficient rank induces the following constraints:

\[ \sum_{k=1}^{5} [\emptyset_k(a_{11}) - \emptyset_k(a_4)].w_k > 0, \quad \sum_{k=1}^{5} [\emptyset_k(a_{11}) - \emptyset_k(a_5)].w_k > 0, \quad \sum_{k=1}^{5} [\emptyset_k(a_5) - \emptyset_k(a_3)].w_k > 0, \quad \sum_{k=1}^{5} [\emptyset_k(a_9) - \emptyset_k(a_1)].w_k > 0, \quad \sum_{k=1}^{5} [\emptyset_k(a_1) - \emptyset_k(a_{10})].w_k > 0. \]

According to linear system written in (6.4), the related constraints are as follows:
This system, which includes inequalities and one equality, considers the existing compatibility between DEA analysis and PROMETHEE II ranking in problem of ranking hypermarkets with the aim of generating the GAIA brain (Step 4).

The VEA, as explained in Section 6.2.1, is used to solve this system. In this example, the problem is bounded; thus, it is not necessary to enter into the discussion of unboundedness. In the case of encountering the unboundedness, as already explained, it should be defined enough large constraints boxes on the variable $w_k$.

Finally, in step 5, the computed weight vectors via VEA system are projected on the GAIA plane and the related convex hull is determined as illustrated in Figure 6.4.

Figure 6.4 GAIA Plane of Numerical example (The screen shot from MATLAB)

The $\delta$ value is 82%. It shows that the amount of lost information is acceptable. Moreover, it can be seen that the GAIA brain has a reasonable size and does not encompass the centre of the
graph. This brain illustrates the space of freedom for DM. Clearly, it indicates that the set of alternatives can be split into two subsets. It is unlikely that, given these weights values, alternatives \( a_1, a_2, a_6, a_7, a_8, \) and \( a_9 \) are chosen. Among them, it seems that \( a_1 \) and \( a_2 \) are not good enough on \( C_1, C_4, \) and \( C_5; \) however, they act quiet well on \( C_2 \) and \( C_3. \) Alternatives \( a_6, a_7, a_8, \) and \( a_9 \) perform very well on population \( (C_3) \) [the hypermarkets in Brussels]. The other alternatives show a higher degree of compatibility with the DEA ranking as presented in Table 6.4 (alternatives \( a_{11}, a_{12}, a_5 \)). Almost all alternatives \( a_3, a_4, a_5, a_{10}, a_{11}, \) and \( a_{12} \) performs well on \( C_3. \) Alternative \( a_{12}, \) relatively, behaves well enough on \( C_4 \) (it has reasonable access to parking place). This alternative does not act well on competition and population \( (C_2 \) and \( C_3). \) It should be paid attention that however, in GAIA plane, \( a_{12} \) is close enough to \( C_5 \) but, in fact, it does not act well in this criterion according to Tables 6.2 and 6.3 (because of losing information in projection process). Some other alternatives such as \( a_3, a_5, \) and \( a_{11} \) appear well in criterion \( C_5. \) The dispersion of criteria on the GAIA plane shows us three different directions. Obviously, \( C_1, C_4, \) and \( C_5 \) act more similar than other criteria.

The last step of algorithm proposes a weight vector. Solving the system (6.4), when inequalities considered as equalities, gives also a matrix of weights (the possible vertices). There exist different methods in computing more homogenous weights from a given weight matrix. Taking average from appropriate criteria weights (inputs and outputs weights) is one of the proposed methods. Here, a geometric mean (Borowski and Borwein, 1989) can be applied to compute a common weight vector for all criteria (Jablonsky, 2012). The use of a geometric mean normalizes the ranges being averaged. Hence, no range dominates the weighting, and a given rate change in any of the properties has the same effect on the geometric mean. In comparison with the arithmetic mean, the geometric mean is said to be “not overly influenced by the very large values in a skewed distribution” (Kirkwood and Sterne, 2003). The idea of using geometric mean is come from the Common Set of Weights (CSW) (Cook et al., 1990; Roll et al, 1991; Roll and Gollany, 1993) problem in DEA. CSW has been suggested to reduce the flexibility in choosing inputs and outputs weights in evaluating the efficiency of units. The use of CSW makes possible ranking different units on the same basis and a homogenous set of weights (Wang et al., 2011). Therefore, the geometric mean can be applied to make a connection between DEA and PROMETHEE by generating a CSW. It decreases the bias of criteria weight vectors (Jablonsky, 2012).

The vector of weights resulted by geometric mean can be suggested to DMs as a first proposal in parameterizing PROMETHEE; further the area of GAIA brain is the area of DM’s freedom in changing weights.

The result of constraints satisfaction in (6.8) gives a weight matrix \( W = [w_{kj}] \) where \( k = 1, \ldots, q \) is the number of criteria (inputs + outputs) and \( j = 1, \ldots, n \) is the number of constraints in the feasible space \( (n \leq M \text{ or } M'). \) This matrix is demonstrated in Table A5 in Appendix 1. The geometric mean of each column of this matrix is computed as follows (Borowski and Borwein, 1989):
Applying (6.9) on the weight matrix presented in Table A5 gives following weight vector: 

\[ W = (0.2930, 0.2474, 0.0666, 0.1615, 0.1760) \]

This weight vector is normalized to 1 by dividing each weight to sum of all weights: 

\[ (0.3102, 0.2620, 0.0705, 0.1710, 0.1863) \]

Thus, \( w_1 = 31.02 \% \), \( w_2 = 26.20 \% \), \( w_3 = 7.05 \% \), \( w_4 = 17.10 \% \) and \( w_5 = 18.63 \% \). This weight vector can be communicated to DM. She/he can start from these preferences to parametrize the decision-making problem in PROMETHEE.

6.2.2. Advantages and limitations

The main limitation in the proposed algorithm can be the computational difficulties in large-scale problems. With increasing the number of criteria and alternatives, naturally, problem size is increased (the number of constraints is increased as well). Consecutively, the possibility of infeasibility is augmented. However, it has not been studied on the limitation of the problem size. It can be a future research finding an \( n \), which for the value greater than \( n \) the problem becomes infeasible. Some other limitations are listed as follows:

1- Possibility of infeasibility in the DEA analysis. However, infeasibility is a rare option in the CCR classical model.

2- Using the CCR classical model does not ensure the reasonable weight values. The full flexibility in weights may allow the CCR model assigns zero (or very low) values to some inputs and/or outputs. Therefore, the first classification of DEA (the 2nd step of algorithm) may be not enough attentive. As discussed in Chapter 4, this limitation can be removed by using a weight restricted DEA model instead of the classical CCR model. It is a light of future researches. Moreover, adding constraints to weights may increase the discrimination power of DEA between efficient units.

3- Possibility of having unbounded constraints in LP (6.4). It can be solved in programming level by defining the large box constraints on them. When these boxes are as large that they do not change the definition of the problem, make LP bounded.

Despite the above limitations, this algorithm is the first one proposing the weight vector in PROMETHEE, based on DEA. It gives the first light to DMs in choosing importance of criteria in the complex decision-making problems when there is no clear distinction among criteria.

6.3. Conclusion

In this contribution, a tool is proposed to help the DM in determination of weights for PROMETHEE VI. The underlying idea consists to identify possible weight values that are compatible with the DEA analysis. When the projected area is small and outlying, it allows to quickly identifying alternatives that are compatible with DEA and those, which are not. It is
worth noting that these values are proposed to the DM as a first suggestion to parameterize PROMETHEE VI. Later, she/he remains free to refine them.

Among the future research directions, the integration of this model in a PROMETHEE based software like D-Sight can be proposed. Naturally, other investigations can be conducted on the use of DEA to elicit weights in the PROMETHEE method. One important future light for this investigation is taking into account the limitation of DEA in weight freedom. The flexibility of weight values may assign extremely low (near to zero) weights to some unfavorable input/output factors of a DMU under evaluation. This causes excluding these factors from the evaluation process, which is not desirable. Alternatively, a DMU may become efficient for just a single ratio of an output to an input, while ignoring other inputs and outputs. It is also not favorable. Consequently, the classical DEA models with unbounded weights may generate some unlikely results. The use of weight restricted DEA models helps solve these issues. Furthermore, in some cases, in order to evaluate the performance of DMUs, specific outputs should be related with specific inputs and it is expected to find a link between the weights assigned to these factors. For this aim, using some types of weight restrictions in DEA, e.g. ARs can be practical. Evaluating the efficiency of perinatal care units in the U.K is such a case (Thanassoulis et al., 1995).
General Conclusion

General conclusion

In this thesis, the possibility of integration between Data Envelopment Analysis (DEA) and Multi-Criteria Decision Aid (MCDA) has been investigated. For years, researchers have discussed the similarities, with regards to the mathematical structure. They combined DEA and MCDA with the purpose of helping Decision Makers (DMs) to solve decision-making problems (Belton and Vickers, 1993; Stewart, 1996; Belton and Stewart, 1999 and 2002; Bouyssou, 1999; Nakayama et al., 2002). Different methods in the MCDA family such as AHP have been used with DEA (Shang et Sueyoshi, 1995; Zhu, 1996; Sinuany-Stern et al., 2000; Premachandra, 2001; Entani et al., 2004; Li and Ma, 2008; Ho and Oh, 2010; Jablonsky, 2007 and 2012; Fulop and Markovits, 2012; Pakkar, 2014-2016). In the present thesis, for the first time, PROMETHEE as a MCDA method has been integrated with DEA through two synergies: 1- applying PROMETHEE in DEA, and 2- applying DEA in PROMETHEE.

As Belton and Stewart (1999 and 2002) stressed, DEA and MCDA can be competing. However, they are complementary methods also. The highlight of their differences is the concept of objectivity and subjectivity. While DEA is a method to extract information, as much as possible, from objective historical data, MCDA tries to understand and manage value judgements (Belton and Stewart, 1999). DEA is suggested as a method for monitoring and recognizing the group of efficient units, whereas MCDA is most appropriate for ranking and choosing perspectives. Nevertheless, there are many applications, which pass these differences and apply DEA and MCDA as complementary methods. There is a similarity between the classifying issue in DEA and ranking problems in MCDA. Furthermore, using value judgements and a priori information of DMs in DEA are milestones in the connection between DEA and MCDA.

The standing point to investigate the synergies between DEA and MCDA is based on two actual views: 1- Seeing the Decision Making Units (DMUs) as a set of alternatives to be evaluated, and 2- Seeing the set of inputs and outputs as multiple criteria to be collected. In this manner, DEA is considered as a MCDA tool to support DMs in eliciting some preferences such as weights of criteria. As well, MCDA is used in DEA; e.g., the outputs of MCDA can be used as a tool to control the flexibility of weights in DEA (Bagherikahvarin and De Smet, 2016§). Therefore, it is possible using DEA and MCDA in parallel to solve multicriteria decision-making problems.

There are a number of applications from DEA and MCDA integration. Each one of them is applied for a particular purpose. As mentioned earlier, ranking is one of the possible interactions between these two methods. However, mostly, DEA classical models are unable to fully rank units. DEA divides units into two groups of efficient and inefficient when there is no discrimination among efficient units. As discussed in Chapter 4, one of the main reasons for this disability of
DEA is the weights freedom. During years, researchers proposed weight restricted DEA models to solve this issue. Different MCDA methods have been used to restrict weights in DEA, such as AHP and MACBETH (Shang and Sueyoshi, 1995; Zhu, 1996; Papagapiou et al., 1997; Sarrico et al., 1997; Premachandra, 2001; Takamura and Tone, 2003; Entani et al., 2004; Junior, 2008; Pakkar, 2014-2016). In this thesis, for the first time, one employs PROMETHEE to restrict freedom of weights in DEA. Some pros and cons of using PROMETHEE compared with AHP are discussed in Chapter 4.

The new weight restricted DEA model (PIIWCCR) engages two outputs of PROMETHEE in DEA: 1- the weight stability intervals and 2- the unicriterion net flow scores. Using weight stability intervals of PROMETHEE as weight restrictions in DEA causes the increase in the DEA discrimination power, and so the decrease in the number of efficient DMUs. Furthermore, the unicriterion net flow scores of PROMETHEE replace the output of DEA. These make possible to partly integrate the preference information of DMs in an extended DEA model. The PIIWCCR model is a response to the first synergy: using PROMETHEE in DEA.

To fully rank units in DEA, several techniques have been proposed (Adler et al., 2002; Hosseinzadeh Lotfi et al., 2013). In Chapter 5, a new ranking technique is developed. This technique profits of the advantages of both DEA and PROMETHEE to rank units. It uses DEA to compare each pair of units. The results of the pairwise comparisons is kept in a matrix to be used by means of PROMETHEE to compute the final scores and ranking of units. As discussed, classical DEA models are not appropriate to rank units, while the new technique provides a complete ranking. Furthermore, when there is not enough a priori information, and/or when eliciting preferences in complicated decision-making problems is not an easy task, DMs may encounter some problems. To face these issues, DMs need some tools. As several researchers discussed, DEA can be such a tool (Thompson et al., 1986; Golany, 1988; Cook and Kress, 1990; Cook et al., 1990, 1992 and 1993; Belton, 1992; Belton and Vickers, 1993; Doyle and Green, 1993; Stewart, 1994 and 1996; Shang et Sueyoshi, 1995; Sinuany-Stern et al., 2000; Ho and Oh, 2010; Jie, et al., 2010; Fulop and Markovits, 2012). In this DEA-PROMETHEE algorithm, DEA is used to generate a pairwise comparison matrix without applying a priori information of DMs in eliciting the parameters in PROMETHEE. From the perspective of a DM, the less-subjective approaches can be more applicable (Stewart, 1994; Sinuany-Stern et al., 2000). This new algorithm serves both synergies: using DEA in PROMETHEE and using PROMETHEE in DEA.

One issue of PROMETHEE in comparison with an MCDA method like AHP is the lack of a tool to compute weights of criteria (Mahmoud and Garcia, 2000; Macharis et al., 2004; Turcksin et al., 2011; Balali et al., 2014; Mursanto and Halim, 2014). Chapter 6 suggests a new technique to propose weights in PROMETHEE. This technique is based on DEA. The underlying idea is to detect possible weight values, which are compatible with a DEA analysis. The ranking (classifying) induced by DEA constraints the weights used in building an extension of GAIA brain. Indeed, the rank of alternatives in PROMETHEE II is forced to be the same as DEA analysis. If
alternative/DMU $a_i$ has better position in comparison with $a_j$, weight values of the PROMETHEE II ranking, in computing the net flow scores, are restricted such that $a_i$ has better position than $a_j$. For this aim, a linear system should be solved, including defined constraints compatible by the DEA result. These constraints define a polyhedron of PROMETHEE weights values. Then, the polyhedron is projected on the GAIA plane to make an extension of the GAIA brain. This brain is not the same as GAIA brain in PROMETHEE VI; nevertheless, it may provide some similar information as GAIA brain: it can be an evidence of alternatives importance when they are lying in the same direction with this brain. Finally, the weight values computed in this way are communicated to DM as a first idea about importance of criteria. Then, she/he can modify these values in order to better present her/his preferences. This algorithm is the first effort in proposing weights to PROMETHEE. It lies in the second synergy: using DEA in PROMETHEE.

In this thesis, one investigates for the first time the possibility of integration between DEA and PROMETHEE. This would ultimately lead to further perspectives and future developments.

**Perspectives**

In this section, several tracks for further research in this domain are considered. Among the many possibilities, the following are described:

- Rank reversal;
- Infeasibility of DEA models;
- Developing new ranking algorithms: weight restricted DEA models based on PROMETHEE.

**Rank reversal**

Rank reversal is a common problem in DEA and PROMETHEE. It can be considered as the third synergy in this domain and discussed in the future researches.

As defined in Definition 2.9, rank reversal is a phenomenon that occurs in the ranking order of alternatives when for example, the set of alternatives/criteria and/or the method of ranking change (Belton and Gear, 1983). It means that the relative ranking of two decision alternatives could be reversed when an alternative is added or deleted. The same issue can happen by adding or deleting a criterion. Such a phenomenon was first noticed and discussed in AHP by Belton and Gear (1983). Since then, several researchers studied this issue in different decision-making methods such as PROMETHEE (Keyser and Peeters, 1996; Verly and De Smet, 2013) and DEA (Wang and Luo, 2009; Soltanifar and Shahghobadi, 2014).

With regards to the concept of rank reversal, some research panels can be opened in the proposed algorithm of Chapter 4 (PIIWCCR):

- Considering the ranking order of alternatives after adding a new alternative or a copy of existing alternative to the set of alternatives;
General Conclusion

- Considering the ranking order of alternatives after adding/deleting the best/worst alternative;
- Considering the ranking order of alternatives after adding/deleting a discriminant/ non-discriminant criterion.

Additionally, rank reversal phenomenon can be studied in the DEA-PROMETHEE II ranking algorithm proposed in Chapter 5. It is already discussed the absence of rank reversal in the presence of a single input and a single output. However, rank reversal might be possible in the presence of multiple inputs and multiple outputs.

The first step in studying the behavior of rank reversal in the mentioned algorithms is conducting empirical tests based on the artificial data sets. Then, one possible future discussion would be providing the conditions to avoid rank reversal.

**Infeasibility of DEA models**

An LP is said to be “infeasible” if no solution exists, which satisfies all the constraints. DEA as a specific type of LP can be infeasible. However, infeasibility is a rare option in the CCR classical models with CRS. The source of infeasibility is often not clear. It may arise from some of the constraints in the model, from some not well-defined/wrong numbers in the data set, from a combination of variables, or etc. Actually, bounding variables may cause infeasibility in DEA. As several researchers studied, weight restricted DEA models can be infeasible, since adding constraints decreases the space of feasible solutions (Podinovski and Athanassopoulos, 1998; Sarrico and Dyson, 2004; Podinovski, 1999-2005; Estellitas Lins et al., 2007; Podinovski and Bouzdine-Chameeva, 2013 and 2015).

The model PIIWCCR (proposed in Chapter 4), as a weight restricted DEA model, can also be infeasible depends on the data set (e.g. size of data set, some not well-defined data), specifically when the number of constraints is increased (Estellitas Lins et al., 2007). Further, replacing CCR by BCC model may increase the possibility of infeasibility. One prospect of future research is to restructure the proposed model and/or to provide some conditions to avoid infeasibility.

**Developing new ranking algorithms: weight restricted DEA models based on PROMETHEE**

As briefly presented in Chapter 4, there are several techniques used to impose weight restrictions into CCR-I-O multiplier model. This has the purpose of using the preferences and value judgements of DMs, and increasing the discrimination power of DEA. Allen et al (1997) categorized these techniques in three major groups. In this thesis, the absolute weight restrictions are used to limit weights in DEA. The absolute weight restrictions can be replaced by several other weight restriction techniques, such as ARs types I and II or virtual weight restrictions. Moreover, depends on the context of the decision-making problem, the constraints can limit just specific variables or the ratio of specific variables instead of all variables. In DEA, in some cases, specific
outputs are directly dependent on specific inputs (or other outputs). It is expected to find a link between the weights assigned to these factors.

Besides the stressed perspectives, some other future researches can be conducted with regards to this thesis. In the 2nd contribution (Chapter 5), cross efficiency technique is used for pairwise comparisons between actions. The cross efficiency can be replaced by some other techniques used in DEA to avoid the problems of cross efficiency (some problems are discussed in Chapter 5: Section 5.3.1). Further, in each pairwise comparison, the incomparability between actions can be considered, since it is a main feature in PROMETHEE method.

In the 3rd contribution (Chapter 6), a weight restricted DEA model can be proposed to replace CCR classical model: in order to avoid undesirable weight values and engaging a priori information of DMs into model.
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Appendix 1 (Tables)
### Appendix 1

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Table A2. The weight matrix resulted by model PIIWCCR when \( r=1 \), for localization of the solid waste management system
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Table A4. The weight matrix resulted by model PIHWCCR when $(W_k + 5.6\%, W_k - 5.6\%)$, for localization of the solid waste management system.
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Table A5. The weight matrix resulted by system (6.8) in Example 6.3
Appendix 2 (Core thesis publication reprints)
A ranking method based on DEA and PROMETHEE II (a rank based on DEA & PR.II)

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**Article info**

*Article history:*

Received 13 February 2015

Received in revised form 6 April 2016

Accepted 12 April 2016

Available online 13 April 2016

**Keywords:**

Data envelopment analysis Multiple criteria decision aid PROMETHEE

Weight Stability Intervals Weight restrictions

**Abstract**

In this paper we present an integrated Data Envelopment Analysis-Multiple Criteria Decision Aid (DEA-MCDA) model which can be applied to increase the discrimination power of DEA. The aim is to restrict weight values of a DEA model by using tools from MCDA. This model leads to more reasonable inputs/outputs weights while in classic DEA models, some inputs/outputs may be characterized by very low or high weight values. To achieve this goal we use the stability intervals based on PROMETHEE II (Preference Ranking Organization METHod for Enrichment of Evaluations) as weight constraints in DEA. Furthermore, the unicriterion net flow scores matrix is used instead of the initial evaluation matrix. By doing so, we already integrate preferential information in the DEA process. By construction, the best results are compatible with the PROMETHEE II ranking. Additional comparisons with the outputs of other decision making techniques are provided based on two examples.

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1. Introduction

Data Envelopment Analysis (DEA) is a subfield of Operations Research (OR) and Management Sciences (MS) that has been developed to empirically measure the relative efficiency of Decision Making Units (DMUs). The full weight flexibility is one of the main distinctive features of the original multiplier DEA models where flexibility in choosing the endogenous weights puts the evaluated unit in the best possible light. In some cases, this flexibility causes two main problems: weakness in discrimination power and unreasonable weight assignments. When the number of DMUs is not large enough in comparison with the total number of inputs and outputs, many DMUs are identified as efficient; thus the discrimination power of DEA is decreased. The second problem leads to the situation where some DMUs can be evaluated as efficient just because of some extremely low/high weight values of their inputs or outputs. These values may be senseless or undesirable. Furthermore, the assigned input and output weights may be in contradiction with a priori knowledge offered by the Decision Maker (DM). During the last 3 decades, researchers have proposed different weight restriction techniques to address these problems and increase the discrimination power of DEA [1, 2]. In this regard, we refer the interested reader to [3-15].

Thompson et al. [3] were among the first researchers to suggest the use of weight restrictions in DEA. They introduced Assurance Region (AR) methodology to tackle the problem of assessing the efficiency of a group of physics laboratory. Dyson and Thanassoulis [5] attempted to eliminate the use of zero weights by applying regression analysis. Wong and Beasley [6] proposed the so-called virtual weights model when imposing weight restrictions in DEA based on lower and/or upper bounds. Unfortunately, the large number of restrictions that should be imposed on the weights can increase the likelihood of infeasibility (in the optimization problem) [16]. Roll and Golany [7] run a basic DEA model to use its generated weights in determining bounds. Dimitrov and Sutton [8] proposed a symmetric weight assignment technique. Hatami-Marbini et al. [13] extended another symmetric weight assignment technique by applying dual weight restrictions. In this regard, they proposed four new DEA models. Saati et al. [14] developed a two-phase algorithm based on a common weight restriction method. In the first phase, they created an ideal DMU that is a DMU consuming the least inputs to generate the most outputs. In the second phase, they applied the common weight restriction method to calculate the efficiency measures of DMUs. Lotfi et al. [15] used a common weight restriction method for allocating fixed resources and setting targets.

From our point of view, restricting weights values in DEA models is a first step to integrate the preferences of the DM in the evaluation process. Several researchers have investigated the combination of Analytic Hierarchy Process (AHP) [17] and DEA (e.g. [9-11]). Takamura and Tone [9] integrated an AR-DEA model with AHP to manage the problem of relocating Japanese government agencies out of Tokyo. They used the judgments of people who are familiar with the characteristics of the evaluated locations to propose weights restrictions. Shang et Sueyoshi [10] developed a unified DEA-AHP framework in selecting a Flexible Manufacturing System (FMS). They applied AHP to make a pairwise comparison matrix of the relative importance of each input and output. The obtained matrix served as a guideline to determine bounds in a virtual weight restrictions DEA model. Liu [11] focused on the integration of two subjective and objective weight restrictions method. Two weights were derived for each input/output: one from AHP and other from a classical DEA model. The author created a new weight by mixing these two values and ranked the DMUs. Junior [12] used the judgment matrix of MACBETH [18] as a MCDA tool to generate the weights were the minimum and maximum of these values are the lower and upper bounds in the virtual weight restrictions DEA model [6]. Hatami-Marbini et al. [19] considered the relation between TOPSIS (Technique for Order Preference by Similarity to the Ideal Solution) and DEA.

Determining subjective bounds for weight values is usually a complex cognitive task for DMs. In this work, we first propose a DEA model where PROMETHEE [20] (as a MCDA tool) is applied to automatically determine bounds for input and output weights. This method that can be considered as weight restricted DEA enables us to integrate preferential information of the decision maker and the results are
consistent with PROMETHEE II. We then extend a ranking method with a single constant input where the outputs are the unicriterion net flow scores and the objective function is to maximize the PROMETHEE II net flow scores.

To the best of our knowledge, our contribution is the first investigating the combination of PROMETHEE and DEA with the aim of improving the discrimination power of DEA.

The paper is organized as follows: in the second section, a brief outline of DEA basic models is provided. It is followed by a short description of PROMETHEE II and the stability intervals method in section three. The fourth section is dedicated to the presentation of the methodology (weight restricted DEA approach). Finally, in section five, we illustrate the model using two numerical examples.

2. Data Envelopment Analysis

DEA was first introduced in 1978 by Charnes, Cooper and Rhodes (so-called CCR) [21]. They explained how a fractional linear measure of multiple weighted outputs compared to multiple weighted inputs can be formulated in order to evaluate the relative efficiency of a set of DMUs. DEA is a widely used non-parametric and non-statistical method [22-24] that is based on a Linear Program (LP). In this regard, input and output weights are determined for the DMU under evaluation in order to find the best possible efficiency scores.

Let \( E_o \) denotes the efficiency measure of \( DMU_o \). This DMU is supposed to be characterized by \( s \) outputs, \( y_{ro}, r = 1,2, \ldots, s \) and \( m \) inputs, \( x_{io}, i = 1,2, \ldots, m \). We have:

\[
E_o = \text{Max} \sum_{r=1}^{s} u_r y_{ro} / \sum_{i=1}^{m} v_i x_{io}
\]  

where \( u_r \) and \( v_i \) are the non-negative weights. Let us assume that the efficiency of \( DMU_o \) should not exceed 1 for every DMU. We therefore obtain the following traditional CCR model:

\[
E_o = \text{Max} \sum_{r=1}^{s} u_r y_{ro} \\
\text{s.t.} \\
\sum_{i=1}^{m} v_i x_{io} = 1, \tag{2} \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1,2, \ldots, n \\
\quad u_r, v_i \geq 0, \forall i,r
\]

This problem is solved \( n \) times to calculate the efficiency score of \( DMU_j, j = 1,2, \ldots, n \). In this context, it is worth noting that the production function is assumed to satisfy the Constant Returns to Scale (CRS) assumption.

The method is composed of two steps; first, it determines a frontier by identifying DMUs that are efficient (i.e. \( E_o = 1 \)). Then, the relative efficiency of other DMUs (inefficient units) is estimated based on their distance from this frontier. Note that model (2) is input oriented multiplier model since it determines how to improve the inputs of a unit to become efficient while remaining at the same output level [25].

Finally, let us point out the Super Efficiency (SE) model to generate a complete ranking of DMUs. Anderson and Peterson in 1993 [26] presented SE model in order to rank the efficient DMUs. The technique removes the constraint from model (2) to enable an efficient unit to achieve an efficiency score that is greater than or equal to 1. It can be expressed as follows:

\[
E_o = \text{Max} \sum_{r=1}^{s} u_r y_{ro} \\
\text{s.t.} \\
\sum_{i=1}^{m} v_i x_{io} = 1, \tag{3} \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1,2, \ldots, n, j \neq o
\]
During the last five decades, several Multiple Criteria Decision Aid (MCDA) techniques like MAUT [27], AHP [17], AHP-TOPSIS [28], ELECTRE [29], fuzzy ELECTRE models [30, 31] and PROMETHEE [20] have been developed with the aim of supporting DMs in selection of compromise solution(s) or ranking and sorting of alternatives. In this paper, we focus ourselves on PROMETHEE II.

The family of PROMETHEE methods is known thanks to their simplicity, an important number of applications in different fields such as finance, business, education, health care, etc. [32] and the existence of user friendly software, D-Sight [33].

The PROMETHEE method that is based on pair wise comparisons has been initiated by J. P. Brans in 1982 [20] and developed by Brans and Vincke in 1985 [34]. PROMETHEE II allows a DM to rank a finite set of actions (DMUs in DEA) that are evaluated over a set of criteria.

In what follows, we assume (without loss of generality) that criteria have to be maximized.

The first step of the method consists of computing differences between every pair of actions on all criteria as presented below:

\[ d_k(a_i, a_j) = f_k(a_i) - f_k(a_j), \forall a_i, a_j \in A, \forall k = 1, ..., q \]  

This MCDA method works in the framework of preference functions to integrate intra-criterion information. Therefore one has to associate a generalized criterion \( \{f_k(\cdot), P_k(a_i, a_j)\} \) to each criterion where \( P_k(a_i, a_j) \) provides the preference strength of action \( a_i \) over \( a_j \). \( P_k : \mathbb{R} \rightarrow [0, 1] \) is a positive non-decreasing preference function. The concept of preference function is used to transform the difference to a unicriterion preference degree as follows:

\[ \pi_k(a_i, a_j) = P_k[d_k(a_i, a_j)] \]  

The method provides the DM with a set of predefined preference functions for which at most two parameters have to be defined (the indifference and preference thresholds). Details about preference functions are explained in (Brans and Mareschal, 2002) [35].

Usually, 6 main different types of preference functions are used. In this paper we limit ourselves to the linear preference functions. This particular function is the most suited one since it involves an area of indifference and strict preference. Moreover, it covers four other types of preference functions [35].

![Figure 1- Linear preference function](image-url)
\[
P_k(d_k(a_i, a_j)) = \begin{cases} 
0 & |d_k(a_i, a_j)| \leq q_k \\
\frac{|d_k(a_i, a_j) - q_k|}{p_k - q_k} & q_k < |d_k(a_i, a_j)| < p_k \\
1 & |d_k(a_i, a_j)| > p_k 
\end{cases}
\]

where \(q_k\) and \(p_k\) are respectively the indifference and preference thresholds associated to criterion \(k\). If \(d_k \in [0, q_k]\), \(a_i, a_j\) are considered indifferently on criterion \(k\) and if \(d_k\) is greater than \(p_k\), \(a_i\) is strictly preferred to \(a_j\). Between these 2 thresholds, the preference function is assumed to linearly increase.

The global preference degree of \(a_i\) over \(a_j\) is computed as follows:

\[
\pi(a_i, a_j) = \sum_{k=1}^{q} \pi_k(a_i, a_j). w_k
\]

where \(w_k\), \(k = 1, \ldots, q\) are normalized positive weights associated to the different criteria by DM. This degree varies between 0 and 1. Obviously we have:

\[
\pi(a_i, a_j) \geq 0 \ , \ \pi(a_i, a_j) + \pi(a_j, a_i) \leq 1
\]

The positive and negative outranking flow scores are defined as follows:

\[
\varnothing^+(a_j) = \frac{1}{n-1} \sum_{x \in A} \pi(a_j, x)
\]

\[
\varnothing^-(a_j) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a_j)
\]

The PROMETHEE I ranking is the intersection of the two rankings induced by these flows. Moreover, a complete pre-order, called PROMETHEE II can be obtained on the basis of the net flow score:

\[
\varnothing(a_j) = \varnothing^+(a_j) - \varnothing^-(a_j)
\]

Let us stress that the net flow score can also be computed as follows:

\[
\varnothing(a_j) = \sum_{k=1}^{q} \varnothing_k(a_j). w_k
\]

Such that:

\[
\varnothing_k(a_j) = \frac{1}{n-1} \sum_{x \in A} (\pi_k(a_j, x) - \pi_k(x, a_j)), \ k = 1, \ldots, q
\]

The quantity \(\varnothing_k \in [-1,1]\) is called the unicriterion net flow score of action \(a_j\). At this point, it is worth noting that the multicriteria problem can be viewed as an evaluation table (and associated parameters) or a matrix \(\varnothing = (\varnothing_k(a_j))\). These values already integrate intra-criterion preference information and are all lying in the same range.

The determination of precise weight values is often a cognitive complex task for the DM. When the ranking is computed a question can be raised: what is the impact of changing a given weight value? The purpose of the Weight Stability Intervals (WSI) technique [36] is to preserve the preference ranking of a subset of alternatives within the intervals of criteria weights. Its main strength is the automated generation of intervals limits that confirms the robustness of PROMETHEE II outputs (typically maintaining the subset of alternatives within the intervals of criteria weights). In this paper, we only consider the case of preference between alternatives \(a_i \rightarrow a_j \leftrightarrow \varnothing(a_i) > \varnothing(a_j)\) for a full stability of ranking. We refer the interested reader to [36] for complete explanations.

In a complete pre-order the structure of \((P, I)\) (\(P\): preference and \(I\): indifference) between a pair of actions \((a_i, a_j)\) is:
\[\begin{align*}
(a_i P a_j & \iff \emptyset(a_i) > \emptyset(a_j) \\
(a_i I a_j & \iff \emptyset(a_i) = \emptyset(a_j))
\end{align*}\]  

In this context, we want to investigate the potential modification in the \((P, I)\) structure when the weights are changed. Let us denote by \((P', I')\) the complete pre-order of modified weights \((w'_k)\). To maintain the normalization of modified weights, all the other weights are rearranged as \(w'_i = \alpha w_i\) when \(l \neq k\).

We represent the following notations:
\[
\Delta(a_i, a_j) = \emptyset(a_i) - \emptyset(a_j)
\]
(14)

\[
\Delta'(a_i, a_j) = \emptyset'(a_i) - \emptyset'(a_j)
\]
(15)

\[
\Delta_k(a_i, a_j) = \emptyset_k(a_i) - \emptyset_k(a_j)
\]
(16)

It is easy to show that:
\[
\Delta'(a_i, a_j) = \alpha \Delta(a_i, a_j) + (1 - \alpha) \Delta_k(a_i, a_j)
\]
(17)

In other words, if we hold the relative position between a pair of actions, we need to satisfy the following constraint:
\[
\Delta(a_i, a_j)\Delta'(a_i, a_j) > 0 \quad s.t. \quad \Delta(a_i, a_j) \neq 0
\]
(18)

Thus:
\[
\alpha[\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)] < \Delta(a_i, a_j) \Delta_k(a_i, a_j)
\]
(19)

In this position, we investigate 3 different cases:

4. When \(f_k\) is in disagreement with ranking of \(a_i\) and \(a_j \leftrightarrow \Delta(a_i, a_j) \Delta_k(a_i, a_j) < 0\), (20) gives the lower bound of \(\alpha (\alpha^-_k)\). This bound has to be satisfied for all pairs of actions that fall in this category. Therefore let us define \(\Omega^-\) as follows:
\[
\Omega^- = \{(a_i, a_j) \in A \times A, s.t. \Delta(a_i, a_j) \Delta_k(a_i, a_j) < 0\}
\]
(20)

We have:
\[
\alpha^-_k = \max_{(a_i, a_j) \in \Omega^-} \frac{\Delta(a_i, a_j) \Delta_k(a_i, a_j)}{\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)}
\]
(21)

5. When \(f_k\) is in agreement with ranking of \(a_i\) and \(a_j \leftrightarrow \Delta(a_i, a_j) \Delta_k(a_i, a_j) > \Delta^2(a_i, a_j)\), (20) gives the upper bound of \(\alpha (\alpha^+_k)\). As before, this bound has to be satisfied for all pairs of actions that fall in this category. Therefore let us define \(\Omega^+\) as follows:
\[
\Omega^+ = \{(a_i, a_j) \in A \times A, s.t. \Delta(a_i, a_j) \Delta_k(a_i, a_j) > \Delta^2(a_i, a_j)\}
\]
(22)

And the upper bound of \(\alpha\) is:
\[
\alpha^+_k = \min_{(a_i, a_j) \in \Omega^+} \frac{\Delta(a_i, a_j) \Delta_k(a_i, a_j)}{\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)}
\]
(23)

6. Finally when \(0 \leq \Delta(a_i, a_j) \Delta_k(a_i, a_j) \leq \Delta^2(a_i, a_j)\), then inversion between \(a_i\) and \(a_j\) is not possible. As before let us define:
\[
\Omega^0 = \{(a_i, a_j) \in A \times A, s.t. \Delta(a_i, a_j) = 0 \text{ and } \Delta_k \neq 0\}
\]
(24)
According to (20) this situation shows that there exist 2 states for set $\Omega^0$:

(III) If $\Omega^0 \neq \emptyset$, then value of $\alpha$ is fixed on 1. It means no change of weights is allowed (no inversions between preferences and only changing from indifference to preference can happen)

(IV) If $\Omega^0 = \emptyset$ then $\alpha^- < \alpha < \alpha^+$. The final step is to determine the WSI of criterion $f_k$:

$$WSI_k = (W_k^-, W_k^+) = (1 - (1 - w_k)\alpha_k^-, 1 - (1 - w_k)\alpha_k^+).$$

In the next section we use the PROMETHEE II WSI as weight constraints in DEA. In addition, let us note that these conditions can relaxed when the DM wants to focus himself on a subset of alternatives (and not the whole ranking). For instance, he/she can focus himself/herself on the top $h$ alternatives.

### 4. Methodology

The purpose of using the PROMETHEE II Weight Stability Intervals (WSI) tool in multiplier DEA models is to restrain full freedom of weights. This process commonly decreases the size of production possibility set. Thus, the production frontier is moved away from the detected DMUs and their efficiency scores are decreased. As a result, the discrimination power of DEA is increased. Besides, this allows avoiding undesirable results such as allocating very low or high weight values to some inputs or outputs. In addition, preferences of the DM can be partly integrated in the DEA analysis. In this work, we automatically generate the WSI in PROMETHEE II and use these intervals in an extended CCR input-oriented multiplier DEA model. This ensures that the compromise solution(s) identified in PROMETHEE is characterized by an efficiency score equal to 1 in DEA. It is important to mention that in the first step of computing WSI in PROMETHEE, the following conditions should be respected: the inputs and the outputs are considered as criteria or attributes in a DEA problem, with minimization of inputs and/or maximization of outputs as associated objectives [22]. Further, a DM chooses the pre-determined criteria weights due to additional preferences. In the particular case: in the absence of distinct priorities between criteria, DM chooses all weights equally. In this case the criteria are considered equally important.

A first classic weighted DEA model, in the presence of WSI, can be defined as follows:

$$E_o = \text{Max } \sum_{r=1}^{s} u_r y_{ro}$$

s.t.

$$\sum_{i=1}^{m} v_i x_{io} = 1,$$

$$\sum_{r=1}^{s} u_r y_{rf} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, 2, \ldots, o, \ldots, n$$

$$w_i^- \leq v_i \leq w_i^+; \quad v_i \geq 0, \forall i$$

$$w_r^- \leq u_r \leq w_r^+; \quad u_r \geq 0, \forall r$$

where $w_i^-, w_i^+, w_r^-, w_r^+$ are the PROMETHEE WSI for the cost criteria (inputs) and the profit criteria (outputs). This simple weight restricted DEA model is referred as Weighted CCR (WCCR) in numerical examples. Therefore as the main contribution of this work, we consider a ranking method with a single input for all alternatives where we maximize the net flow scores and outputs are the unicriterion net flow scores of the PROMETHEE II.

Thus (26) is changed to:

$$E_o = \text{Max } [\varnothing(a_o) = \sum_{k=1}^{q} w_k \varnothing_k(a_o)]$$

s.t.

$$\sum_{k=1}^{q} w_k \varnothing_k(a_j) \leq 1; j = 1, 2, \ldots, n$$

(27)
\(W_k^- \leq w_k \leq W_k^+\)
\(w_k \geq 0, \forall k\)

where \(W_k^-\) and \(W_k^+\) are the weights derived from PROMETHEE WSI for the outputs. The first constraint in (27) imposes that the efficiency scores are less than or equal to 1. The presented model is CCR/CRS model. In what follows, we will denote it: PROMETHEE II Weighted CCR (PIIWCCR). One should pay attention that applying additional weight restrictions in DEA multiplier models may cause infeasibility. In this regard, some researchers developed approaches to resolve this problem [1, 37, 38].

The standard DEA models are formulated based on input and output data of DMUs. Nevertheless, in some cases data sets are considered without inputs such as index data [39, 40] or pure output data. Different approaches and modified DEA models are used to resolve these types of problems [25, 39-43]. In this paper, a dummy input that has a value of 1 for all DMUs is used. The lower and upper bounds of dummy input are taken 0 and 1 in PIIWCCR. Figure 2 compares the unicriterion net flow scores matrix of PROMETHEE and the proposed view of DEA.

Unicriterion Net flow Score

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<th>Criteria 2</th>
<th>(\ldots)</th>
<th>Criteria q</th>
</tr>
</thead>
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<tr>
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<td>(\emptyset_2(a_2))</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>Alternative n</td>
<td>(\emptyset_1(a_n))</td>
<td>(\emptyset_2(a_n))</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

Proposed view of DEA

<table>
<thead>
<tr>
<th>Output 1</th>
<th>Output 2</th>
<th>(\ldots)</th>
<th>Output q</th>
<th>Dummy Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>DMU 1</td>
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<tr>
<td>DMU 2</td>
<td>(\emptyset_1(a_2))</td>
<td>(\emptyset_2(a_2))</td>
<td>(\ldots)</td>
<td>(\emptyset_p(a_2))</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>DMU n</td>
<td>(\emptyset_1(a_n))</td>
<td>(\emptyset_2(a_n))</td>
<td>(\ldots)</td>
<td>(\emptyset_p(a_n))</td>
</tr>
</tbody>
</table>

Figure 2- A comparison of the unicriterion net flow scores and the extended view of DEA

We use a super efficiency CCR model to generate a complete ranking in PIIWCCR as follows:

\[E_o = Max \ \ [\emptyset(a_o) = \sum_{k=1}^{q} w_k \emptyset_k(a_o)]\]

s.t.

\[\sum_{k=1}^{q} w_k \emptyset_k(a_j) \leq 0; \ j = 1, 2, \ldots, n, \ j \neq o\]  \(\text{(28)}\)

\[W_k^- \leq w_k \leq W_k^+;\]
\[w_k \geq 0, \forall k\]

In the next section, we will compare the results between different combinations of DEA and MCDA methods. Moreover, we will restrict the stability intervals level to the first position (we will fix the position of the 1st alternative).

5. Numerical examples

In this section, we illustrate the application of the proposed model using two different data sets. In the first example, Hokkanen and Salminen [44] evaluated data involving economic, environmental, political, employment and resource recovery perspectives to localize a solid waste management system in Oulu, Finland. Finally they applied ELECTRE III outranking method (ELimination Et Choix Traduisant la REalité) to rank different locations and choose among them to construct the solid waste management
systems. Table 1 presents the related data set. Further, we compare our results with the results of several decision making methods. The second data set is provided from a research report of IWEPS (Institut Wallon de l'Evaluation, de la Prospective et de la Statistique) [45]. We evaluate the well-being level in different municipalities of Wallonia (Belgium).

**Example 1.**

This example is based on the normalized data set of Hokkanen and Salminen [44]. This problem includes 22 alternatives and 8 criteria. The alternatives obtained by combination of 3 key factors in different municipalities of Oulu:

1. Cooperation level;
2. Treatment method and
3. Number of treatment sites.

For example, the combination of decentralized cooperation level, landfill method and 17 landfills give the first alternative.

The criteria are classified into two groups where the groups (1) and (2) are the inputs and outputs, respectively:

1. C₁ (net cost per ton), C₂ (global effects), C₃ (local and regional health effects), C₄ (Acidificative releases) and C₅ (surface water dispersed releases);
2. C₆ (technical reliability), C₇ (number of employees) and C₈ (amount of recovered waste).

The represented criteria’ weights by collaboration of 113 DMs are: w₁: 0.2700, w₂: 0.0160, w₃: 0.0960, w₄: 0.0470, w₅: 0.0900, w₆: 0.2600, w₇: 0.0500 and w₈: 0.1400. The data set and details in choosing alternatives, criteria and their weights are described in [44].

We consider ranking results of different methods: CCR, WCCR and the proposed methodology: PIIWCCR. The best alternatives obtained by these models are compared with the results of some multi criteria methods in Salminen et al. [46]. They considered ELECTRE III, SMART (Simple Multi-Attribute Rating Technique), PROMETHEE I and II [38]. Furthermore we use a Super Efficiency (SE) model in CCR, WCCR and PIIWCCR to improve the discrimination power between alternatives scores (SECCR, SEWCCR, SEPIIWCCR).

<table>
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<tr>
<th>Inputs</th>
<th>Outputs</th>
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<td>0.9626</td>
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<td>6</td>
<td>1.1372</td>
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<td>0.7908</td>
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<td>8</td>
<td>0.9299</td>
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<td>9</td>
<td>1.1426</td>
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<td>10</td>
<td>0.7895</td>
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<td>11</td>
<td>0.9381</td>
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<td>12</td>
<td>1.1426</td>
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<td>13</td>
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<td>14</td>
<td>0.9667</td>
</tr>
<tr>
<td>15</td>
<td>1.1576</td>
</tr>
</tbody>
</table>
Table 1- The normalized data set of Hokkanen and Salminen (localization of a solid waste management system in Oulu, Finland)

In this example the parameterizations of ELECTRE III and SMART are fixed as determined in [46]. Table 2 reports the PROMETHEE parameters. We use given weights of [44] by normalizing sum of them to 1. In the process of choosing thresholds, we fix indifference and preference thresholds to the first and third quartile of differences in each criterion, respectively. The unicriterion net flow scores, $\varnothing_k(a_j)$, are used as the output in the DEA analysis. To avoid using negative values of $\varnothing_k(a_j)$ in DEA, the unicriterion scores are transformed by using a linear transformation as $[\varnothing_k(a_j) + 1]/2$.

Table 2- PROMETHEE parameters (localization of a waste management system)

Table 3 illustrates PROMETHEE II WSI where the stability level is set to 1.

Table 3- Weight Stability Intervals of PROMETHEE II in the level 1 (localization of a waste management system)
Table 4- The efficiency scores of DEA models (localization of a solid waste management system in Oulu, Finland)

The efficiency scores of the different DMUs are summarized in Table 4. The bold numbers show the efficiency scores that are equal to or higher than 1. Obviously, this table presents DMUs 7 and 10 as the best efficient units in all mentioned DEA models. In the classic CCR model the number of equal efficient DMUs (Efficiency=1) is 16 out of 22. Alternatively, the SECCR model gives a complete ranking. This shows that CCR alone is not a good discriminator among DMUs. As expected, the number of equal efficient units in WCCR and PIIWCCR are reduced to 9 and 2 (which is clearly less than CCR). Though, incorporating WSI in DEA models can be done as a strategy not only to partly integrate the DMs preferences but also to improve the DEA discrimination power by decreasing the number of efficient units. Figure 3 illustrates the number of efficient units according to their DEA models:

![Figure 3-The number of efficient DMUs in different DEA methods (localization of a solid waste management system)](image)

Table 5 represents the ranking results of different decision making methodologies. The rank orders are not similar in different approaches but it can be seen that alternatives/DMUs 10 and 7 are nearly always between the most preferred ones and 1 and 2 among the least preferred ones. The first two alternatives in PROMETHEE I, II, SMART and PIIWCCR/SEPIIWCCR are exactly the same (alts. 10, 7), however in ELECTRE III and SEWCCR alternative 7 is tied at rank 7. From this table, the DM can determine the best and worst choice by analysing rank order of different methods. Moreover, the rankings induced by different methods can be used to determine more robust choices.
Table 5- Increasing rank order of alternatives (localization of a solid waste management system in Oulu, Finland)

Table 6 shows the Kendall’s tau correlation between different methods. The correlation values are significant at the 0.01 level.

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<tr>
<th></th>
<th>SEPIIWCCR</th>
<th>SEWCCR</th>
<th>SECCR</th>
<th>PR. II</th>
<th>EL. III</th>
<th>SMART</th>
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<tr>
<td>SEPIIWCCR</td>
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<td>0.2468</td>
<td>0.1602**</td>
<td>0.8576*</td>
<td>0.1169</td>
<td>0.7056</td>
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<td>0.4978</td>
<td>0.4372</td>
<td>0.4372</td>
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<td>SECCR</td>
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<td>0.1439</td>
<td>0.2987</td>
<td>0.2641</td>
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<td>PR II</td>
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<td>0.3247</td>
<td>0.9307</td>
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<td></td>
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<td>EL. III</td>
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<td></td>
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<td>SMART</td>
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According to Table 6, incorporating the PROMETHEE II WSI into an extended DEA model seems to provide some results, which are correlated to other methods. That is why the correlation between SEPIIWCCR and PROMETHEE II* (0.8576) is much more than SEPIIWCCR and SECCR** (0.1602) (maximizing the net flow score of PR.II in an extended DEA model leads to generate more similar ranking with PR.II).

Example 2.

In this example, we compare the level of well-being in different municipalities of Wallonia (Belgium) as reported in [45]. They evaluated 262 municipalities according to 20 different well-being indices. Here we focus on 132 municipalities and 13 criteria. The criteria are health-care, accommodation, education and training, employment, income and purchasing power, mobility, quality of life and environment, access to shopping centers, security of life and environment, administrative institutions, the situation of marital and family, happiness and the revenue of municipality. We minimize the last criterion (revenue of municipality) and maximize others. Tables 7 and 8 give details of the PROMETHEE II parameters and the related stability intervals at level 1. Finally, let us stress that the weights are chosen to be equal.
The integration of weight intervals to DEA model allows to decrease the number of efficient units. This number in CCR model is equal to 42 whereas in WCCR and PIIWCCR are 21 and 25 as illustrated in Figure 4. Table 9 displays the ranking results of SECCR, SEWCCR, PROMETHEE II and SEPIIWCCR. It is noticeable that the municipalities 118 and 92, Tintigny and Ottignies-LLN, are placed almost in the top of the different ranking methods.

Table 7- PROMETHEE parameters (weights are equal) (well-being level in Wallonia)

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<th>Criteria</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
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Table 8- Stability intervals in level 1 (well-being level in Wallonia)

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Min weight</th>
<th>Value</th>
<th>Max weight</th>
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<tr>
<td>C1</td>
<td>0</td>
<td>0.077</td>
<td>0.112</td>
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<td>C2</td>
<td>0.061</td>
<td>0.077</td>
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<td>C3</td>
<td>0.037</td>
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Table 9- Stability intervals in level 1 (well-being level in Wallonia)

Figure 4-The number of efficient DMUs in different DEA methods (well-being level in Wallonia)
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<th>Rank</th>
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<th>SE-PIIWCCR</th>
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The main point that can be seen in Table 10 is that the basic DEA model (SE
CCR) has very low correlation with PROMETHEE II* (0.0946) while the correlation between SEPIIWCCR with
PROMETHEE II are quite high** (0.6926).

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<td>SECCR</td>
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Table 10- Kendall's tau correlation (well-being level in Wallonia)

6. Conclusion

In this work, we have proposed a new ranking method based on DEA and PROMETHEE II. This model can be considered as weight restricted DEA. On the one hand, it reduces the number of DMUs that are characterized by an efficiency score equal to 1 and thus increases the discrimination power of DEA. On the other hand, it makes possible to partly integrate the preferential information of the decision maker in an extended DEA model. This model is a ranking method with a constant single input where the outputs are the unicriterion net flow scores and the objective function is to maximize the net flow scores of PROMETHEE II.

The underlying idea is to use the outputs of the sensitivity analysis of PROMETHEE, namely the Weight Stability Intervals, in order to reduce the weights’ degrees of freedom in an extended DEA analysis. These intervals are generated automatically (depending on the acceptance of the PROMETHEE ranking and given a certain stability level).

The new proposed model is named PIIWCCR and its output has approximately high correlation with the outputs of some other DEA and MCDA methods. This has been applied using 2 real case studies. Clearly, the number of efficient DMUs is decreased in weighted DEA models and the discrimination power of DEA is increased.

To the best of our knowledge, this is a first work in which one investigates the potential synergies between PROMETHEE and DEA.
References:


[38] Podinovsky, V. V. and Bouzdine-Chameeva, T. (2013). Weight restrictions and free production in Data Envelopment Analysis, EJOR, Vol. 61, No. 2, PP. 426-437, DOI: 10.1287/opre.1120.1122.


A DEA-PROMETHEE approach for complete ranking of units

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Abstract
Data Envelopment Analysis (DEA) and Multiple Criteria Decision Aid (MCDA) are two well-known approaches to rank so-called Decision Making Units (DMUs) or alternatives. In this contribution, a two-step model is presented to completely rank units according to multiple inputs and outputs. In the first step, DEA is applied between each pair of DMUs independently to generate a pairwise comparison matrix. In the second step, the obtained matrix is exploited by means of PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) to completely rank units. We show the compatibility between the resulting ranking of DEA and DEA-PROMETHEE methods while there exist just one input and one output. We also discuss the monotonicity property of the method. We compare DEA-PROMETHEE with an integrated DEA-AHP approach on a numerical example.

Keywords: Data Envelopment Analysis (DEA); Multiple Criteria Decision Aid (MCDA); PROMETHEE; Efficiency; Ranking; Decision Making (DM)

1. Introduction
Data Envelopment Analysis is a sub field of Operations Research (OR) that was initiated when Charnes, Cooper and Rhodes (CCR) (1978) showed how to transfer a fractional measure of efficiency into a Linear Program (LP). As a result, DEA allows classifying DMUs into efficient and inefficient units according to multiple inputs and multiple outputs. During years, DEA has been developed by many researchers in various directions and applications. Liu and his colleagues (Liu, J. S. et al., 2013) reviewed the related literature. They adopted the ISI Web Of Science (WOS) data base as the source of this study: a total of 4936 papers were listed.

In the basic DEA models, the so-called efficient units have equal scores (equal to 1). Thus, the best-performing DMUs can be identified but further distinction within classical DEA models is not possible. In order to overcome this limit and improve the discrimination power of DEA, researchers have developed methods to present a full ranking of DMUs. Adler and his colleagues (Adler, N. et al., 2002), reviewed the DEA ranking methods in six general groups. The first group is cross efficiency ranking methods, in which units are self and peer evaluated. It was first proposed by Sexton et al. (1986). It is based on the computation of the efficiency score of each DMU by applying the optimal generated weights in a DEA problem. The result of these computations can be outlined in a cross efficiency matrix. Doyle and Green (1994) and Liang et al. (2008a,b) developed this method by working on the definition of non-uniqueness optimized weights in DEA. Further, Wang and Luo (2009) considered rank reversal problem in the process of cross efficiency evaluation when a DMU is added or removed. Lim and Zhu (2015) evaluates units by a DEA cross efficiency model under Variable Return to Scale (VRS). The second group is referred to super efficiency methods. It
was developed by Anderson and Peterson (1993) to rank the efficient DMUs through the exclusion of the unit under evaluation in the objective function of a dual LP. It computes the distance between the evaluated unit and the Pareto frontier. Chen, Y. (2005), Seiford, L. M., (1996), Seiford, L. M. et Zhu, J., (1999 and Thrall, R. M., (1996). Hadi-Vencheh and Esmaeilzadeh (2013) considered a new super-efficient model in the presence of negative data. Amirteimoori et al. (2014) initiated a super-efficient model with a common set of weights. The third group is the benchmark ranking models. In this context an efficient DMU is highly ranked while it is regularly chosen as a reference for other units. A complete ranking between efficient units can be built by measuring their importance as a benchmark for inefficient units (Torgersen, A.M. et al., 1996). Zhu (2015) summarized benchmarking DEA models in a chapter book of DEA. Multivariate statistical techniques such as canonical correlation analysis and discriminant analysis are introduced in the fourth group by Sinuany-Stern and Friedman (1998). The fifth category is named slack-adjusted DEA models (Bardhan et al., 1996). Bardhan and his colleagues created an index based on the slacks in order to rank the DMUs. In this category, the inefficient DMUs are ranked through an index, which is called Measures of Inefficiency Dominance (MID). Finally the last group discusses the integration of MCDA and DEA methods. This approach needs additional preferential information from DMs (Cook, W. D. et al., 1990, 1992, 1993, Fulop and Markovits, 2012, Golany, B., 1988, Ho and Oh, 2010, Jie, W. et al., 2010, Shang et Sueyoshi, 1995, Sinuany-Stern et al., 2000, Thompson, R. G. et al., 1986). It should be mentioned that several researchers have proposed approaches in the literature, which cannot be easily categorized, into one or other groups. For instance, we can mention the work of Alirezaee and Afsharian (2007). They presented a Balance Index to completely rank DMUs. In modifying this model to give a more stable index, Guo et Wu (2013) proposed the Maximal Balance Index (MBI), which can determine a unique ranking, using restrictions in DEA models. Finally, the application of Fuzzy Logic which is one of the interesting paths for complete ranking of units was introduced by Wen and Li (2009). Alem et al. (2013) constructed a new fuzzy-DEA-AHP model to rank DMUs completely.

Many researchers have studied the connection between DEA and MCDA (Belton, V. et Vickers, S. P., 1993). Indeed, the main difference between them is that DEA assigns weights automatically to criteria (Belton, V., 1992, Cook, W. D. et Kress, M., 1990, Cook et al., 1992, Doyl, J. et Green, R., 1993, Stewart, T. J., 1994, 1996). Ranking is a common issue in both methods. DEA is based on the concept of Pareto optimality, efficient frontier or dominant solution (Charsnes, A., et al., 1981). However, the Pareto concept has its own shortcomings. To exemplify this issue, let us compare two vectors: $(10,60,5)$ and $(10^{10},60^{10},5 - \varepsilon)$ where $\varepsilon$ is a non-Archimedean number. According to Pareto optimality theories, these two vectors are efficient. Nevertheless, to the point of view of most DMs, the second vector is preferred. This simple example underlines the limitation of DEA and pleads in favor of the integration of preferences in the analysis.

Shang et Sueyoshi (1995), Sinuany-Stern et al. (2000), Li and Ma (2008), Ho and Oh (2010) and Fulop and Markovits (2012) have proposed an AHP/DEA methodology for ranking DMUs. The Analytical Hierarchical Process (AHP) was first presented by Saaty (1980). AHP is a multi-criteria measurement technique based on pairwise comparisons and relies on experts judgments. Shang et Sueyoshi (1995) ranked and selected flexible manufacturing systems by using the subjective AHP results in DEA. Sinuany-Stern et al. (2000) presented a two stage AHP/DEA ranking model to remove the pitfalls of Shang et Sueyoshi. In the first stage, DEA is used for each pair of units separately in order to generate a pairwise comparison matrix. This matrix is used to rank the units based on a single level AHP. Fulop and Markovits (2012) used a variant of the CCR model to build a non-reciprocal pairwise comparison matrix and then use AHP to obtain ranking values of DMUs. Li and Ma (2008) developed an iterative method by integrating DEA, AHP and Gower plot techniques. Ho and Oh (2010) constructed a stock selection framework by integrating DEA and AHP. Further, Rocio Guede M. and his colleagues (2012) attempted to analyze the innovation efficiency in Spain.
They used ELECTER Tri (Roy, B., 1991) to give a robustness analysis of the efficient activity branches to improve the DEA discrimination ability.

In the current work, we use PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) (Brans, J. P., 1982) to rank units. This method is based on pairwise comparisons of the actions. The new integrated DEA-PROMETHEE method works in two steps. In the first phase, DEA is run between each pair of DMUs. In the second phase, the PROMETHEE method is used to aggregate scores from the generated pairwise comparison matrix. In PROMETHEE, DM subjectively chooses the preference functions, related thresholds and weights to generate the pairwise comparison matrix whilst the new integrated method relies on DEA to perform this task. Finally, while via classical DEA, the efficient DMUs cannot be discriminated, the new model avoids this problem.

Let us point out that there also exists a few works between DEA and PROMETHEE. Bagherikahvarin and De Smet used the PROMETHEE stability intervals as weight restrictions in DEA problems to improve its discrimination power (Bagherikahvarin, M. et De Smet, Y., 2016). In a second contribution, they tried to determine new possible weight values in PROMETHEE VI, based on a DEA methodology (Bagherikahvarin, M. et De Smet, Y., 2015).

This paper is organized as follows. In section 2, we describe a classic DEA model. In section 3, we briefly present the PROMETHEE II method. In section 4 we explain the proposed approach. Further, we discuss the possibility of compatibility between DEA and the new model as well as the monotonicity of the new model in the case of 1 input and 1output. Finally, we illustrate our methodology on a particular example and compare it with a DEA-AHP model (Sinuany-Stern et al., 2000).

2. DEA

Data Envelopment Analysis (DEA) is a mathematical model which was first introduced by Charnes, Cooper and Rhodes in 1978. It is a non-parametric and non-statistical optimization technique for measuring the relative efficiency of Decision Making Units (DMUs) with multiple inputs and multiple outputs. One of the basic models is CCR (Charnes, A. et al., 1978) which is based on the assumption of Constant Return to Scale (CRS). Its relative efficiency is defined as the ratio of the total weighted outputs to the total weighted inputs (with a range of zero to one that specifies how efficient the DMU under evaluation is in comparison to other DMUs). Therefore, the best performing DMUs can be identified as efficient (with efficiency score equal to 1). Other units are ranked according to their efficiency scores.

The efficiency score of DMU \( o \) is denoted \( E_o \). It is characterized by \( s \) outputs, \( y_{ro}, r = 1,2,...,s \) and \( m \) inputs, \( x_{io}, i = 1,2,...,m \).

\[
E_o = \text{Max} \sum_{r=1}^{s} \frac{u_r y_{ro}}{\sum_{i=1}^{m} v_{i} x_{io}} \tag{1}
\]

While \( u_r \) and \( v_i \) are non-negative weights.

As it has been mentioned above the efficiency score of a DMU is assumed to be less than or equal to 1. Hence, the CCR formulation is:

\[
E_o = \text{Max} \sum_{r=1}^{s} \frac{u_r y_{ro}}{\sum_{i=1}^{m} v_{i} x_{io}}
\]

\[
\sum_{j=1}^{n} u_j y_{rj} \leq \sum_{i=1}^{m} v_i x_{ij} \Rightarrow u_r, v_i \geq 0, \forall i = 1,2,...,m, r = 1,2,...,s \tag{2}
\]
(ν*, u*), the optimal solution of (3), indicates the most favorable weights for the DMU₀, while maximizing the ratio score.

The result of DEA is the determination of a Pareto frontier. DMUs which lie on this frontier are efficient (with efficiency scores equal to 1) whilst those which do not, are inefficient and have efficiency scores lower than 1.

In order to modify the CRS formulation in (3) to Variable Return to Scale (VRS) (Banker, R. D. et al., 1984), a constant variable (u) should be added:

\[
\begin{align*}
& s.t. \\
& \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& \sum_{r=1}^{s} u_r y_{rf} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, 2, \ldots, n \\
& u_r, v_i \geq 0, \forall i = 1, 2, \ldots, m, r = 1, 2, \ldots, s
\end{align*}
\]  

(4)

This model is referred to as BCC.

Let us note that the dual version of this problem can be written as follows:

\[
\begin{align*}
& s.t. \\
& \sum_{i=1}^{m} v_i x_{i0} = 1, \\
& \sum_{r=1}^{s} u_r y_{rf} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leq 0, j = 1, 2, \ldots, n \\
& u_r, v_i \geq 0, \forall i = 1, \ldots, m, r = 1, \ldots, s, u_0 \text{ free in sign}
\end{align*}
\]  

(5)

By means of the duality theorem in LP we have \( E_o^* = \theta^* \). Hence both problems (4) and (5) can be used to compute efficiency scores. Later, we will use formulation (5) to explain some features of our new model (see appendix).

Models (3), (4) and (5) are input-oriented (Cooper, W. W. et al., 2011). These LP problems show how to enhance the input aspects of a unit to put it on the efficient frontier with the same output level. For more details about these models, we refer the interested readers to (Charnes, A. et al., 2006), (Cooper, W. W. et al., 2005) and (Cooper, W. W. et al., 2011).

3. Multiple Criteria Decision Aid (MCDA)

Multiple Criteria Decision Aid (MCDA) is a sub field of Operations Research (OR) which has evolved rapidly during the last five decades. The aim of MCDA is to support DMs in ranking, choosing and sorting various actions evaluated on multiple conflicting criteria. Several MCDA techniques like MAUT (Keeney, R. and Raiffa, H., 1993), AHP (Saaty, T. L., 1980), ELECTRE (Roy, B., 1991) and PROMETHEE (Brans, J. P., 1982) have been developed in this context. In this contribution, we focus on applying PROMETHEE II as an additional tool to compute a complete ranking of the actions. The family of PROMETHEE methods is known thanks to their simplicity, the existence of user friendly software D-
sight (Hayez, Q. et al., 2012) and an important number of applications (Behzadian, M. et al., 2010) in different fields such as finance, business, education, health care, insurance, etc.

3-1. PROMETHEE

PROMETHEE is an outranking method which is based on pairwise comparisons of the actions. It has been initiated by J. P. Brans in 1982. We consider a finite set of \( n \) actions \( A = \{ a_1, a_2, ..., a_n \} \) that are evaluated over a set of \( q \) criteria, \( F = \{ f_1(a_i), ..., f_q(a_i) \} \). \( f_k: A \rightarrow \mathbb{R}: a_i \rightarrow f_k(a_i), \forall k \epsilon \{ 1, 2, ..., q \}. f_k(a_i) \) indicates the evaluation of action \( a_i \) on criterion \( f_k \). PROMETHEE II allows a DM to completely rank these \( n \) actions (alternatives). In what follows, we assume without loss of generality that criteria have to be maximized.

The first step consists to compute differences between every pairs of actions for all criteria as follows:

\[
d_k(a_i, a_j) = f_k(a_i) - f_k(a_j), \forall a_i, a_j \in A, \forall k = 1, ..., q
\]  

In PROMETHEE, there exists a set of predefined preference functions for which at most two parameters (indifference and preference thresholds) have to be defined by the DM (Brans and Mareschal, 2002): \( P_k: A \times A \rightarrow \mathbb{R} \cdot [0,1] \) which is a positive non-decreasing function of \( d_k(a_i, a_j) \) and provides the preference strength of action \( a_i \) over \( a_j \) for each criterion individually. The concept of preference function is used to transform the differences into a unicriterion preference degree; hence, we have:

\[
\pi_k(a_i, a_j) = P_k[d_k(a_i, a_j)]
\]

In the third step, a global preference degree is computed as follows:

\[
\pi(a_i, a_j) = \sum_{k=1}^{q} P_k(a_i, a_j)w_k
\]

Where \( w_k \) (\( k = 1, ..., q \)) is a normalized positive weight associated to criterion \( k \).

Obviously we have:

\[
\pi(a_i, a_j) \geq 0 \quad , \quad \pi(a_i, a_j) + \pi(a_j, a_i) \leq 1 \quad , \quad \pi(a_i, a_i) = 0
\]

The positive and negative outranking flow scores are defined in the forth step as follows:

\[
\varnothing^+(a_i) = \frac{1}{n-1} \sum_{x \epsilon A} \pi(a_i, x)
\]

\[
\varnothing^-(a_i) = \frac{1}{n-1} \sum_{x \epsilon A} \pi(x, a_i)
\]

The PROMETHEE I ranking is the intersection of the two rankings induced by these flow scores. Moreover, a complete pre-order, called PROMETHEE II can be obtained on the basis of the net flow score:

\[
\varnothing(a_i) = \varnothing^+(a_i) - \varnothing^-(a_i) = \frac{1}{n-1} [\sum_{x \epsilon A} \pi(a_i, x) - \sum_{x \epsilon A} \pi(x, a_i)]
\]

This complete pre-order respects the following relations: (a) \( \varnothing(a_i) > \varnothing(a_j) \) iff \( a_i \) is preferred to \( a_j \), (b) \( \varnothing(a_i) = \varnothing(a_j) \) iff \( a_i \) and \( a_j \) are indifferent.

4. The DEA-PROMETHEE ranking model

In this section, we propose an integrated model between DEA and PROMETHEE II. It is based on two steps. In the first phase, DEA is applied to compare each pair of units and to generate a pairwise comparison matrix. In the second phase, PROMETHEE II is used to compute scores from the generated pairwise comparison matrix and to provide a complete ranking.

Let us consider there are just two units, denoted \( A \) and \( B \). The formulation (3) is changed to the following form:
Problem (a)

\[
E_{AB} = \max \sum_{r=1}^{s} u_r y_r A
\]

s.t.

\[
\sum_{t=1}^{m} v_t x_{iA} = 1,
\]

\[
\sum_{r=1}^{s} u_r y_r A + s_1 = 1
\]

\[
\sum_{r=1}^{s} u_r y_r B - \sum_{t=1}^{m} v_t x_{iB} + s_2 = 0
\]

\[
u_r, v_t \geq 0, \forall \ i = 1, 2, ..., m, r = 1, 2, ..., s
\]

Problem (a) is the CCR input-oriented DEA model for two DMUs. \(E_{AB}\) is the optimal value (efficiency score) of unit A in comparison with unit B (similarly \(E_{BA}\) is the optimal value of unit B in comparison with unit A). As it is clear there exist \(m + s + s_1 + s_2\) variables ((\(s_1\) and \(s_2\) slack variables of constraints (13) and (14), respectively)). If unit A is efficient, \(s_1 = 0\) and \(s_2 \geq 0\). In the same way, if B is efficient, \(s_2 = 0\) and \(s_1 \geq 0\). Sinuany-Stern and her colleagues (Sinuany-Stern, Z. et al., 2000) proposed a simple solution for this problem. They showed that in Problem (a), if there exists any pair of inputs and outputs (\(i_h, r_h\)) such that \(\frac{y_{rhA}}{x_{ihA}} > \frac{y_{rhB}}{x_{ihB}}\), then \(E_{AB} = 1\) and unit A is efficient, if not \(E_{AB} = \max(\frac{x_{iA}}{y_r A} / \frac{x_{iB}}{y_r B}) < 1\) and unit B is efficient.

A drawback with the DEA technique is that if all units can implement their most favourable weights, they may all appear to be efficient. This especially happens when the number of inputs and outputs increases. In such a case, we use the cross efficiency methodology, initiated by Sexton et al. (1986), to be able to discriminate each pair of units. We compute the cross efficiency scores based on cross efficiency matrix of Doyl and Green (1994). This method is a two steps process. The first step is the self-evaluation step where DEA weights (\(v_r^i, u_r^i\)) are generated by model (Problem (a)) and efficiency scores are calculated. In the second step, the obtained weights (from the first step) are applied to all peer DMUs in order to evaluate the cross efficiency scores:

\[
E^A_B = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} v_i^j u_j^i y_r B}{\sum_{i=1}^{n} \sum_{j=1}^{m} v_i^j x_{iB}}
\]

\(E^A_B\) is the cross efficiency of DMU \(B\) using the optimal generated weights of DMU \(A\) (\(v_r^i, u_r^i\)) in Problem (a). \(E^A_A\), \(E^A_B\) and \(E^B_B\) are calculated in the same way. \(E^A_A\) and \(E^B_B\) are the DEA efficiency scores of DMU \(A\) and DMU \(B\). It is expected that a good DMU has relatively high values for both efficiency scores. Finally, to discriminate between two DMUs, the average of these efficiency scores is computed as follows: \(E^*_A = (E^A_A + E^B_A)/2\) and \(E^*_B = (E^A_B + E^B_B)/2\). For the sake of simplicity, we put \(E^*_A = E^*_{AB}\) and \(E^*_B = E^*_{BA}\), the cross efficient scores of DMU \(A\) and DMU \(B\), respectively. In addition, we suppose \(E^*_{AB} \neq E^*_{BA}\) which is reasonable in practice.

Based on these results, the pairwise comparison matrix is generated as follow \(\forall j = 1, 2, ..., n\) and \(j \neq i\):

\[
a_{ij} = \begin{cases} 
E^*_i - E^*_j & \text{if } E^*_i > E^*_j \\
0 & \text{else}
\end{cases}
\]

As in PROMETHEE, we interpret the quantity \(a_{ij}\) as being the advantage (preference) of \(a_i\) over \(a_j\). Furthermore, for \(i = j\): \(a_{ii} = 0\). This gives us a pairwise comparison matrix that will then be exploited by means of the net flow scores.

In accordance with the PROMETHEE method (9) we have: \(a_{ij} + a_{ji} \leq 1 \left( \pi(a_i, a_j) + \pi(a_j, a_i) \right) \leq 1 \), \(a_{ij} \geq 0 \left( \pi(a_i, a_j) \geq 0 \right)\) and \(a_{ii} = 0 \left( \pi(a_i, a_j) = 0 \right)\).
In the second phase, net flow scores of the different units are computed from the new pairwise comparison matrix. Two cases may happen: either $E_{ij}^* > E_{ji}^*$ or else. We consider $I_{ij} = \begin{cases} 1 & \text{if } E_{ij}^* > E_{ji}^* \\ 0 & \text{else} \end{cases}$. It is obvious that $I_{ii} = 1 - I_{ij}$.

As a consequence, and according to (11), the net flow scores ($\Phi(a_i)$) are calculated as follows:

$$
\Phi(a_i) = \frac{1}{n-1} \sum_{j=1}^{n} [\pi(a_i, a_j) - \pi(a_j, a_i)]
$$

$$
= \frac{1}{n-1} \sum_{j=1}^{n} (a_{ij} - a_{ji})
$$

$$
= \frac{1}{n-1} \sum_{j=1}^{n} (E_{ij}^* - E_{ji}^*)I_{ij} - \sum_{j=1}^{n} (E_{ji}^* - E_{ij}^*)I_{ji}
$$

$$
= \frac{1}{n-1} \sum_{j=1}^{n} [(E_{ij}^* - E_{ji}^*)I_{ij} - (E_{ji}^* - E_{ij}^*)I_{ji}]
$$

Therefore:

$$
\Phi(a_i) = \frac{1}{n-1} \sum_{j=1}^{n} (E_{ij}^* - E_{ji}^*)
$$

(17)

Let us point out some properties of the proposed method:

- **The compatibility between the resulting ranks of the DEA and DEA-PROMETHEE**

We assume that model (3) is run between $n$ units with one input and one output. We aim to show that if $E_{i}^* > E_{i+k}^*$, $i = 1, 2, ..., n - 1$ and $k = 1, ..., n - i$, then $\Phi(a_i) > \Phi(a_{i+k})$. For this reason, we show if $E_{i}^* > E_{i+k}^*$, then $E_{i,i+k}^* > E_{i+k,i}^*$. We demonstrate this relation in appendix (from (A-1) to (A-11)).

Due to formulation (17): $\Phi(a_i) = \frac{1}{n-1} \sum_{j=1}^{n} (E_{ij}^* - E_{ji}^*)$ and $\Phi(a_{i+k}) = \frac{1}{n-1} \sum_{j=1}^{n} (E_{i+k,j}^* - E_{j,i+k}^*)$. By (16), when $E_{i,i+k}^* > E_{i+k,i}^*$ for $i = 1, 2, ..., n - 1$ and $k = 1, ..., n - i$, we have:

$$
\begin{aligned}
& \begin{cases}
E_{ij}^* - E_{ji}^* = E_{i+k,j}^* - E_{j,i+k}^* = 0, & \forall j: 1 \leq j \leq i \quad (a) \\
E_{ij}^* - E_{ji}^* > 0, & \forall j: i + 1 \leq j \leq i + k \quad (b) \\
E_{ij}^* - E_{ji}^* > E_{i+k,j}^* - E_{j,i+k}^*, & \forall j: i + k + 1 \leq j \leq n \quad (c)
\end{cases}
\end{aligned}
$$

(a) and (b) are evident. We already showed in (A-11) that $E_{i,i+k}^* > E_{i+k,i}^*$. Consecutively, we have $E_{i,i+k+1}^* > E_{i+k+1,i}^*$, ..., $E_{i,n}^* > E_{n,i}^*$. For showing (c), based on (A-9) and (A-10) $\forall j: i + k + 1 \leq j \leq n$ we can write:

$$
E_{ij}^* - E_{ji}^* = (E_{i,i+k+1}^* - E_{i+1,i+k+1}^*) + \cdots + (E_{i,n}^* - E_{n,i}^*) = \left(1 - \frac{y_{i+k+1}x_i}{x_{i+k+1}y_i}\right) + \cdots + \left(1 - \frac{y_nx_i}{x_{n}y_i}\right)
$$

(19)

(A-11) also claims that $E_{i,i+k+1}^* > E_{i+k+1,i+k}^*$, ..., $E_{i,n}^* > E_{n,i}^*$:

$$
E_{i,i+k+1}^* - E_{i+k+1,i}^* = (E_{i,i+k+1}^* - E_{i+k+1,i+k+1}^*) + \cdots + (E_{i,n}^* - E_{n,i}^*) = \left(1 - \frac{y_{i+k+1}x_i}{x_{i+k+1}y_i}\right) + \cdots + \left(1 - \frac{y_nx_i}{x_{n}y_i}\right)
$$

(20)

From (A-7) we have $\frac{y_i}{x_i} > \frac{y_{i+k}}{x_{i+k}}$ thus:

$$
\frac{y_{i+k}x_{i+k+1}}{x_{i+k}y_{i+k+1}} > \frac{y_{i+k+1}x_{i+k+1}}{x_{i+k+1}y_{i+k+1}} \Rightarrow \frac{x_{i+k}y_{i+k+1}}{y_{i+k+1}x_{i+k+1}} < \frac{x_{i+k}y_{i+k+1}}{y_{i+k+1}x_{i+k+1}} \Rightarrow 1 - \frac{x_{i+k}y_{i+k+1}}{y_{i+k+1}x_{i+k+1}} > 1 - \frac{x_{i+k}y_{i+k+1}}{y_{i+k+1}x_{i+k+1}}
$$

(21)
For other terms of (19) and (20) we can build the same relations as (21). Obviously, \( \forall j: i + k + 1 \leq j \leq n \) we have \( E_{ij}^* - E_{ji}^* > E_{i+k,j}^* - E_{j,i+k}^* \).

Therefore \( \emptyset(a_i) > \emptyset(a_{i+k}) \).

Actually, in this situation, it is not any more necessary using the cross efficiency since in each pairwise comparison \( E_{i,i+k}^* > E_{i+k,i}^* \).

Therefore, we can observe that both methods are compatible in the presence of just one input and one output. On the other hand, in the presence of several inputs and outputs, the resulting rankings of DEA and DEA-PROMETHEE are not compatible. This will be outlined in section 5.

- The monotonicity between the resulting ranks of the DEA and DEA-PROMETHEE

A DEA ranking result is said to be monotone if by adding a constant to the output of all DMUs (input of all DMUs), in an input-oriented model (output-oriented), the ranking order of DMUs does not change. As a consequence, the ranking in the new model is also monotone.

To show the monotonicity of the new model in the presence of just 1 input and 1 output, it is sufficient to show that the DEA model between \( n \) units is monotone. We run model (3) between units with single input and single output ((A-1) in appendix). We suppose that \( E_{i1}^* > E_{i+k}^* \) for \( i = 1,2,\ldots,n-1 \) and \( k = 1,\ldots,n-i \). From (A-7) we have:

\[
\frac{y_i}{x_i} > \frac{y_{i+k}}{x_{i+k}} \Rightarrow \frac{y_i}{x_i} > \frac{x_i}{x_{i+k}} \tag{22}
\]

We put \( y_i’ = y_i + \alpha \) and \( y_{i+k}’ = y_{i+k} + \alpha \) while \( \alpha \) is a constant. We aim to show \( E_{i1}'' > E_{i+k}'' \) (\( E_{i1}'' \) and \( E_{i+k}'' \) are efficiency scores of \( DMU_i \) and \( DMU_{i+k} \) after adding \( \alpha \) to their outputs, respectively):

\[
\frac{y_i’+\alpha}{x_i} > \frac{y_{i+k}’+\alpha}{x_{i+k}} \Rightarrow \frac{y_i’+\alpha}{x_i} > \frac{x_i}{x_{i+k}} \tag{23}
\]

To show (23), we should determine a condition. From (22) and (23) we can deduce:

\[
\frac{y_i’+\alpha}{y_{i+k}’} > \frac{y_i}{y_{i+k}} \Rightarrow (y_i’+\alpha)y_{i+k} > (y_{i+k}’+\alpha)y_i \Rightarrow y_iy_{i+k} + \alpha y_{i+k} > y_{i+k}y_i + \alpha y_i \Rightarrow \alpha y_i > \alpha y_{i+k} \tag{24}
\]

Therefore to have monotonicity in DEA ranking order, after adding \( \alpha \) to outputs of problem, following condition should be satisfied:

\[
y_{i+k} > y_i \quad \forall \alpha > 0 \tag{25}
\]

According to condition (25) \( E_{i1}'' > E_{i+k}'' \).

We already show in appendix if \( E_{i1}'' > E_{i+k}'' \), then \( E_{i,i+k}'' > E_{i+k,i}'' \) for \( i = 1,2,\ldots,n-1 \) and \( k = 1,\ldots,n-i \). Consecutively, if \( E_{i1}'' > E_{i+k}'' \), then \( E_{i,i+k}' > E_{i+k,i}' \); thus \( \emptyset'(a_i) > \emptyset'(a_{i+k}) \) and the new model is also monotone.

Furthermore, the input oriented BCC model is Translation Invariance (Cooper, W. W. et al., 2005, 2011); therefore, it is monotone also (we refer to it in appendix).

- No rank reversal in the new DEA-PROMETHEE method with adding a copy of the worst unit

In the PROMETHEE method, adding a copy of an existing alternative may result to rank reversal (Verly, C. and De Smet, Y., 2013). Here we show that adding a copy of the worst alternative does not cause rank reversal.

Let us, one more time, suppose that the primary ranking is \( E_1^* > E_{i+k}^* \), \( i = 1,2,\ldots,n-1 \) and \( k = 1,\ldots,n-i \). It shows that \( E_1^* > E_2^* > \cdots > E_n^* \). If we add a copy of the last unit to the existed units:
such that $a_n = a_{n+1}$ we have $E_n = E_{n+1}^*$. Obviously, there is no rank reversal in DEA. Respectively and with attention to (17), in the new pairwise comparison matrix, $\emptyset(a_n) = \emptyset(a_{n+1})$ and the preference of other alternatives does not change.

5. Numerical example

We first study the well-known problem of hypermarkets localization in Belgium (Brans, J. P. and Mareschal, B., 2002). Then, we consider an example from (Sinuany-Stern, Z. et al., 2000). We compare the result of the proposed model (DEA-PII) with the result of Sinuany-Stern and her colleagues.

**Example 1** - In the problem of hypermarkets localization in Belgium, 12 sites in 4 cities are considered. The problem is characterized by 2 inputs (minimized criteria in PROMETHEE): construction cost ($C_1$) and competition ($C_2$) and 3 outputs (maximized criteria in PROMETHEE): population ($C_3$), parking availability ($C_4$) and network access ($C_5$). We refer interested readers to (Brans, J. P. and Mareschal, B., 2002) for further details. Data are provided in Table 1.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Inputs (min)</th>
<th>Outputs (max)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_1$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>Alternatives/DMUs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a1 Anvers 1</td>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>a2 Anvers 2</td>
<td>21,3</td>
<td>0</td>
</tr>
<tr>
<td>a3 Bruges 1</td>
<td>8,2</td>
<td>2</td>
</tr>
<tr>
<td>a4 Bruges 2</td>
<td>6,6</td>
<td>1</td>
</tr>
<tr>
<td>a5 Bruges3</td>
<td>4,9</td>
<td>3</td>
</tr>
<tr>
<td>a6 Bruxelles 1</td>
<td>21,3</td>
<td>5</td>
</tr>
<tr>
<td>a7 Bruxelles 2</td>
<td>17,9</td>
<td>5</td>
</tr>
<tr>
<td>a8 Bruxelles 3</td>
<td>17,3</td>
<td>5</td>
</tr>
<tr>
<td>a9 Bruxelles 4</td>
<td>14,2</td>
<td>6</td>
</tr>
<tr>
<td>a10 Namur 1</td>
<td>10,4</td>
<td>3</td>
</tr>
<tr>
<td>a11 Namur 2</td>
<td>12,9</td>
<td>2</td>
</tr>
<tr>
<td>a12 Namur 3</td>
<td>9,6</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1- Evaluation table based on (Brans, J. P. and Mareschal, B., 2002)

As explained in the previous section, each pair of sites is compared by using a DEA approach (Problem (a)) to construct the required pairwise comparison matrix based on (16). In the case where Problem (a) gives two equal efficient units, the cross efficiency (15) is applied between each pair to compute $E_{ij}^*$ and $E_{ji}^*$. See Table 2 as a resulted matrix:

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
<th>$a_7$</th>
<th>$a_8$</th>
<th>$a_9$</th>
<th>$a_{10}$</th>
<th>$a_{11}$</th>
<th>$a_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0</td>
<td>0</td>
<td>0.3018</td>
<td>0.3279</td>
<td>0.4531</td>
<td>0.2106</td>
<td>0.3263</td>
<td>0.3271</td>
<td>0.2677</td>
<td>0.3018</td>
<td>0.0072</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.4967</td>
<td>0</td>
<td>0.2277</td>
<td>0.458</td>
<td>0.3826</td>
<td>0.4313</td>
<td>0.3875</td>
<td>0.4728</td>
<td>0.4673</td>
<td>0.413</td>
<td>0.3201</td>
</tr>
<tr>
<td>$a_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0056</td>
<td>0.2256</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.4288</td>
<td>0.0021</td>
</tr>
<tr>
<td>$a_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2401</td>
<td>0.0072</td>
</tr>
<tr>
<td>$a_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0275</td>
<td>0</td>
<td>0</td>
<td>0.1412</td>
<td>0.0555</td>
<td>0</td>
<td>0.0811</td>
<td>0.0081</td>
</tr>
</tbody>
</table>
Table 2 – Generated pairwise comparison matrix by DEA

For example, in comparing \( a_1 \) and \( a_2 \), \( E_{21}^* = E_{12}^* = 1 \). We apply cross efficiency scores to discriminate between them. Finally, \( E_{21}^* > E_{12}^* \), thus, \( a_{12} = 0 \) and \( a_{21} = E_{21}^* - E_{12}^* = 1 - 0.5033 = 0.4967 \). In the second step, \( \Phi(a_1) \) can be computed as follows (based on (17)):

\[
\Phi(a_1) = \frac{1}{12-1} \sum_{j=1}^{12}(E_{1j}^* - E_{j1}^*) = \frac{1}{11} (0.3013 + 0.3279 + 0.4531 + 0.2106 + 0.3263 + 0.3271 + 0.2677 + 0.3018 + 0.0072 + 0.1072 - 0.4967) = 0.194.
\]

In Table 3, we provide the scores and the ranking results of the DEA model (3) and the new method (DEA-PII).

<table>
<thead>
<tr>
<th>DEA-PII</th>
<th>DMUs Rank</th>
<th>DEA Scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3969</td>
<td>a_2</td>
<td>1</td>
</tr>
<tr>
<td>0.2282</td>
<td>a_11</td>
<td>1</td>
</tr>
<tr>
<td>0.1940</td>
<td>a_1</td>
<td>0.822</td>
</tr>
<tr>
<td>0.0174</td>
<td>a_6</td>
<td>1</td>
</tr>
<tr>
<td>-0.0108</td>
<td>a_9</td>
<td>1</td>
</tr>
<tr>
<td>-0.0438</td>
<td>a_12</td>
<td>1</td>
</tr>
<tr>
<td>-0.0478</td>
<td>a_3</td>
<td>1</td>
</tr>
<tr>
<td>-0.0559</td>
<td>a_8</td>
<td>0.8659</td>
</tr>
<tr>
<td>-0.1092</td>
<td>a_4</td>
<td>1</td>
</tr>
<tr>
<td>-0.1370</td>
<td>a_5</td>
<td>1</td>
</tr>
<tr>
<td>-0.2091</td>
<td>a_10</td>
<td>0.6059</td>
</tr>
<tr>
<td>-0.2228</td>
<td>a_7</td>
<td>0.9683</td>
</tr>
</tbody>
</table>

Table 3 – Comparison between DEA and DEA-PII rankings

DEA gives 8 efficient units. In this example, the DEA and DEA-PROMETHEE rankings are not perfectly compatible. Nevertheless, the 6 units out of 8 efficient units in DEA are ranked in top places in the new model. Site \( a_1 \) (Anvers 1), which belongs to inefficient units in DEA, has the 3rd rank however it has a good rank in the DEA-PII. Site \( a_4 \) and \( a_5 \) (Bruges 2 and 3) are two efficient units that do not have good places in the new ranking (rank 9 and 10). On the other hand, \( a_4 \) and \( a_5 \) have no distinct ranks in DEA. Site \( a_{10} \) (Namur 1) in both methods has a low rank. While DEA ranks 8 units out of 12 as efficient units, the DEA-PII provides a complete rank.

Example 2- Table 4 shows the data used by Sinuany-Stern and her colleagues. The problem is characterized by 4 DMUs according to 2 inputs and 2 outputs. They proposed a similar two steps approach that integrates DEA and AHP to completely rank units. In the first phase they construct the pairwise comparison matrix.
with a different method. Each member of this matrix is 

\[ a_{jk} = \frac{E_{jj} + E_{kj}}{E_{kk} + E_{kj}}, \]

\[ a_{jj} = 1 \] and \[ j, k = 1, \ldots, n \] is the number of DMUs. In AHP, \( a_{kj} = 1/a_{jk} \). As we explained in (16), \( a_{jk} \) shows the preference of unit \( j \) over unit \( k \). \( E_{jj} \) and \( E_{kj} \) are the evaluations given to unit \( j \) and \( E_{kk} \) and \( E_{kj} \) are the evaluations set to unit \( k \). We refer the interested readers to (Saaty, 1980, 1986) for details about AHP and to (Sinuany-Stern, Z. et al., 2000) for the DEA-AHP integrated approach.

### Table 4 - Database of Sinuany-Stern and her colleagues (2000)

<table>
<thead>
<tr>
<th></th>
<th>Input1</th>
<th>Input2</th>
<th>Output1</th>
<th>Output2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>50</td>
<td>55</td>
<td>10</td>
<td>56</td>
</tr>
<tr>
<td>B</td>
<td>130</td>
<td>60</td>
<td>12</td>
<td>78</td>
</tr>
<tr>
<td>C</td>
<td>68</td>
<td>96</td>
<td>45</td>
<td>9</td>
</tr>
<tr>
<td>D</td>
<td>45</td>
<td>30</td>
<td>35</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 5 presents the pairwise comparison matrix in DEA-AHP and DEA-PROMETHEE methods.

### Table 5a

<table>
<thead>
<tr>
<th></th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0,8508</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>1</td>
<td>1</td>
<td>1,1754</td>
<td>1</td>
</tr>
</tbody>
</table>

### Table 5b

<table>
<thead>
<tr>
<th></th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>0</td>
<td>0,1803</td>
<td>0,3571</td>
<td>0,1056</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0</td>
<td>0</td>
<td>0,0412</td>
<td>0</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>0</td>
<td>0,3585</td>
<td>0,1492</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 6 - Ranking results of DEA, DEA-AHP and DEA-PII

<table>
<thead>
<tr>
<th>DMUs</th>
<th>DEA-PII Scores</th>
<th>DMUs Rank</th>
<th>DEA Scores</th>
<th>DEA-AHP Scores</th>
<th>DMUs rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0,2143</td>
<td>1</td>
<td>1</td>
<td>0,4994</td>
<td>2-3</td>
</tr>
<tr>
<td>B</td>
<td>-0,1658</td>
<td>3</td>
<td>1</td>
<td>0,4994</td>
<td>2-3</td>
</tr>
<tr>
<td>C</td>
<td>-0,1825</td>
<td>4</td>
<td>0,851</td>
<td>0,4800</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>0,1340</td>
<td>2</td>
<td>1</td>
<td>0,5204</td>
<td>1</td>
</tr>
</tbody>
</table>

The DMUs does not have a complete rank in DEA and DEA-AHP methods. In DEA, 3 units out of 4 have equal efficiency scores and in DEA-AHP, units \( A \) and \( B \) have the same scores. But DEA-PII gives the complete rank of units. \( DMU_A \) has the first rank in DEA-PII but in DEA-AHP, it has the 2\textsuperscript{nd} or 3\textsuperscript{rd} rank in common with \( DMU_B \). In DEA, it is an efficient unit and has the same rank as \( B \) and \( D \). \( DMU_C \) in all mentioned ranking is in the last place.

### 6. Conclusion
In this work, we suggest a two steps integrated DEA-PROMETHEE II method to rank different units in the presence of multiple inputs and outputs. In the first phase, we compare each pair of units in the context of DEA (CCR). In this step, in the case of existing equal efficiency scores among two units, we used cross efficiency method to discriminate them. In the second phase we use PROMETHEE II to generate the net flow scores and a complete ranking of units. In a specific sample, we compare our results with a DEA-AHP model. In the DEA-PROMETHEE model we do not observe any tie in ranking while DEA-AHP gives a tied ranking.

For the particular case of one input and one output, we show a perfect compatibility between DEA and DEA-PROMETHEE II ranking results. In this case, we also show the monotonicity (A DEA ranking result is said to be monotone if by adding a constant to the output of all DMUs in an input-oriented model, the ranking order of DMUs does not change) in the DEA result and consecutively in the new model. Further, in adding a copy of the worst unit, there is no rank reversal in the proposed model.

We compare our methodology with a DEA-AHP approach on a numerical example. While DEA-AHP model gives the same ranking for some units, our approach presents a complete ranking.

The new integrated DEA-PROMETHEE method has two main characteristics: On the one hand, the PROMETHEE II pairwise comparison matrix is provided directly from input/output data, by applying DEA between each pair of units. On the other hand, the classic DEA models classify units into two main groups of efficient and inefficient (all efficient units have efficiency equal to 1). Whereas, in the new model, a complete ranking is provided based on PROMETHEE II. It should be mentioned that this integrated model does not replace DEA or PROMETHEE and it just supplies an additional analysis to complete rank of different units in the presence of multiple inputs and outputs. Discussing in Rank reversal of DEA-PROMETHEE method can be a future light for this research.

Acknowledgement
Hereby, I would like to express my appreciation to Prof. Dr. Yves De Smet for his precise assistance and comments that greatly improved this article.

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Appendix

We run model (3) between $n$ units when there is just a single input and a single output. We suppose that $E^*_i > E^*_{i+k}$ for $i = 1, 2, ..., n - 1$ and $k = 1, ..., n - i$. We aim to show that $E^*_{i(i+k)} > E^*_{i+k,i}$.

Model (3) with just one input and one output for $DMU_i$ is changed to:

$$E_i = \text{Max } u_i y_i$$
$$s.t.$$
$$v_i x_i = 1,$$
$$u_i y_i \leq 1,$$
$$u_i y_j - v_i x_j \leq 0, \forall j \neq i, j = 1, 2, ..., n,$$
$$u_i, v_i \geq 0.$$

By solving (A-1) the optimal solutions for $DMU_i$ are $v_i^* = 1/x_i$ and $u_i^* = \text{Min}_j \left\{ \frac{1}{v_j} \times \frac{x_j}{y_j} \right\}$ for $j = 1, ..., n, j \neq i$. The hypothesis is $E^*_i > E^*_{i+k}$ for $i = 1, 2, ..., n - 1$ and $k = 1, ..., n - i$. In other words we have $E^*_1 > E^*_2 > \cdots > E^*_n$ and $DMU_1$ is the best unit (it has an efficiency score equal to 1: $E^*_1 = 1$). As a consequence the successive efficiency scores can be written as follows:

$$\frac{y_1}{x_1} \text{Min}_{j \neq 1} \left\{ \frac{x_j}{y_j} \right\} > \cdots > \frac{y_j}{x_j} \text{Min}_{j \neq i} \left\{ \frac{x_j}{y_j} \right\} > \cdots > \frac{y_{i+k}}{x_j} \text{Min}_{j \neq 1} \left\{ \frac{x_j}{y_j} \right\} > \cdots > \frac{y_n}{x_n} \text{Min}_{j \neq 1} \left\{ \frac{x_j}{y_j} \right\}$$

(A-2)

It is evident that:
$$\{ \text{A}_i \subset \text{A}_1 \text{ since } \text{A}_i = \text{A}_1 \setminus \{ i \} \}$$

(A-3)

Thus:
$$\text{Min} \text{A}_1 = \cdots = \text{Min} \text{A}_i = \cdots = \text{Min} \text{A}_{i+k} = \cdots = \text{Min} \text{A}_n = \frac{y_1}{y_1}$$

(A-4)

Therefore the efficiency scores of $DMU_i$ and $DMU_{i+k}$ are as follows:

$$E_i^* = u_i^* y_i = \frac{y_1}{x_1} \times \text{Min} \text{A}_i = \frac{y_1 x_1}{x_1 y_1}$$

(A-5)

and

$$E_{i+k}^* = u_{i+k}^* y_{i+k} = \frac{y_{i+k}}{x_{i+k}} \times \text{Min} \text{A}_{i+k} = \frac{y_{i+k} x_1}{x_{i+k} y_1}$$

(A-6)

When $E_i^* > E_{i+k}^*$ for $i = 1, 2, ..., n - 1$ and $k = 1, ..., n - i$, we have $\frac{y_{i+k}}{x_{i+k}} > \frac{y_{i+k} x_1}{x_{i+k} y_1}$, thus:

$$\frac{y_i}{x_i} > \frac{y_{i+k}}{x_{i+k}}$$

(A-7)

Running (A-1) between just two units, $DMU_i$ and $DMU_{i+k}$, gives us:

$$E_{i(i+k)} = \text{Max } u_i y_i$$
$$s.t.$$ 
$$v_i x_i = 1,$$
$$u_i y_i \leq 1,$$
$$u_i y_{i+k} - v_{i+k} x_{i+k} \leq 0,$$
$$u_i, v_i \geq 0.$$
Since from (A-7) we have \( \frac{x_i}{y_i} < \frac{x_{i+k}}{y_{i+k}} \) therefore the efficiency scores of \( DMU_i \) and \( DMU_{i+k} \) are:

\[
E^*_i = u_i^* y_i = \text{Min} \left\{ \frac{1}{\frac{y_i}{x_i} \frac{y_{i+k}}{x_{i+k}}} \right\} \times y_i = \text{Min} \left\{ \frac{x_i}{y_i} \frac{x_{i+k}}{y_{i+k}} \right\} \times \frac{y_i}{x_i} = \frac{x_i}{y_i} = 1 \quad (A-9)
\]

\[
E^*_{i+k} = u_{i+k}^* y_{i+k} = \text{Min} \left\{ \frac{1}{\frac{y_i}{x_i} \frac{y_{i+k}}{x_{i+k}}} \right\} \times y_{i+k} = \text{Min} \left\{ \frac{x_i}{y_i} \frac{x_{i+k}}{y_{i+k}} \right\} \times \frac{y_{i+k}}{x_{i+k}} = \frac{x_i}{y_i} \frac{x_{i+k}}{y_{i+k}} < 1 \quad (A-10)
\]

So:

\[
\text{If } E^*_i > E^*_{i+k} \quad \forall i = 1, 2, ..., n-1, k = 1, ..., n-i: E^*_i > E^*_{i+k}, i \quad \blacksquare \quad (A-11)
\]

The input oriented BCC model is Translation Invariance (Cooper, W. W. et al., 2005, 2011) [23, 24]. Two constraints of (5) is changed as follows:

\[
\begin{align*}
\sum_{j=1}^{n} x_{ij} \lambda_j + s^-_i &= \theta x_{i0}, i = 1, 2, ..., m; \\
\sum_{j=1}^{n} y_{rj} \lambda_j - s^+_r &= y_{r0}, r = 1, 2, ..., s; \\
\end{align*}
\]

(A-12)

To show this characteristic of BCC models, we put \( y'_{rj} = y_{rj} + \alpha_r \) and \( y'_{ro} = y_{ro} + \alpha_r \) in the second constraint of (A-12) (the other parts do not be changed):

\[
\sum_{j=1}^{n} (y'_{rj} - \alpha_r) \lambda_j - s^+_r = y'_{ro} - \alpha_r. \\
\]

(A-13)

The equality of (A-13) is the result of considering the convexity condition \( \sum_{j=1}^{n} \lambda_j = 1 \). The optimal solution of both problems is \( (\lambda^*_j, s^*_j, s^*_r) \). Thus this model is Translation Invariance and consecutively monotone. As it can be seen, the model is not more limited with single input and single output. In this working paper we do not entering in discussions of BCC cross efficient models.\[\blacksquare\]
Determining new possible weight values in PROMETHEE: a procedure based on Data Envelopment Analysis

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Abstract
Operational Research (OR) offers efficient tools to support managers in strategic decision making processes. Data Envelopment Analysis (DEA) and Multiple Criteria Decision Aid (MCDA) are two important research areas in OR. These two domains are both based on the evaluation of “objects” according to multiple “points of views”. Within the MCDA framework, choosing appropriate weights for the different criteria often arises as a problem itself for Decision Makers (DMs). As a consequence, researchers have developed original methodologies to help them during this elicitation phase. In this work we aim to investigate how DEA can be used to propose weights in the context of the PROMETHEE II method. More precisely we suggest an extension of the so-called “decision maker brain” used in the GAIA plane (also known as PROMETHEE VI) but in the base of DEA to enrich the understanding of the problem. The underlying idea is based on the computation of weights in PROMETHEE (GAIA brain) which are compatible with the DEA analysis. We end this paper with a numerical example.

Keywords: Data Envelopment Analysis, Multiple Criteria Decision Aid, Weights, PROMETHEE II, GAIA plane, GAIA brain.

1. Introduction
Nowadays, researchers in Multi-Criteria Decision Aid (MCDA) are more and more interested in integrated approaches. Among others, the combined use of Data Envelopment Analysis (DEA) and MCDA has already received a lot of attention.

The main objective of DEA models is to provide a scalar measure of efficiency for, so-called, Decision Making Units (DMUs). It relies on direct comparisons between DMUs in an input-output system. In the end, the computed score allows to rank the different DMUs from the least to the most efficient ones. For the last four decades, the initial work of Charnes, Cooper and Rhodes (1978), CCR, has been extensively enriched leading to dynamic research areas in DEA (Seiford, L. M., 1997; Subhash, C. R., 2004; Avkiran, N. K. and Rowlands, T., 2006; Cheng, E. W. L. et al., 2007).

During the same period, MCDA also knew a tremendous evolution. Its aim is to provide support to Decision Makers (DMs) involved in problems where possible choices are evaluated on several conflicting criteria. Usually, one distinguishes three main approaches (Vincke, Ph., 1992): interactive methods...
In this context, the notion of “optimal solution” (which is at the core of traditional Operational Research (OR) models) disappears and leaves room for the concept of “best compromise solution” which heavily relies on the preferences of DMs. Obviously there is a strong analogy between the problems tackled by DEA and the decision making problems in multicriteria analysis (Roy, B., 1985). Indeed, inputs and outputs in DEA may be viewed as attributes or criteria in MCDA. Moreover, DMUs might be seen as alternatives. Doyle and Green introduced DEA as an aid to MCDA methods (Doyle, J. and Green, R., 1993). Stewart (Stewart, T. J., 1996) mentioned the connection of the two fields on the basis of the objective function. While in DEA, the efficiency frontier is determined by optimizing the weighted sum of outputs over the weighted sum of inputs, in MCDA, assessing and ranking alternatives is based on the conflicting set of criteria and subjective judgments. Similarly, Ishizaka and Nemery (Ishizaka, A. and Nemery, P., 2013) indicated the differences between DEA and MCDA based on their mechanism of comparing actions. Sarkis (Sarkis, J., 2000) called DEA as thoughtless approach for MCDA to evaluate alternatives objectively. Through others, this similarity has been clearly pointed out in the works of Belton and Vickers (Belton V. and Vickers, S.P., 1993) and has led to the creation of special interest groups to study the interactions between DEA and different approaches of MCDA. Among these interactions, we can cite the Generalized Data Envelopment Analysis (GDEA) models. The purpose of GDEA, in general, is to present an extension of existing DEA models such as CCR and BCC in a unified formulation. The GDEA models show that the idea of DEA can be applied in MCDA problems by incorporating value judgments of DMs in decision support systems (Yu G. et al., 1996; Nakayama, H. et al., 2002; Kleine, A., 2004; Jahanshahloo G.H. et al., 2009). Multiple Criteria Data Envelopment Analysis (MCDEA) is presented in the paper of (Zhao, M. Y. et al., 2006). They proposed a new two-stage methodology to fully rank alternatives according to various inputs and outputs. This model can incorporate both qualitative and quantitative criteria. In the first stage, they compared the qualitative performance of alternatives. Then, in the second stage, MCDEA was used to rank the alternatives due to the relative degree of qualitative factors. Kao (Kao, Ch., 2010) worked on a common-weight DEA model to rank alternatives in a MCDA problem. He proposed a measure of relative distance and computed the relative position of an alternative between the ideal and anti-ideal measures. Yilmaz and Yurdusev (Yilmaz, B. and Yurdusev, M. A., 2011) used a DEA method as a tool to solve a MCDA problem. They deployed an Assurance Regions (AR) approach to integrate DMs preferential judgments in solving an irrigation management problem and improve the discriminating power of DEA. ARs are one of the popular weight restrictions methodologies, which impose weights ratios to be within certain ranges.

One of the main differences between DEA and MCDA is the role played by DMs. Indeed, in a multicriteria framework, the preferences of DMs have to be determined cautiously. Among others, weight values have to be assessed and tested carefully whereas in DEA, it is the model itself which generates the weights automatically. It is worth noting that some papers have been proposed to use MCDA methods in DEA in order to restrict possible weights values. Shang et Sueyoshi (Shang, J. and Sueyoshi, T., 1995), Sinuany-Stern et al. (Sinuany-Stern, Z., Mehrez, A. and Hada, Y., 2000), Liu (Liu, C.C., 2003) and Takamura and Tone (Takamura, Y. and Tone, K., 2003) are the researchers investigated the combination of AHP (Saaty, T. L., 1980) and DEA for this purpose. AHP is a multi-criteria measurement technique based on pairwise comparisons and relies on experts judgments. Liu (2003) combined DEA and AHP to integrate two objective and subjective weight restrictions methods. He derived two weights for each input/output: one from AHP and other from a classical DEA model. Then these weight values are aggregated to obtain a complete ranking. Takamura and Tone (Takamura, Y. and Tone, K., 2003) integrated AR and AHP. Junior (Junior, H. V., 2008) used MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) (Bana et Costa et al., 2003) as a MCDA method to give restrictions to DEA weights in the Wong and Beasley virtual Weights Restrictions DEA model (Wong Y. H. and Beasley J. E., 1990). He constructed a judgment matrix by using MACBETH to determine the bounds of weights. This matrix is built by asking the DMs to judge the variety of attractiveness among each pair of actions. The resulted weights are normalized such that the
minimum and maximum of these values set as lower and upper bounds to use in the virtual weights restrictions DEA model.

This paper reversely investigates how a traditional DEA model can be applied in the specific context of the MCDA, PROMETHEE II method. As far as we know, it is the first time one investigates the application of DEA in PROMETHEE to help DMs. In the PROMETHEE method, a complementary visual tool, called GAIA (Geometrical Analysis for Interactive Aid) (Mareschal, B. and Brans, J. P., 1988) is often used in practice. For instance, it may help the DM to perform a sensitivity analysis based on weight values and enrich the perceptive of the problem. The so-called “decision maker brain” (GAIA brain) module (or PROMETHEE VI) allows assessing the impact of different weight values on the final ranking. In this context, the DM is supposed to provide weight intervals to be tested. More specifically in this work, we analyse how the ranking induced by DEA may constraint the weights used in constructing an extension of the GAIA brain. This extension is not the same as the GAIA brain in PROMETHEE VI but it provides DM the same information about alternatives and criteria and the space of freedom in changing weights. Regarding to this brain (so-called DM brain), we are able to analyse a MCDA problem. The originality of this contribution is using DEA to identify and illustrate possible weight values in PROMETHEE II that are simultaneously consistent with the induced DEA ranking. The underlying idea is to define a polyhedron of weight values (GAIA brain) that are compatible with the DEA analysis. This will be characterized by a linear system. The VEA (Vertex Enumeration Algorithm) will be used to enumerate possible vertices. Then, these points will be projected on the GAIA plane and the associated convex hull will be determined. Finally a geometric mean will applied on the resulted weight matrix (vertices) to propose an initial weight vectors to DM.

The paper is structured as follows: in the next section we will summarize the traditional DEA models. Section 3 will detail PROMETHEE II and GAIA. In section 4 we will describe the proposed methodology. This will be then applied in an illustrative example.

2. Data Envelopment Analysis (DEA)

DEA is based on an input-output system to evaluate the relative efficiency of different DMUs. The computed scores allow ranking the DMUs from the least to the most efficient ones. The efficiency score of a DMU is assessed by computing the ratio of total weighted of outputs divided by the total weighted of inputs (Cooper, W. W. et al., 2011).

The first model was proposed by Charnes, Cooper & Rhodes (1978). Intuitively, the method is based on mathematical programming in order to first determine an efficient frontier and to identify DMUs that belong to it. Then, the relative efficiency of other inefficient units is estimated based on their distances from this frontier. A DMU is expected to have an input and an output vector. Formally, let us assume there are \( n \) DMUs to be rated and each DMU uses \( m \) inputs and \( s \) outputs. A DMU \( j \) is characterized by an input vector \( X_j = (x_{i_1j}; x_{i_2j}; \ldots; x_{i_mj})^{T} \), \( x_{ij} \) is the \( i_\text{th} \) input of DMU \( j \) and an output vector \( Y_j = (y_{r_1j}; y_{r_2j}; \ldots; y_{r_sj})^{T} \), \( y_{rj} \) is the \( r_\text{th} \) output of DMU \( j \) where \( j = 1,2,\ldots,n \), \( i = 1,2,\ldots,m \) and \( r = 1,2,\ldots,s \).

DEA assigns weights automatically to the inputs and outputs of a given DMU in order to assess the best possible efficiency score. The DEA model is a non-statistical and non-parametric method (Doyle, J. and Green, R., 1993) that has been applied to assess the efficiency of DMUs in a huge number of sectors. Among them we can cite: bank branches (Ioannis, E. T., 2011), the educational institutions (Salerno, C., 2006), hospitals (Elokou, A., 2010), insurance companies (Hwang, Sh. and Kao, T. L., 2008), electric utility sectors (Vaninsky, A. Y., 2008), cities, countries, etc.

One of the initial DEA model is the CCR model (Charnes, A., Cooper, W. W. and Rhodes, E., 1978):

\[
Max \ Z = \frac{\sum_{r=1}^{s} w_{r} y_{rj}}{\sum_{i=1}^{m} w_{i} x_{ij}}
\]
subject to
\[ \sum_{r=1}^{s} u_r y_{rj} \leq \sum_{i=1}^{n} v_i x_{ij}, j = 1, 2, ..., n \] (2)
\[ v_i, u_r \geq 0, \forall i, r \]

The above formulation supposes that \( x_{ij}, y_{rj} \geq 0, \forall i, r, j \) when \( x_{ij} \) and \( y_{rj} \) are the quantity of input \( i \) and output \( r \) in \( DMU_j \), subsequently. \( v_i \) and \( u_r \) are related weights of each input and output in \( DMU_j \). The scalar \( Z \) in (1) is the efficiency indicator of \( DMU_o \) which is less or equal to 1. All variables are assumed to be non-negative.

Finally, let us note that when the denominator (sum of weighted inputs for \( DMU_o \)) in (1) is forced to be equal to 1, we obtain the following formulation:

\[ \text{Max} z = \sum_{r=1}^{s} u_r y_{ro} \]

subject to
\[ \sum_{i=1}^{m} v_i x_{io} = 1 \]
\[ \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, 2, ..., n \] (3)
\[ v_i, u_r \geq 0, \forall i, r \]

This problem is solved \( n \) times in order to obtain the efficiency scores of all DMUs.

Model (3) is an Input-Oriented CCR model. It determines how to improve the inputs of a unit to become efficient while remaining at the same output level (Cooper, W. W. et al., 2011). Later, in section 4, we will assume the inputs and the outputs of DEA as criteria or attributes of the multi-criteria problem, with maximization of outputs and/or minimization of inputs as associated objectives (Doyle, J. and Green, R., 1993).

The CCR model generally characterizes a Constant Returns to Scale (CRS) (Banker, R. D., 2004). By adding a constant to the objective function (1) and relatively, changing the constraints, the CCR model is changed to Variable Returns to Scale (VRS) (Banker, R. D., 2004). The constant is free in sign. This leads to BCC formulation (Banker, R. D., Charnes, A. and Cooper, W. W., 1984). For two reasons, in this paper, we avoid putting BCC formulation and working with them. First, the BCC model generates more ties (efficient units with efficiency scores equal to 1) rather than CCR models; since, the DMUs with the lowest input or highest output levels are assessed efficient in the BCC model, thus, efficiency scores are always at least equal to the one given by the CCR model (Cooper, W. W. et al., 2011). On the other hand, however, the Super-Efficient CCR model may be infeasible (Thrall, R. M., 1996), but the possibility of infeasibility in the classic BCC Super-Efficient (Andersen, P. and Petersen, N. C., 1993) model is more than Super-Efficient CCR model (Zhu, J., 1996, Lovell, C. A. K. and Rouse, A. P. B., 2003). Developing our proposed algorithm in section 4 to a VRS model may be a possible light for future works.

Furthermore let us point out the Super-Efficient model to show a complete ranking of DMUs. It was developed by Andersen and Petersen to rank the efficient DMUs (Andersen, P. and Petersen, N. C., 1993). The \( oth \) constraint will be removed in the formulation to enable an extreme efficient unit \( o \). It computes the distance between the evaluated unit \( o \) and the Pareto frontier that evaluated without unit \( o \) for \( j = 1, 2, ..., n, j \neq o \).

The result is efficiency greater than 1. It can be appeared as follows:

\[ \text{max} z = \sum_{r=1}^{s} u_r y_{ro} \]

subject to
\[ \sum_{i=1}^{m} v_i x_{io} = 1, \]
3. PROMETHEE and GAIA

Many decision making problems are based on the simultaneous optimization of several conflicting criteria to sort, rank and choose between alternatives. During the last five decades, several MCDA techniques like MAUT (Keeney, R. and Raiffa, H., 1976), AHP (Saaty, T. L., 1980), ELECTRE (Roy, B., 1968) families and PROMETHEE (Brans, J.P. and Vincke, Ph., 1985) have been developed to facilitate the decision making process. In this paper, we will focus ourselves on the PROMETHEE II outranking method. The family of PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) methods is known thanks to their simplicity, the existence of user friendly softwares: D-Sight (Hayez, Q. et al., 2012), Visual PROMETHEE and Smart Picker and an important number of applications in different fields including business, financial management, environment management, water management, education, health care centres, logistics, transportation, manufacturing, energy management, insurance companies, chemistry, social and other fields (Behzadian, M. et al., 2010).

3-1. PROMETHEE II

The PROMETHEE method was initiated by Brans (Brans J. P., 1982) and extended by Vincke and Brans (Brans, J.P. and Vincke, Ph., 1985). It is based on pair wise comparisons. PROMETHEE II allows a DM to full rank a finite set of n actions \( A = \{a_1, ... a_j, ..., a_n\} \), \( j = 1, ..., n \) that are evaluated over a set of q criteria \( F = \{f_1(a_j), ..., f_k(a_j), ..., f_q(a_j)\} \). Let \( f_k(a_j) \) denotes the evaluation of action \( a_j \) on criterion \( f_k \).

In what follows, we will assume (without loss of generality) that criteria have to be maximized.

The first step of the method is related to the determination of performance differences. More formally:

\[
d_k(a_i, a_j) = f_k(a_i) - f_k(a_j), \quad \forall a_i, a_j \in A, \forall k = 1, ..., q
\]

The framework of PROMETHEE methods is based on the definition of preference functions to aggregate the related information on each criterion. Therefore one has to associate a generalized criterion \( \{f_k(a_i), P_k(a_i, a_j)\} \) to each criterion, \( \forall a_i, a_j \in A, \forall k = 1, ..., q \). \( P_k(a_i, a_j) \) provides the preference strength of action \( a_i \) over \( a_j \). This is characterized by the function \( P_k(d_k(a_i, a_j)) \). It is assumed to be a non-decreasing function of \( d_k(a_i, a_j) \). The method provides the DM with a set of predefined preference functions for each of which at most two parameters have to be defined (preference and indifference thresholds). There exists 6 main different types of preference functions (Brans, J.P. and Vincke, Ph., 1985) but in this paper we limit ourselves to linear type since it involves an area of indifference and preference; further, it covers four other types of functions. Figure 1 shows a linear preference function:

![Linear preference function](image)

Fig. 1- Linear preference function. \( q_k \) and \( p_k \) are respectively indifference and preference thresholds for criterion \( k \) when in a maximization problem \( d_k(a_i, a_j) = f_k(a_i) - f_k(a_j) \). If \( d_k \in [0, q_k] \), \( a_i, a_j \) are considered indifferently on criterion \( k \). If \( d_k \) is greater than \( p_k \), \( a_i \) is strictly preferred to \( a_j \). In the distance between these 2 thresholds, preference function linearly increases.
The concept of preference function is used to transform the differences into unicriterion preference degrees, thus:
\[
\pi_k(a_i, a_j) = P_k[d_k(a_i, a_j)], \forall a_i, a_j \in A, \forall k = 1, \ldots, q
\]  
(6)

The third step is the computation of outranking (preference) degree as follows:
\[
\pi(a_i, a_j) = \sum_{k=1}^{q} \pi_k(a_i, a_j) w_k, \forall a_i, a_j \in A, \forall k = 1, \ldots, q
\]  
(7)

Where \(w_k\) is the normalized positive weight associated to criterion \(k\). This degree varies between 0 and 1. We have:
\[
\pi(a_i, a_j) \geq 0, \pi(a_i, a_j) + \pi(a_j, a_i) \leq 1, \forall a_i, a_j \in A
\]  
(8)

In step four, each alternative \(a_j\) when compared with \((n-1)\) other alternatives in \(A\), the positive and negative outranking flows are defined as follows:
\[
\emptyset^+(a_j) = \frac{1}{n-1} \sum_{x \in A} \pi(a_j, x) \quad \text{the positive outranking flow}
\]  
(9)

\[
\emptyset^-(a_j) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a_j) \quad \text{the negative outranking flow}
\]  
(10)

The PROMETHEE I ranking is the intersection of the two rankings induced by these flows. As a consequence, it allows obtaining a partial pre-order of the alternatives.

Moreover, a complete pre-order, called PROMETHEE II, can be obtained on the basis of the net flow score as follows:
\[
\emptyset(a_j) = \emptyset^+(a_j) - \emptyset^-(a_j), \forall a_j \in A
\]  
(11)

Mareschal and Brans (Mareschal, B. and Brans, J. P., 1988) have proposed a complementary geometrical tool that helps the DM to better understand the results presented by the PROMETHEE rankings. This tool, called GAIA, is based on a Principal Component Analysis (PCA) of the unicriterion net flow scores.

### 3-2. GAIA (Geometrical Analysis for Interactive Aid)

In the previous section equation (10) leads us point out that the net flow score can also be computed as follows:
\[
\emptyset(a_j) = \sum_{k=1}^{q} \emptyset_k(a_j) w_k, \forall a_j \in A
\]  
(12)

Thus, we have:
\[
\emptyset_k(a_j) = \frac{1}{n-1} \sum_{x \in A} (P_k(a_j, x) - P_k(x, a_j)), \forall a_j \in A, k = 1, \ldots, q
\]  
(13)

The quantity \(\emptyset_k\) is called the unicriterion net flow of action \(a_j\) and is such that \(-1 \leq \emptyset_k(a_j) \leq 1\). At this point, it is worth noting that the multicriteria problem can be viewed as an evaluation table (and associated parameters) or a matrix \(\emptyset = (\emptyset_k(a_j))\). These values already integrate intra-criterion parameters and are all lying within the same range.

In order to obtain a two dimensional visualization of the \(k\)-dimensional space of unicriterion net flows matrix (\(\emptyset\)), a principal component analysis is applied. The obtained projection plane is called GAIA.
The two vectors, \( u \) and \( v \), characterizing the GAIA plane, are the eigenvectors associated to the two highest eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of the matrix selected \( nC = \emptyset \emptyset \). The quality of the projection can be computed as follows:

\[
\delta = \frac{\lambda_1 + \lambda_2}{2} \sum_{k=1}^{\lambda_k}
\]  

(13)

A complete description of the GAIA plane and its interpretation can be found in (Mareschal, B. and Brans, J. P., 1988).

The GAIA plane facilitates the decision making process. Indeed, several interpretations can be done according to the relative positions of the alternatives between themselves, in comparison with the different criteria, between the criteria and with the decision stick. Let us recall that the decision stick is the projection of the weights vector on the GAIA plane. For instance, it allows detecting groups of similar and/or incomparable alternatives, conflicting and/or redundant criteria, best compromise solutions, rankings of alternatives according to different criteria, etc.

A further use of the GAIA plane relies on sensitivity analysis and robustness. Indeed a DM may hesitate to assign precise weight values to the different criteria. Therefore, one may investigate the impact of weight variations in the final ranking. On the one hand, the DM may explicitly change the values, one by one, and control their impacts. This can be done with the so-called “walking weights” tool. On the other hand, he/she may want to define weight intervals within which the values are likely to vary:

\[
w_k^- \leq w_k \leq w_k^+, k = 1, ..., q
\]

(14)

Where \( w_k^- \) and \( w_k^+ \) are determined values by DM. These intervals can also be verified by tolerating a percentage \( (\alpha_k, k = 1, ..., q) \) in this value:

\[
w_k^+ \pm \alpha_k, w_k^-, k = 1, ..., q
\]

(15)

\( \alpha_k \) is a value which is determined by DM(s). This information will determine a hypercube of possible weight values in the k-dimensional space of unicriterion net flow scores; the projection of this hypercube on the GAIA plane will permit to identify a region where the decision stick will belong. Depending on the position of this region with respect to the alternatives, one may determine those that could eventually become good compromise solutions and those that will never be considered as good candidates. This tool is referred to as the “GAIA brain” (PROMETHEE VI) (Brans, J. P. and Mareschal, B., 2002).

4. Methodology

In the GAIA brain tool, the DM is assumed to freely propose weights intervals to be tested (Brans, J. P. and Mareschal, B., 2002). In some cases, he/she may find this task to be difficult. The idea of the proposed approach is to suggest weights intervals for the PROMETHEE VI method that are compatible with the DEA analysis. More precisely, if the DEA analysis leads to state that the efficiency of alternative \( a_i \) is higher than alternative \( a_j \), we restrict weight values of the PROMETHEE II ranking such that \( a_i \) has a lower rank than \( a_j \) (a lower rank is assumed to be better than a higher rank). The weight intervals computed in this way are communicated to the DM as a starting point. Then, he/she remains free to modify these values in order to better represent the weights he/she wants to investigate. It should be noted that the aim of forcing alternatives in PROMETHEE to be in the same order with the DEA analysis is creating an extension of GAIA brain, without knowledge in initial weights values, to deepen in the conception of the multi-criteria problem and help to DMs.

The methodology can basically be summarized in the following steps:

7- We assume that the evaluation table, preference functions and parameters (indifference and preference thresholds and weights) are available;

8- Criteria are partitioned into two groups: inputs (criterias to be minimized) and outputs (criterias to be maximized);
9- A DEA analysis is performed: DMUs are ranked according to their efficiency scores; whether they can be divided into efficient and non-efficient units or be ranked completely via a Super-Efficient model;
10- The PROMETHEE unicriterion flows, \( \phi_k(a_j) \) are computed for all criteria and alternatives;
The ranking computed at step (3) is used to generate weight constraints on the PROMETHEE II ranking. More precisely, if the efficiency of \( a_i \) is better than the efficiency of \( a_j \) in DEA then the following linear constraint is defined in PROMETHEE II:

\[
\sum_{k=1}^{q} w_k \left[ \phi_k(a_i) - \phi_k(a_j) \right] > 0
\]  

(16)

If we put \( \phi_k(a_i) - \phi_k(a_j) = \Delta_{ij} \), the following linear system should be solved:

\[
\begin{align*}
\sum_{k=1}^{q} w_k \Delta_{ij} &> 0 \\
\sum_{k=1}^{q} w_k &= 1, w_k \geq 0
\end{align*}
\]  

(17)

The second equation in (17) shows that the sum of the nonnegative weights is normalized to 1.

11- These constraints define a polyhedron of PROMETHEE weights values that induce rankings which are compatible with the DEA analysis;
12- This polyhedron is projected on the GAIA plane;

Let us illustrate the mentioned compatibility in step 4 with an artificial example (in this case: \( k = 3 \) criteria and \( n = 6 \) alternatives). Table 1 summarizes the data associated to this instance.

<table>
<thead>
<tr>
<th>Min / Max Functions Type</th>
<th>Criteria</th>
<th>Inputs</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alts/DMUs</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a1</td>
<td>1</td>
<td>1.7</td>
<td>12</td>
</tr>
<tr>
<td>a2</td>
<td>2</td>
<td>4.1</td>
<td>23</td>
</tr>
<tr>
<td>a3</td>
<td>3</td>
<td>30</td>
<td>21</td>
</tr>
<tr>
<td>a4</td>
<td>3</td>
<td>5.5</td>
<td>4.5</td>
</tr>
<tr>
<td>a5</td>
<td>5</td>
<td>2.2</td>
<td>6.7</td>
</tr>
<tr>
<td>a6</td>
<td>8</td>
<td>14</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 1- Illustrative example, evaluation table and preference parameters

Figure 2 illustrates the potential Kendall’s correlation between the PROMETHEE II rankings and the DEA analysis for different values of \( w_1 \) and \( w_2 \) (since weights are normalized we have \( 1 - w_1 - w_2 = w_3 \)). In order to identify weight values that are compatible with the DEA analysis, 1000 instantiations (1000 normalized weight combinations) are randomly drawn from the hypercube \( \sum_{k=1}^{3} w_k = 1 \) and tested (1000 PROMETHEE II rankings are generated in these weights to compare with the DEA result). The compatible values help us to determine the area where should be projected on the GAIA plane. The lighter area shows
more correlation. It means that in lighter parts more weights induce the same ranking. High correlation values correspond to the lighter part of the graph. Clearly, in this particular instance, a reasonable part of the weights domain is compatible with the DEA analysis.

![The area with more compatible rankings](image)

Fig. 2- Kendall’s correlation between PROMETHEE II rankings and DEA ranking for an instance based on 3 criteria and 6 alternatives (n=6, k=3).

Each constraint (that imposes the compatibility) limits the area of the compatible domain. Conceptually, the features of the compatible weights domain can be represented as in figure 3.

![3a- Possible weight values which are compatible between the two rankings](image)

Fig. 3- Area of DEA-PROMETHEE compatible weights in a 3 criteria instance with 6 alternatives.

The central white convex polytope is the result of the constraints’ satisfaction of all related constraints in the instance. For this sample, we do not show the DEA analysis and resulted constraints but in the numerical example, all the steps and tables will be explained in details. In this contribution, we propose to project this area on the GAIA plane as shown on figure 4.
In this illustrative example, the δ value is equal to 90%. It means the amount of preserved information is highly significant. The dispersion of all criteria in different direction shows the presence of conflicts. Their lengths are long enough, thus all criteria are important in differentiation between alternatives. Alternatives a5 and a6 seem to be ranked before other alternatives since they are in the same direction with the GAIA brain. Alternative a5 is powerful on the C1 and alternative a6 is quite good on the C2. Alternative a3 acts good on the criteria C3 but not on other criteria.

The reasonable size of projected polyhedron in this plane indicates that few weight values are compatible with the DEA analysis. Moreover, it shows that all DEA compatible alternatives always lie in the right top part of the graph. This eventually leads to hesitations between a4, a5 and a6 but not with other options. A detailed example will be provided in the next section.

More formally, the intersections of constraints should be found by solving a system of linear equations. There exist different algorithms to solve such a system and find all its vertices. One of the most famous one is the VEA (Avis, D. and Fukuda, K., 1992) which we summarize it in the next sub section.

4-1. VEA (Vertex Enumeration Algorithm)

A convex polyhedron is defined as: \( P = \{x \in \mathbb{R}^q : Ax \leq b\} \) such that \( A = (a_{ij}) \) is a given matrix, \( i = 1,2, ..., m \) and \( j = 1,2, ..., q \), and \( b \) is a \( m \) dimensional vector. Polyhedron \( P \) may be empty (\( P = \emptyset \)) or unbounded (e.g. \( P = \{x \in \mathbb{R}^q : x \geq 0\} \)). A bounded polyhedron is a polytope. It is normally assumed a full dimensional situation for a polytope, i.e., there exists an interior point \( x \) that strictly satisfies all inequalities of \( P \). A vertex (\( v \)) is a point \( x \in P \) if and only if it is the single solution of a subset of \( q \) inequalities solved as equations. The vertex enumeration problem is to generate all vertices of a polytope \( P \).

The problem of enumerating all vertices of a polyhedron was first investigated by Mattheiss and Rubin (Matheiss, T.H. and Rubin, D. S., 1980) and Dyer (Dyer, M.E., 1983). There are two classes of methods: first, the "double description" method of Motzkin et al. (Motzkin, T.S., Raiffa, H., Thompson, G.L. and Thrall, R. M., 1953) and the second method is based on pivoting (Dyer, M.E., 1983). David Avis and Komei Fukuda (Avis, D, and Fukuda, K., 1992) presented a pivot-based algorithm to find \( v \) vertices of a polyhedron in \( \mathbb{R}^q \) defined by a non-degenerate system of \( m \) inequalities (or dually to find \( v \) facets). A vertex lying on more than \( q \) hyperplanes in \( \mathbb{R}^q \) is called degenerate. In this algorithm, each vertex is contained on exactly \( q \) hyperplanes. The related path to find each vertex is found by pivoting. It means interchanging one of the
equations defining the vertex with one not currently used. Details about this method can be seen in (Avis, D, and Fukuda, K., 1992). The basic idea is based on following steps:

4- Find any starting point as a vertex \( \nu \);

5- Create edges from \( \nu \) subsequently;

6- Move to any unsearched vertex adjacent to \( \nu \); If none back to previous step.

There exist different software packages and programs to solve this problem. In this work, we use an extension of a MATLAB code (Kelder, M, 2005) to enumerate all vertices of a polyhedron while \( m \geq q \) (\( m \) constraints, \( q \) variables). This program converts the polyhedron (convex polygon, polytope, etc.) defined by the system of inequalities \( Ax \leq b \) and equalities into a list of vertices \( v \). It employs a primal-dual polytope method. The feasible region should have some finite extent in all dimensions. It also identifies the list of redundant constraints. Further, it finds the redundant vertices in dimensions higher than 2, detect their redundancies at up to 6 digits of precision and finally returns the unique vertices.

Another problem that may arise is the existence of unbounded constraints. Each inequality outlining the polyhedron \( P \) is bounded by a hyperplane to make a polytope. When these hyperplanes do not limit entirely the polyhedron (polyhedron \( P \) is unbounded), the program returns an error. In this case, to induce bounding and continue to solve the problem, bounds should be determined. We may detect this problem by defining the large box constraints on the variables. For example if \( A = [0 \ 1; \ 1 \ 0; \ 1 \ 1] \) and \( b = [1 \ 1 \ 1] \), the MATLAB code returns an error for unboundedness. We define \( A = [A; 0 -1; 0 \ 1; -1 \ 0; 1 \ 0] \) and \( b = [b; 2; 1000; 2; 1000] \). Actually, we determine the bounding box of constraints on variables within \([-1,1000]\). Indeed, the constraints boxes are determined as large that do not modify tangibly the space as the definition of problem changes but give bounds to variables. As a toy sample, if \( x \) is a tree’s height in meter, the box constraint \([-1,1000]\) for \( x \) can be a reasonable choice to create boundedness, since all possible solutions for \( x \) are possible within this box.

The generated intersection points are the compatible weights between DEA analysis and PROMETHEE II ranking. In the final step these points are projected on the GAIA plane and the related convex hull is determined.

5. Numerical example
In order to illustrate the proposed method, we consider the problem of ranking 12 hyper markets in Belgium according to five criteria [Brans, J. P. and Mareschal, B., 2002].

We summarize the first step of algorithm (section 4) in Table 2. It shows the alternatives, the considered criteria, the evaluations as well as preference functions and parameters. In the second step, we partitioned the criteria into two groups. Criteria include: construction cost \( (C_1) \) and competition \( (C_2) \) (minimization) and population \( (C_3) \), parking availability \( (C_4) \) and network access \( (C_5) \) (maximization). The minimized and the maximized criteria are supposed to be the inputs and outputs in DEA, respectively. As it has been mentioned, in this work, we limit ourselves to the linear preference functions. The indifference \( (q) \) and preference \( (p) \) thresholds are determined by DMs according to their behavior.
In step 3, we analyse the multi-criteria problem (Table 2) via the DEA model (3). Further, we may use the Super-Efficient model (4) to rank completely all units. Table 3 shows the DEA results:

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>Linear</td>
</tr>
<tr>
<td>q=0.5, p=3.2</td>
<td>q=0, p=75</td>
<td>q=0, p=225</td>
</tr>
<tr>
<td>C2</td>
<td>Min</td>
<td>Max</td>
</tr>
<tr>
<td></td>
<td>Linear</td>
<td>Linear</td>
</tr>
<tr>
<td>a1</td>
<td>Anvers 1</td>
<td>21</td>
</tr>
<tr>
<td>a2</td>
<td>Anvers 2</td>
<td>21.3</td>
</tr>
<tr>
<td>a3</td>
<td>Bruges 1</td>
<td>8.2</td>
</tr>
<tr>
<td>a4</td>
<td>Bruges 2</td>
<td>6.6</td>
</tr>
<tr>
<td>a5</td>
<td>Bruges 3</td>
<td>4.9</td>
</tr>
<tr>
<td>a6</td>
<td>Bruxelles 1</td>
<td>21.3</td>
</tr>
<tr>
<td>a7</td>
<td>Bruxelles 2</td>
<td>17.9</td>
</tr>
<tr>
<td>a8</td>
<td>Bruxelles 3</td>
<td>17.3</td>
</tr>
<tr>
<td>a9</td>
<td>Bruxelles 4</td>
<td>14.2</td>
</tr>
<tr>
<td>a10</td>
<td>Namur 1</td>
<td>10.4</td>
</tr>
<tr>
<td>a11</td>
<td>Namur 2</td>
<td>12.9</td>
</tr>
<tr>
<td>a12</td>
<td>Namur 3</td>
<td>9.6</td>
</tr>
</tbody>
</table>

Table 2- Evaluation table and preference parameters, based on (Brans, J. P. and Mareschal, B., 2002)

As result of DEA analysis (Table 3), DMUs are divided into two groups: efficient and inefficient. Later in step 4, we use these preferences to generate weight constraints on the PROMETHEE II ranking.
In step 4, we compute the PROMETHEE unicriterion flows, \( \varphi_k(a_j) \) for all criteria and alternatives. Table 4 presents the unicriterion net flows matrix.

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>-0.815</td>
<td>0.576</td>
<td>0.121</td>
<td>-0.701</td>
<td>-0.455</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>-0.818</td>
<td>0.818</td>
<td>0.286</td>
<td>-0.656</td>
<td>-0.455</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>0.596</td>
<td>0.333</td>
<td>-0.505</td>
<td>0.397</td>
<td>0.636</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>0.808</td>
<td>0.576</td>
<td>-0.869</td>
<td>-0.045</td>
<td>-0.182</td>
</tr>
<tr>
<td>( a_5 )</td>
<td>0.949</td>
<td>0</td>
<td>-0.835</td>
<td>0.897</td>
<td>0.455</td>
</tr>
<tr>
<td>( a_6 )</td>
<td>-0.818</td>
<td>-0.606</td>
<td>1</td>
<td>0.36</td>
<td>-0.182</td>
</tr>
<tr>
<td>( a_7 )</td>
<td>-0.37</td>
<td>-0.606</td>
<td>0.818</td>
<td>-1</td>
<td>-0.455</td>
</tr>
<tr>
<td>( a_8 )</td>
<td>-0.357</td>
<td>-0.606</td>
<td>0.495</td>
<td>0.13</td>
<td>-0.455</td>
</tr>
<tr>
<td>( a_9 )</td>
<td>0.364</td>
<td>0</td>
<td>-0.699</td>
<td>-0.216</td>
<td>0.455</td>
</tr>
<tr>
<td>( a_{10} )</td>
<td>0.051</td>
<td>0.333</td>
<td>-0.139</td>
<td>0.147</td>
<td>0.636</td>
</tr>
<tr>
<td>( a_{11} )</td>
<td>0.441</td>
<td>0</td>
<td>-0.224</td>
<td>0.848</td>
<td>-0.455</td>
</tr>
</tbody>
</table>

Table 4- Unicriterion net flows scores matrix

The ranking computed at step (3) (Table 3) generates weight constraints on the PROMETHEE II ranking. When the efficiency of \( a_i \) is better than the efficiency of \( a_j \) in DEA then the linear constraint in (16) \( \sum_{k=1}^{q} w_k \cdot [\varphi_k(a_i) - \varphi_k(a_j)] > 0 \) is used in PROMETHEE II. Generally it can be said that the number of constraints in considering the complete ranking via Super-Efficient model (4) is:

\[
M = c(n, 2)
\]

\( c(n, 2) \) is the binomial coefficient. Thus, the number of possible resulted hyperplanes (constraints) is: \( M = c(12,2) = 66 \). Between these constraints, the positive ones are considered in constructing the central polytope (GAIA brain). Further the normalization equation is also added to the system (\( \sum_{k=1}^{5} w_k = 1 \)) (17).

In a classic DEA model (column 1 in Table 3), the number of constraints is computed as follows:

\[
M' = c(n, 2) - c(N(\text{efficiency} = 1, 2)
\]

(19)

When \( n \) is the number of alternatives/DMUs and \( N(\text{efficiency} = 1) \) is the number of equal efficient DMUs. Literally, the value of \( M' \) shows the number of absolute preferences between each pair of actions in DEA (not considering preferences between equal efficient units). Hence \( M' = c(12,2) - c(8,2) = 38 \).

In this example, we study all possible constraints, \( M + 1 = 67 \), (the case of Super-Efficient rank) to take into consideration all aspects of problem in generating GAIA brain. In applying (19), just the number of constraints is reduced. As a result the size of GAIA brain is also reduced slightly. As mentioned, the priority in PROMETHEE II rankings should be compatible with the DEA analysis. For instance in Table 3, the Super-Efficient rank induces the following constraints: \( \sum_{k=1}^{5} [\varphi_k(a_{11}) - \varphi_k(a_4)], w_k > 0, \sum_{k=1}^{5} [\varphi_k(a_{14}) - \varphi_k(a_5)], w_k > 0, \sum_{k=1}^{5} [\varphi_k(a_{13}) - \varphi_k(a_3)], w_k > 0, ..., \sum_{k=1}^{5} [\varphi_k(a_8) - \varphi_k(a_1)], w_k > 0 \) and \( \sum_{k=1}^{5} [\varphi_k(a_{10}) - \varphi_k(a_{10})], w_k > 0 \). According to linear system written in (17), the related constraints can be written as follows:
This system, which includes inequalities and one equality, considers the existing compatibility between DEA analysis and PROMETHEE II ranking in problem of ranking hyper markets with the aim of generating the GAIA brain (Step 5).

The VEA, as explained in section 4-1, is used to solve this system. In this example, the problem is bounded; thus we do not get into the discussion of unboundedness. If we encounter the unboundedness, as already explained, we should define enough large constraints boxes on the variables \( w_k \).

Finally, in step 6, the computed weight vectors via VEA system are projected on the GAIA plane and the related convex hull is determined as illustrated in figure 5.

The \( \delta \) value is 82%. It means the amount of lost information is acceptable. Moreover, we see that the GAIA brain has a reasonable size and does not encompass the centre of the graph. This brain illustrates the space of freedom for DM. Clearly it indicates that the set of alternatives can be split into two subsets. It is unlikely that, given these weights values, alternatives \( a_1, a_2, a_6, a_7, a_8 \) and \( a_9 \) will be chosen. Among them, it seems that \( a_1 \) and \( a_2 \) have not at all importance in any criteria but \( a_6, a_7, a_8 \) and \( a_9 \) are quiet good on population (C_3). The other alternatives show a higher degree of compatibility with the DEA ranking as presented in table 4 (alternatives \( a_{11}, a_{12}, a_5 \)). Alternatives \( a_3, a_4, a_8, a_{10}, a_{11} \) and \( a_{12} \) act well on construction cost (C_1).
Alternative a_{12}, relatively, performs well enough on C_4 (it has reasonable good access to parking place). This alternative does not act well on competition and population (C_2 and C_3). It should be paid attention that however, in GAIA plane, a_{12} is close enough to C_5 but, in fact, it does not perform well in this criterion according to Tables 2 and 3 (because of losing information in projection process). Some other alternatives such as a_3, a_5 and a_{11} perform well in criterion C_5. The dispersion of criteria on GAIA plane shows us three different directions. Obviously, C_1, C_4 and C_5 act more similar than other criteria.

Solving the system (17), when inequalities considered as equalities, gives also a matrix of weights (the possible vertices). There exist different methods in computing more homogenous weights from a weight matrix. Taking average from appropriate criteria weights (inputs and outputs weights) is one of the proposed methods. Here, a geometric mean can be applied to compute a common weight for each criterion (Jablonsky, J., 2012). These weights can be suggested to DMs as a first proposal in parameterizing PROMETHEE; further the area of GAIA brain is the area of DM’s freedom in changing weights. The idea of geometric mean is come from the Common Set of Weights (CSW) (Cook, W. D. et al., 1990 and Roll, Y. et al, 1991, Roll, Y., 1993) problem in DEA. CSW has been suggested to reduce the flexibility in choosing inputs and outputs weights in evaluating the efficiency of units. The use of CSW makes possible ranking different units on the same basis and a homogenous set of weights (Wang, Y. M. et al., 2011). The geometric mean can be the connection between DEA and PROMETHEE to decrease the bias of criteria weight vectors and offer a common weight for each criterion (Jablonsky, J., 2012).

The result of constraints satisfaction in (20) gives a weight matrix. The geometric mean of each column gives following weight vector: \( W = (0.2566, 0.1505, 0.1898, 0.1446, 0.1772) \). Thus \( w_1 = 25.6\% \), \( w_2 = 15\% \), \( w_3 = 19\% \), \( w_4 = 14.5\% \) and \( w_5 = 17.7\% \). This weight can be communicated to DM. He/she can start from these preferences to parametrize the decision making problem in PROMETHEE.

6. Conclusion

In this contribution, we have proposed a tool to help the DM in the determination of weights for PROMETHEE VI. The underlying idea consists to identify possible weight values that are compatible with a DEA analysis. When the projected area is small and outlying, it allows to quickly identifying alternatives that are compatible with DEA and those which are not. It is worth noting that these values are proposed to the DM as a first suggestion to parameterize PROMETHEE VI. Later, he/she remains free to refine them.

Among the future research directions, the integration of this model in a PROMETHEE based software like D-SIGHT is proposed. Naturally, other investigations will be conducted on the use of DEA to elicit weights in the PROMETHEE methods. One important future light for this investigation is taking into account the problem of DEA in weight freedom (some factors may be disregarded from the assessment because DMUs can assign extremely low (near to zero) weights to some factors), applying some weighted DEA model to solve this problem. Furthermore, in DEA, in some cases, specific outputs are directly dependent on specific inputs and it is expected to find a link between the weights assigned to these factors.

In this contribution, we have investigated a first way to apply DEA to a particular multicriteria method, namely PROMETHEE. Let us point out that we could also consider the problem in the other way; how can we apply PROMETHEE to enhance classic tools of DEA. This is the topic of another paper by the same authors.
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Use of DEA and PROMETHEE II to Assess the Performance of Older Drivers

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Abstract

In recent years, there has been an increasing concern regarding the safety and mobility of elderly drivers. This study aims to evaluate the overall performance and ranking of a sample of 55 drivers, aged 70 and older, based on data from an assessment battery and a fixed-based driving simulator, by using the concept of composite indicators and multi criteria approach. To do so, drivers completed tests of an assessment battery of psychological and physical aspects as well as knowledge of road signs. Moreover, they took part in a driving simulator test in which scenarios that are known to be difficult for older drivers were included. Composite indicators (CIs) are becoming increasingly recognized as a tool for performance evaluation, benchmarking and policy analysis by summarizing complex and multidimensional issues. One of the essential steps in the construction of composite indicators is aggregation and assignment of weights to each sub-indicator which directly affect the quality and reliability of the calculated CIs. In this regard, Data Envelopment Analysis (DEA) and Multi Criteria Decision Aiding (MCDA) have been acknowledged as two popular methods for aggregation and problem solving: ranking, sorting and choosing.

In this case study, we apply a DEA model to calculate the optimal performance index score for each driver. On the other hand, we apply a MCDA method to enrich the analysis of this problem by considering preferential information from Decision Makers (DM) using both the raw and the normalized data. Applying GAIA plane and Spider web, as two strong graphic tools in PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) can help Decision Makers (DM) to put managerial preferences in problem, analyze drivers’ profiles, give complementary advices to them and deepen the results.

The results of this study show that the best and the worst drivers identified by the two models are similar. Respectively, this complementary approach leads us to global improvement of our analysis and suggest older drivers more insight in characterizing their driving performance. These observations point out the interest of using PROMETHEE II and DEA. The significant correlation between these results confirms the robustness of our answers.
Keywords. Multiple Criteria Decision Aiding; PROMETHEE II; Data Envelopment Analysis; Composite Indicator; Older drivers; Driving Performance

1. Introduction

The number of elderly drivers is increasing as a result of demographic changes (Mathieson et al. 2013). Although driving helps elderly to maintain their independence and autonomy, ageing is associated with decline in sensory, motor and cognitive abilities which affects the ability to drive safely. To help elderly drivers to be aware of their own abilities and weaknesses and to regularly check their driving performance, there is an increasing need for developing a reliable assessment procedure to determine whether a person is still fit to drive.

The aim of this study is to provide a method to screen older drivers and to assess their relative performance, using data from an assessment battery and a fixed-based driving simulator. Within a performance improvement framework, performance evaluation plays a critical role in identifying weaknesses and planning goals for improvement. In this regard, composite indicators (CIs) are increasingly recognized as a valuable tool for performance evaluation, benchmarking and policy analysis by summarizing complex and multidimensional issues such as driving performance. One of the critical steps in the construction of a CI is weighting and aggregation which directly affects the quality and reliability of the calculated CI (OECD 2008). In this respect, Data Envelopment Analysis (DEA) and Multi Criteria Decision Aiding (MCDA) have been considered as two popular methods for problem solving. In the road safety context, the technique of DEA has been applied for input-output data sets (Hermans et al. 2008; Shen et al. 2011) and composite indicator research (e.g. Shen et al. 2012; Bax et al., 2012). We may also cite the work of Bao about the use of the TOPSIS method (Technique for Order Preference by Similarity to Ideal Solution) to combine safety performance indicators into an overall index for a set of European countries (Bao, 2010). Recently, a combination of the PROMETHEE and Analytical Hierarchy Process (AHP) were used to assess the performances of marked and unmarked pedestrian crosswalks at unsignalized intersections (Zhao, 2013). Raquel Morte and her colleagues applied PROMETHEE and MMASSI (Multi-criteria Methodology for Supporting the Selection of Information Systems) to develop a method for evaluating truck’s driver performance. They finally appraised and ranked 31 drivers according to 13 criteria.

Evidently, there is a strong correspondence between the problems tackled by DEA and the ranking problems in multicriteria analysis (Roy, B., 1985). Indeed, inputs and outputs in DEA should be viewed as criteria in MCDA. Moreover, DMUs are considered as alternatives. Through others, this similarity has been clearly pointed out in the works of Belton and Vickers (Belton V. and Vickers, S.P., 1993) and has led to the creation of special interest groups to study the interactions between DEA and different approaches of MCDA. Between them we can cite the Doyle and Green work illustrated DEA as an aid MCDA (Doyle, J. and Green, R., 1993). Similarly, Ishizaka and Nemery (Ishizaka, A. and Nemery, P., 2013) indicated the differences between DEA and MCDA based on their mechanism of comparing actions. Yilmaz and Yurdusev (Yilmaz, B. and Yurdusev, M. A., 2011) used a DEA method as a tool to resolve a MCDA problem. They explained that DEA can be a supportive method to be used beside to MCDA. Shang and Sueyoshi (Shang, J. and Sueyoshi, T., 1995), Liu (Liu, C.C., 2003) and Takamura and Tone (Takamura, Y. and Tone, K., 2003) worked between DEA and AHP (Saaty, T. L., 1980). Baltazar and her colleagues (2014) used DEA and a MCDA method, MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique), to evaluate airports performance and efficiency.

In this paper, we aim to evaluate the overall performance of a sample of 55 drivers aged 70 and older, and to rank them from the best to the worst. In practical, we apply a DEA model to calculate the optimal performance index score for each driver based on data from an assessment battery of psychological and physical aspects and a fixed-based driving simulator. Based on the results of our analysis, the best
performers, as benchmarks, are distinguished from underperforming ones, and all drivers are ranked by computing their cross index scores.

On the other hand, we apply a MCDA method to enrich the analysis of this problem by considering preferential information from Decision Makers (DM). PROMETHEE II outranking method is used to generate a complete ranking of drivers by pairwise comparisons of all the individuals under study. This comparison is done for the raw and normalized data to quantify to what extent the normalization of the evaluations is impacting drivers’ ranking. Consecutively, we compute the correlation between the results. Finally, we analyse the profile of each driver by using visualisation tools such as the GAIA plane resulting from the PROMETHEE methods and the Spider web representation. These complementary analyses allow us to identify more specifically how each driver could improve his/her performance in driving.

The rest of this paper is organized as follows: In the next section we will introduce the indicators and related data. Section 3 will detail data analysis based on DEA and PROMETHEE II. In section 4 we will summarize and discuss the results. This paper ends with conclusions (section 5).

2. Indicators and Data

2.1. Participants

Subjects aged 70 and older were recruited through the Geriatrics department of the Jessa Hospital with flyers distributed in the hospitals, senior associations and senior flats via local media. Participants had to hold a valid driver's license and still be active car drivers, with no stroke in the last four months and without any indication for dementia as assessed with the Amsterdam Dementia Screening (ADS) test. They had to have the physical ability to complete tests of a clinical assessment battery and simulator driving. 77 volunteers agreed to participate. Among them, 22 participants were excluded due to simulator sickness. Thus, 55 participants remained in the sample (mean age = 76.49; standard deviation = 5.40).

2.2. Procedure

The test procedure consisted of two parts: First, a validated neuropsychological test battery including psychological and physical tests, as well as knowledge of road signs was administered at the Jessa Hospital. Next, a driving simulator test was conducted at the Transportation Research Institute of Hasselt University. For the purpose of this study, the following tests from the battery were incorporated in the analysis.

2.2.1. Psychological ability

The Mini Mental State Examination (MMSE)

The mini mental state examination (MMSE) is the most commonly used test for screening cognitive function. It is an 11-questions measure that investigates different areas of cognitive function: orientation to time and place, short term memory, registration (immediate memory), recall, constructional ability as well as language functioning (Folstein et al. 1975). The possible score ranges from 0 to 30. Scores of 25-30 out of 30 are considered normal. The higher the score, the better the psychological ability.

The Digit Span Forward (DSF)

DSF is originally part of the Digit Span Subtest of the Wechsler Adult Intelligence Scale (Wechsler 1955) where a random sequence of numbers is read by the examiner and the examinee recalls the numbers back. It assesses attention and working memory, as well as short-term verbal memory (Clark et al. 2011). Scores on this task are divided into four categories (0 = impaired, 1= beneath average, 2 = average, 3 = above average). The more numbers a person can repeat correctly, the better the psychological ability.
**UFOV**

It is a computer-based test of functional vision and visual attention, which consists of three subtests measuring visual processing speed (UFOV 1), divided attention (UFOV 2), and selective attention (UFOV 3) (Edwards et al. 2005). It is recommended for use as a screening measure in conjunction with a clinical examination of cognitive functioning or fitness to drive. Scores are expressed in milliseconds for each subtest and range from 16.7ms to 500ms. Lower scores correspond with improved visual attention.

### 2.2.2. Physical ability

**The Snellen Chart**

This test is used for measuring visual acuity and is one of the most common clinical measurements of visual function (Rosser et al. 2001). Participants have to stand 6m from the whiteboard with several lines of black letters and read the lines. The more lines a person can read, the better the visual acuity (maximum score = 1.2).

**The Get-Up-and-Go test**

The Get-Up-and-Go test, also known as Timed Up-and-Go or Rapid Pace Walk, assesses mobility and balance of older adults (Carr et al., 2010). It measures, in seconds, the time taken by an individual to stand up from a standard arm chair, walk a distance of 3m, turn around, return and sit down again (Clark et al. 2011). Scores on this task are divided into three categories (0 = more than 20 seconds, 1= between 11 and 20 seconds, 2= less than 11 seconds). The faster one can complete the task, the better the motor ability.

**The Four-test Balance Scale**

This test is also used to assess motor abilities; more specifically, lower limb muscle strength and balance, with a maximum score of 1. An individual has to stand on 4 different foot positions of increasing difficulty - standing feet together, standing semi-tandem, standing tandem and one leg standing - for at least 10 seconds without an assistive device (Gardner et al. 2001).

### 2.2.3. Knowledge of Road Signs

The Road Sign Recognition (RSR) test is used to measure the knowledge of participants regarding road signs with a maximum score of 12 (Lundberg et al. 2003).

### 2.2.4. Driving ability evaluation

Driving performance was measured in a fixed-based medium-fidelity driving simulator (STISIM M400; Systems Technology Incorporated) with a force-feedback steering wheel, brake pedal, and accelerator. The visual virtual environment was presented on a large 180° field of view seamless curved screen, with rear view and side-view mirror images. Three projectors offer a resolution of 1024 × 768 pixels on each screen and a 60 Hz refresh rate. Data were collected at frame rate.

A 10 minutes practice session preceding the evaluation was implemented to allow participants to become familiar with the driving simulator. Information on the crash types of older drivers was taken into account and situations were included in the scenarios that are known to be difficult for older drivers. These situations are well documented in the literature. For instance, older drivers are over-represented in crashes occurring while turning off at intersections, where typically the older driver turns against oncoming traffic with right of way on the main road (Hakamies-Blomqvist, 1993; Zhang et al., 1998), gap acceptance while turning left at an intersection (Langford and Koppel, 2006; Yan et al., 2007) and response to signs, signals and road hazards (Horswill et al., 2010). A detailed description of the driving scenario is mentioned in Cuenen et al.
The rides took place at inner-city (50 km/h) sections, outer-city (70-90 km/h) sections and highway (120 km/h) sections, in daylight and good weather conditions. The speed limit was indicated by the appropriate sign at the start of each outer-city and inner-city segment and repeated 30 meters after each intersection.

A total of 3 driving measures or indicators are used in the analysis: 1) Mean-Complete Stop which is computed from 200 meters before reaching the stop sign until the location of the stop sign. 2) Average Following Distance, between the driver and a lead vehicle in a road with a speed of 50 km/h and 70 km/h. 3) Mean driving Speed which is measured across separate road segments (i.e., 4.8 km) without any events (Trik et al., 2010) with the posted speed limits of 50, 70, 90 and 120 km/h.

3. Data analysis

In this study, to measure the multi-dimensional concept of driving performance, a composite indicator is created with respect to all aforementioned indicators for older drivers (see Figure 1). Simplistically, the composite indicator synthesizes the information included in the selected set of indicators in one score (Nardo et al., 2005). It should be mentioned that the driving measures and road sign recognition are considered in the group of driving ability.

Before the CI construction, normalization is carried out to tackle the different measurement units of the indicators. Among existing methods (Freudenberg 2003), the distance to a reference approach (OECD, 2008) is used in this study since the ratio of two numbers is best kept by this approach. Thereafter, for the older-driver performance index construction, DEA is first applied. In doing so, the multiple layer model is adopted to take the hierarchical structure of the indicators into account and the cross index method is used for the ranking of the drivers. On the other hand, a MCDA method is applied to enrich the analysis of this problem by considering preferential information from DMs. PROMETHEE II (Preference Ranking Organization METHod for Enrichment of Evaluations) outranking method is used to generate a complete ranking of drivers by pairwise comparison of all the drivers under study. Then, the simultaneous analysis of the results of the DEA and PROMETHEE II models would allow us to better describe the profile of each driver and to identify methodological opportunities about a complementary use of these methodologies. Both of them are presented in the following sections.

3.1. DEA for CI construction

Data Envelopment Analysis (Cooper et al. 2007) is one of the most commonly used techniques for performance evaluation. It is a non-parametric optimization technique using a linear programming tool to measure the relative efficiency of a set of Decision Making Units (DMUs), or drivers in our study. Recently, there has been an increasing interest to the application of DEA in the construction of CIs (Despotis 2005; Cherchy et al. 2008; Hermans et al. 2008). By solving a linear programming problem, the best possible indicator weights are determined, and an optimal index score between zero and one is obtained for each unit, with a higher value indicating a better relative performance.
In this study, to evaluate the driving performance of each older driver by combining all the 16 hierarchically structured indicators in one index score, a multiple layer DEA based composite indicator model (MLDEA-CI) developed by Shen et al. (2011, 2012) is adopted.

**Multiple Layer DEA-based CI model (MLDEA-CI)**

Unlike the DEA-WEI model in which all the indicators are equally treated as they belong to the same layer which may lead to weak discriminating power and unrealistic weight allocations, a multiple layer DEA-based CI model (Shen et al., 2011) is able to take the layered hierarchy of indicators into account that often exists in reality. The main idea of this model is to first aggregate the values of the indicators within a particular category of a particular layer by the weighted sum approach in which the sum of the internal weights equals one. Then, for the first layer, the weights for all the sub-indexes are determined using the basic DEA approach. Figure 2 depicts a general hierarchical structure of MLDEA-CI with \( K \) layers and \( s \) indicators (\( y \)).

\[ \text{figure 1: Hierarchical structure of older driver’s performance indicators.} \]

---

1 The sum-up-to-one requirement for the internal weights is necessary for the linear transformation of the model. In doing so, normalized data should be used before aggregation so as to remove scale differences.
More specifically, suppose that a set of \( n \) DMUs is to be evaluated in terms of \( s \) indicators \((y)\) with a \( K \) layered hierarchy, the MLDEA-CI model can be formulated as follows:

\[
\begin{align*}
\max \quad & G_{\theta} = \sum_{k=1}^{s(K)} \left( \sum_{r_{k-1} \in A_{k}^{(K-1)}} \left( \cdots \left( \sum_{r_{1} \in A_{1}^{(2)}} \left( \sum_{r_{1} \in A_{1}^{(1)}} \left( \sum_{r_{1} \in A_{1}^{(1)}} p_{r_{1}}^{(1)} y_{r_{1},s} \right) \right) \right) \right) \right) \\
\text{s.t.} \quad & \sum_{r_{k} \in A_{k}^{(K)}} \left( \cdots \left( \sum_{r_{1} \in A_{1}^{(2)}} \left( \sum_{r_{1} \in A_{1}^{(1)}} p_{r_{1}}^{(1)} y_{r_{1},s} \right) \right) \right) \leq 1, \quad j = 1, \ldots, n \\
& \sum_{r_{k} \in A_{k}^{(K)}} p_{r_{k}}^{(K)} = 1, \quad p_{r_{k}}^{(K)} \geq \zeta, \quad r_{k} = 1, \ldots, s^{(K)}, \quad k = 1, \ldots, K - 1 \\
& u_{r_{k}} \geq \varepsilon, \quad r_{k} = 1, \ldots, s^{(K)}
\end{align*}
\]  

Where \( \zeta \) is a small given number with the same purpose as the \( \varepsilon \) in DEA models. This way, the existing hierarchical structure between the indicators is reflected in the model and weights on one layer can be treated differently from the ones on another layer. However, these weights are not given directly but to be deduced from the mathematical model.
Since all the weights mentioned above are not given directly, their multiplication will lead up to a nonlinear model, and the more indicators to consider, the longer the iteration times and the harder to derive an optimal solution. To handle this problem, Shen et al. linearized the model by using some variable substitutions (e.g., \( \hat{u}_{r_1} = \prod_{k=1}^{K-1} p_r^{(k)}_{r_k \in A_r^{(k+1)}} u_{r_k} \) see Shen et al., 2011 for more details). The linear MLDEA-CI model is as follows:

\[
\begin{align*}
CL_o &= \max \sum_{r_1=1}^{s} \hat{u}_{r_1} y_{r_1o} \\
\text{s.t.} & \\
\sum_{r_1=1}^{s} \hat{u}_{r_1} y_{r_1j} & \leq 1, \quad j = 1, ..., n \\
\frac{\sum_{r_1 \in A_r^{(k+1)}} \hat{u}_{r_1}}{\sum_{r_1 \in A_r^{(k+1)}} \hat{u}_{r_1}} &= p_r^{(k)}_{r_k \in A_r^{(k+1)}} \in \Theta, \quad r_k = 1, ..., s^{(k)}, \quad k = 1, ..., K - 1 \\
\hat{u}_{r_1} & \geq 0, \quad r_1 = 1, ..., s
\end{align*}
\]

where

\( \hat{u}_{r_1} \) is the unknown weight for \( r_1^{th} \) indicator of DMU\(_0\),

\( \Theta \) indicates the restrictions imposed to the corresponding internal weights.

The subscript \( o \) refers to the driver whose index score is to be obtained by solving the constrained optimization problem, which maximizes the index value of the driver and satisfies the imposition restrictions. The first set of restrictions guarantees an intuitive interpretation of the composite indicator and implies that no driver in the data set can be assigned an index value larger than one under these weights. The model involves \( n \) normalization constraints (\( \sum_{r_1=1}^{s} u_{r_1} y_{r_1j} \leq 1, \quad j = 1, ..., n \)). With respect to the second set of restrictions, the layered hierarchy of the indicators is reflected by specifying the weights in each category of each layer and further restricting their flexibility. In doing so, obtainment of realistic and acceptable weights is guaranteed. In addition, by the third restriction, all weights are restricted to be non-negative.

Note that prior to the application of the MLDEA-CI model, the raw data should be first normalized so as to eliminate the scale differences of the indicators and the effects of the measurement unit, and moreover, to ensure that all the indicators are expressed in the same direction with respect to their expected safety performance impact, i.e., a higher indicator value should always correspond to a better performance.

### 3.2. Cross index score

In addition, to fully rank all the drivers, the cross index method is employed. The main idea of this method is to evaluate the performance of a DMU using not only its own optimal weights, but also the ones of all other DMUs (Sexton et al., 1986). It means that a cross index matrix is to be developed in a way that the element in the \( i^{th} \) row and \( j^{th} \) column represents the index score of DMU\(_j\) using the optimal weights of DMU\(_i\) as shown in Table 1. The basic DEA efficiencies are thus located in the leading diagonal.
Table 1. A generalized Cross Index Matrix (CIM)

<table>
<thead>
<tr>
<th>DMU</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$l_{11}$</td>
<td>$l_{12}$</td>
<td>$l_{13}$</td>
<td>...</td>
<td>$l_{1n}$</td>
</tr>
<tr>
<td>2</td>
<td>$l_{21}$</td>
<td>$l_{22}$</td>
<td>$l_{23}$</td>
<td>...</td>
<td>$l_{2n}$</td>
</tr>
<tr>
<td>3</td>
<td>$l_{31}$</td>
<td>$l_{32}$</td>
<td>$l_{33}$</td>
<td>...</td>
<td>$l_{3n}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>$l_{n1}$</td>
<td>$l_{n2}$</td>
<td>$l_{n3}$</td>
<td>...</td>
<td>$l_{nn}$</td>
</tr>
</tbody>
</table>

| Mean | $\bar{l}_1$ | $\bar{l}_2$ | $\bar{l}_3$ | ... | $\bar{l}_n$ |

To rank the DMUs using the cross index method, the average of each column is calculated to obtain the mean cross index score. Mathematically, the index score of DMU$_j$ using the weights produced by evaluation model of DMU$_o$ is defined as follows:

$$CIS_{oj} = \sum_{r=1}^{s} u_r y_{rj} \quad j = 1, ..., n$$  \hspace{1cm} (3)

Then cross-index score for DMU$_j$ is defined as follow:

$$\overline{CIS}_j = \frac{\sum_{o=1}^{n} CIS_{oj}}{n} \quad j = 1, ..., n$$  \hspace{1cm} (4)

### 3.3. PROMETHEE II

Multiple Criteria Decision Aid (MCDA) techniques like MAUT (Keeney et al. 1979), AHP (Saaty 1980), ELECTRE (Roy 1991) and PROMETHEE (Brans, 1982) have been developed during the last five decades. Their objective is supporting DM in the selection of most compromise solution(s) and the ranking or sorting of alternatives. In this work, we focus on PROMETHEE II. The family of PROMETHEE methods is known thanks to their simplicity, number of applications in different fields such as finance, business, education, health care centers, insurance companies, etc. (Behzadian et al. 2010) and the existence of user friendly software, D-sight (Hayez, Q. et al., 2012).

The PROMETHEE method has been created by J. P. Brans in 1982 (Brans, 1982) and developed by Vincke and Brans (Brans, J.P. and Vincke, Ph., 1985). It is based on pairwise comparisons. PROMETHEE II allows a DM to full rank a finite set of $n$ actions (DMUs in DEA) $A = \{a_1, ..., a_j, ..., a_n\}$ that are evaluated over a set of $q$ criteria (like indicators in DEA) $F = \{f_1(a_j), ..., f_k(a_j), ..., f_q(a_j)\}$. We suppose that $f_k(a_j)$ is the evaluation of action $a_j$ on the criterion $f_k$. In what follows, we assume without loss of generality that criteria have to be maximized. First, the differences between every pair of actions on all criteria are computed as follows:

$$d_k(a_i, a_j) = f_k(a_i) - f_k(a_j), \forall a_i, a_j \in A, \forall k = 1, ..., q$$  \hspace{1cm} (5)
The preference functions contribute the main framework of PROMETHEE to integrate intra-criterion information. Thus in the second step, a generalized criterion \( f_k(\cdot), P_k(a_i, a_j) \) is associated to each criterion. \( P_k(a_i, a_j) \) represents the preference strength of action \( a_i \) over \( a_j \). \( P_k(d_k(a_i, a_j)) \) is assumed to be a positive non-decreasing function of \( d_k(a_i, a_j) \). The concept of preference function is used to transform the difference into a unicriterion preference degree; hence:

\[
\pi_k(a_i, a_j) = P_k(d_k(a_i, a_j))
\] (6)

Therefore one has to associate a generalized criterion \( f_k(a_i), P_k(a_i, a_j) \) to each criterion. \( P_k(a_i, a_j) \) provides the preference strength of action \( a_i \) over \( a_j \). This is characterized by the function \( P_k(d_k(a_i, a_j)) \). It is assumed to be a non-decreasing function of \( d_k(a_i, a_j) \). The method provides the DM with a six different predefined preference functions for each of which at most two parameters have to be determined (indifference \( q \) and preference \( p \) thresholds) (Brans, J.P. and Vincke, Ph., 1985). The details of preference functions are also explained in Brans and Mareschal (2002). But in this paper we limit ourselves to three types between them: the usual (without \( q \) and \( p \))\( U \)-shape (with just \( q \)) and Linear type (with \( q \) and \( p \)).

![Figure 3: Illustration of piecewise linear preference function](image)

Figure 3 shows the piecewise linear preference function where \( q_k \) and \( p_k \) are indifference and preference thresholds consecutively for criterion \( k \) in a maximization problem: \( d_k(a_i, a_j) = f_k(a_i) - f_k(a_j) \). \( a_i \) and \( a_j \) are considered indifferent on criterion \( k \) when \( d_k \in [0, q_k] \). In the distance between these 2 thresholds, preference function linearly increases. \( a_i \) is strictly prefered to \( a_j \), for the value of \( d_k \) which is greater than \( p_k \).

The following relation is used to show the transforming of differences into unicriterion preference degrees:

\[
\pi_k(a_i, a_j) = P_k(d_k(a_i, a_j)), \forall a_i, a_j \in A, \forall k = 1, ..., q
\] (7)

In the next step, the global preference degree between \( a_i \) and \( a_j \), which varies between 0 and 1, is computed as follows:
\[
\pi(a_i, a_j) = \sum_{k=1}^{q} \pi_k(a_i, a_j)w_k
\]  
(8)

Where \( \pi(a_i, a_j) \geq 0 \), \( \pi(a_i, a_j) + \pi(a_j, a_i) \leq 1 \) and \( w_k \) \((k = 1, \ldots, q)\) are normalized positive weights associated to the criterion \( k \). The positive and negative outranking flows, when alternative \( a_i \) is compared to \((n - 1)\) other alternatives in \( A \), are defined as follows:

\[
\phi^+(a_i) = \frac{1}{n-1} \sum_{x \in A} \pi(a_i, x)
\]
\[
\phi^-(a_i) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a_i)
\]  
(9)

The intersection of the two rankings induced by these flows is the PROMETHEE I ranking. Accordingly, a partial pre-order of the alternatives is obtained. A complete pre-order, called PROMETHEE II can be obtained on the basis of the net flow score:

\[
\phi(a_i) = \phi^+(a_i) - \phi^-(a_i)
\]  
(10)

To have a clear interception from PROMETHEE outranking method and help DMs, a complementary geometrical tool has been proposed by Mareschal and Brans (Mareschal, B. and Brans, J. P., 1988). This useful tool, named GAIA, is based on a Principal Component Analysis (PCA) of the unicriterion net flow scores.

### 3.3.1. GAIA (Geometrical Analysis for Interactive Aid)

Let us stress that the net flow score in (10) can also be computed as follows:

\[
\phi(a_j) = \sum_{k=1}^{q} \phi_k(a_j)w_k
\]  
(11)

Such that:

\[
\phi_k(a_j) = \frac{1}{n-1} \sum_{x \in A} \{ \pi_k(a_j, x) - \pi_k(x, a_j) \}, \quad k = 1, \ldots, q
\]  
(12)

The value \( \phi_k \) is called the unicriterion net flow score of action \( a_j \) and is such that \(-1 \leq \phi_k(a_j) \leq 1\). It should be noticed that a multicriteria problem can be considered as an evaluation table and related parameters (or a matrix \( \phi = (\phi_k(a_j)) \)). These values, which are all lying in the same range, integrate intra-criterion parameters.

A principal component analysis is applied in order to obtain a two dimensional visualization of the \( k \)-dimensional space of unicriterion net flows matrix \( \phi \). The gained projection is called GAIA plane. The eigenvectors, \( u \) and \( v \), associated to the two highest eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of the matrix \( nC = \phi' \phi \), characterize the GAIA plane. The quality of this projection can be calculated as follows:

\[
\delta = \frac{\lambda_1 + \lambda_2}{\sum_{k=1}^{q} \lambda_k}
\]  
(13)

A complete description of the GAIA plane and its interpretation can be found in (Mareschal, B. and Brans, J. P., 1988).
When

Then the WSI can be determined as follows. If \( \Omega^0 = 0 \) the WSI of criterion \( f_k \) is:

\[
WSI_k = (w_k^-, w_k^+) = (1 - (1 - w_k)\alpha_k^+), 1 - (1 - w_k)\alpha_k^-).
\]

Such that:

\[
\alpha_k^- = \max_{(a_i, a_j) \in \Omega^-} \frac{\Delta(a_i, a_j) \Delta_k(a_i, a_j)}{\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)}
\]

With \((a_i, a_j) \in \Omega^-\)

\[
\alpha_k^+ = \min_{(a_i, a_j) \in \Omega^+} \frac{\Delta(a_i, a_j) \Delta_k(a_i, a_j)}{\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)}
\]

With \((a_i, a_j) \in \Omega^+\)

3.3.2. Weight Stability Intervals

As mentioned above, allocating exact weight values to criteria can be a complex task for DM. Therefore, the impact of weight variations in the final ranking may be investigated. The objective of the Weight Stability Intervals (WSI) technique is to maintain the preference ranking of alternatives within the intervals of criteria weights. Its main characteristic is automated generation of weight intervals during solving phase of PROMETHEE. It validates the robustness of PROMETHEE II complete rank-order.

To determine WSI three sets should be defined:

\[
\Omega^0 = \{(a_i, a_j) \in A \times A, \text{s.t. } \Delta(a_i, a_j) = 0 \text{ and } \Delta_k \neq 0\},
\]

\[
\Omega^- = \{(a_i, a_j) \in A \times A, \text{s.t. } \Delta(a_i, a_j) \Delta_k(a_i, a_j) < 0\},
\]

\[
\Omega^+ = \{(a_i, a_j) \in A \times A, \text{s.t. } \Delta(a_i, a_j) \Delta_k(a_i, a_j) > \Delta^2(a_i, a_j)\}
\]

Then the WSI can be determined as follows. If \( \Omega^0 = 0 \) the WSI of criterion \( f_k \) is:

\[
WSI_k = (w_k^-, w_k^+) = (1 - (1 - w_k)\alpha_k^+), 1 - (1 - w_k)\alpha_k^-).
\]

To determine WSI three sets should be defined:

\[
\Omega^0 = \{(a_i, a_j) \in A \times A, \text{s.t. } \Delta(a_i, a_j) = 0 \text{ and } \Delta_k \neq 0\},
\]

\[
\Omega^- = \{(a_i, a_j) \in A \times A, \text{s.t. } \Delta(a_i, a_j) \Delta_k(a_i, a_j) < 0\},
\]

\[
\Omega^+ = \{(a_i, a_j) \in A \times A, \text{s.t. } \Delta(a_i, a_j) \Delta_k(a_i, a_j) > \Delta^2(a_i, a_j)\}
\]

Then the WSI can be determined as follows. If \( \Omega^0 = 0 \) the WSI of criterion \( f_k \) is:

\[
WSI_k = (w_k^-, w_k^+) = (1 - (1 - w_k)\alpha_k^+), 1 - (1 - w_k)\alpha_k^-).
\]

Such that:

\[
\alpha_k^- = \max_{(a_i, a_j) \in \Omega^-} \frac{\Delta(a_i, a_j) \Delta_k(a_i, a_j)}{\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)}
\]

With \((a_i, a_j) \in \Omega^-\)

\[
\alpha_k^+ = \min_{(a_i, a_j) \in \Omega^+} \frac{\Delta(a_i, a_j) \Delta_k(a_i, a_j)}{\Delta(a_i, a_j) \Delta_k(a_i, a_j) - \Delta^2(a_i, a_j)}
\]

With \((a_i, a_j) \in \Omega^+\)

When
\[ \Delta(a_i, a_j) = \varnothing(a_i) - \varnothing(a_j) \quad \text{and} \quad \Delta_k(a_i, a_j) = \varnothing_k(a_i) - \varnothing_k(a_j). \] (18)

If \( \Omega^0 \neq 0 \), the WSI cannot be defined. It means that no change of weights is allowed.

For more comprehensive information about WSI and mathematically proving the formulation can be referred to [Mareschal, B. (1988)]. In this working paper we use the D-sight software to generate the WSI (Hayez, Q. et al., 2012). The aim of presenting PROMETHEE II WSI in this work is using them to limit weights in the DEA problem.

In the next section, different results will be presented (DEA and PROMETHEE methods) and the sensitivity analysis will be shown by PROMETHEE graphical tools. A detailed explanation of these graphics will be given. Further, according to these analyses, the practical suggestions will be provided for old drivers in this group to improve their performance abilities in different indices and behave as an ideal driver (behave better as a driver?). Even, by standardization of an ideal driver’s indices, the result may be transmitted to larger society of older drivers and decreased the risk of accidents by respect to their abilities (and not the security of ways).

4. Results

In order to assess the robustness of the obtained results with the DEA model, we have modeled the problem with PROMETHEE II. We applied two different approaches in order to guarantee the independency of our results with the modelling strategy. We used both the raw data of the initial problem and then the normalized data from the DEA model to define the preferences functions and the corresponding thresholds. As Chatterjee and Chakraborty considered in their work (2014), the PROMETHEE method is very robust against normalization scheme and the result with raw and normalization data are almost the same. Therefore in this work we used just the normalized data to present the PROMETHEE ranking. As expected, the two resulted rankings in PROMETHEE II are almost the same. The high value of spearman correlation (0.9923) indicates that the use of normalized data rather than raw data to model the problem with PROMETHEE does not really impact the final results. Then, it seems adequate to use these normalized data as in the DEA model.

This comparative analysis with PROMETHEE would allow us to enrich the analysis of the best and the worst solutions highlighted with the previous model.

4.1. PROMETHEE model via normalized data versus DEA

Here, we modeled the problem in PROMETHEE by using normalized data which is in agree with DEA model also. In order to limit the complexity of the model and to respect the nature of each criterion, we used the usual preference function for all the criteria in this problem. In this case, there is no need to choose the indifference and preference thresholds (q, p).
Table 2. Results obtained with DEA model and PROMETHEE II (normalized data, usual functions)

<table>
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<tr>
<th>Rank</th>
<th>Driver’s ID</th>
<th>Cross index scores (DEA)</th>
<th>Driver’s ID</th>
<th>Net flow scores (PII)</th>
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The details in choosing preference functions and their thresholds are discussed in Brans and Mareschal (2002). We defined equal weights for the three categories of criteria introduced previously (i.e. Psychological Ability, Physical Ability, Driving Performance). Then, we allocated equal positive weights to each sub-criterion of the category. Subsequently, we calculated the Spearman’s correlation coefficient to compare the ranking results of the DEA and PROMETHEE models. We obtained a value of 0.9871 which indicates a high correlation between the two rankings. We also tested the Kendall’s correlation coefficient (cf. Figure 4). We gained a value of 0.8653.

As we expected, this value is smaller than Spearman’s correlation but again significantly high to express a good correlation between the two rankings. This high replicability of the results points out the robustness of the best and the worst solutions. When focusing on the drivers ranked at the best positions with the DEA model, we observe that they are all ranked in the top positions of the PROMETHEE II ranking (cf. Table 2 and Figure 4). Among the best solutions of the two rankings, the strongest difference concerns the alternative ID 24 that is ranked at the 3rd position with DEA but at the 8th position with the PROMETHEE II model. Concerning the worst solutions, the correlation is very high between the two rankings. The main differences concern the middle-ranked solutions, such as the alternatives ID 13 (respectively ranked at the 39th and 30th positions), ID 38 (34th and 24th ranks) or ID 39 (19th and 27th ranks). Based on these observations, the best and the worst drivers seem very robust. Consequently, we may consider the drivers that belong to these two categories as representative alternatives of the problem. By analyzing more precisely their evaluations, we could identify the profiles of these representative drivers. In the case of the worst drivers, it may allow us to define strategies and actions to apply in order to improve some of their inabilities.

4.2. Visual analysis of the results

In order to characterize the problem more precisely, we could use the GAIA plane to conduct a visual analysis of the solutions. In Error! Source du renvoi introuvable., we clearly observe that several solutions are located in the same area of the GAIA (e.g. ID 21-42-54, ID 11-12), which means that they are very similar in terms of performance on the 3 categories of criteria (Psychological, Physical and Driving
Performance). In the PROMETHEE II ranking with normalized data, the alternatives ID 1 ($\phi = 0.1151$, 18th rank) and ID 35 ($\phi = 0.0943$, 19th rank) are close to each other. Even in the DEA ranking, they are both performing as good solutions (CIS (ID1) = 0.8610; CIS (ID35) = 0.8240). However, the analysis of the GAIA plane shows that in fact they have very different profiles. The alternative ID 1 performs very well on the Psychological category, while it obtains average and bad evaluations on other categories. On the contrary, the alternative ID 35 performs well on the Driving performance and Physical categories while it obtains weak evaluations on the Psychological category. Another example of highly dissimilar alternatives which are close to each other in the DEA and PROMETHEE rankings concerns the ID 51 (resp. 35th and 32nd rank) and ID 7 (33rd rank in both rankings).

Consequently, using a visualization tool such as the GAIA plane appears to be particularly relevant and efficient to compare and distinguish alternatives which seem apparently similar. In this case study, it may help the DM to refine the analysis of the best and worst drivers.

Moreover, the observation of the Decision Brain on the GAIA plane is very interesting to assess the impact of the weights on the final ranking. It corresponds to the area in red (polytope) on the Erreur ! Source du renvoi introuvable., around the Decision stick (red line). The Decision Brain shows all the direction the Decision Stick could take when modifying the weights of the criteria. Its shape and position
on the GAIA plane illustrate that the best solutions are quite robust and will remain the same with different set of weights. Only the order of the best solutions in the ranking will slightly change.

To compare more precisely the best solutions of the problem, we could use the spider graph representation. The following figures illustrate the profiles of the alternatives ID 50, 54 and 21 on the 13 criteria of the problem (cf. Figure 6).

When focusing on the two best solutions of the DEA ranking (i.e. ID 50 (CSI = 1.000) and 21 (CSI = 0.9783)), we observe that they have quite different profiles. Particularly, the ID 50 obtains better results on the criteria MS_90, MS_120, MMSE, UFOV3, Snellen chart and Mean complete stop while the ID 21 is only better than ID 50 on the criteria UFOV1 and UFOV2. Consequently, the complementary use of this Spider web with the DEA ranking may allow the DM to refine his analysis of the results.

Additionally, the comparison of the profiles of the ID 21 (rank 2 in DEA, rank 4 in PROMETHEE) and the ID 54 (resp. rank 6 and rank 2) seems to be significantly in favor of the ID 54. The alternative 54 obtains better or equal evaluations on every criterion except UFOV2, UFOV3, RSR and 70_Ave_FD. Then, independently to the weights assigned to each criterion, the analysis of the Figure 6 might lead to the conclusion that the ID 54 should be preferred to the ID 21. However, the alternative ID 21 has the best
“average rank” when comparing the DEA and PROMETHEE rankings. This observation points out that the choice of the method used to solve the multicriteria problem may impact the final results, so that the information given by a single ranking might be quite restrictive. Hence, the complementary use of the DEA and PROMETHEE methodologies might provide a more detailed insight about the nature of the case study.

4.3. Sensitivity analysis

In order to assess the robustness of the results obtained with the PROMETHEE II method, a sensitivity analysis was conducted. In particular, the robustness of the first position of the ranking (i.e. ID 50) was analyzed. Table 3 contains the stability intervals for each criterion of the problem, i.e. the range of values each weight can take without affecting the first position of the ID 50 in the PROMETHEE ranking. Most of the intervals are quite wide so that the first position of the alternative ID 50 seems to be very robust.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Min Weight</th>
<th>Max Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSE</td>
<td>0.00%</td>
<td>6.70%</td>
</tr>
<tr>
<td>DSF</td>
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</tr>
<tr>
<td>UFOV1</td>
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</tr>
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<td>UFOV2</td>
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<td>6.70%</td>
</tr>
<tr>
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</tr>
<tr>
<td>Snellen chart</td>
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<td>11.10%</td>
</tr>
<tr>
<td>Up and go</td>
<td>0.00%</td>
<td>11.10%</td>
</tr>
<tr>
<td>4-test balance</td>
<td>0.00%</td>
<td>11.10%</td>
</tr>
<tr>
<td>RSR</td>
<td>0.00%</td>
<td>4.20%</td>
</tr>
<tr>
<td>Mean Complete Speed</td>
<td>0.00%</td>
<td>4.20%</td>
</tr>
<tr>
<td>50_Ave_FD</td>
<td>0.00%</td>
<td>4.20%</td>
</tr>
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<td>70_Ave_FD</td>
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<tr>
<td>MS-120</td>
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Then, considering the robustness of the alternative ID 50 and given that it is identified as the best alternative by the DEA and PROMETHEE rankings, we may consider the alternative ID 50 as a “reference driver” and then use the stability intervals from the Table 3 as weight restrictions in DEA. The ranking obtained with these new weight restriction ranges is available in Table 4.
Table 4. DEA rankings obtained with the new (left) and previous (right) weight restriction ranges

<table>
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<tr>
<th>Rank</th>
<th>Driver’s ID</th>
<th>Cross index scores (new weight restrictions)</th>
<th>Driver’s ID</th>
<th>Cross index scores (previous weight restrictions)</th>
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<td>1</td>
<td>50</td>
<td>1</td>
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</table>
An interesting observation from this new DEA ranking is that 4 alternatives are now equally considered as the best ones (i.e. ID 50, 42, 21 and 24). This can be explained by the fact that the stability intervals from PROMETHEE gives wider weight restriction ranges than previously (for 8 criteria over 16). Nevertheless, the information given by the DEA ranking remains globally the same. The Spearman’s correlation coefficient between the new and the previous DEA ranking is equal to 0.9863 while the Kendall tau rank correlation is equal to 0.8801.

Consequently, the use of the outputs of the sensitivity analysis of PROMETHEE as the inputs of the DEA model seems particularly interesting and it points out the interest of using complementary these two methods.

5. Conclusions

In this study, we applied a multiple layer DEA based composite indicator model to assess the driving performance of 55 drivers aged 70 and older. This model allowed us to aggregate the values of 16 indicators within a particular category of a particular layer by using a weighted sum approach. Then, the cross index method was used to rank all the drivers with respect to their global performances. In order to quantify the robustness of the ranking, we modelled the problem with PROMETHEE II and we compared the results. The calculation of the Pearson’s correlation coefficient pointed out the high replicability of the results with the PROMETHEE models and the robustness of the final solutions. In addition, the analysis of the GAIA plane and Spider web representation of the results allowed us to study more precisely the profiles of the best alternatives of the multicriteria problem. Finally, the computation of a sensitivity analysis pointed out the robustness of the best solution and allowed us to identify an interest in using the stability intervals of PROMETHEE II as the weight restrictions of the DEA model.

To conclude, this study has shown the value of using a DEA model beside a MCDA method (PROMETHEE) for drivers’ evaluation. This approach allowed us to rank all the drivers based on their performances and to assess the robustness of the best and worst candidates. In future works, we will improve this study by considering the combination of DEA and PROMETHEE as an analyzing tool to give older drivers more insight in characterizing their driving performance. Applying GAIA plane (Brans and Mareschal, 2002) can be a case in this analysis. Further, the PROMETHEE II weight stability intervals can be used as assurance regions in the DEA model to improve the discrimination power of DEA. It also would be valuable to incorporate an artificially created, ideal driver in the analysis, so that instead of a relative comparison, an evaluation of drivers in an absolute manner would be possible.

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