The Gribov problem in presence of background field for SU(2) Yang–Mills theory

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A B S T R A C T

The Gribov problem in the presence of a background field is analyzed: in particular, we study the Gribov copies equation in the Landau–De Witt gauge as well as the semi-classical Gribov gap equation. As background field, we choose the simplest non-trivial one which corresponds to a constant gauge potential with non-vanishing component along the Euclidean time direction. This kind of constant non-Abelian background fields is very relevant in relation with (the computation of) the Polyakov loop but it also appears when one considers the non-Abelian Schwinger effect. We show that the Gribov copies equation is affected directly by the presence of the background field, constructing an explicit example. The analysis of the Gribov gap equation shows that the larger the background field, the smaller the Gribov mass parameter. These results strongly suggest that the relevance of the Gribov copies (from the path integral point of view) decreases as the size of the background field increases.

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1. Introduction

The main tool to compute observable quantities in QFT is perturbation theory. In gauge theories, and in Yang–Mills (YM) theory in particular, a fundamental problem to solve in order to compute physical quantities is the over-counting of degrees of freedom related to gauge invariance (for a detailed analysis see [1]). The Faddeev–Popov (FP) gauge fixing procedure is the cornerstone which allows using the Feynman rules and Feynman diagrams in all applications of the standard model. The obvious fundamental hypothesis is that the gauge-fixing condition must intersect once and only once every gauge orbit. Locally, in the space of gauge fields, this hypothesis requires that the FP operator should not have zero modes so that the FP determinant is different from zero. The reason is that the existence of a proper gauge transformation preserving the gauge-fixing would spoil the whole quantization procedure since it would imply that the FP recipe does not completely eliminate the over-counting of degrees of freedom.

However, in [2], Gribov showed that in non-Abelian gauge theories (in flat, topologically trivial space–times) the FP procedure fails at non-perturbative level. The reason is that a proper gauge fixing is not possible due to the appearance of Gribov copies: namely, gauge equivalent configurations satisfying the Coulomb gauge. Later, Singer [3] showed that if Gribov ambiguities occur for the Coulomb gauge, they occur for all gauge fixing conditions involving derivatives of the gauge field.

Naïvely, one could expect to completely avoid the Gribov problem by simply choosing algebraic gauge fixings like the axial gauge or the temporal gauge, which are free of Gribov copies. However, these choices have their own, and even worse, problems (for a detailed reviews see [4,5]). Here it is just worth mentioning one serious issue: any loop computation in the algebraic gauge-fixings mentioned above are very difficult already beyond two-loop. Hence, from the practical point of view, linear covariant gauge-fixings are far more convenient: here we will only consider this kind of gauge-fixing.

On the other hand, the existence of Gribov copies is not just a problem since, as Gribov himself argued, the natural way to solve such a problem is able to shed considerable light on the infrared (IR) region of YM theory. Such solution is to restrict the path-

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[1] The origin of these problems is that in all these algebraic gauges the free propagator of the gauge field is more singular than in linear covariant ones owing to the presence of additional “spurious” singularities [6].
integral only to a region $\Omega$, which is called Gribov region, where FP operator is definite-positive [2,7–10] (detailed reviews are [11] and [5]) so that there are no Gribov copies connected to the identity.\(^2\)

In order to restrict the path integral to the Gribov region, one can use the Gribov–Zwanziger (GZ) approach [12,13]. When the space-time geometry is flat and the topology trivial,\(^3\) this method is able to reproduce the usual perturbation theory encoding, at the same time, the effects related to the elimination of the Gribov copies. For instance, it allows the computation of the glueball masses in excellent agreement with the lattice data [22–24]. Within the same framework, it is also possible to solve the sign problem for the Casimir energy and force in the MIT-bag model [25].

This scheme works very well also at finite-temperature [26–30] (this is also supported by the results in [31]). Moreover, at one-loop order, it is possible to compute the vacuum expectation value for the Polyakov loop [32]: these results are in a good agreement with the expected behavior for the deconfinement phase transition [33]. Within the GZ approach, the non-perturbative correction to the gluon propagator is encoded in the Gribov mass which is determined in a self-consistent way by solving the so-called Gribov gap equation. Therefore, the analysis of the dependence of Gribov mass on the temperature (as well as on other relevant external parameters) is very useful to determine the phase-diagram of Yang–Mills theory.

Thus, it is natural to wonder whether or not this approach works so well also in the presence of a background gauge field. From the theoretical point of view, this analysis is very important as it discloses how strongly the presence of a background field can affect the Gribov region and the whole issue of Gribov copies. One of the most relevant applications of the background field method is the computation of the (vacuum expectation value of the) Polyakov loop [32] in which the presence of the Polyakov loop manifests itself as a constant background field with component along the Euclidean time.\(^4\) Another very important non-perturbative phenomenon in which the presence of a background gauge field plays a key role is the (both Abelian and non-Abelian) Schwinger effect [36–38]. Also in the case of the non-Abelian Schwinger effect, the relevant background gauge fields are constant $A_\mu$, which have components both along time and space directions. From the point of view of applications, such an analysis can also be quite relevant in relation with quark-gluon plasma [39,40], color superconductivity in QCD [41], astrophysics [42,40], and cosmology [43,44].

The idea of the present paper is precisely to begin the study of the following very relevant and broad question: how the presence of a background field affects the (appearance of) Gribov copies as well as the gap equation form. To the best of authors knowledge, such issue has not been deeply analyzed so far.

The Background Field Method (BFM) [45–47] together with the techniques developed in [14–20] are adopted in the present paper. The results of these references on the Gribov problem on curved space strongly suggest (taking into account that the background metric can play the role of an external field) that background fields can play a prominent role within the GZ approach to YM theory. Here we show that the Gribov copies equation is affected directly by the presence of a background field. In particular, explicit examples will be constructed in which the “relevance” of the allowed Gribov copies decreases as the background field is increased. Moreover, the analysis of the semi-classical Gribov gap equation shows that the Gribov mass parameter decreases as the size of the background field is increased.

The paper is organized as follows. In the second section, the Gribov problem in the Landau–De Witt gauge is introduced. In the third section, explicit examples of Gribov copies in the Landau–De Witt gauge are studied. In the fourth section, the Gribov gap equation within a background field is analyzed. Some conclusions and discussions are drawn at the end.

## 2. A brief review of the Gribov–Zwanziger action

In this section we present an outlook of the GZ-approach without background consider background fields, which is the aim of Section 3. The Euclidean Yang–Mills action

$$S_{YM} = \frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu},$$

is invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu = U^{-1} A_\mu U + U^{-1} \partial_\mu U,$$

with $U \in SU(N)$. In order to take into account the existence of Gribov copies due to the this gauge transformation, Gribov proposed [2] to restrict the domain of integration in the path integral to a region in functional space where the eigenvalues of the FP operator $M^{ab}$ are strictly positive. This region is known as the Gribov region $\Omega$, and is defined as

$$\Omega = \{ A_\mu^a | \partial_\mu A_\mu^a = 0; \}

$$

$$M^{ab} = -\partial_\mu (\partial^\mu \delta^{ab} - g f^{abc} A^c_\mu) = -\partial_\mu D^{ab}_\mu > 0),$$

where $D^{ab}_\mu = i\mu^{ab} - g f^{abc} A_\mu^c$ is the usual covariant derivative, which depends on $A_\mu$. The boundary of this region is called the first Gribov horizon. Later on, Zwanziger [12] implemented the Gribov region $\Omega$ in Euclidean Yang–Mills theories in the Landau gauge, by means of the following action

$$S_b = S_{YM} + \int d^4x \left( b^2 \partial_\mu A_\mu^a + c^2 \partial_\mu D^{ab}_\mu c \right) + \gamma^4 \int d^4x h(x),$$

with $S_{YM}$ the Euclidean version of the Yang–Mills action defined in (1), and where $h(x)$ is the so-called horizon function

$$h(x) = g^2 f^{abc} A^b_\mu (M^{-1})^{cd} f^{dec} A^c_\mu.$$

The $\gamma$ parameter, known as Gribov mass parameter, at a semi-classical level, provides a detailed description of the confinement as the poles of the propagators are imaginary when $\gamma^2 \neq 0$ [11], and is determined by a self-consistent horizon condition

$$(h(x)) = d(N^2 - 1),$$

where $d$ is the number of the space–time dimensions and we understand for (…) functional integral over the fields. The local version of the horizon function $h(x)$ can be achieved through a suitable set of additional fields, which belong to a BRST doublet. Then, the local full action reads,

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\(^2\) Some Gribov copies are still left within the Gribov region [7]. One can define a modular region which is completely free of Gribov copies (both small and large). However, how to implement the restriction to the modular region is not known yet. Thus, we will work within the Gribov region as it is usually done.

\(^3\) On the other hand, on curved spaces the pattern of appearance of Gribov copies can be considerably more complicated (for instance, even Abelian gauge theories can have ‘duced’ gauge copies [14–21]). Thus, in the following only the standard flat case will be considered.

\(^4\) Formally, a constant background gauge field with only the timelike component non-vanishing is related to a bosonic chemical potential [34,35]. On the other hand, the physical interpretation of such Bosonic chemical potential is rather obscure in the case of non-perturbative gluons and so it will not be discussed in the present case.
\[ S_{GZ} = S_{FP} + S_{\gamma} + S_0, \]
\[ S_{FP} = \int d^4x \left( \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} + i b^\sigma \partial_\sigma A^\rho + \bar{c}^\alpha M^{\alpha\dot{\alpha}} c^\dot{\alpha} \right), \]
\[ S_{\gamma} = \int d^4x \left( \gamma^2 g f^{abc} (\bar{\psi}_\rho^a - \bar{\psi}_\rho^b) + \gamma^4 \int d^4x \phi(x) \right), \]
\[ S_0 = \int d^4x (\bar{\psi}^a_{\alpha} M^{ab} \psi^b_\beta + \bar{\omega}^\alpha_\rho M^{\alpha\dot{\alpha}} \omega^\dot{\alpha}_\rho + g f^{\alpha\beta\gamma} (\delta_\sigma \bar{\omega}_\rho^\gamma (D^\rho_m e)^\alpha \psi^\beta_\rho)) \]

where Greek indexes run from \( \mu = 1 \ldots d \) and Latin indexes from \( a = 1 \ldots N^2 - 1 \). The fields \((\bar{\psi}^a_{\alpha}, \psi^b_\beta)\) are a pair of complex conjugate bosonic fields, while \((\bar{\omega}^\alpha_\rho, \omega^\dot{\alpha}_\rho)\) are anti-commuting fields.

Now, if we consider the relation between the local action \( S_{GZ} \) and the non-local action \( S_{\gamma} \)
\[
\int [dA][db][dc][d\check{c}] e^{-S_{GZ}} = \int [dA][db][dc][d\phi][d\bar{\phi}][d\omega][d\bar{\omega}] e^{-S_{vac}}
\]

and we take the partial derivative of both sides w.r.t \( \gamma^2 \) (with \( \gamma \neq 0 \)), we obtain
\[
\langle g f^{abc} A_{\mu}^a \phi^{bc}_\mu \rangle + \langle g f^{abc} A_{\mu}^a \bar{\phi}^{bc}_\mu \rangle + 2\gamma^2 d(N^2 - 1) = 0.
\]

which it is precisely the horizon condition (6). On the other hand, we know that the effective action \( \varepsilon_{\text{vac}} \) is obtained through
\[
\varepsilon_{\text{vac}} = \int [d\Phi] e^{-S_{GZ}},
\]

where \( [d\Phi] \) stands for the integrations over all the fields contained in the action \( S_{GZ} \). From this last expression, the \( \gamma \) parameter can also be determined by a self-consistent way by the following gap equation
\[
\frac{\partial \varepsilon_{\text{vac}}}{\partial \gamma^2} = 0.
\]

Therefore, equation (14) represents the horizon condition formula which will allow us to determine the Gribov parameter later on.

3. Gribov–Zwanziger action in a background field

As we present a brief introduction in Section 2 of the GZ-approach, now we analyze what happens if we take into account a background field. We consider the SU(N) Yang–Mills theory in \( d = 4 \) Euclidean dimensions defined in Eq. (4). In the BFM (see [4,48] for more details), one introduces a fixed background gauge field configuration \( B_\mu \) through the splitting
\[
A_\mu \rightarrow a_\mu \equiv A_\mu + B_\mu,
\]

where \( A_\mu \) and \( B_\mu \) play completely different roles. On the one hand, \( A_\mu \) represents the quantum fluctuations of the gauge field. On the other hand, the background field \( B_\mu \) plays the role of a classical background. (This approach is quite relevant in the case of the Polyakov loop computation [32]). The gauge symmetry (2) changes with this background field as
\[
\delta A_\mu^a = \delta B_\mu^a = 0.
\]

In this case, the symmetry transformation in Eq. (16) can be written as
\[
A_\mu \rightarrow A_\mu^U = U^{-1} a_\mu U + U^{-1} (A_\mu + B_\mu) U.
\]

where it has been explicitly taken into account that \( B_\mu \) is not affected by the gauge transformation. At the infinitesimal level, \( U \approx 1 + \alpha^2 \tau_\alpha, \omega \ll 1 \), one recovers the usual infinitesimal gauge transformations with a background gauge field [4,48]
\[
\delta A_\mu^a = f^{abc} \omega^b (A_\mu^c + B_\mu^c) + \frac{1}{g} \delta_\mu \omega^a ,
\]

\[
\delta B_\mu^a = 0.
\]

Correspondingly, the Landau gauge-fixing condition is also modified. In the presence of a background field, the most convenient gauge-fixing condition takes the form
\[
C_G[B] \equiv D_\mu A_\mu^a = 0, \quad D_\mu = \partial_\mu + g f^{abc} B_\mu^c,
\]

known as the Landau–DeWitt (LDW) gauge fixing condition. The FP procedure in the presence of a background gauge field leads to the following action (see [49] for a detailed discussion)
\[
S_B^f = \int d^4x \left( \frac{1}{4} F_{\mu\sigma}^a F^{\mu\sigma} + e^2 D_{\mu}(B) D_\mu (a) e^a - \frac{(D_\mu (B) A_\mu^a)^2}{2 g^2} \right)
\]

with \( c \) and \( \check{c} \) denoting the ghost and antighost fields, respectively. The LDW gauge is actually recovered in the limit \( g \rightarrow 0 \), taken at the very end of each computation, and is also plagued by Gribov copies, as we will show in the following sections. On the other hand, the GZ method can be applied to this situation by means a suitable choice of the background field \( B_\mu \) in order to the new FP operator \( M_\mu^a \equiv - D_\mu (B) D_\mu^a (a) \) is invertible inside the Gribov region \( \Omega \). Following the lines of [50] and [32] (in the case of a fixed background), the GZ action under the LDW gauge acquires the form
\[
S_{GZ} = \int d^4x \left( \frac{1}{4} F_{\mu\sigma}^a F^{\mu\sigma} + e^2 D_{\mu}(B) D_\mu (a) e^a - \frac{(D_\mu (B) A_\mu^a)^2}{2 g^2} + \bar{\psi}^a_\alpha M^{\alpha\dot{\alpha}} \psi^\beta_\rho + \bar{\omega}^\alpha_\rho M^{\alpha\dot{\alpha}} \omega^\dot{\alpha}_\rho - \frac{\alpha_\mu^a D_\mu (B) D_\mu (a) e^a}{g^2} - g f^{abc} A_\mu^a (\bar{\psi}^b_\alpha + \bar{\psi}^c_\rho) \right) - g^4 d(N^2 - 1)).
\]
to construct examples in which it is not satisfied. However, in the cases of the background fields considered in this paper (which are relevant in relation with both the Polyakov loop and the Schwinger effect computations) the above condition is satisfied.

The second key requirement (which does change in the presence of a background field) is the validity of the Dell’Antonio–Zwanziger theorem [10]. In the case of the Landau gauge, such a theorem provides the whole Gribov–Zwanziger idea with solid bases since it shows that one does not lose any relevant information when the restriction to the Gribov region is implemented (since Every Gauge Orbit Passes Inside the Gribov Horizon). Remarkably, Gribov based its idea of the restriction to the Gribov region on this local version of the Dell’Antonio–Zwanziger theorem. In the presence of a background gauge field, many of the technical assumptions of [10] do not hold in general. Consequently, the generalization of the Dell’Antonio–Zwanziger theorem appears to be a very difficult problem in non-linear functional analysis (on which we hope to come back in a future publication). On the other hand, the local argument by Gribov can be repeated step by step in the case of the LDW gauge provided the background field is constant and commutes with itself (as is the case for the background field considered in the next section).

4. The simplest non-trivial background field

In order to describe the effects of a background field both on the Gribov copies equation and on the gap equation avoiding unnecessary technical complications, we will consider the simplest non-trivial background gauge field (which is relevant in the computation of the Polyakov loop [32]):

$$B^a_\mu = -\frac{r_0}{g} \delta^{a3} \delta_{\mu 0}.$$ (23)

Constant background non-Abelian gauge fields are very relevant in the analysis of the non-Abelian Schwinger effect [37,38] too. However, the most interesting configurations considered in these works have both time-like and space-like components turned on at the same time. Here we have chosen the above background gauge field with only Euclidean time component since it allows to construct explicitly analytic examples of Gribov copies as well as to solve the semi-classical Gribov gap equations (which, quite consistently, shows that the Gribov mass decreases with the increase of $r_0$).

On the other hand, the background gauge potentials considered in such works would still allow a complete study of the semi-classical Gribov gap equation (along the lines of the present analysis) but make extremely difficult to construct explicit examples of Gribov copies. As we believe that, when analyzing the Gribov problem with an external background field, it is very instructive to analyze both the Gribov copies equation and, at the same time, the corresponding Gribov gap equation (which, in a sense, are the two sides of the same coin) we consider here the background gauge field in Eq. (23).

In this case, the LDW gauge fixing reads

$$\tilde{G}^a_\mu[B] = 0,$$ (24)

so that the Gribov copies equation becomes

$$\partial^\mu A^U_\mu + g[B^\mu, A^U_\mu] = 0,$$ (25)

where $A^U_\mu$ is defined in Eq. (17). It is worth emphasizing that the background gauge field $B^\mu$ identically satisfies the LDW gauge-fixing (as it should):

$$\partial^\mu B_\mu + g[B^\mu, B_\mu] = 0.$$

The following standard parametrization of the SU(2)-valued functions $U(x')$ is useful

$$U = Y^0 1 + Y^a Y_a, (Y^0)^2 + Y^a Y_a = 1,$$

$$(Y^a)^2 + Y^a Y_a = 1.$$ (26)

where $Y^0$ and $Y^a$ are functions on the coordinates $x'$, and the sum over repeated indices is understood also in the case of the group indices (in which case the indices are raised and lowered with the flat metric $\delta_{ab}$). The SU(2) generators $\tau^a$ satisfy

$$\tau_a \tau_b = -\delta_{ab} 1 - \epsilon_{abc} \tau^c.$$ (27)

where $1$ is the identity $2 \times 2$ matrix and $\epsilon_{abc}$ are the components of the totally antisymmetric Levi-Civita tensor with $\epsilon^{123} = \epsilon_{123} = 1$.

4.1. Gribov copies of the vacuum

In the present case, the gauge transformations of the vacuum have the expression (see Eq. (17))

$$0 \rightarrow U^{-1} \partial_\mu U \left( U^{-1} B_\mu U - B_\mu \right).$$

Correspondingly, the equation for the Gribov copies of the vacuum in the presence of a background field reads

$$\partial^\mu \left( U^{-1} \partial_\mu U + (U^{-1} B_\mu U - B_\mu) \right)$$

$$+ g[B^\mu, U^{-1} \partial_\mu U + (U^{-1} B_\mu U - B_\mu)] = 0.$$ (28)

This, actually, is a system of coupled non-linear partial differential equations. In order to reduce it consistently to a single differential equation a particular hedgehog ansatz can be used [17] (see Appendix A for the details on the vacuum case). This corresponds to the following ansatz for the gauge copy

$$U = Y^0(x') 1 + Y^a(x') \tau_a,$$ (29)

where

$$Y^0(x') = \cos \alpha(x'), \quad Y^a(x') = \hat{n}_a \sin \alpha(x')$$

being $\hat{n}_a$ normalized with respect to the internal metric $\delta_{ab}$ as

$$\delta_{ab} \hat{n}_a \hat{n}_b = 1.$$ (30)

4.2. Vacuum Gribov copies with $T^3$ topology

Let us analyze the Gribov copies equation in a flat spatial space with $T^3$-topology. Such choice of topology can be very useful in relation with lattice studies [51,52]. We take the metric

$$ds^2 = \sum_{i=1}^{3} \lambda_i^2 d^2 \phi_i,$$ (31)

where the $\lambda_i \in \mathbb{R}$ represents the length of the torus along the $i$-axis and the coordinates $\phi_i \in [0, 2\pi)$ corresponds to the $i$-th factor $S^1$ in $T^3$. In the $T^3$ case, the gauge transformation $U$ is independent of the Euclidean temporal coordinate $x^0$ and is proper when [17]

$$U(\phi_i + 2m_i \pi) = U(\phi_i), \quad m_i \in \mathbb{Z}, \quad i = 1, 2, 3.$$ (32)
The generalized hedgehog ansatz adapted to this topology reads
\[
\alpha = \alpha(\phi_1), \quad \hat{n}^1 = \cos(p\phi_2 + q\phi_3), \\
\hat{n}^2 = \sin(p\phi_2 + q\phi_3), \quad \hat{n}^3 = 0,
\]
with \(p, q\) arbitrary integers. From this ansatz, the equation (48) is reduce to the following single scalar non-linear differential equation (see Appendix A for details),
\[
\frac{d^2\alpha}{d\phi_1^2} = \xi \sin(2\alpha),
\]
where
\[
\xi = \frac{\lambda_1^2}{2} \left( \frac{p^2}{\lambda_2^2} + \frac{q^2}{\lambda_3^2} + \frac{4\lambda_0^2}{g} \right),
\]
and, according to (33), the condition
\[
\alpha(\phi_1 + 2\pi) = \alpha(\phi_1) + 2\pi k,
\]
must be fulfilled. The equation (35) can be reduced to a first order conservation law
\[
V = \frac{1}{2} \left[ \left( \frac{d\alpha}{d\phi_1} \right)^2 + \xi \cos(2\alpha) \right] \Rightarrow \phi_1 - \phi_0
\]
\[
= \pm \frac{\alpha(\phi_1) - \alpha(\phi_0)}{\sqrt{2V - \xi \cos(2\phi_1)}},
\]
where \(\phi_0\) and \(V\) are integration constants. However, the integration constant \(\phi_0\) is not relevant as it just corresponds to a shift of the origin. Consequently, the relevant integration constants which labels different solutions of the Gribov copies equation in Eq. (35) are \(\xi\) and, through the boundary condition (33), \(V\) in Eq. (38).

On the other hand, not any solution of Eq. (35) is allowed Gribov copy as the boundary conditions in Eq. (33) must be required. Since \(\phi_1\) belongs to the range \((0, 2\pi)\), let us take \(\phi_1 = 2\pi n\) and \(\phi_0 = 0\). The condition (33) implies \(\alpha(\phi_1) = \alpha(\phi_0) + 2\pi n\), where \(n \in \mathbb{Z}\). Taking this into account, we have for (38) the following expression
\[
2\pi = \xi \left[ \frac{\alpha(0) + 2\pi n}{\xi(0)} \right] \int \frac{dy}{\sqrt{Z - \cos(2y)}}, \quad Z = \frac{2V}{\xi} > 1,
\]
where \(Z > 1\) since the integrand must be well defined in the range \(y \in (0, 2\pi n)\).

The present analysis shows that already the Gribov copies of the vacuum depend very substantially on the background field. The different Gribov copies which can be constructed with the present ansatz are in 1-to-1 correspondence with the solutions of Eq. (39).

As it is well known (see the detailed discussion in [53]), the weight of a given copy \(U\) is related to its norm
\[
N[U] = \int d^4x \sqrt{g} \text{Tr} \left[ \left( U^{-1} \partial_x U + U^{-1} \partial_x U - B_{\mu} \right)^2 \right]
\]
where in this case \(g\) refers to the determinant of the metric associated to the line element (32), setting the coupling constant to be zero. In particular, the bigger is \(N[U]\), the less relevant the copy is from the path integral point of view. As in this case there is a background potential, the integral (40) can be written as
\[
\text{Fig. 1. The norm of the copies } \frac{N[U]}{\sqrt{g}}, \text{ according to (41), in the case } p = q = \lambda = 1 \text{ versus the background } \lambda_0 \text{ for } k = 1 \text{ (in red) and } k = 2 \text{ (in green). The solutions of } \alpha(\phi_1) \text{ fulfill the condition (37). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)}
\]

\[
N[U] = \frac{(2\pi)^2 \lambda_2 \lambda_3}{\lambda_1} \int_0^{2\pi} d\phi_1 \left( \frac{d\alpha}{d\phi_1} \right)^2 + 2\xi \sin^2 \alpha
\]

\[
= \frac{(2\pi)^2 \lambda_2 \lambda_3}{\lambda_1} \int_0^{2\pi} d\phi_1 \left( 2V + 3\xi \sin^2 \alpha(\phi_1) - \xi \cos^2 \alpha(\phi_1) \right),
\]

where in the last equality we used the definition (38) of the constant \(V\). In Fig. 1, we show the norm \(N[U]\) for \(p = q = \lambda = 1\) increases when \(\lambda_0\) grows both for \(k = 1\) and \(k = 2\), at least in the range \(\lambda_0 \in (0.0, 1.0)\), for solutions \(\alpha(\phi_1)\) such that fulfill the condition (37) and \(\alpha(0) = 0\). Consequently, in this region, the bigger \(\lambda_0\) the smaller the importance of Gribov copies of the form considered here. It is necessary more computational power to see how is the behavior of the norm outside the region studied here (for instance, \(|p| > 1\) and \(|q| > 1\)). The above considerations suggest that the Gribov gap equation should also be affected non-trivially by the background field. In the next section, it will be shown that this is indeed the case.

5. Solving the GZ gap equation for SU(2) with constant background field

In order to determine the gap equation, we will proceed first to show the effective potential to GZ action at one-loop approximation for the SU(2) internal gauge group in the presence of a background potential discussed in Section 3. We will work at zero temperature as, in this case, there is no need neither to introduce two different form factors in the LDW propagator for the gluon as in [40] nor to consider the plasma approximation as in [26,27]. Thus, the present computation is able to disclose in a very clean way the effects of the background gauge potential on the Gribov mass parameter. In order to obtain the vacuum energy at one loop, we consider only from (22) the quadratic terms in the fields which are functionally integrated.6 We find [32]

6 Higher order corrections to (42) are obtained if we consider connected diagrams, see details in [54].
Fig. 2. (a) The gap equation (47) as a function of $\lambda^2$ for different values of $r_0$. The value of $\lambda$ which corresponds the curve intersects the $y$-axis is the solution of (47). (b) The zeros of the gap equation (47) as a function of the background field $r_0$. We clearly see that the Gribov mass parameter decrease when $r_0$ grows. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

\[
\epsilon_v(r_0) = -\frac{d(N^2-1)}{2N\beta^2}\lambda^4 \gamma + \frac{1}{2V}(d-1)\text{Tr}\ln B^2 + \frac{\lambda^4}{\Lambda^4} \\
- \frac{d}{2V}\text{Tr}\ln \frac{d^2}{\Lambda^2},
\]

where $V$ is the Euclidean spacetime volume, $\lambda^4 = 2N\beta^2\gamma^4$, being $\gamma$ the Gribov parameter, $B$ is the covariant background derivative in the adjoint representation defined in (20), and $\Lambda^2$ is a scale parameter in order to regularize the result. We can rewrite (42), taking into account the Cartan subalgebra of $SU(2)$ is one-dimensional, as

\[
\epsilon_v(r_0) = -\frac{d(N^2-1)}{2N\beta^2}\lambda^4 \\
+ \frac{1}{2}(d-1) \sum_{s=1}^{s=1} \left[ I(sr_0, -i\lambda^2) + I(sr_0, i\lambda^2) \right] \\
- \frac{d}{2} \sum_{s=1}^{s=1} I(sr_0, i\lambda^2),
\]

(43)

where $s$ is the isospin $SU(2)$, and we defined the function

\[
I(r, m^2) = \int \frac{d^4 q}{(2\pi)^4} \ln \frac{q^2 + r^2 + m^2}{\Lambda^2},
\]

(44)

passing to the Fourier space in the second equality. We will compute first (44) using similar techniques which were already applied in GZ approach (see [32,55]). In this case the computations are easier because are made at zero temperature (see details in Appendix B), and gives us

\[
I(r, m^2) = \frac{(m^2 + r^2)^2}{32\pi^2} \ln \left( \frac{m^2 + r^2}{\Lambda^2} \right) - \frac{3}{2}.
\]

(45)

Inserting (45) into (43), we have

\[
\sum_{q} \rightarrow V \int \frac{d^4 q}{(2\pi)^4} \text{Tr} F_{\mu\nu} F^{\mu\nu} [11].
\]

\[
\epsilon_v(r_0) = -\frac{d(N^2-1)}{2N\beta^2}\lambda^4 \\
- \frac{(d-1)\lambda^4}{32\pi^2} \ln \left( \frac{\lambda^4 + r_0^4}{\Lambda^4} \right) + \ln \left( \frac{\lambda^4}{\Lambda^4} \right) - 6 \\
+ \frac{(d-1)r_0^4}{32\pi^2} \ln \left( \frac{\lambda^4 + r_0^4}{\Lambda^4} \right) - 3 \\
- \frac{dr_0^4}{32\pi^2} \ln \left( \frac{r_0^4}{\Lambda^4} \right) - 3 \\
- \frac{(d-1)r_0^4}{8\pi^2} \arctan \left( \frac{\lambda^2}{r_0^2} \right).
\]

(46)

where in the last term we took the principal branch of the log in the complex plane [56]. In order to normalize the last equation, we shall choose $\Lambda^2$ in order that for $r_0 = 0$ the solution is $\lambda_0 = 1$. Because we are interested in solving the gap equation (14), we can re-scale it in the following way

\[
\frac{\partial \epsilon_v(r_0, \lambda)}{\partial \lambda^2} = \frac{\lambda^2}{\lambda_0^2} \frac{\partial \epsilon_v(r_0 = 0, \lambda)}{\partial \lambda^2} \\
= -\frac{(d-1)\lambda^2}{16\pi^2} \ln \left( \frac{\lambda^4 + r_0^4}{\Lambda_0^4} \right) + \ln \left( \frac{\lambda^4}{\Lambda_0^4} \right) \\
- \frac{(d-1)r_0^4}{8\pi^2} \arctan \left( \frac{\lambda^2}{r_0^2} \right) = 0.
\]

(47)

The scaled gap equation (47) can be solved using numerical techniques. In Fig. 2 (a), it is plotted the left hand side of the gap equation (47) as a function of $\lambda^2$ for different values of $r_0$. We see clearly the Gribov mass parameter decrease when the background $r_0$ grows, as it is shown more clearly in Fig. 2 (b), where it is shown the parameters $\lambda$ which are solution of gap equation at zero temperature versus $r_0$. We could interpret this as the theory becomes less confined as the Gribov parameter reduces (see Section 6).
6. Conclusions and perspectives

In the present paper, it has been shown that the Gribov copies equation is affected directly by the presence of a background gauge field. In particular, explicit examples have been constructed in which the norm of the Gribov copies satisfying the usual boundary conditions increases when the size of the background field is very large.

The analysis of the semi-classical Gribov gap equation in the chosen background gauge potential and of the dependence of the Gribov mass on the background potential itself, quite consistently, confirms the above results. Namely, we have shown that the larger is the size of the background gauge potential, the smaller is the corresponding Gribov mass.

It is worth emphasizing the importance of the chosen constant background gauge field is related to the fact that it appears in the analysis of the computation of the Polyakov loop. Moreover, constant background gauge potentials are very important also in relation with the non-Abelian Schwinger effect [37,38]. Although the constant gauge potentials considered in that references allow a complete analysis of the semi-classical Gribov gap equation, they make extremely difficult to construct explicit examples of Gribov copies. We hope to come back on the more general configurations considered in these works and on the relations between the non-Abelian Schwinger effect and the Gribov problem in background gauge fields in a future publication.

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Appendix A. Gribov copies with a constant background field

In this appendix we consider the derivations and properties of the equation of gauge-equivalent fields satisfying the LDW gauge in the presence of background field. Our aim is to calculate the condition for existence of Gribov copies in the vacuum. Thus, we must compute the following expression

\[ U^{-1} \partial_\mu U + U^{-1} B_\mu U - B_\mu = \left( Y^0 \partial_\mu Y^c - Y^c \partial_\mu Y^0 + \epsilon_{abc} Y^a \partial_\mu Y^b \right) \tau_c - \frac{2r_0}{g} \delta_{\mu 0} \left( \epsilon_{abc} Y^a Y^0 + Y^3 Y^c \right) \tau_c + \frac{2r_0}{g} \delta_{\mu 0} Y^a Y^0 \tau_3. \]

The next step is to apply to this last expression the covariant background derivative and set it to be zero according to (25). This results in the following expression

\[ \left( -Y^0 \square Y^c - Y^c \square Y^0 + \epsilon_{abc} Y^a \square Y^b \right) \tau_c - \frac{2r_0}{g} \left( Y^c Y^3 + Y^3 Y^c + \epsilon_{abc} \left[ Y^0 \partial^0 Y^a + \partial^0 Y^a \right] \right) \tau_c - 2r_0 \epsilon_{3bc} \left( Y^0 \partial^0 Y^b - Y^0 \partial^0 Y^b - Y^c \partial^0 Y^3 + Y^3 \partial^0 Y^c \right) - \frac{4r_0^2}{g} \left( Y^0 \partial^0 Y^c + \epsilon_{abc} Y^3 \partial^0 Y^b \right) \tau_c + \frac{4r_0}{g} \partial^0 Y^a \tau_3 = 0. \]

where \( \square (\ldots) = \partial_\mu \partial^\mu (\ldots) \), and \( \partial_\mu \) represents the derivative with respect to the component which the background field belongs. In the particular case of flat spatial space \( T^3 \) for the \( Y^\mu \), for the hedgehog ansatz [34] the set of equations (48) reduces to these three equations

\[ \hat{n}^c \square_\alpha + \frac{1}{2} \sin(2\alpha) \square \hat{n}^c - \frac{2r_0^2}{g} \sin(2\alpha) \hat{n}^c = 0. \]

Taking into account the \( T^3 \)-metric (32), we end up with the following ordinary differential equation

\[ \frac{d^2 \alpha}{d\phi_1^2} - \frac{\beta(p,q)}{2} \sin(2\alpha) - \frac{2r_0^2}{g} \sin(2\alpha) = 0, \]

where we defined \( \beta(p,q) = \lambda^2 \left( \frac{p^2}{r_0^2} + \frac{q^2}{r_0^2} \right) \). If we introduce the \( \xi \) definition (36), we get the result (35).

Appendix B. Computation of the I-function

In this Appendix we will derive in the detail equation (45), following the lines of [32,55]. The quantity we would like to compute is

\[ I(r_0, m^2) = \int \frac{d^4 q}{(2\pi)^4} \ln \left( \frac{(r_0^2 + m^2 + \bar{q}^2)}{\Lambda^2} \right). \]

where \( r_0 \) is the background field, \( m^2 \) is the square mass (could be complex), \( \Lambda \) is a quantity we used to regularize the divergence. We can write \( I(r_0, m^2) \) as the derivative respect of some auxiliary variable \( \epsilon \) and then taking the limit \( \epsilon \to 0 \):

\[ I(r_0, m^2) = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \left( -\Lambda^{2\epsilon} \int \frac{d^4 q}{(2\pi)^4} (r_0^2 + m^2 + \bar{q}^2)^{-\epsilon} \right). \]

Defining a new variable \( t = \frac{\sqrt{r_0^2 + m^2}}{\bar{q}} \) and passing to spherical coordinates, we have

\[ \int d^4 q = 2\pi^2 \int_0^{+\infty} dt t^3 \left( r_0^2 + m^2 \right)^2, \]

which gives us

\[ I(r_0, m^2) = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \left( \frac{\Lambda^{2\epsilon}}{8\pi^2} \left( r_0^2 + m^2 \right)^{2-\epsilon} \int_0^{+\infty} dt t^3 (1 + t^2)^{-\epsilon} \right). \]

We can write the last integral as

\[ \int_0^{+\infty} dt t^3 (1 + t^2)^{-\epsilon} = \frac{1}{2(\epsilon - 1)(\epsilon - 2)}. \]

where in the equality we used Mathematica 9.0 and we take the analytical continuation because strictly must be \( \Re(\epsilon) > 2 \) in the real domain. So, we have

\[ I(r_0, m^2) = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \left( -\Lambda^{2\epsilon} \left( r_0^2 + m^2 \right)^{2-\epsilon} \right) \]

Finally, a direct computation of the derivatives, and taken the limit \( \epsilon \to 0 \), gives us the result

\[ I(r_0, m^2) = \left( \frac{r_0^2 + m^2}{32\pi^2} \right)^2 \ln \left( \frac{s^2 r_0^2 + m^2}{\Lambda^2} \right) - \frac{3}{2}. \]
References