A General Extension Result with Applications to Convexity, Homotheticity and Monotonicity

Thomas Demuynck*

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Abstract

A well known result in the theory of binary relations states that a binary relation has a complete and transitive extension if and only if it is consistent (Suzumura (1976), theorem 3). A relation is consistent if the elements in the transitive closure are not in the inverse of the asymmetric part. We generalize this result by replacing the transitive closure with a more general function. Using this result, we set up a procedure which leads to existence results for complete extensions satisfying various additional properties. We demonstrate the usefullness of this procedure by applying it to the properties of convexity, homotheticity and monotonicity.

1 Introduction

Consider a universal set of alternatives, X, and a binary relation, R, on X, with asymmetric part $P(R)$. An extension R^* of R is a binary relation on X for which $R \subseteq R^*$ and $P(R) \subseteq P(R^*)$. The concept was initiated by Szpilrajn (1930), who showed that every transitive relation has an ordering (complete and transitive) extension.

Since then, the concept of ordering extensions has drawn a lot of attention within various research areas (e.g. Dushnik and Miller (1941), Suzumura (1976), Donaldson and Weymark (1998) and Duggan (1999)). A fundamental contribution to the theory of ordering extensions is due to Suzumura (Suzumura (1976), theorem 3), who showed that:

A binary relation R has an ordering extension if and only if it is consistent, i.e. if and only if $T(R) \cap P^{-1}(R) = \emptyset$, where $T(R)$ is the transitive closure of R.

This paper generalizes Suzumura's result by replacing the transitive closure, T, with a more general function F. Theorem 2, in section 2, shows that under certain restrictions on the function F:

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A binary relation R has a complete extension R', satisfying $R' = F(R')$, if and only if $F(R) \cap P^{-1}(R) = \emptyset$.

This result allows the use of a uniform procedure which verifies the existence of complete extensions satisfying various additional properties. The procedure takes the following steps:

i. Construct a function F.

ii. Show that a complete relation R' equals $F(R')$ if and only if R' satisfies some desired properties, such as convexity, homotheticity, monotonicity or transitivity.

iii. Verify that F satisfies the requirements of theorem 2 in section 2,

iv. Apply theorem 2 to conclude that a relation R has a complete extension $R' = F(R')$ if and only if $F(R) \cap P^{-1}(R) = \emptyset$.

We use this procedure to establish existence results of ordering extensions satisfying the properties mentioned in step ii above.
In section 2, we introduce notation and basic definitions and derive the main result of the paper. In section 3, we apply this result to specific properties, i.e. transitivity, convexity, homotheticity and monotonicity. In section 4, we build a bridge between the revealed preference literature and our result and in section 5, we present conclusions.

2 The General Extension Result

Consider a universal set X of alternatives. A set $R \subseteq X \times X$ is called a *binary relation* on X. We denote the set of all binary relations on X by $\mathcal{R}$. Given a relation R, we define its *inverse* R^{-1} by $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$. The *symmetric part* of R is given by $R \cap R^{-1}$ and is denoted by $I(R)$, the *asymmetric part* $R - I(R)$ is denoted by $P(R)$ and the *non-comparable* part $X \times X - (R \cup R^{-1})$ by $N(R)$.
A binary relation R is *complete* if for all $x, y \in X : (x, y) \in R$ or $(y, x) \in R$. A binary relation R is *transitive* if for all $x, y, z \in X$: if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Definition 1. *A relation R' is an extension of the relation R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

The relation $\preceq$ is symmetric: for all $R : R \preceq R$, and transitive: if $R \preceq R'$ and $R' \preceq R''$, then $R \preceq R''$.

Consider a function $F : \mathcal{R} \to \mathcal{R}$ and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. In the next section, we will present several examples for F. Until then, it may be usefull to think of the transitive closure[1] T as a possible candidate.

We have the following result.

[1]For a relation $R \in \mathcal{R}$, we say that $(x, y) \in T(R)$ if there exists elements $x_1, ..., x_n$ in X such that $x_1 = x$, $x_n = y$ and for all $i = 1, ..., n-1$: $(x_i, x_{i+1}) \in R$.

Lemma 1. *Let $R \subseteq F(R)$. Then in order that $R \in \mathcal{R}^*$ it is necessary and sufficient that $F(R) \cap P^{-1}(R) = \emptyset$.*

Proof. ($\rightarrow$) Let $R \preceq F(R)$ and assume on the contrary that there are elements x and $y \in X$ such that $(x, y) \in F(R)$ and $(y, x) \in P(R)$. As $R \preceq F(R)$, we must have that $(y, x) \in P(F(R))$. This contradicts with $(x, y) \in F(R)$.

($\leftarrow$) Let $R \subseteq F(R)$ and assume that $F(R) \cap P^{-1}(R) = \emptyset$. In order to show that $R \preceq F(R)$ it suffices to demonstrate that $P(R) \subseteq P(F(R))$. Assume that there are elements x and $y \in X$ such that $(x, y) \in P(R)$. From $F(R) \cap P^{-1}(R) = \emptyset$, we conclude that $(y, x) \notin F(R)$. Therefore, $(x, y) \in P(F(R))$ and $R \preceq F(R)$. □

Recall from the introduction that Suzumura characterized the set of relations $R \in \mathcal{R}$, for which there exists a complete and transitive relation R' such that $R \preceq R'$, by the condition that $T(R) \cap P^{-1}(R) = \emptyset$.

Likewise, we would like to characterize the relations $R \in \mathcal{R}$, for which there exists a complete relation $R' \in \mathcal{R}^*$ (or equivalently $R' = F(R')$) such that $R \preceq R'$, by the condition $F(R) \cap P^{-1}(R) = \emptyset$.

Lemma 2. *Let $F : \mathcal{R} \rightarrow \mathcal{R}$ and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. If F satisfies the following conditions:*

- *C1: for every well-ordered chain $R_0 \subset R_1 \subset ... \subset R_\alpha \subset ...$ of relations in $\mathcal{R}^*$, the union $\bigcup_{0 \leq \alpha} R_\alpha$ is also in $\mathcal{R}^*$, and,*
- *C2: for every relation $R \in \mathcal{R}^*$ such that $N(R) \neq \emptyset$, there exists a non-empty subset T of $N(R)$ such that $R \cup T \in \mathcal{R}^*$,*

then in order for a relation $R \in \mathcal{R}$ with $R \subseteq F(R)$, to have a complete extension $R' = F(R')$ it is sufficient that $F(R) \cap P^{-1}(R) = \emptyset$.

Before we give the proof, let us first outline the intuition behind the conditions $C1$ and $C2$. Recall from lemma 1 that if $R \subseteq F(R)$, the condition $F(R) \cap P^{-1}(R) = \emptyset$ is equivalent to the condition $R \in \mathcal{R}^*$. The idea is to enlarge R by repeatedly adding elements of $N(R)$, such that these enlarged relations remain in $\mathcal{R}^*$. This is exactly what condition $C2$ allows to do. If X is finite, $C2$ is sufficient to end up with a complete extension. However, if X is infinite, this is no longer true. For these cases we added the, rather technical, condition $C1$.

Proof of lemma 2. Let Ω be the set $\{R' \in \mathcal{R}^* | R \preceq R'\}$. By lemma 1, we know that $R \in \Omega$. Consider a well-ordered chain $R_0 \subset R_1 \subset ... \subset R_\alpha \subset ...$ in Ω and consider the relation $B = \bigcup_{0 \leq \alpha} R_\alpha$. We will prove that $B \in \Omega$. From condition $C1$, we know that B is in $\mathcal{R}^*$, so we are left to show that $R \preceq B$. Clearly, $R \subseteq B$. If on the contrary there are elements x and $y \in X$ for which $(x, y) \in P(R)$ and $(y, x) \in B$, we have that there must be a relation R_α in the well-ordered chain for which $(y, x) \in R_\alpha$. This contradicts with

$R \preceq R_\alpha$. Conclude that $B \in \Omega$. By application of Zorn's lemma, the set Ω has a maximal element. Let R' be such a maximal element.

In order to show that R' is complete, assume on the contrary that $N(R') \neq \emptyset$. We can use condition $C2$ and conclude that there exists a subset $T \subseteq N(R')$ for which $R' \cup T \in \mathcal{R}^*$. Let us now show that $R' \cup T \in \Omega$. For this, we must prove that $R \preceq R' \cup T$. Clearly $R \subset R' \cup T$. To show that $P(R) \subseteq P(R' \cup T)$, assume on the contrary that $(x, y) \in P(R)$ and $(y, x) \in T$. This implies that $(x, y) \in N(R')$, in contradiction with $P(R) \subseteq P(R')$. Therefore, $R \preceq R' \cup T$ and $R' \cup T \in \Omega$. This contradicts with the maximality of R'. Conclude that R' is complete.

We finish the proof by demonstrating that $R' = F(R')$. As $R' \preceq F(R')$, we immediately have that $R' \subseteq F(R')$. To see the reverse, assume that there are elements x and $y \in X$ for which $(x, y) \in F(R')$. If $(y, x) \in P(R')$, we would derive that $(y, x) \in P(F(R'))$, a contradiction. Therefore, we must have that $(y, x) \notin P(R')$. From completeness of R', we conclude that $(x, y) \in R'$. Hence, $F(R') \subseteq R'$. □

Lemma 3. *Let $F : \mathcal{R} \to \mathcal{R}$ and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. If F satisfies the following condition:*

C3: for all R and $R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$,

then in order that a relation $R \in \mathcal{R}$ with $R \subseteq F(R)$, has a complete extension $R' = F(R')$ it is necessary that $F(R) \cap P^{-1}(R) = \emptyset$.

Proof. Let $R \subseteq F(R)$ and assume that $R' = F(R')$ is a complete extension of R. Assume,on the contrary, that there exists elements $x, y \in X$ for which $(x, y) \in F(R)$ and $(y, x) \in P(R)$. The relation R' extends R. Therefore $(y, x) \in P(R')$. If we apply condition $C3$ to, $R \subseteq R'$, we derive that $F(R) \subseteq F(R')$. Hence, $(x, y) \in F(R') = R'$, a contradiction. □

The combination of lemma 1, lemma 2 and lemma 3 leads to the following result.

Theorem 1. *Let $F : \mathcal{R} \to \mathcal{R}$ satisfy the conditions $C1$, $C2$, $C3$. Then in order that a relation $R \in \mathcal{R}$ with $R \subseteq F(R)$ has a complete extension $R' = F(R')$ it is necessary and sufficient that $F(R) \cap P^{-1}(R) = \emptyset$.*

In the remaining part of this section, we will impose restrictions on the function F beyond $C1$, $C2$ and $C3$. There are several reasons for this. First of all, it allows us to impose a more familiar structure on F: although the conditions $C1$, $C2$ and $C3$ are fairly general, they do not correspond to a particular known class of functions. Second, we have been unable to find any economically interesting applications for which the function F satisfies conditions $C1$, $C2$ and $C3$, but not these extra conditions. Finally, imposing these additional restrictions here allows us to simplify and shorten the proofs in the next section.

Definition 2. *The function F is a closure operator if it satisfies condition $C3$,*

C4: for all $R \in \mathcal{R} : R \subseteq F(R)$, and, *C5: for all $R \in \mathcal{R} : F(F(R)) \subseteq F(R)$.*

The function F is an algebraic closure operator if, in addition, it satisfies:

C6: for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$.

Algebraic closure operators are found mainly within the field of algebra whereas closure operators are found in various fields of mathematics: e.g. topology, set theory, lattice theory, etc. Let us first show how algebraic closure operators relate to the conditions $C1, C2$ and $C3$.

Lemma 4. *Let $F : \mathcal{R} \to \mathcal{R}$ be an algebraic closure operator and let $\mathcal{R}^* = \{R \in \mathcal{R} | R \preceq F(R)\}$. If F satisfies,*

C7: for all $R \in \mathcal{R}$, if $R = F(R)$ and $N(R) \neq \emptyset$, there exists a non-empty subset T of $N(R)$ such that $R \cup T \in \mathcal{R}^*$

then F satisfies the conditions $C1$, $C2$ and $C3$.

Proof. The function F is a closure operator, hence it automatically satisfies condition $C3$.

Let us begin by showing that F satisfies condition $C2$. First, notice that $C4$ and $C5$ together, imply that $F(F(R)) = F(R)$ for all $R \in \mathcal{R}$. Consider a relation $R \in \mathcal{R}^*$ for which $N(R) \neq \emptyset$.

If $R = F(R)$, we have from $F(F(R)) = F(R)$, that $F(R) \in \mathcal{R}^*$. Therefore, in this case, condition $C2$ is equivalent to condition $C7$. If $R \subset F(R)$, we consider the set $T = F(R) - R$. Again, by $F(F(R)) = F(R)$, we have that $R \cup T \in \mathcal{R}^*$. The proof is complete if we can show that $T \subseteq N(R)$. Therefore, assume on the contrary that there are elements x and $y \in X$ such that $(x, y) \in T$ and $(x, y) \notin N(R)$. If $(x, y) \in R$, we derive, from the construction of T, that $(x, y) \notin T$, a contradiction. Therefore, it must be that $(y, x) \in P(R)$. As $T \subseteq F(R)$, we must also have that $(x, y) \in F(R)$. This contradicts $R \preceq F(R)$, i.e. $P(R) \subseteq P(F(R))$. Conclude that $T \subseteq N(R)$.

Finally, we need to show that F satisfies $C1$. Consider a well-ordered chain $R_0 \subseteq R_1 \subseteq ... \subseteq R_\alpha \subseteq ...$ in $\mathcal{R}^*$ and let $B = \bigcup_{\alpha \geq 0} R_\alpha$. We have to show that $B \in \mathcal{R}^*$. Applying condition $C4$, we derive that $B \subseteq F(B)$. Hence, from lemma 1, we only need to verify that $F(B) \cap P^{-1}(B) = \emptyset$. Therefore, assume on the contrary that there exists elements x and $y \in X$ for which $(x, y) \in F(B)$ and $(y, x) \in P(B)$. The closure F is algebraic, hence there exists a finite subset $B' \subseteq B$ for which $(x, y) \in F(B')$. Consider a relation R_α in the well-ordered chain for which $B' \subseteq R_\alpha$. The existence of such relation is guaranteed by finiteness of B'. From condition $C3$, we know that $(x, y) \in F(R_\beta)$ for all $\beta \geq \alpha$. Also, from the construction of B, we know that there is an integer $\alpha' \geq 0$ such that $(y, x) \in P(R_{\beta'})$ for all $\beta' \geq \alpha'$. Conclude that $(x, y) \in F(R_{\alpha''}) \cap P^{-1}(R_{\alpha''})$ for all $\alpha'' \geq \max\{\alpha, \alpha'\}$. This contradicts with the assumption that $R_{\alpha''} \in \mathcal{R}^*$ for all $R_{\alpha''}$ in the well-ordered chain. $\square$

Theorem 1, together with lemma 1 and lemma 4, gives us the following result:

Theorem 2. *Consider an algebraic closure operator $F : \mathcal{R} \to \mathcal{R}$ that satisfies condition $C7$. Then a relation $R \in \mathcal{R}$ has a complete extension $R' = F(R')$ if and only if $F(R) \cap P^{-1}(R) = \emptyset$.*

We finish this section by providing a characterization for closure operators which will be usefull in the next section. This characterization is well-known, but we prove it for completeness.

Lemma 5. *Assume that F satisfies $F(X \times X) = X \times X$. Then F is a closure operator if and only if for all $R \in \mathcal{R}$:*

$$F(R) = \bigcap\{Q \supseteq R | Q = F(Q)\}.$$

Proof. Let $F(R) = \bigcap\{Q \supseteq R|Q = F(Q)\}$ for all $R \in \mathcal{R}$. To show that F satisfies condition $C4$, assume that $(x, y) \in R$. Then, $(x, y) \in Q$ for all $Q \supseteq R$. Hence also for those relations Q that satisfy $Q = F(Q)$. Therefore $(x, y) \in F(R)$. To see condition $C3$ let $R \subseteq R'$ and assume that $(x, y) \in F(R)$. Then $(x, y) \in Q$ for all $Q \supseteq R$ that satisfy $Q = F(Q)$. As $R' \supseteq R$, we must have that $(x, y) \in Q'$ for all $Q' \supseteq R'$ that satisfy $Q' = F(Q')$. Hence, $(x, y) \in F(R')$. To see condition $C5$, let $(x, y) \in F(F(R))$. Then we have that $(x, y) \in Q$ for all $Q \supseteq F(R)$ that satisfy $Q = F(Q)$. If on the contrary $(x, y) \notin F(R)$ there must be a $Q' \supseteq R$ for which $(x, y) \notin Q'$ and $Q' = F(Q')$. As $Q' \supseteq R$, we derive from condition $C3$ that $F(Q') \supseteq F(R)$. Together with $Q' = F(Q')$, we derive that $Q' \supseteq F(R)$. This, however, contradicts with the assumption that $(x, y) \in F(F(R))$.

Let F satisfy conditions $C3$, $C4$ and $C5$. From $C3$ and $C5$, we know that $F(R) = F(F(R))$ for all $R \in \mathcal{R}$. Hence if $(x, y) \in Q$ for all $Q \supseteq R$ that satisfy $Q = F(Q)$, we must also have that $(x, y) \in F(R)$. Therefore $\bigcap\{Q \supseteq R|Q = F(Q)\} \subseteq F(R)$. To see the reverse, assume that $(x, y) \in F(R)$. By condition $C3$, we have that $(x, y) \in F(Q)$ for all $Q \supseteq R$. In particular, this must also hold for all Q that satisfy $Q = F(Q)$. Therefore $F(R) \subseteq \bigcap\{Q \supseteq R|Q = F(Q)\}$. □

3 Transitive, convex, homothetic and monotonic extensions

This section applies theorem 2 to several properties. The procedure that we will follow for any of these properties takes the following steps:

i. Define a function F.

ii. Show that a (complete) relation R' equals $F(R')$ if and only if R' satisfies the desired properties.

iii. Verify that F is an algebraic closure operator that satisfies condition $C7$. We do this in three steps:

iii.1 Show that F is a closure operator, i.e. $F(R) = \bigcap\{Q \supseteq R | Q = F(Q)\}$.

iii.2 Show that the closure operator F is algebraic, i.e. F satisfies condition $C6$.

iii.3 Show that F satisfies condition $C7$.

iv. Apply theorem 2 to conclude that a relation R, with $R \subseteq F(R)$, has a complete extension $R' = F(R')$ if and only if $R \cap P^{-1}(R) = \emptyset$.

3.1 Transitive extensions

In this section, we reproduce the result of Suzumura (1976) that every relation has a complete and transitive extension if and only if it is consistent[2].

We begin by introducing some notation and definitions.

Definition 3. *A finite sequence s in X of length $n_s \in \mathbb{N}$ is a function*

$$s : \{1, ..., n_s\} \to X : i \to s(i).$$

Let S collect all finite sequences in X. Sometimes, it will be convenient to define the sequence $s \in S$ by the enumeration of its image: $s = s(1), ..., s(n_s)$.
For two sequences s^1 and $s^2 \in S$ we can construct the compound sequence $s' \in S$ of length $(n_{s^1} + n_{s^2})$, given by $s' = s^1(1), ..., s^1(n_{s^1}), s^2(1), ..., s^2(n_{s^2})$. We denote this sequence by $s' = s^1 \oplus s^2$.

Definition 4. *A relation R in X is transitive if for all x, y and $z \in X$:*

$$(x, y) \in R \text{ and } (y, z) \in R \to (x, z) \in R.$$

Now, we are ready to apply steps (i)-(iv) mentioned in the introductory paragraph of section 3. We start by defining the function T.

i. Define the function T

The function $T : \mathcal{R} \to \mathcal{R}$ is given by $(x, y) \in T(R)$ if and only if there is a sequence $s \in S$ such that $s(1) = x$, $s(n_s) = y$ and for all $i = 1, ..., n_s - 1$:

$$(s(i), s(i+1)) \in R.$$

In the second step, we relate the function T to the property of transitivity.

ii. For all $R \in \mathcal{R} : R = T(R) \leftrightarrow R$ is transitive.

[2]A relation R is consistent if for any sequence $x_1, ..., x_n$ of elements in X, if $x_1 = x$, $x_n = y$ and for all $i = 1, ..., n-1$: $(x_i, x_{i+1}) \in R$, then $(y, x) \notin P(R)$.

Proof. ($\rightarrow$) Consider a relation $R = T(R)$ and elements x, y and $z \in X$. Consider the sequence $s = x, y, z$. If $(x, y) \in R$ and $(y, z) \in R$, we can apply the definition of T to the sequence s and conclude that $(x, z) \in T(R) = R$. Therefore, R is transitive.

($\leftarrow$) Assume that R is transitive. First of all, notice that for all $R \in \mathcal{R} : R \subseteq T(R)$, i.e. the function T satisfies condition $C4$. To see that $T(R) \subseteq R$, let $(x, y) \in T(R)$. From the definition of T, we know that there exists a sequence $s \in S$ such that $s(1) = x, s(n_s) = y$ and for all $i = 1, ..., n_s - 1$, $(s(i), s(i+1)) \in R$. Let us show that $(x, y) \in R$ by induction on n_s. For $n_s = 2$, we immediately have that $(x, y) \in R$. For the induction step, assume that the property holds for all $n_s \leq k$ and consider the case where $n_s = k + 1$. Consider the subsequence s' of s given by $s' = s(1), ..., s(n_s - 1)$. By the induction hypothesis, we have that $(x, s(n_s - 1)) \in R$. Also, by assumption, $(s(n_s - 1), y) \in R$. By transitivity of R, we conclude that $(x, y) \in R$. Hence, $R = T(R)$. $\square$

In the third step, we show that T satisfies the conditions of theorem 2.

iii. The function T is an algebraic closure operator which satisfies property $C7$.

First we show that T is a closure operator.

iii.1 For all $R \in \mathcal{R} : T(R) = \bigcap\{Q \supseteq R | Q = T(Q)\}$.

Proof. ($\subseteq$) Let $(x, y) \in T(R)$ and assume that $Q \supseteq R$ and $Q = T(Q)$. From the definition of T, we know that there exists a sequence $s \in S$ for which $s(1) = x$, $s(n_s) = y$ and for all $i = 1, ..., n_s - 1$: $(s(i), s(i+1)) \in R$. As $Q \supseteq R$, we have that for all $i = 1, ..., n_s - 1$, also $(s(i), s(i+1)) \in Q$. Applying the definition of T to the relation Q and the sequence s, we see that $(x, y) \in T(Q)$. Conclude that $(x, y) \in Q$, hence $(x, y) \in \bigcap\{Q \supseteq R | Q = T(Q)\}$.

($\supseteq$) Assume that $(x, y) \in \bigcap\{Q \supseteq R | Q = T(Q)\}$. We first show that $T(R)$ is transitive. Consider elements x, y and $z \in X$ such that $(x, y) \in T(R)$ and $(y, z) \in T(R)$. From the definition of T, we know that there must exist sequences s and $s' \in S$ such that $s(1) = x, s(n_s) = y, s'(1) = y, s'(n_{s'}) = z$ and for all $i = 1, ..., n_s - 1$ and $j = 1, ..., n_{s'} - 1$: $(s(i), s(i + 1)) \in R$ and $(s(j), s(j + 1)) \in R$. Consider the sequence $s'' = s(1), ..., s(n_s), s'(2), s'(3), ..., s'(n_{s'})$. If we apply the definition of T to this sequence, we derive that $(x, z) \in T(R)$, establishing that the relation $T(R)$ is transitive. Applying the result from section 3.1.ii, we derive that $T(T(R)) = T(R)$. Therefore $T(R) \in \{Q \supseteq R | Q = T(Q)\}$. Conclude that $(x, y) \in T(R)$. $\square$

Next, we show that the closure operator T is algebraic.

iii.2 The function T satisfies condition $C6$.

Proof. Consider a relation R and an element $(x, y) \in T(R)$. From the definition of T, we must have that there is a sequence $s \in S$ for which $s(1) = x$, $s(n_s) = y$ and for all $i = 1, ..., n_s - 1$: $(s(i), s(i + 1)) \in R$. Consider the set $D = \{s(1), ..., s(n_s)\}$ and construct the relation $R' = R \cap (D \times D)$. If we apply the definition of T to the sequence s and the finite relation R' we can conclude that $(x, y) \in T(R')$. Therefore, the function T satisfies condition $C7$. $\square$

Finally, we show that T satisfies condition $C7$.

iii.3 The function T satisfies condition $C7$.

Proof. Consider a relation $R = T(R)$ and assume that $N(R) \neq \emptyset$. Take an element $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y)\}$. Let us show that $R' \preceq T(R')$. The function T satisfies condition $C4$, hence: $R' \subseteq T(R')$. Therefore, by lemma 1, we only need to show that $T(R') \cap P^{-1}(R') = \emptyset$. Assume, on the contrary, that there are elements z and $w \in X$ for which $(z, w) \in P(R')$ and $(w, z) \in T(R')$ and consider first the case where $(z, w) \neq (x, y)$. From the definition of T, we know that there must exist a sequence $s \in S$ such that $s(1) = w$, $s(n_s) = z$ and for all $i = 1, ..., n_s - 1$: $(s(i), s(i + 1)) \in R'$. Clearly, there is an i such that $(s(i), s(i + 1)) = (x, y)$. Otherwise, we would have that $(w, z) \in T(R) = R$, contradicting $(z, w) \in P(R')$. Let l be the highest integer such that $(s(l-1), s(l)) = (x, y)$ and let f be the smallest integer such that $(s(f), s(f+1)) = (x, y)$. Construct the sequence $s' = s(l), s(l + 1), ..., s(n_s), s(1), ..., s(f - 1), s(f)$. If we apply the definition of T to this sequence, we have that $(y, x) \in T(R) = R$, a contradiction. If $(z, w) = (x, y)$, we construct the sequence $s' = s(l), ..., s(n_s)$ (if there is no $i = 1, ..., n_s - 1$ for which $(s(i), s(i + 1)) = (x, y)$, we set $l = 1$). If we apply the definition of T to this sequence, we have the same contradiction: $(y, x) \in T(R) = R$. $\square$

iv. Conclusion

The function T is an algebraic closure operator that satisfies condition $C7$. We can apply theorem 2 and conclude that R has a complete and transitive extension if and only if $T(R) \cap P^{-1}(R) = \emptyset$.

3.2 Convex extensions

In this section, we look for the existence of complete, transitive and convex extensions (see also Bossert and Sprumont (2001) and Scapparone (1999)). We assume that our universal space, X, is a convex[3] subset of $\mathbb{R}^n$.

For any finite set $A \subseteq X$, we denote by $V(A)$ the interior of the convex hull spanned by the elements of A:

$$V(A) = \left\{ x \in X \,\middle|\, x = \sum_{y_i \in A} \alpha_i y_i \right\},$$

where for all $i, \alpha_i > 0$ and $\sum_i \alpha_i = 1$.

[3]It is possible to reproduce the results of this section without this condition. However, this would drastically increase the notational complexity without really adding something fundamental to the analysis.

The property of convexity has many forms, depending on the additional requirements imposed on the relation under consideration (e.g. completeness). The definition we will use is the following:

Definition 5. *A relation R is convex if for all finite sets $A \subseteq X$ and all $y \in X$:*

- *if $(y_i, y) \in R$ for all $y_i \in A$, then for all $z \in V(A) : (z, y) \in R$ and*
- *if $(y_i, y) \in R$ for all $y_i \in A$ and there is an $y_j \in A$ for which $(y_j, y) \in P(R)$, then for all $z \in V(A) : (z, y) \in P(R)$.*

We are ready to go through the steps (i)-(iv) of the introductory section. We start by defining the function C.

i. Define the function C

Consider a finite number of sequences $s^1, ..., s^m \in S$. For an element $s^j(i)$, $i < n_{s^j}$, we say that the set A is compatible with $s^j(i)$ if

- $A \subseteq \{s^k(v)|k \in \{1, ..., m\}, v \in \{1, ..., n_{s^k}\}\}$ and,
- $s^j(i + 1) \in A$.

Given the sequences $s^1, ..., s^m$, we denote by $\mathcal{A}(s^j(i); s^1, .., s^m)$ the collection of all sets A which are compatible with $s^j(i)$.

To simplify notation, we also write $\mathcal{A}(s^j(i))$ instead of $\mathcal{A}(s^j(i); s^1, ..., s^m)$.

The function $C : \mathcal{R} \to \mathcal{R}$ is defined in the following way: Given a relation $R \in \mathcal{R}$ we have that $(x, y) \in C(R)$ if there exists a finite number of sequences $s^1, ..., s^m \in S$ such that for all $j = 1, ..., m$: $s^j(1) = x, s^j(n_{s^j}) = y$ and for all $j = 1, ..., m$ and $i = 1, ..., n_{s^j} - 1$:

- $(s^j(i), s^j(i + 1)) \in R$ or
- there is a set $A \in \mathcal{A}(s^j(i))$ such that $s^j(i) \in V(A)$.

We will show that C is an algebraic closure operator which satisfies condition $C7$, but let us first show how C relates to the the property of convexity.

ii. If R is complete, then $R = C(R)$ if and only if R is transitive and convex.

Proof. Necessity is straightforward, hence we only show sufficiency.

Assume that the relation R is complete, transitive and convex. As R is convex, we know that for all finite sets $A \subseteq X$ and all $z \in V(A)$, it cannot be the case that:

- $(y_i, z) \in R$ for all $y_i \in A$ and $(y_j, z) \in P(R)$ for at least one element $y_j \in A$.

Otherwise, we would conclude that $(z, z) \in P(R)$, a contradiction. From completeness of R, we can rewrite this conditions in the following way:
For all $A \subseteq X$, if $z \in V(A)$ then:

- there is an $y_j \in A$ for which $(z, y_j) \in P(R)$ or $$\tag{1}$$
- for all $y_i \in A : (z, y_i) \in R.$ $$\tag{2}$$

Now, assume that $(x, y) \in C(R)$. We need to show that $(x, y) \in R$. From the definition of C, we know that there must exist a finite number of sequences $s^1, ..., s^m$ such that for each sequence $j = 1, ..., m$: $s^j(1) = x$, $s^j(n_{s^j}) = y$ and for each $i = 1, ..., n_{s^j} - 1$ either $(s^j(i), s^j(i+1)) \in R$ or $s^j(i) \in V(A)$ for some $A \in \mathcal{A}(s^j(i))$.

In order to show that $(x, y) \in R$, we construct a sequence $s' = s'(1), ..., s'(n_{s'})$ such that $s'(1) = x$, $s'(n_{s'}) = y$ and for all $i = 1, ..., n_{s'} - 1$: $(s'(i), s'(i+1)) \in R$. The result then follows from transitivity of R and the result in section 3.1.ii. Consider the following algorithm:

1. Initiate $s'(1) = s^1(1)$ and set $k = 1$,
2. if $s'(k) = y$, we stop. Otherwise, we increase k by one, i.e. $k := k + 1$,
3. for $s'(k-1) = s^j(i)$, if $(s^j(i), s^j(i+1)) \in R$, we set $s'(k) = s^j(i+1)$ and return to step 2,
4. for $s'(k-1) = s^j(i)$, if $s^j(i) \in V(A)$ for some $A \in \mathcal{A}(s^j(i))$, we know from the first part of the proof that there are two cases to consider:
   (a) if (2) holds, we have that $(s^j(i), s^j(i+1)) \in R$. Then we put $s'(k) = s^j(i+1)$, and we return to step 2,
   (b) if (1) holds, we have that $(s^j(i), s^w(v)) \in P(R)$, for some element $s^w(v)$ in some sequence s^w. Then we put $s'(k) = s^w(v)$ and we return to step 2,

The algorithm can only terminate at the value y (step 2). Therefore, the algorithm is well-defined if it reaches this value after a finite number of steps. If, on the contrary, the algorithm does not terminate after finite number of steps, then, by finiteness of the sequences $s^1, ..., s^m$, there must be a loop in the sequence $s'(1), s'(2), ..., s'(f), ..., s'(l), ...$ Suppose that $s'(f)$ and $s'(l)$ correspond to the same element in the same sequence. This can only occur if the algorithm passes through step 4.b. Therefore, there must be a strict relation involved, for example $(s'(v), s'(v+1)) \in P(R)$ (with $f \leq v \leq l$). Using transitivity of R, we can apply the result from section 3.1.ii and conclude that $(s'(v+1), s'(v)) \in R$, a contradiction. Consequently, the algorithm must terminate after a finite number of steps, at the value of y. Conclude that $(x, y) \in R$. $\square$

iii. The function C is an algebraic closure operator which satisfies condition $C7$

We start by showing that C is a closure operator.

iii.1 For all $R \in \mathcal{R}$: $C(R) = \bigcap\{Q \supseteq R | Q = C(Q)\}$.

Proof. It is easy to see that $C(X \times X) = X \times X$. Therefore, for all $R \in \mathcal{R}$, the set $\{Q \supseteq R | Q = C(Q)\}$ is non-empty.

($\subseteq$) Assume that $(x, y) \in C(R)$. From the definition of C, we know that there exists a finite number of sequences $s^1, ..., s^m$ in S such that for all $j = 1, ..., m$: $s^j(1) = x$, $s^j(n_{s^j}) = y$ and for all $i = 1, ..., n_{s^j} - 1$: $(s^j(i), s^j(i+1)) \in R$ or there is an $A \in \mathcal{A}(s^j(i))$ for which $s^j(i) \in V(A)$. It follows that for every $Q \supseteq R$, $(x, y) \in C(Q)$. Therefore $(x, y) \in \bigcap\{Q \supseteq R | Q = C(Q)\}$.

($\supseteq$) Let us first show that $C(C(R)) = C(R)$. Evidently $C(R) \subseteq C(C(R))$. To see the reverse, consider elements x and $y \in X$ and assume that $(x, y) \in C(C(R))$. From the definition of C, there must be a finite number of sequences $s^1, ..., s^m$ in S such that for all $j = 1, ..., m$: $s^j(1) = x$, $s^j(n_{s^j}) = y$, and for all $i = 1, ..., n_{s^j} - 1$ either $(s^j(i), s^j(i+1)) \in C(R)$ or $s^j(i) \in V(A)$, where $A \in \mathcal{A}(s^j(i))$.

For each sequence s^j $(j = 1, ..., m)$ and element $s^j(i)$ in this sequence, for which $(s^j(i), s^j(i+1)) \in C(R)$, there must be a finite number of sequences $s^1_{(j,i)}, ..., s^{m_{(j,i)}}_{(j,i)}$, such that all these sequences start with $s^j(i)$, end with $s^j(i+1)$ and for each sequence $s^v_{(j,i)}$ and each nonterminal element $s^v_{(j,i)}(w)$ in this sequence, we have that either $\left(s^v_{(j,i)}(w), s^v_{(j,i)}(w+1)\right) \in R$ or $s^v_{(j,i)}(w) \in V(A)$, with $A \in \mathcal{A}(s^v_{(j,i)}(w))$.

For each sequence s^j $(j = 1, ..., m)$ and element $s^j(i)$ in in this sequence, for which $s^j(i) \in V(A)$ we construct the sequence $s^1_{(j,i)} = s^1_{(j,i)}(1), s^1_{(j,i)}(2)$ with $s^1_{(j,i)}(1) = s^j(i)$ and $s^1_{(j,i)}(2) = s^j(i+1)$.

Let $q^v_{(j,i)}$ be the sequence $s^v_{(j,i)}$ without its last element. Now, for each $j = 1, ..., m$; each $i = 1, ..., n_{s^j} - 1$ and each $v = 1, ..., m_{j,i}$, consider the following compound sequence:

$$s_{(j,i,v)} = q^1_{(j,1)} \oplus q^1_{(j,2)} \oplus ... \oplus q^1_{(j,i-1)} \oplus q^v_{(j,i)} \oplus q^1_{(j,i+1)} \oplus ... \oplus q^1_{(j,n_{s^j}-2)} \oplus s^1_{(j,n_{s^j}-1)}.$$

Using these sequences in the definition of C, we see that $(x, y) \in C(R)$. Hence $C(C(R)) = C(R)$. Therefore, $C(R) \in \{Q \supseteq R | Q = C(Q)\}$. Conclude that $\bigcap\{Q \supseteq R | Q = C(Q)\} \subseteq C(R)$. □

Now we show that the closure operator C is algebraic.

iii.2 The function C satisfies condition $C6$.

Proof. Consider a relation R and assume that $(x, y) \in C(R)$. From the definition of C, we know that there exists a finite number of sequences $s^1, ..., s^m$ such that for all $j = 1, ..., m$: $s^j(1) = x$, $s^j(n_{s^j}) = y$ and for all $i = 1, ..., n_{s^j} - 1$: $(s^j(i), s^j(i+1)) \in R$ or $s^j(i) \in V(A)$ for some set $A \in \mathcal{A}(s^j(i))$. Consider the set $D = \{s^j(i) | j = 1, ..., m; i = 1, ..., n_{s^j}\}$ and construct the relation $R' = R \cap (D \times D)$. This relation is finite and it is easy to see that $(x, y) \in C(R')$. Therefore, the function C satisfies condition $C6$. □

Finally we verify condition $C7$.

iii.3 The function C satisfies condition $C7$.

Proof. Let $R = C(R)$ and assume that $N(R) \neq \emptyset$. We have to find a nonempty subset T of $N(R)$ such that $R \cup T \in \mathcal{R}^*$. Let $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y)\}$. We will show that $R' \preceq C(R')$. By condition $C4$: $R' \subseteq C(R')$. Using lemma 1, we know that we can finish the proof if we show that $P^{-1}(R') \cap C(R') = \emptyset$. Now, assume on the contrary that there are elements z and $w \in X$ for which $(z, w) \in P(R')$ and $(w, z) \in C(R')$ and consider first the case where $(z, w) \neq (x, y)$.
From the definition of C, we know that there exists a finite number m of sequences $s^1, ..., s^m \in S$ such that for all sequences s^j: $s^j(1) = w$, $s^j(n_{s^j}) = z$, and for $i = 1, ..., n_{s^j} - 1$ either $(s^j(i), s^j(i+1)) \in R'$ or $s^j(i) \in V(A)$ for some $A \in \mathcal{A}(s^j(i))$. If for all $s^j(i)$ where $(s^j(i), s^j(i+1)) \in R'$ also $(s^j(i), s^j(i+1)) \in R$, then $(w, z) \in C(R) = R$. This contradicts with $(z, w) \in P(R')$. Therefore, there must be at least one $s^j(i)$ for which $(s^j(i), s^j(i+1)) = (x, y)$.

For any sequence s^j $(j = 1, ..., m)$ there are two cases to consider.

i. There is an $i = 1, ..., n_{s^j} - 1$ for which $(s^j(i), s^j(i+1)) = (x, y)$.

ii. There is no $i = 1, ..., n_{s^j} - 1$ for which $(s^j(i), s^j(i+1)) = (x, y)$.

As argued above, the set of sequences that fall under case 1 is not empty. Furthermore, for all sequences s^j that fall under case 1 and for all $i = 1, ..., n_{s^j} - 1$ for which $(s^j(i), s^j(i+1)) = (x, y)$ there are again two cases to consider:

1.1 There is a $v \geq i$ such that $(s^j(v), s^j(v+1)) = (x, y)$.

1.2 There is no $v \geq i$ such that $(s^j(v), s^j(v+1)) = (x, y)$.

For each sequence s^j under case 1 and for each $i \in \{1, ..., n_{s^j} - 1\}$ under case 1.1, we consider the smallest integer $w > i$ such that $(s^j(w), s^j(w+1)) = (x, y)$ and we construct the sequence:

$$s^j(i+1), s^j(i+2), ..., s^j(w-1), s^j(w). \tag{1}$$

For each sequence s^j under case 1 and for each $i \in \{1, ..., n_{s^j} - 1\}$ under case 1.2, we consider the smallest integer $w \geq 0$ such that $(s^j(w), s^j(w+1)) = (x, y)$ and we construct the sequence:

$$s^j(i+1), ..., s^j(n_{s^j}), s^j(1), ..., s^j(w-1), s^j(w). \tag{2}$$

Consider a sequence s^k that falls under case 1. Assume that l is the largest integer for which $(s^k(l), s^k(l)) = (x, y)$ and assume that f is the smallest integer such that $(s^k(f), s^k(f+1)) = (x, y)$.

For each sequence s^j that falls under case 2, we construct the sequence:

$$s^k(l+1), ..., s^k(n_{s^k}), s^j(1), s^j(2), ..., s^j(n_{s^j}), s^k(1), s^k(2), ..., s^k(f). \tag{3}$$

Applying the definition of C to the finite number of sequences that are constructed by (1), (2) and (3), we see that $(y, x) \in C(R) = R$, a contradiction. The proof for the case where $(z, w) = (x, y)$ is very similar and is left to the reader. □

iv Conclusion.

We see that the function C is an algebraic closure operator which satisfies condition $C7$. If we apply theorem 2 to the function C, we can conclude that a relation R has a convex, transitive and complete extension if and only if $C(R) \cap P^{-1}(R) = \emptyset$.

3.3 Homothetic extensions

In order to define the concept of homotheticity, we assume that our universal set X is a subset of $\mathbb{R}^m$. Further, we assume that X is a cone, i.e., if $x \in X$, then for all $\alpha \in \mathbb{R}_{++}$: $\alpha x \in X$, where $\mathbb{R}_{++}$ is the set of strict positive reals. For two elements x and $y \in X$, we say that $x \geq y$ if every coordinate of x is at least as large as every corresponding coordinate of y. We write $x > y$ if $x \geq y$ and $x \neq y$.

Definition 6. *A relation R is homothetic if for all elements x and $y \in X$ and all $\alpha \in \mathbb{R}_{++}$*

$$(x, y) \in R \leftrightarrow (\alpha x, \alpha y) \in R.$$

It turns out that homotheticity is a lot easier to analyze together with monotonicity:

Definition 7. *A relation R is monotonic if*

$$x \geq y \text{ implies } (x, y) \in R.$$

In this section, we look for the necessary and sufficient conditions such that a relation R has a complete, transitive, homothetic and monotonic extension.

We start by defining a function H.

i. Define the function H.

The function H is defined as, $(x, y) \in H(R)$ if there is a sequence $s \in S$ such that $s(1) = x$, $s(n_s) = y$ and for all $i = 1, ..., n_s - 1$:

- $s(i) \geq s(i+1)$ or
- there is an $\alpha_i \in \mathbb{R}_{++}$, such that $(\alpha_i s(i), \alpha_i s(i+1)) \in R$.

In the second step we relate the function H to the properties of homotheticity and monotonicity.

ii. For all $R \in \mathcal{R} : R = H(R) \leftrightarrow R$ is transitive, homothetic and monotonic.

Proof. That $R = H(R)$ implies that R is transitive, homothetic and monotonic is straightforward. To see the reverse, assume that R is transitive, homothetic and monotonic. It is easy to see that H satisfies condition $C4$: for all $R \in \mathcal{R}, R \subseteq H(R)$. Therefore, we are left to show that $H(R) \subseteq R$. Let $(x, y) \in H(R)$. This implies that there is a sequence $s \in S$ such that for all $i = 1, ..., n_s - 1$, $s(i) \geq s(i+1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ such that $(\alpha_i s(i), \alpha_i s(i+1)) \in R$. The relation R is monotonic and homothetic, so we have that for all $i = 1, ..., n_s - 1$, $(s(i), s(i+1)) \in R$. Using transitivity of R and applying the result from subsection 3.1.ii we see that $(x, y) \in R$. $\square$

iii. The function H is an algebraic closure operator which satisfies condition $C7$.

We start by showing that H is a closure operator.

iii.1 For all $R \in \mathcal{R} : H(R) = \bigcap\{Q \supseteq R | Q = H(Q)\}$.

Proof. From condition $C4$: $H(X \times X) = X \times X$. This implies that $\{Q \supseteq R | Q = H(Q)\}$ is non-empty for all $R \in \mathcal{R}$.

($\subseteq$) Let $(x, y) \in H(R)$. From the definition of H, we know that there exists a sequence $s \in S$ for which $s(1) = x$, $s(n_s) = y$ and for all $i = 1, ..., n_s - 1$, $s(i) \geq s(i+1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ such that $(\alpha_i s(i), \alpha_i s(i+1)) \in R$. From the definition of H, we immediately have that for all $Q \supseteq R$, $(x, y) \in H(Q)$. Therefore $(x, y) \in \bigcap\{Q \supseteq R | Q = H(Q)\}$.

($\supseteq$) Let us first show that $H(R)$ is transitive, homothetic and monotonic. To see transitivity, assume that there are elements x, y and $z \in X$ such that $(x, y) \in H(R)$ and $(y, z) \in H(R)$. Then there are sequences s and $s' \in S$ such that $s(1) = x$, $s(n_s) = s'(1) = y$, $s'(n_{s'}) = z$, for all $i = 1, ..., n_s - 1$: $s(i) \geq s(i+1)$ or there is an $\alpha_i > 0$ for which $(\alpha_i s(i), \alpha_i s(i+1)) \in R$ and for all $j = 1, ..., n_{s'} - 1$: $s'(j) \geq s'(j+1)$ or there is an $\alpha_j > 0$ for which $(\alpha_j s'(j), \alpha_j s'(j+1)) \in R$. Consider the compound sequence $s'' = s(1), ..., s(n_s), s'(2), ..., s'(n_{s'})$. If we apply the definition of H to the sequence s'', we have that $(x, z) \in H(R)$. Hence, $H(R)$ is transitive.

To show homotheticity, assume that $(x, y) \in H(R)$ and let $\beta > 0$. From the definition of H, we know that there is a sequence $s \in S$ such that $s(1) = x$, $s(n_s) = y$ and for every $i = 1, ..., n_s - 1$: $s(i) \geq s(i+1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i+1)) \in R$. Consider the sequence $s' = \beta s(1), \beta s(2), ..., \beta s(n_s)$. If $s(i)$ satisfies $s(i) \geq s(i+1)$, then $s'(i) \geq s'(i+1)$, and if $s(i)$ satisfies $(\alpha_i s(i), \alpha_i s(i+1)) \in R$ we can construct $\alpha'_i = \frac{\alpha_i}{\beta} > 0$, to derive that $(\alpha'_i s'(i), \alpha'_i s'(i+1)) \in R$. Therefore $(\beta x, \beta y) \in H(R)$. Conclude that $H(R)$ is homothetic.

Finally, it is easy to see that $H(R)$ is monotonic.

From the result in subsection 3.3.ii, we derive that $H(H(R)) = H(R)$. Therefore $H(R) \in \{Q \supseteq R | H(Q) = Q\}$. Conclude that $\bigcap\{Q \supseteq R | Q = H(Q)\} \subseteq H(R)$. $\square$

Now we show that the closure H is algebraic.

iii.2 The function H satisfies condition $C6$.

Proof. Consider elements x and $y \in X$ for which $(x, y) \in H(R)$. From the definition of H, we know that there exists a sequence $s \in S$ for which $s(1) = x$, $s(n_s) = y$ and for all $i = 1, ..., n_s - 1$, $s(i) \geq s(i + 1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$. Let $D = \{s(1), s(2), ..., s(n_s)\}$ and consider the relation $R' = R \cap (D \times D)$. It is easy to see that R' is finite and that $(x, y) \in H(R')$. Therefore H satisfies condition $C6$. $\square$

Finally, we show that H satisfies condition $C7$.

iii.3 The function H satisfies condition $C7$.

Proof. Assume that $R = H(R)$ and let $(x, y) \in N(R)$. Consider the relation $R' = R \cup \{(x, y)\}$. We prove that $R' \in \mathcal{R}^*$. From condition $C4$ and lemma 1, we know that this is equivalent to the condition that $H(R') \cap P^{-1}(R') = \emptyset$. Therefore, assume on the contrary that $(z, w) \in P(R')$ and $(w, z) \in H(R')$. From the definition of H, we know that there exists a sequence $s \in S$ for which $s(1) = w$, $s(n_s) = z$ and for all $i = 1, ..., n_s - 1$: $s(i) \geq s(i + 1)$ or there is an $\alpha_i \in \mathbb{R}_{++}$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R'$.

If for all $i = 1, ..., n - 1$ for which $(\alpha_i s(i), \alpha_i s(i + 1)) \in R'$ also $(\alpha_i s(i), \alpha_i s(i + 1)) \in R$, then $(w, z) \in H(R) = R$, a contradiction. Hence, there must be at least one $i = 1, ..., n - 1$ such that $(\alpha_i s(i), \alpha_i s(i + 1)) = (x, y)$.

From finiteness of s, it follows that there is a number $q \in \mathbb{N}$ and a finite set $I = \{\beta_1, ..., \beta_q\}$ of elements in $\mathbb{R}_{++}$ such that for all $i = 1, ..., q - 1$:$\left(\frac{1}{\beta_i} y, \frac{1}{\beta_{i+1}} x\right) \in H(R) = R$, and $\left(\frac{1}{\beta_q} y, \frac{1}{\beta_1} x\right) \in H(R) = R$. Consider the smallest value from the set I, say β_j. If $j > 1$, by homotheticity of R, we get $\left(y, \frac{\beta_{j-1}}{\beta_j} x\right) \in R$ and by monotonicity, $\left(\frac{\beta_{j-1}}{\beta_j} x, x\right) \in R$. By transitivity of R, we derive that $(y, x) \in R$, a contradiction. If $j = 1$, we have that $\left(y, \frac{\beta_q}{\beta_1} x\right) \in R$ and $\left(\frac{\beta_q}{\beta_1} x, x\right) \in R$. Again by transitivity: $(y, x) \in R$, a contradiction. Conclude that H satisfies condition $C7$. $\square$

iv. Conclusion.

The function H is an algebraic closure operator that satisfies condition $C7$. We can apply theorem 2 to the function H and conclude that a relation R has a homothetic, monotonic, complete and transitive extension if and only if $H(R) \cap P^{-1}(R) = \emptyset$.

3.4 Monotonic extensions

The last part of this section focusses on the properties of monotonicity and strict monotonicity. Again, we assume that X is a subset of $\mathbb{R}^m$.

We recall from definition 7 that a relation R is monotonic if for all $x, y \in X$:

$$x \geq y \to (x, y) \in R.$$

Definition 8. *A relation R on X is strict monotonic if R is monotonic and for all $x, y \in X$:*

$$x > y \text{ implies } (x, y) \in P(R).$$

Given a relation R, we define the relation $\bar{R}$ as:

$$\bar{R} = R \cup \{(x, y) \,|x \geq y\}.$$

Consider a function $F : \mathcal{R} \to \mathcal{R}$ and assume that F is an algebraic closure operator that satisfies condition $C7$, e.g. the function T or C.

In this section, we will characterize the relations R which have a complete and (strict) monotonic extension $R' = F(R')$.

We begin by defining the function $\bar{F}$.

i. Define the function $\bar{F}$.

For any function $F : \mathcal{R} \to \mathcal{R}$, we can define the function $\bar{F}$ such that for all $R \in \mathcal{R}$:

$$\bar{F}(R) = F\left(\bar{R}\right).$$

Let us now show how $\bar{F}$ relates to the property of monotonicity.

ii. $R = \bar{F}(R)$ if and only if R is monotonic and $R = F(R)$

Proof. ($\to$). To show that R is monotonic, observe first that $R \subseteq \bar{R}$. Furthermore, we have that, by condition $C4$: $\bar{R} \subseteq \bar{F}(R) = R$. If $x \geq y$, immediately $(x, y) \in \bar{R}$, hence also $(x, y) \in R$. To see that $R = F(R)$, we first notice that by condition $C4$: $R \subseteq F(R)$. Now, we have that $R \subseteq \bar{R}$, and F satisfies $C3$, hence $F(R) \subseteq \bar{F}(R) = R$. From this $F(R) \subseteq R$ and we are done.

($\leftarrow$). Let $R = F(R)$ and assume that R is monotonic. Clearly, $R \subseteq F(R)$. Monotonicity implies that $R = \bar{R}$. If we combine this with $R = F(R)$, we derive that $R = F(\bar{R}) = \bar{F}(R)$. $\square$

iii. The function $\bar{F}$ is an algebraic closure operator that satisfies condition $C7$.

iii.1 For all $R \in \mathcal{R} : \bar{F}(R) = \bigcap\{Q \supseteq R | Q = \bar{F}(R)\}$.

Proof. Clearly $\bar{F}(X \times X) = X \times X$. Therefore, the set $\{Q \supseteq R | Q = \bar{F}(Q)\}$ is not empty.

As F is a closure operator, we know that $F(\bar{R}) = \{Q \supseteq \bar{R} | Q = F(Q)\}$. Therefore, it suffices to show that $\{Q \supseteq R | Q = \bar{F}(Q)\} = \{Q \supseteq \bar{R} | Q = F(Q)\}$.

($\subseteq$) Let $R' \in \{Q \supseteq R | Q = \bar{F}(Q)\}$. From section 3.4.ii, R' is monotonic and $R' = F(R')$. From monotonicity: $R' \supseteq \bar{R}$. Conclude that $R' \in \{Q \supseteq \bar{R} | Q = F(Q)\}$.

($\supseteq$) Let $R' \in \{Q \supseteq \bar{R} | Q = F(Q)\}$. As $R' \supseteq \bar{R}$, we have that R' is monotonic. Together with $R' = F(R')$, we know from section 3.4.ii that $R' = \bar{F}(R')$. Conclude that $R' \in \{Q \supseteq R | R' = \bar{F}(R')\}$. □

iii.2 The function F satisfies condition $C6$.

Proof. Let $(x, y) \in \bar{F}(R)$. Then from the definition of $\bar{F}$, we have that $(x, y) \in F(\bar{R})$. As F satisfies condition $C6$, we know that there exists a finite subset R' of $\bar{R}$ such that $(x, y) \in F(R')$. As $R' \subseteq \bar{R}'$, we derive from condition $C3$ that $(x, y) \in F(\bar{R}') = \bar{F}(R')$. Conclude that $\bar{F}$ satisfies condition $C6$. □

iii.3 The function $\bar{F}$ satisfies condition $C7$.

Proof. Let $R = \bar{F}(R)$ and assume that $N(R) \neq \emptyset$. As $R \subseteq \bar{R}$ and $\bar{R} \subseteq F(\bar{R}) = R$ we have that $R = \bar{R}$. Application of condition $C7$ to the function F shows the existence of a set $T \subseteq N(R)$ for which $R \cup T \preceq F(R \cup T)$. Clearly, $R \cup T = \overline{R \cup T}$. Therefore $F(R \cup T) = F(\overline{R \cup T}) = \bar{F}(R \cup T)$. This implies that $R \cup T \preceq \bar{F}(R \cup T)$. □

iv. Conclusion.

We know that $\bar{F}$ is an algebraic closure operator that satisfies condition $C4$. Therefore, from theorem 2, we know that a relation R has a complete and monotonic extension $R' = F(R')$ if and only if $\bar{F}(R) \cap P^{-1}(R) = \emptyset$.

We can derive a similar result regarding the property of strict monotonicity:

If F is an algebraic closure operator satisfying $C7$, then a relation R has a strict monotonic and complete extension $R' = F(R')$ if and only if $P^{-1}(R) \cap \bar{F}(R) = \emptyset$ and for all $y > x$:

$$(x, y) \notin \bar{F}(R).$$

Proof. ($\leftarrow$) Notice first that $P^{-1}(R) \cap \bar{F}(R) = \emptyset$ is a necessary condition to have a monotonic and complete extension $R' = F(R')$, so that it is also necessary to have a strict monotonic and complete extension $R' = F(R')$. Second, if on the contrary $y > x$ and $(x, y) \in \bar{F}(R)$, we have by condition $C3$ and $R \subseteq R'$, that $(x, y) \in F(R') = R'$, a contradiction.

($\rightarrow$) Assume that $\bar{F}(R) \cap P^{-1}(R) = \emptyset$ and for all $y > x$, $(x, y) \notin \bar{F}(R)$. From the first result in this section, we know that $\bar{F}(R)$ has a complete extension $R' = \bar{F}(R')$ which is also an extension of R. We now show that R' is strict monotonic. Consider two elements x and $y \in X$ for which $x > y$. We have that $(x, y) \in \bar{F}(R)$ and $(y, x) \notin \bar{F}(R)$. Therefore, $(x, y) \in P(\bar{F}(R))$. From $\bar{F}(R) \preceq R'$, we conclude that $(x, y) \in P(R')$, hence R' is strict monotonic. $\square$

4 F-rationalizability

Let X be a universal set of alternatives and let Σ be a set of nonempty subsets of X. A choice function K is a correspondence

$$K : \Sigma \rightarrow\rightarrow X : S \rightarrow K(S) \subseteq S,$$

such that for all $S \in \Sigma$, $K(S)$ is non-empty. We assume that this function contains the notion of transitivity, i.e. for all $R \in \mathcal{R}$, if $(x, y) \in T(R)$, then $(x, y) \in F(R)$.

Definition 9. *A choice function K is said to be F-rationalizable if there exists a complete relation $R^* = F(R^*)$, such that for all $S \in \Sigma$:*

$$K(S) = \{x \in S \,|(x, y) \in R^* \text{ for all } y \in S\},$$

i.e. the elements chosen from S are top-ranked according to R^*.

Given a choice function K, we define the revealed preference relation R_v by $(x, y) \in R_v$ if there is a set $S \in \Sigma$ such that $x \in K(S)$ and $y \in S$. If also $y \notin K(S)$, we say that x is strictly revealed preferred to y and write $(x, y) \in P_v$.

We can now give the characterization result for F-rationalizability.

Theorem 3. *If the function F satisfies property $C1$, $C2$ and $C3$ then a choice function K is F-rationalizable if and only if $R_v \cap P_v^{-1} = \emptyset$.*

Proof. First of all, notice that by $R \subseteq T(R)$ and $T(R) \subseteq F(R)$, we have that F satisfies condition $C4$: for all $R \in \mathcal{R} : R \subseteq F(R)$.

($\rightarrow$) If K is F-rationalizable, there exists a complete relation R^* such that $R^* = F(R^*)$, and $x \in K(S)$ implies that $(x, y) \in R^*$ for all $y \in S$. As $T(R^*) \subseteq F(R^*)$, we have that R^* is also transitive. Now, assume on the contrary that there is an element $(x, y) \in F(R_v) \cap P_v^{-1}$. It is easy to see that $R_v \subseteq R^*$, hence, by condition $C3$, we have that $F(R_v) \subseteq F(R^*) = R^*$. So $(x, y) \in R^*$. Now, as $(y, x) \in P_v$, there is a $S \in \Sigma$ such that $y \in K(S)$ and $x \in S - K(S)$. Let us show that we must have that $(y, x) \in P(R^*)$.

From $R_v \subseteq R^*$, we have that $(y, x) \in R^*$. If on the contrary also $(x, y) \in R^*$, then by transitivity of R^*, $(x, z) \in R^*$ for all $z \in S$, which implies, from rationalizability of K, that $x \in K(S)$. This concludes the contradiction.

($\leftarrow$) To see the reverse, let $F(R_v) \cap P_v^{-1} = \emptyset$. It is easy to see that this implies that $P_v = P(R_v)$. Hence, by lemma 1 and condition $C4$: $R_v \preceq F(R_v)$. By theorem 2, R_v has a complete extension, $R^* = F(R^*)$. Let us show that R^* rationalizes F. If $x \in K(S)$, by definition $(x, y) \in R_v$ for all $y \in S$ and hence $(x, y) \in R^*$ for all $y \in S$. On the other hand, if $x \notin K(S)$, by non-emptiness of K, there must be an $y \in S$ such that $y \in K(S)$. By definition: $(y, x) \in P_v = P(R_v)$. As R^* is an extension of R_v, we must have that $(y, x) \in P(R_v)$, hence not $(x, y) \in R^*$ for all $y \in S$. □

This result is immediately applicable to the functions T, C, H, $\bar{T}$ and $\bar{C}$ defined in section 3.

5 Conclusion

In this paper, we discussed the existence of complete extensions satisfying additional properties. Our main result, theorem 1, states that: if F satisfies conditions $C1$, $C2$, $C3$, then a relation $R \subseteq F(R)$ with $R \subseteq F(R)$, has a complete extension $R' = F(R')$ if and only if $F(R) \cap P^{-1}(R) = \emptyset$.

Then, we added additional structure on the function F and showed (cfr theorem 2) that if F is an algebraic closure operator that satisfies condition $C7$, then a relation $R \in \mathcal{R}$ has a complete extension $R' = F(R')$ if and only if $F(R) \cap P^{-1}(R) = \emptyset$.

We demonstrated the usefullness of these result by providing characterizations for the existence of complete extensions satisfying the properties of transitivity, convexity, homotheticity and monotonicity. Finally, we showed how it can be used within a choice theoretical framework to derive interesting rationalizability results.

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