The computational complexity of rationalizing Pareto optimal choice behavior*

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Abstract

We consider a setting where a coalition of individuals chooses one or several alternatives from each set in a collection of choice sets. We examine the computational complexity of Pareto rationalizability. Pareto rationalizability requires that we can endow each individual in the coalition with a preference relation such that the observed choices are Pareto efficient. We differentiate between the situation where the choice function is considered to select all Pareto optimal alternatives from a choice set and the situation where it only contains one or several Pareto optimal alternatives. In the former case we find that Pareto rationalizability is an NP-complete problem. For the latter case we demonstrate that, if we have no additional information on the individual preference relations, then all choice behavior is Pareto rationalizable. However, if we have such additional information, then Pareto rationalizability is again NP-complete. Our results are valid for any coalition of size greater or equal than two.
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1 Motivation

We determine the computational complexity of validating whether a choice function is consistent with Pareto optimal choice behavior. In concreto, we ask whether there exists an efficient algorithm that can verify whether a given data set on observed choices from a collection of choice sets is consistent with the choices from a coalition of individuals that selects only Pareto optimal alternatives? In general we find that Pareto optimal choice behavior has either no testable restrictions or that its testable implications are very difficult to verify. For the latter case, this is established by showing that the problem is NP-complete. Our findings bear important empirical implications. The fact that the verification of Pareto consistent choice behavior is either trivial or NP-complete demonstrates that empirical refutation or acceptance of Pareto optimal choice behavior might be extremely difficult. In fact, all known algorithms to solve NP-complete problems suffer from exponential worst–case time complexity.

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Consider an individual who selects from every set in a collection of choice sets one or several alternatives. These choices are rationalizable if it is possible to endow this individual with a nicely behaved (i.e. transitive and complete) preference relation over the set of alternatives such that for every choice set, the set of chosen alternatives coincides with the set of all maximal elements according to this preference relation. In this single person setting rationalizability is easily verified. In a seminal contribution to the literature Richter (1966) demonstrates that a choice function is 'individually' rationalizable if it satisfies the congruence condition. This condition requires that the transitive part of the revealed preference relation, which can be computed efficiently using, for example, Warshall's algorithm (Warshall, 1962), does not conflict with the strict revealed preference relation.

Now, consider a setting where multiple individuals in a coalition jointly choose one or several alternatives from every set in a collection of choice sets. It is well known that in a multi–person setting the observed choices do not always coincide with the set of maximal elements from a single preference relation. Moreover, the outcome of the joint decision will largely depend on the underlying decision process. One of the most straightforward extensions of individual rationality to a multi–person setting is rendered by the notion of Pareto optimality. Pareto optimality requires that if an alternative is chosen then there is no other feasible alternative that was preferred to this chosen alternative by all individuals (We take a preference relation to be asymmetric, transitive and complete. As such, an alternative is Pareto optimal if it is not unanimously dominated by another feasible alternative.)

The principle of Pareto optimality is one of the cornerstones of normative economic analysis and it is beyond any doubt the most frequently used concept in welfare economics and cooperative game theory. Apart from this normative perspective Pareto optimality is also frequently used to explain actual cooperative behavior (e.g. models of household behavior, firm-union wage negotiations, job-matching and job-search models, international trade negotiation models and models of cartel formation in oligopolistic competition). Nevertheless, despite its wide prevalence as a normative and behavioral principle there are relatively few researches that look at its testable implications.

In this research we look at the computational complexity of verifying whether a given choice function is consistent with Pareto efficient choice behavior from a coalition of individuals. Towards this end we distinguish between two concepts.

The first concept, **Pareto rationalizable**, requires that there exist preference relations, one for each member in the coalition, such that the observed choices from a choice set coincide with the entire set of Pareto optimal alternatives from this set. The concept of Pareto rationalizability has previously been analyzed by Sprumont (2000) in the setting of a normal form game. More recently, Echenique and Ivanov (2011) looked at Pareto rationalizability in a general 2 agent choice theoretic setting. Other relevant research imposes more structure on the underlying framework. For example, Bossert and Sprumont (2002) characterize consistency of a choice function with Pareto efficiency (and individual rationality) in the setting of a two person exchange economy.

Although Pareto rationalizability is a useful concept, it suffers from the fact that it is difficult to apply in reality. Indeed, in most real life settings the choices from a coalition do not coincide with the entire set of Pareto optimal allocations. For example, if the coalition chooses by means of a bargaining model (e.g. Nash bargaining or Raiffa–Kalai–Smorodinsky) then the observed outcome will be Pareto optimal but the chosen alternative(s) will not necessarily coincide with the entire set of Pareto optimal outcomes. In this perspective, we say that a choice function is **weak Pareto rationalizable** if there exist individual preference relations such that the chosen elements are a subset of the set of Pareto efficient alternatives. In other words, a choice function is weak Pareto rationalizable if every chosen alternative is not Pareto dominated by another feasible alternative. Although Weak Pareto rationalizability is the more reasonable concept when faced with real life choice situations, this is, to our knowledge, the first research that looks at this property in a general choice theoretic setting.[1]

In section 3, we derive the computational complexity of Pareto and weak Pareto rationalizability using a general choice theoretic setting. We show that Pareto rationalizability is NP–complete for all coalitions with at least two individuals. On the other hand we show that, in general, the notion of weak Pareto rationalizability has no testable constraints on observed choice behavior. In fact, it is quite trivial to show that any choice function is weak Pareto rationalizable by a coalition with two individuals (see Proposition 3.1). Although this result is well known[2] it nevertheless emphasizes that from an empirical viewpoint Pareto optimality is a very weak concept. In order to restore empirical refutability we introduce the concept of a dominance relation. Simply said, a dominance relation is a known subrelation of the Pareto dominance relation: if an alternative a is better than the alternative b according to the dominance relation we know that all individuals in the coalition prefer a over b. We provide several settings where such a dominance relation appears naturally. Further, we show that the inclusion of a dominance can lead to non–trivial restrictions on observed choice behavior.[3] Next, we demonstrate that the inclusion of such dominance relation implies that the problem of weak Pareto rationalizability becomes NP–complete for coalitions with at least two individuals.

The demonstration that there is no complexity gap between (weak) Pareto rationalizability by two individuals and (weak) Pareto rationalizability by more than two individuals is a key result of this paper. Indeed, many decision problems have such a gap. For example, the recognition of 3-colorable graphs is NP-complete, while 2-colorable graphs can be recognized in polynomial time. The same is true for 3-SAT versus 2-SAT and 3-dimensional matching versus 2-dimensional matching.

By establishing the computational complexity of rationalizing (weakly) Pareto efficient choice behavior we also contribute to the small but growing literature that establishes NP-completeness results for various economic problems. Particularly relevant to our results is the line of research within this literature that looks at the computational complexity of various (individual or collective) rationalizability problems. Galambos (2009) employs the setting of Sprumont (2000) and shows that the problem of rationalizing a choice function as the outcome of a noncooperative Nash equilibrium in a normal form game is an NP-complete problem. Next, Apesteguia and Ballester (2010) consider the model of choice by multiple rationales from Kalai, Rubinstein, and Spiegler (2002) and demonstrate that computing the minimal number of rationales that rationalizes a given choice function is an NP-complete problem. Demuynck (2011) establishes similar NP-completeness results for the sequential choice model of Manzini and Mariotti (2007) and the model of choice by game trees from Xu and Zhou (2007).

[1]On the other hand, there has been a growing stream of research that looks at the testable implications of weak Pareto rationalizability in a household consumption setting with private and public goods (see, for example, Apps and Rees (1988); Chiappori (1988, 1992) and Cherchye, De Rock, and Vermeulen (2007).

[2]In fact, this result (and its proof) is very similar to the result of Sprumont (2000, Proposition 1) who showed that weak Pareto rationalizability has no testable implications in the setting of a normal form game.

[3]A trivial restriction would be, for example, that a cannot be chosen from $\{a, b\}$ when b is better than a according to the dominance relation.

Finally, Talla Nobibon and Spieksma (2010) find that verifying the revealed preference conditions for weak Pareto rationalizable choice behavior for a two person coalition as derived by Cherchye, De Rock, and Vermeulen (2007) is an NP-complete problem. This setting differs from ours in the sense that these conditions are obtained from a revealed preference analysis *a là* Afriat (1967) and Varian (1982) (i.e. in a household consumption setting). On the other hand, our paper focuses on the more general choice theoretic setting. Also, while we introduce a dominance relation in order to obtain testable implications, the revealed preference setting obtains its testable implications from the fact that utility is strict monotonic. Rather interestingly, Talla Nobibon and Spieksma (2010) use a reduction from the NP-complete problem not all equal 3-SAT while we use a reduction from its monotone variant. Apart from this similarity, however, the proofs of the two results are rather different.

Section 2 provides a short introduction into the theory of computational complexity. The readers who are familiar with this theory may safely skip this section. Section 3 introduces the main notation and establishes the computational complexity results for Pareto and weak Pareto rationalizability. Section 4 provides the proofs.

2 NP-completeness

This section provides a short introduction to the theory of computational complexity. For the readers who are familiar with the notion of NP-completeness this section may be skipped. For compactness, we only provide a very quick introduction alas at the cost of completeness. For a detailed introduction into the theory of computational complexity and NP-completeness in particular we refer to Papadimitriou (1994) and Garey and Johnson (1979).

The theory of computational complexity attempts to answer how much time (and memory) is needed to solve a decision problem. A decision problem is composed of a collection of instances which are the input to the problem and a Yes/No question.

The collection of instances $\mathcal{I}$ give the inputs of the decision problem. Normally, it is assumed that these instances are encoded in some convenient way. This is done by using a suitable set of symbols Σ (e.g. $\Sigma = \{0, 1\}$) and by defining $\mathcal{I}$ as a subset of all finite strings of symbols from Σ, i.e. $\mathcal{I} \subseteq \bigcup_{n=1}^{\infty} \Sigma^n$. For a particular instance $I \in \mathcal{I}$ we call the smallest n such that $I \in \Sigma^n$ the length or size of I. In general, the particular encoding of the instances are not really important and does not really change the results as long as they are 'reasonable'. Formally, encoding schemes are said to be 'reasonable' if they are (i) concise, i.e. they are not 'padded' with unnecessary information or symbols, (iii) the symbols occurring in the encoding are in a fixed base (other than 1) and (ii) the encoding should be decodable in polynomial time.

The Yes/No question of a decision problem maps each instance $I \in \mathcal{I}$ to a Yes or a No depending on whether the particular instance I satisfies a certain property. Formally, one could think of the Yes/No question as a function f from the set of all instances $\mathcal{I}$ to the binary set $\{0, 1\}$. Then, we say that an instance satisfies the particular property or is a Yes instance if $f(I) = 1$, and it does not satisfy the property and is a No instance when $f(I) = 0$.

The theory of computational complexity classifies decision problems according to the time it takes to compute the value of $f(I)$ given the instance I. Here, time is expressed with respect to the size of the instance. The two most important classes of decision problems are the classes P and NP. The class P (polynomial) contains all decision problems which are easy to solve. These problems can be solved using an algorithm that computes the solution in a polynomial number of steps in terms of the size of the instance. The class NP (nondeterministic polynomial) contains all problems that might be difficult to solve (i.e. it might take exponential time) but which are easy to verify. In particular, any solution to the problem can be verified in polynomial time.[4]

Of course, any decision problem in the class P is also in NP. At present, it is not known if the converse also holds. The general accepted belief is that P $\neq$ NP. A decision problem which is as least as difficult to solve as any problem in the class NP is called NP-hard. A decision problem is NP-complete if it is both NP-hard and in NP. NP-complete problems are among the most difficult problems in the class NP. They are considered to be computationally intractable especially for large instances. In fact, all known solution methods applicable to NP-complete problems suffer from exponential worst-time complexity.

In order to understand the proofs in section 4, it might be interesting to have a quick overview of how NP-completeness results are established. In principle, for a candidate decision problem to be NP-complete it suffices to demonstrate two things. First, one must demonstrate that the problem is in the class NP. In other words, it must be shown that given a proposed polynomial sized solution to the problem it can be efficiently verified (i.e. in polynomial time) that this proposed solution is indeed a solution. Second, it must be shown that the NP-complete problem is at least as hard as any other problem in NP (i.e. the problem is NP-hard). The way by which this is established is by taking a known NP-hard problem and showing that this problem is a special case of the candidate problem. As such, the candidate problem will be at least as difficult as the NP-hard problem. This demonstrates that the candidate problem is also NP-hard. Usually, this second step is proved by the technique of polynomial reduction. Let the known NP-complete problem be represented by a collection of instances $\mathcal{I}$ and the function $f : \mathcal{I} \to \{0, 1\}$ and let the candidate problem be represented by the collection of instances $\mathcal{I}'$ and the function $g : \mathcal{I}' \to \{0, 1\}$. In order to show that $(g, \mathcal{I}')$ is NP-hard it suffices to demonstrate that there exists a function γ from $\mathcal{I}$ to $\mathcal{I}'$ such that (i) γ is computable in polynomial time and (ii) an instance $I \in \mathcal{I}$ of the NP-complete problem provides a solution to this problem (i.e. $f(I) = 1$) if and only if the instance $\gamma(I)$ provides a solution to the candidate problem (i.e. $g(\gamma(I)) = 1$). The idea behind this construction is that any algorithm that efficiently computes the function g can also be used to efficiently compute the function f by means of the intermediate function γ (i.e. in order to know the value of $f(I)$ it is always possible to compute $g(\gamma(I))$). In this sense, the problem $(f, \mathcal{I})$ is at least as easy to solve as the problem $(g, \mathcal{I}')$.

3 Pareto and weakly Pareto rationalizability

In this section we establish the computational complexity of Pareto and weak Pareto rationalizability. We start with introducing the necessary notation and definitions. Next, we present the relevant decision problems. We end the section by stating our complexity results.

[4]The exact way by which this is defined is that there exists a polynomial time algorithm (function) g and for each instance I for which $f(I) = 1$, there exists a certificate $C(I)$ of polynomial size such that $g(C(I), I) = 1$ and for all instances I for which $f(I) = 0$ and all certificates C it is always the case that $g(C, I) = 0$.

3.1 Notation and definitions

Consider a finite set of alternatives X and a finite collection $\mathcal{D}$ of nonempty subsets of X. We call $\mathcal{D}$ the **domain** of the decision problem. A **choice function** c corresponds to each choice set A from $\mathcal{D}$ a nonempty set $c(A) \subseteq A$.

A binary relation $\succ$ on X is **transitive** if for all a, b and $c \in X$, $a \succ b$ and $b \succ c$ implies $a \succ c$. The relation is **complete** or total if for every two distinct elements a and $b \in X$ either $a \succ b$ or $b \succ a$. The relation $\succ$ is **asymmetric** if for all distinct a and $b \in X$ not ($a \succ b$ and $b \succ a$). A **partial order** is a transitive and asymmetric relation. A linear order or **preference relation** is a transitive, asymmetric and complete relation.[5]

Given a relation $\succ$ on X and a subset A from X we denote by $M(\succ, A)$ the set of **maximal elements** of A according to the relation $\succ$. Formally,

$$M(\succ, A) = \{a \in A | \forall b \in A, b \not\succ a\}.$$

For a strictly positive number $K \in \mathbb{N}$ and a profile (list) of preference relations $\{\succ_k\}_{k \leq K}$ we say that $\succ$ is the **Pareto dominance relation** of $\{\succ_k\}_{k \leq K}$ if for all $a, b \in X$,

$$a \succ b \text{ if and only if } a \succ_k b \text{ for all } k \leq K.$$

Equivalently, we can write:

$$\succ = \bigcap_{k \leq K} \succ_k .$$

A Pareto dominance relation is always a partial order. On the other hand, Dushnik and Miller (1941) proved that for any partial order $\succ$ there exists a number K and a profile $\{\succ_k\}_{k \leq K}$ for which $\succ$ is the corresponding Pareto dominance relation. As such, any partial order is the Pareto dominance relation for some coalition. They defined the **dimension** of a partial order as the smallest number of linear orders whose Pareto dominance relation coincides with this partial order. Formally, a partial order $\succ$ has **dimension** less or equal than K if there exists a profile $\{\succ_k\}_{k \leq K}$ of K preference relations such that $\succ$ is the corresponding Pareto dominance relation. In the same article, Dushnik and Miller also provide two characterizations for a partial order to have a dimension smaller than or equal to 2.[6]

Let us now turn to the definitions of Pareto and weak Pareto rationalizability. We define a choice function to be Pareto rationalized by a given profile of preference relations if the choices from each choice set coincide with the set of all Pareto efficient alternatives from this profile.

Definition 1 (Pareto rationalizability). *A choice function c is Pareto rationalized by the profile of preference relations $\{\succ_k\}_{k \leq K}$ iff for all $A \in \mathcal{D}$,*

$$c(A) = M(\succ, A),$$

where $\succ$ is the Pareto dominance relation of $\{\succ_k\}_{k \leq K}$.

[5]Sometimes, partial and linear orders are also referred to as strict partial and strict linear orders, in order to make a distinction between the non-asymmetric variants. In this paper we use 'partial order' and 'linear order' as referring to the strict varieties.

[6]See also Sprumont (2001) for a different but simpler characterization for a partial order to be of dimension 2, provided some regularity conditions are satisfied.

As mentioned in the introduction, the notion of Pareto rationalizability is probably too restrictive from an empirical point of view. It is difficult to imagine a real life example where a group of individual selects all Pareto optimal alternatives from a choice set. A more reasonable assumption is that a chosen alternative is not Pareto dominated by another available alternative, i.e. the chosen alternatives are a subset of the set of Pareto optimal allocations. We call a choice function that satisfies this condition weak Pareto rationalizable.

Definition 2. *The choice function c on a domain $\mathcal{D}$ is weak Pareto rationalized by the profile of preference relations $\{\succ_k\}_{k\leq K}$ iff for all $A \in \mathcal{D}$.*

$$c(A) \subseteq M(\succ, A),$$

where $\succ$ is the Pareto dominance relation for the profile $\{\succ_k\}_{k\leq K}$.

Consider a choice function c which is Pareto rationalizable and let $\succ_R$ be defined by $a \succ_R b$ iff there is a set $A \in \mathcal{D}$ such that $\{a\} = c(A)$ and $b \in A$. From the definition of Pareto rationalizability, we immediately see that $\succ_R$ is a subrelation of the Pareto dominance relation $\succ$ (i.e. $\succ_R \subseteq \succ$). As such, $\succ_R$ should be acyclic. This shows that the concept of Pareto rationalizability has some testable implications (i.e. it can be rejected). On the other hand, the following result shows that this is not true for the concept of weak Pareto rationalizability.

Proposition. *For any choice function c, there exist a profile of preference relations $\{\succ_1, \succ_2\}$ that weakly Pareto rationalizes c.*

By replicating the preferences $\succ_1$ and $\succ_2$, this result extends to coalitions with more than two individuals.

The proof of the proposition is quite trivial. Consider an arbitrary ranking of the alternatives in X which we represent by the preference relation $\succ_1$. Next, for all a and $b \in X$ define $a \succ_2 b$ if and only if $b \succ_1 a$. The preference relation $\succ_2$ is the inverse relation of $\succ_1$. It follows that the Pareto dominance relation is empty. This in turn implies that for any choice set $A \in D$, the set of Pareto optimal elements from A is the set A itself, $M(\succ, A) = A$.

Above proposition shows that we need to include additional information in order to reject the notion of weak Pareto rationalizability. We proceed by introducing the concept of a dominance relation as a subrelation of the Pareto dominance relation. Consider a preference profile $\{\succ_k\}_{k\leq K}$ with a Pareto dominance relation $\succ$. A binary relation $\rhd$ is a **dominance relation** of the preference profile if $\rhd\ \subseteq \succ$. In other words, for all $a, b \in X$, if $a \rhd b$, then $a \succ_k b$ for all $k \leq K$. In order to motivate the idea of a dominance relation we provide several examples where such relation appears naturally.

Example 1. Consider the setting where X is a finite set of bundles of public goods. Then, we could impose $a \rhd b$ if and only if $a > b$. If the bundle a has at least as much of every good as the bundle b and if $a \neq b$ then a is considered better than b for all individuals in the coalition. This will be the case if individual preferences are monotone.

Example 2. Consider a finite set of outcomes O and let X be a finite subset of the power set of O. Every alternative in X consists of a finite number of outcomes. If all outcomes are desirable we can assume that $a \rhd b$ whenever $b \subset a$, i.e. if all outcomes in b are also contained in a and a contains some outcomes which are not in b then a is better than b for all individuals.

Example 3. As a final example, consider the setting where X is a finite set of income distributions. The individuals in the coalition can be thought of as a group of government representatives who must decide on the most favorable income distribution (for example, by implementing a certain tax policy). In this setting it is logical to assume that $a \rhd b$ if the distribution a first order stochastically dominates the distribution b.

The following definition combines the notion of weakly Pareto rationalizability with the idea of a dominance relation.

Definition 3. *The profile $\{\succ_k\}_{k \leq K}$ weakly Pareto rationalizes the choice function c with the dominance relation $\rhd$ if there exists a profile of preferences $\{\succ_k\}_{k \leq K}$ such that for all $A \in \mathcal{D}$,*

$$c(A) \subseteq M(\succ, A),$$

where $\succ$ is the Pareto dominance relation for the profile $\{\succ_k\}_{k \leq K}$ and for all $a, b \in X$, $a \rhd b$ implies that $a \succ_k b$ for all $k \leq K$ (i.e. $\rhd \subseteq \succ$).

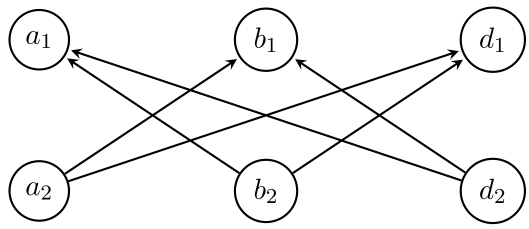
The inclusion of a dominance relation into the definition of weakly Pareto rationalizability immediately imposes some restrictions on the joint choice behavior. For example, if $a \rhd b$ then it should not be the case that $b \in c(A)$ while $a \in A$. If b was chosen over a then at least one individual should prefer b over a. As such, we see that not every choice function will be weakly Pareto rationalizable. However, above example is a rather trivial restriction which has no bite if the domain contains only choice sets with alternatives that are incomparable according to the dominance relation $\rhd$.

As an example of a less trivial restriction, consider the set of alternatives $X = \{a_1, a_2, b_1, b_2, d_1, d_2\}$. Define the dominance relation $\rhd$ by the comparisons $a_2 \rhd b_1$, $a_2 \rhd d_1$, $b_2 \rhd a_1$, $b_2 \rhd d_1$, $d_2 \rhd a_1$ and $d_2 \rhd b_1$. See Figure 1 for an illustration of the relation $\rhd$. The domain $\mathcal{D}$ consists of the sets $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{d_1, d_2\}$. Observe that none of the choice sets contains elements that are comparable according to $\rhd$. The choice function is given by $c(\{a_1, a_2\}) = \{a_1\}$, $c(\{b_1, b_2\}) = \{b_1\}$ and $c(\{d_1, d_2\}) = \{d_1\}$. If $\{\succ_1, \succ_2\}$ weakly Pareto rationalizes this choice function it is necessary that the following three conditions are satisfied.

$$(a_1 \succ_1 b_1 \text{ and } a_1 \succ_1 d_1) \text{ or } (a_1 \succ_2 b_1 \text{ and } a_1 \succ_2 d_1)$$
$$(b_1 \succ_1 a_1 \text{ and } b_1 \succ_1 d_1) \text{ or } (b_1 \succ_2 a_1 \text{ and } b_1 \succ_2 d_1)$$
$$(d_1 \succ_1 b_1 \text{ and } d_1 \succ_1 a_1) \text{ or } (d_1 \succ_2 b_1 \text{ and } d_1 \succ_2 a_1)$$

It is easy to see that these three conditions are incompatible. Therefore, the choice function is not weakly Pareto rationalizable by 2 preference relations.

Figure 1: Illustration of $\rhd$



Above example already gives us a hint why the problem of weak Pareto rationalizability with a dominance relation might be a difficult problem. Observe that the (necessary) conditions for weak Pareto rationalizability in the example are given in terms of three exclusive OR conditions that have to be jointly satisfied. This gives us a total of $2^3 = 8$ combinations that have to be verified in total. Given the limited size of the problem this can still be done quite rapidly for the setting at hand. However, it is easy too see that the number of combinations increases exponentially in the number of OR conditions. This indicates that the verification may become increasingly difficult even for moderately sized problems.

3.2 The decision problems

Let us now turn to the presentation of the decision problems. We begin by presenting the decision problem corresponding to the notion of weak Pareto rationalizability.

K-weak Pareto rationalizability (K-WPRAT): *Given a set of alternatives X, a domain $\mathcal{D}$, a choice function c on $\mathcal{D}$ and a partial order $\rhd$ on X, does there exist a profile of preference relations $\{\succ_k\}_{k\leq K}$ such that this profile provides a weak Pareto rationalization of the choice function c and such that for all $a, b \in X$, $a \rhd b$ implies that $a \succ_k b$ for all $k \leq K$?*

In terms of the formulation in section 2, we have that each instance I of the decision problem K-WPRAT is determined by a quadruple $(X, \mathcal{D}, c, \rhd)$ which contains a set of alternatives, a domain, a choice function on this domain and dominance relation. The function f that determines the decision problem K-WPRAT maps an instance $(X, \mathcal{D}, c, \rhd)$ to 1 if and only if it is weakly Pareto rationalizable by a profile of K preference relations that satisfy the dominance relation $\rhd$.

Let us denote by n the size of the set X and by m the size of the domain $\mathcal{D}$. Any instance $(X, \mathcal{D}, c, \rhd)$ of K-WPRAT can be encoded by first enumerating every set A in $\mathcal{D}$, subsequently enumerating the chosen elements from these sets, i.e. $c(A)$ and subsequently enumerating the elements of $\rhd$. We can do this using $2\,m\,n + n^2$ bits.[7] As such, if we set $k = \max\{n, m\}$, then the size of the instance is $O(k^2)$.

Let us now turn to the decision problem corresponding to the notion of Pareto rationalizability.

K-Pareto rationalizability (K-PRAT): *Given a finite universal set X, a domain $\mathcal{D}$ and a choice function c, does there exist a profile of K preference relations $\{\succ_k\}_{k\leq K}$ that Pareto rationalizes the choice function c?*

Instances of K-PRAT are determined by a triple $(X, \mathcal{D}, c)$ and the function f that determines this decision problem maps any instance $(X, \mathcal{D}, c)$ to 1 if and only if the instance is Pareto rationalizable by profile of K preference relations. Similarly as for the problem K-WPRAT, we can show that an instance of K-WPRAT can be encoded in $O(k^2)$ bits.

Observe that the problem K-PRAT does not depend on a dominance relation. To ameliorate this, we could also consider the following variation of K-PRAT.

K-Pareto rationalizability-2 (K-PRAT-2): *Given a finite universal set X, a domain $\mathcal{D}$, a choice function c and a partial order $\rhd$, does there exist a profile of K preference relations $\{\succ_k\}_{k\leq K}$ that Pareto rationalizes the choice function c and such that for all $a, b \in X$, $a \rhd b$ implies that $a \succ_k b$ for all $k \leq K$?*

[7] Given the elements $x_1, \ldots, x_n$ of X, every set $A \in \mathcal{D}$ and $c(A)$ can be encoded by an n dimensional array which has a one in position i if $a_i \in A$ (or $c(A)$) and a zero in position i if $a_i \notin A$ (or $c(A)$). Next, $\rhd$ can be encoded as a matrix B of n^2 bits where entry $b_{i,j}$ is equal to 1 if and only if $a_i \rhd a_j$. As such, every instance of K-WPRAT can be encoded using a total of $2mn + n^2$ bits.

The problem K-PRAT corresponds to the special case where $\rhd = \emptyset$. As such K-PRAT is more restrictive than the problem K-PRAT-2. As a consequence, NP-completeness of K-PRAT will also imply that K-PRAT-2 is NP-completene (although the reverse need not hold). This is the reason why we focus on the more restrictive problem K-PRAT.

Finally, in both problems K-PRAT and K-WPRAT, the size of the coalition K is a parameter of the decision problem. As such, we actually obtain an infinite number of decision problems, one for each value of $K \in \mathbb{N}$. This setting is more restrictive than when we would take the number of individuals in the coalition K as an additional parameter of the instance. For example, we could define the following problems, PRAT and WPRAT.

Pareto rationalizability (PRAT): *Given a universal set X, a domain $\mathcal{D}$, a choice function c and a number K, does there exist a profile of K preference relations $\{\succ_k\}_{k\leq K}$ that Pareto rationalizes the choice function c?*

Weak Pareto rationalizability (WPRAT): *Given a universal set X, a domain $\mathcal{D}$, a choice function c, a dominance relation $\rhd$ and a number K, does there exist a profile of K preference relations $\{\succ_k\}_{k\leq K}$ that weakly Pareto rationalizes the choice function c and for which $\rhd \subseteq \succ_k$ $(k \leq K)$?*

Observe that the instances of the problem PRAT consists of quadruples $(X, \mathcal{D}, c, K)$ and instances of WPRAT are of the form $(X, \mathcal{D}, c, \rhd, K)$. Hence, for these problems, the number of individuals K in the coalition is a part of the input to the problem. The problem PRAT and WPRAT are NP-complete as soon as there exists at least one value of K for which K-PRAT and K-WPRAT is NP-complete. However, the converse does not necessarily hold, i.e. it is possible that PRAT or WPRAT are NP-complete while K-PRAT or K-WPRAT are in P for some value of K. For example, this is obviously the case when K = 1.

3.3 Main results

We derive the computational complexity of K-PRAT and K-WPRAT (for $K \geq 2$) in two steps. First we focus on the case where the size of the coalition, K, is greater or equal to three. Subsequently, we take on the setting where the coalition has size two. Consider the decision problem of establishing the dimension of a partial order.

K-dimension (K-Dim): *Given a partial order $\succ$ on a set X, is this relation of dimension K or less?*

Yannakakis (1982) proved that the decision problem K-Dim is NP-complete for all K $\geq$ 3. On the other hand, it is known that K-Dim is efficiently solvable, i.e. in the class P, if K is less than or equal to two. We refer to Spinrad (1994) for an overview of the different algorithms that can be applied in this case.

Using the result of Yannakakis (1982) we can show that K-PRAT and K-WPRAT are NP-complete for all K $\geq$ 3. The proof uses a reduction from the NP-compete problem K-Dim.

Corollary. *The decision problems K-PRAT and K-WPRAT are NP-complete for all $K \geq 3$.*

The proof is quite easy, so we state it here. For K-PRAT we construct from the partial order $\succ$, an instance $(X, \mathcal{D}, c)$ for which the Pareto dominance relations coincides with $\succ$. For this, it suffices to consider the instance where the domain $\mathcal{D}$ coincides with all two element subsets from X. Then, if $a \succ b$ we determine $c(\{a, b\}) = \{a\}$ and if $a \not\succ b$ and $b \not\succ a$ we set $c(\{a, b\}) = \{a, b\}$. As such, we have that the partial relation $\succ$ has dimension less than or equal to K if and only if the corresponding instance $(X, \mathcal{D}, c)$ is rationalizable by a profile of no more than K preference relations. This shows that K-PRAT is at least as difficult to solve as K-Dim. The NP-completeness of K-PRAT for K ≥ 3 then follows immediately from the NP-completeness of K-Dim for K ≥ 3.

For K-WPRAT we construct an instance, $(X', \mathcal{D}, c, \rhd)$ from the instance $(X, \succ)$ of K-Dim in the following way. The set of alternatives X' is defined by $X \cup \{d\}$ with d a new alternative not in X. Next, we set $\rhd = \succ$ and we consider the domain $\mathcal{D} = \{\{a, b\}, \{a, b, d\} | \neg(a \succ b) \text{ and } \neg(b \succ a)\}$. We define the choice function by $c(\{a, b, d\}) = \{a\}$ and $c(\{a, b\}) = \{b\}$. It is easy to see that the dimension of $\succ$ is equal to K if and only if the choice function is weakly Pareto rationalizable by a profile of K preferences. This shows that K-WPRAT is NP-complete.

When K $= 2$, above reductions from K-Dim can no longer be used to prove NP-completeness because we have that 2-Dim is efficiently solvable. Rather surprisingly, however, we find that both 2-PRAT and 2-WPRAT are also NP-complete. For 2-PRAT, our proof relies on a reduction from the NP-complete problem 3-SAT. For 2-WPRAT, we use a reduction from monotone not all equal 3-SAT. Although the proof of the latter result considers the general case where the dominance relation $\rhd$ is some unrestricted partial ordering, the proof can easily be adjusted such that the dominance relation coincides with a more specific partial relation like in the examples given above. The proof of the theorem is in the next section.

Theorem. *The decision problems 2-PRAT and 2-WPRAT are NP-complete.*

We end this section with several remarks.

First of all, if the domain $\mathcal{D}$ is binary (i.e. if $\mathcal{D}$ contains all two element subsets of X), we can use the efficiency of 2-Dim to show that 2-PRAT is also efficiently solvable. In order to do this, we devise an algorithm that verifies 2-PRAT in three steps. First one constructs the partial order $\succ$ such that $a \succ b$ if and only if $\{a\} = c(\{a, b\})$. This relation is well defined because the domain $\mathcal{D}$ is binary. In a second step, it is verified whether $\succ$ is of dimension less than or equal to 2. Finally, it is verified that for all $A \in \mathcal{D}$, $c(A) = M(\succ, A)$. It is easy to see that an instance satisfies 2-PRAT if and only if it passes this algorithm. Also, all three steps in this algorithm can be verified in polynomial time. Therefore, 2-PRAT is in P for all instances with a binary domain.

Although this shows that 2-PRAT is in P if the domain is binary, it is not known whether this result is also valid for the problem 2-WPRAT. The difficulty lies in the fact that for 2-WPRAT it is not possible to recover the Pareto dominance relation from the choices on all two element subsets.

Second, if we consider the case where the domain under consideration is complete, i.e. $\mathcal{D}$ contains all nonempty subsets of X, it is possible to show that both the problems K-PRAT and K-WPRAT are quasi-polynomially bounded.[8] Given this result, it is highly unlikely that the problems K-PRAT or K-WPRAT are NP-complete when restricted to universal domains. Otherwise, it would follow that all NP-complete problems are also quasi-polynomially bounded. However, such a bound has never been found and there is a strong conviction that this will never happen.

[8] A proof of this can be constructed along the lines of the proof of Theorem 4 of Apesteguia and Ballester (2010).

On the other hand, notice that the complete domain assumption is quite restrictive in the sense that the size of an instance is exponential in the size of the set X. For example, if the universal set contains 10 alternatives, then the domain must contain no less than 1.023 choice sets.

Finally, we end this section with an open problem. In our analysis, we departed from the assumption that we know the size of the coalition. However, we could also imagine a situation where we do not have this kind of information. In such a setting, the problem of Pareto rationalizability can be rephrased in the following way:

Unrestricted Pareto rationalizability (∞-PRAT): *Given a set of alternatives X, a domain $\mathcal{D}$ and a choice function c on $\mathcal{D}$ does there exist a strictly positive number K and a profile of preference relations $\{\succ_k\}_{k\leq K}$ such that this profile provides a Pareto rationalization of the choice function c?*

Using the result of Dushnik and Miller (1941) presented in 3.1, this problem can be shown to be equivalent to the problem of maximal element rationalizability by a partial order (see for example Bossert, Sprumont, and Suzumura (2005) and Bossert and Suzumura (2010)). It would be interesting to know whether this problem is also NP-complete.

4 Proof of the main theorem

We begin with the proof that 2-PRAT is NP-complete. Next, we show that 2-WPRAT is NP-complete.

4.1 2-PRAT is NP-complete

Membership in NP is easily verified. For the second step of the proof we use a reduction from the problem 3-SAT. An instance of 3-SAT consists of a finite set of binary variables $u_1, \ldots, u_n$ and a finite set of clauses $C_1, \ldots, C_m$. Each variable can either take the value TRUE or FALSE. Every clause contains three literals and each literal is either equal to a variable u_i (i.e. the literal is TRUE if u_i is TRUE and FALSE if u_i is FALSE) or its negation $\bar{u}_i$ (i.e. the literal is FALSE if u_i is TRUE or the literal is TRUE if u_i is FALSE). The following defines the decision problem 3-SAT.

3-Satisfiability (3-SAT): *Given a finite set of variables $u_1, \ldots, u_n$ and a finite set of clauses $C_1, \ldots, C_m$, does there exist an assignment to the variables, either TRUE or FALSE, such that each clause contains at least one literal with the value TRUE.*

Let $u_1, \ldots, u_n$ be a list of variables and let $C_1, \ldots, C_m$ be a list of clauses corresponding to an instance of 3-SAT. From these, we construct an instance of 2-PRAT: a set of alternatives X, a domain $\mathcal{D}$ and a choice function c. We begin by defining the set X.

- We construct two alternatives a and b.
- For each clause C_ℓ ($\ell = 1, \ldots, m$) we construct an alternative d_ℓ.
- For each variable u_i ($i = 1, \ldots, n$) we construct four alternatives $y_i, \bar{y}_i, w_i$ and $\bar{w}_i$.

For each clause C_ℓ and each literal $l_{k,\ell}$ ($k = 1, 2, 3$; $\ell = 1, \ldots, m$) from this clause we consider the alternatives $z_{k,\ell}$ and $\bar{z}_{k,\ell}$ in X such that if $l_{k,\ell} = u_i$ then $z_{k,\ell} = y_i$ and $\bar{z}_{k,\ell} = \bar{y}_i$,

Table 1: Choice sets and choice function

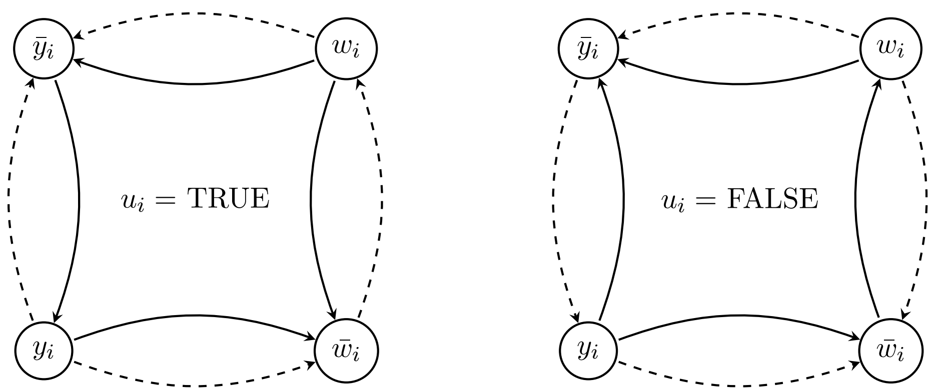
choice set, A	choice, $c(A)$	range	
$\{a,b\}$	$\{a,b\}$		(1)
$\{d_\ell,b\}$	$\{d_\ell\}$	$\ell=1,\ldots,m$	(2)
$\{d_\ell,a\}$	$\{d_\ell,a\}$	$\ell=1,\ldots,m$	(3)
$\{y_i,\bar{y}_i\}$	$\{y_i,\bar{y}_i\}$	$i=1,\ldots,n$	(4)
$\{w_i,\bar{w}_i\}$	$\{w_i,\bar{w}_i\}$	$i=1,\ldots,n$	(5)
$\{\bar{y}_i,w_i\}$	$\{w_i\}$	$i=1,\ldots,n$	(6)
$\{\bar{w}_i,y_i\}$	$\{y_i\}$	$i=1,\ldots,n$	(7)
$\{d_\ell,\bar{z}_{k,\ell}\}$	$\{d_\ell\}$	$\ell=1,\ldots,m; k=1,2,3$	(8)
$\{z_{1,\ell},z_{2,\ell},z_{3,\ell},a\}$	$\{z_{1,\ell},z_{2,\ell},z_{3,\ell}\}$	$\ell=1,\ldots,m$	(9)

and if $l_{k,\ell}=\bar{u}_i$ then $z_{k,\ell}=w_i$ and $\bar{z}_{k,\ell}=\bar{w}_i$. The construction of the domain $\mathcal{D}$ and the value of the choice function c is given in Table 1. This construction can be performed in polynomial time.

We begin by showing that, if this instance satisfies 2-PRAT, the corresponding 3-SAT problem has a solution. Consider a rationalization $\{\succ_1,\succ_2\}$ of the instance $(X,\mathcal{D},c)$. First, consider the comparison between a and b. We can assume, without loss of generality that $a\succ_1 b$ and $b\succ_2 a$. Otherwise, we can exchange the preferences $\succ_1$ and $\succ_2$ everywhere.

Now, consider the alternatives $y_i,\bar{y}_i,w_i$ and $\bar{w}_i$ whose comparisons are determined by conditions (4)–(7) in Table 1. One can verify that comparisons between these alternatives must take one of two mutually exclusive configurations. We determine the values of the variables u_i $(i=1,\ldots,n)$ according to which configuration prevails. The two configurations are given by Figure 2 where a dashed arrow determines the relation $\succ_1$ and a solid arrow the relation $\succ_2$. From the figure, we see that $u_i=$ TRUE whenever $y_i\succ_1\bar{y}_i$ (and not $w_i\succ_1\bar{w}_i$) and $u_i=$ FALSE when $w_i\succ_1\bar{w}_i$ (and not $y_i\succ_1\bar{y}_i$).

Figure 2: Value of the variable u_i

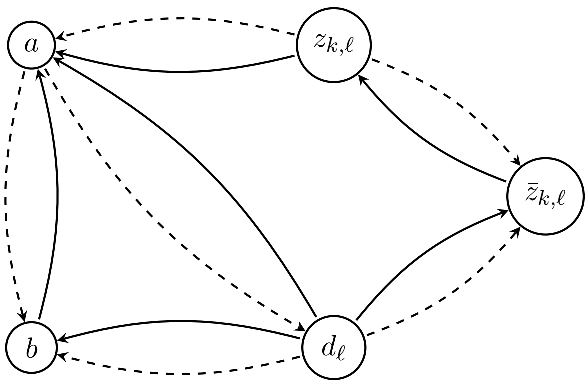


We see that for no variable u_i $(i=1,\ldots,n)$ we have that both $u_i=$ TRUE and $u_i=$ FALSE.

Now, consider the choice set $\{z_{1,\ell}, z_{2,\ell}, z_{3,\ell}, a\}$, with choices $\{z_{1,\ell}, z_{2,\ell}, z_{3,\ell}\}$ (see condition (9) in Table 1). As the instance satisfies 2-PRAT, we see that there must be at least one alternative in the set $\{z_{1,\ell}, z_{2,\ell}, z_{3,\ell}\}$ that Pareto dominates a (because, a is not retained). Let $z_{k,\ell}$ be this alternative. We will show that the literal $l_{k,\ell}$ equals TRUE.

The reasoning is illustrated in Figure 3. First of all from $d_\ell \succ_2 b$ (comparison (2)) and $b \succ_2 a$ we see that $d_\ell \succ_2 a$. As such, $a \succ_1 d_\ell$ (follows from transitivity and (3)). Then from $z_{k,\ell} \succ_1 a \succ_1 d_\ell \succ_1 \bar{z}_{k,\ell}$ we have that $z_{k,\ell} \succ_1 \bar{z}_{k,\ell}$ (this follows from transitivity and (8)). As such, if $z_{k,\ell} = y_i$ then $y_i \succ_1 \bar{y}_i$ and consequently $u_i =$ TRUE, and if $z_{k,\ell} = w_i$ then $w_i \succ_1 \bar{w}_i$ and $u_i =$ FALSE. In both cases, we have that the literal $l_{k,\ell}$ is TRUE.

Figure 3: Demonstration that $z_{k,\ell}$ equals one.



Let us now assume that 3-SAT is satisfied. We need to show that the instance of 2-PRAT is a Yes-instance. In other words, we need to show the existence of two preferences that provide a Pareto rationalization. We do this by constructing two acyclic relations $\succ_1$ and $\succ_2$ that Pareto rationalize every choice set. These relations can always be extended to complete, transitive and asymmetric relations (by using, for example, a finite analogue of Szpilrajn (1930)'s Lemma). Table 2 provides a first set of comparisons conditional on the values of u_i $(i = 1, \ldots, n)$.

Table 2: First set of comparisons

$u_i =$ TRUE		$u_i =$ FALSE		unconditional	
$y_i \succ_1 \bar{y}_i$	$\bar{y}_i \succ_2 y_i$	$\bar{y}_i \succ_1 y_i$	$y_i \succ_2 \bar{y}_i$	$a \succ_1 b$	$b \succ_2 a$
$\bar{w}_i \succ_1 w_i$	$w_i \succ_2 \bar{w}_i$	$w_i \succ_1 \bar{w}_i$	$\bar{w}_i \succ_2 w_i$	$d_\ell \succ_1 b$	$d_\ell \succ_2 b$
				$a \succ_1 d_\ell$	$d_\ell \succ_2 a$
$y_i \succ_1 a$	$y_i \succ_2 a$	$a \succ_1 y_i$	$y_i \succ_2 a$	$y_i \succ_1 \bar{w}_i$	$y_i \succ_2 \bar{w}_i$
$a \succ_1 w_i$	$w_i \succ_2 a$	$w_i \succ_1 a$	$w_i \succ_2 a$	$w_i \succ_1 \bar{y}_i$	$w_i \succ_2 \bar{y}_i$
				$d_\ell \succ_1 \bar{z}_{k,\ell}$	$d_\ell \succ_2 \bar{z}_{k,\ell}$

Table 3 provides a second set of comparisons conditional on the values of two variables u_i and u_j $(1 \leq i < j \leq n)$. It is an easy but cumbersome exercise to verify that these relations rationalize the choice function. Let us now demonstrate that they are acyclic. We focus on the relation $\succ_1$. The proof that $\succ_2$ is also acyclic is very similar and left to the reader.

Table 3: Second set of comparisons for $i < j$

		$u_i = \text{TRUE}$		$u_i = \text{FALSE}$	
		y_i	w_i	y_i	w_i
$u_j = \text{TRUE}$	y_j	$y_i \succ_1 y_j$ $y_j \succ_2 y_i$	$y_j \succ_1 w_i$ $w_i \succ_2 y_j$	$y_j \succ_1 y_i$ $y_i \succ_2 y_j$	$w_i \succ_1 y_j$ $y_j \succ_2 w_i$
	w_j	$y_i \succ_1 w_j$ $w_j \succ_2 y_i$	$w_i \succ_1 w_j$ $w_j \succ_2 w_i$	$y_i \succ_1 w_j$ $w_j \succ_2 y_i$	$w_i \succ_1 w_j$ $w_j \succ_2 w_i$
$u_j = \text{FALSE}$	y_j	$y_i \succ_1 y_j$ $y_j \succ_2 y_i$	$w_i \succ_1 y_j$ $y_j \succ_2 w_i$	$y_i \succ_1 y_j$ $y_j \succ_2 y_i$	$w_i \succ_1 y_j$ $y_j \succ_2 w_i$
	w_j	$y_i \succ_1 w_j$ $w_j \succ_2 y_i$	$w_j \succ_1 w_i$ $w_i \succ_2 w_j$	$w_j \succ_1 y_i$ $y_i \succ_2 w_j$	$w_i \succ_1 w_j$ $w_j \succ_2 w_i$

For a contradiction, assume that $\succ_1$ contains a cycle. We proceed by sequentially excluding elements from this cycle.

Fact 1. *For all $i = 1, \ldots, n$,*

- *if u_i = FALSE, then $\bar{w}_i$ is not in the cycle of $\succ_1$.*
- *if u_i = TRUE, then $\bar{y}_i$ is not in the cycle of $\succ_1$.*

This follows from the observation that there is no alternative that is dominated by $\bar{w}_i$ (for u_i = FALSE) or $\bar{y}_i$ (for u_i = TRUE).

Fact 2. *For all $i = 1, \ldots, n$,*

- *if u_i = FALSE, then y_i is not in the cycle of $\succ_1$.*
- *if u_i = TRUE, then w_i is not in the cycle of $\succ_1$.*

The proof is by (reverse) induction on i, i.e. starting from $i = n$. Assume that y_n (with u_n = FALSE) or w_n (with u_n = TRUE) is in the cycle. The next element in the cycle is then given by $\bar{w}_n$ (if u_n = FALSE) or $\bar{y}_n$ (if u_n = TRUE). However, this contradicts the previous fact.

Now, assume that the fact holds for all i with $i \geq t$. Then let us look at the case where $i = t - 1$. If u_i = FALSE then the alternative following y_i is either $\bar{w}_i$ (according to table 2), w_j with $j > i$ and u_j = TRUE, or y_j with $j > i$ and u_j = FALSE (according to table 3). All these cases either contradict the previous fact or the induction hypothesis. This shows that y_i is not in the cycle if u_i = FALSE.

On the other hand, if u_i = TRUE then the alternative following w_i is either $\bar{y}_i$ (according to table 2), w_j (with $j > i$ and u_j = TRUE), or y_j (with $j > i$ and u_i = FALSE) (according to table 3). These cases either either contradict the previous fact or the induction hypothesis. This shows that w_i is not in the cycle if u_i = TRUE.

The proof is completed by induction.

Fact 3. *For all $i = 1, \dots, n$,*

- *b is not in the cycle of $\succ_1$.*
- *If u_i = TRUE, then $\bar{w}_i$ is not in the cycle of $\succ_1$.*
- *If u_i = FALSE, then $\bar{y}_i$ is not in the cycle of $\succ_1$.*
- *d_ℓ ($\ell = 1, \dots, m$) is not in the cycle of $\succ_1$*
- *a is not in the cycle of $\succ_1$.*

We will proof each of the items separately:

- The alternative b dominates no other element according to $\succ_1$, hence it must be a terminal node. As such, it cannot be part of a cycle.
- If u_i = TRUE and the cycle contains $\bar{w}_i$, then the next element in the cycle must be w_i. However, this contradicts Fact 2.
- If u_i = FALSE and the cycle contains $\bar{y}_i$ then the next element in the cycle must be y_i which contradicts Fact 2.
- If the cycle contains d_ℓ then the following alternative in the cycle must be either $\bar{w}_i$ or $\bar{y}_i$ (see Table 2). However, this either contradicts the previous finding or Fact 1.
- If the cycle contains a then the following element in the cycle is either b, d_ℓ ($\ell = 1, \dots, m$) , w_i (with u_i = TRUE) or y_i (with u_i = FALSE). All these cases contradict previous findings.

Fact 4. *For all $i = 1, \dots, n$,*

- *if u_i = TRUE, then y_i is not in a cycle of $\succ_1$.*
- *if u_i = FALSE, then w_i is not in a cycle of $\succ_1$.*

The proof is again by reverse induction on i starting with $i = n$. If y_n with u_n = TRUE or w_n with u_n = FALSE is in a cycle of $\succ_1$ then the next element in the cycle is either $\bar{y}_n$, $\bar{w}_n$, a or w_j (with u_j = TRUE), neither of which can be part of the cycle given previous facts.

For the induction hypothesis, assume that the fact holds for all $i \geq t$ and take the case where $i = t - 1$. Then if u_i = TRUE and y_i is in the cycle then the next element in the cycle is either $\bar{y}_i$, a, $\bar{w}_i$ (according to table 2), y_j (with u_j = TRUE and $j > i$), w_j (with u_j = FALSE and $j > i$), w_j (with u_j = TRUE), or y_j (with u_j = FALSE) (according to table 3). All these cases are either excluded by previous facts or by the induction hypothesis.

Next assume that u_i = FALSE and that w_i is in the cycle. Then the next element in the cycle is either $\bar{w}_i$, a, $\bar{y}_i$ (according to table 2), w_j (with u_j = FALSE and $j > i$), y_j (with u_j = TRUE and $j > i$), w_j (with u_j = TRUE), or y_j (with u_j = FALSE) (according to table 3). Again, all these cases are either excluded by previous facts or by the induction hypothesis.

The proof is completed by induction.

We have shown that no element can be part of the cycle of $\succ_1$. From this, it follows that $\succ_1$ is acyclic, which concludes the proof.

4.2 2-WPRAT is NP-complete

First of all, notice that 2-WPRAT is in NP. The proof uses a reduction from the NP-complete problem Monotone Not All Equal 3-SAT (M-NAE-3SAT). [9]

An instance of M-NAE-3-SAT consists of a set of binary variables $u_1, \ldots, u_n$ and a finite list of clauses $C_1, \ldots, C_m$. Each clause contains three variables.

Monotone Not All Equal 3-SAT (M-NAE-3-SAT): *Does there exist an assignment to the variables (either TRUE or FALSE) such that each clause contains at least one TRUE variable and at least one FALSE variable?*

Consider an instance of Monotone Not All Equal 3-SAT, i.e. a set of variables $u_1, \ldots, u_n$ and a set of clauses $C_1, \ldots, C_m$. We first construct the instance $(X, \mathcal{D}, c, \rhd)$ of 2-WPRAT. We begin with the definition of the set X.

- For each variable u_i we construct two alternatives a_i and $\bar{a}_i$.
- For each clause C_ℓ, we construct 12 alternatives: $z_{1,\ell}, z_{2,\ell}, z_{3,\ell}$, $t_{1,\ell}, t_{2,\ell}, t_{3,\ell}$, $v_{1,\ell}, v_{2,\ell}, v_{3,\ell}$ and $w_{1,\ell}, w_{2,\ell}, w_{3,\ell}$.
- We construct an additional alternative d.

The domain $\mathcal{D}$ and the choice function is given in Table 4. This construction can be performed in polynomial time.

Table 4: Construction of choice sets and choice function

choice sets	choices	range	
$\{a_i, \bar{a}_i\}$	$\{a_i\}$	$i = 1, \ldots, n$	(1)
$\{a_i, d, \bar{a}_i\}$	$\{\bar{a}_i\}$	$i = 1, \ldots, n$	(2)
$\{z_{k,\ell}, t_{k,\ell}\}$	$\{z_{k,\ell}\}$	$\ell = 1, \ldots, m; k = 1, 2, 3$	(3)
$\{z_{k,\ell}, d, t_{k,\ell}\}$	$\{t_{k,\ell}\}$	$\ell = 1, \ldots, m; k = 1, 2, 3$	(4)

We define two functions $f(k, \ell)$ and $\bar{f}(k, \ell)$ ($k = 1, 2, 3; \ell = 1, \ldots, m$). If the kth variable in the ℓth clause is equal to the variable u_i then we set $f(k, \ell) = a_i$ and $\bar{f}(k, \ell) = \bar{a}_i$. Further, we denote by $k \oplus 1$ the number $(k + 1) \mod 3$.

Next, we construct the dominance relation $\rhd$ as in Table 5. The structure of the relation $\rhd$ is illustrated in Figure 4.

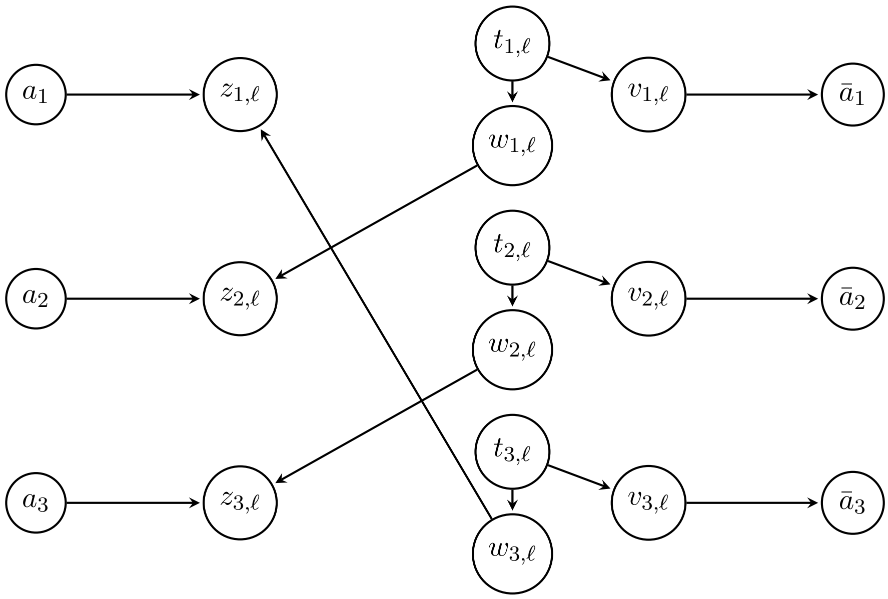
Let us first show that a solution to the weakly Pareto rationalization problem leads to a solution of M-NAE-3SAT. First of all, we see that the two individuals must differ on their preference of a_i over $\bar{a}_i$ (from (1) and (2)). Now, if $a_i \succ_1 \bar{a}_i$ (and $\bar{a}_i \succ_2 a_i$) we set $u_i =$ TRUE and if $a_i \succ_2 \bar{a}_i$ (and $\bar{a}_i \succ_1 a_i$) we set $u_i =$ FALSE. Let us show that this provides a solution to M-NAE-3SAT. First we demonstrate that if $f(k, \ell) = a_i$ and $u_i =$ TRUE then $z_{k,\ell} \succ_1 t_{k,\ell}$. Otherwise we would have that $\bar{a}_i \succ_2 a_i \succ_2 z_{k,\ell} \succ_2 t_{k,\ell} \succ_2 v_{k,\ell} \succ_2 \bar{a}_i$ which is a contradiction. Similarly, we can show that if $f(k, \ell) = a_i$ and $u_i =$ FALSE then $z_{k,\ell} \succ_2 t_{k,\ell}$.

[9] Monotone-not-all-equal-3SAT can be obtained from the NP-complete problem Not-all-equal-3SAT (Garey and Johnson, 1979) by replacing all literals of the form $\bar{u}_i$ by a variable r_i and adding an additional clause of the form $\{r_i, u_i, u_i\}$.

Table 5: Construction of relation $\rhd$

comparisons	range
$t_{k,\ell} \rhd v_{k,\ell}$	$\ell = 1, \ldots, m; k = 1, 2, 3$
$t_{k,\ell} \rhd w_{k,\ell}$	$\ell = 1, \ldots, m; k = 1, 2, 3$
$w_{k,\ell} \rhd z_{k\oplus 1,\ell}$	$\ell = 1, \ldots, m; k = 1, 2, 3$
$v_{k,\ell} \rhd \bar{f}(k,\ell)$	$\ell = 1, \ldots, m; k = 1, 2, 3$
$f(k,\ell) \rhd z_{k,\ell}$	$\ell = 1, \ldots, m; k = 1, 2, 3$

Figure 4: Dominace relation $\rhd$ for clause $\{u_1, u_2, u_3\}$



Now assume, towards a contradiction, that M-NAE-3SAT is not satisfiable. Then there must be a clause C_ℓ where each variable is either TRUE or FALSE. If all variables are FALSE then $z_{k,\ell} \succ_1 t_{k,\ell}$ for all $k = 1, 2, 3$. This produces the cycle $z_{1,\ell} \succ_1 t_{1,\ell} \succ_1 w_{1,\ell} \succ_1 z_{2,\ell} \succ_1 t_{2,\ell} \succ_1 w_{2,\ell} \succ_1 z_{3,\ell} \succ_1 t_{3,\ell} \succ_1 w_{3,\ell} \succ_1 z_{1,\ell}$. The case where all variables are equal to TRUE gives an identical cycle for the relation $\succ_2$. This shows that M-NAE-3SAT has a solution.

To see the converse, assume that M-NAE-3SAT has a solution. We need to show that the choice function is weakly Pareto rationalizable. If $u_i =$ TRUE we set $a_i \succ_1 \bar{a}_i$ and $\bar{a}_i \succ_2 a_i$. Otherwise, if $u_i =$ FALSE we set $a_i \succ_2 \bar{a}_i$ and $\bar{a}_i \succ_1 a_i$. If $u_i =$ TRUE and $f(k,\ell) = a_i$ then we set $z_{k,\ell} \succ_1 t_{k,\ell}$ and $t_{k,\ell} \succ_2 z_{k,\ell}$. Otherwise, if $f(k,\ell) = a_i$ and $u_i =$ FALSE we set $z_{k,\ell} \succ_2 t_{k,\ell}$ and $t_{k,\ell} \succ_1 z_{k,\ell}$. Further, we include into $\succ_1$ and $\succ_2$ all the comparisons of $\rhd$. Finally, let d be bottom ranked for both the relations $\succ_1$ and $\succ_2$.

Observe that these preferences rationalize the choice function. We still need to show that they can be extended to complete and transitive relations. For this it suffices to show that $\succ_1$ and $\succ_2$ are acyclic. Here, we focus on the relation $\succ_1$. The proof that $\succ_2$ is acyclic is very similar.

Assume, on the contrary that $\succ_1$ contains a cycle. We proceed by sequentially excluding all elements from this cycle.

Fact 5. *$\bar{a}_i$ is not in the cycle.*

If it is, then the next element in the cycle can only be a_i. This implies that u_i = FALSE. The third element in the cycle is an alternative $z_{k,\ell}$ with $f(k,\ell) = a_i$. Finally, the fourth element in the cycle then equals $t_{k,\ell}$. This implies that u_i = TRUE, a contradiction.

Fact 6. *$v_{k,\ell}$ ($k = 1, 2, 3; \ell = 1, \dots, m$) is not in the cycle.*

If it is then the next element in the cycle must be $\bar{a}_i$ (with $f(k,\ell) = a_i$). This contradicts the previous fact.

Fact 7. *$z_{k,\ell}$ ($k = 1, 2, 3; \ell = 1, \dots, m$) is not in the cycle.*

If it is then from the previous facts we must have that this cycle coincides with $z_{1,\ell} \succ_1 t_{1,\ell} \succ_1 w_{1,\ell} \succ_1 z_{2,\ell} \succ_1 t_{2,\ell} \succ_1 w_{2,\ell} \succ_1 z_{3,\ell} \succ_1 t_{3,\ell} \succ_1 w_{3,\ell} \succ_1 z_{1,\ell}$. This implies that all literals in the clause C_ℓ are true, which is a contradiction.

Fact 8. *a_i is not in the cycle.*

If it is then the next element in the cycle is either $\bar{a}_i$ or $z_{k,l}$ (with $f(k,l) = a_i$), both of which are ruled out by previous facts.

Observe that we can also exclude all alternatives $w_{k,\ell}$ (because the next element is $z_{k\oplus 1,\ell}$) and $t_{k,\ell}$ (because the next element is either $w_{k,\ell}$, $v_{k,\ell}$ or $z_{k,\ell}$) from the cycle. As such, we have shown that the cycle in $\succ_1$ contains no elements, hence, $\succ_1$ is acyclic. This concludes the proof.

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