The computational complexity of rationalizing boundedly rational choice behavior*

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Abstract

We determine the computational complexity of various choice models that use multiple rationales to explain observed choice behavior. First, we demonstrate that the notion of rationalizability by K rationales, introduced by Kalai, Rubinstein, and Spiegler (2002), is **NP**-complete for K greater or equal to two. Then, we show that the question of sequential rationalizability by K rationales, introduced by Manzini and Mariotti (2007), is **NP**-complete for K greater or equal to three. Finally, we focus on the computational complexity of two models that refine this model of sequential choice behavior. We establish that the model of choice by game trees, from Xu and Zhou (2007), is **NP**-complete while the status-quo bias model, from Masatlioglu and Ok (2005), can be verified in polynomial time.

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1 Introduction

Motivation Neoclassical rational choice theory departs from the assumption that a decision maker selects among the available alternatives the ones that are highest ranked according to her preference relation. In general, this preference relation is assumed to be complete, transitive, stable over time and stable across different choice environments. The neoclassical model is not only convenient from a theoretical perspective but it also has strong and easily verifiable testable implications; e.g. Richter (1966)'s congruence condition. Unfortunately, these testable implications are frequently rejected by empirical research. A first kind of rejection bears on the property of transitivity. Cyclical choice behavior has been observed by, for example, Tversky (1969); Loomes, Starmer, and Sugden (1991); Loomes and Taylor (1992) and Roelofsma and Read (2000). A second kind of refutation pertains to the concept of contraction consistency (or independence of irrelevant alternatives) which requires that the chosen element from a set is also selected from every subset that contains it (e.g. Seidl and Traub (1996) and Kroll and Vogt (2008)). A violation of contraction consistency results in, so called, menu-dependent or context dependent choice behavior.[1]

As a resolution to these empirical findings several alternative boundedly rational choice models have been put forward. These models explain choice behavior by rendering a more realistic and more explicit description of how a decision maker actually makes choices. An interesting subcollection of these models explains choice behavior by utilizing multiple rationales (selves). In this research, we concentrate on several popular models from this collection. Our two benchmark models are the model of choice by multiple rationales, introduced by Kalai, Rubinstein, and Spiegler (2002), and the model of sequential choice by multiple rationales from Manzini and Mariotti (2007). We infer the computational complexity for verifying consistency of observed choice behavior with these models, given the number of rationales.

As shown by Apesteguia and Ballester (2009), the sequential choice model contains several other important choice models as specific cases. In particular, they demonstrated that both the model of choice by game trees, from Xu and Zhou (2007), and the model of choice with status quo bias, from Masatlioglu and Ok (2005) refine the sequential choice model. Inspired by this result, we also determine the computational complexity of these two refinements.

[1]Interestingly, under the full domain assumption, contraction consistency imposes acyclic choice behavior (see, for example, Suzumura (1983)).

Our results fit into the recent literature that utilize insights from computational complexity theory in the study of interesting economic subjects.[2] Even so, as noted by Apesteguia and Ballester (2010), there is still relatively little work on the relationship between computational complexity and (bounded) rationality. We contribute to this literature by characterizing the computational complexity of several well-known choice models. Although this paper is purely theoretical, our results have also important empirical consequences. The fact that the verification of a certain choice model is NP-complete demonstrates that empirical refutation or acceptance of these models might be extremely difficult.[3] The remaining part of this section motivates the various choice models and summarizes our main contribution.

Choice by multiple rationales The first boundedly rational choice framework that we focus on is the model of choice by multiple rationales, introduced by Kalai, Rubinstein, and Spiegler (2002). The model departs from the idea that choices are context dependent. More specific, the model of choice by multiple rationales set forth a collection of rationales (preference relations) which is said to rationalize the observed choice behavior if each choice maximizes at least one rationale from the collection. In this way, the model permits for context dependent choice behavior. Kalai, Rubinstein, and Spiegler (2002) provide several results relating to the minimal number of rationales that are needed in order to rationalize a given choice function. Recently, Apesteguia and Ballester (2010) prove that computing this minimal number of rationales is a difficult problem (i.e. it is **NP**-complete). In Section 2 we strengthen this result by establishing that **NP**-completeness also arises if we know the number of rationales and if this is larger or equal than two.

Sequential choice with multiple rationales: The second boundedly rational choice model is the model of sequential choice by multiple rationales from Manzini and Mariotti (2007). This model assumes that choices are

[2]See, among many others, Gilboa and Zemel (1989); Chu and Halpern (2001); Cechlarova and Hajdukova (2002); Fang, Zhu, Cai, and Deng (2002); Woeginger (2003); Baron, Durieu, Haller, and Solal (2004); Baron, Durieu, Haller, Savani, and Solal (2008); Brandt and Fisher (2008); Conitzer and Sandholm (2008); Kalyanaraman and Umans (2008); Procaccia and Rosenschein (2008); Cherchye, Demuynck, and De Rock (2009); Galambos (2009); Hudry (2009); Brandt, Fisher, Harrenstein, and Mair (2010); Talla Nobibon, Cherchye, De Rock, Sabbe, and Spieksma (2010); Deb (2010) and Apesteguia and Ballester (2010)

[3]We refer to the working paper version of this paper (Demuynck, 2010) for a more thorough discussion on this topic.

made by solving a sequence of intermediate smaller choices. The model formalizes the intuition that the decision maker pursues a step-wise procedure that gradually constricts the set of viable alternatives. More specifically, the decision maker is endowed with a fixed number of rationales (preferences) which are sequentially applied to remove dominated elements from the remaining set of alternatives. Manzini and Mariotti (2007) characterize the choice functions that are sequentially rationalizable by two and three rationales. Several other researches expand the theory of sequential choice behavior. Houy (2007) characterizes the lists of rationales that lead to nonempty choice functions or choice correspondences. Apesteguia and Ballester (2009) examine the choice functions that are sequentially rationalizable by an (arbitrarily long) list of acyclic relations. Garcia-Sanz and Alcantud (2010) analyze the set of choice correspondences (mutli-valued choice functions) that can be rationalized by the sequential application of two rationales.

Recently, Apesteguia and Ballester (2009) proved that the model of sequential choice contains two other interesting models as special cases. The first is the model of choice by game trees, introduced by Xu and Zhou (2007). In this model, different selves (or different decision makers) compete in a sequential game of perfect information for which the resulting choices conform with the sub-game perfect Nash equilibrium of this game. The model of Xu and Zhou (2007) differs from other game-theory based choice models[4] in the sense that it abstains from assuming any knowledge pertaining to the number of selves (decision makers) or the underlying rules of the game (i.e. the form of the game tree). In other words, the underlying decision making process is unknown to the observer. The second special case is the model of choice with a status-quo bias, characterized by Masatlioglu and Ok (2005). The status-quo bias model formalizes the idea a decision maker typically values an alternative more highly when it is the status-quo.[5] More precisely, the model presumes that if the decision maker is confronted with a choice set without a status-quo, then she simply selects the best alternative according to her preference ordering. On the other hand, if there is a status-quo, then this status-quo is overruled only if there is an alternative that performs better than the status-quo on several criteria. However, if there is no such alternative, then the status-quo is maintained.

Given the relationship between the three sequential choice models one

[4]See, for example, Sprumont (2000), Ray and Zhou (2001), Lee (2009), Galambos (2009) and Demuynck and Lauwers (2009).

[5]This effect was first discovered by Samuelson and Zeckhauser (1988). The presence of a status-quo bias has been repeatedly supported by experimental studies. See, for example, the paper of Kahneman, Knetch, and Thaler (1991) for an overview.

could ask the question whether they differ with respect to their computational complexity. We demonstrate that this is indeed the case. In Section 3 we demonstrate that the issue of rationalizability by sequential choice is **NP**-complete as soon as the number of rationales are greater or equal to three. Further, we show that the issue of rationalizing choice functions by game trees is also **NP**-complete but that the problem of rationalizing choice functions by the status-quo bias model is decidable in polynomial time.

All proofs can be found in Section 4.

2 Choice by multiple rationales

In this section we concentrate on the model of choice by multiple rationales. In doing so, we also introduce the necessary notation and concepts for the other sections. Throughout this section, we will try to compare our findings to the complementary paper of Apesteguia and Ballester (2010).

Preliminaries Take a finite set of alternatives X with cardinality $n \in \mathbb{N}$. We denote by $\mathcal{U}$ the collection of all nonempty subsets of X. A choice function c is a function from a collection of sets $\mathcal{D} \subseteq \mathcal{U}$ to $\mathcal{U}$ such that for all sets $A \in \mathcal{D}$, $c(A) \subseteq A$. We say that $\mathcal{D}$ is the **domain** of the choice problem and we call the elements in $\mathcal{D}$ the **choice sets** of the choice problem.

The domain $\mathcal{D}$ is **binary** if it admit all 2 element subsets of X: for all $x, y \in X$, $\{x, y\} \in \mathcal{D}$. The choice function c is **single valued** if for all choice sets $A \in \mathcal{D}$, $|c(A)| = 1$.

Denote by $\succeq$ a binary relation or **rationale** on X, i.e. $\succeq \subseteq X \times X$. We denote its asymmetric part by $\succ$.[6] The binary relation $\succeq$ is **transitive** if for all x, y and $z \in X$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$. It is **asymmetric** if for all distinct x and $y \in X$, it is not the case that $x \succeq y$ and $y \succeq x$. Finally, we say that $\succeq$ is **complete** (or total) if for all x and $y \in X$ we have that $x \succeq y$ or $y \succeq x$.

For a choice set $A \in \mathcal{D}$ and a rationale $\succeq$ on X, we denote by $\mathcal{M}(A, \succeq)$ the set of **maximal elements** in A according to $\succeq$. Formally, $x \in \mathcal{M}(A, \succeq)$ if for all $y \in A$ it is not the case that $y \succ x$.

Computational complexity To be compact, we will only provide a quick introduction to the concepts of computational complexity, alas at the cost of accuracy. For a detailed introduction into the theory of computational

[6] Formally, $x \succ y$ if $x \succeq y$ and $\neg(y \succeq x)$.

complexity and **NP**-completeness in particular, we refer to the seminal work of Garey and Johnson (1979).

The theory of computational complexity examines how much resources (time and memory) are needed to solve a given **decision problem**. Every decision problem is composed of a collection of **instances**, which are the inputs for the problem, and a 'yes'/'no' question, which inquires whether the instances satisfy a certain property. In other words, a decision problem maps to each of its instances either a 'yes' or a 'no' as an output depending on whether the instance satisfies the property.

The class **P** of decision problems comprises all problems which are easy to solve, i.e. they can be solved by an algorithm which computes the solution in a polynomial number of steps. A second class of problems, denoted by **NP**, holds all problems that might be difficult to solve (i.e. it might take exponential time) but are easy to verify (i.e. a solution can be verified in polynomial time).

A decision problem that is at least as hard (as difficult) as any other problem from **NP** is said to be **NP**-hard. Finally, a decision problem is **NP**-complete if it is both **NP**-hard and in the class **NP**. In other words, an **NP**-complete problem is among the most difficult problems in the class **NP**.

A profound open question in computational complexity (and in all of mathematics) is whether the class of decision problems in **P** is equal to the class of decision problems in **NP**. By definition, it holds that **P**$\subseteq$ **NP**. Even so, it is not known if all problems in **NP** can be solved in polynomial time. The general accepted belief is that **P**$\neq$ **NP**. The class **NP** contains many of the computable real world problems, hence, **NP**-complete problems are considered to be computationally intractable (especially for large instances). As such, all known solution methods applicable to **NP**-complete problems suffer from exponential worst time complexity.

Rationalization by multiple rationales To bring in the concept of rationalization by multiple rationales we depart from a list of complete and transitive rationales $\{\succeq_k\}_{k\leq K}$. A choice function c is rationalizable by the the list $\{\succeq_k\}_{k\leq K}$ if for every choice set A in the domain $\mathcal{D}$ we can find at least one rationale $\succeq_k$ in this list such that the choice $c(A)$ matches the set of $\succeq_k$-maximal elements in A.

Definition 1 (K-Rationalizable by Multiple Rationales)**.** *A list of K transitive and complete relations $\{\succeq_k)_{k\leq K}$, K-rationalizes the choice function c on the domain $\mathcal{D}$ if for all choice sets $A \in \mathcal{D}$ there exist at least one $k \leq K$*

such that:

$$c(A) = \mathcal{M}(A, \succeq_k).$$

When the number of rationales, K, is equal to one, then K-rationalizability by multiple rationales boils down to the problem of rationalizing a choice function by a single transitive, and complete preference relation. It is well-known that this problem can be solved in polynomial time (see for example Apesteguia and Ballester (2010, Observation 1)). The interesting question is whether this result still binds if we concentrate on settings with more than a single rationale. Apesteguia and Ballester (2010) prove that computing the minimum number of rationales, K, such that a choice function is rationalizable by K rationales is an **NP**-complete problem.

In order to present this result, let us first define the relevant decision problem. We specify an instance by a triplet $(X, \mathcal{D}, c)$ composed of a finite set of alternatives, X, a domain $\mathcal{D}$ of nonempty subsets of X and a choice function, c, on $\mathcal{D}$. Apesteguia and Ballester (2010) look at the following decision problem.

Rationalization (RAT): *Given an instance $(X, \mathcal{D}, c)$ and a number K, can we find a list of $k \leq K$ transitive and complete rationales, $\{\succeq_k\}_{k \leq K}$ such that this list K-rationalizes the choice function c?*

Their result is summarized in the next theorem.

Theorem 1 (Apesteguia and Ballester (2010, theorem 2 and corollary 3))**.** *The decision problem RAT is* **NP***-complete. This* **NP***-completeness result also holds in the subclass of single valued choice correspondences.*

Let us now regard the decision problem when we hold the number of rationales fixed.

K-rationalization (K-RAT): *Given an instance $(X, \mathcal{D}, c)$, can we find a list of K transitive and complete rationales, $\{\succeq_k\}_{k \leq K}$ such that this list K-rationalizes the choice function c?*

Notice that, for the decision problem K-RAT, the number of rationales K is a parameter in the decision problem. Put differently, there are an infinite number of decision problems, one for each value of $K \in \mathbb{N} - \{0\}$.[7] Of course, **NP**-completeness of the decision problem RAT does not inevitably imply that K-RAT is also **NP**-complete for all values of K. (This is the case, for

[7] In fact, if the set of alternatives has size n, there are only n-1 relevant decision problems that are really relevant. As shown by Kalai, Rubinstein, and Spiegler (2002), no choice function needs more than such a number of rationales.

example, when K = 1). However, as the following theorem shows, as soon as K is greater or equal to two, **NP**-completeness prevails.

Theorem 2. *The decision problem K-RAT is* **NP***-complete for all* $K \geq 2$*. This* **NP***-completeness result also holds in the subclass of single valued choice correspondences.*

Apesteguia and Ballester (2010, Theorem 4) carry on by considering the subclass of instances that satisfy the universal domain condition, i.e. the instances for which $\mathcal{D} = \mathcal{U}$, and for which the choice functions are single valued. They show that in this case, the problem RAT, is quasi-polynomially bounded, thus, no longer **NP**-complete.[8] Following the same reasoning as in their proof we can prove that if we restrict the instances to the subclass that satisfies the universal domain condition, then K-RAT is also quasi-polynomially bounded. Interestingly, however, this result no longer requires the choice function to be single-valued.[9]

3 Sequential choice

Our second choice model is the model of sequential choice by multiple rationales. This framework departs from a list of asymmetric, not necessary complete, rationales $\{\succ_k\}_{k \leq K}$ and enforces each of these rationales sequentially, eliminating in each round the alternatives that are dominated.

Definition 2 (K-Sequential Rationalizability)**.** *A choice function c is K-sequentially rationalizable whenever there exists an ordered list of K asymmetric rationales $\{\succ_k\}_{k \leq K}$, such that, defining recursively,*

$$M_0(A) = A,$$
$$M_k(A) = \mathcal{M}(M_{k-1}(A), \succ_k), \; k = 1, \ldots, K,$$

we have,

$$c(A) = M_K(A),$$

for all $A \in \mathcal{D}$.

[8] In particular, they show that RAT is of the order $O(n^{\log n \log\log n})$

[9] The intuition behind this difference lies in the fact that the problem K-RAT automatically sets an upper bound on the number of rationales that should rationalize the choice function (i.e. K). On the other hand, for the problem RAT, this upper bound is not always polynomially bounded in the number of alternatives.

The intuition is the following. First, the decision maker considers a sequence of rationales $\succ_1, \ldots, \succ_K$. Then for all choice sets A the decision maker computes the undominated elements of A according to her first rationale $\succ_1$, $M_1(A) = \mathcal{M}(A, \succ_1)$. Next, she looks for the maximal elements of $M_1(A)$ according to her second rationale $\succ_2$. $M_2(A) = \mathcal{M}(M_1(A), \succ_2)$. In the third step, she retrieves the undominated elements of $M_2(A)$ according to the rationale $\succ_3$, $M_3(A) = \mathcal{M}(M_2(A), \succ_3)$. This routine is applied sequentially until the last set $M_K(A)$ is computed.

In order to keep focussed, we restrict ourselves to rationalizations for which the resulting choice function is single valued on all binary choice sets,[10] i.e. we only look for rationales $\{\succ_k\}_{k\leq K}$ such that for all binary choice sets $\{x, y\} \subseteq X$: $|M_K(\{x, y\})| = 1$. This implies, among other things, that the observed choice function must be single valued.[11] Now we are ready to introduce the relevant decision problem.

K-sequential rationalization (K-SR): *Given an instance (X, Σ, c), does there exist a list of K asymmetric relations $\{\succ_k\}_{k\leq K}$ that provides a K-sequential rationalization of c?*

Before we present the computational complexity result for this decision problem, we consider two other choice models which refine the sequential choice model. The first model is the model of rationalization by game trees introduced by Xu and Zhou (2007). We denote by $(G, \{\succ_k\}_{k\leq K})$ an extensive form game with perfect information. It is composed of a game tree G, that has all the alternatives in X as terminal nodes with the additional restriction that each alternative in X occurs once and only once as a terminal node, and a list of preferences relations $\{\succ_k\}_{k\leq K}$ for the different players in the game. It is assumed that these preferences are asymmetric and complete rationales on X. Let $G|A$ be the reduced tree that retains all the branches of G leading to terminal nodes in A, and let $SPNE(G|A, \{\succ_k\}_{k\leq K})$ be the unique sub-game perfect Nash equilibrium outcome of this reduced game.

Definition 3 (Rationalizability by game trees)**.** *A game $(G, \{\succ_k\}_{k\leq K})$ rationalizes the choice function c on $\mathcal{D}$ if for all $A \in \mathcal{D}$,*

$$c(A) = SPNE(G|A, \{\succ_k\}_{k\leq K}).$$

[10] This is also the setting in the paper of Manzini and Mariotti (2007).

[11] If we relax the model to include choice correspondences, then the problem of rationalization by sequential choice turns out to be **NP**-complete for all possible values of K. For a proof of this result, we refer to the to the working paper version of this article (Demuynck, 2010).

The next decision problem corresponds to the notion of rationalizability by game trees.

Rationalization by game trees (RGT): *Given an instance* $(X, \mathcal{D}, c)$*, does there exist an extensive form game* $(G, \{\succ_k\}_{k \leq K})$ *that rationalizes the choice function* c*?*

The second refinement is the choice model with status-quo bias, introduced by Masatlioglu and Ok (2005). In order to define this concept, it is necessary to broaden the concept of a choice set. First of all, we introduce the symbol $\diamond$ to denote an object that does not belong to the set X. Next, we define a choice problem by a pair (A, x) where A is a nonempty subset of X and either $x \in A$ or $x = \diamond$. If $x \in A \in \mathcal{U}$, then the pair (A, x) refers to the situation of choosing an element from A where the status-quo is given by the alternative $x \in A$. On the other hand, if $x = \diamond$, then (A, x) relates to the problem of choosing an element out of A without a status-quo. We define $\mathcal{D}_{sq}$ as the new domain of choice problems, i.e. $\mathcal{D}_{s,q} \subseteq \{(A, x) | A \in \mathcal{U} \text{ and } x \in A \text{ or } x = \diamond\}$.

Definition 4. *A choice function* c *is status-quo biased if and only if there exist a number* $q \in \mathbb{N}$*, an injective function* $u : X \to \mathbb{R}^q$ *and a strictly increasing map* $f : u(X) \to \mathbb{R}$ *such that for all* $(A, x) \in \mathcal{D}_{sq}$*, if* $x = \diamond$*, then:*

$$c(A, \diamond) = \arg\max\{f(u(y)) | y \in A\}$$

and if $x \neq \diamond$*, then*

$$c(A, x) = \begin{cases} x & \text{if for all } y \in A: \ u(y) \not> u(x) \\ \arg\max\{f(u(y)) | u(y) > u(x)\} & \text{else.} \end{cases}$$

In the absence of a status-quo, i.e. when $x = \diamond$, the decision maker simply maximizes the utility function $f(u(.))$ over the set A. Alternatively, when a status-quo is present, she maximizes the same utility function, but now only among the elements that dominate the status-quo in all attributes of the q-dimensional function u. If there are no such dominating alternatives, then she retains the status-quo. We can now define the relevant decision problem for the status-quo bias model.

Status-quo bias (SQB): *Given an instance* $(X, \mathcal{D}_{sq}, c)$*, is the choice function,* c*, status-quo biased?*

Apesteguia and Ballester (2009) showed that all choice functions that are are rationalizable by game trees are also sequential rationalizable; that all choice functions that are status quo biased are also rationalizable by game trees; and that all these inclusions are strict.[12] Given this knowledge, it is of particular interest to see how the computational complexity varies among these different choice models. The results are given in the following theorem.

Theorem 3.

- *If $K \leq 2$, then K-SR is in the class* **P**.
- *For all $K \geq 3$, K-SR is* **NP***-complete. This result also holds in the subclass of problems for which the domain is binary and if the rationalizations are required to be transitive or acyclic.*
- *RGT is* **NP***-complete. This result also holds if the game tree is restricted to have no more than 3 stages and the number of rationales (agents) is smaller or equal to two.*
- *SQB is in the class* **P**.

This theorem clearly shows where the **NP**-completeness results stops. First of all, if we restrict the number of rationales to be less than or equal to two, the issue of rationalization by sequential choice becomes easy to verify. Next, from the moment where the choice model reduces to the model with a status-quo bias, the rationalization problem is in **P**. Interestingly, however, the confinement from sequential rationalization to the rationalization by game trees is not sufficient to reduce its complexity.

4 Proofs

4.1 Proof of Theorem 2

We consider a candidate solution of K-RAT to be a list of K transitive and complete relations. Each of these relations can be described by no more than n^2 elements. As such, every certificate is of polynomial size (for fixed K). Further, one can verify in polynomial time whether this list rationalizes the instance. This shows that K-RAT is in **NP**.

For the second part of the proof, we need to show that a known **NP**-complete problem is polynomial time reducible to K-RMR. First, we restrict ourselves to the case where $K = 2$. The known **NP**-complete problem

[12] To be precise, Apesteguia and Ballester (2009) considered a slightly more restricted version of the status quo bias model. We refer to their paper for further details.

that we use for the reduction is 'Monotone not all equal 3SAT' (M-NAE-3SAT).[13]. An instance of M-NAE-3SAT consists of a finite set of variables $\{x_1, \ldots, x_t\}$ and a collection of clauses $\{C_1, \ldots, C_r\}$ such that each clause is composed of three variables. The question corresponding to M-NAE-3SAT is the following.

Monotone not all equal 3SAT (M-NAE-3SAT): *Does there exist an assignment to the variables $x_1, \ldots, x_t$ (either 1 or 0) such that each clause contains at least one variable with the value 1 and at least one variable with the value 0.*

Consider an instance of M-NAE-3SAT with a set of variables $\{x_1, \ldots, x_t\}$ and a set of clauses $\{C_1, \ldots, C_r\}$. We start by creating the corresponding instance of 2-RAT.

- For each variable x_i ($i = 1, \ldots, t$) we construct two alternatives a_i and $\bar{a}_i$.
- For each clause C_ℓ ($\ell = 1, \ldots, r$), we construct three alternatives $z_{1,\ell}, z_{2,\ell}$ and $z_{3,\ell}$.

Consider the function f from the set of variables $z_{k,\ell}$ ($k = 1, 2, 3$ and $\ell = 1, \ldots, r$) to the set $\{1, \ldots, t\}$, such that $f(z_{k,\ell}) = i$ if and only if the k-th variable in the clause C_ℓ is equal to x_i. Further, for each $k = 1, 2, 3$, we denote by $k \oplus 1$ the number $(k + 1) \mod 3$. The construction of the choice domain $\mathcal{D}$ and the choice function c is given in table 1.

Table 1: Instance for 2-RAT

choice domain $\mathcal{D}$	choice function c	range
$\{a_i, \bar{a}_i\}$	$\{a_i\}$	$i = 1, \ldots, t$
$\{a_i, \bar{a}_i, z_{k,\ell}\}$	$\{\bar{a}_i\}$	$\ell = 1, \ldots, r; k = 1, 2, 3; i = f(z_{k,\ell})$
$\{z_{k,\ell}, z_{k\oplus 1,\ell}, \bar{a}_i\}$	$\{z_{k,\ell}\}$	$\ell = 1, \ldots, r; k = 1, 2, 3; i = f(z_{k,\ell})$

Evidently, this construction can be performed in polynomial time.

Next, let us prove that when this problem satisfies 2-RAT, then there must be a truth assignment that satisfies M-NAE-3SAT. Let $\succeq_1$ and $\succeq_2$ be the two rationales that solve the problem 2-RAT. To each choice set in $\mathcal{D}$,

[13] Monotone-not-all-equal-3SAT can be reduced from the **NP**-complete problem Not-all-equal-3SAT (Garey and Johnson, 1979) by replacing all literals of the form $(1 - x_i)$ by a variable y_i and adding an additional clause of the form $\{y_i, x_i, x_i\}$.

we can correspond a rationale ($\succeq_1$ or $\succeq_2$) that rationalizes the choice made from this set. It is easy to establish, by asymmetry of $\succ_1$ and $\succ_2$, that for all $i = 1, \ldots, t$ and all $z_{k,\ell}$ with $i = f(z_{k,\ell})$, there are only two mutually exclusive configurations possible. These are listed in table 2:

Table 2: Configurations 2-RAT

choice sets	choice function	configuration 1	configuration 2
$\{a_i, \bar{a}_i\}$	$\{a_i\}$	$\succ_1$	$\succ_2$
$\{a_i, \bar{a}_i, z_{k,\ell}\}$	$\{\bar{a}_i\}$	$\succ_2$	$\succ_1$
$\{z_{k,\ell}, z_{k\oplus 1,\ell}, \bar{a}_i\}$	$\{z_{k,\ell}\}$	$\succ_1$	$\succ_2$

Now, if configuration 1 prevails for $i \in \{1, \ldots, t\}$, we fix $x_i = 1$ and if configuration 2 prevails, we set $x_i = 0$. All we need to show is that this solution provides a 'yes' instance of M-NAE-3SAT. Assume, on the contrary, that there is a clause C_ℓ for which all variables are equal to 1. In that case, we have that $z_{1,\ell} \succ_1 z_{2,\ell}$, $z_{2,\ell} \succ_1 z_{3,\ell}$ and $z_{3,\ell} \succ_1 z_{1,\ell}$, contradicting acyclicity of $\succ_1$. On the other hand, if all variables in C_ℓ are equal to zeros we must have that: $z_{1,\ell} \succ_2 z_{2,\ell}$, $z_{2,\ell} \succ_2 z_{3,\ell}$ and $z_{3,\ell} \succ_2 z_{1,\ell}$, contradicting acyclicity of $\succ_2$. Conclude that M-NAE-3SAT must be satisfied.

Finally, we also need to demonstrate that any 'yes' instance of M-NAE-3SAT corresponds to a 'yes' instance of 2-RAT. Towards this end notice that, from the single-valuedness of c, that it is sufficient to to demonstrate the existence of two acyclic and asymmetric relations $\succ_1$ and $\succ_2$ such that for each choice set, $A \in \Sigma$ with $b \in c(A)$ either $b \succ_1 d$ for all $d \in F$ or $b \succ_2 d$ for all $d \in F$. The relations $\succ_1$ and $\succ_2$ can always be extended to complete and transitive relations in a polynomial number of steps (using for example a finite analogue of Szpilrajn (1930)'s lemma). We assign $\succ_1$ and $\succ_2$ to the choice sets as presented in table 3, depending on the value of x_i ($i = 1, \ldots, t$) that solve M-NAE-3SAT: In other words, if $x_i = 1$ we set

Table 3: Construction of $\succ_1$ and $\succ_2$

choice sets	choice function	$x_i = 1$	$x_i = 0$
$\{a_i, \bar{a}_i\}$	$\{a_i\}$	$\succ_1$	$\succ_2$
$\{a_i, \bar{a}_i, z_{k,\ell}\}$	$\{\bar{a}_i\}$	$\succ_2$	$\succ_1$
$\{z_{k,l}, z_{k\oplus 1,\ell}, \bar{a}_i\}$	$\{z_{k,\ell}\}$	$\succ_1$	$\succ_2$

$a_i \succ_1 \bar{a}_i$, $z_{k,\ell} \succ_1 z_{k\oplus 1,\ell}$ and $z_{k,\ell} \succ_1 \bar{a}_i$ and we fix $\bar{a}_i \succ_2 a_i$ and $\bar{a}_i \succ_2 z_{k,\ell}$ (given that $f(z_{k,\ell}) = i$). On the other hand, if $x_i = 0$ we define $a_i \succ_2 \bar{a}_i$, $z_{k,\ell} \succ_2 z_{k\oplus 1,\ell}$ and $z_{k,\ell} \succ_2 \bar{a}_i$ and we set $\bar{a}_i \succ_1 a_i$ and $\bar{a}_i \succ_1 z_{k,\ell}$ (given that $f(z_{k,\ell}) = i$).

We must still establish that the relations $\succ_1$ and $\succ_2$ are acyclic. Towards a contradiction, assume that there exist alternatives $b_1, \ldots, b_q$ such that for each $s = 1, \ldots, q-1$, $b_s \succ_1 b_{s+1}$ and $b_q \succ_1 b_1$. (The case of a cycle in the relation $\succ_2$ is similar and is left to the reader.) We distinguish different cases depending on the the different possible values for b_1.

- Case 1. If $b_1 = a_i$ for some $i = 1, \ldots, t$, then there must exists alternatives b_q and b_2 such that $b_q \succ_1 a_i$ and $a_i \succ_1 b_2$. This is impossible as it imposes that both $x_i = 1$ and $x_i = 0$.

- Case 2. If $b_1 = \bar{a}_i$ for some $i = 1, \ldots, t$, then there must exist alternatives b_q and b_2 such that $\bar{a}_i \succ_1 b_2$ and $b_q \succ_1 \bar{a}_i$. Again, this would imply that both $x_i = 1$ and $x_i = 0$.

- Case 3. If $b_1 = z_{k,\ell}$ for some $k = 1, 2, 3$ and $\ell = 1, \ldots, r$, then b_2 cannot be equal to $\bar{a}_i$ because this would bring us back to case 2 (replacing b_1 by b_2, b_2 by b_3 and b_q by b_1.). As such, the cycle under consideration must be the cycle $z_{1,\ell} \succ_1 z_{2,\ell}$, $z_{2,\ell} \succ_1 z_{3,\ell}$, $z_{3,\ell} \succ_1 z_{1,\ell}$. However, this restricts all variables in C_l to be equal to one, a contradiction.

Conclude that 2-RAT is satisfied.

Until present, we demonstrated that 2-RAT is **NP**-complete. To show that K-RAT is **NP**-complete for all $K > 2$ we use an induction argument. We know that K-RAT it is **NP**-complete for $K = 2$. Assume that it is **NP**-complete for $K = M$ and consider the case $K = M + 1$. First, we construct for each instance $(X, \mathcal{D}, c)$ of M-RAT, an instance $(X', \mathcal{D}', c')$ of $(M+1)$-RAT.

- For each $x \in X$, we create an alternative $x \in X'$. Further, we create two additional alternatives $a', b' \in X'$.

- For each $A \in \mathcal{D}$, create the choice set $A' = A \cup \{a'\}$ and impose that $c'(A') = c(A)$.

- Create one additional choice set $Z = X'$ and impose that $c'(Z) = \{a'\}$.

Of course, the instance $(X', \mathcal{D}', c')$ can be constructed in a polynomial number of steps.

Next, assume that $(\succeq_k)_{k\leq M}$ solves M-RAT. Construct for each $i = 1,\ldots,M$, the relation $\succeq_i' = \succeq_i \cup \{(x,a'),(x,b')|x \in X' - \{a'\}\} \cup \{a',a'\}$ and let $\succeq_{M+1}'$ be an arbitrary (transitive and complete) relation for which a' is top ranked among all alternatives. It is clear to see that $(\succeq_i')_{i\leq M+1}$ provides a solution for $(M+1)$-RAT, where $\succeq_{M+1}'$ rationalizes the choice set Z and $\succeq_i'$ rationalizes the choice set A' if and only if $\succeq_i$ rationalizes the choice set A.

Finally, let $(\succeq_i')_{i\leq M+1}$ provide a solution to $(M+1)$-RAT and let $\succeq_{M+1}'$ be the relation that rationalizes the choice function Z. It must be that a' is top ranked in this relation (this follows from the fact that $Z = X'$ and $c'(Z) = \{a'\}$). For any other choice set, A', it must be that $\succeq_{M+1}'$ does not rationalize this set (this is because a' is in A' but not chosen). Now, let $\succeq_k$ $(k = 1,\ldots,M)$ be the relation that is equal to $\succeq_k'$ less the comparisons involving the alternatives a' and b'. Evidently, $\succeq_k$ rationalizes the choice set A if and only if $\succeq_k'$ rationalizes the choice set A'. Therefore, it follows that $\{\succeq_k\}_{k\leq M}$ rationalizes the choice function c.

4.2 Proof of Theorem 3

We split this proof into 4 parts corresponding to the items in the theorem.

4.2.1 If $K \leq 2$, then K-SR is in the class P.

We first focus on the case with $K = 1$. Consider the definition of Weakened WARP.

Definition 5 (Weakened WARP)**.** *A choice function satisfies Weakened WARP if for all $x, y \in X$ and $A \in \mathcal{D}$ with $x \in c(A)$ and $y \in A - c(A)$ there does not exist a set $B \in \mathcal{D}$ such that $y \in c(B)$ and $x \in B - c(B)$.*

Weakened WARP was introduced by Ehlers and Sprumont (2008). These authors show that Weakened WARP characterizes the choice functions that are rationalizable by an upper-class rule. The following result uses this property to characterize the instances that satisfy 1-SR.

Proposition 1. *An instance $(X,\mathcal{D},c)$ is a 'yes' instance for 1-SR if and only if c is single valued and satisfies Weakened WARP.*

Proof. Assume that $(X,\mathcal{D},c)$ is a 'yes' instance for 1-SR. Let $\succ$ rationalize c. It is easy to see that $\succ$ is a tournament, i.e. $\succ$ is complete and asymmetric. Let us prove that c satisfies Weakened WARP. Let $x \in c(A)$ and $y \in A - c(A)$ for some $A \in \mathcal{D}$. This implies that $\neg(y \succ x)$. The relation $\succ$ is complete, hence, it follows that $x \succ y$. If, on the contrary, $y \in c(B)$ and $x \in B - c(B)$ for some $B \in \mathcal{D}$, we see that for all $z \in B$, $\neg(z \succ y)$, contradicting $x \succ y$ and $x \in B$. Conclude that c satisfies Weakened WARP.

For the reverse, let c be a single valued choice function satisfying Weakened WARP and let us construct a rationalization $\succ$ of c. Consider any two elements $x, y \in X$. If there is a set $A \in \mathcal{D}$ such that $x \in c(A)$ and $y \in A$, set $x \succ y$. Else, select a random element $z \in \{x, y\}$. If $z = x$, fix $x \succ y$ and if $z = y$, set $y \succ x$. As we considered every pair of elements, the relation $\succ$ is complete. Moreover, Weakened WARP implies that $\succ$ is asymmetric. It is easy to see that that $\succ$ rationalizes c. $\square$

Weakened WARP can be verified in polynomial time. Therefore, 1-SR is in the class **P**. Let us now focus on the decision problem 2-SR. Manzini and Mariotti (2007) provide a characterization of 2-SR for the case where the domain $\mathcal{D}$ is binary.

Theorem 4 (Manzini and Mariotti (2007))**.** *An instance $(X, \mathcal{D}, c)$ with $\mathcal{D}$ a binary domain is a 'yes' instance for 2-SR if and only if it satisfies WWE: c is single valued and if $x = c(S_i)$ in a class and $x = c(\{x, y\})$ then $y \neq c(R)$ for all $R \in \mathcal{D}$ with $\{x, y\} \subset R \subseteq \bigcup_i S_i$.*

The property WWE can be verified in a polynomial number of steps. For our prove, we need to relax the condition of the binary domain. The following proposition is a slight adaptation of Theorem 4 for this more general case.

Proposition 2. *An instance $(X, \mathcal{D}, c)$ is a 'yes' instance for 2-SR if and only if it satisfies NB-WWE: c is single-valued and for all x and y in X and $R, T \in \mathcal{D}$, if $x \in c(S_i)$ in a class and $y \in c(V_i)$ in a class, $R \subseteq \bigcup_i S_i$ and $T \subseteq \bigcup_i V_i$, then not $x \in c(T)$ and $y \in c(R)$.*

Proof. Necessity is obvious. For sufficiency, notice that it is sufficient to demonstrate that for every instance $(X, \mathcal{D}, c)$ that is a 'yes' instance of 2-SR, there exists a choice function c' which is single valued on the binary domain $\mathcal{D}' = \mathcal{D} \bigcup \{\{x, y\} | x, y \in X\}$ and for which,

- $(X, \mathcal{D}', c')$ is a 'yes' instance of 2-SR, i.e. $(X, \mathcal{D}', c')$ satisfies WWE, and
- c' agrees with c on the domain $\mathcal{D}$, i.e. for all $A \in \mathcal{D}$, $c(A) = c'(A)$.

Now, let us construct such choice function c'. Consider a pair of alternatives $x, y \in X$ for which $\{x, y\} \notin \mathcal{D}$. If $x \in c(S_i)$ for a class and there exist a choice set $R \in \mathcal{D}$ such that $y \in c(R)$ and $R \subseteq \bigcup_i S_i$, we impose that $\{y\} = c'(\{x, y\})$. Similarly, if $y \in c(V_i)$ for a class and there exist a choice set $T \in \mathcal{D}$ such that $x \in c(T)$ and $T \subseteq \bigcup_i V_i$, we set $\{x\} = c'(\{x, y\})$. If none of above two conditions are satisfied, we pick at random an element out of $\{x, y\}$, say z and we determine $c'(\{x, y\}) = \{z\}$. Condition NB-WWE guarantees that for no pair of alternatives x and y, we have that both $\{x\} = c(\{x, y\})$ and $\{y\} = c(\{x, y\})$. It is easily verified that the instance $(X, \mathcal{D}', c')$ satisfies WWE. □

The condition NB-WWE can be verified in a polynomial number of steps, hence, 2-SR is in **P**.

4.2.2 For all $K \geq 3$, K-SR is NP-complete.

We construct a certificate of K-SR as be a list of K asymmetric rationales. Each of this relation can be described by no more than n^2 elements. As such, every certificate is of polynomial size (for fixed K). Further, it is easily verified that given a list of K rationales, one can verify in polynomial time whether this list rationalizes the instance. This shows that K-SR is in **NP**.

For the second part of the proof, we use a reduction from the **NP**-complete problem 3SAT. An instance for 3SAT consists of a finite set of binary variables $\{x_1, \ldots, x_t\}$ and a finite list of clauses $\{C_1, \ldots, C_r\}$. Each clause, C_ℓ exists of three literals $l_{1,\ell}, l_{2,\ell}$ and $l_{3,\ell}$ and each literal either equals a certain variable, x_i, or its negation, $(1 - x_i)$. The question corresponding to 3SAT is the following.

3 satisfiability (3SAT): *Does there exist an assignment to the variables $\{x_1, \ldots, x_t\}$ (either 1 or 0) such that for every clause C_ℓ ($\ell = 1, \ldots, r$) at least one literal has the value 1?*

Consider an instance of 3SAT with a set of variables $\{x_1, \ldots, x_t\}$ and a set of clauses $\{C_1, \ldots, C_r\}$. First we create the instance of K-SR.

- For each variable x_i, $i = 1, \ldots, t$, we create two alternatives a_i and $\bar{a}_i$.
- We create 3 other additional alternatives v_1, v_2 and q.

Consider the function f from the set of elements (k, ℓ) ($k = 1, 2, 3$ and $\ell = 1, \ldots, r$) to the set of alternatives, X, such that $f(k, \ell) = a_i$ if the kth literal in the ℓ-th clause, C_ℓ, equals x_i and $f(k, \ell) = \bar{a}_i$ if the k-th literal in the ℓ-th clause equals $(1 - x_i)$. The choice domain, $\mathcal{D}$ and the choice function, c, are given in table 4. Notice that the domain is binary. Obviously, this instance of K-SR can be constructed in polynomial time.

Table 4: Instance for K-SR

choice domain $\mathcal{D}$	choice function c	range	
$\{v_1, v_2\}$	$\{v_1\}$		(1)
$\{v_1, q\}$	$\{q\}$		(2)
$\{v_1, a_i\}$	$\{v_1\}$	$\forall i = 1, \ldots, t$	(3)
$\{v_1, \bar{a}_i\}$	$\{v_1\}$	$\forall i = 1, \ldots, t$	(4)
$\{v_2, q\}$	$\{q\}$		(5)
$\{v_2, a_i\}$	$\{v_2\}$	$\forall i = 1, \ldots, t$	(6)
$\{v_2, \bar{a}_i\}$	$\{v_2\}$	$\forall i = 1, \ldots, t$	(7)
$\{q, a_i\}$	$\{a_i\}$	$\forall i = 1, \ldots, t$	(8)
$\{q, \bar{a}_i\}$	$\{\bar{a}_i\}$	$\forall i = 1, \ldots, t$	(9)
$\{a_i, \bar{a}_i\}$	$\{a_i\}$	$\forall i = 1, \ldots, t$	(10)
$\{a_i, a_j\}$	$\{a_i\}$	$\forall i < j; i, j = 1, \ldots, t$	(11)
$\{a_i, \bar{a}_j\}$	$\{a_i\}$	$\forall i, j = 1, \ldots, t$	(12)
$\{\bar{a}_i, \bar{a}_j\}$	$\{\bar{a}_i\}$	$\forall i < j; i, j = 1, \ldots, t$	(13)
$\{v_1, v_2, a_i, q\}$	$\{q\}$	$\forall i = 1, \ldots, t$	(14)
$\{v_1, v_2, \bar{a}_i, q\}$	$\{q\}$	$\forall i = 1, \ldots, t$	(15)
$\{v_1, a_i, \bar{a}_i, q\}$	$\{v_1\}$	$\forall i = 1, \ldots, t$	(16)
$\{v_2, a_i, \bar{a}_i, q\}$	$\{v_2\}$	$\forall i = 1, \ldots, t$	(17)
$\{v_2, f(1,\ell), f(2,\ell), f(3,\ell), q\}$	$\{v_2\}$	$\forall \ell = 1, \ldots, r$	(18)

Next, assume that 3SAT is satisfiable. We prove that we can find a list of three rationales $\{\succ_1, \succ_2, \succ_3\}$ that rationalizes the instance of K-SR.

For all $i = 1, \ldots, t$ with $x_i = 1$, set $v_1 \succ_1 a_i$ and $v_2 \succ_1 \bar{a}_i$. If $x_i = 0$, we set $v_1 \succ_1 \bar{a}_i$ and $v_2 \succ_1 a_i$. These are the only elements in $\succ_1$.

For all $i = 1, \ldots, t$, set $a_i \succ_2 q$ and $\bar{a}_i \succ_2 q$. These are the only elements in $\succ_2$. The elements of $\succ_3$ are listed in table 5. Notice that the relations $\succ_1, \succ_2$ and $\succ_3$ are acyclic and that the rationalization is single-valued. We could also make them transitive by taking their transitive closure. One can easily verify that these three relations rationalize the instance.

Finally, assume that the instance of K-SR is rationalizable by the list $\{\succ_k\}_{k \leq K}$. We need to show that 3SAT has a solution. We begin by introducing some new notation. Consider 4 alternatives a, b, c and d. We write $ab \trianglerighteq cd$ if there exist rationales $\succ_j$ and $\succ_k$ such that $a \succ_j b$, $c \succ_k d$ and,

$$\min\{i | a \succ_i b\} \leq \min\{i | c \succ_i d\}.$$

Table 5: Construction of $\succ_1$ and $\succ_2$

elements of $\succ_3$	range
(v_1, v_2)	
$(q, v_1), (q, v_2)$	
$(a_i, \bar{a}_i)$	$\forall i = 1, \ldots, t$
$(v_2, a_i), (v_2, \bar{a}_i)$	$\forall i = 1, \ldots, t$
$(v_1, a_i), (v_2, \bar{a}_i)$	$\forall i = 1, \ldots, t$
$(a_i, a_j), (\bar{a}_i, \bar{a}_j)$	$\forall i < j; i, j = 1, \ldots, t$
$(a_i, \bar{a}_j)$	$\forall i, j = 1, \ldots, t$

In other words, we have that $ab \trianglerighteq cd$ if the first rationale in the list that contains (a, b) is not after the first rationale that contains (c, d). Similarly, we write $ab \triangleright cd$ if there exist rationales $\succ_j$ and $\succ_k$ such that $a \succ_j b$, $c \succ_k d$ and:

$$\min\{i | a \succ_i b\} < \min\{i | c \succ_i d\}.$$

Consider the following lemma:

Lemma 1. *If the instance $(X, \mathcal{D}, c)$ is a 'yes' instance of K-SR then for all $i = 1, \ldots, t$, either*

$$(v_2\bar{a}_i \triangleright \bar{a}_i q) \tag{C.1}$$

or (exclusively),

$$(v_2 a_i \triangleright a_i q) \tag{C.2}$$

Proof. First of all, from (8) and (9) it follows that there must be a $\succ_j$ and $\succ_l$ in the list such that $a_i \succ_j q$ and $\bar{a}_i \succ_l q$. From (14) and (15), it follows that:

$$(v_1 a_i \triangleright a_i q) \qquad \text{or} \qquad (v_2 a_i \triangleright a_i q)$$

and

$$(v_1 \bar{a}_i \triangleright \bar{a}_i q) \qquad \text{or} \qquad (v_2 \bar{a}_i \triangleright \bar{a}_i q)$$

A negation of one of these conditions would imply that $\{q\} \neq c(\{v_1, v_2, a_i, q\})$ or $\{q\} \neq c(\{v_1, v_2, \bar{a}_i, q\})$.

From (2) and (5) it follows that there must be a $\succ_j$ and $\succ_l$ such that $q \succ_j v_1$ and $q \succ_l v_2$. Combined with (16) and (17), it follows that:

$$\text{not } [(v_1 a_i \rhd a_i q) \qquad \text{and} \qquad (v_1 \bar{a}_i \rhd \bar{a}_i q)]$$

and

$$\text{not } [(v_2 a_i \rhd a_i q) \qquad \text{and} \qquad (v_2 \bar{a}_i \rhd \bar{a}_i q)]$$

The proof is completed by taking those combinations that do not lead to a contradiction. □

Now, consider a solution $\{\succ_k\}_{k \leq K}$ for the decision problem K-SR and set $x_i = 1$ if the first case (C.1) of the lemma is satisfied, i.e. $(v_2 \bar{a}_i \rhd \bar{a}_i q)$ or equivalently: $\neg(v_2 a_i \rhd a_i q)$. On the other hand, we set $x_i = 0$ if (C.2) is satisfied, i.e. $(v_2 a_i \rhd a_i q)$ or equivalently: $\neg(v_2 \bar{a}_i \rhd \bar{a}_i q)$. Consider a clause C_ℓ with three literals $l_{1,\ell}, l_{2,\ell}$ and $l_{3,\ell}$. We need to show that for each clause C_ℓ, at least one of its literals hold. Consider the choice:

$$c(\{v_2, f(1,\ell), f(2,\ell), f(3,\ell), q\}) = \{v_2\}.$$

Then it cannot be the case that:

$$(v_2 f(1,\ell) \rhd f(1,\ell)q) \text{ and } (v_2 f(2,\ell) \rhd f(2,\ell)q) \text{ and } (v_2 f(3,\ell) \rhd f(3,\ell)q)$$

As such, there must be at least one literals that has a value of one.

4.2.3 RGT is NP-complete.

One can easily verify that RGT is **NP**. For the reduction, we again use the **NP**-complete problem 3SAT

Consider an instance of 3SAT with variables $\{x_1, \ldots, x_t\}$ and clauses $\{C_1, \ldots, C_r\}$. First we create the corresponding instance of RGT.

- For each variable x_i, $i = 1, \ldots, t$, we create two alternatives a_i and $\bar{a}_i$.
- We create 4 other alternatives y, n, z_1 and z_2.

Again, consider the function f from the set of elements (k, ℓ) ($k = 1, 2, 3$ and $\ell = 1, \ldots, r$) to the set of alternatives, X, such that $f(k, \ell) = a_i$ if the k-th literal in the ℓ-th clause, C_ℓ, equals x_i and $f(k, \ell) = \bar{a}_i$ if the k-th literal in the ℓt-h clause equals $(1 - x_i)$. The choice domain, $\mathcal{D}$ and the choice function, c, are given in table 6.

Table 6: Instance of RGT

choice domain $\mathcal{D}$	choice function c	range	
$\{a_i, y, n\}$	$\{y\}$	$\forall i = 1, \ldots, t$	(1)
$\{\bar{a}_i, y, n\}$	$\{y\}$	$\forall i = 1, \ldots, t$	(2)
$\{a_i, \bar{a}_i, y, z_1\}$	$\{z_1\}$	$\forall i = 1, \ldots, t$	(3)
$\{y, z_1\}$	$\{y\}$		(4)
$\{a_i, \bar{a}_i, n, z_2\}$	$\{z_2\}$	$\forall i = 1, \ldots, t$	(5)
$\{n, z_2\}$	$\{n\}$		(6)
$\{f(1,\ell), f(2,\ell), f(3,\ell), y, z_1\}$	$\{z_1\}$	$\forall \ell = 1, \ldots, r$	(7)

Observe that this instance can be constructed in polynomial time. Next, let us show that if this choice function is rationalizable by game trees then the corresponding 3SAT problem has a solution. For the proof, we will repeatedly make use of the fact that if $c(\{b, c\}) = \{b\}$ for all c in a certain set A, then $c(\{b\} \cup A) = \{b\}$.

Observe that for all $i = 1, \ldots, t$, it holds that,

$$c(\{a_i, y\}) = \{y\} \text{ or } c(\{a_i, n\}) = \{n\} \text{ and, } \qquad c(\{\bar{a}_i, y\}) = \{y\} \text{ or } c(\{\bar{a}_i, n\}) = \{n\}.$$

Otherwise, we would have that either $\{a_i\} = c(\{a_i, y, n\})$ or $\{\bar{a}_i\} = c(\{\bar{a}_i, y, n\})$, which contradicts (1) and (2). Further, we also see that:

$$c(\{a_i, y\}) = \{a_i\} \text{ or } c(\{\bar{a}_i, y\}) = \{\bar{a}_i\} \text{ and, } \qquad c(\{a_i, n\}) = \{a_i\} \text{ or } c(\{\bar{a}_i, n\}) = \{\bar{a}_i\}.$$

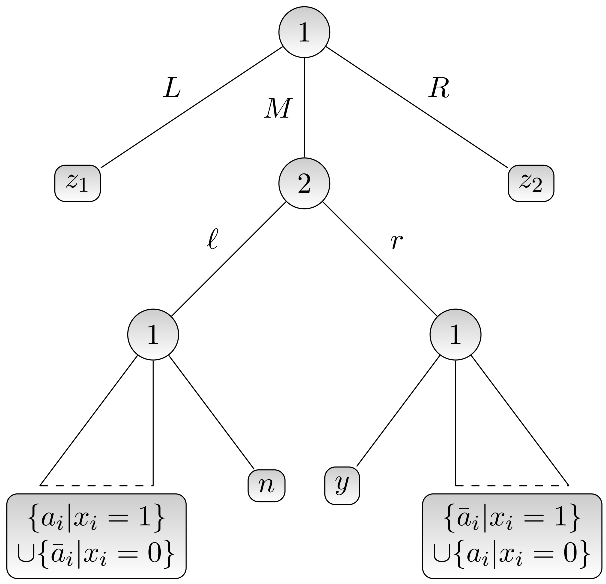
Otherwise, we would have that either $\{y\} = c(\{a_i, \bar{a}_i, y, z_1\})$ or $\{n\} = c(\{a_i, \bar{a}_i, n, z_2\})$, which contradicts (3) and (5).

Above two restriction imply that for all $i = 1, \ldots, t$ either $c(\{a_i, y\}) = \{a_i\}$ or (exclusively) $c(\{\bar{a}_i, y\}) = \{\bar{a}_i\}$. Now, for all $i = 1, \ldots, t$, let $x_i = 1$ if $c(\{a_i, y\}) = \{a_i\}$ (or equivalently $c(\{\bar{a}_i, y\}) = \{y\}$) and set $x_i = 0$ if $c(\{a_i, y\}) = \{y\}$ (or equivalently $c(\{\bar{a}_i, y\}) = \{\bar{a}_i\}$).

Let us show that this assignment satisfies 3SAT. Consider a clause C_ℓ with literals $l_{1,\ell}$, $l_{2,\ell}$ and $l_{3,\ell}$. Then, from $c(\{f(1,\ell), f(2,\ell), f(3,\ell), y, z\}) = \{z\}$ and $c(\{y, z\}) = \{y\}$ (conditions (4) and (7)) it follows that there is at least one $k = 1, 2, 3$ such that $c(\{f(k,\ell), y\}) = f(k,\ell)$. As such, if $f(k,\ell) = x_i$, then $x_i = 1$ and if $f(k,\ell) = 1 - x_i$, then $x_i = 0$. Therefore, each clause contains at least one true literal. This shows that 3SAT is satisfiable.

Finally, consider a solution to 3SAT. We need to show that the corresponding choice function is rationalizable by game trees. We construct the following three stage game, with two players (see also figure 1):

Figure 1: Game tree



Stage I Player 1 has three strategies: L(eft), M(iddle) or R(ight). If she chooses L, then the game ends with the outcome z_1. If she picks R, then the game ends with the outcome z_2. If she opts for strategy M, the game proceeds to stage II.

Stage II Player 2 has two strategies ℓ(eft) or r(ight) both of which lead to a third stage.

Stage III Depending on the choice of player 2 in stage II, player 1 faces the following strategies which lead to a final outcome in X:

- If player 2 has chosen ℓ then player 1 can either choose a strategy that leads to outcome n or she may choose for each element in the set $\{a_i|x_i = 1\} \cup \{\bar{a}_i|x_i = 0\}$ a strategy leading to the corresponding outcome.

- If player 2 has chosen r the second strategy then player 1 can either choose a strategy that leads to outcome y or she may choose for each element in the set $\{\bar{a}_i | x_i = 1\} \cup \{a_i | x_i = 0\}$ a strategy leading to the corresponding outcome.

We define the acyclic and asymmetric relations $\succ_1$ and $\succ_2$ as in table 7, depending on the specific value of x_i. Of course, these acyclic comparisons

Table 7: Construction of $\succ_1$ and $\succ_2$

$x_i = 1$	$x_i = 0$	unconditional
$n \succ_1 a_i$	$n \succ_1 \bar{a}_i$	$n \succ_1 z_2$
$y \succ_1 \bar{a}_i$	$y \succ_1 a_i$	$y \succ_1 z_1$
$z_1 \succ_1 a_i$	$z_1 \succ_1 \bar{a}_i$	
$z_2 \succ_1 \bar{a}_i$	$z_2 \succ_1 a_i$	
$a_i \succ_2 y$	$\bar{a}_i \succ_2 y$	$y \succ_2 n$
$\bar{a}_i \succ_2 n$	$a_i \succ_2 n$	

can be extended to complete, asymmetric and transitive preference relations. It is easy to verify that these preference relations, together with the game tree defined above, rationalize the choice function.

4.2.4 SQB is in the class P

Before we start the proof, we need to define several rationales. We begin with the rationale $\succeq_1$.

$$x \succeq_1 y \text{ if and only if there is a pair } (A, y) \in \mathcal{D}_{sq} \text{ with } y \neq \diamond \text{ such that } x \in c(A, y).$$

Let $\succeq_{1,t}$ be the transitive closure of $\succeq_1$.[14] Now, define the rationale $\succeq_2$ by $x \succeq_2 y$ if and only if one of the following three conditions hold.

1. $x \succeq_{1,t} y$.
2. There is a pair $(A, z) \in \mathcal{D}_{sq}$ with $z \neq \diamond$ such that $x \in c(A, z)$, $y \in A$ and $y \succeq_{1,t} z$.

[14] Formally, $x \succeq_{1,t} y$ if there exist a (possibly empty) sequence of alternatives $x_1, \ldots, x_n$ such that $x = x_1$, $x_n = y$ and for all $i = 1, \ldots, n-1$: $x_i \succeq x_{i+1}$.

3. There is a pair $(A, \diamond) \in \mathcal{D}_{sq}$ such that $x \in c(A, \diamond)$ and $y \in A$.

Let $\succeq_{2,t}$ be the transitive closure of $\succeq_2$. Finally, we define the rationale $\succeq_3$ by the conditions that $x \succeq_3 y$ if and only if one of the following three conditions hold.

1. $x \succeq_{1,t} y$.
2. There is a pair $(A, z) \in \mathcal{D}$ with $z \neq \diamond$ such that $x \in c(A, z)$, $y \in A - c(A, z)$ and $y \succeq_{1,t} z$.
3. There is a pair $(A, \diamond) \in \mathcal{D}$ such that $x \in c(A, \diamond)$ and $y \in A - c(A, .)$.

The following lemma provides necessary and sufficient conditions for status quo bias rationalizability.

Lemma 2. *The choice function c is status quo rationalizable if and only if:*

$$\succeq_1 \text{ is acyclic.} \tag{D.1}$$

$$\begin{aligned} &\textit{If } y \succeq_{1,t} x \textit{ and } y \in A, \textit{ then for all } (A,x) \in \mathcal{D} \textit{ with,} \\ &\qquad\qquad\qquad\qquad\qquad x \neq \diamond : x \notin c(A,x). \end{aligned} \tag{D.2}$$

$$\textit{If } x \succeq_{2,t} y \textit{ then it is not the case that } y \succeq_3 x. \tag{D.3}$$

$$\textit{If } x \succeq_{1,t} y \textit{ then it is not the case that } y \succeq_2 x. \tag{D.4}$$

Proof. **necessity.** By construction, we have that $x \succeq_1 y$ implies $u(x) > u(y)$, $x \succeq_2 y$ implies $f(u(x)) \geq f(u(y))$ and $x \succeq_3 y$ implies $f(u(x)) > f(u(y))$. The four conditions D.1-D.4 follow immediately.

sufficiency. By condition (D.1), the relation $\succeq_{1,t}$ is asymmetric and transitive. A such, we can use Szpilrajn (1930)'s lemma to show that it has a complete, asymmetric and transitive extension. Denote by Σ the finite set of all these transitive, asymmetric and complete extensions. A result from Dushnik and Miller (1941) shows that,

$$\succ_{1,t} = \bigcap_{\succ \in \Sigma} \succ . \tag{1}$$

Each relation in the finite set Σ can be represented by an injective function $u_{\succ} : X \to \mathbb{R}$. Let u be the function $X \to \mathbb{R}^{|\Sigma|}$ that stacks all these functions. Observe that $u(x) > u(y)$ if and only if $x \succeq_{1,t} y$. This constructs the function u.

Now, let us focus on the construction of the function f. From the definitions, it follows that $\succ_2 \subseteq \succeq_3$. As such, condition (D.3) implies that $\succeq_2$

is consistent.[15] From Suzumura (1976, theorem 3) it follows that $\succeq_2$ has a complete and transitive extension. Further, condition (D.4) together with the definition of $\succeq_2$ shows that $\succeq_2$ extends the relation $\succeq_1$. Therefore, the complete and transitive extension of $\succeq_2$ also extends $\succeq_1$. This extension can be represented by a function from X to $\mathbb{R}$, say g. Also, it is possible to write g as a function of u, because $u(x) = u(y)$ (which implies $x = y$) implies $g(x) = g(y)$. Let $f(u(.))$ be this function. It follows that $u(x) > u(y)$ implies $g(x) > g(y)$, hence, $f(u(.))$ is strictly increasing. Let us now show that the functions u and f rationalize the choice function.

If $y \in C(A, \diamond)$ we have that, by construction, $y \succeq_2 x$ for all $x \in A$, hence $f(u(y)) \geq f(u(x))$. If $x \in c(A, x)$ and, on the contrary, there is an alternative $y \in A$ such that $u(y) > u(x)$, we also have that $y \succeq_{1,t} x$. This violates condition (D.2). If $y \in c(A, x)$ with $y \neq x$, then, by definition, $y \succ_1 x$. Then, if, on the contrary, there is a $z \in A$, with $u(z) > u(x)$ (i.e. $z \succeq_{1,t} x$ and $f(u(z)) > f(u(y))$, we must have, by definition, that $z \succeq_2 y$. However, this violates the fact that f was derived from an extension of $\succeq_2$, i.e. $f(u(z)) > f(u(y))$ implies $\neg(y \succeq_2 z)$. $\square$

The construction of the relations $\succeq_1$, $\succeq_2$ and $\succeq_3$ can be performed in polynomial time. The construction of the transitive closures $\succeq_{1,t}$ and $\succeq_{2,t}$ can also be established in polynomial time (using, for example, the algorithm by Warshall (1962)). Finally, the verification of the four conditions in the lemma can also takes a polynomial number of steps. As such, the problem of deciding whether a choice function is rationalizable by a status-quo biased choice rule is in **P**.

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[15]A relation $\succeq$ is consistent if for all $x \succeq_t y$ it is not the case that $y \succ x$.

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