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Generalized Optimal Pilot Allocation for Channel Estimation in Multicarrier Systems

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***Abstract*—This paper addresses the design of MSE-optimal preambles for multicarrier channel estimation under a maximum likelihood or minimum mean squared error criterion. The derived optimality condition gives insight on how to allocate the pilots that compose the preamble. While many papers show that equispaced and equipowered allocation is optimal, the generalized condition demonstrates that there exist many different configurations that offer the same optimal performance. Furthermore, the condition applies not only to maximum likelihood but also to minimum mean squared error channel estimation. An application of the generalized condition in the presence of inactive subcarriers (virtual subcarriers problem) is shown such that a non equispaced allocation can achieve the same optimal performance as if an equispaced one could be used.**

***Keywords*—Multicarrier transmission, channel estimation, optimal pilot allocation.**

I. Introduction

Multicarrier systems aim at dividing a wideband signal into multiple narrowband signals centered around different subcarriers [1]. If the number of subcarriers is large with respect to the delay spread of the channel, each narrowband channel can be considered as frequency flat, which greatly simplifies the equalization task at the receiver. It is mainly for their ability to effectively cope with the channel frequency selectivity that multicarrier systems have been so popular.

Channel estimation for multicarrier systems has been extensively studied; see, for instance, [2], [3] for recent review papers concerning channel estimation for orthogonal frequency division multiplexing (OFDM) or [4] for a review in offset QAM-based filterbank multicarrier (FBMC/OQAM) modulations. Specifically, the problem of optimizing the pilot allocation (including their position in frequency and their relative power) to minimize the mean squared error (MSE) of the channel estimate has been addressed in a number of works. Most of them focus on the case of least squares (LS) channel estimation, which corresponds to the maximum likelihood (ML) estimator under the Gaussian noise assumption and their main common conclusion is that the pilots should be equispaced and equipowered [5], [6]. The authors in [7] extend this result to the MIMO case and show that optimal pilot sequences are equipowered, equispaced, and phase shift orthogonal.

Surprisingly, there are only few works addressing optimal pilot allocation for the minimum mean squared error (MMSE) estimator, even in the single-antenna case. In [8], the pilot allocation that maximizes the capacity for a MMSE channel estimator is shown to also correspond to equipowered, equispaced pilots. In [9], the authors derive the MSE-optimal training condition for the MMSE estimator in the context of a two-way relay OFDM-based network.

Very few papers actually investigate if other pilot configurations also satisfy the optimality condition and therefore offer the same performance as the equispaced configuration. In this paper, we find a sufficient and necessary condition for optimal pilot allocation that holds for both the maximum likelihood (ML) and the MMSE channel estimators (for a general channel correlation matrix). It is shown that the condition can be simplified if the channel taps are assumed uncorrelated. This result can be seen as generalizing the classical equispaced pilot configuration. Indeed, it is shown that a much wider family of pilot allocations may be optimal. Furthermore, the optimality condition can be directly used to solve the problem of allocating the pilots in the more realistic (and more challenging) case where inactive (virtual) subcarriers are present. The problem of virtual subcarriers is widely studied in the literature; see, for example, [10] for the single antenna case and [11] for the multiple antennas case. The originality of the work presented here is that it gives a methodology for finding optimal allocations, when the problem is feasible, which attains the same performance as if no null subcarriers were present.

The rest of the paper is organized as follows. Section II first serves as a reminder of channel estimation in multicarrier systems using the ML and the MMSE criteria. Section III derives a general optimality condition for pilot allocation, which is then simplified for the ML case. The simplified condition also holds for the MMSE estimator if the channel taps are assumed uncorrelated. Section IV shows a possible application of the generalized condition to the virtual subcarriers problem. Simulation results are presented in Section V. Section VI concludes the paper.

Notation: Vectors and matrices are denoted by bold lowercase and uppercase letters, respectively. Superscripts *, ^{T} and ^{H} stand for conjugate, transpose and Hermitian transpose operators. tr, $\mathbb{E}$, $\Im$ and $\Re$ denote trace, expectation, imaginary and real parts, respectively.

II. Channel Estimation in Multicarrier Systems

We consider a multicarrier system with M subcarriers. Let us assume that the different subchannels can be considered flat and orthogonal to each other. A preamble of one multicarrier symbol is transmitted. The preamble vector composed of the transmitted pilots at the M subcarriers is denoted by $\mathbf{d} \in \mathbb{C}^{M\times 1}$. The channel impulse response $\mathbf{h} \in \mathbb{C}^{L_h\times 1}$ is assumed to be quasi-static[1] and L_h is the channel length. The vector of samples received after demodulation can be written as

$$\mathbf{y} = \mathbf{D}\mathbf{F}\boldsymbol{\Sigma}\mathbf{h} + \boldsymbol{\eta}$$

where $\mathbf{F} \in \mathbb{C}^{M\times M}$ is the unitary discrete Fourier transform (DFT) matrix, $\mathbf{D} = \mathrm{diag}(\mathbf{d}) \in \mathbb{C}^{M\times M}$ contains the vector of transmitted pilot symbols $\mathbf{d}$ on its diagonal and $\boldsymbol{\eta} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_\eta)$ is additive Gaussian noise. Moreover, the noise is assumed to be white, i.e. $\mathbf{C}_\eta = \sigma^2 \mathbf{I}_M$.[2] $\boldsymbol{\Sigma} = \begin{pmatrix}\mathbf{I}_{L_h} & \mathbf{0}^H_{(M-L_h)\times L_h}\end{pmatrix}^H$ can be seen either as a selection matrix of the first L_h columns of $\mathbf{F}$ or as a zero padding matrix that appends $M - L_h$ 0's to the vector $\mathbf{h}$.

Note that this model fits both an OFDM system if the cyclic prefix is at least as long as the channel order [2] and an FBMC/OQAM system if the channel frequency selectivity is sufficiently mild and the noise correlation is neglected [4]. Then $\mathbf{d}$ represents the so-called pseudo-pilots (for fully loaded preambles) or the pilots (otherwise) [12]–[14].

A. Maximum likelihood channel estimator

We here assume that $\mathbf{h}$ is deterministic and unknown. Denoting by $f(\mathbf{y}|\mathbf{h})$ the conditional probability density function of $\mathbf{y}$ given $\mathbf{h}$, the ML estimator of the channel is given by [15]

$$\begin{aligned}\hat{\mathbf{h}}_{\mathrm{ML}} &= \arg\max f(\mathbf{y}|\mathbf{h}) \\ &= \left(\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{D}^H\mathbf{y}\end{aligned}$$

where $\mathbf{P} = \mathbf{D}^H\mathbf{D}$ is a diagonal matrix containing the power of each pilot on its diagonal. We define $\mathbf{p} \in \mathbb{C}^{M\times 1}$ as the vector containing the power of each pilot, such that $p_k = |d_k|^2$ and $\mathbf{P} = \mathrm{diag}(\mathbf{p})$. One can note that the ML estimator coincides here with the weighted LS estimator. The MSE is given by

$$\mathrm{MSE}_{\mathrm{ML}}(\mathbf{P}) = \sigma^2 \mathrm{tr}\left[\left(\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\boldsymbol{\Sigma}\right)^{-1}\right].$$

This expression only depends on the power of the transmitted pilots $\mathbf{p}$ and not on their phase, which can be appropriately chosen for other purposes such as e.g., peak-to-average-power-ratio (PAPR) reduction [16].

B. MMSE channel estimator

We here consider that $\mathbf{h}$ follows a zero mean Gaussian distribution with correlation matrix denoted by $\mathbf{C}_h$, i.e. $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_h) \in \mathbb{C}^{L_h\times 1}$. The MMSE estimate $\hat{\mathbf{h}}_{\mathrm{MMSE}}$ is the one that minimizes $\mathbb{E}(\|\mathbf{h} - \hat{\mathbf{h}}_{\mathrm{MMSE}}\|^2)$. It can be shown to be equal to [15]

$$\begin{aligned}\hat{\mathbf{h}}_{\mathrm{MMSE}} &= \mathbb{E}(\mathbf{h}|\mathbf{y}) \\ &= \left(\mathbf{C}_h^{-1} + \frac{1}{\sigma^2}\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\boldsymbol{\Sigma}\right)^{-1}\frac{1}{\sigma^2}\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{D}^H\mathbf{y}\end{aligned}$$

and the corresponding MSE is

$$\mathrm{MSE}_{\mathrm{MMSE}}(\mathbf{P}) = \sigma^2 \mathrm{tr}\left[\left(\sigma^2\mathbf{C}_h^{-1} + \boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\boldsymbol{\Sigma}\right)^{-1}\right]. \quad (1)$$

One can check that $\mathrm{MSE}_{\mathrm{MMSE}}(\mathbf{P}) \leq \mathrm{MSE}_{\mathrm{ML}}(\mathbf{P})$ and they will eventually converge at high signal-to-noise ratio (SNR), i.e. when $\sigma^2 \to 0$.

III. Optimality Condition for Channel Estimation

This section aims at deriving a sufficient and necessary condition for pilot allocations that solve the following pilot allocation problem subject to a training power constraint:

$$\min_{\mathbf{P}} \quad \mathrm{MSE}_{\mathrm{ML/MMSE}}(\mathbf{P}) \quad \text{s.t.} \quad \mathrm{tr}\left[\mathbf{P}\right] = P_T. \quad (2)$$

First, using the fact that the function $f(\mathbf{A}) = tr\left[\mathbf{A}^{-1}\right]$ is a convex function for Hermitian positive definite matrices $\mathbf{A}$ [17, Problem 3.18], it is trivial to check that $\mathrm{MSE}_{\mathrm{MMSE}}(\mathbf{P})$ (and hence $\mathrm{MSE}_{\mathrm{ML}}(\mathbf{P})$ since it is a special case of the latter) is a convex function of the power allocation $\mathbf{P}$. One can then find the optimality conditions of (2) by using the well known KKT conditions [17]. We first form the Lagrangian that includes the training power and the positiveness of the pilot powers constraints,

$$\begin{aligned}L(\mathbf{P}, \mu, \alpha_1, \ldots, \alpha_M) = \mathrm{tr}&\left[\left(\mathbf{C}_h^{-1} + \frac{1}{\sigma^2}\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\boldsymbol{\Sigma}\right)^{-1}\right] \\ &+ \mu\left(\mathrm{tr}\left[\mathbf{P}\right] - P_T\right) - \sum_{k=1}^{M}\alpha_k p_k\end{aligned}$$

Setting the derivative of $L(\mathbf{P}, \mu, \alpha_1, \ldots, \alpha_M)$ with respect to each diagonal element of $\mathbf{P}$ to zero, we find the following conditions

$$\frac{dL}{d[\mathbf{P}]_{kk}} = 0 \quad k = 0, 1, ..., M-1 \quad (3)$$

$$\mu = \mathbf{e}_k^H\frac{1}{\sigma^2}\mathbf{F}\boldsymbol{\Sigma}\left(\mathbf{C}_h^{-1} + \frac{1}{\sigma^2}\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\boldsymbol{\Sigma}\right)^{-2}\boldsymbol{\Sigma}^H\mathbf{F}^H\mathbf{e}_k + \alpha_k$$

where $\mathbf{e}_k$ is the $k+1$-th column of the $M\times M$ identity matrix and the values of $\alpha_k, k = 0, 1, \ldots, M-1$ are fixed by the M equations $\alpha_k p_k = 0, \alpha_k \geq 0$. This set of equations gives the conditions under which a certain allocation $\mathbf{P}$ is optimal, i.e. that the matrix on the right hand side has equal diagonal elements of value μ and the value of μ is set to meet the transmit power constraint.

Optimal values of $\mathbf{P}$ can be efficiently computed directly by minimizing (2) using algorithms for classical convex problem solving. However, this does not give any better intuitive idea of optimal solutions. In the following, we show an alternative necessary and sufficient condition that characterizes the optimal power allocations for the ML case. The condition also

[1] We assume, as usual, that the channel remains invariant in the duration of a multicarrier symbol.

[2] The whiteness of the noise samples makes sense in an OFDM system while this is a stronger assumption in an FBMC system where correlation exists in both time and frequency [4].

holds for the MMSE case under the (common) assumption that the channel taps are uncorrelated, i.e. $\mathbf{C}_h$ is a diagonal matrix with elements $\mathbf{C}_h = \text{diag}(\lambda_h^1, \dots, \lambda_h^{L_h})$.

Proposition III.1. *Under the previous assumptions and non correlation of the channel taps, any pilot allocation $\mathbf{p} \in \mathbb{R}_+^{M\times 1}$ is optimal in the sense of the minimum* MSE *for the ML and the MMSE estimators under a training power constraint if and only if (iff) $\mathbf{p}$ satisfies*

$$\sqrt{M}\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{p} = \begin{pmatrix} P_T \\ \mathbf{0} \end{pmatrix}. \tag{4}$$

Proof: The matrix $\mathbf{F}^H\mathbf{P}\mathbf{F}$ is circulant and has thus equal diagonal elements that we denote by x. Given the training power constraint, the value of x is independent of the structure of $\mathbf{P}$ and always equals

$$\text{tr}\left[\mathbf{F}^H\mathbf{P}\mathbf{F}\right] = Mx$$
$$x = \frac{P_T}{M}$$

where we used the cyclic trace property and the fact that $\text{tr}[\mathbf{P}] = P_T$. The matrix $\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\mathbf{\Sigma}$ is the upper left $L_h \times L_h$ submatrix of $\mathbf{F}^H\mathbf{P}\mathbf{F}$ and therefore has diagonal elements equal to x. Using the fact that for a positive definite matrix $\mathbf{A}$, the following inequality [15, p. 65] holds,

$$\text{tr}\left(\mathbf{A}^{-1}\right) \geq \sum_{i=1}^{L_h} a_{ii}^{-1} \tag{5}$$

where a_{ii} is the i-th diagonal element of $\mathbf{A}$ and equality holds iff $\mathbf{A}$ is diagonal, (1) can be lower bounded by

$$\text{MSE}_{\text{MMSE}}(\mathbf{P}) \geq \sum_{l=1}^{L_h}\left(\frac{1}{\lambda_h^l} + \frac{1}{\sigma^2}\frac{P_T}{M}\right)^{-1} \tag{6}$$

where equality holds iff $\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\mathbf{\Sigma}$ is diagonal. This condition can be rewritten as

$$[\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\mathbf{\Sigma}]_{l,m} = \frac{1}{M}\sum_{k=0}^{M-1} p_k e^{j\frac{2\pi}{M}k(l-m)} = \begin{cases} \frac{P_T}{M} & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases}$$

where $l = 0, 1, \dots, L_h - 1$ and $m = 0, 1, \dots, L_h - 1$. $\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\mathbf{\Sigma}$ is a Toeplitz Hermitian matrix. Then, it is sufficient to impose that its first column is 0 except for its first entry, i.e.

$$\sum_{k=0}^{M-1} p_k e^{j\frac{2\pi}{M}kl} = 0 \quad \forall l = 1, 2, \dots, L_h - 1$$

which means that the inverse Fourier transform of the power allocation should be zero at indexes between 1 and $L_h - 1$ or in matrix form,

$$\sqrt{M}\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{p} = \begin{pmatrix} P_T \\ \mathbf{0} \end{pmatrix}.$$

To conclude, if $\mathbf{p}$ satisfies (4), $\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\mathbf{\Sigma}$ is diagonal and reaches the lower bound in (6) and hence, $\mathbf{p}$ is optimal. In the other direction, if $\mathbf{p}$ is optimal, the lower bound in (6) should be satisfied and $\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{P}\mathbf{F}\mathbf{\Sigma}$ has to be diagonal (by (5)), which is achieved only if (4) is satisfied. The previous derivations can be particularized to the case of the ML estimator setting $\mathbf{C}_h^{-1} = \mathbf{0}$ and the same result holds. This concludes the proof. ■

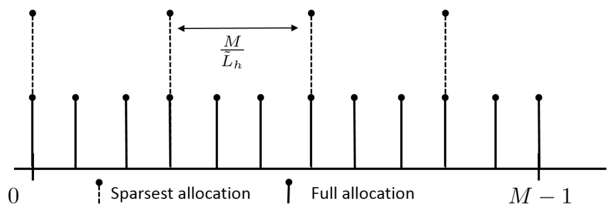

Fig. 1. Two examples of optimal allocation: equispaced and equipowered allocation and full equipowered allocation.

The result of Proposition III.1 and the following remarks can be related to the work in [16] which takes also the CP energy and PAPR into account. However, the approach in [16] is only concerned with the LS estimator whereas here the MMSE case is also addressed. Some remarks related to Proposition III.1:

- It is straightforward to see that condition (4) will satisfy (3) if particularized to the ML case ($\mathbf{C}_h^{-1} = \mathbf{0}$) and to the MMSE case with non correlated taps ($\mathbf{C}_h^{-1} = \text{diag}(\frac{1}{\lambda_h^1}, \dots, \frac{1}{\lambda_h^{L_h}})$).
- If $\mathbf{C}_h$ is diagonal, the optimal pilot allocation is independent of the power delay profile (PDP) $\lambda_h^1, \dots, \lambda_h^{L_h}$ and the noise level σ^2.
- Condition (4) imposes that the inverse Fourier transform (IFT) of the pilot allocation should only have zero coefficients between indexes 1 and $L_h - 1$ while the coefficient at index 0 represents the training power. This can somehow be seen as a requirement that $\mathbf{p}$ should not vary too slowly over the subarriers, except for the average value which represents the total training power. For instance, adding a cosine wave of frequency $\frac{2\pi l}{M}$ (and of amplitude such that the power allocation remains positive) to an optimal allocation will not affect the optimality if $l > L_h - 1$. Moreover, every optimal pilot allocation can be cyclically rotated arbitrarily in the frequency domain since this would simply correspond to multiplying by a complex exponential in the other domain, not affecting condition (4).
- There may be an infinite number of pilot allocations depending on L_h and M. Two classical allocations (see Fig. 1) are first the sparsest equispaced and equipowered allocation. Define $\tilde{L}_h$ as the smallest integer that divides M and such that $\tilde{L}_h \geq L_h$. Then, the sparsest allocation is given by $p_k = \frac{P_T}{\tilde{L}_h}, \forall k = 0, \frac{M}{\tilde{L}_h}, \dots, (\tilde{L}_h - 1)\frac{M}{\tilde{L}_h}$ which is optimal since its IFT will have all elements at indexes smaller than $\tilde{L}_h$ equal to 0 (with $\tilde{L}_h \geq L_h$) except for the first one. A second is the full equipowered pilot allocation, i.e. $p_k = \frac{P_T}{M}, \forall k = 0, 1, \dots, M - 1$ since its IFT is a delta at 0. Furthermore, let us denote the two last power allocations $\mathbf{p}_1$ and $\mathbf{p}_2 \in \mathbb{R}_+^{M\times 1}$ respectively. Then, the convex combination $\mathbf{p}_3 = 0.5(\mathbf{p}_1 + \mathbf{p}_2)$ is an optimal allocation too.

IV. Application in The Virtual Subcarriers Problem

In typical systems, a frequency mask should be respected imposing a limited out-of-band radiation. To ensure that the spectrum respects the mask, it is usual to keep a number of subcarriers inactive at the edges of the band. Furthermore, in LTE-like systems, the time-frequency resources are "boxed" into different physical channels that may be transmitted simultaneously [18]. All of this imposes constraints on the possible pilot locations. Due to those multiple constraints, an equispaced pilot allocation may not always be possible, which complicates the problem [10], [11]. We here show that thanks to our generalized condition, optimality can still be reached in certain situations, which means that we can reach the same performance as if an equispaced or full equipowered allocation was possible.

Let us consider a system where no pilots can be transmitted at certain frequencies, i.e. $p_k = 0, \forall k \in \mathcal{K}$ where $\mathcal{K}$ is the set of inactive subcarriers. The number of remaining pilot positions is denoted by $N = M - |\mathcal{K}|$ and $\mathbf{p}_f \in \mathbb{R}^{N\times 1}$ is a vector made of the powers transmitted at those available subcarriers. Taking the virtual subcarriers into consideration, the optimality condition can then be rewritten as

$$\mathbf{A}\mathbf{p}_f = \mathbf{b} \quad \text{s.t.} \quad \mathbf{p}_f \in \mathbb{R}_+^{N\times 1} \tag{7}$$

where $\mathbf{A} = \sqrt{M}\mathbf{\Sigma}^H\mathbf{F}^H\mathbf{S}$, $\mathbf{b} = \begin{pmatrix} P_T \\ \mathbf{0} \end{pmatrix}$ and $\mathbf{S} \in \mathbb{R}^{M\times N}$ is formed by an identity matrix $\mathbf{I}_M$ where we removed the $|\mathcal{K}|$ columns corresponding to the virtual subcarriers frequencies. The constraint ensures that any element of $\mathbf{p}_f$ is real and greater than or equal to 0. Note that too many or badly placed virtual subcarriers may increase the ill-conditionning of the channel sensing [12]. In that case, there may not exist a solution to (7), i.e. an allocation that can still reach the optimal MSE. Finding a possible solution to (7) can be seen as a classical feasibility problem in the linear programming literature [19, Chap. 10]. Solving (7) is similar to finding an initial feasible point of a linear program. This can be solved efficiently in polynomial time outputting either a feasible point or the infeasibility of the problem.

V. Simulation Results

Fig. 2 shows an example of the virtual subcarriers problem. The simulation parameters are $M = 128$ subcarriers, a channel length $L_h = 10$ and the following sets of subcarriers, [0,5], [22,30], [43,48], [57,65], [123,M-1] are inactive, i.e. they cannot be used for training. The feasibility problem of (7) was solved using the Matlab function *linprog* and the solution is plotted in Fig. 2.

Following the IEEE 802.11a standard [20], the number of subcarriers is fixed to $M = 64$, 12 of which are inactive due to the DC frequency and the edge guard bands, i.e. $\mathcal{K} = \{0\} \bigcup [27, 37]$. With these constraints, the problem in (7) remains feasible for a channel length up to $L_h = 5$, which is shown in Fig. 3. Note that no equispaced and equipowered allocation could reach the same result since in that case, $\tilde{L}_h = 8$, which would require a pilot spacing of $\frac{M}{\tilde{L}_h} = 8$ subcarriers, not possible with the given frequency mask.

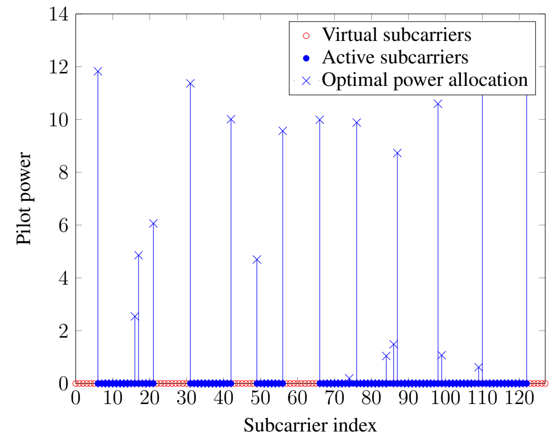

Fig. 2. Due to system constraints, a large part of the band can not be allocated for training. However, there is still a feasible optimal allocation.

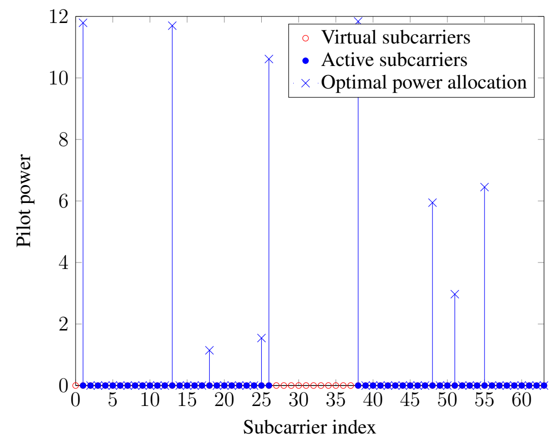

Fig. 3. Application of the generalized optimality condition to the IEEE802.11a standard frequency mask requirements.

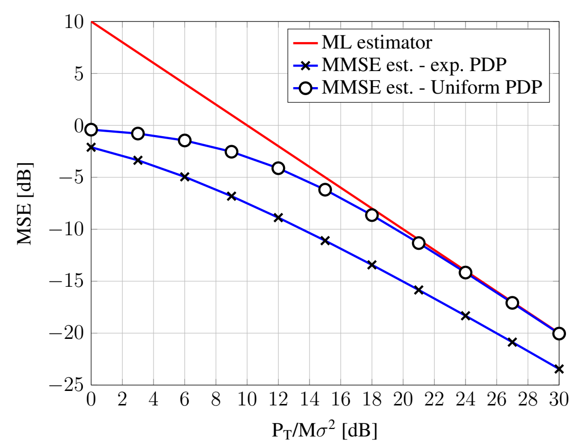

Fig. 4. Comparison of the ML and MMSE estimators with optimal pilot allocation and uncorrelated channel taps.

To model the correlation matrices of the channel, we first define the matrix $\mathbf{\Lambda}_h = \text{diag}(\lambda_h^1, \dots, \lambda_h^{L_h})$. Second, we define the matrix $\mathbf{R}$ whose (i, j) element equals $[\mathbf{R}]_{i,j} = r^{|i-j|}$ with $r < 1$ the correlation coefficient.

In Fig. 4, the performance of the optimal pilot allocation with the ML and MMSE estimators of Fig. 4 is plotted. For both estimators, the power allocation is optimized so as to

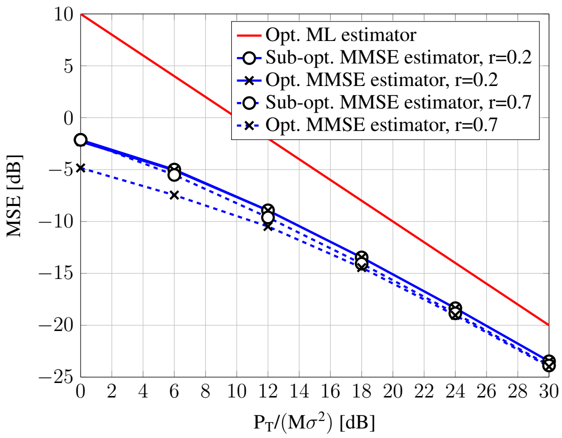

Fig. 5. Comparison of optimal allocation for ML and MMSE estimators, with correlated channel.

meet (4). A uniform PDP, i.e. $\lambda_h^l = \frac{1}{L_h}$, and an uncorrelated exponentially decaying delay profile is considered, i.e. $\lambda_h^l = \alpha 10^{-\frac{2(l-1)}{L_h}}$ such that $\text{tr}\left[\mathbf{\Lambda}_h\right] = 1$ and $\mathbf{C}_h = \mathbf{\Lambda}_h$ in both cases. As expected, the MMSE estimator outperforms the ML one and the gap decreases at high SNR. The performance gap is larger for the exponentially decaying PDP than for the uniform PDP. This comes from the larger regularization effect of $\mathbf{C}_h^{-1}$ in (1) for the exponentially decaying PDP.

Fig. 5 refers to the case of correlated channel. In that case, condition (4) is only sufficient in the ML case but not anymore for the MMSE estimator and should be replaced by condition (3). However, solving (3) requires an iterative convex optimization algorithm. We then aim at checking how far from optimality is a preamble that only meets (4), as the correlation of the channel increases. The channel covariance is defined as $\mathbf{C}_h = \mathbf{R}^{1/2}\mathbf{\Lambda}_h\mathbf{R}^{1/2}$ with $\mathbf{\Lambda}_h$ defined as the exp. decaying PDP in Fig. 4. In Fig. 5, for $r = 0.2$, one can see that the gap between the true optimal pilot allocation (which meets (3)) and the sub-optimal which only meets (4) is negligible. For a higher correlation ($r = 0.7$), a gap at low SNR is obserbed, which reduces very quickly as the SNR increases. This suggests that even in correlated channels, an allocation that only meets (4) is very close to optimality, especially at high SNR values.

VI. Conclusion

In summary, this paper addressed optimal pilot allocation for multicarrier systems for channel estimation under the ML and MMSE criteria. The optimality conditions for a general channel correlation matrix are given and further simplified for uncorrelated channel taps. The obtained condition generalizes the commonly adopted equispaced pilot configuration, allowing for a much wider family of optimal allocations. This proves useful in practice, where null (virtual) subcarriers are present as well. The reported simulation results demonstrated the value of the optimality condition in such a context. This paper was concerned with single antenna systems. Extending these results to the multiple antennas case is a possible subject of future research.

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